The Uncertain OWA Operator

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The ordered weighted averaging (OWA) operator was introduced by Yager[1] to provide a method for aggregating several inputs that lie between the max and min operators. In this article, we investigate the uncertain OWA operator in which the associated weighting parameters cannot be specified, but value ranges can be obtained and each input argument is given in the form of an interval of numerical values. The problem of ranking a set of interval numbers and obtaining the weights associated with the uncertain OWA operator is studied. © 2002 Wiley Periodicals, Inc.

1. INTRODUCTION

Yager[1] introduced the ordered weighted averaging (OWA) operator to provide a method for aggregating several inputs that lie between the max and min operators. Some new families of OWA operators were introduced in Refs. 2–5. In the short time since their first appearance, the OWA operators have been investigated in many documents,[1–13] and used in an astonishingly wide range of applications (See, for example, Ref. 14 and references therein).

An OWA operator of dimension n is a mapping $f : R^n \to R$ that has an associated n vector $w = (w_1, w_2, \ldots, w_n)^T$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Furthermore, $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{j=1}^{n} w_j b_j$, where b_j is the jth largest of the a_i.

The fundamental aspect of the OWA operator is the re-ordering step, in particular, an argument, a_i, is not associated with a particular weight w_i, but rather a weight, w_i, is associated with a particular ordered position, i, of the arguments. One important issue in the theory of OWA aggregation is the determination of the associated weights. A number of approaches have been suggested for obtaining the weights.[2,5,17] All of these methods, however, have been used in situations in which the input arguments are the exact values.

The aim of this article is to investigate the uncertain OWA aggregation in which the associated weighting parameters cannot be specified, but value ranges can be

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obtained, and each input argument is given in the form of an interval of numerical values. In Section 2 we propose a formula for comparing two interval numbers, and rank all the input arguments by using this formula. In Section 3 we give the definition of the uncertain OWA operator, and establish a linear objective-programming model to generate the uncertain OWA weights.

2. A FORMULA FOR COMPARING INTERVAL NUMBERS

DEFINITION 2.1. *Let* $a = [a^L, a^U] = \{x \mid a^L \leq x \leq a^U\}$, *then a is called an interval number. Especially, a is a real number, if* $a^L = a^U$.

Let $N = \{1, 2, \ldots, n\}$, and let Ω be the set of all interval numbers.

DEFINITION 2.2. *Let* $a = [a^L, a^U]$, $b = [b^L, b^U]$, *and* $\lambda \geq 0$; *then*:

(1) $a = b$, *if* $a^L = b^L$ *and* $a^U = b^U$.
(2) $a + b = [a^L + b^L,\ a^U + b^U]$.
(3) $\lambda a = [\lambda a^L, \lambda a^U]$. *Especially*, $\lambda a = 0$, *if* $\lambda = 0$.

DEFINITION 2.3. *Let* $a = [a^L, a^U]$ *and* $b = [b^L, b^U]$, *and let* $l_a = a^U - a^L$ *and* $l_b = b^U - b^L$; *then the degree of possibility of* $a \geq b$ *is defined as*:

$$p(a \geq b) = \max\left\{1 - \max\left(\frac{b^U - a^L}{l_a + l_b}, 0\right), 0\right\} \tag{1}$$

Similarly, the degree of possibility of $b \geq a$ *is defined as*:

$$p(b \geq a) = \max\left\{1 - \max\left(\frac{a^U - b^L}{l_a + l_b}, 0\right), 0\right\} \tag{2}$$

From Definition 2.3, we can get the following results easily:

THEOREM 2.1. *Let* $a = [a^L, a^U]$ *and* $b = [b^L, b^U]$; *then*:

(1) $0 \leq p(a \geq b) \leq 1$, $0 \leq p(b \geq a) \leq 1$.
(2) $p(a \geq b) = 1$ *if and only if* $b^U \leq a^L$. *Similarly*, $p(b \geq a) = 1$ *if and only if* $a^U \leq b^L$.
(3) $p(a \geq b) = 0$ *if and only if* $a^U \leq b^L$. *Similarly*, $p(b \geq a) = 0$ *if and only if* $b^U \leq a^L$.
(4) $p(a \geq a) = \frac{1}{2}$.
(5) $p(a \geq b) + p(b \geq a) = 1$.
(6) $p(a \geq b) \geq \frac{1}{2}$ *if and only if* $a^U + a^L \geq b^U + b^L$. *Especially*, $p(a \geq b) = \frac{1}{2}$ *if and only if* $a^U + a^L = b^U + b^L$.
(7) *Let a*, *b*, *and c be interval numbers*; *if* $p(a \geq b) \geq \frac{1}{2}$ *and* $p(b \geq c) \geq \frac{1}{2}$, *then* $p(a \geq c) \geq \frac{1}{2}$.

We now suppose that there are n input arguments a_i $(i \in N)$ having the forms of interval numbers; that is, $a_i = [a_i^L, a_i^U]$, $i \in N$. To rank these arguments, we first compare each argument a_i with all arguments a_j, $j \in N$ by using Equation 1; that is:

$$p(a_i \geq a_j) = \max\left\{1 - \max\left(\frac{a_j^U - a_i^L}{l_{a_i} + l_{a_j}}, 0\right), 0\right\}, \quad j \in N \tag{3}$$

For simplicity, we let $p_{ij} = p(a_i \geq a_j)$. Then we can construct a complementary matrix as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & & \vdots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, and $i, j \in N$.

Summing all elements in each line of matrix P, we have:

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad i \in N \tag{4}$$

Then we can rank the arguments a_i, $i \in N$ in descending order in accordance with the value of p_i, $i \in N$.

Example 2.1. Given the arguments

$$a_1 = [3, 5], \quad a_2 = [4, 6], \quad a_3 = [4, 7], \quad \text{and} \quad a_4 = [3, 6]$$

By Equation 3, we have:

$$P = \begin{bmatrix} 0.50 & 0.25 & 0.20 & 0.40 \\ 0.75 & 0.50 & 0.40 & 0.60 \\ 0.80 & 0.60 & 0.50 & 0.80 \\ 0.60 & 0.40 & 0.20 & 0.50 \end{bmatrix}$$

Thus:

$$\begin{aligned} p_1 &= 0.50 + 0.25 + 0.20 + 0.40 = 1.35 \\ p_2 &= 0.75 + 0.50 + 0.40 + 0.60 = 2.25 \\ p_3 &= 0.80 + 0.60 + 0.50 + 0.80 = 2.70 \\ p_4 &= 0.60 + 0.40 + 0.20 + 0.50 = 1.70 \end{aligned}$$

Here, $p_3 > p_2 > p_4 > p_1$. Therefore, the re-ordered arguments a_i, $i \in N$ in descending order are:

$$b_1 = a_3, \quad b_2 = a_2, \quad b_3 = a_4, \quad \text{and} \quad b_4 = a_1$$

3. A LINEAR OBJECTIVE-PROGRAMMING MODEL

DEFINITION 3.1. *An uncertain OWA operator of dimension n is a mapping $g : \Omega^n \to \Omega$ that has an associated n vector $v = (v_1, v_2, \ldots, v_n)^T$, such that $v_i \in [0, 1]$ and $\sum_{i=1}^{n} v_i = 1$. Furthermore, $g(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} v_j b_j$, where b_j is the jth largest of the a_i, and all of the $a_i (i \in N)$ are interval numbers.*

In this section we shall give a linear objective programming model that can be used to obtain the weights associated with the uncertain OWA operator by utilizing the given partial weight information and the arguments.

Given are a collection of m samples (observations) each comprised of an n-tuple of arguments $(a_{k1}, a_{k2}, \ldots, a_{kn})$, an associated aggregated value, s_k, where $a_{ki} = [a_{ki}^L, a_{ki}^U]$, $s_k = [s_k^L, s_k^U]$, $i \in N$, and $k = 1, 2, \ldots, m$, and partial weight information:

$$v = \left\{ (v_1, v_2, \ldots, v_n)^T \,|\, 0 \leq \alpha_i \leq v_i \leq \beta_i \leq 1, i \in N, \sum_{i=1}^{n} \alpha_i \leq 1, \sum_{i=1}^{n} \beta_i \geq 1 \right\}$$

We need an uncertain OWA operator, a weighting vector w, such that for the entire collection of data we satisfy the following condition as faithfully as possible:

$$g(a_{k1}, a_{k2}, \ldots, a_{kn}) = s_k, \quad k = 1, 2, \ldots, m$$

We can utilize Equation 3 to compare the kth sample arguments a_{ki}, $i \in N$, and utilize Equation 4 to obtain p_{ki}, $i \in N$. Then we can rank the arguments of the kth sample by $b_{k1}, b_{k2}, \ldots, b_{kn}$ in descending order in accordance with the value of p_i, $i \in N$. Using these re-ordered arguments, we need to find a vector of the uncertain OWA weights $v = (v_1, v_2, \ldots, v_n)^T$ such that:

$$\sum_{j=1}^{n} v_j\, b_{kj} = s_k, \quad k = 1, 2, \ldots, m$$

that is:

$$\sum_{j=1}^{n} v_j b_{kj}^L = s_k^L, \quad \sum_{j=1}^{n} v_j b_{kj}^U = s_k^U, \quad k = 1, 2, \ldots, m$$

We will relax the above equations by looking for a vector of the uncertain OWA weights $v = (v_1, v_2, \ldots, v_n)^T$ that approximates the aggregation operator by minimizing the instantaneous errors e_{1k} and e_{2k}, where:

$$e_{1k} = \left| \sum_{j=1}^{n} v_j b_{kj}^L - s_k^L \right|, \quad k = 1, 2, \ldots, m$$

$$e_{2k} = \left| \sum_{j=1}^{n} v_j b_{kj}^U - s_k^U \right|, \quad k = 1, 2, \ldots, m$$

with respect to weights v_k. Thus, we can construct the following multi-objective programming model (MOP):

$$\min e_{1k} = \left| \sum_{j=1}^{n} v_j b_{kj}^L - s_k^L \right|, \quad k = 1, 2, \ldots, m$$

$$\min e_{2k} = \left| \sum_{j=1}^{n} v_j b_{kj}^U - s_k^U \right|, \quad k = 1, 2, \ldots, m$$

$$s.t. \quad \alpha_j \leq v_j \leq \beta_j, \quad j \in N, \quad \sum_{j=1}^{n} v_j = 1$$

Solution to the above minimization problem is found by solving the following linear objective programming model (LOP):

$$\min J = \sum_{k=1}^{m} [(e_{1k}^{+} + e_{1k}^{-}) + (e_{2k}^{+} + e_{2k}^{-})]$$

$$s.t. \quad \sum_{j=1}^{n} b_{kj}^{L} v_j - s_k^L - e_{1k}^{+} + e_{1k}^{-} = 0, \quad k = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} b_{kj}^{U} v_j - s_k^U - e_{2k}^{+} + e_{2k}^{-} = 0, \quad k = 1, 2, \ldots, m$$

$$\alpha_j \leq v_j \leq \beta_j, \quad j \in N, \quad \sum_{j=1}^{n} v_j = 1$$

$$e_{1k}^{+} \geq 0, e_{1k}^{-} \geq 0, e_{1k}^{+} \cdot e_{1k}^{-} = 0, \quad k = 1, 2, \ldots, m$$

$$e_{2k}^{+} \geq 0, e_{2k}^{-} \geq 0, e_{2k}^{+} \cdot e_{2k}^{-} = 0, \quad k = 1, 2, \ldots, m$$

where, e_{1k}^{+} and e_{1k}^{-} are the upper and lower deviation variables of s_k^L, respectively, while e_{2k}^{+} and e_{2k}^{-} are the upper and lower deviation variables of s_k^U, respectively.

By solving the LOP model, we can obtain the vector of the uncertain OWA weights $v = (v_1, v_2, \ldots, v_n)^T$.

Example 3.1. We assume the collection of samples of data as listed in Table I. Each sample consists of three arguments and the relevant aggregated value, all having the forms of interval numbers. The known partial weight information is as follows:

$$0.2 \leq v_1 \leq 0.6, \quad 0.3 \leq v_2 \leq 0.5, \quad 0.1 \leq v_3 \leq 0.4$$

By Equations 3 and 4, we get the arguments in samples 1–4 in descending order as follows:

$$b_{11} = [0.7, 0.8], b_{12} = [0.4, 0.7], b_{13} = [0.2, 0.5]$$
$$b_{21} = [0.6, 0.8], b_{22} = [0.3, 0.5], b_{23} = [0.3, 0.4]$$
$$b_{31} = [0.5, 0.8], b_{32} = [0.3, 0.4], b_{33} = [0.2, 0.6]$$
$$b_{41} = [0.5, 0.8], b_{42} = [0.3, 0.5], b_{43} = [0.3, 0.4]$$

Table I.

Sample	Argument values	Aggregated value
1	[0.4, 0.7] [0.2, 0.5] [0.7, 0.8]	[0.3, 0.7]
2	[0.3, 0.4] [0.6, 0.8] [0.3, 0.5]	[0.4, 0.5]
3	[0.2, 0.6] [0.3, 0.4] [0.5, 0.8]	[0.3, 0.6]
4	[0.5, 0.8] [0.3, 0.5] [0.3, 0.4]	[0.4, 0.6]

Utilizing the LOP model, we have:

$$
\begin{aligned}
& \min J = \sum_{k=1}^{4} [(e_{1k}^{+} + e_{1k}^{-}) + (e_{2k}^{+} + e_{2k}^{-})] \\
s.t.\quad & 0.7v_1 + 0.4v_2 + 0.2v_3 - 0.3 - e_{11}^{+} + e_{11}^{-} = 0 \\
& 0.6v_1 + 0.3v_2 + 0.3v_3 - 0.4 - e_{12}^{+} + e_{12}^{-} = 0 \\
& 0.5v_1 + 0.3v_2 + 0.2v_3 - 0.3 - e_{13}^{+} + e_{13}^{-} = 0 \\
& 0.5v_1 + 0.3v_2 + 0.3v_3 - 0.4 - e_{14}^{+} + e_{14}^{-} = 0 \\
& 0.8v_1 + 0.7v_2 + 0.5v_3 - 0.7 - e_{21}^{+} + e_{21}^{-} = 0 \\
& 0.8v_1 + 0.5v_2 + 0.4v_3 - 0.5 - e_{22}^{+} + e_{22}^{-} = 0 \\
& 0.8v_1 + 0.4v_2 + 0.6v_3 - 0.6 - e_{23}^{+} + e_{23}^{-} = 0 \\
& 0.8v_1 + 0.5v_2 + 0.4v_3 - 0.6 - e_{24}^{+} + e_{24}^{-} = 0 \\
& 0.2 \le v_1 \le 0.6, \quad 0.3 \le v_2 \le 0.5, \quad 0.1 \le v_3 \le 0.4 \\
& v_1 + v_2 + v_3 = 1 \\
& e_{1k}^{+} \ge 0,\, e_{1k}^{-} \ge 0,\, e_{1k}^{+} \cdot e_{1k}^{-} = 0, \quad k = 1, 2, 3, 4 \\
& e_{2k}^{+} \ge 0,\, e_{2k}^{-} \ge 0,\, e_{2k}^{+} \cdot e_{2k}^{-} = 0, \quad k = 1, 2, 3, 4
\end{aligned}
$$

By solving the above model, we obtain the vector of the uncertain OWA weights: $v = (0.3, 0.3, 0.4)^T$. Therefore, we get the estimated aggregated values $\hat{s}_k$ of s_k as follows:

$$
\begin{aligned}
\hat{s}_1 &= 0.3 \times b_{11} + 0.3 \times b_{12} + 0.4 \times b_{13} = [0.41, 0.65] \\
\hat{s}_2 &= 0.3 \times b_{21} + 0.3 \times b_{22} + 0.4 \times b_{23} = [0.39, 0.55] \\
\hat{s}_3 &= 0.3 \times b_{31} + 0.3 \times b_{32} + 0.4 \times b_{33} = [0.32, 0.60] \\
\hat{s}_4 &= 0.3 \times b_{41} + 0.3 \times b_{42} + 0.4 \times b_{43} = [0.36, 0.57]
\end{aligned}
$$

Using Equations 3 and 4, we can rank the estimated aggregated values, $\hat{s}_k$, in descending order:

$$\hat{s}_1 \succ \hat{s}_2 \succ \hat{s}_4 \succ \hat{s}_3$$

4. CONCLUSION

We have investigated the uncertain OWA operator in which the associated weighting parameters cannot be specified, but value ranges can be obtained, and each input argument is given in the form of an interval of numerical values, which develops the theory of the OWA operator by Yager.

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