1. Random Variables

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Outline

Lesson 1.1 — Introduction

Definition: A **random variable** (RV) is a function from the sample space to the real line. $X : S \to \mathbb{R}$.

Example: Flip 2 coins. $S = \{HH, HT, TH, TT\}$.

Suppose X is the RV corresponding to the number of H's.

$X(TT) = 0$, $X(HT) = X(TH) = 1$, $X(HH) = 2$.

$P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$. $\square$

Notation: Capital letters like X, Y, Z, U, V, W usually represent RVs.

Small letters like x, y, z, u, v, w usually represent particular values of the RVs.

Thus, you can speak of $P(X = x)$.

Example: Let X be the sum of two dice rolls. Then, e.g., $(4,6)$ is an outcome from the sample space, and of course $X((4,6)) = 10$.

In addition,

$$P(X = x) = \begin{cases} 1/36 & \text{if } x = 2 \\ 2/36 & \text{if } x = 3 \\ \vdots & \\ 6/36 & \text{if } x = 7 \\ \vdots & \\ 1/36 & \text{if } x = 12 \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

Example: Flip a coin.

$$X \equiv \begin{cases} 0 & \text{if } T \\ 1 & \text{if } H. \end{cases}$$

Example: Roll a die.

$$Y \equiv \begin{cases} 0 & \text{if } \{1,2,3\} \\ 1 & \text{if } \{4,5,6\}. \end{cases}$$

For our purposes, X and Y are the same, since $P(X=0)=P(Y=0)=\frac{1}{2}$ and $P(X=1)=P(Y=1)=\frac{1}{2}$.

Example: Select a real number at random between 0 and 1.

There are an *infinite* number of "equally likely" outcomes.

Conclusion: $P(\text{we choose the } \textit{individual point } x) = P(X = x) = 0$, believe it or not!

But $P(X \leq 0.65) = 0.65$ and $P(X \in [0.3, 0.7]) = 0.4$.

If A is any *interval* in [0,1], then $P(X \in A)$ is the length of A.

Definition: If the number of possible values of a RV X is finite or countably infinite, then X is a **discrete** RV. Otherwise,. . . .

A **continuous** RV is one with probability 0 at every point.

Example: Flip a coin — get H or T. Discrete.

Example: Pick a point at random in $[0, 1]$. Continuous.

Example: The amount of time you wait in a line is either 0 (with positive probability) or some positive real number — a *combined* discrete-continuous random variable!

Outline

Lesson 1.2 — Discrete Random Variables

Definition: If X is a discrete RV, its **probability mass function (pmf)** is $f(x) \equiv P(X = x)$. Note that $0 \le f(x) \le 1$, $\sum_x f(x) = 1$.

Example: Flip 2 coins. Let X be the number of heads.

$$f(x) = \begin{cases} 1/4 & \text{if } x = 0 \text{ or } 2 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example: **Uniform distribution** on integers $1, 2, \ldots, n$. X can equal $1, 2, \ldots, n$, each with prob $1/n$. $f(i) = 1/n$, $i = 1, 2, \ldots, n$.

Example: A discrete RV can have any values. For instance, let X denote the possible profits from an inventory policy, where $f(-5.1) = 0.2$ (lose money), $f(1.3) = 0.5$ (break even), and $f(11) = 0.3$ (big bucks!).

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Example/Definition: Let X denote the number of "successes" from n independent trials such that the $P(\text{success})$ at each trial is p $(0 \le p \le 1)$. Then X has the **Binomial distribution** with parameters n and p.

The trials are referred to as **Bernoulli trials**.

Notation: $X \sim \text{Bin}(n, p)$. "X has the Binomial distribution."

Example: Roll a die 3 independent times. Find

$$P(\text{Get exactly two 6's}).$$

"success" (6) and "failure" (1,2,3,4,5).

All 3 trials are indep, and $P(\text{success}) = 1/6$ doesn't change from trial to trial.

Let $X =$ number of 6's. Then $X \sim \text{Bin}(3, \frac{1}{6})$. $\square$

Theorem: If $X \sim \text{Bin}(n,p)$, then the probability of k successes in n trials is

$$P(X=k) \;=\; \binom{n}{k} p^k q^{n-k}, \quad k = 0,1,\ldots,n,$$

where $q = 1-p$.

Proof: Consider the particular sequence of successes and failures:

$$\underbrace{SS\cdots S}_{k \text{ successes}}\ \underbrace{FF\cdots F}_{n-k \text{ failures}} \quad (\text{probability} = p^k q^{n-k}).$$

The number of ways to arrange the sequence is $\binom{n}{k}$. Done. $\square$

Example (cont'd): Back to the dice example, where $X \sim \text{Bin}(3, \frac{1}{6})$, and we want $P(\text{Get exactly two 6's})$.

$n = 3$, $k = 2$, $p = 1/6$, $q = 5/6$.

$$P(X = 2) = \binom{n}{k} p^k q^{n-k} = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 = \frac{15}{216}.$$

k	0	1	2	3
$P(X = k)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

$\square$

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Example: Roll 2 dice and get the sum. Repeat 12 times.

Find $P(\text{Sum will be 7 or 11 exactly 3 times})$.

Let $X =$ the number of times we get 7 or 11.

$P(7 \text{ or } 11) \; = \; P(7) + P(11) \; = \; \frac{6}{36} + \frac{2}{36} \; = \; \frac{2}{9}$.

So $X \sim \text{Bin}(12, 2/9)$, and then

$$P(X = 3) \; = \; \binom{12}{3} \left(\frac{2}{9}\right)^3 \left(\frac{7}{9}\right)^9. \quad \square$$

Definition: If $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \ldots$, $\lambda > 0$, we say that X has the **Poisson distribution** with parameter λ.

Notation: $X \sim \text{Pois}(\lambda)$.

Example: Suppose the number of raisins in a cup of cookie dough is Pois(10). Find the probability that a cup of dough has at least 4 raisins.

$$\begin{aligned} P(X \geq 4) &= 1 - P(X = 0, 1, 2, 3) \\ &= 1 - e^{-10}\left(\frac{10^0}{0!} + \frac{10^1}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!}\right) \\ &= 0.9897. \quad \square \end{aligned}$$

Outline

Lesson 1.3 — Continuous Random Variables

Example: Pick a point X randomly between 0 and 1, and define the continuous function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, if $0 \le a \le b \le 1$, then

$$\begin{aligned} P(a < X < b) &= \text{the "area" under } f(x) \text{ from } a \text{ to } b \\ &= b - a. \end{aligned}$$

Definition: Suppose X is a continuous RV. The magic function $f(x)$ is the **probability density function (pdf)** if

- $\int_{\mathbb{R}} f(x)\,dx = 1$ (area under $f(x)$ is 1).
- $f(x) \geq 0,\ \forall x$ (always non-negative).
- If $A \subseteq \mathbb{R}$, then $P(X \in A) = \int_A f(x)\,dx$ (probability that X is in a certain region A).

Remarks: If X is a continuous RV, then

$$P(a < X < b) = \int_a^b f(x)\,dx.$$

An individual point has probability *zero*, i.e., $P(X = x) = 0$.

Note that $f(x)$ denotes both pmf (**discrete** case) and pdf (**continuous** case) — but they are *different*:

If X is *discrete*, then $f(x) = P(X = x)$ and must have $0 \leq f(x) \leq 1$.

If X is *continuous*, then

- $f(x)$ *isn't a probability* — but it's used to *calculate* probabilities.
- Instead, think of $f(x)\,dx \approx P(x < X < x + dx)$.
- Must have $f(x) \geq 0$ (and possibly > 1).
- Calculate the probability of an event by integrating, $P(X \in A) = \int_A f(x)\,dx$.

Example: If X is "equally likely" to be anywhere between a and b, then X has the **uniform distribution** on (a, b).

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Notation: $X \sim \text{Unif}(a, b)$.

Remark: $\int_{\mathbb{R}} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = 1$ (as desired).

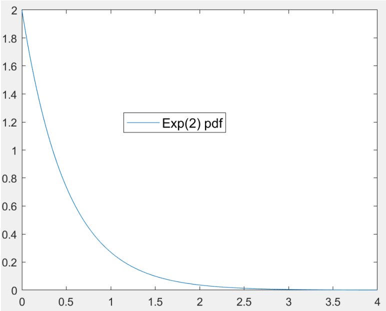
Example: If $X \sim \text{Unif}(-2, 8)$, then

$$P(-1 < X < 6) = \int_{-1}^{6} \frac{1}{8-(-2)}\,dx = 0.7. \quad \Box$$

Example: X has the **exponential distribution** with parameter $\lambda > 0$ if it has pdf $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.

Notation: $X \sim \text{Exp}(\lambda)$.

Remark: $\int_{\mathbb{R}} f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx = 1$ (as desired).



Example: Suppose $X \sim \text{Exp}(1)$. Then

$$P(X \leq 3) \;=\; \int_0^3 e^{-x}\,dx \;=\; 1 - e^{-3}.$$

$$P(X = 3) \;=\; \int_3^3 e^{-x}\,dx \;=\; 0.$$

$$P(X \geq 5) \;=\; \int_5^\infty e^{-x}\,dx \;=\; e^{-5}.$$

$$P(2 \leq X < 4) \;=\; P(2 \leq X \leq 4) \;=\; \int_2^4 e^{-x}\,dx \;=\; e^{-2} - e^{-4}. \quad \Box$$

Example: Suppose X is a continuous RV with pdf

$$f(x) = \begin{cases} cx^2 & \text{if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

First of all, let's find c. Noting that the pdf must integrate to 1, we have

$$1 = \int_{\mathbb{R}} f(x)\,dx = \int_0^2 cx^2\,dx = 8c/3,$$

so that $c = 3/8$.

Now we can calculate any reasonable probabilities, e.g.,

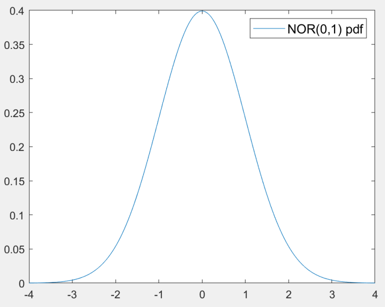
$$P(0 < X < 1) = \int_0^1 \frac{3}{8}x^2\,dx = 1/8.$$

And more-complicated ones, e.g.,

$$\begin{aligned}
P\left(0 < X < 1 \,\middle|\, \frac{1}{2} < X < \frac{3}{2}\right) & \\
&= \frac{P(0 < X < 1 \text{ and } \frac{1}{2} < X < \frac{3}{2})}{P(\frac{1}{2} < X < \frac{3}{2})} \\
&= \frac{P(\frac{1}{2} < X < 1)}{P(\frac{1}{2} < X < \frac{3}{2})} \\
&= \frac{\int_{1/2}^{1} \frac{3}{8}x^2\,dx}{\int_{1/2}^{3/2} \frac{3}{8}x^2\,dx} \\
&= 7/26. \quad \square
\end{aligned}$$

Example: X has the **standard normal distribution** if its pdf is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, for all $x \in \mathbb{R}$.

This is the famous "bell curve" distribution.



Outline

Lesson 1.4 — Cumulative Distribution Functions

Definition: For any RV X (discrete or continuous), the **cumulative distribution function (cdf)** is defined for all x by

$$\boxed{F(x) \equiv P(X \le x).}$$

For X discrete,

$$F(x) = \sum_{\{y|y \le x\}} f(y) = \sum_{\{y|y \le x\}} P(X = y).$$

For X continuous,

$$F(x) = \int_{-\infty}^{x} f(y)\, dy.$$

Discrete cdf's

Example: Flip a coin twice. Let X = number of H's.

$$X = \begin{cases} 0 \text{ or } 2 & \text{w.p. } 1/4 \\ 1 & \text{w.p. } 1/2. \end{cases}$$

Then the cdf is the following *step function*:

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/4 & \text{if } 0 \le x < 1 \\ 3/4 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2. \end{cases}$$

Warning! For *discrete* RVs, you have to be careful about "$\le$" vs. "$<$" at the endpoints of the intervals (where the step function jumps).

Continuous cdf's

Theorem: If X is a *continuous* RV, then $f(x) = F'(x)$ (assuming the derivative exists).

Proof: $F'(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$, by the Fundamental Theorem of Calculus. $\Box$

Example: $X \sim \text{Unif}(0,1)$. The pdf and cdf are

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1. \end{cases} \qquad \Box$$

Example: $X \sim \text{Exp}(\lambda)$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0. \end{cases}$$

We can use the cdf to find the **median** of X, that is, the point m such that

$$0.5 = F(m) = 1 - e^{-\lambda m}.$$

Solving, we obtain $m = (1/\lambda)\ell\text{n}(2)$. $\square$

Properties of all cdf's

$F(x)$ is *non-decreasing* in x, i.e., $a < b$ implies that $F(a) \leq F(b)$.

$\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$.

$F(x)$ is *right-continuous* at every point x.

Theorem: $P(X > x) = 1 - F(x)$.

Proof: By complements, $P(X > x) = 1 - P(X \leq x) = 1 - F(x)$. $\square$

Theorem: $a < b \Rightarrow P(a < X \le b) = F(b) - F(a)$.

Proof: Since $a < b$, we have

$$\begin{aligned} & P(a < X \le b) \\ &= P(X > a \cap X \le b) \\ &= P(X > a) + P(X \le b) - P(X > a \cup X \le b) \\ &= 1 - F(a) + F(b) - 1. \quad \Box \end{aligned}$$

Outline

Lesson 1.5 — Great Expectations

The Next Few Lessons:

- Mean (Expected Value)
- Law of the Unconscious Statistician
- Variance
- Probability Inequalities
- Approximations

Definition: The **mean** or **expected value** or **average** of a random variable X is

$$\boxed{\mu \;\equiv\; \mathrm{E}[X] \;\equiv\; \begin{cases} \sum_x x f(x) & \text{if } X \text{ is discrete} \\ \int_{\mathbb{R}} x f(x)\,dx & \text{if } X \text{ is continuous.} \end{cases}}$$

The mean gives an indication of a RV's *central tendency*. Think of it as a weighted average of the possible x's, where the weights are given by $f(x)$.

Example: Suppose X has the **Bernoulli distribution** with parameter p, i.e., $P(X = 1) = p$, $P(X = 0) = q = 1 - p$. Then

$$\mathrm{E}[X] \;=\; \sum_x x f(x) \;=\; (1 \cdot p) + (0 \cdot q) \;=\; p.$$

Example: Die toss. $X = 1, 2, \ldots, 6$, each w.p. 1/6. Then

$$\mathrm{E}[X] \;=\; \sum_x x f(x) \;=\; 1 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} \;=\; 3.5.$$

Extended Example: Suppose X has the **geometric distribution** with parameter p, i.e., X is the number of Bern(p) trials until you obtain your first success (e.g., $FFFFS$ gives $X = 5$). Then X has pmf

$$f(x) \;=\; (1-p)^{x-1}p,\;\; x = 1, 2, \ldots.$$

Notation: $X \sim \text{Geom}(p)$.

Here's an application that requires me to admit that I'm not a very good basketball player. ☹ Suppose I take independent foul shots, but the chance of making any particular shot is only 0.4. What's the probability that it'll take me at least 3 tries to make a successful shot?

Answer: The number of tries until my first success is $X \sim \text{Geom}(0.4)$. Thus,

$$\begin{aligned} P(X \geq 3) &= 1 - P(X \leq 2) \\ &= 1 - P(X = 1) - P(X = 2) \\ &= 1 - 0.4 - (0.6)(0.4) \;=\; 0.36. \quad \Box \end{aligned}$$

Now, let's find the expected value of $X \sim \text{Geom}(p)$.

$$\begin{aligned}
\text{E}[X] &= \sum_x x f(x) = \sum_{x=1}^{\infty} x q^{x-1} p \quad \text{(where } q = 1-p\text{)} \\
&= p \sum_{x=1}^{\infty} \frac{d}{dq} q^x \\
&= p \frac{d}{dq} \sum_{x=1}^{\infty} q^x \quad \text{(swap derivative and sum)} \\
&= p \frac{d}{dq} \frac{q}{1-q} \quad \text{(geometric sum)} \\
&= p \left[\frac{(1-q) - q(-1)}{(1-q)^2} \right] \quad \text{(ho de hi minus hi dee ho over ho ho)} \\
&= 1/p. \quad \Box
\end{aligned}$$

So it'll take, on average, $\text{E}[X] = 1/p = 1/0.4 = 2.5$ shots. $\Box$

Example: $X \sim \text{Exp}(\lambda)$. $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$. Then

$$\begin{aligned}
\text{E}[X] &= \int_{\mathbb{R}} x f(x)\, dx \\
&= \int_0^\infty x\lambda e^{-\lambda x}\, dx \\
&= -xe^{-\lambda x}\Big|_0^\infty - \int_0^\infty (-e^{-\lambda x})\, dx \quad \text{(by parts)} \\
&= \int_0^\infty e^{-\lambda x}\, dx \quad \text{(L'Hôspital's rule)} \\
&= 1/\lambda. \quad \square
\end{aligned}$$

Outline

Lesson 1.6 — LOTUS, Moments, and Variance

Law of the Unconscious Statistician (LOTUS)

Theorem: The expected value of a function of X, say $h(X)$, is

$$\boxed{\mathrm{E}[h(X)] \equiv \begin{cases} \sum_x h(x) f(x) & \text{if } X \text{ is discrete} \\ \int_{\mathbb{R}} h(x) f(x)\,dx & \text{if } X \text{ is continuous.} \end{cases}}$$

$\mathrm{E}[h(X)]$ is a weighted function of $h(x)$, where the weights are the $f(x)$'s.

Remark: It looks like a definition, but it's really a theorem — that's why they call it LOTUS!

Examples: $\mathrm{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx, \quad \mathrm{E}[\sin X] = \int_{\mathbb{R}} (\sin x) f(x)\,dx.$

Just a moment please...

Definition: The kth **moment** of X is

$$\boxed{\mathrm{E}[X^k] \;=\; \begin{cases} \sum_x x^k f(x) & \text{if } X \text{ is discrete} \\ \int_{\mathbb{R}} x^k f(x)\,dx & \text{if } X \text{ is continuous.} \end{cases}}$$

Example: Suppose $X \sim \mathrm{Bern}(p)$, so that $f(1) = p$ and $f(0) = q$.

$$\mathrm{E}[X^k] \;=\; \sum_x x^k f(x) \;=\; (0^k \cdot q) + (1^k \cdot p) \;=\; p \quad \text{for } \textit{all } k! \qquad \square$$

Example: $X \sim \text{Exp}(\lambda)$. $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$. Then

$$\begin{aligned}
\text{E}[X^k] &= \int_{\mathbb{R}} x^k f(x)\,dx \\
&= \int_0^\infty x^k \lambda e^{-\lambda x}\,dx \\
&= \int_0^\infty (y/\lambda)^k \lambda\, e^{-y} (1/\lambda)\,dy \quad \text{(substitute } y = \lambda x\text{)} \\
&= \frac{1}{\lambda^k} \int_0^\infty y^{(k+1)-1} e^{-y}\,dy \\
&= \frac{\Gamma(k+1)}{\lambda^k} \quad \text{(by definition of the gamma function)} \\
&= \frac{k!}{\lambda^k}. \quad \Box
\end{aligned}$$

Definition: The kth **central moment** of X is

$$\mathrm{E}[(X-\mu)^k] \;=\; \begin{cases} \sum_x (x-\mu)^k f(x) & X \text{ is discrete} \\ \int_{\mathbb{R}} (x-\mu)^k f(x)\,dx & X \text{ is continuous.} \end{cases}$$

Definition: The **variance** of X is the second central moment, i.e., the expected squared deviation of X from its mean,

$$\boxed{\mathrm{Var}(X) \;\equiv\; \mathrm{E}\big[(X-\mu)^2\big].}$$

Variance is a measure of spread or dispersion.

Notation: $\sigma^2 \equiv \mathrm{Var}(X)$.

Definition: The **standard deviation** of X is $\sigma \equiv +\sqrt{\mathrm{Var}(X)}$.

Example: $X \sim \text{Bern}(p)$, so that $f(1) = p$ and $f(0) = q = 1 - p$. Recall that $\mu = \text{E}[X] = p$. Then

$$\begin{aligned}
\text{Var}(X) &= \text{E}[(X-\mu)^2] \\
&= \sum_x (x-p)^2 f(x) \\
&= (0-p)^2 q + (1-p)^2 p \\
&= p^2 q + q^2 p \quad = \quad pq(p+q) \\
&= pq. \quad \square
\end{aligned}$$

The next results establish the fact that the expected value operator can pass through certain linear functions of X, and then can be used to obtain pleasant expressions for other expected values and variances.

Theorem: For any $h(X)$ and constants a and b — "shift" happens!

$$\mathrm{E}[ah(X) + b] \;=\; a\mathrm{E}[h(X)] + b.$$

Proof (just do continuous case):

$$\begin{aligned}
\mathrm{E}[ah(X) + b] &= \int_{\mathbb{R}} (ah(x) + b) f(x)\, dx \\
&= a \int_{\mathbb{R}} h(x) f(x)\, dx + b \int_{\mathbb{R}} f(x)\, dx \\
&= a\mathrm{E}[h(X)] + b. \quad \square
\end{aligned}$$

Corollary: In particular,

$$\boxed{\mathrm{E}[aX + b] \;=\; a\mathrm{E}[X] + b.}$$

Similarly, $\mathrm{E}[g(X) + h(X)] = \mathrm{E}[g(X)] + \mathrm{E}[h(X)]$.

Theorem (easier way to calculate variance):

$$\boxed{\mathrm{Var}(X) \;=\; \mathrm{E}[X^2] - (\mathrm{E}[X])^2.}$$

Proof: By the above results,

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2] \\
&= \mathrm{E}[X^2 - 2\mu X + \mu^2] \\
&= \mathrm{E}[X^2] - 2\mu\mathrm{E}[X] + \mu^2 \\
&= \mathrm{E}[X^2] - \mu^2. \quad \square
\end{aligned}$$

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Example: Suppose $X \sim \text{Bern}(p)$. Recall that $\text{E}[X^k] = p$, for all $k = 1, 2, \ldots$. Then

$$\text{Var}(X) \;=\; \text{E}[X^2] - (\text{E}[X])^2 \;=\; p - p^2 \;=\; pq. \quad \square$$

Example: $X \sim \text{Unif}(a, b)$. $f(x) = \frac{1}{b-a}$, $a < x < b$. Then

$$\text{E}[X] \;=\; \int_{\mathbb{R}} x f(x)\, dx \;=\; \int_a^b \frac{x}{b-a}\, dx \;=\; \frac{a+b}{2}$$

$$\text{E}[X^2] \;=\; \int_{\mathbb{R}} x^2 f(x)\, dx \;=\; \int_a^b \frac{x^2}{b-a}\, dx \;=\; \frac{a^2+ab+b^2}{3}$$

$$\text{Var}(X) \;=\; \text{E}[X^2] - (\text{E}[X])^2 \;=\; \frac{(a-b)^2}{12} \quad \text{(algebra)}. \quad \square$$

Theorem: Variance doesn't put up with any "shift" b,

$$\boxed{\mathrm{Var}(aX+b) \;=\; a^2\mathrm{Var}(X).}$$

Proof:

$$\begin{aligned}
\mathrm{Var}(aX+b) &= \mathrm{E}[(aX+b)^2] - (\mathrm{E}[aX+b])^2 \\
&= \mathrm{E}[a^2X^2 + 2abX + b^2] - (a\mathrm{E}[X]+b)^2 \\
&= a^2\mathrm{E}[X^2] + 2ab\mathrm{E}[X] + b^2 \\
&\qquad - \left\{a^2(\mathrm{E}[X])^2 + 2ab\mathrm{E}[X] + b^2\right\} \\
&= a^2\left(\mathrm{E}[X^2] - (\mathrm{E}[X])^2\right) \\
&= a^2\mathrm{Var}(X). \quad \square
\end{aligned}$$

Example: $X \sim \text{Bern}(0.3)$. Recall that

$$\text{E}[X] = p = 0.3 \quad \text{and} \quad \text{Var}(X) = pq = (0.3)(0.7) = 0.21.$$

Let $Y = h(X) = 4X + 5$. Then

$$\text{E}[Y] \;=\; \text{E}[4X+5] \;=\; 4\text{E}[X]+5 \;=\; 6.2$$

and

$$\text{Var}(Y) \;=\; \text{Var}(4X+5) \;=\; 16\text{Var}(X) \;=\; 3.36. \quad \square$$

Outline

Lesson 1.7 — Approximations to $\mathrm{E}[h(X)]$ and $\mathrm{Var}(h(X))$

Sometimes $Y = h(X)$ is messy, and we may have to approximate $\mathrm{E}[h(X)]$ and $\mathrm{Var}(h(X))$ via a Taylor series approach.

Let $\mu = \mathrm{E}[X]$ and $\sigma^2 = \mathrm{Var}(X)$, and note that

$$Y \;=\; h(\mu) + (X-\mu)h'(\mu) + \frac{(X-\mu)^2}{2}h''(\mu) + R,$$

where R is a remainder term that we'll ignore. Then

$$\mathrm{E}[Y] \;\doteq\; h(\mu) + \mathrm{E}[X-\mu]h'(\mu) + \frac{\mathrm{E}[(X-\mu)^2]}{2}h''(\mu) \;=\; h(\mu) + \frac{h''(\mu)\sigma^2}{2}$$

and (now an even-cruder approximation)

$$\mathrm{Var}(Y) \;\doteq\; \mathrm{Var}\big(h(\mu) + (X-\mu)h'(\mu)\big) \;=\; [h'(\mu)]^2\sigma^2.$$

Example: Suppose X has pdf $f(x) = 3x^2$, $0 \le x \le 1$, and we want to test out our approximations on the "complicated" RV $Y = h(X) = X^{3/4}$.

Well, it's not really that complicated, since we can calculate the *exact moments*:

$$\begin{aligned}
\mathrm{E}[Y] &= \int_{\mathbb{R}} x^{3/4} f(x)\,dx = \int_0^1 3x^{11/4}\,dx = 4/5 \\
\mathrm{E}[Y^2] &= \int_{\mathbb{R}} x^{6/4} f(x)\,dx = \int_0^1 3x^{7/2}\,dx = 2/3 \\
\mathrm{Var}(Y) &= \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = 2/75 = 0.0267.
\end{aligned}$$

Before we can do the approximation, note that

$$\begin{aligned}
\mu &= \mathrm{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^1 3x^3\,dx = 3/4 \\
\mathrm{E}[X^2] &= \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^1 3x^4\,dx = 3/5 \\
\sigma^2 &= \mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 3/80 = 0.0375.
\end{aligned}$$

Further,

$$\begin{aligned} h(\mu) &= \mu^{3/4} = (3/4)^{3/4} = 0.8059 \\ h'(\mu) &= (3/4)\mu^{-1/4} = (3/4)(3/4)^{-1/4} = 0.8059 \\ h''(\mu) &= -(3/16)\mu^{-5/4} = -0.2686. \end{aligned}$$

Thus,

$$\mathrm{E}[Y] \doteq h(\mu) + \frac{h''(\mu)\sigma^2}{2} = 0.8059 - \frac{(0.2686)(0.0375)}{2} = 0.8009$$

and

$$\mathrm{Var}(Y) \doteq [h'(\mu)]^2\sigma^2 = (0.8059)^2(0.0375) = 0.0243,$$

both of which are reasonably close to their true values (0.8 and 0.0267, respectively). $\square$

Outline

Lesson 1.8 — Moment Generating Functions

Recall that $\mathrm{E}[X^k]$ is the kth **moment** of X.

Definition: The **moment generating function** (mgf) of the RV X is

$$\boxed{M_X(t) \equiv \mathrm{E}\left[e^{tX}\right].}$$

Remark: $M_X(t)$ is a function of t, *not* of X!

Example: $X \sim \mathrm{Bern}(p)$, so that $X = 1$ w.p. p and 0 w.p. q. Then

$$M_X(t) = \mathrm{E}[e^{tX}] = \sum_x e^{tx} f(x) = e^{t \cdot 1} p + e^{t \cdot 0} q = pe^t + q. \quad \square$$

Example: If $X \sim \text{Exp}(\lambda)$, then

$$
\begin{aligned}
M_X(t) &= \mathrm{E}[e^{tX}] \\
&= \int_{\mathbb{R}} e^{tx} f(x)\, dx \quad \text{(LOTUS)} \\
&= \int_0^\infty e^{tx}\, \lambda e^{-\lambda x}\, dx \\
&= \lambda \int_0^\infty e^{(t-\lambda)x}\, dx \\
&= \frac{\lambda}{\lambda - t} \quad \text{if } \lambda > t. \quad \square
\end{aligned}
$$

Big Theorem: Under certain technical conditions (e.g., $M_X(t)$ must exist for all $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$), we have

$$\boxed{\mathrm{E}[X^k] \;=\; \frac{d^k}{dt^k} M_X(t)\Big|_{t=0}, \quad k = 1, 2, \ldots.}$$

Thus, you can *generate* the moments of X from the mgf. (Sometimes, it's easier to get moments this way than directly.)

"Proof" (a little non-rigorous):

$$\begin{aligned} M_X(t) &= \mathrm{E}[e^{tX}] \text{ ``$=$'' } \mathrm{E}\left[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right] \text{ ``$=$'' } \sum_{k=0}^{\infty} \mathrm{E}\left[\frac{(tX)^k}{k!}\right] \\ &= 1 + t\,\mathrm{E}[X] + \frac{t^2\,\mathrm{E}[X^2]}{2} + \cdots . \end{aligned}$$

This implies

$$\frac{d}{dt} M_X(t) \text{ ``$=$'' } \mathrm{E}[X] + t\,\mathrm{E}[X^2] + \cdots$$

and so

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = \mathrm{E}[X].$$

Same deal for higher-order moments. $\square$

Example: $X \sim \text{Bern}(p)$. Then $M_X(t) = pe^t + q$, and

$$\mathrm{E}[X] \;=\; \frac{d}{dt}M_X(t)\Big|_{t=0} \;=\; \frac{d}{dt}(pe^t + q)\Big|_{t=0} \;=\; pe^t\big|_{t=0} \;=\; p.$$

In fact, it's easy to see that $\mathrm{E}[X^k] = \frac{d^k}{dt^k}M_X(t)\big|_{t=0} = p$, for all k. $\square$

Example: $X \sim \text{Exp}(\lambda)$. Then $M_X(t) = \frac{\lambda}{\lambda - t}$ for $\lambda > t$. So

$$\mathrm{E}[X] \;=\; \frac{d}{dt}M_X(t)\Big|_{t=0} \;=\; \frac{\lambda}{(\lambda - t)^2}\Big|_{t=0} \;=\; 1/\lambda$$

$$\mathrm{E}[X^2] \;=\; \frac{d^2}{dt^2}M_X(t)\Big|_{t=0} \;=\; \frac{2\lambda}{(\lambda - t)^3}\Big|_{t=0} \;=\; 2/\lambda^2$$

$$\mathrm{Var}(X) \;=\; \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \;=\; \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \;=\; 1/\lambda^2. \quad \square$$

Other Applications of mgf's

You can do lots of nice things with mgf's....

- Find the mgf of a linear function of X. (Just a second!)
- Identify distributions. (Just a minute!)
- Probability inequality applications. (Next lesson!)
- Find the mgf of the sum of independent random variables. (Later!)
- Convergence of random variables proofs. (Another course!)

Theorem (mgf of a linear function of X): Suppose X has mgf $M_X(t)$ and let $Y = aX + b$. Then

$$M_Y(t) \; = \; e^{tb} M_X(at).$$

Proof:

$$M_Y(t) \; = \; \mathrm{E}[e^{tY}] \; = \; \mathrm{E}[e^{t(aX+b)}] \; = \; e^{tb}\mathrm{E}[e^{(at)X}] \; = \; e^{tb} M_X(at). \quad \Box$$

Example: Let $X \sim \mathrm{Exp}(\lambda)$ and $Y = 3X + 2$. Then

$$M_Y(t) \; = \; e^{2t} M_X(3t) \; = \; e^{2t} \frac{\lambda}{\lambda - 3t}, \quad \text{if } \lambda > 3t. \quad \Box$$

Theorem (identifying distributions): *In this class,* each distribution has a unique mgf.

Proof: Not here!

Example: Suppose that Y has mgf

$$M_Y(t) \;=\; e^{2t}\frac{\lambda}{\lambda - 3t}, \quad \text{for } \lambda > 3t.$$

Then by a previous example and the uniqueness of mgf's, it *must* be the case that $Y \sim 3X + 2$, where $X \sim \text{Exp}(\lambda)$. $\Box$

Outline

Lesson 1.9 — Some Probability Inequalities

Goal: Give results that provide general probability bounds.

Theorem (Markov's Inequality): If X is a nonnegative random variable and $c > 0$, then $P(X \geq c) \leq \mathrm{E}[X]/c$. (This is a very crude upper bound.)

Proof: Because X is nonnegative, we have

$$\begin{aligned}
\mathrm{E}[X] &= \int_{\mathbb{R}} x f(x)\,dx \\
&= \int_0^\infty x f(x)\,dx \\
&\geq \int_c^\infty x f(x)\,dx \\
&\geq c \int_c^\infty f(x)\,dx \\
&= cP(X \geq c). \quad \square
\end{aligned}$$

Theorem (Chebychev's Inequality):[1] Suppose $\mathrm{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Then, for any $c > 0$,

$$\boxed{P\big(|X - \mu| \geq c\big) \;\leq\; \sigma^2/c^2.}$$

Proof: By Markov with $|X - \mu|^2$ in place of X and c^2 in place of c, we have

$$P\big(|X - \mu| \geq c\big) \;=\; P\big((X - \mu)^2 \geq c^2\big) \;\leq\; \frac{\mathrm{E}[(X - \mu)^2]}{c^2} \;=\; \sigma^2/c^2. \quad \square$$

Remarks: Can also write $P(|X - \mu| < c) \geq 1 - \sigma^2/c^2$.

Or, if $c = k\sigma$, then $P(|X - \mu| \geq k\sigma) \leq 1/k^2$.

Chebychev gives a bound on the probability that X deviates from the mean by more than a constant, in terms of the constant and the variance. You can always use Chebychev, but it's crude.

[1] There are many, many ways to spell "Chebychev." See https://en.wikipedia.org/wiki/Talk%3APafnuty_Chebyshev

Example: Suppose $X \sim \text{Unif}(0,1)$. $f(x) = 1$ for $0 < x < 1$.

Recall that $\text{E}[X] = 1/2$, $\text{Var}(X) = 1/12$.

Then Chebychev implies

$$P\left(\left|X - \frac{1}{2}\right| \geq c\right) \;\leq\; \frac{1}{12c^2}.$$

In particular, for $c = 1/3$,

$$P\left(\left|X - \frac{1}{2}\right| \geq \frac{1}{3}\right) \;\leq\; \frac{3}{4} \quad \text{(upper bound)}.$$

Example (cont'd): Let's compare the above bound to the *exact* answer.

$$\begin{aligned} P\left(\left|X - \frac{1}{2}\right| \geq \frac{1}{3}\right) &= 1 - P\left(\left|X - \frac{1}{2}\right| < \frac{1}{3}\right) \\ &= 1 - P\left(-\frac{1}{3} < X - \frac{1}{2} < \frac{1}{3}\right) \\ &= 1 - P\left(\frac{1}{6} < X < \frac{5}{6}\right) \\ &= 1 - \int_{1/6}^{5/6} f(x)\,dx \\ &= 1 - \frac{2}{3} = 1/3. \end{aligned}$$

So Chebychev bound of 3/4 was pretty high by comparison. □

Bonus Theorem (Chernoff's Inequality): For any c,

$$P(X \geq c) \;\leq\; e^{-ct} M_X(t).$$

Proof: By Markov with e^{tX} in place of X and e^{tc} in place of c, we have

$$P(X \geq c) \;=\; P(e^{tX} \geq e^{tc}) \;=\; e^{-ct}\mathrm{E}[e^{tX}] \;=\; e^{-ct} M_X(t). \quad \square$$

Example: Suppose X has the **standard normal distribution** with pdf $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, for all $x \in \mathbb{R}$, i.e., the famous "bell curve" distribution.

It is easy to show (via a little calculus elbow grease involving completing a square) that the mgf of the standard normal is

$$M_X(t) \;=\; \mathrm{E}[e^{tX}] \;=\; \int_{\mathbb{R}} e^{tx}\phi(x)\,dx \;=\; e^{t^2/2}.$$

Then using Chernoff with $t = c$ immediately yields the tail probability

$$P(X \geq c) \;\leq\; e^{-c^2} M_X(c) \;=\; e^{-c^2/2}. \quad \square$$

Outline

Lesson 1.10 — Functions of a Random Variable

The Next Few Lessons:

- Problem Statement with Examples
  - Discrete Case
  - Continuous Case
- Inverse Transform Theorem
- Some Honors Topics

Problem: You have a RV X and you know its pmf/pdf $f(x)$.

Define $Y \equiv h(X)$ (some function of X).

Find $g(y)$, the pmf/pdf of Y.

Remark: Recall that LOTUS gave us results for $\mathrm{E}[h(X)]$. But this is much more general than LOTUS, because now we're going to get the *entire distribution* of $h(X)$.

We'll start with the case in which X is a discrete RV, and then we'll go to the continuous X case.

Discrete Case: X discrete implies Y discrete implies

$$g(y) = P(Y = y) = P(h(X) = y) = P(\{x | h(x) = y\}) = \sum_{x|h(x)=y} f(x).$$

Example: X is the number of H's in 2 coin tosses. We want the pmf for $Y = h(X) = X^3 - X$.

x	0	1	2
$f(x) = P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$y = x^3 - x$	0	0	6

$g(0) = P(Y = 0) = P(X = 0 \text{ or } 1) = 3/4$ and
$g(6) = P(Y = 6) = P(X = 2) = 1/4$.

$$g(y) = \begin{cases} 3/4 & \text{if } y = 0 \\ 1/4 & \text{if } y = 6. \end{cases} \quad \square$$

Example: X is discrete with

$$f(x) = \begin{cases} 1/8 & \text{if } x = -1 \\ 3/8 & \text{if } x = 0 \\ 1/3 & \text{if } x = 1 \\ 1/6 & \text{if } x = 2. \end{cases}$$

Let $Y = X^2$ (so Y can only equal 0, 1 or 4).

$$g(y) = \begin{cases} P(Y = 0) = f(0) = 3/8 \\ P(Y = 1) = f(-1) + f(1) = 11/24 \\ P(Y = 4) = f(2) = 1/6. \quad \Box \end{cases}$$

Continuous Case: X continuous implies Y can be continuous *or* discrete.

Example: $Y = X^2$ (clearly continuous).

Example: $Y = \begin{cases} 0 & \text{if } X < 0 \\ 1 & \text{if } X \geq 0 \end{cases}$ is *not* continuous.

Method: Compute $G(y)$, the cdf of Y.

$$G(y) = P(Y \leq y) = P(h(X) \leq y) = \int_{\{x|h(x)\leq y\}} f(x)\, dx.$$

If $G(y)$ is continuous, construct the pdf $g(y)$ by differentiating.

Example: $f(x) = |x|$, $-1 \le x \le 1$.

Find the pdf of the RV $Y = h(X) = X^2$.

$$G(y) \;=\; P(Y \le y) \;=\; P(X^2 \le y) \;=\; \begin{cases} 0 & \text{if } y \le 0 \\ 1 & \text{if } y \ge 1 \\ (\star) & \text{if } 0 < y < 1, \end{cases}$$

where

$$(\star) \;=\; P(-\sqrt{y} \le X \le \sqrt{y}) \;=\; \int_{-\sqrt{y}}^{\sqrt{y}} |x|\,dx \;=\; y.$$

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Thus,

$$G(y) = \begin{cases} 0 & \text{if } y \le 0 \\ 1 & \text{if } y \ge 1 \\ y & \text{if } 0 < y < 1. \end{cases}$$

This implies

$$g(y) = G'(y) = \begin{cases} 0 & \text{if } y \le 0 \text{ or } y \ge 1 \\ 1 & \text{if } 0 < y < 1. \end{cases}$$

This means that Y has the Unif(0,1) distribution! $\Box$

Example: Suppose $U \sim \mathrm{Unif}(0,1)$. Find the pdf of $Y = -\ell\mathrm{n}(1-U)$.

$$\begin{aligned} G(y) &= P(Y \le y) \\ &= P(-\ell\mathrm{n}(1-U) \le y) \\ &= P(1-U \ge e^{-y}) \\ &= P(U \le 1-e^{-y}) \\ &= \int_0^{1-e^{-y}} f(u)\,du \\ &= 1-e^{-y} \quad (\text{since } f(u)=1). \end{aligned}$$

Taking the derivative, we have $g(y) = e^{-y}$, $y > 0$.

Wow! This implies $Y \sim \mathrm{Exp}(\lambda = 1)$. $\Box$

We can generalize this result. . . .

Outline

Lesson 1.11 — Inverse Transform Theorem

Here's a terrific result that has lots of applications.

Inverse Transform Theorem:[2] Suppose X is a continuous random variable having cdf $F(x)$. Then the *random variable* $F(X) \sim \mathrm{Unif}(0, 1)$.

Proof: Let $Y = F(X)$. Then the cdf of Y is

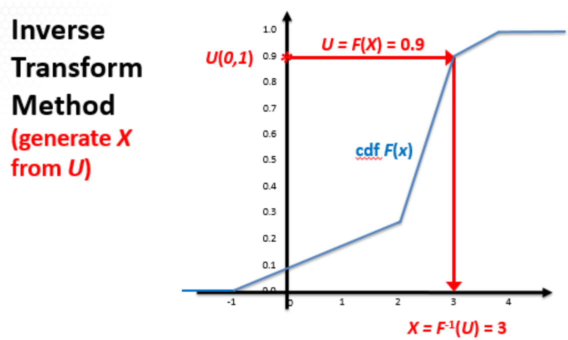
$$\begin{aligned} G(y) &= P(Y \leq y) \\ &= P(F(X) \leq y) \\ &= P(X \leq F^{-1}(y)) \text{ (the cdf is mono. increasing)} \\ &= F(F^{-1}(y)) \text{ ($F(x)$ is the cdf of X)} \\ &= y. \quad \text{Uniform!} \quad \square \end{aligned}$$

[2] Also known as the Probability Integral Transform.

Remark: This is a great theorem, since it applies to all continuous RVs X.

Corollary: $X = F^{-1}(U)$, so you can plug a Unif(0,1) RV into the inverse cdf to generate a realization of a RV having X's distribution.

Method: Set $F(X) = U$ and solve for $X = F^{-1}(U)$ to generate X.

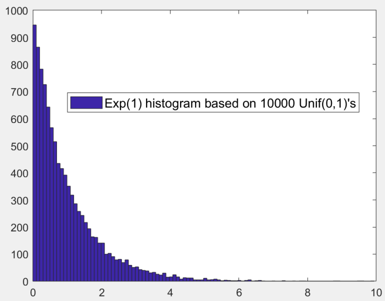


Example: Suppose X is Exp(λ), so that it has cdf $F(x) = 1 - e^{-\lambda x}$. Similar to a previous example, set $F(X) = 1 - e^{-\lambda X} = U$ and generate an Exp(λ) RV by solving for

$$X \;=\; F^{-1}(U) \;=\; -\frac{1}{\lambda}\ell\mathrm{n}(1-U) \;\sim\; \mathrm{Exp}(\lambda). \quad \Box$$

Remark: If you'd like to generate a nice, beautiful Exp(λ) pdf on a computer, then all you have to do is...

- Generate 10000 Unif(0,1)'s (e.g., use the `rand` function in Excel, or `unifrnd` in Matlab),
- Plug those 10000 into the equation for X above, and
- Plot the histograms of the X's.



Remark: This trick has tremendous applications in simulation, where we need to generate random variables all the time (e.g., customer arrival times, service times, machine breakdown times, etc.).

Outline

Lesson 1.12 — Honors Bonus Results

Another Way to Find the pdf of a Function of a Continuous RV

Suppose that $Y = h(X)$ is a monotonic function of a continuous RV X having pdf $f(x)$ and cdf $F(x)$. Let's get the pdf $g(y)$ of Y directly.

$$\begin{aligned} g(y) &= \frac{d}{dy}G(y) \;=\; \frac{d}{dy}P(Y \le y) \\ &= \frac{d}{dy}P(h(X) \le y) \\ &= \frac{d}{dy}P(X \le h^{-1}(y)) \quad (h(x) \text{ is monotonic}) \\ &= \frac{d}{dy}F(h^{-1}(y)) \\ &= f(h^{-1}(y)) \left| \frac{dh^{-1}(y)}{dy} \right| \quad \text{(chain rule)}. \quad \square \end{aligned}$$

Example: Suppose that $f(x) = 3x^2$, $0 < x < 1$. Let $Y = h(X) = X^{1/2}$, which is monotone increasing. Then the pdf of Y is

$$\begin{aligned} g(y) &= f(h^{-1}(y)) \left| \frac{dh^{-1}(y)}{dy} \right| \\ &= f(y^2) \left| \frac{d(y^2)}{dy} \right| \\ &= 3y^4(2y) \\ &= 6y^5, \quad 0 < y < 1. \quad \Box \end{aligned}$$

Theorem (why LOTUS works): Let's assume that $h(\cdot)$ is monotonically increasing. Then

$$
\begin{aligned}
\mathrm{E}[h(X)] &= \mathrm{E}[Y] \\
&= \int_{\mathbb{R}} y g(y)\, dy \\
&= \int_{\mathbb{R}} y f(h^{-1}(y)) \left| \frac{dh^{-1}(y)}{dy} \right| dy \\
&= \int_{\mathbb{R}} h(x) f(x) \left| \frac{dx}{dy} \right| dy \\
&= \int_{\mathbb{R}} h(x) f(x)\, dx. \quad \Box
\end{aligned}
$$

Next up: Bivariate generalizations of the material in this module!