STAT7017 Big Data Statistics **Semester 2 2018**

Lecture 5: 20 August

Lecturer: Dr Dale Roberts *Scribes: Rui Qiu*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this class only with the permission of the Instructor.*

Now that we have some integration tools at our disposal, we can consider some integrals fro moments and statistics of the MP distribution.

Moments of the MP distribution

Proposition 5.1 *For the standard MP distribution F_y with index $y > 0$ and $\sigma^2 = 1$, if holds for any analytic function f on a domain containing interval $[a, b] = [(1 \pm \sqrt{y})^2]$,*

$$\int f(x) dF_y(x) = -\frac{1}{4\pi i} \oint_{|z|=1} \frac{f(|1+\sqrt{y}z|^2)(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})} dz.$$

Proof: (We will prove a stronger case later.) ■

Let's look at some applications.

Example 1: Logarithms of eigenvalues are often used in multivariate analysis. Set

$$f(x) = \log(x).$$

Assume $0 < y < 1$ so that we don't get zero eigenvalues.

$$\begin{aligned} \int \log(x) dF_y(x) &= -\frac{1}{4\pi i} \oint_{|z|=1} \frac{\log(|1+\sqrt{y}z|^2)(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})} dz \\ &= -\frac{1}{4\pi i} \oint_{|z|=1} \frac{\log(1+\sqrt{y}z)(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y}z)} dz - \frac{1}{4\pi i} \oint_{|z|=1} \frac{\log(1+\sqrt{y}\bar{z})(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{z})} dz \\ &= I_1 + I_2 \end{aligned} \tag{5.1}$$

Note $z \in \mathbb{C}$:

$$\begin{aligned} |1+\sqrt{y}z|^2 &= (1+\sqrt{y}z)\overline{(1+\sqrt{y}z)} \\ &= (1+\sqrt{y}z)(1+\sqrt{y}\bar{z}) \end{aligned} \tag{5.2}$$

Q: When do these integrals have singularities?

There is one at the point $z = 0$, due to the $\frac{1}{z^2}$ term ("order 2 pole").

Another at $z = -\sqrt{y}$.

Both within contour $|z| = 1$.

By Cauchy residue theorem,

$$\int_C f(z)dz = 2\pi i \sum_{a\in C} Res(f;a)$$

where a are points of singularity.

We need to find the residues at the points $z = 0$ and $z = -\sqrt{y}$. We could expand and find the Laurent series but there is an easier way.

Proposition 5.2 *If f has a pole of order $n \geq 1$ at a. Define $g(z) = (z-a)^n f(z)$ then*

$$Res(f;a) = \frac{1}{(n-1)!}\lim_{z\to a} g^{(n-1)}(z).$$

Proof: Remember that the residue is the term c_{-1} in the Laurent series expansion of $f(z)$:

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \cdots + \frac{c_{-1}}{z-a} + a_0 + \cdots$$

so $g(z) = c_{-n} + \cdots + c_{-1}(z-a)^{n-1} + c_0(z-a)^n + \cdots$ and $g^{(n-1)}(z) = (n-1)!c_{-1} + n(n-1)\cdots 2\cdot c_0(z-a) + \cdots$

Hence, $\lim_{z\to a} g^{(n-1)}(z) = g^{(n-1)}(a) = (n-1)!c_{-1}$. ■

Applying this proposition at $a = -\sqrt{y}$:

$$\lim_{z\to-\sqrt{y}} \frac{\log(1+\sqrt{y}z)(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})}(z-(-\sqrt{y})) = \frac{\log(1-y)(1-y)^2}{y(1-y)} = \log(1-y)\frac{(1-y)}{y}.$$

The singularity at $a = 0$ is of order 2, so

$$g(z) = z^2\frac{\log(1+\sqrt{y}z)(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})} = \frac{\log(1+\sqrt{y}z)(1-z^2)^2}{(1+\sqrt{y}z)(z+\sqrt{y})}$$

$$g'(z) = \frac{\sqrt{y}(1-z^2)^2}{(\sqrt{y}+z)(1+\sqrt{y}z)^2} - \frac{4z(1-z^2)\log(1+\sqrt{y}z)}{(\sqrt{y}+z)(1+\sqrt{y}z)} - \frac{\sqrt{y}(1-z^2)^2\log(1+\sqrt{y}z)}{(\sqrt{y}+z)(1+\sqrt{y}z)^2} - \frac{(1-z^2)^2\log(1+\sqrt{y}z)}{(\sqrt{y}+z)^2(1+\sqrt{y}z)}$$

$$g'(0) = \frac{\sqrt{y}}{\sqrt{y}} - 0 - 0 - 0 = 1.$$

So by the residue theorem

$$
\begin{aligned}
I_1 &= -\frac{1}{4\pi i}\left[2\pi \cdot \left(\log(1-y)\frac{(1-y)}{y} + 1\right)\right] \\
&= -\frac{1}{2}\left(\log(1-y)\frac{(1-y)}{y} + 1\right)
\end{aligned}
\tag{5.3}
$$

Now for I_2 we have

$$I_2 = -\frac{1}{4\pi i}\oint_{|z|=1}\frac{\log(1+\sqrt{y}\bar{z})(1-z^2)^2}{z^2(1+\sqrt{y}z)(z+\sqrt{y})}dz.$$

We shall make the change of variable $s = \bar{z}$ and notice that since $|z| = 1$, we have

$$\frac{1}{z} = \frac{1}{e^{i\theta}} = e^{-i\theta} = \bar{z}$$

So

$$I_2 = -\frac{1}{4\pi i}\oint_{|s|=1}\frac{\log(1+\sqrt{y}s)\left(1-\left(\frac{1}{s}\right)^2\right)^2}{\left(\frac{1}{s}\right)^2(1+\sqrt{y}\left(\frac{1}{s}\right))(\frac{1}{s}+\sqrt{y})}(-\frac{1}{s^2})ds.$$

and this can be shown to be

$$I_2 = I_1.$$

hence, $I = -\log(1-y)\frac{(1-y)}{y} - 1$.

Example 2: We can calculate the mean of the MP distribution. For all $y > 0$,

$$\int x dF_y(x) = 1.$$

Proof: This can be shown in the same way as Example 1. ■

For any monomial function $f(x) = x^k$ for $k \in \mathbb{N}$, the residue approach becomes tedious. There is a direct proof as well. (Bai & Silverstein 2010; Lemma 3.1).

Proposition 5.3 *The moments of the standard MP distribution*

$$\beta_k := \int x^k dF_y(x) = \sum_{r=0}^{k-1}\frac{1}{r+1}\binom{k}{r}\binom{k-1}{r}y^r$$

Proof:

$$P_y(x) = \begin{cases} \frac{1}{2\pi xy}\sqrt{(b-x)(x-a)}, a \le x \le b \\ 0, \text{ otherwise.} \end{cases}$$

$a = (1-\sqrt{y})^2, b = (1+\sqrt{y})^2$.

$$\beta_k = \frac{1}{2\pi y}\int_a^b x^{k-1}\sqrt{(b-x)(x-a)}dx.$$

Set $x = 1+y+z, dx = dz, x = a \implies (1-\sqrt{y})^2 = 1+y+z, z = (1-\sqrt{y})^2 - 1 - y = -2\sqrt{y}$.

$(b-x)(x-a) = (2\sqrt{y}-z)(2\sqrt{y}+z) = (4y-z^2), x = b \implies z = 2\sqrt{y}$.

So

$$\beta_k = \frac{1}{2\pi y}\int_{-2\sqrt{y}}^{2\sqrt{y}}(1+y+z)^{k-1}\sqrt{4y-z^2}dz.$$

Recall $(a+b)^\alpha = \sum_{k=0}^{\alpha}\binom{\alpha}{k}a^k b^{\alpha-k}, \binom{\alpha}{k} := \frac{\alpha!}{k!(\alpha-k)!}$

So here $a = 1+y, b = z, \alpha = k-1$.

$$\begin{aligned}\beta_k &= \frac{1}{2\pi y}\int_{-2\sqrt{y}}^{2\sqrt{y}}\sum k-1_{l=0}\binom{k-1}{l}(1+y)^{k-1-l}z^l\sqrt{4y-z^2}dz \\ &= \frac{1}{2\pi y}\sum_{l=0}^{k-1}\binom{k-1}{l}(1+y)^{k-1-l}\int_{-2\sqrt{y}}^{2\sqrt{y}}z^l\sqrt{4y-z^2}dz.\end{aligned} \tag{5.4}$$

We continue to set $z = 2\sqrt{y}u, dz = 2\sqrt{y}du \implies 4y - z^2 = 1-u^2$.

$z = -2\sqrt{y} \implies u = -1; z = 2\sqrt{y} \implies u = 1$.

$$\begin{aligned}\beta_k &= \frac{1}{2\pi y}\sum_{l=0}^{k-1}\binom{k-1}{l}(1+y)^{k-1-l}2\sqrt{y}(2\sqrt{y})^l\int_{-1}^1 u^l\sqrt{1-u^2}du \\ &= \frac{1}{2\pi y}\sum_{l=0}^{k-1}\binom{k-1}{l}(1+y)^{k-1-l}(4y)^{\frac{l+1}{2}}\int_{-1}^1 u^l\sqrt{1-u^2}du \\ &= \frac{1}{2\pi y}\sum_{l=0}^{\frac{k-1}{2}}\binom{k-1}{2l}(1+y)^{k-1-2l}(4y)^{l+1}\int_{-1}^1 u^{2l}\sqrt{1-u^2}du\end{aligned} \tag{5.5}$$

Set $u = \sqrt{w}, du = \frac{1}{2}\frac{1}{\sqrt{w}}dw, u = -1, w = 1$.

$$\beta_k = \frac{1}{2\pi u}\sum_{l=0}^{\frac{k-1}{2}}\binom{k-1}{2l}(1+y)^{k-1-2l}(4y)^{l+4}\int_0^1 w^{l-\frac{1}{2}}\sqrt{1-w}dw. \tag{5.6}$$

As $\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}dx, \Gamma(n) := (n-1)!, \Gamma(l+\frac{1}{2}) = \frac{(2l)!}{4^l l!}\sqrt{\pi}$.

As $\int_0^1 w^{l-\frac{1}{2}}\sqrt{1-w}dw = \frac{\sqrt{\pi}\Gamma(l+\frac{1}{2})}{2\Gamma(2+l)}$ if $l > \frac{1}{2}$. We continue:

$$\begin{aligned}\beta_k &= \sum_{l=0}^{[(k-1)/2]} \frac{1}{2\pi y} \frac{(k-1)!}{(2l)!((k-1)-2l)!} \frac{\sqrt{\pi}}{2} \frac{(2l)!\sqrt{\pi}}{4^l l!(l+1)!} 4^{l+1} y^{l+1} (1+y)^{k-1-2l} \\ &= \sum_{l=0}^{[(k-1)/2]} \frac{(k-1)!}{l!(l+1)!(k-1-2l)!} y^l (1+y)^{k-1-2l}\end{aligned} \tag{5.7}$$

As $(1+y)^{k-1-2l} = \sum_{s=0}^{k-1-2l} \binom{k-1-2l}{s} y^s = \sum_{s=0}^{k-1-2l} \frac{(k-1-2l)!}{s!(k-1-2l-s)!} y^s$, continue:

$$\begin{aligned}\beta_k &= \sum_{l=0}^{[(k-1)/2]} \frac{(k-1)!}{l!(l+1)!(k-1-2l)!} y^l \sum_{s=0}^{k-1-2l} \frac{(k-1-2l)!}{s!(k-1-2l-s)!} y^s \\ &= \sum_{l=0}^{[(k-1)/2]} \sum_{s=0}^{k-1-2l} \frac{(k-1)!}{l!(l+1)!s!(k-1-2l-s)!} y^{l+s}\end{aligned} \tag{5.8}$$

Substitute $r = l+s, s = 0 \implies r = l, s = k-1-2l \implies r = k-1-l$. Continue:

$$\begin{aligned}\beta_k &= \sum_{l=0}^{[(k-1)/2]} \sum_{r=l}^{k-1-l} \frac{(k-1)!}{l!(l+1)!s!(k-1-r-l)!} y^r \\ &= \frac{1}{k} \sum_{r=0}^{k-1} \binom{k}{r} y^r \sum_{l=0}^{\min(r,k-1-r)} \binom{r}{l} \binom{k-r}{k-r-l-1} \\ &= \frac{1}{k} \sum_{r=0}^{k-1} \binom{k}{r} \binom{k}{r+1} y^r = \sum_{r=0}^{k-1} \frac{1}{r+1} \binom{k}{r} \binom{k-1}{r} y^r.\end{aligned} \tag{5.9}$$

■

<u>Fubini theorem for sequences:</u> If $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_{nm}| < \infty$ then

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}.$$

<u>How did I use that?</u>

$$
\begin{aligned}
\sum_{l=0}^{[(k-1)/2]} \sum_{r=l}^{k-1-l} \frac{(k-1)!}{l!(l+1)!(r-l)!(k-1-r-l)!} y^r &= \sum_{l=0}^{\infty} \mathbf{1}_{(l \le [(k-1)/2])} \sum_{r=0}^{\infty} \mathbf{1}_{(r \ge l)} \mathbf{1}_{(r \le k-1-l)} \\
&= \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \mathbf{1}_{(l \le [(k-1)/2])} \mathbf{1}_{(l \le r)} \mathbf{1}_{(l \le k-1-r)} \mathbf{1}_{(r \le k-1)} \\
&= \sum_{r=0}^{k-1} \sum_{l=0}^{\min(r,k-1-r)} \square \\
\square &= \frac{k!}{r!(k-r)!} y^r \frac{(k-1)!r!(k-r)!}{k!l!(l+1)!(r-l)!(k-1-r-l)!} \\
&= \binom{k}{r} y^r \frac{1}{k} \cdot \frac{r!}{l!(r-l)!} \cdot \frac{(k-r)!}{(l+1)!(k-1-r-l)!} \\
&\text{note: } k-r-(k-1-r-l) = l+1 \\
&= \frac{1}{k} \binom{k}{r} y^r \binom{r}{l} \binom{k-r}{k-1-r-l}
\end{aligned}
\tag{5.10}
$$

So we plug $\square$ back in to have

$$
= \sum_{r=0}^{k-1} \frac{1}{k} \binom{k}{r} y^r \sum_{l=0}^{\min(r,k-1-r)} \binom{r}{l} \binom{k-r}{k-1-r-l}.
$$

Generalized MP distribution

Previously, we've seen the case where the population covariance matrix has the simple form $\sum = \sigma^2 \mathbf{I}_p$.

We can consider a slightly more general case if we make the assumption that the observation vectors $\{y_k\}_{1 \le k \le n}$ can be represented as

$$
y_k := \sum^{1/2} x_k, x_k \text{ iid}, \sum \text{ nonnegative square root of } \sum.
$$

This gives the associated covariance matrix

$$
\begin{aligned}
\tilde{\mathbf{B}}_n &= \frac{1}{n} \sum_{k=1}^{n} \mathbf{y}_k \mathbf{y}_k^* = \sum^{1/2} \left(\frac{1}{n} \sum_{k=1}^{n} \mathbf{x}_k \mathbf{x}_k^* \right) \sum^{1/2} \\
&= \sum^{1/2} \mathbf{S}_n \sum^{1/2}
\end{aligned}
\tag{5.11}
$$

S_n is the sample covariance matrix with iid components.

The eigenvalues of $\tilde{B}_n$ are the same as $\mathbf{S}_n \sum$.

The following result holds for $\mathbf{B}_n = \mathbf{S}_n \mathbf{T}_n$ for general nonnegative definite matrix $\mathbf{T}_n$. ($\mathbf{T}_n = \sum$ is a special case.)

Theorem 5.4 *Let $\mathbf{S}_n$ be the sample covariance matrix $\mathbf{S}_n = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i \mathbf{x}^*$ with iid components and let $(\mathbf{T}_n)$ be a sequence of nonnegative definite Hermitian matrices of size $p \times p$.*

Define $\mathbf{B}_n = \mathbf{S}_n \mathbf{T}_n$ and assume:

1. *The entries (x_{jk}) of the data matrix $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ are iid with mean zero and variance $\mathbf{1}$.*
2. *The data dimension to sample size ratio $p/n \to y > 0$ as $n \to \infty$.*
3. *The sequence $(\mathbf{T}_n)$ is either deterministic or independent of $(\mathbf{S}_n)$.*
4. *Almost surely, the sequence $(H_n = F^{\mathbf{T}_n})$ of the ESD of $(\mathbf{T}_n)$ weakly converges to a non-random probability measure H.*

Then, almost surely, F (???) weakly converges to a non-random probability measure $F_{y,H}$. Its Stieltjes transform is given by

$$S(z) = \int \frac{1}{t(1 - y - yzs(z)) - z} dH(t), z \in \mathbb{C}^+ \tag{5.12}$$

Notice that the ST of $F_{y,H}$ is implicitly defined. It can be shown that a unique solution exists, but, unfortunately, no closed-form solution exists. (see Silverstein & Combettes 1992)

There is a better way to represent the ST of $F_{y,H}$. Consider for $\mathbf{B}_n$ a companion matrix.

$$\underline{\mathbf{B}_n} = \frac{1}{n}\mathbf{X}^*\mathbf{T}\mathbf{X} \qquad \text{size } n \times n$$

Both matrices share the same nonzero eigenvalues so their ESD satisfy

$$nF(???) - pF^{\mathbf{B}_n} = (n-p)\mathbf{S}_0(???).$$

Note: Given two matrices $\mathbf{A}_{p\times q}$ and $\mathbf{B}_{q\times p}$ where $p \geq q$, eigenvalues of $\mathbf{AB}$ is that of $\mathbf{BA}$ augmented by $p-q$ zeros.

$$\mathbf{B}_n = \mathbf{S}_n\mathbf{T}_n = \frac{1}{n}\mathbf{X}\mathbf{X}^*\mathbf{T}_n, \underline{\mathbf{B}_n} = \frac{1}{n}\mathbf{X}^*\mathbf{T}\mathbf{X}, \text{ where } \mathbf{X} \text{ is a } p \times n \text{ matrix.}$$

When $p/n \to y > 0, F^{\mathbf{B}_n}$ has limit $F_{c,H}$ if and only if $F^{\underline{\mathbf{B}_n}}$ has limit $\underline{F}_{c,H}$. In this case, the limit satisfies

$$\underline{F}_{c,H} - yF_{c,H} = (1-y)S_0(???).$$

and their ST are related by

$$\underline{s}(z) = -\frac{1-y}{z} + ys(z).$$

Now substituting $\underline{s}$ for s in (5.12) yields

$$\underline{s}(z) = \left(z - y\int \frac{t}{1+ts(z)}dH(t)\right)^{-1}$$

Solving in z gives:

$$z = -\frac{1}{\underline{s}(z)} + y\int \frac{t}{1+t\underline{s}(z)}dH(t) \tag{5.13}$$

which defines the inverse function of $\underline{s}$.

- (5.12) is called the Marchenko-Pastur equation and
- (5.13) is the Silverstein equation.

Limiting spectral distribution for Random Fisher matricecs

In the univariate case, when we need to test equality between the variances of 2 Gaussian populations, a Fisher statistic of the form S_1^2/S_2^2 is used where S_i^2 are estimators of the unknown variances in the two populations.

The equivalent for the multivariate setting is:

Take two independent samples $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ both from p-dimensional population with iid components and finite second moment.

$$\mathbf{S}_1 = \frac{1}{n_1}\sum_{k=1}^{n_1} \mathbf{x}_k\mathbf{x}_k^*$$

$$\mathbf{S}_2 = \frac{1}{n_2}\sum_{k=1}^{n_2} \mathbf{y}_k\mathbf{y}_k^*$$

Then $\mathbf{F}_n := \mathbf{S}_1\mathbf{S}_2^{-1}$ is called a Fisher matrix, $\mathbf{n} = (n_1, n_2)$.

(Note: need $p \leq n_2$ so that $\mathbf{S}_2$ invertible.)

Let $s > 0$ and $0 < t < 1$. The Fisher LSD $F_{s,t}$ is the distribution with density function

$$P_{s.t}(x) = \frac{1-t}{2\pi x(s+tx)}\sqrt{(b-x)(x-a)}, a \leq x \leq b$$

with $a = a(s,t) = \frac{(1-h)^2}{(1-t)^2}, b = b(s,t) = \frac{(1+h)^2}{(1-t)^2}, h = h(s,t) = (s+t-st)^{1/2}$.

When $s > 1, F_{s,t}$ has a mass at $x = 0$ of value $1 - \frac{1}{s}$ with the total mass of the rest of the distribution for $x > 0$ is equal to $1/s$.

The Fisher LSD has many similarities to the standard MP distribution. This is not a coincidence as the MP LSD F_y is the Fisher LSD $F_{y,0}$ (i.e. $s, y = y, 0$)

Also note $t \to 1, a(s,t) \to \frac{1}{2}(1-s)^2, b(s,t) \to \infty$ ($Supp(F_{s,t})$ becomes unbounded).

Theorem 5.5 *For an analytic function f on a domain containing $[a,b]$ (as above). We have*

$$\int_a^b f(x)dF_{s,t}(x) = -\frac{h^2(1-t)}{4\pi i}\oint_{|z|=1}\frac{f\left(\frac{|1+hz|^2}{(1-t)^2}\right)(1-z^2)^2dz}{z(1+hz)(z+h)(tz+h)(t+hz)}$$

Proof: Using the density $P_{s,t}(x)$.

$$I = \int_a^b f(x)dF_{s,t}(x) = \int_a^b f(x)\frac{1-t}{2\pi x(s+xt)}\sqrt{(x-a)(b-x)}dx$$

Make change of variable $x = \frac{1+h^2+2h\cos\theta}{(1-t)^2}, \theta \in (0,\pi)$.

$$\begin{aligned}
x-a &= \frac{1+h^2+2h\cos\theta}{(1-t)^2} - \frac{(1-h)^2}{(1-t)^2} = \frac{2h+2h\cos\theta}{(1-t)^2}\\
b-x &= \frac{(1+h)^2}{(1-t)^2} - \frac{1+h^2+2h\cos\theta}{(1-t)^2} = \frac{2h-2h\cos\theta}{(1-t)^2}\\
\sqrt{(x-a)(b-x)} &= \sqrt{\frac{(2h)^2}{(1-t)^4}(1-\cos\theta)(1+\cos\theta)}\\
&= \frac{2h}{(1-t)^2}\sin\theta\\
x = a &\implies \cos\theta = \frac{a(1-t)^2-(1+h^2)}{2h} = 0 \implies \theta = 0\\
x = b &\implies \theta = \pi\\
dx &= \frac{-2h\sin\theta}{(1-t)^2}d\theta
\end{aligned} \tag{5.14}$$

Hence

$$\begin{aligned}
I &= \frac{2h^2(1-t)}{\pi}\int_0^\pi \frac{f\left(\frac{1+h^2+2h\cos\theta}{(1-t)^2}\right)\sin^2\theta d\theta}{(1+h^2+2h\cos\theta)(s(1-t)^2+t(1+h^2+2h\cos\theta))}\\
&= \frac{h^2(1-t)}{\pi}\int_0^{2\pi} \frac{f\left(\frac{1+h^2+2h\cos\theta}{(1-t)^2}\right)\sin^2\theta d\theta}{(1+h^2+2h\cos\theta)(s(1-t)^2+t(1+h^2+2h\cos\theta))}
\end{aligned} \tag{5.15}$$

Now let $z = e^{i\theta}$,

$$\begin{aligned}
1+h^2+2h\cos\theta &= |1+hz|^2\\
\sin\theta &= \frac{z-z^{-1}}{2i}\\
\log(z) = i\theta \implies \theta &= \frac{1}{i}\log(z) \implies d\theta = \frac{1}{iz}dz.\\
I &= -\frac{h^2(1-t)}{4\pi i}\oint_{|z|=1}\frac{f\left(\frac{|1+hz|^2}{(1-t)^2}\right)(1-z^2)^2dz}{z^3|1+hz|^2(s(1-t)^2+t|1+hz|^2)}
\end{aligned} \tag{5.16}$$

On $|z| = 1$, we have $|1 + hz|^2 = (1 + hz)(1 + hz^{-1})$.

So expanding denominator and simplifying we have the result. ■

Example: Take $(s, t) = (y, 0)$ and we get the result for MP distribution.

Example 2: The first two moments are

$$\int x dF_{s,t}(x) = \frac{1}{1-t}, \int x^2 dF_{s,t}(x) = \frac{h^2 + 1 - t}{(1-t)^3}.$$

Hence the variance equals $\frac{h^2}{(1-t)^3}$.