Probability of 2H or 2T Coin Toss in a Row

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1 Introduction

This document analyzes the probability of obtaining two consecutive heads (HH) or tails (TT) in a sequence of coin tosses and calculates the expected number of tosses to achieve this outcome using a fair coin.

2 Methodology

2.1 Base Case Analysis

The probability P_n is defined as the probability of getting two consecutive same outcomes (HH or TT) on the nth flip. For $n = 2$, the outcomes can be HH, TT, HT, or TH. Out of these, HH and TT meet the criteria, so:

$$P_2 = \frac{2}{4} = \frac{1}{2}$$

2.2 Generalization to nth Toss

For $n > 2$, the sequence leading up to the nth toss must alternate (like HTHTHT... or THTHTH...) for $n - 1$ tosses, so that the nth toss can match the previous one to form either HH or TT. The probability of a specific alternating sequence up to $n - 1$ is $\left(\frac{1}{2}\right)^{n-1}$ since each flip independently has a 1/2 chance of being either H or T. There are two such sequences (starting with H or T), so the probability of reaching the $n - 1$th toss in an alternating pattern is:

$$2 \times \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-2}}$$

2.3 Calculation of P_n

Given that we have an alternating sequence up to $n - 1$, the probability that the nth toss matches the $n - 1$th toss is $\frac{1}{2}$ (since the coin is fair). Thus:

$$P_n = \frac{1}{2^{n-2}} \times \frac{1}{2} = \frac{1}{2^{n-1}}$$

2.4 Expected Number of Tosses

The expected number of tosses to get two consecutive same results is calculated using the expected value formula, where you sum up $n \times P_n$ for all $n \geq 2$:

$$E(X) = x_1 p_1 + x_2 p_2 + ... + x_k p_k$$

$$E(X) = \sum_{n=2}^{\infty} n \times \frac{1}{2^{n-1}}$$

2.5 Understanding the Series

Shift the Index: Since it starts from $n = 2$, we can shift the index by 1 for simplification. Let $m = n - 1$, then $n = m + 1$ and our sum becomes:

$$E(X) = \sum_{m=1}^{\infty} (m + 1) \times \frac{1}{2^m}$$

This can be rewritten as:

$$E(X) = \sum_{m=1}^{\infty} m \times \frac{1}{2^m} + \sum_{m=1}^{\infty} \frac{1}{2^m}$$

The first part is a weighted geometric series and the second part is a standard geometric series.

2.6 What is a Geometric Series?

A geometric series is a series of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. For instance, in the series

$$1, r, r^2, r^3, \ldots$$

each term is produced by multiplying the previous term by r.

Why the Above Formula a Geometric Series

The series discussed:

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2^n} \right)$$

is a form of a weighted geometric series. Each term of the series is of the form

$$n \left(\frac{1}{2^n} \right)$$

where the n serves as a weighting factor applied to each term of a basic geometric series. The "base" part of each term,

$$\frac{1}{2^n}$$

follows the standard form of a geometric series with a common ratio of

$$\frac{1}{2}$$

.

Properties of a Geometric Series

- **Common Ratio:** The common ratio in this series is $\frac{1}{2}$. This ratio is consistent throughout the series, defining it as geometric.
- **Form:** The series can be written as:

  $$\frac{1}{2} + 2\left(\frac{1}{2^2}\right) + 3\left(\frac{1}{2^3}\right) + 4\left(\frac{1}{2^4}\right) + \cdots$$

  Here, each term after the first is derived by multiplying the previous term by $\frac{1}{2}$ and then adjusting for the index weighting factor n.

Geometric Series Sum Formula

For an unweighted geometric series

$$a, ar, ar^2, ar^3, \ldots$$

the sum is given by:

$$S = \frac{a}{1-r}$$

where a is the first term and r is the common ratio, assuming $|r| < 1$ for convergence.

For the weighted series

$$nr^n$$

we use a derived formula:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

This is derived by differentiating the sum formula of the basic geometric series with respect to r, and then multiplying by r. This operation effectively weights each term by its index n, which matches the structure of the series we are considering.

2.7 Standard Geometric Series Calculuation

Let's break down the calculation for the standard geometric series part in detail. The series in question is:

$$\sum_{m=1}^{\infty} \frac{1}{2^m}$$

This is a geometric series with the first term starting from $m = 1$ (not from $m = 0$).

Geometric Series Formula

The standard formula for the sum of an infinite geometric series starting from $m = 0$ is:

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

where x is the common ratio of the series. For this particular series, $x = \frac{1}{2}$.

Adjusting for $m = 1$

However, our series starts from $m = 1$. The formula given calculates from $m = 0$. If we compute

$$\sum_{m=0}^{\infty} \frac{1}{2^m}$$

it includes the term $\frac{1}{2^0}$ which is 1. Therefore, to adjust for the series starting at $m = 1$, we need to subtract the $m = 0$ term:

$$\sum_{m=1}^{\infty} \frac{1}{2^m} = \left(\sum_{m=0}^{\infty} \frac{1}{2^m} \right) - 1$$

Substituting the Series Sum from $m = 0$

$$\sum_{m=0}^{\infty} \frac{1}{2^m} = \frac{1}{1 - \frac{1}{2}} = 2$$

Now, subtract the $m = 0$ term:

$$\sum_{m=1}^{\infty} \frac{1}{2^m} = 2 - 1 = 1$$

Explanation

This result shows that when you sum the terms from $m = 1$ to infinity, where each term is $\frac{1}{2^m}$, you get a total of 1. This calculation hinges on the adjustment for the index starting from 1 instead of 0, by subtracting the $m = 0$ term from the full series sum.

2.8 Weighted Geometric Series Calculation

To solve the weighted infinite series

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2^n}\right),$$

we need to analyze the series where each term is weighted by its index n. This is a bit more complex than a straightforward geometric series.

This series is a form of the weighted geometric series where each term is the product of the term number n and a geometric ratio $\frac{1}{2^n}$.

2.9 Approach

To find the sum of this series, we use the formula for the sum of an infinite series where each term is multiplied by its index:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

where x is the common ratio of the series. For this specific series, $x = \frac{1}{2}$.

2.10 Applying the Formula

Substituting $x = \frac{1}{2}$ into the formula gives:

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2^n}\right) = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

2.11 Explanation

The formula $\frac{x}{(1-x)^2}$ is derived from differentiating the standard geometric series sum formula $\frac{1}{1-x}$ with respect to x and then multiplying by x. This differentiation brings down the index n, which acts as a weighting factor, indicating how each term's contribution increases linearly with n. The substitution and simplification then calculate the exact sum as 2, indicating the expected sum of the series where each term is increasingly weighted by its index.

$$\sum_{m=1}^{\infty} \frac{1}{2^m} = 1$$

2.12 Solve the Weighted Geometric Series

The first part $\sum_{m=1}^{\infty} m \times \frac{1}{2^m}$ is a weighted geometric series. Calculating this requires understanding the infinite sum of $m \times x^m$ which for $x = \frac{1}{2}$ simplifies to:

$$\sum_{m=1}^{\infty} m \times \left(\frac{1}{2}\right)^m = 2$$

2.13 Conclusion of Series

Summing these results:

$$E(X) = 2 + 1 = 3$$

3 Conclusion

The analysis concludes that, on average, three tosses are needed to achieve two consecutive heads or tails with a fair coin.