Chapter 2

Boolean Formulas and Equations

Symbolic logic, like other parts of mathematics, starts from a small collection of axioms and employs rules of inference to find additional propositions consistent with those axioms. This chapter will define a grammar of logic formulas, specify a few equations stating that certain formulas carry the same meaning as others, and derive new equations using substitution of equals for equals as the rule of inference.

You will probably find this familiar from your experience with numeric algebra, but we will attend carefully to details, and this formality may extend beyond what you are accustomed to. What it buys us is mechanization. That is, our logic formulas and our reasoning about them will amount to mechanized computation, and this will make it possible to computers to check that our reasoning follows all the rules, without exception. This gives us a higher level of confidence in our conclusions than would otherwise be possible.

We will be doing all of this in the domain of symbolic logic, which includes operations like "logical or" and "logical negation", rather than arithmetic operations, such as addition and multiplication. Logical operations put our formulas in the domain of Boolean algebra, rather than numeric algebra, but our rule of inference, which is substitution of equals for equals, applies equally well in both the Boolean and the numeric domains. To illustrate the level of formality that we are shooting for, let's see how it works with a problem in numeric algebra.

You are surely familiar with the equation $(-1) \times (-1) = 1$, but you may

Hold on to your seat! This section illustrates an essential method used throughout the book. It introduces the notion of "formality" in mathematical argumentation. This is "formality" in the sense of being "based on formulas." The formulas involved have a prescribed grammar similar to the one for numeric formulas that you have used for many years. This grammar is how you know that the formula "$x + 3 \times (y + z)$" is well formed and the non-formula "$x + 3 \times (y+) \times z$" is not well formed.
Things may seem overly simple at the very beginning. Then, suddenly, you may find yourself thrashing around in deep water. Take a deep breath, and slowly work through the material. It provides a basis for everything to follow. It calls for careful study and frequent review when things start to go off track. Fold the corner of this page down. You will probably need to come back to it later.

Aside 3: Hold on to Your Seat

$$
\begin{array}{ll}
x + 0 = x & \{+ \text{ identity}\} \\
(-x) + x = 0 & \{+ \text{ complement}\} \\
x \times 1 = x & \{\times \text{ identity}\} \\
x \times 0 = 0 & \{\times \text{ null}\} \\
x + y = y + x & \{+ \text{ commutative}\} \\
x + (y + z) = (x + y) + z & \{+ \text{ associative}\} \\
x \times (y + z) = (x \times y) + (x \times z) & \{\text{distributive law}\}
\end{array}
$$

Figure 2.1: Equations of Numeric Algebra

not know that it is a consequence of some basic facts about arithmetic calculation. That is, the fact that multiplying two negative numbers produces a positive one is not independent of other facts about numbers. Instead, it is an inference one can draw from an acceptance of other familiar equations.

The equations in Figure 2.1 (page 14) express some standard rules of numeric computation. In those equations, the letters stand in place of numbers, or in place of other formulas expressed in terms of common numeric operations (addition, multiplication, etc). We refer to letters used in this way as "variables" even though, within a particular equation, they stand for a fixed number or for a particular formula. Their values, though unspecific, do not vary.

If we accept the equations of Figure 2.1 (pare 14) as axioms, we can apply one of them to transform the formula $(-1) \times (-1)$ to a new formula that stands for the same number. Then, we can apply another axiom to transform that formula to a new one, and so on. We apply the axiomatic equations in such a way that, at some point, we arrive at the formula "1". At every stage, we know that the new formula stands for the same value as the old one, so in the end we know that $(-1) \times (-1) = 1$.

Figure 2.2 (page 16) displays this sort of equation-by-equation derivation of the formula "1" from the formula "$(-1) \times (-1)$". To understand Figure 2.2, you must remember that each variable can denote any grammatically correct formula. For example, in the {+ identity} equation, $x+0 = x$, the variable x could stand for a number, such as 3, or it could stand for a more complicated formula, such as $(1+3)$. It could even stand for a formula with variables in it, such as $(a + (b \times c))$ or $(((-1) \times (x + 3)) + (x + y))$.

Another crucial point is that each step cites exactly one equation from Figure 2.1 (page 14) to justify the transformation from the formula in the previous step. We are so accustomed to calculating numeric formulas that we often combine many basic steps into one. When we reason formally, we must not do this. We must justify each step citing an equation from a list of known equations. In our proof of $(-1) \times (-1) = 1$, we will justify steps by citing equations from Figure 2.1 and from no other source. We will not skip steps. Think of that as you go through the proof, line by line.

The first step in the proof (Figure 2.2, page 16) uses a version of the {+ identity} equation in which the variable x stands for the formula $((-1) \times (-1))$. The second step reads the {+ complement} equation backwards (equations go both ways), but in a form where the variable x stands for the number 1. And so on. The transformations, step by step, finally confirm that the two formulas $(-1) \times (-1)$ and 1 stand for the same number.

Pay particular attention to the last three lines. Most people tend to jump from the formula $0 + 1$ to the formula 1 in one step. This requires knowing the equation $0+1 = 1$. However, that equation is not among those listed in Figure 2.1 (page 14). To do the proof without citing any equations other than those in Figure 2.1, we need two steps, and those are the last two steps in the proof.

One of the things we hope you will glean from this derivation is that the equation $(-1)\times(-1) = 1$ does not depend on vague, philosophical assertions like "two negatives make a positive." Instead, the equation $(-1) \times (-1) = 1$ is a consequence of some basic arithmetic equations. If you accept the basic equations and the idea of substituting equals for equals, you must, as a

$$
\begin{array}{lll}
 & (-1) \times (-1) & \\
= & ((-1) \times (-1)) + 0 & \{+ \text{ identity}\} \\
= & ((-1) \times (-1)) + ((-1) + 1) & \{+ \text{ complement}\} \\
= & (((-1) \times (-1)) + (-1)) + 1 & \{+ \text{ associative}\} \\
= & (((-1) \times (-1)) + ((-1) \times 1)) + 1 & \{\times \text{ identity}\} \\
= & ((-1) \times ((-1) + 1)) + 1 & \{\text{distributive law}\} \\
= & ((-1) \times 0) + 1 & \{+ \text{ complement}\} \\
= & 0 + 1 & \{\times \text{ null}\} \\
= & 1 + 0 & \{+ \text{ commutative}\} \\
= & 1 & \{+ \text{ identity}\}
\end{array}
$$

Figure 2.2: Why $(-1) \times (-1) = 1$

rational consequence, accept the equation $(-1) \times (-1) = 1$.

Using this same kind of reasoning, we will derive equations between formulas in logic from a few, simple equations postulated as axioms. We will also learn that digital circuits are physical representations of logic formulas, and we will be able to parlay this basic idea to derive behavioral properties of computer components.

Likewise, because a computer program is, literally, a logic formula, we will be able to derive properties of those programs directly from the programs, themselves. This makes it possible for us to be entirely certain about some of the behavioral characteristics of software, and of computer hardware, too, since a hardware component is also a logic formula. Our certainty stems from the mechanistic formalism that we insist on from the beginning, which can be checked to the last detail with automated computation.

2.1 Boolean equations

Let's start with the Boolean equations in Figure 2.3 (page 17). If they look strange to you, try not to worry. Everything new seems strange for a while. Try to view them as ordinary, algebraic equations, but with a different collection of operators. A formula in numeric algebra contains operations like addition (+) and multiplication ($\times$). Boolean formulas employ logic operations: logical-and ($\wedge$), logical-or ($\vee$), logical-negation ($\neg$), and implication ($\rightarrow$).

$x \vee False = x$	$\{\vee$ identity$\}$
$x \vee True = True$	$\{\vee$ null$\}$
$x \vee y = y \vee x$	$\{\vee$ commutative$\}$
$x \vee (y \vee z) = (x \vee y) \vee z$	$\{\vee$ associative$\}$
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$\{\vee$ distributive$\}$
$x \rightarrow y = (\neg x) \vee y$	{implication}
$\neg(x \vee y) = (\neg x) \wedge (\neg y)$	$\{\vee$ DeMorgan$\}$
$x \vee x = x$	$\{\vee$ idempotent$\}$
$x \rightarrow x = True$	{self-implication}
$\neg(\neg x) = x$	{double negation}

Figure 2.3: Basic Boolean equations (axioms)

Furthermore, Boolean formulas stand for logic values ($True$, $False$), rather than for numbers (. . . -1, 0, 1, 2 . . .). So, in Boolean algebra there are just two basic values ($True$, $False$), not an infinite collection of numbers. That does not limit the potential domain of discourse, however. By aggregating the basic $True/False$ elements in sequences, we will find that we can deal with numbers and with the full range of things that numbers can represent.

When we derive new equations from equations we already know, we refer to the derived equations as "theorems" to distinguish them from axioms. We call the derivation a "proof" of the theorem.

The first equation in the following theorem $\{\vee$ truth table$\}$ is a special case of the $\{\vee$ identity$\}$ axiom in Figure 2.3 (see page 17), and the proof of that equation consists of that observation. The proof of the second equation is equally short, but cites a different equation in the axioms of Figure 2.3 to justify the transformation from $False \vee True$ to $True$. For practice, try to prove the other two equations in the $\{\vee$ truth table$\}$ theorem by citing axioms in a similar way.

Theorem 1 ($\{\vee$ truth table$\}$)**.**

- $False \vee False = False$
- $False \vee True = True$
- $True \vee False = True$

- $True \vee True = True$

Proof.

$$
\begin{array}{lll}
 & False \vee False & \\
= & False & \{\vee \text{ identity}\} \ \ldots\text{taking } x \text{ in the axiom to stand for False}
\end{array}
$$

$$
\begin{array}{lll}
 & False \vee True & \\
= & True & \{\vee \text{ null}\} \ \ldots\text{taking } x \text{ in the axiom to stand for False}
\end{array}
$$

...for practice, prove the other two equations yourself ...

Q.E.D.

We are serious about that. Did you prove the other two equations? No? Well ...go back and do it, then. Without participation, there is no learning.

Finished now? Good for you. You cited the {∨ identity} axiom in your proof of the third equation in the theorem and the {∨ null} axiom in your proof of the fourth equation, right? We knew you could do it.

A "truth table" for a formula is a list of the values the formula represents, with one entry in the list for every possible combination of the values of its variables. If there is only one variable in the formula, there will be two entries in its truth table, one for the case when the variable has the value *True* and one for the case when the variable has the value *False*. If there are two variables in the formula, there will be four entries in the truth table because for each choice of the first variable, there are choices of the other. Three variables lead to eight entries. The number of entries doubles with each new variable.
A truth table for a logical operator is the truth table for the formula that has variables in place of the operands. For example, the truth table for the logical-or operator (∨) is the truth table for the formula $x \vee y$. That formula has two variables, so the truth table has four entries.

Aside 4: Truth Tables

Derivations are usually more complicated, of course. For example, the following {∨ complement} theorem is not a special case of any of the axioms, but has a two-step proof, citing implication and self-implication.

Theorem 2 ({$\vee$ complement})**.** $(\neg x) \vee x = True$

Proof.

$$
\begin{array}{lll}
 & (\neg x) \vee x & \\
= & x \to x & \{\text{implication}\} \\
= & True & \{\text{self-implication}\}
\end{array}
$$

Q.E.D.

The {$\vee$ complement} theorem is often referred to as the "law of the excluded middle" because it states that any logical statement, together with its negation, comprises all of the possibilities. A logical statement is either true of false. There is no middle ground.

All of the logical operators have truth tables, and we can derive the equations in those truth tables from the axioms. The following theorem provides the truth table for the negation ($(\neg x)$) operator. The proof of the first equation in the {$\neg$ truth table} theorem has a four step proof. To beef up your comprehension of the ideas, construct your own proof of the second equation in the theorem.

Theorem 3 ({$\neg$ truth table})**.**

- $\neg True = False$
- $\neg False = True$

Proof.

$$
\begin{array}{lll}
 & \neg True & \\
= & \neg(False \to False) & \{\text{self-implication}\} \\
= & \neg((\neg False) \vee False) & \{\text{implication}\} \text{ (taking both } x \text{ and } y \text{ in the axiom to stand for } False) \\
= & \neg(\neg False) & \{\vee \text{ identity}\} \text{ (taking } x \text{ in the axiom to stand for } \neg False) \\
= & False & \{\text{double negation}\} \text{ (taking } x \text{ in the axiom to stand for False)}
\end{array}
$$

$$
\begin{array}{ll}
 & \neg False \\
= & \ldots \text{you fill in the details here} \ldots \\
= & True
\end{array}
$$

Q.E.D.

An important facet of these proofs is that they are entirely syntactic. That is, they apply axioms by matching the grammar of a formula f in the proof with the formula g in an equation from the axioms. This matching associates the variables in g with certain sub-formulas in the formula f. Then, the formula h on the other side of the equation, with the same association between its variables and sub-formulas of f, becomes the new, derived formula. We know that the derived formula stands for the same value as the original formula because the axiom asserts this relationship, and we are assuming that axioms are right.

Let's do another truth-table theorem, partly to practice reasoning with equations, but also to discuss a common point of confusion about logic. The implication operator ($\to$) is a cornerstone logic in real-world problems, but many people misunderstand its meaning—that is, its truth table.

The {$\to$ truth table} theorem provides the truth table for the implication operator. An important aspect of the proof is that it cites not only axioms from Figure 2.3 (see page 17), but also equations from the {$\neg$ truth table} theorem. This is the way mathematics goes. Once we have derived a new equation from the axioms, we can cite the new equation to derive still more equations.

Theorem 4 ({$\to$ truth table})**.**

- $False \to False = True$
- $False \to True = True$
- $True \to False = False$
- $True \to True = True$

Proof.

	$False \to False$	
$=$	$(\neg False) \vee False$	{implication} …taking both x and y in the axiom to stand for $False$
$=$	$\neg False$	{$\vee$ identity}
$=$	$True$	{$\neg$ truth table}

…for practice, prove the other equations yourself…

Q.E.D.

In day to day life outside the sphere of symbolic logic, the interpretation of the logical implication $x \to y$ is that we can conclude that y is true if we know that x is true. However, the implication says nothing about y when

Another way to prove that two formulas stand for the same value is to build truth tables for both formulas. A truth table lists all possible combinations of values for the variables in a formula, and displays the value that the formula denotes for each of those combinations. (Theorem {$\vee$ truth table} provides the truth table for the logical-or operation, and theorem {$\neg$ truth table} provides the truth table for the logical-negation operation.) Two truth tables that list identical values of the corresponding formulas for all combinations of values for the variables demonstrate that the formulas always stand for the same value. This proof method works well for formulas with only a few variables. In that case, there are only a few combinations of values for the variables, and the comparison can be completed quickly.

On the other hand, if there are many variables in the formulas, things get out of hand. With two variables, as in the truth table for logical-or, there are four combinations of values (two choices for each variable, True or False, so two times two combinations in all). With three variables, there are eight (2^3) combinations, which makes the truth-table method tedious, but not infeasible. After that, it gets rapidly out of hand. Ten variables produce 1,024 (2^{10}) combinations of values. That makes it difficult for people, but no real problem for a computer. Even twenty variables (a little more than a million combinations) also can be checked quickly by computers.

However, the formula specifying a computing component, hardware or software, has hundreds of variables. Our goal is to be able to reason about computing components, and there is no hope of doing that when the formulas have hundreds of variables. The number of combinations of values for the variables in a formula with, say, 100 variables is 2^{100}, and that number is so large that no computer could finish checking for equality before the sun runs out of fuel.

So, it is definitely not feasible to know the full meaning of computing components by analyzing the truth tables of the formulas that comprise their designs. Reasoning based on grammatical form makes it feasible to deal with realistic computing components because the reasoning process can be split into parts small enough to manage, and those parts can be reintegrated, based on their grammatical relationships, to produce a full analysis. It will take some diligence to reach this goal, and we will need a few more tools, but the proof methods of this chapter will get us started in the right direction.

Aside 5: Truth Tables and Feasibility

Citing proven theorems to prove new ones is similar an idea known as "abstraction" that is employed in engineering design. At the point where we cite an old theorem to prove a new one, we could, instead, copy the proof of the old theorem into new proof. However, that would make the proof longer, harder to understand, and more likely to contain errors.

Computer programs are built from components that are, themselves, other computer programs. As the project proceeds, more and more components become available, and they are used to build more complex ones. Sometimes, a component has almost the right form to be used in a new program, but not quite. Maybe the existing component doubles a number where the new program would need to triple it. Most software engineers are tempted to make a copy the old component and change the $2 \times x$ formula to $3 \times x$ at the point where a doubling should be a tripling. *This practice the single most common cause of errors in computer software.*

What the engineer should do is to make a new component in which the 2 is replaced by a variable, say m. This is known as creating an "abstraction" of the component ("abstract" as opposed to "specific" or "concrete"). The new component can be used for both doubling and tripling, simply by specifying 2 for m in one case and 3 for m in the other. That way, if the component turns out later to have an error in it, the error can be fixed in one place instead of two (or maybe ten or a hundred places, depending on how many engineers have made copies of the original component to change the 2 to 7 or 9 or whatever.

This notion of abstraction is one of the most important methods in all of engineering design. Citing old theorems to prove new ones, instead of copying their proofs into the new proof with appropriate choices for the variables, is part of that tradition.

Aside 6: Abstraction

x is not true. In particular, it does not say that y is not true whenever x is not true. A quick look at theorem $\{\rightarrow$ truth table$\}$ shows that the formula $False \rightarrow y$ has the value $True$ when y is $True$ and also when y is $False$. In other words, the truth of the formula $x \rightarrow y$ in the case where the hypothesis, x, of the implication is $False$ has provides no information about the conclusion, y.

A common mistake in everyday life is to assume that if the implication $x \rightarrow y$ is true, then the implication $(\neg x) \rightarrow (\neg y)$ is also true. Sometimes this leads to bad results, even in everyday life. In symbolic logic, it is worse than that. Such a conclusion puts an inconsistency into the mathematical system, and that renders the system useless.

Over half of the basic Boolean equations in Figure 2.3 (see page 17) have names associated with the logical-or ($\vee$) operation. One of them, the $\{\vee$ DeMorgan$\}$ equation establishes a connection between logical-or and logical-and. It converts the negation of a logical-or to the logical-and of two negations: $\neg(x \vee y) = (\neg x) \wedge (\neg y)$. We can use this connection to prove a collection of equations for logical-and that are similar to the basic ones for logical-or. An example is the null law for logical-and.

Theorem 5 ($\{\wedge$ null$\}$). *$x \wedge False = False$*

Proof.

	$x \wedge False$	
$=$	$x \wedge (\neg True)$	$\{\neg$ truth table$\}$
$=$	$(\neg(\neg x)) \wedge (\neg True)$	{double negation}
$=$	$\neg((\neg x) \vee True)$	$\{\vee$ DeMorgan$\}$ …taking x in the axiom to stand for $(\neg x)$ and y in the axiom to stand for $True$
$=$	$\neg True$	$\{\vee$ null$\}$
$=$	$False$	$\{\neg$ truth table$\}$

Q.E.D.

This regime of theorem after theorem, proof after proof, is a little tiresome, isn't it? Nevertheless, let's push through one more. Then we'll go on to other topics, and give you a few to work out on your own, later.

Some equations simplify the target formula when used in one direction, but make it more complicated when used in the other direction. For example, applying the null law for logical-or, $\{\vee$ null$\}$, from left to right simplifies a logical-or formula to $True$. The formula on the right in the $\{\wedge$ null$\}$ equation discards both the logical-or operation and the variable

x. When you use this equation in the other direction, you can make the formula as complicated as you like because the variable x stands for any formula you want to make up (as long as it's grammatically correct). It can have hundreds of variables and thousands operations. This may seem perverse, but if that's what it takes to complete the proof, so be it.

The null law for logical-and, $\{\wedge$ null$\}$, is also asymmetric. It goes from complicated to simple in one direction and from simple to complicated in the other. A particularly interesting and important asymmetric equation is the absorption law for logical-and. It has two variables and two operations on one side, but only one variable and no operations on the other.

Theorem 6 ($\{\wedge$ absorption$\}$)**.** $(x \vee y) \wedge y = y$

Proof.

$$
\begin{aligned}
& (x \vee y) \wedge y & \\
= \quad & (x \vee y) \wedge (y \vee False) & \{\vee \text{ identity}\} \\
= \quad & (y \vee x) \wedge (y \vee False) & \{\vee \text{ commutative}\} \\
= \quad & y \vee (x \wedge False) & \{\vee \text{ distrubutive}\} \\
= \quad & y \vee False & \{\wedge \text{ null}\} \\
= \quad & y & \{\vee \text{ identity}\}
\end{aligned}
$$

Q.E.D.

We hope the gauntlet of theorems and proofs we have run you through (or, more likely, asked you to plow through, to the point of exhaustion) helps you understand how to derive a new equation from equations you already know. The technique requires matching the formula to one side of a known equation, then replacing it by the corresponding formula on the other side.

The "matching" process is a crucial step. It involves replacing the variables in the known equation by constituents of the formula you are trying to match. This is based in the mechanics of a formal grammar. It is surprisingly easy have a lapse of concentration and make a mistake.

Fortunately, it is easy for computers to verify correct matchings and report erroneous ones. A computer system that does this is called a "mechanized logic." After you have enough practice to have a firm grasp on the process, we will begin to use a mechanized logic to make sure our reasoning is correct.

. . . put exercises here? at end of chapter? . . .

2.2 Boolean formulas

We have been doing proofs based on the grammatical elements of formulas, but instead of taking the time to put together a precise definition of that grammar, we have been relying on your experience with formulas of numeric algebra, such as $2(x + y) + 3z$. Surely it would be better to have a precise definition, so we can determine whether or not a formula is grammatically correct. We are going to define a grammar by starting with the most basic elements, and working up from there to more complicated ones.

The simplest Boolean formulas are the basic constants, $True$ and $False$, and variables. We normally use ordinary, lower-case letters, such as x and y for variables. Sometimes variables are letters with subscripts, such as x_3, y_i, or z_n. This gives us sufficient variety for all of the situations we will encounter, but we will assume there is an infinite pool of names for variables. If we run out of Roman letters, we can use Greek letters. If we run out of those, we can start making up recognizable squiggles like Dr Seuss did in some of his stories.

So, if you write $True$, or $False$, or a lower-case letter, you have composed a grammatically correct Boolean formula. This is the first rule of Boolean grammar. Formulas conforming to this rule have no substructure, so we call them "atomic" formulas.

To make more complicated formulas, we use Boolean operators. We refer to the operators that require two operands as the "binary operators" ($\wedge$, $\vee$, and $\to$). These operators lead to the second rule of Boolean grammar: If a and b are grammatically correct Boolean formulas, and $\circ$ is a binary operator (that is, $\circ$ is one of the symbols $\wedge$, $\vee$, or $\to$), then $(a \circ b)$ is also a grammatically correct Boolean formula.

For example, the first rule confirms that x and $True$ are grammatically correct Boolean formulas. Since $\wedge$ is a binary operator, $(x \wedge True)$ is a grammatically correct Boolean formula by the second rule of grammar. Furthermore, since $\to$ is a binary operator, and y is a grammatically correct Boolean formula (by the first rule), $((x \wedge True) \to y)$ must be a grammatically correct Boolean formula (by the second rule). As you can see, the first two rules of Boolean grammar provide an infinite variety of grammatically correct Boolean formulas.

The third rule of Boolean grammar shows how to incorporate the negation operator in formulas. The rule is that if a is a grammatically correct Boolean formula, then so is $(\neg a)$.

These three rules cover the full range of grammatically correct Boolean formulas, but there is a fine point to discuss about parentheses. Parentheses are important because they make it easy to define the grammar and to explain the meaning of grammatically correct formulas.

The formulas covered by the three rules are fully parenthesized, including a top level of parentheses enclosing the entire formula. Top level parentheses are often omitted in informal presentations, and we have been omitting them on a regular basis.

For example, we have been writing formulas like "$x \vee y$", even though they do not have the top-level parentheses required in the grammar of binary operations. To conform to the grammar, we would have to write the formula with its top-level parentheses: "$(x \vee y)$". Because we have often omitted top-level parentheses, requiring them probably comes as a surprise. But, allowing non-atomic formulas without top-level parentheses requires additional rules of grammar, and the extra complexity is not compensated by added value.

Here is less surprising example of incorrect grammar: $x \wedge y \vee z$. This formula is missing two levels of parentheses. Even worse, there are two options for the inner parentheses. Does $x \wedge y \vee z$ mean $((x \wedge y) \vee z)$ or $(x \wedge (y \vee z))$? There is a way to deal with formulas that omit some of the parentheses, but to avoid confusion, we are not going to allow such formulas. The same problem occurs with formulas in numeric algebra. We know that $x \times y + z$ means $((x \times y) + z)$ and not $(x \times (y + z))$ because we know the convention that gives multiplicative operations a higher precedence than additive operations. But that gets some getting used to, and since Boolean formulas may be new to you, we want to minimize the possibility of misinterpretation.

We will sometimes be informal enough to omit the top level of parentheses around the whole formula, but we will not omit any interior parentheses. We will, however, allow redundant parentheses in grammatically correct formulas. For example, we the formula $(x \vee ((x \wedge y)))$ is grammatically correct and has the same meaning as the formula $(x \vee (x \wedge y))$. The first formula has redundant parentheses, but the second one doesn't.

Allowing redundant parentheses requires a fourth rule of grammar: If a is a grammatically correct Boolean formula, then so is (a).

With the four rules of Figure 2.4 (see page 27), we can determine whether or not any given sequence of symbols is a grammatically correct Boolean formula. The definition of the grammar is circular, but in a useful way that shows how to build more complicated formulas from simpler ones.

Grammatically correct Boolean formulas

v	{atomic}
$(a \circ b)$	{bin-op}
$(\neg a)$	{negation}
(a)	{group}

Requirements on symbols

- v is a variable or $True$ or $False$
  (a variable is a letter or a letter with a subscript)
- a and b are grammatically correct Boolean formulas
- $\circ$ is a binary operator

Figure 2.4: Rules of Grammar for Boolean Formulas

To verify that a formula is grammatically correct, find the rule of grammar that matches it, then verify that each part of the formula that matches with a letter in the rule of grammar is also grammatically correct. Atomic formulas have no substructure, so they require no further verification. A line of verification terminates when it arrives at an atomic formula.

For example, consider the formula $((x \vee (\neg y)) \wedge (x \rightarrow z))$. It matches with the {bin-op} rule. The symbols in the rule match with the elements of the formula in the following way.

symbol from {bin-op} rule	*matching element in* $((x \vee (\neg y)) \wedge (x \rightarrow z))$
a	$(x \vee (\neg y))$
$\circ$	$\wedge$
b	$(x \rightarrow z)$

The only other symbols in the rule are the top-level parentheses, and these match identically with the outer parentheses in the target formula. Therefore, the target formula is grammatically correct if the formulas $(x \vee (\neg y))$ and $(x \rightarrow z)$ are grammatically correct. We use the same approach to verify the grammatical correctness of those formulas.

The first one, $(x \vee (\neg y))$, again matches with the {bin-op} rule, but this time the matchings of the elements in the target formula with symbols in the rule are as follows:

symbol from {bin-op} rule	*matching element in* $(x \vee (\neg y))$
a	x
$\circ$	$\vee$
b	$(\neg y)$

This reduces the verification of the grammatical correctness of $(x \vee (\neg y))$ to the verification of the two formulas x and $(\neg y)$. Since x matches with the {atomic} rule, it must be grammatically correct. The $(\neg y)$ element matches with the {negation} rule, with y from the formula matching a in the rule. So, $(\neg y)$ is grammatically correct if y is, and y is grammatically correct because it matches with the {atomic} rule. That completes the verification of the $(x \vee (\neg y))$ formula.

The second element of the original formula, $(x \to z)$, is easier to verify. It matches the {bin-op} rule with x corresponding to a in the rule, y corresponding to b, and $\to$ corresponding to $\circ$ in the rule. Since x and z match the {atomic} rule, they are grammatically correct. This completes the verification of the grammatical correctness of the formula $((x \vee (\neg y)) \wedge (x \to z))$.

Let's look at another example: $(x \vee (\wedge y)$. This formula matches with the {bin-op} rule, with x corresponding to a in the rule, $\vee$ corresponding to $\circ$, and $(\wedge y)$ corresponding to b. So, the formula is grammatically correct if x and $(\wedge y)$ are. However, there is no rule that matches $(\wedge y)$. The only place the symbol $\wedge$ could match a rule in the table is in the {bin-op} rule. In the {bin-op} rule, there must be a grammatically correct formula between the opening parenthesis and the operator. Since there is no such element present between the opening parenthesis and the $\wedge$ operator in the target formula, it cannot be grammatically correct.

That covers the grammar of Boolean formulas. What about meaning? Every grammatically correct Boolean formula denotes, in the end, either the constant $True$ or the constant $False$. To find out which, you only need to know which value ($True$ or $False$) each of the variables in the formula stands for. Each of the binary operators, given specific operands ($True$ or $False$), delivers a specific result ($True$ or $False$). We worked out what the delivered values would be for some of the operators when we proved truth-table theorems for them (see page 17).

$(a \vee False) = a$	$\{\vee$ identity$\}$
$(x \vee True) = True$	$\{\vee$ null$\}$
$(a \vee b) = (b \vee a)$	$\{\vee$ commutative$\}$
$(a \vee (b \vee c)) = ((a \vee b) \vee c)$	$\{\vee$ associative$\}$
$(a \vee (b \wedge c)) = ((a \vee b) \wedge (a \vee c))$	$\{\vee$ distributive$\}$
$(a \rightarrow b) = ((\neg a) \vee b)$	$\{$implication$\}$
$(\neg(a \vee b)) = ((\neg a) \wedge (\neg b))$	$\{\vee$ DeMorgan$\}$
$(a \vee a) = a$	$\{\vee$ idempotent$\}$
$(a \rightarrow a) = True$	$\{$self-implication$\}$
$(\neg(\neg a)) = a$	$\{$double negation$\}$
$((a)) = (a)$	$\{$redundant grouping$\}$
$(v) = v$	$\{$atomic release$\}$

Requirements on symbols

- a, b, and c are grammatically correct Boolean formulas
- v is a variable or $True$ or $False$
  (a variable is a letter or a letter with a subscript)

Figure 2.5: Axioms of Boolean Algebra

In the process of deriving the truth tables, we used the Boolean equations of Figure 2.3 (see page 17). We can use this same method to derive the meaning ($True$ or $False$) of any grammatically correct formula that contains no variables. However, to deal with parentheses in a completely mechanized way, we need to add two equations to those of Figure 2.3. Figure 2.5 (see page 29) provides all of the information needed to determine the $True/False$ value of any grammatically correct formula that contains no variables. In fact it has even more general applicability. It provides all the information needed to verify not only whether a given formula has the same meaning as the formula $True$ or the formula $False$, but also to verify whether or not any two given, grammatically correct, formulas have the same meaning.

Ex. 5 — Use the rules of grammar for Boolean formulas (Figure 2.4, see page 27) to determine which of the following formulas are grammatically correct.

$$((x \wedge y) \vee y)$$
$$((x \rightarrow y) \wedge (x \rightarrow (\neg y)))$$
$$((False \rightarrow (\neg y))\neg(x \vee True))$$

Ex. 6 — Derive the truth tables (see page 18) of the grammatically correct formulas from the previous exercise.

Ex. 7 — Use the axioms of Boolean algebra (Figure 2.5, see page 29) to prove that the following equations are valid. After you have proved the validity of an equation, you may cite it in subsequent proofs of other equations.

$$
\begin{array}{ll}
(a \rightarrow False) = (\neg a) & \{\neg \text{ as } \rightarrow\} \\
(\neg(a \wedge b)) = ((\neg a) \vee (\neg b)) & \{\wedge \text{ DeMorgan}\} \\
(a \vee (\neg a)) = True & \{\vee \text{ complement}\} \\
(a \wedge (\neg a)) = False & \{\wedge \text{ complement}\} \\
(\neg True) = False & \{\neg True\} \\
(\neg False) = True & \{\neg False\} \\
(True \rightarrow a) = a & \{\rightarrow \text{ identity}\} \\
(a \wedge True) = a & \{\wedge \text{ identity}\} \\
(a \wedge b) = (b \wedge a) & \{\wedge \text{ commutative}\} \\
(a \wedge (b \wedge c)) = ((a \wedge b) \wedge c) & \{\wedge \text{ associative}\} \\
(a \wedge (b \vee c)) = ((a \wedge b) \vee (a \wedge c)) & \{\wedge \text{ distributive}\} \\
(a \wedge a) = a & \{\wedge \text{ idempotent}\} \\
(a \rightarrow b) = ((\neg b) \rightarrow (\neg a)) & \{\text{contrapositive}\} \\
(a \rightarrow (b \rightarrow c)) = ((a \wedge b) \rightarrow c)) & \{\text{Currying}\} \\
((a \wedge b) \vee b) = b & \{\vee \text{ absorption}\} \\
((a \rightarrow b) \wedge (a \rightarrow (\neg b))) & \{\text{absurdity}\}
\end{array}
$$

2.3 Digital Circuits

Logic formulas provide a mathematical notation for concepts in symbolic logic. These same concepts can be materialized as electronic devices. The basic operators of logic represented in the form of electronic devices are called "logic gates". There are logic gates for the logical-and, the logical-or, negation, and several other operators that we have not yet discussed.

A logic gate takes input signals that correspond to the operands of logical operators and delivers output signals that correspond to the values

One of the operators we have discussed at length, implication ($\rightarrow$), is not among the operators normally represented in the form of logic gates. This does not restrict the kinds of operations that can be performed by digital logic because, as we know from the {implication} axiom of Boolean algebra (Figure 2.5, page 29), the formula $(a \rightarrow b)$ has the same meaning as the formula $((\neg a) \vee b)$. So, anything we can do with the implication operator, we can also do with by combining negation and logical-or in a particular way. Therefore, since we have logic gates for logical-or and negation, there is no loss in the range of behavior of logic gates, compared to that of the basic logical operators.

The lack of a conventional logic gate is ironic for at least three reasons. One is that George Boole himself, the inventor of Boolean algebra, called implication the queen of logical operators. Another is that the implication operator is one of only a few binary operators that are sufficient for representing any formula in logic. That is, given any grammatically correct logic formula, there is a formula using only implication operators (no logical-and, logical-or, or negation operators) that produces the same result as the original formula, given the same values for the variables in the formula.

But, the most dramatic reason for the irony in having implication turn up missing from the conventional collection of basic logic gates is that recent discoveries make it possible that the implication operator will in the future be the only gate used in large-scale digital circuits. This is because implication can be materialized as an electronic component in a form that allows three-dimensional stacking of circuits in a way that has never before been feasible. This makes it possible to fabicate digital circuits with vastly greater computational capacity than present-day circuits. If this intrigues you, view the online video of R. Stanley Williams on “memristor chips” (`http://www.youtube.com/watch?v=bKGhvKyjgLY`).

Aside 7: No Gate for Implication

delivered by logical operations. A logic gate with two inputs is a physical representation of a binary operator. The negation operator is represented by a logic gate with one input.

There are only two kinds of operands for logical operators: $True$ and $False$. And, they can deliver only those two kinds of values. Similarly, a logic gate can interpret only two kinds of input signals and deliver only two kinds of output signals (the same two kinds, of course). Those signals could be called $True$ and $False$, but conventionally they are written as 1 (for $True$) and 0 (for $False$). Of course, logic gates are electronic devices, so 0 and 1 are just labels for the signals. Any two different symbols could be used to represent them in writing. The choice of 1 and 0 is more-or-less arbitrary.

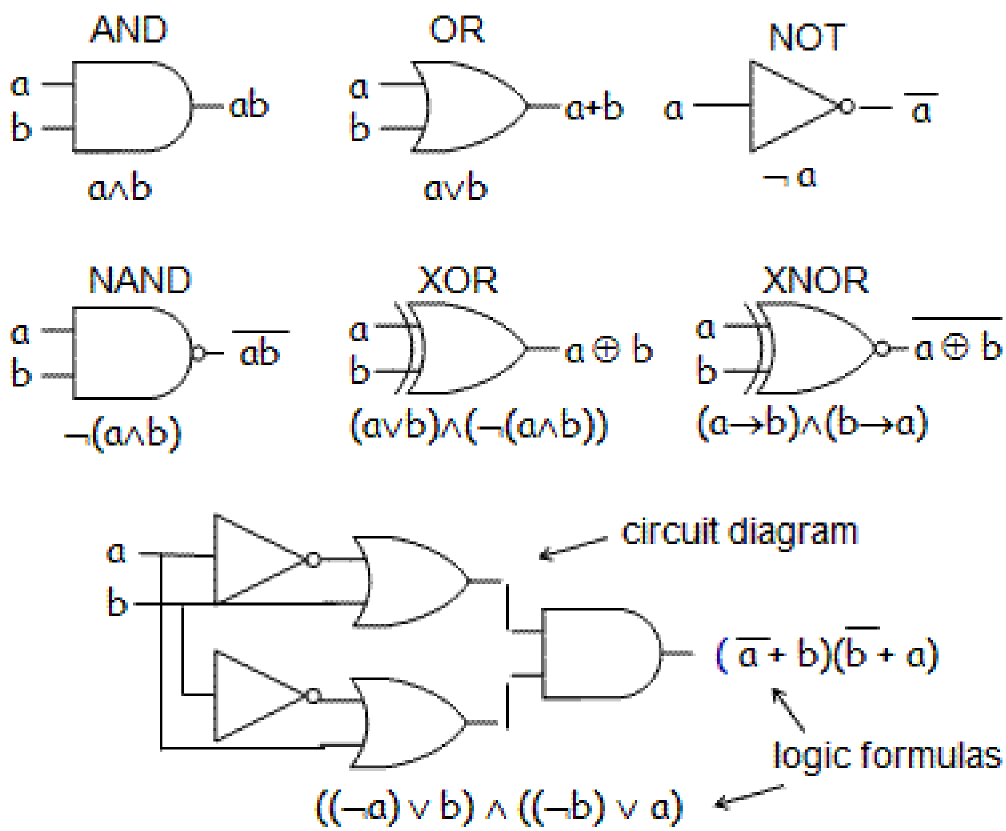
There are many ways to handle the electronics. We are going to leave the physics to the electrical engineers. If you want to, you can imagine representing the signal 1 as a small electrical current on a wire and the signal 0 as the lack of electrical current. That is one way to do it. But, we are going to focus on the logic and trust that the electronic hardware can faithfully represent the two kinds of signals we need.

Engineers who design circuits represent their designs as wiring diagrams in which "wires" (represented by lines) carry signals between gates. They use distinctive shapes to represent the different logic gates in circuit diagrams, and they use an algebraic notation similar to, but not the same as, the notation logicians normally use when they want to represent their circuits as formulas.

In the algebraic notation commonly used by circuit designers, the logical-and is represented by the juxtaposition of the names being used for the signals (in the same way that juxtaposition of variables is used to denote multiplication in numeric algebra formulas). Logical-or is represented by a plus sign (+), and negation is represented by a bar over the variable being negated (for example, $\bar{a}$).

Figure 2.6 (see page 33) summarizes the relationships between the symbols for logic gates used in circuit diagrams (that is, wiring diagrams that route signals between a collection of logic gates), the algebraic notation used by circuit designers, and the logic formulas we have use up to now.

The important fact to remember is this: All three notations represent the same concepts in logic. Circuit diagrams, logic formulas, and the algebraic notation used by circuit designers are three different notations for exactly the same mathematical objects. In this sense, digital circuits, and,



TODO: I did not have Visio with me. Improvised with PowerPoint. Figure will need to be redrawn.

Figure 2.6: Digital Circuits = Logic Formulas

therefore, computers, are materializations of logic formulas. Computers truly are "logic in action".

The logical operators that we have been using are logical-and, logical-or, implication, and negation. These are sufficient to write formulas that have any possible input/output relationship between the variables in the formula and the value it delivers.

The {implication} axiom (Figure 2.5, see page 29) expresses implication in terms of logical-and, logical-or, and negation, which means we lose no expressive power by discarding implication from the set of logical operations.

Surprisingly, the reverse is also true. That is, for any given input/output relationship that can be expressed in a formula using logical-and, logical-or, and negation, there is a logic formula using implication as the only operator that has the same input/output relationship. The $\{\neg \text{ as } \rightarrow\}$ equation (see page 30) provides at start in this direction by showing how to express negation in terms of the implication operator. Logical-and and logical-or are a little trickier. You will will get a chance to look into that in the exercises at the end of this section.

Furthermore, implication is not the only one-operator basis for the entire system of logic. Another one is the negation of logical-and, which is called "nand".

The nand gate is one of the standard logic gates, and the fact that all of the other logical operators can be expressed in terms of nand alone makes the nand gate especially important. It happens that the nand gate is the basis for designing most large-scale digital circuits, such as computer chips. Part of the reason for this is that the physics of putting gates on chips is simplified when all of the gates are the same.

Let's see how nand can be a basis for the whole system. We will express the operators in the basis we have at this point ($\wedge$, $\vee$, and $\neg$) in terms of nand, starting with negation. Negation has only one input signals, and nand has two. Feeding the same signal into both inputs of a nand gate produces the behavior of the negation operator.

It is easy to verify this from what we already know about logical operators because there is a one-step proof of the following equation. The proof cites the $\{\wedge$ idempotent$\}$ theorem (see page 30).

$$(\neg a) = (\neg(a \wedge a)) \quad \{\neg \text{ as nand}\}$$

The $\{\neg$ as nand$\}$ equation expresses negation as a nand operation (the negation of a logical-and). That takes care of negation. What about logical-and?

That's easy, too, because we only need to negate the output from a nand gate, and we already know we can use a nand gate to do that negation. So, we can construct a circuit with the same behavior as the logical-and by feeding the output signal from one nand gate into both inputs of a second nand gate.

Algebraically, this circuit corresponds to the following equation. It takes a two-step proof to verify the equation. The first step converts the outside nand to negation using the $\{\neg$ as nand$\}$ equation, and the second step cites the {double negation} axiom from Figure 2.5 (page 29).

$$(a \wedge b) = (\neg((\neg(a \wedge b)) \wedge (\neg(a \wedge b)))) \quad \{\wedge \text{ as nand}\}$$

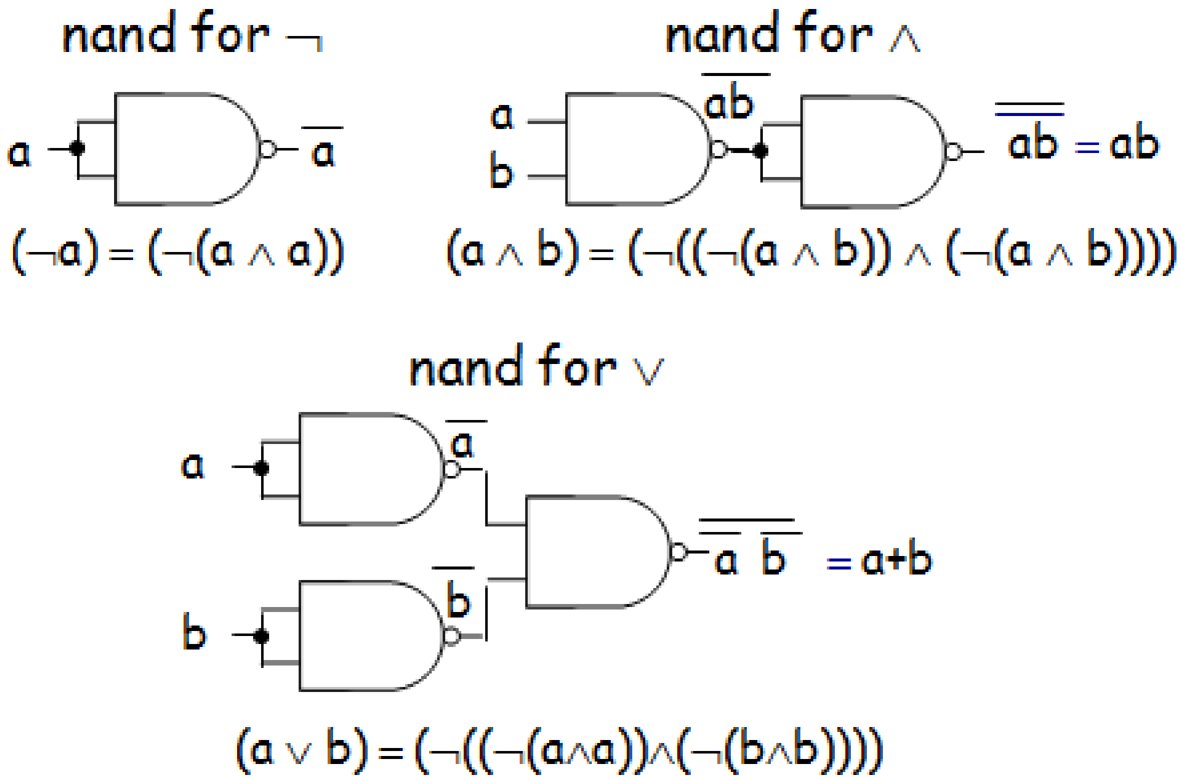
That takes care of logical-and, which brings us to logical-or. Expressing logical-or in terms of nand is a little trickier than the other two. We got negation using one nand gate and logical-and using two nand gates. We will need three nand gates for logical-or, and to verify the equation we will need a multi-step proof involving the $\{\neg$ as nand$\}$ equation, DeMorgan's laws, and double negation.

$$(a \vee b) = (\neg((\neg(a \wedge a))) \wedge ((\neg(b \wedge b))))) \quad \{\vee \text{ as nand}\}$$

We will rely on you to carry out the proofs of the $\{\neg$ as $\wedge\}$, $\{\wedge$ as nand$\}$, and $\{\vee$ as nand$\}$ equations. Figure 2.7 (page 36) displays the digital circuits that correspond to the equations verifying that nand is the only gate we really need.

Ex. 8 — Using a negation-gate and an or-gate, draw a circuit diagram with the input/output behavior of the implication operator. We refer to this circuit diagram as an "implication circuit". Hint: Follow the example of the {implication} axiom (**??**). One of the inputs will need to be a constant rather then a variable.

Ex. 9 — For each of the following logic formulas, draw an equivalent circuit diagram. Implication operators will need to be represented in the form of the circuit diagram from the previous exercise.



TODO: I did not have Visio with me. Improvised with PowerPoint. Figure will need to be redrawn.

Figure 2.7: Nand is All You Need

$$((a \vee (b \wedge (\neg a))) \vee (\neg(a \vee b)))$$
$$(((\neg a) \wedge (\neg b)) \wedge (b \wedge (\neg c)))$$
$$(a \to (b \to c))$$
$$((a \wedge b) \to c)$$

Ex. 10 — Rewrite each of the formulas in the previous exercise in the algebraic notation used by electrical engineers: juxtaposition for $\wedge$, + for $\vee$, and $\bar{a}$ for $(\neg a)$.

Ex. 11 — Draw circuit diagrams with behavior of the and-gate, or-gate, and negation-gate, but in these diagrams use implication circuits only—no other kinds of logic gates.

TODO: do we still want to add half-adder and/or full-adder circuits as examples? problem: they have two outputs, so need to talk about tapping outputs from subformulas to show correspondence to algebraic form

2.4 Deduction

We have been reasoning with equations, which means we are reasoning in two directions at the same time, since equations go both ways. Deductive reasoning is one-directional. It derives a conclusion from hypotheses using one-directional rules of inference. A proof shows that the conclusion is true whenever the hypotheses are true, but provides no information about the conclusion when the truth of one or more of the hypotheses is unknown.

Another way to say this is that a deductive proof of the formula c (a conclusion) from the formula h (the hypothesis) guarantees that the formula $(h \to c)$ is true. Whenever we want to use deduction to prove that a formula with implication as its top-level operator, $(h \to c)$, is true, this is the way we will do it. We will construct a deductive proof of the conclusion c, assuming that the hypothesis h is true.

It is important to realize that proving an implication formula does not require a proof of the hypothesis, only a derivation of the conclusion from the hypothesis. This is because an implication formula is always true if its hypothesis is false (by the $\{\to$ truth table$\}$ theorem, page 20). So, we can be sure that the implication formula is true as long as we know that the only combination of values that makes the implication $(h \to c)$ false, that is $h = True$ and $c = False$, cannot arise.

There might, of course, be any number of hypotheses, combined with the $\wedge$ operator. If for example there are three hypotheses, h_1, h_2, and h_3, then to use deduction to prove $((h_1 \wedge h_2 \wedge h_3) \rightarrow c)$, we need to derive, through the rules of deductive reasoning, the formula c from the formula $(h_1 \wedge h_2 \wedge h_3)$.

By the way, we have been a little cagey with the formula $(h_1 \wedge h_2 \wedge h_3)$. It is not fully parenthesized. Does it mean $((h_1 \wedge h_2) \wedge h_3)$ or $(h_1 \wedge (h_2 \wedge h_3))$? The answer is, it doesn't matter because of the $\{\wedge$ associative$\}$ theorem (page 30). Both formulas have the same meaning.

There might even be no hypotheses at all, in which case a proof of the conclusion c would guarantee the truth of the formula c. That is, the proof would guarantee the equation $c = True$.

We are not going to discuss deductive reasoning in great detail because most of the things we will prove will come from reasoning with equations. A formal treatment of deductive reasoning makes it possible to prove all of the axioms of Boolean algebra (Figure 2.5, page 29) from a small set of inference rules, each of which is easily seen to be consistent with the use of logic in everyday life.

So, in a sense deductive reasoning gets closer to the fundamentals of logic than the axioms of Boolean algebra, but we are more interested in getting at how computers work than in building a formal system of logic from the ground up. If you are interested in fundamentals, an accessible treatment can be found in a text byO'Donnell, Hall, and Page (*Discrete Mathematics Using a Computer*, Springer, 2006). See the sections on "natural deduction" (an invention of Gerhard Gentzen).

To give you a light introduction to how it works, consider the rules of inference for deductive reasoning in Figure 2.8 (page 39). One of the rules, $\{\rightarrow$ introduction$\}$, is a formal statement of the discussion at the beginning of this section about proofs of implication formulas.

The $\{\vee$ elimination$\}$ rule supports case-by-case proofs. That is, if you can make a list of cases that covers all of the possibilities, you can prove that a formula is true if you derive it from each of the cases separately.

The {modus ponens} rule, probably the most famous one because of the well known "Socrates was a man" application, says that if you know that the formula a is true and that the formula $a \rightarrow b$ is true, you can conclude that b is true.

The {reductio ad absurdum} rule supports "proof by contradiction". It says that if you can derive $False$ from $(\neg a)$, then $(\neg a)$ must be $False$,

$$\begin{array}{l} \text{Prove } a \\ \text{Prove } a \to b \\ \hline \text{Infer } b \end{array} \quad \{\text{modus ponens}\}$$

$$\begin{array}{l} \text{Prove } a \\ \text{Prove } b \\ \hline \text{Infer } a \wedge b \end{array} \quad \{\wedge \text{ introduction}\}$$

$$\begin{array}{l} \text{Assume } a \\ \text{Prove } b \\ \hline \text{Infer } a \to b \end{array} \quad \{\to \text{ introduction}\}$$

$$\begin{array}{l} \text{Prove } a \vee b \\ \text{Prove } a \to c \\ \text{Prove } b \to c \\ \hline \text{Infer } c \end{array} \quad \{\vee \text{ elimination}\}$$

$$\begin{array}{l} \text{Prove } (\neg a) \to False \\ \hline \text{Infer } a \end{array} \quad \{\text{reductio ad absurdum}\}$$

$$\begin{array}{l} \text{Prove } a \wedge b \\ \hline \text{Infer } a \end{array} \quad \{\wedge \text{ elimination}\}$$

Figure 2.8: Rules of Inference for Reasoning by Deduction

which means that a must be $True$.

The $\{\wedge$ elimination$\}$ rule says that if you know $a \wedge b$, you can conclude that a must be $True$. The same goes for b of course, but that is another rule. We have not included it in the table because our goal is to give you the general idea, not to put together a complete set of rules. Following the approach we have stated here, we would need to add five more rules (the other version of $\{\wedge$ elimination$\}$ being one of them) to have a complete set that would allow us to derive all of the equations of Boolean algebra.

Fortunately, citing equations from Boolean algebra to justify steps is consistent with proof by deduction. That is, a formula in a deductive proof may be replaced by an equivalent formula justified by an equation known from Boolean algebra. So, if we accept the $\{\wedge$ commutative$\}$ equation (see page 30), we can derive the other $\{\wedge$ elimination$\}$ from Figure 2.8 (page 39).

The rule would be stated as follows.

$$\frac{\text{Prove } a \wedge b}{\text{Infer } b} \quad \{\wedge \text{ elimination-2}\}$$

The derivation of the new rule, $\{\wedge$ elimination-2$\}$, proceeds by deductive reasoning as follows.

$$\frac{\dfrac{\text{Prove } a \wedge b}{\text{Prove } b \wedge a} \quad \{\wedge \text{ commutative}\}}{\text{Infer } b} \quad \{\wedge \text{ elimination}\}$$

Just as with Boolean equations, once a new rule is proven, it can be cited to justify steps in proofs. So, at this point we could use $\{\wedge$ elimination-2$\}$ in a proof by deductive reasoning.

To firm up the idea at least a little, we will do one more proof using deductive reasoning. The theorem is known as the implication chain rule.

$$((a \to b) \wedge (b \to c)) \to (a \to c) \quad \{\to \text{chain}\}$$

The proof proceeds as follows.

$$\begin{array}{ll}
\text{Assume } ((a \to b) \wedge (b \to c)) & \\
\hline
\text{Infer } (a \to b) & \{\wedge \text{ elimination}\} \\
\text{Assume } a & \\
\hline
\text{Infer } b & \{\text{modus ponens}\} \\
\text{Assume } ((a \to b) \wedge (b \to c)) & \\
\hline
\text{Infer } (b \to c) & \{\wedge \text{ elimination-2}\} \\
\hline
\text{Infer } c & \{\text{modus ponens}\} \\
\hline
\text{Infer } (a \to c) & \{\text{modus ponens}\} \\
\hline
\text{Infer } ((a \to b) \wedge (b \to c)) \to (a \to c) & \{\text{modus ponens}\}
\end{array}$$

This theorem is a tautology. That is, the {→ chain} theorem confirms that a particular formula is equivalent to $True$. In the form of an equation, this theorem would be stated as follows.

$$(((a \to b) \wedge (b \to c)) \to (a \to c)) = True \quad \{\to \text{chain}\}$$

If you want to get a little more practice in reasoning with equations, construct a prooof of the {→ chain} equation based on the axioms of Boolean algebra (Figure 2.5, page 29).

TODO: Not sure whether we need a section on one-directional reasoning or not ... seems like we do, but a full-fledged Gentzen-style treatment gets tedious ... how do we keep it lively, but still provide the necessary apparatus? ... Maybe better to introduce inference rules when we introduce induction? ... how many rules do we need? Would modus ponens and or-elim (plus induction) be enough? How about reductio-ad-absurdum? law of excluded middle? Just the rules we will be using in doing inductive proofs about software and circuits

TODO: after writing this, I'm not sure it does any good. I'm especially not sure the proof notation I've used is clearly explained. I went through all the lectures, homeworks, and exams in the existing applied logic course, and did not find any theorems proved by deductive reasoning that were not more easily handled by stating them as implications and proving them as equations of the form $(a \to b) = True$. *I guess we could leave this stuff in, but give it short shrift, and refer back to it if necessary. We will use deduction when we come to induction, which is a deductive inference rule, but I'm not sure we need to make a big deal out of it*

TODO: Check to make sure predicates and quantifiers are covered somewhere. Show how to convert between forall and exists.