

Trig without Tears

or, How to Remember Trigonometric Identities

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Summary: Faced with the large number of trigonometric identities, students tend to try to **memorize them all**. That way lies **disaster**. When you memorize a formula by rote, you have no way to know whether you're remembering it correctly. I believe it is **much more effective** (and, in the long run, **much easier**) to understand thoroughly how the trig functions work, memorize **half a dozen formulas**, and work out the rest as needed. That's what these pages show you how to do.

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Trig without Tears Part 1:

Introduction

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About Trigonometry

Trigonometry is fascinating! It started as the measurement (Greek *metron*) of triangles (Greek *trigonon*), but now it has been formalized under the influence of algebra and analytic geometry and we talk of **trigonometric functions**, not just **sides and angles of triangles**.

Trig is almost the **ideal math subject**. Big and complex enough to have all sorts of interesting odd corners, it is still small and regular enough to be taught thoroughly in a semester. (You can easily master the essential points in a week or so.) It has lots of obvious practical uses, some of which are actually taught in the usual trig course. And trig extends plenty of tentacles into other fields like complex numbers, logarithms, and calculus.

If you'd like to learn some of the history of trigonometry and peer into its dark corners, I recommend *Trigonometric Delights* by Eli Maor (Princeton University Press, 1998).

The computations in trigonometry used to be a big obstacle. But now that we have calculators, that's no longer an issue.

Would you believe that when I studied trig, back when dinosaurs ruled the earth (actually, in the 1960s), to solve any problem we had to look up function values in long tables in the back of the book, and then multiply or divide those five-place decimals *by hand*? The "better" books even included tables of logs of the trig functions, so that we could save work by adding and subtracting five-place decimals instead of multiplying and dividing them. My *College Outline Series* trig book covered all of plane and spherical trigonometry in 188 pages—but then needed an additional 138 pages for the necessary tables!

Though calculators have freed us from tedious computation, there's still one big stumbling block in the way many trig courses are taught: all those identities. They're just too much to memorize. (Many students despair of understanding what's going on, so they just try to memorize everything and hope for the best at exam time.) Is it $\tan^2 A + \sec^2 A = 1$ or $\tan^2 A = \sec^2 A + 1$? (Actually, it's neither—see [equation 39](#)!)

Fortunately, you don't *need* to memorize them. This paper shows you the few that you do need to memorize, and how you can produce the others as needed. I'll present some ideas of my own, and a [wonderful insight by W.W. Sawyer](#).

About Trig without Tears

I wrote *Trig without Tears* to show that **you need to memorize very little**. Instead, you learn how all the pieces of trigonometry hang together, and you get used to combining identities in different ways so that you can derive most **results on the fly in just a couple of steps**.

You might like to read some ideas of mine on [the pros and cons of memorizing](#).

To help you find things, I'll number the most important equations and other facts. (Don't worry about the gaps in the numbering. I've left those to make it easier to add information to these pages.)

A very few of those, which you need to memorize, will be marked "**memorize**". Please **don't memorize the others**. The whole point of *Trig without Tears* is to teach you how to derive them as needed without memorizing them. If you