

Lecture 5: 20 August

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Now that we have some integration tools at our disposal, we can consider some integrals for moments and statistics of the MP distribution.

Moments of the MP distribution

Proposition 5.1 *For the standard MP distribution F_y with index $y > 0$ and $\sigma^2 = 1$, it holds for any analytic function f on a domain containing interval $[a, b] = [(1 \pm \sqrt{y})^2]$,*

$$\int f(x) dF_y(x) = -\frac{1}{4\pi i} \oint_{|z|=1} \frac{f(|1 + \sqrt{y}z|^2)(1 - z^2)^2}{z^2(1 + \sqrt{y}z)(z + \sqrt{y})} dz.$$

Proof: (We will prove a stronger case later.) ■

Let's look at some applications.

Example 1: Logarithms of eigenvalues are often used in multivariate analysis. Set

$$f(x) = \log(x).$$

Assume $0 < y < 1$ so that we don't get zero eigenvalues.

$$\begin{aligned} \int \log(x) dF_y(x) &= -\frac{1}{4\pi i} \oint_{|z|=1} \frac{\log(|1 + \sqrt{y}z|^2)(1 - z^2)^2}{z^2(1 + \sqrt{y}z)(z + \sqrt{y})} dz \\ &= -\frac{1}{4\pi i} \oint_{|z|=1} \frac{\log(1 + \sqrt{y}z)(1 - z^2)^2}{z^2(1 + \sqrt{y}z)(z + \sqrt{y}z)} dz - \frac{1}{4\pi i} \oint_{|z|=1} \frac{\log(1 + \sqrt{y}\bar{z})(1 - z^2)^2}{z^2(1 + \sqrt{y}z)(z + \sqrt{y}\bar{z})} dz \\ &= I_1 + I_2 \end{aligned} \quad (5.1)$$

Note $z \in \mathbb{C}$:

$$\begin{aligned} |1 + \sqrt{y}z|^2 &= (1 + \sqrt{y}z)\overline{(1 + \sqrt{y}z)} \\ &= (1 + \sqrt{y}z)(1 + \sqrt{y}\bar{z}) \end{aligned} \quad (5.2)$$

Q: When do these integrals have singularities?

There is one at the point $z = 0$, due to the $\frac{1}{z^2}$ term (“order 2 pole”).

Another at $z = -\sqrt{y}$.

Both within contour $|z| = 1$.

By Cauchy residue theorem,

$$\int_C f(z)dz = 2\pi i \sum_{a \in C} \text{Res}(f; a)$$

where a are points of singularity.

We need to find the residues at the points $z = 0$ and $z = -\sqrt{y}$. We could expand and find the Laurent series but there is an easier way.

Proposition 5.2 *If f has a pole of order $n \geq 1$ at a . Define $g(z) = (z - a)^n f(z)$ then*

$$\text{Res}(f; a) = \frac{1}{(n-1)!} \lim_{z \rightarrow a} g^{(n-1)}(z).$$

Proof: Remember that the residue is the term c_{-1} in the Laurent series expansion of $f(z)$:

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \cdots + \frac{c_{-1}}{z-a} + a_0 + \cdots$$

so $g(z) = c_{-n} + \cdots + c_{-1}(z-a)^{n-1} + c_0(z-a)^n + \cdots$ and $g^{(n-1)}(z) = (n-1)!c_{-1} + n(n-1) \cdots 2 \cdot c_0(z-a) + \cdots$

Hence, $\lim_{z \rightarrow a} g^{(n-1)}(z) = g^{(n-1)}(a) = (n-1)!c_{-1}$. ■

Applying this proposition at $a = -\sqrt{y}$:

$$\lim_{z \rightarrow -\sqrt{y}} \frac{\log(1 + \sqrt{y}z)(1 - z^2)^2}{z^2(1 + \sqrt{y}z)(z + \sqrt{y})} (z - (-\sqrt{y})) = \frac{\log(1 - y)(1 - y)^2}{y(1 - y)} = \log(1 - y) \frac{(1 - y)}{y}.$$

The singularity at $a = 0$ is of order 2, so

$$g(z) = z^2 \frac{\log(1 + \sqrt{y}z)(1 - z^2)^2}{z^2(1 + \sqrt{y}z)(z + \sqrt{y})} = \frac{\log(1 + \sqrt{y}z)(1 - z^2)^2}{(1 + \sqrt{y}z)(z + \sqrt{y})}$$

$$g'(z) = \frac{\sqrt{y}(1 - z^2)^2}{(\sqrt{y} + z)(1 + \sqrt{y}z)^2} - \frac{4z(1 - z^2)\log(1 + \sqrt{y}z)}{(\sqrt{y} + z)(1 + \sqrt{y}z)} - \frac{\sqrt{y}(1 - z^2)^2 \log(1 + \sqrt{y}z)}{(\sqrt{y} + z)(1 + \sqrt{y}z)^2} - \frac{(1 - z^2)^2 \log(1 + \sqrt{y}z)}{(\sqrt{y} + z)^2(1 + \sqrt{y}z)}$$

$$g'(0) = \frac{\sqrt{y}}{\sqrt{y}} - 0 - 0 - 0 = 1.$$

So by the residue theorem

$$\begin{aligned}
I_1 &= -\frac{1}{4\pi i} \left[2\pi \cdot \left(\log(1-y) \frac{(1-y)}{y} + 1 \right) \right] \\
&= -\frac{1}{2} \left(\log(1-y) \frac{(1-y)}{y} + 1 \right)
\end{aligned} \tag{5.3}$$

Now for I_2 we have

$$I_2 = -\frac{1}{4\pi i} \oint_{|z|=1} \frac{\log(1 + \sqrt{y}\bar{z})(1 - z^2)^2}{z^2(1 + \sqrt{y}z)(z + \sqrt{y})} dz.$$

We shall make the change of variable $s = \bar{z}$ and notice that since $|z| = 1$, we have

$$\frac{1}{z} = \frac{1}{e^{i\theta}} = e^{-i\theta} = \bar{z}$$

So

$$I_2 = -\frac{1}{4\pi i} \oint_{|s|=1} \frac{\log(1 + \sqrt{y}s) \left(1 - \left(\frac{1}{s}\right)^2\right)^2}{\left(\frac{1}{s}\right)^2 (1 + \sqrt{y}\left(\frac{1}{s}\right)) \left(\frac{1}{s} + \sqrt{y}\right)} \left(-\frac{1}{s^2}\right) ds.$$

and this can be shown to be

$$I_2 = I_1.$$

hence, $I = -\log(1-y) \frac{(1-y)}{y} - 1$.

Example 2: We can calculate the mean of the MP distribution. For all $y > 0$,

$$\int x dF_y(x) = 1.$$

Proof: This can be shown in the same way as Example 1. ■

For any monomial function $f(x) = x^k$ for $k \in \mathbb{N}$, the residue approach becomes tedious. There is a direct proof as well. (Bai & Silverstein 2010; Lemma 3.1).

Proposition 5.3 *The moments of the standard MP distribution*

$$\beta_k := \int x^k dF_y(x) = \sum_{r=0}^{k-1} \frac{1}{r+1} \binom{k}{r} \binom{k-1}{r} y^r$$

Proof:

$$P_y(x) = \begin{cases} \frac{1}{2\pi xy} \sqrt{(b-x)(x-a)}, & a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$$

$$a = (1 - \sqrt{y})^2, b = (1 + \sqrt{y})^2.$$

$$\beta_k = \frac{1}{2\pi y} \int_a^b x^{k-1} \sqrt{(b-x)(x-a)} dx.$$

$$\text{Set } x = 1 + y + z, dx = dz, x = a \implies (1 - \sqrt{y})^2 = 1 + y + z, z = (1 - \sqrt{y})^2 - 1 - y = -2\sqrt{y}.$$

$$(b-x)(x-a) = (2\sqrt{y}-z)(2\sqrt{y}+z) = (4y-z^2), x = b \implies z = 2\sqrt{y}.$$

So

$$\beta_k = \frac{1}{2\pi y} \int_{-2\sqrt{y}}^{2\sqrt{y}} (1+y+z)^{k-1} \sqrt{4y-z^2} dz.$$

$$\text{Recall } (a+b)^\alpha = \sum_{k=0}^\alpha \binom{\alpha}{k} a^k b^{\alpha-k}, \binom{\alpha}{k} := \frac{\alpha!}{k!(\alpha-k)!}$$

$$\text{So here } a = 1+y, b = z, \alpha = k-1.$$

$$\begin{aligned} \beta_k &= \frac{1}{2\pi y} \int_{-2\sqrt{y}}^{2\sqrt{y}} \sum_{l=0}^{k-1} \binom{k-1}{l} (1+y)^{k-1-l} z^l \sqrt{4y-z^2} dz \\ &= \frac{1}{2\pi y} \sum_{l=0}^{k-1} \binom{k-1}{l} (1+y)^{k-1-l} \int_{-2\sqrt{y}}^{2\sqrt{y}} z^l \sqrt{4y-z^2} dz. \end{aligned} \tag{5.4}$$

$$\text{We continue to set } z = 2\sqrt{y}u, dz = 2\sqrt{y}du \implies 4y - z^2 = 1 - u^2.$$

$$z = -2\sqrt{y} \implies u = -1; z = 2\sqrt{y} \implies u = 1.$$

$$\begin{aligned} \beta_k &= \frac{1}{2\pi y} \sum_{l=0}^{k-1} \binom{k-1}{l} (1+y)^{k-1-l} 2\sqrt{y} (2\sqrt{y})^l \int_{-1}^1 u^l \sqrt{1-u^2} du \\ &= \frac{1}{2\pi y} \sum_{l=0}^{k-1} \binom{k-1}{l} (1+y)^{k-1-l} (4y)^{\frac{l+1}{2}} \int_{-1}^1 u^l \sqrt{1-u^2} du \\ &= \frac{1}{2\pi y} \sum_{l=0}^{\frac{k-1}{2}} \binom{k-1}{2l} (1+y)^{k-1-2l} (4y)^{l+1} \int_{-1}^1 u^{2l} \sqrt{1-u^2} du \end{aligned} \tag{5.5}$$

$$\text{Set } u = \sqrt{w}, du = \frac{1}{2} \frac{1}{\sqrt{w}} dw, u = -1, w = 1.$$

$$\beta_k = \frac{1}{2\pi u} \sum_{l=0}^{\frac{k-1}{2}} \binom{k-1}{2l} (1+y)^{k-1-2l} (4y)^{l+1} \int_0^1 w^{l-\frac{1}{2}} \sqrt{1-w} dw. \tag{5.6}$$

$$\text{As } \Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx, \Gamma(n) := (n-1)!, \Gamma(l + \frac{1}{2}) = \frac{(2l)!}{4^l l!} \sqrt{\pi}.$$

$$\text{As } \int_0^1 w^{l-\frac{1}{2}} \sqrt{1-w} dw = \frac{\sqrt{\pi} \Gamma(l+\frac{1}{2})}{2\Gamma(2+l)} \text{ if } l > \frac{1}{2}. \text{ We continue:}$$

$$\begin{aligned}
\beta_k &= \sum_{l=0}^{[(k-1)/2]} \frac{1}{2\pi y} \frac{(k-1)!}{(2l)!((k-1)-2l)!} \frac{\sqrt{\pi}}{2} \frac{(2l)! \sqrt{\pi}}{4^l l! (l+1)!} 4^{l+1} y^{l+1} (1+y)^{k-1-2l} \\
&= \sum_{l=0}^{[(k-1)/2]} \frac{(k-1)!}{l!(l+1)!(k-1-2l)!} y^l (1+y)^{k-1-2l}
\end{aligned} \tag{5.7}$$

As $(1+y)^{k-1-2l} = \sum_{s=0}^{k-1-2l} \binom{k-1-2l}{s} y^s = \sum_{s=0}^{k-1-2l} \frac{(k-1-2l)!}{s!(k-1-2l-s)!} y^s$, continue:

$$\begin{aligned}
\beta_k &= \sum_{l=0}^{[(k-1)/2]} \frac{(k-1)!}{l!(l+1)!(k-1-2l)!} y^l \sum_{s=0}^{k-1-2l} \frac{(k-1-2l)!}{s!(k-1-2l-s)!} y^s \\
&= \sum_{l=0}^{[(k-1)/2]} \sum_{s=0}^{k-1-2l} \frac{(k-1)!}{l!(l+1)!s!(k-1-2l-s)!} y^{l+s}
\end{aligned} \tag{5.8}$$

Substitute $r = l + s, s = 0 \implies r = l, s = k-1-2l \implies r = k-1-l$. Continue:

$$\begin{aligned}
\beta_k &= \sum_{l=0}^{[(k-1)/2]} \sum_{r=l}^{k-1-l} \frac{(k-1)!}{l!(l+1)!s!(k-1-r-l)!} y^r \\
&= \frac{1}{k} \sum_{r=0}^{k-1} \binom{k}{r} y^r \sum_{l=0}^{\min(r, k-1-r)} \binom{r}{l} \binom{k-r}{k-r-l-1} \\
&= \frac{1}{k} \sum_{r=0}^{k-1} \binom{k}{r} \binom{k}{r+1} y^r = \sum_{r=0}^{k-1} \frac{1}{r+1} \binom{k}{r} \binom{k-1}{r} y^r.
\end{aligned} \tag{5.9}$$

■

Fubini theorem for sequences: If $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_{nm}| < \infty$ then

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}.$$

How did I use that?

$$\begin{aligned}
\sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \sum_{r=l}^{k-1-l} \frac{(k-1)!}{l!(l+1)!(r-l)!(k-1-r-l)!} y^r &= \sum_{l=0}^{\infty} \mathbf{1}_{(l \leq \lfloor (k-1)/2 \rfloor)} \sum_{r=0}^{\infty} \mathbf{1}_{(r \geq l)} \mathbf{1}_{(r \leq k-1-l)} \\
&= \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \mathbf{1}_{(l \leq \lfloor (k-1)/2 \rfloor)} \mathbf{1}_{(l \leq r)} \mathbf{1}_{(l \leq k-1-r)} \mathbf{1}_{(r \leq k-1)} \\
&= \sum_{r=0}^{k-1} \sum_{l=0}^{\min(r, k-1-r)} \square \\
\square &= \frac{k!}{r!(k-r)!} y^r \frac{(k-1)!r!(k-r)!}{k!l!(l+1)!(r-l)!(k-1-r-l)!} \\
&= \binom{k}{r} y^r \frac{1}{k} \cdot \frac{r!}{l!(r-l)!} \cdot \frac{(k-r)!}{(l+1)!(k-1-r-l)!} \\
&\text{note: } k-r-(k-1-r-l) = l+1 \\
&= \frac{1}{k} \binom{k}{r} y^r \binom{r}{l} \binom{k-r}{k-1-r-l}
\end{aligned} \tag{5.10}$$

So we plug \square back in to have

$$= \sum_{r=0}^{k-1} \frac{1}{k} \binom{k}{r} y^r \sum_{l=0}^{\min(r, k-1-r)} \binom{r}{l} \binom{k-r}{k-1-r-l}.$$

Generalized MP distribution

Previously, we've seen the case where the population covariance matrix has the simple form $\Sigma = \sigma^2 \mathbf{I}_p$.

We can consider a slightly more general case if we make the assumption that the observation vectors $\{y_k\}_{1 \leq k \leq n}$ can be represented as

$$y_k := \sum_{k=1}^{1/2} x_k, x_k \text{ iid}, \sum \text{ nonnegative square root of } \Sigma.$$

This gives the associated covariance matrix

$$\begin{aligned}
\tilde{\mathbf{B}}_n &= \frac{1}{n} \sum_{k=1}^n \mathbf{y}_k \mathbf{y}_k^* = \sum_{k=1}^{1/2} \left(\frac{1}{n} \sum_{k=1}^n \mathbf{x}_k \mathbf{x}_k^* \right)^{1/2} \\
&= \sum_{k=1}^{1/2} \mathbf{S}_n \sum_{k=1}^{1/2}
\end{aligned} \tag{5.11}$$

\mathbf{S}_n is the sample covariance matrix with iid components.

The eigenvalues of $\tilde{\mathbf{B}}_n$ are the same as $\mathbf{S}_n \Sigma$.

The following result holds for $\mathbf{B}_n = \mathbf{S}_n \mathbf{T}_n$ for general nonnegative definite matrix \mathbf{T}_n . ($\mathbf{T}_n = \Sigma$ is a special case.)

Theorem 5.4 Let \mathbf{S}_n be the sample covariance matrix $\mathbf{S}_n = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^*$ with iid components and let (\mathbf{T}_n) be a sequence of nonnegative definite Hermitian matrices of size $p \times p$.

Define $\mathbf{B}_n = \mathbf{S}_n \mathbf{T}_n$ and assume:

1. The entries (x_{jk}) of the data matrix $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ are iid with mean zero and variance $\mathbf{1}$.
2. The data dimension to sample size ratio $p/n \rightarrow y > 0$ as $n \rightarrow \infty$.
3. The sequence (\mathbf{T}_n) is either deterministic or independent of (\mathbf{S}_n) .
4. Almost surely, the sequence $(H_n = F^{\mathbf{T}_n})$ of the ESD of (\mathbf{T}_n) weakly converges to a non-random probability measure H .

Then, almost surely, F (???) weakly converges to a non-random probability measure $F_{y,H}$. Its Stieltjes transform is given by

$$S(z) = \int \frac{1}{t(1-y-yzs(z))-z} dH(t), z \in \mathbb{C}^+ \quad (5.12)$$

Notice that the ST of $F_{y,H}$ is implicitly defined. It can be shown that a unique solution exists, but, unfortunately, no closed-form solution exists. (see Silverstein & Combettes 1992)

There is a better way to represent the ST of $F_{y,H}$. Consider for \mathbf{B}_n a companion matrix.

$$\underline{\mathbf{B}}_n = \frac{1}{n} \mathbf{X}^* \mathbf{T} \mathbf{X} \quad \text{size } n \times n$$

Both matrices share the same nonzero eigenvalues so their ESD satisfy

$$nF(???) - pF^{\mathbf{B}_n} = (n-p)\mathbf{S}_0(???)$$

Note: Given two matrices $\mathbf{A}_{p \times q}$ and $\mathbf{B}_{q \times p}$ where $p \geq q$, eigenvalues of \mathbf{AB} is that of \mathbf{BA} augmented by $p-q$ zeros.

$$\mathbf{B}_n = \mathbf{S}_n \mathbf{T}_n = \frac{1}{n} \mathbf{X} \mathbf{X}^* \mathbf{T}_n, \underline{\mathbf{B}}_n = \frac{1}{n} \mathbf{X}^* \mathbf{T} \mathbf{X}, \text{ where } \mathbf{X} \text{ is a } p \times n \text{ matrix.}$$

When $p/n \rightarrow y > 0$, $F^{\mathbf{B}_n}$ has limit $F_{c,H}$ if and only if $F^{\underline{\mathbf{B}}_n}$ has limit $\underline{F}_{c,H}$. In this case, the limit satisfies

$$\underline{F}_{c,H} - yF_{c,H} = (1-y)S_0(???)$$

and their ST are related by

$$\underline{s}(z) = -\frac{1-y}{z} + ys(z).$$

Now substituting \underline{s} for s in (5.12) yields

$$\underline{s}(z) = \left(z - y \int \frac{t}{1 + t\underline{s}(z)} dH(t) \right)^{-1}$$

Solving in z gives:

$$z = -\frac{1}{\underline{s}(z)} + y \int \frac{t}{1 + t\underline{s}(z)} dH(t) \quad (5.13)$$

which defines the inverse function of \underline{s} .

- (5.12) is called the Marchenko-Pastur equation and
- (5.13) is the Silverstein equation.

Limiting spectral distribution for Random Fisher matrices

In the univariate case, when we need to test equality between the variances of 2 Gaussian populations, a Fisher statistic of the form S_1^2/S_2^2 is used where S_i^2 are estimators of the unknown variances in the two populations.

The equivalent for the multivariate setting is:

Take two independent samples $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ both from p -dimensional population with iid components and finite second moment.

$$\mathbf{S}_1 = \frac{1}{n_1} \sum_{k=1}^{n_1} \mathbf{x}_k \mathbf{x}_k^*$$

$$\mathbf{S}_2 = \frac{1}{n_2} \sum_{k=1}^{n_2} \mathbf{y}_k \mathbf{y}_k^*$$

Then $\mathbf{F}_n := \mathbf{S}_1 \mathbf{S}_2^{-1}$ is called a Fisher matrix, $\mathbf{n} = (n_1, n_2)$.

(Note: need $p \leq n_2$ so that \mathbf{S}_2 invertible.)

Let $s > 0$ and $0 < t < 1$. The Fisher LSD $F_{s,t}$ is the distribution with density function

$$P_{s,t}(x) = \frac{1-t}{2\pi x(s+tx)} \sqrt{(b-x)(x-a)}, a \leq x \leq b$$

with $a = a(s, t) = \frac{(1-h)^2}{(1-t)^2}$, $b = b(s, t) = \frac{(1+h)^2}{(1-t)^2}$, $h = h(s, t) = (s + t - st)^{1/2}$.

When $s > 1$, $F_{s,t}$ has a mass at $x = 0$ of value $1 - \frac{1}{s}$ with the total mass of the rest of the distribution for $x > 0$ is equal to $1/s$.

The Fisher LSD has many similarities to the standard MP distribution. This is not a coincidence as the MP LSD F_y is the Fisher LSD $F_{y,0}$ (i.e. $s, y = y, 0$)

Also note $t \rightarrow 1$, $a(s, t) \rightarrow \frac{1}{2}(1-s)^2$, $b(s, t) \rightarrow \infty$ ($\text{Supp}(F_{s,t})$ becomes unbounded).

Theorem 5.5 For an analytic function f on a domain containing $[a, b]$ (as above). We have

$$\int_a^b f(x) dF_{s,t}(x) = -\frac{h^2(1-t)}{4\pi i} \oint_{|z|=1} \frac{f\left(\frac{|1+hz|^2}{(1-t)^2}\right) (1-z^2)^2 dz}{z(1+hz)(z+h)(tz+h)(t+hz)}$$

Proof: Using the density $P_{s,t}(x)$.

$$I = \int_a^b f(x) dF_{s,t}(x) = \int_a^b f(x) \frac{1-t}{2\pi x(s+xt)} \sqrt{(x-a)(b-x)} dx$$

Make change of variable $x = \frac{1+h^2+2h\cos\theta}{(1-t)^2}$, $\theta \in (0, \pi)$.

$$\begin{aligned} x-a &= \frac{1+h^2+2h\cos\theta}{(1-t)^2} - \frac{(1-h)^2}{(1-t)^2} = \frac{2h+2h\cos\theta}{(1-t)^2} \\ b-x &= \frac{(1+h)^2}{(1-t)^2} - \frac{1+h^2+2h\cos\theta}{(1-t)^2} = \frac{2h-2h\cos\theta}{(1-t)^2} \\ \sqrt{(x-a)(b-x)} &= \sqrt{\frac{(2h)^2}{(1-t)^4} (1-\cos\theta)(1+\cos\theta)} \\ &= \frac{2h}{(1-t)^2} \sin\theta \\ x=a &\implies \cos\theta = \frac{a(1-t)^2 - (1+h^2)}{2h} = 0 \implies \theta = 0 \\ x=b &\implies \theta = \pi \\ dx &= \frac{-2h\sin\theta}{(1-t)^2} d\theta \end{aligned} \tag{5.14}$$

Hence

$$\begin{aligned} I &= \frac{2h^2(1-t)}{\pi} \int_0^\pi \frac{f\left(\frac{1+h^2+2h\cos\theta}{(1-t)^2}\right) \sin^2\theta d\theta}{(1+h^2+2h\cos\theta)(s(1-t)^2+t(1+h^2+2h\cos\theta))} \\ &= \frac{h^2(1-t)}{\pi} \int_0^{2\pi} \frac{f\left(\frac{1+h^2+2h\cos\theta}{(1-t)^2}\right) \sin^2\theta d\theta}{(1+h^2+2h\cos\theta)(s(1-t)^2+t(1+h^2+2h\cos\theta))} \end{aligned} \tag{5.15}$$

Now let $z = e^{i\theta}$,

$$\begin{aligned} 1+h^2+2h\cos\theta &= |1+hz|^2 \\ \sin\theta &= \frac{z-z^{-1}}{2i} \\ \log(z) = i\theta &\implies \theta = \frac{1}{i} \log(z) \implies d\theta = \frac{1}{iz} dz. \\ I &= -\frac{h^2(1-t)}{4\pi i} \oint_{|z|=1} \frac{f\left(\frac{|1+hz|^2}{(1-t)^2}\right) (1-z^2)^2 dz}{z^3 |1+hz|^2 (s(1-t)^2+t|1+hz|^2)} \end{aligned} \tag{5.16}$$

On $|z| = 1$, we have $|1 + hz|^2 = (1 + hz)(1 + hz^{-1})$.

So expanding denominator and simplifying we have the result. ■

Example: Take $(s, t) = (y, 0)$ and we get the result for MP distribution.

Example 2: The first two moments are

$$\int x dF_{s,t}(x) = \frac{1}{1-t}, \int x^2 dF_{s,t}(x) = \frac{h^2 + 1 - t}{(1-t)^3}.$$

Hence the variance equals $\frac{h^2}{(1-t)^3}$.