

## Lecture 4: 13 August

*Lecturer: Dr Dale Roberts**Scribes: Rui Qiu*

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**Tools for Integration**

Working with complex numbers and the sophisticated machinery of contour integration will be needed in this course.

This week we are going to look at this beautiful area of Mathematics.

These tools will be very useful for calculations of the form

$$\int f(x) dF_y(x)$$

where  $F_y$  is the MP distribution, for example.

**Complex numbers and elementary functions**

With  $i^2 := -1$ , a complex number is an expression of the form  $z = x + iy$ . We write  $Re(z) = x$  and  $Im(z) = y$ .

We can also write complex numbers in Polar form  $(r, \theta)$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $r \geq 0$ .

A complex number  $z$  can be written

$$z = x + iy = r(\cos \theta + i \sin \theta) \quad (4.1)$$

The radius  $r = \sqrt{x^2 + y^2} = |z|$  (aka. modulus of  $z$ ) and the angle  $\theta$  is the argument of  $z$ , denoted  $\arg z$ .

When  $z \neq 0$ , we can find  $\theta$  by trigonometry

$$\tan \theta = \frac{y}{x}.$$

$\theta = \arg z$  is multivalued as  $\tan \theta$  is a periodic function of  $\theta$  with period  $\pi$ .

Example:  $z = -1 + i$ ,  $|z| = r = \sqrt{2}$ ,  $\theta = \frac{3\pi}{4} + 2n\pi$ ,  $n = 0, \pm 1, \pm 2, \dots$

We can define the (polar) exponential.

$$\cos \theta + i \sin \theta = e^{i\theta} \implies \text{by (4.1)} z = r \cdot e^{i\theta}$$

Some beautiful formulas:

$$\begin{aligned} e^{2\pi i} &= 1, e^\pi = -1, \dots \\ e^{i\theta_1} e^{i\theta_2} &= e^{i(\theta_1 + \theta_2)}, (e^{i\theta})^m = e^{im\theta}, (e^{i\theta})^{1/n} = e^{i\theta/n} \end{aligned} \quad (4.2)$$

The complex conjugate of  $z$  is  $\bar{z} = x - iy = re^{-i\theta}$ .

The usual rules apply:

$$\begin{aligned} z_1 \pm z_2 &= (x_1 \pm x_2) + i(y_1 \pm y_2) \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned} \quad (4.3)$$

Notice that  $z\bar{z} = \bar{z}z = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

so that

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \end{aligned} \quad (4.4)$$

We can define same elementary functions of complex argument. The simplest is

$$f(z) = z^n, n = 0, 1, 2, \dots, \text{ "Power function"}$$

A polynomial of order  $n$ :

$$P_n(z) = \sum_{j=0}^n a_j z^j = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n \quad (4.5)$$

where  $a_j$  are complex numbers.

A rational function:

$$R(z) = \frac{P_n(z)}{Q_m(z)}, P_n(z), Q_m(z) \text{ are polynomials.} \quad (4.6)$$

In general, the function  $f(z)$  is complex-valued and can be written

$$f(z) = u(x, y) + iv(x, y) = \operatorname{Re} f + i \operatorname{Im} f.$$

Example:  $z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$  where  $u(x, y) = x^2 - y^2, v(x, y) = 2xy$ .

We can define the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

and it is easy to show  $e^z = e^x(\cos y + i \sin y)$ .

$$e^{z_1+z_2} = e^{z_1} e^{z_2}, (e^z)^n = e^{nz}, n = 1, 2, \dots$$

$$|e^z| = |e^z| |\cos y + i \sin y| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x$$

$$\overline{(e^z)} = e^{\bar{z}} = e^x(\cos y - i \sin y)$$

### Trig functions:

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2i} \end{aligned} \tag{4.7}$$

$$\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z}, \sec z = \frac{1}{\cos z}, \csc z = \frac{1}{\sin z}.$$

### Hyperbolic functions:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}, \tanh z = \frac{\sinh z}{\cosh z},$$

Note that

$$\begin{aligned} \sinh iz &= i \sin z, \quad \sin iz = i \sinh z \\ \cosh iz &= \cos z, \quad \cos iz = \cosh z \end{aligned} \tag{4.8}$$

Most of the definitions could have been introduced through the concept of power series.

The power series of  $f(z)$  around the point  $z = z_0$  is

$$f(z) = \lim_{n \rightarrow \infty} \sum_{j=0}^n a_j (z - z_0)^j = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

where  $a_j, z_0$  are constants.

Remember convergence only occurs within some radius, i.e., within some circle  $|z - z_0| = R$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

We have the power series representations:

$$\begin{aligned} e^z &= \sum_{j=0}^{\infty} \frac{z^j}{j!}, \\ \sin z &= \sum_{j=1}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}, \\ \cos z &= \sum_{j=1}^{\infty} \frac{(-1)^j z^{2j}}{(2j)!}, \\ \sinh z &= \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}, \\ \cosh z &= \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}. \end{aligned}$$

Let  $f(z)$  be defined in some region  $R$  containing the neighbourhood of a point  $z_0$ . The derivative of  $f(z)$  at  $z = z_0$ , denoted by  $f'(z_0)$  or  $\frac{d}{dz}f(z_0)$  is defined by

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \left( \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right).$$

provided the limit exists. Alternatively, writing  $\Delta z = z - z_0$ .

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Caution: A continuous function is not necessarily differentiable as complex functions have a two-dimensional character.

For example,  $f(z) = \bar{z}$ .

$$\lim_{\Delta z \rightarrow 0} \frac{\overline{(z_0 + \Delta z)} - \bar{z}_0}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = 0$$

and a unique value for  $c$  cannot be found!

NOT DIFFERENTIABLE!

Differentiable complex functions are called analytic.

If  $f$  and  $g$  have derivatives:

$$\begin{aligned}(f + g)' &= f' + g', \quad (fg)' = f'g + g'f \\ \left(\frac{f}{g}\right)' &= (f'g - fg')/g^2 \quad (g \neq 0)\end{aligned}\tag{4.9}$$

If  $f'(g(z))$  and  $g'(z)$  exist, then

$$[f(g(z))]' = f'(g(z))g'(z)$$

Since  $\frac{(z+\Delta z)^n - z^n}{\Delta z} = nz^{n-1} + a_1z^{n-2}\Delta z + a_2z^{n-3}(\Delta z)^2 + \dots + (\Delta z)^{n-1}$  as  $\Delta z \rightarrow 0$ , where  $a_j$  are binomial coefficients of  $(a+b)^n$ .

We have  $\frac{d}{dz}(z^n) = nz^{n-1}$ .

It follows (formally) that

$$\frac{d}{dz} \left( \sum_{n=0}^{\infty} a_n z^n \right) = \sum_{n=0}^{\infty} n a_n z^{n-1} \text{ inside radius of convergence}$$

Writing  $f(z) = u(x, y) + iv(x, y)$ ,

$$\begin{aligned}f'(z) &= \lim_{\Delta x \rightarrow 0} \left( \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right) \\ &= u_x(x, y) + i v_x(x, y) \\ u_x &:= \frac{\partial u}{\partial x}, v_x := \frac{\partial v}{\partial x} \\ f'(z) &= \lim_{\Delta y \rightarrow 0} \left( \frac{u(x, y + \Delta y) - u(x, y)}{i \Delta y} + i \frac{v(x, y + \Delta y) - v(x, y)}{i \Delta y} \right) \\ &= -i u_y(x, y) + v_y(x, y)\end{aligned}\tag{4.10}$$

Hence, equating these two expressions we have the Cauchy-Riemann equations:

$$u_x = v_y, v_x = -u_y$$

that are necessarily satisfied if  $f(z)$  is differentiable.

**Theorem 4.1** *The function  $f(z) = u(x, y) + iv(x, y)$  is differentiable at a point  $z = x + iy$  of a region in the complex plane if and only if  $u_x, u_y, v_x, v_y$  are continuous and satisfy the Cauchy Riemann equations.*

We shall now look at how to evaluate integrals of complex-valued functions along curves in the complex plane.

First, consider the case of the complex-valued function  $f$  of the real variable  $t$ , on the interval  $a \leq t \leq b$ .

$$f(t) = u(t) + iv(t)$$

We say that  $f$  is integrable if  $u$  &  $v$  are integrable, then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt.$$

Usual rules apply:

$$\frac{d}{dt} \int_a^t f(t)dt = f(t) \text{ text "fundamental theorem of calculus"}$$

and if  $f'(t)$  is continuous,  $\int_a^b f'(t)dt = f(b) - f(a)$ .

### Integration along a curve

A curve in  $\mathbb{C}$  can be described by the parametrization

$$z(t) = x(t) + iy(t), a \leq t \leq b.$$

A curve  $c$  is called simple if it does not intersect itself. It is called differentiable curve if  $z'(t) = x'(t) + iy'(t)$  is non-null.

A piecewise differentiable curve (or path) is obtained by joining a finite number of differentiable curves.

Let  $C$  be a path, we call it closed if  $z(a) = z(b)$ .

A closed path is also called a contour.

Example: The unit circle in  $\mathbb{C}$  is a contour and is parametrized by  $z(t) = e^{it}$ ,  $0 \leq t \leq 2\pi$ .

The contour integral of a piecewise continuous function on a smooth contour (i.e. differentiable) is defined to be

$$\int_C f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

Note that  $dz \approx z'(t)dt$ .

Usual properties apply:

$$\int_C [\alpha f(z) + \beta g(z)]dz = \alpha \int_C f(z)dz + \beta \int_C g(z)dz.$$

If we traverse the contour in the opposite direction (i.e. from  $t = b$  to  $t = a$ ) then this is denoted  $-C$  and

$$\int_{-C} f(z)dz = - \int_C f(z)dz.$$

And if  $C = C_1 \cup C_2 \cup C_3 \cup \dots \cup C_n$ , then

$$\int_C f = \sum_{j=1}^n \int_{C_j} f.$$

From the fundamental theorem of calculus we get:

**Theorem 4.2** Suppose  $F(z)$  is an analytic function such that  $f(z) = F'(z)$  is continuous in a domain  $\mathcal{D}$ . Then for the contour  $C$  lying in  $\mathcal{D}$  with endpoints  $z_1$  and  $z_2$

$$\int_C f(z)dz = F(z_2) - F(z_1).$$

**Proof:**

$$\begin{aligned} \int_C f(z)dz &= \int_C F'(z)dz = \int_a^b F'(z(t))z'(t)dt \\ &= \int_a^b \frac{d}{dt}[F(z(t))]dt = F(z(b)) - F(z(a)) = F(z_2) - F(z_1). \end{aligned} \quad (4.11)$$

Note:  $z(a) = z_1, z(b) = z_2$ .

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Q: What happens if  $C$  is a closed contour?

$$\oint_C f(z)dz = \int_C F'(z)dz = 0$$

“ $\oint$ ” denotes integration along a closed contour  $C$ .

Notice that this holds for any closed contour  $C$ . So this integral is independent of the path.

**Theorem 4.3** Let  $f(z)$  be analytic interior to and on a simple closed contour  $C$ . Then at any interior point  $z$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi.$$

This is the Cauchy integral formula.

“The function  $f$  is completely determined by the points  $z \in C$ ”

Further, we can also say something about all the derivatives of  $f$ .

**Theorem 4.4** If  $f(z)$  is analytic interior to and on a simple closed contour  $C$ , then all the derivatives  $f^{(k)}(z), k = 1, 2, \dots$  exist in the domain  $\mathcal{D}$  interior to  $C$  and

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi.$$