Notes on spin-wave analysis

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1 Spin wave on the Heisenberg ferromagnet

For the Heisenberg ferromagnet, we consider a constant Heisenberg ferromagnetic interaction J between any spin pairs, so the ground state of the system consists of spins aligning parallel to each other, independent from the geometrical structure of the lattice, and hence we have the Hamiltonian

$$\begin{aligned}\mathcal{H} &= J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j \\ &= J\sum_{\langle i,j\rangle}\left(S_i^xS_j^x+S_i^yS_j^y+S_i^zS_j^z\right)\end{aligned} \tag{1}$$

with $J<0$. When we adapt the linear expansion of Holstein-Primakoff(HP) transformation

$$S^+=\sqrt{2S}\sqrt{1-\frac{a^\dagger a}{2S}}a\approx\sqrt{2S}a \tag{2}$$

$$S^-=\sqrt{2S}a^\dagger\sqrt{1-\frac{a^\dagger a}{2S}}\approx\sqrt{2S}a^\dagger, \tag{3}$$

we have

$$S^x\approx\sqrt{\frac{S}{2}}(a+a^\dagger) \tag{4}$$

$$S^y\approx -i\sqrt{\frac{S}{2}}(a-a^\dagger) \tag{5}$$

Along with

$$S^z=S-a^\dagger a, \tag{6}$$

we obtain the linear spin wave Hamiltonian from (1) in HP bosons operators

$$\mathcal{H}=JS\sum_{\langle i,j\rangle}(a_ia_j^\dagger+a_i^\dagger a_j-a_i^\dagger a_i-a_j^\dagger a_j) \tag{7}$$

After performing Fourier transform

$$a_i=\frac{1}{\sqrt{N}}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}_i}a_{\mathbf{k}} \tag{8}$$

$$a_i^\dagger=\frac{1}{\sqrt{N}}\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}_i}a_{\mathbf{k}}^\dagger, \tag{9}$$

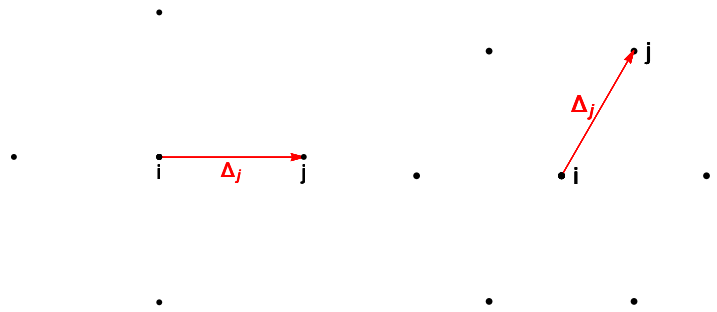
we get

$$\mathcal{H} = JS \sum_{\langle i,j \rangle} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \frac{1}{N} (e^{i\mathbf{k}\cdot\mathbf{r}_i} e^{-i\mathbf{k}'\cdot\mathbf{r}_j} a_{\mathbf{k}} a^\dagger_{\mathbf{k}'} + e^{-i\mathbf{k}\cdot\mathbf{r}_i} e^{i\mathbf{k}'\cdot\mathbf{r}_j} a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} - e^{-i\mathbf{k}\cdot\mathbf{r}_i} e^{i\mathbf{k}'\cdot\mathbf{r}_i} a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} - e^{-i\mathbf{k}\cdot\mathbf{r}_j} e^{i\mathbf{k}'\cdot\mathbf{r}_j} a^\dagger_{\mathbf{k}} a_{\mathbf{k}'}). \tag{10}$$

Noticing for every geometrical structure, we can regard index i running through every lattice site and index j as all the nearest neighboring sites around i, we can rewrite

$$\mathbf{r}_j = \mathbf{r}_i + \mathbf{\Delta}_j, \tag{11}$$

where $\mathbf{\Delta}_j$ is defined as a local position vector pointing towards neighboring lattice site j from site i.



(a) Square lattice (b) Triangular lattice

Figure 1: Nearest neighbors on the square and triangular lattice with their corresponding local position vector $\mathbf{\Delta}_j$

Then we can substitute (10) with (11), and get

$$\mathcal{H} = \frac{JS}{2N} \sum_{i,j,\mathbf{k},\mathbf{k}'} \left[e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i} e^{-i\mathbf{k}'\cdot\mathbf{\Delta}_j} a_{\mathbf{k}} a^\dagger_{\mathbf{k}'} + e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i} \left(e^{i\mathbf{k}'\cdot\mathbf{\Delta}_j} - 1 - e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{\Delta}_j} \right) a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} \right]. \tag{12}$$

With

$$\sum_i e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i} = N\delta_{\mathbf{k},\mathbf{k}'} \tag{13}$$

$$\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{\Delta}_j} = 0 \tag{14}$$

we have

$$\begin{aligned} \mathcal{H} &= \frac{JS}{2} \sum_{j,\mathbf{k}} \left[e^{-i\mathbf{k}\cdot\mathbf{\Delta}_j} a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + \left(e^{i\mathbf{k}\cdot\mathbf{\Delta}_j} - 2 \right) a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \right] \\ &= \frac{JS}{2} \sum_{j,\mathbf{k}} \left[\left(2\cos(\mathbf{k}\cdot\mathbf{\Delta}_j) - 2 \right) a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{\Delta}_j} \right] \\ &= JS \sum_{\mathbf{k}} (\gamma(\mathbf{k}) - n) a^\dagger_{\mathbf{k}} a_{\mathbf{k}}, \end{aligned} \tag{15}$$

where we define $\gamma(\mathbf{k}) = \sum_j \cos(\mathbf{k} \cdot \mathbf{\Delta}_j)$ and n the number of nearest neighbors.

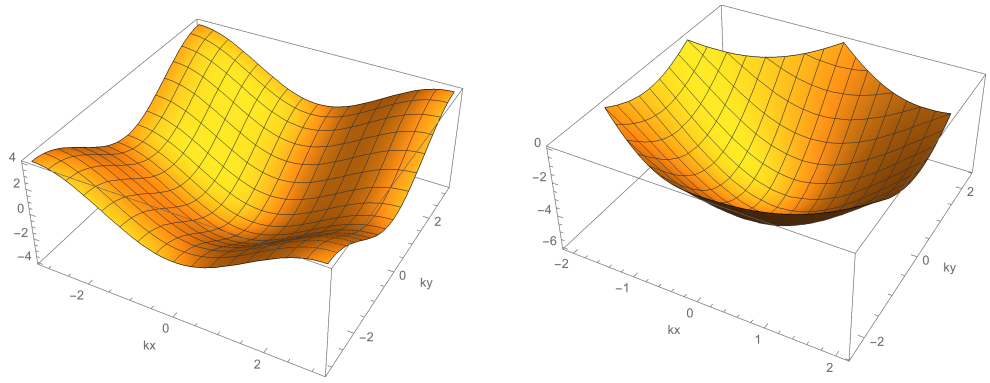
It is clear from (15) that geometrical structure of lattice sites enters the dispersion relation only through the geometric factor $\gamma(\mathbf{k})$. Hence for different lattice configuration, we only need to find the corresponding $\gamma(\mathbf{k})$. For square lattice and triangular lattice with lattice constant $a = 1$, we have

$$\gamma_{\text{sq}}(\mathbf{k}) = 2\left[\cos(k_x) + \cos(k_y)\right] \tag{16}$$

$$\gamma_{\text{tri}}(\mathbf{k}) = 2\left[\cos(k_x) + \cos\left(\frac{k_x}{2} - \frac{\sqrt{3}}{2}k_y\right) + \cos\left(\frac{k_x}{2} + \frac{\sqrt{3}}{2}k_y\right)\right]. \tag{17}$$

Therefore, we can get energy dispersion relation for linear spin wave on both lattices by substituting corresponding $\gamma(\mathbf{k})$ in

$$w(\mathbf{k}) = |JS|(n - \gamma(\mathbf{k})). \tag{18}$$



(a) $w(\mathbf{k})$ for square lattice

(b) $w(\mathbf{k})$ for triangular lattice

Figure 2: Energy dispersion relation for linear spin wave on the square and triangular lattice

2 Spin wave on the Heisenberg Antiferromagnet

Due to the nature of antiferromagnetism, spins on the adjacent sites tend to antiparallel align each other, so a simple ground state like we have under ferromagnetic interaction cannot be achieved given the antiferromagnetic spin interaction. One common method to find the true ground state configuration of a given lattice is Luttinger-Tisza method, which gives the orientation of spins

2.1 Luttinger-Tisza method