DECSAI

**Department of Computer Science**
**and Artificial Intelligence**

# On the Linguistic OWA Operator and Extensions

F. Herrera, E. Herrera-Viedma

Technical Report #DECSAI-96135.
September, 1996

ETS de Ingeniería Informática. Universidad de Granada.

18071 Granada - Spain - Phone +34.58.244019, Fax +34.58.243317

# 1

# ON THE LINGUISTIC OWA OPERATOR AND EXTENSIONS

**F. Herrera,**

**and E. Herrera-Viedma**

*Dept. of Computer Science and Artificial Intelligence,*
*University of Granada, 18071 - Granada, Spain*

## ABSTRACT

A summary on the linguistic OWA operators existing in the literature is presented. To deal with linguistic information with equal importance the LOWA and ordinal OWA operators are studied. To deal with weighted linguistic information two extensions of the LOWA operator are analyzed: the LWA operator when the weights have linguistic nature and the L-WOWA operator if they have numerical nature. The presentation of the operators is completed with a study of their properties and axiomatic.

## 1 INTRODUCTION

Usually, experts express their opinions by means of numerical values. Sometimes, however an expert could have a vague knowledge about this preference valuations, for example the preference degree of any alternative over any other alternative, and cannot estimate his preference with an exact numerical value. Then, a more realistic approach may be to use linguistic assessments instead of numerical values, that is, to suppose that the variables which participate in the problem are assessed by means of linguistic terms [10, 16, 23, 28].

The linguistic approach considers the variables which participate in the problem assessed by means of linguistic terms instead of numerical values [28]. A linguistic variable differs from a numerical one in that its values are not numbers, but words or sentences in a natural or artificial language. Since words, in general, are less precise than numbers, the concept of a linguistic variable serves the purpose of providing a means of approximated characterization of phenomena, which are too complex, or too ill-defined to be amenable its description in conventional quantitative terms. The linguistic approach has been applied to different areas, for instance, we can find applications on information retrieval [2], medical diagnostic [4], control systems [14], decision making [7, 16, 24], etc.

In contexts with multiple information sources an aggregation operator of linguistic labels is needed. Assuming a linguistic context, two main different approaches can be found in order to aggregate linguistic values: the first acts by direct computation on labels [5, 6, 7, 12, 22, 24, 25], and the second uses the associated membership functions [1, 4, 16]. Most available applications use techniques belonging to the latter kind, however, the final results of those methods are fuzzy sets which do not correspond to any label in the original term set. If one finally wants to have a label, then a "linguistic approximation" is needed [1, 4, 16].

In this work, we study aggregation operators of linguistic information which work by means of direct computation on labels. These operators summary the linguistic approaches to the Ordered Weighting Averaging (OWA) operator [21]. To combine non-weighted linguistic information we study the *Linguistic OWA (LOWA)* operator [7, 12] and the *Ordinal OWA* operator [22]. To deal with weighted linguistic information we present two extensions of the LOWA operator: the *Linguistic Weighted Averaging* (LWA) [6] and the *Linguistic Weighted OWA* (L-WOWA) [17] operators. We complete our study presenting the properties and axioms that these operators verify.

To do so, the paper is structured as follows: Section 2 presents the label set used to provide the opinions; Section 3 shows the LOWA and ordinal OWA operators; Section 4 analyzes two extensions of the LOWA operator to combine weighted linguistic information; and finally, some concluding remarks are pointed out.

## 2 PRELIMINARIES

We use label sets with an odd cardinal, representing the middle term an assessment of "approximately 0.5", with the rest of the terms being placed symmetrically around it and the limit of granularity 11 or no more than 13. The semantic of the elements in the label set is given by fuzzy numbers defined on the [0,1] interval, which are described by membership functions. Because the linguistic assessments are just approximate ones given by the experts, we can consider that linear trapezoidal membership functions are good enough to capture the vagueness of those linguistic assessments, since it may be impossible or unnecessary to obtain more accurate values. This representation is achieved by the 4-tuple, $(a_i, b_i, \alpha_i, \beta_i)$, the first two paremeters indicate the interval in which the membership value is 1; the third and fourth parameters indicate the left and right widths. Moreover, the term set, $S = \{s_0, \ldots, s_T\}$, must have the following characteristics:

1) The set is ordered: $s_i \geq s_j$ if $i \geq$ j.

2) There is the negation operator: Neg$(s_i) = s_j$ such that $j = T$-i.

3) Maximization operator: Max$(s_i, s_j) = s_i$ if $s_i \geq s_j$.

4) Minimization operator: Min$(s_i, s_j) = s_i$ if $s_i \leq s_j$.

**Example 1.** For example, this is the case of the following term set [1]:

| | | |
|---|---|---|
| $VH$ | $Very\_High$ | $(1, 1, 0, 0)$ |
| $H$ | $High$ | $(.98, .99, .05, .01)$ |
| $MH$ | $Moreorless\_High$ | $(.78, .92, .06, .05)$ |
| $FFMH$ | $From\_Fair\_to\_Moreorless\_High$ | $(.63, .80, .05, .06)$ |
| $F$ | $Fair$ | $(.41, .58, .09, .07)$ |
| $FFML$ | $From\_Fair\_to\_Moreorless\_Low$ | $(.22, .36, .05, .06)$ |
| $ML$ | $Moreorless\_Low$ | $(.1, .18, .06, .05)$ |
| $L$ | $Low$ | $(.01, .02, .01, .05)$ |
| $VL$ | $Very\_Low$ | $(0, 0, 0, 0)$ |

In what follows, we shall use this set of nine labels in all examples.

# 3 LINGUISTIC VERSIONS OF OWA OPERATORS

Here, we present the two linguistic versions of the OWA operator and their properties.

## 3.1 The LOWA Operator

The *LOWA operator* [7, 12] is based on the *OWA operator* defined by Yager [21] and on the *convex combination of linguistic labels* defined by Delgado et al. [5].

**Definition 1.** *Let $A = \{a_1, \ldots, a_m\}$ be a set of labels to be aggregated, then the LOWA operator, $\phi$, is defined as*

$$\phi(a_1, \ldots, a_m) = W \cdot B^T = \mathcal{C}^m\{w_k, b_k, k = 1, \ldots, m\} =$$

$$= w_1 \odot b_1 \oplus (1 - w_1) \odot \mathcal{C}^{m-1}\{\beta_h, b_h, h = 2, \ldots, m\}$$

*where $W = [w_1, \ldots, w_m]$, is a weighting vector, such that, (i) $w_i \in [0,1]$ and, (ii) $\Sigma_i w_i = 1$,. $\beta_h = w_h / \Sigma_2^m w_k, h = 2, \ldots, m$, and $B = \{b_1, \ldots, b_m\}$ is a vector associated to $A$, such that,*

$$B = \sigma(A) = \{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\}$$

*in which,$a_{\sigma(j)} \leq a_{\sigma(i)}\ \forall\ i \leq j$, with $\sigma$ being a permutation over the set of labels $A$. $\mathcal{C}^m$ is the convex combination operator of $m$ labels and if m=2, then it is defined as*

$$\mathcal{C}^2\{w_i, b_i, i = 1, 2\} = w_1 \odot s_j \oplus (1 - w_1) \odot s_i = s_k, \quad s_j,\ s_i\ \in\ S,\ (j \geq i)$$

*such that, $k = min\{T, i + round(w_1 \cdot (j - i))\}$, where "round" is the usual round operation, and $b_1 = s_j,\ b_2 = s_i$.*

*If $w_j = 1$ and $w_i = 0$ with $i \neq j\ \forall i$, then the convex combination is defined as:*

$$\mathcal{C}^m\{w_i, b_i, i = 1, \ldots, m\} = b_j.$$

Next, we present an extension of the LOWA operator, and Inverse LOWA operator, that will be used in the definition of some weighted operators [6].

**Definition 2.** *An Inverse-Linguistic Ordered Weighted Averaging (I-LOWA) operator, $\phi^I$, is a type of LOWA operator, in which*

$$B = \sigma^I(A) = \{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\}$$

*where,*

$$a_{\sigma(i)} \leq a_{\sigma(j)}\ \forall\ i \leq j.$$

*If m=2, then it is defined as*

$$\mathcal{C}^2\{w_i, b_i, i = 1, 2\} = w_1 \odot s_j \oplus (1 - w_1) \odot s_i = s_k, \quad s_j,\ s_i\ \in\ S,\ (j \leq i)$$

*such that*

$$k \;=\; min\{T, i \;+\; round(w_1 \cdot (j-i))\}.$$

**Example 2.** Suppose that we want to aggregate by means of the LOWA operator the following four labels, $\{L, ML, H, VH\}$. Assuming the weighting vector $W = [0.3, 0.2, 0.4, 0.1]$ the general expression of the aggregation of labels is:

$$\phi(L, ML, H, VH) = [0.3, 0.2, 0.4, 0.1](VH, H, ML, L)$$
$$= \mathcal{C}^4\{(0.3, VH), (0.2, H), (0.4, ML), (0.1, L)\}.$$

Then, we obtain the final result applying the recursive definition of the convex combination, $\mathcal{C}^4$, as follows. Firstly, we develop $\mathcal{C}^4$ until its most simple expression in the following steps:

1. For $m = 4$,
$$\mathcal{C}^4\{(0.3, VH)(0.2, H)(0.4, ML)(0.1, L)\}$$
$$= 0.3 \odot VH \oplus \mathcal{C}^3\{(0.29, H), (0.57, ML), (0.14, L)\}.$$
2. For $m = 3$,
$$\mathcal{C}^3\{(0.29, H), (0.57, ML), (0.14, L)\} = 0.29 \odot H \oplus \mathcal{C}^2\{(0.57, ML), (0.14, L)\}.$$

Now, we are going to go back solving the most simple cases until to obtain the final result:

1. For $m = 2$,
$$\mathcal{C}^2\{(0.57, ML), (0.14, L)\} = 0.57 \odot ML \oplus (1 - 0.57) \odot L = ML \; (s_2),$$
since as $ML = s_2$ and $L = s_1$ then
$$min\{9, 1 + round(0.57 \cdot (2-1))\} \;=\; min\{8, 1 + round(0.57)\} \;=\; min\{8, 2\} = 2.$$
2. For $m = 3$,
$$\mathcal{C}^3\{(0.29, H), (0.57, ML), (0.14, L)\} = 0.29 \odot H \oplus \mathcal{C}^2\{(0.57, ML), (0.14, L)\} = FFML \; (s_3),$$
since as $H = s_7$ and $ML = s_2$ then
$$min\{8, 2 + round(0.29 \cdot (7-2))\} \;=\; min\{8, 2 + round(1.45)\} \;=\; min\{8, 3\} = 3.$$

3. Finally, we obtain the final result for $m = 4$,

$$\mathcal{C}^4\{(0.3, VH)(0.2, H)(0.4, ML)(0.1, L)\} =$$

$$0.3 \odot VH \oplus \mathcal{C}^3\{(0.29, H), (0.57, ML), (0.14, L)\} = FFMH\ (s_5),$$

since as $VH = s_8$ and $FFML = s_3$ then

$$min\{8, 3+round(0.3\cdot(8-3))\} = min\{8, 3+round(1.5)\} = min\{8, 5\} = 5.$$

How to calculate the weighting vector of LOWA operator, $W$, is a basic question to be solved. A possible solution is that the weights represent the concept of fuzzy majority in the aggregation of LOWA operator using fuzzy linguistic quantifiers [29]. Yager proposed an interesting way to compute the weights of the OWA aggregation operator, which, in the case of a non-decreasing proportional fuzzy linguistic quantifier, Q, is given by this expression [21, 23]:

$$w_i = Q(i/n) - Q((i-1)/n), i = 1, \ldots, n,$$

being the membership function of Q, as follows:

$$Q(r) = \begin{cases} 0 & \text{if } r < a \\ \frac{r-a}{b-a} & \text{if } a \leq r \leq b \\ 1 & \text{if } r > b \end{cases}$$

with $a, b, r \in [0, 1]$. Some examples of non-decreasing proportional fuzzy linguistic quantifiers are: "most" $(0.3, 0.8)$, "at least half" $(0, 0.5)$ and "as many as possible" $(0.5, 1)$. When a fuzzy linguistic quantifier, Q, is used to compute the weights of LOWA operator, $\phi$, it is symbolized by $\phi_Q$. Similarly happens for the I-LOWA operator, i.e., in this case it is symbolized by $\phi_Q^I$.

**Example 3.** Suppose that we want to aggregate by means of the LOWA operator the same four above labels, $\{L, ML, H, VH\}$. If we use the fuzzy linguistic quantifier "as many as possible" to calculate the weighting vector, i.e., $W = \{0, 0, 0.5, 0.5\}$ and the aggregation of the labels, working in the similar way, is the following:

$$\phi_Q(L, ML, H, VH) = [0, 0, 0.5, 0.5](VH, H, ML, L)$$

$$= \mathcal{C}^4\{(0, VH), (0, H), (0.5, ML), (0.5, L)\} = ML.$$

The LOWA operator is a rational aggregation operator because it verifies some properties and axioms of any acceptable aggregation operator [9].

## *Properties of LOWA Operator*

**Property 1.-** The LOWA operator is **increasing monotonous** respect to the argument values, in the following sense: let $A = [a_1, a_2, \ldots, a_n]$ be an ordered argument vector, let $B = [b_1, b_2, \ldots, b_n]$ be a second ordered argument vector, such that $\forall\ j,\ a_j \geq bj$ then $\phi(A)\ \geq\ \phi(B)$.

**Property 2.-** The LOWA operator is **commutative**, i.e.,

$$\phi(a_1, a_2, \ldots, a_n)\ =\ \phi(\pi(a_1), \pi(a_2), \ldots, \pi(a_n)),$$

where $\pi$ is a permutation over the set of arguments.

**Property 3.-** The LOWA operator is an **"orand" operator**. That is, for any weighting vector $W$ and ordered labels vector A=$[a_1, a_2, \ldots, a_n]$, then

$$\text{Min}(A)\ \leq\ \phi(A)\ \leq\ \text{Max}(A).$$

## *Axiomatic of the LOWA Operator*

In what follows we are going to study some of the proposed axioms in fuzzy setting considering the LOWA operator working with linguistically valued preferences. Before this, we include the following linguistic notation that we shall use.

Let $A = \{x_1, \ldots, x_n\}$ be a finite non-empty set of alternatives.

Let $E = \{e_1, \ldots, e_m\}$ be a panel of experts.

Let $S = \{s_i,\ i = 0..T\}$ be a label set to voice experts' opinions.

Let $x_{ij} \in S$ be the linguistic rating of alternative $x_i$ by expert $e_j$.

Let $F_j$ be the linguistic rating set over alternatives by expert $e_j$.

Let $\mu_{F_j}$ be the linguistic membership function of $F_j$ such that $x_{ij} = \mu_{F_j}(x_i)$.

Let $F$ be the linguistic rating set such that $F = \phi(F_1, \ldots, F_n)$.

The LOWA operator satisfies these axioms:

**Axiom I**: *Unrestricted domain.* For any set of individual preference patterns $\{F_j,\ j = 1, \ldots, m\}$ there is a social preference pattern $F$, which may be constructed,

$$\forall F_1, \ldots, F_m \in S^n, \exists F \in S^n \text{ such that } F = \phi(F_1, \ldots, F_m).$$

**Axiom II**: *Unanimity or Idempotence.* If everyone agrees on a preference pattern, it must be 3sen as the social 3ice pattern,i.e.,

$$F_j = F, \forall j \Rightarrow F = \phi(F, F, \ldots, F).$$

**Axiom III**: *Positive association of social and individual values.* If an individual increases his linguistic preference intensity for $x_i$ then the social linguistic preference for $x_i$ cannot decrease. This means that if $F_j^{'}$ and $F_j$ are such that $\mu_{F_j} \leq \mu_{F_j'}$, then if $\phi(F_1, \ldots, F_j, \ldots, F_m) = F$ and $\phi(F_1, \ldots, F_j^{'}, \ldots, F_m) = F^{'}$, then $\mu_F \leq \mu_{F_{'}}$.

**Axiom IV**: *Independence of irrelevant alternatives.* The social preference intensity for $x_i$ only depends on the individual preference intensity for $x_i$, and not for $x_k$, $k \neq i$, $\mu_{\phi(F_1,\ldots,F_m)}(x_i) = \varphi(x_{i1}, \ldots, x_{im})$. Clearly this axiom does not extend strictly speaking, since for preference relations the independence of irrelevant alternatives deals with pairs of alternatives.

**Axiom V**: *Citizen sovereignty.* It means that any social preference pattern can be expressed by the society of individuals; in other words

$$\forall F, \exists F_1, \ldots, F_m \text{ such that } F = \phi(F_1, \ldots, F_m).$$

**Axiom VI**: *Neutrality.* The neutrality axiom refers to the invariance properties of the voting procedure. There are several types:

1. *Neutrality with respect to alternatives.* If $x_i$ and $x_k$ are such that $x_{ij} = x_{kj}$, $\forall j$, then $\mu_{\phi(F_1,\ldots,F_m)}(x_i) = \mu_{\phi(F_1,\ldots,F_m)}(x_k)$
2. *Neutrality with respect to voters.* In a homogeneous group, this is the anonymity property, i.e., the commutativity of $\phi$.

Some applications of the LOWA operator guided by fuzzy majority in group decision making problems can be found in [8, 10, 11, 12].

## 3.2 The Ordinal OWA Operator

The ordinal OWA operator was introduced in [22].

**Definition 3.** *Let $A = \{a_1, \ldots, a_m\}$ ($a_i \in S$) be a set of labels to be aggregated, then, the ordinal OWA operator, $\omega$, is defined as*

$$\omega(a_1, \ldots, a_m) = Max_j[w_j \min b_j],$$

*where $W = [w_1, \ldots, w_m]$, is a weighting vector, such that, (i) $w_i \in S$, (ii) $w_j \geq w_i$ if $j > i$, and (iii) $Max_j[w_j] = s_T$; and $B$ according to definition 1.*

Clearly, we may see that two important differences between this operator and the LOWA operator are: the weights in this operator have linguistic nature and in the LOWA operator numerical one, and the diffrent requeriments on the weighting vectors.

### *Properties of the Ordinal OWA*

The ordinal OWA operator can be shown to have the following properties [22]:

**Property 1.-** The ordinal OWA operator is **increasing monotonous** respect to the argument values, in the following sense: let $A = [a_1, a_2, \ldots, a_n]$ be an argument vector, let $B = [b_1, b_2, \ldots, b_n]$ be a second argument vector, such that $\forall\ j,\ a_j \geq bj$ then $\omega(A)\ \geq\ \omega(B)$.

**Property 2.-** The ordinal OWA operator is **commutative**, i.e.,

$$\omega(a_1, a_2, \ldots, a_n)\ =\ \omega(\pi(a_1), \pi(a_2), \ldots, \pi(a_n)).$$

**Property 3.-** The ordinal OWA operator is **an "orand" operator**. That is, for any weighting vector $W$ and ordered labels vector A=$[a_1, a_2, \ldots, a_n]$, then

$$\text{Min}(A)\ \leq\ \omega(A)\ \leq\ \text{Max}(A).$$

**Property 4.-** The ordinal OWA operator present the idempotency property. That is, if we have an ordered vector of labels, A=$[a_1, a_2, \ldots, a_n]$, such that $a_i = a\ \forall i$, then for any weighting vector, $W$, $\omega(A) = a$.

Some applications of ordinal OWA operator in decision making problems can be found in [26].

# 4 EXTENSIONS OF LOWA OPERATOR

In the OWA operators the weights measure the importance of a value (in relation to other values) with independence of the information source. In other operators the weights measure the importance of an information source with independence of the value (for instance, the weighted mean). Different aggregation operators of linguistic information that combines the advantages of both operators types have been proposed in [6, 17]. These operators are based on the LOWA operator [7, 12] and are designed to aggregate weighted linguistic information. In the first, called Linguistic Weighted Averaging (LWA) operator [6], the weights of information sources have linguistic nature, and in the second, called the Linguistic Weighted OWA (L-WOWA) [17], they have numerical nature. In what follows we shall analyze them.

## 4.1 The LWA Operator

Following Cholewa's studies [3] and Montero's aggregation model [15], if we want to aggregate weighted information we have to define two aggregations:

- the aggregation of importance degrees (weights) of information, and
- the aggregation of weighted information (information combined with weights).

The first aspect consists of obtaining a collective importance degree from individual importance degrees that characterizes the final result of aggregation operator. In the LWA operator, as the importance degrees are linguistic values, this is solved using the LOWA operator guided by the concept of fuzzy majority.

The aggregation of weighted information involves the transformation of the weighted information under the importance degrees. The transformation form depends upon the type of aggregation of weighted information being performed [25]. In [19, 20] Yager discussed the effect of the importance degrees in the types of aggregation "MAX" and "MIN" and suggested a class of functions for importance transformation in both types of aggregation. For MIN type aggregation he suggested a family of t-conorms acting on the weighted information and the negation of the weights, which presents the non-increasing monotonic property in the weights. For MAX type aggregation he suggested a family of t-norms acting on weighted information and the weight, which presents the non-decreasing monotonic property in the weights. In [25] Yager proposed a general specification of the requirements that any *importance transformation function*,

$g$, must satisfy for any type of the aggregation operator. The function, $g$, must have the following properties:

1. if $a > b$ then $g(w, a) \geq g(w, b)$
2. $g(w, a)$ is monotone in $w$
3. $g(0, a) = \mathrm{ID}$
4. $g(1, a) = a$.

with $a$, $b \in [0, 1]$ expressing the satisfaction with regards to a criterion, $w \in [0, 1]$ the weight associated to the criterion, and "ID" an identity element, which is such that if we add it to our aggregations it doesn't change the aggregated value. Condition one means that the function $g$ is monotonically non-decreasing in the second argument, that is, if the satisfaction with regards to the criteria is increased the overall satisfaction shouldn't decrease. The second condition may be viewed as a requirement that the effect of the importance be consistent. It doesn't specify whether $g$ is monotonically non-increasing or non-decreasing in the first argument, but must be one of these. It should be noted that conditions three and four actually determine the type of monotonicity obtained from two. If $a > \mathrm{ID}$, the $g(w, a)$ is monotonically non-decreasing in $w$, while if $a < \mathrm{ID}$, then it is monotonically non-increasing. The third condition is a manifestation of the imperative that zero importance items don't effect the aggregation process. The final condition is essentially a boundary condition which states that the assumption of all importances equal to one effectively is like not including importances at all [25].

Considering the aforementioned ideas and assuming a linguistic framework, that is a label set, $S$, to express the information and a label set, $L$, to express the weights, in [6] we proposed the LWA operator, with its respective aggregation operators and transformation functions:

1. Aggregation operator: $LOWA$ or $I - LOWA$.
2. Transformation function: $g_{(\mathrm{LOWA})} = LC^{\rightarrow}(w, a)$ or $g_{(\mathrm{I\text{-}LOWA})} = LI^{\rightarrow}(w, a)$.

Itis based on the combination of the LOWA and I-LOWA operator with several *linguistic conjunction functions* ($LC^{\rightarrow}$) and several *linguistic implication* *functions* ($LI^{\rightarrow}$), respectively. Therefore, the $LWA$ operator is a type of fuzzy majority guided weighted aggregation operator.

Before defining the LWA operator consider the following two families of connectives:

1. Linguistic conjunction functions ($LC^{\rightarrow}$).

   These linguistic conjunction functions, presented in [6], are monotonically non-decreasing t-norms in the weights:

   (a) *The classical MIN operator*:

   $$LC_1^{\rightarrow}(c, a) = MIN(c, a).$$

   (b) *The nilpotent MIN operator*:

   $$LC_2^{\rightarrow}(c, a) = \begin{cases} MIN(c, a) & \text{if } c > Neg(a) \\ s_0 & \text{otherwise.} \end{cases}$$

   (c) *The weakest conjunction*:

   $$LC_3^{\rightarrow}(c, a) = \begin{cases} MIN(c, a) & \text{if } MAX(c, a) = s_T \\ s_0 & \text{otherwise.} \end{cases}$$

2. Linguistic implication functions ($LI^{\rightarrow}$).

   These linguistic implication functions are monotonically non-increasing in the weights:

   (a) *Kleene-Dienes's implication function*:

   $$LI_1^{\rightarrow}(c, a) = MAX(Neg(c), a).$$

   (b) *Gödel's implication function*:

   $$LI_2^{\rightarrow}(c, a) = \begin{cases} s_T & \text{if } c \leq a \\ a & \text{otherwise.} \end{cases}$$

   (c) *Fodor's implication function*:

   $$LI_3^{\rightarrow}(c, a) = \begin{cases} s_T & \text{if } c \leq a \\ MAX(Neg(c), a) & \text{otherwise.} \end{cases}$$

Let $\{(c_1, a_1), \ldots, (c_m, a_m)\}$ be a set of weighted opinions such that $a_i$ shows the opinion of an expert $e_i$, assessed linguistically on the label set, $S$, $a_i \in S$, and $c_i$ the relevance degree of expert $e_i$, assessed linguistically on the same label set $S$, $c_i \in S$.

**Definition 4.** *The aggregation of the set of weighted individual opinions,* $\{(c_1, a_1), \ldots, (c_m, a_m)\}$*, according to the LWA operator,* $\Omega$*, is defined as a weighted collective opinion,* $(c_E, a_E)$*, such that*

$$(c_E, a_E) = \Omega[(c_1, a_1), \ldots, (c_m, a_m)],$$

*where the importance degree of the group opinion,* $c_E$*, is obtained as*

$$c_E = \phi_Q(c_1, \ldots, c_m),$$

*and, the opnion of the group,* $a_E$*, is obtained as*

$$a_E = f[g(c_1, a_1), \ldots, g(c_m, a_m)],$$

*where* $f \in \{\phi_Q, \phi_Q^I\}$ *is an linguistic aggregation operator of transformated information and* $g$ *is a importance transformation function, such that,* $g \in \{LC_1^{\rightarrow}, LC_2^{\rightarrow}, LC_3^{\rightarrow}\}$ *if* $f = \phi_Q$*, and* $g \in \{LI_1^{\rightarrow}, LI_2^{\rightarrow}, LI_3^{\rightarrow}\}$ *if* $f = \phi_Q^I$*.*

In [6] we presented some evidence of its rational aggregation way. It aggregation has been checked examining some of the axioms that an acceptable weighted aggregation operator must verify.

Some applications of the LWA operator in multi-criteria and group decision making can be found in [6, 13].

## 4.2 The L-WOWA Operator

Let $\{(c_1, a_1), \ldots, (c_m, a_m)\}$ be a set of weighted opinions such that $a_i$ shows the opinion of an expert $e_i$, assessed linguistically on the label set, $S$, $a_i \in S$, and $c_i$ the relevance degree of expert $e_i$, such that, (i) $c_i \in [0, 1]$ and (ii) $\sum_i c_i = 1$.

**Definition 5.** *The aggregation of the set of weighted individual opinions,* $\{(c_1, a_1), \ldots, (c_m, a_m)\}$*, according to the L-WOWA operator is defined as*

$$\varphi[(c_1, a_1), \ldots, (c_m, a_m)] = [\lambda_1, \ldots, \lambda_m]B^T = \mathcal{C}^m\{\lambda_i, b_i,\ i = 1, \ldots, m\} =$$

$$= \lambda_1 \odot b_1 \oplus (1 - \lambda_1) \odot \mathcal{C}^{m-1}\{\beta_h, b_h, h = 2, \ldots, m\},$$

*B according to definition 1 and the weights $\lambda_i$ obtained as*

$$\lambda_i = w^*(\sum_{i \geq j} c_{\sigma(j)}) - w^*(\sum_{j<i} c_{\sigma(j)})$$

*with $w^*$ a monotonic increasing function that interpolates the points $(i/m, \sum_{i \geq j} c_j)$ together with the point (0,0).*

Since the difference between the LOWA and L-WOWA operator is the calculation of the weights, the L-WOWA operator satisfies the same properties and axioms of the LOWA operator.

Some applications of the L-WOWA operator are shown in [18]

# 5 CONCLUDING REMARKS

In this paper, we have presented a summary of linguistic OWA operators provided in the literature. We have shown that there are OWA operators to deal with non-weighted and weighted linguistic information. Specifically we have presented the LOWA and ordinal OWA operators for management non-weighted linguistic information and the LWA and L-WOWA operators for weighted one.

# REFERENCES

[1] P.P. Bonissone and K.S. Decker, Selecting Uncertainty Calculi and Granularity: An Experiment in Trading-off Precision and Complexity, in: L.H. Kanal and J.F. Lemmer, Eds., *Uncertainty in Artificial Intelligence* (North-Holland, 1986) 217-247.

[2] G. Bordogna and G. Pasi, A Fuzzy Linguistic Approach Generalizing Boolean Information Retrieval; a Model and Its Evaluation, *Journal of the American Society for Information Science* **44** (1993) 70-82.

[3] W. Cholewa, "Aggregation of Fuzzy Opinions: An Axiomatic Approach," *Fuzzy Sets and Systems* **17** (1985) 249-259.

[4] R. Degani and G. Bortolan, The Problem of Linguistic Approximation in Clinical Decision Making, *Inter. J. of Approximate Reasoning* **2** (1988) 143-162.

[5] M. Delgado, J.L. Verdegay and M.A. Vila, On Aggregation Operations of Linguistic Labels, *Int. Journal of Intelligent Systems* **8** (1993) 351-370.

[6] F. Herrera and E. Herrera-Viedma, Aggregation Operators for Linguistic Weighted Information, *IEEE Transactions on Systems, Man, and Cybernetics* (1996). To appear.

[7] F. Herrera and J.L. Verdegay, Linguistic Assessments in group decision, *Proc. 1th European Congress on Fuzzy and Intelligent Technologies*, Aachen, 1993, 941-948.

[8] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A Sequential Selection Process in Group Decision Making with Linguistic Assessment, *Information Sciences* **85** (1995) 223-239.

[9] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, Aggregating Linguistic Preferences: Properties of LOWA Operator, *Proc. 5th IFSA World Congress*, Sao Paulo, 1995, 153-156.

[10] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A Linguistic Decision Process in Group Decision Making, *Group Decision and Negotiation* **5** (1996) 165-176.

[11] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A Model of Consensus in Group Decision Making under Linguistic Assessments, *Fuzzy Sets and Systems* **78** (1996) 73-87.

[12] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, Direct Approach Processes in Group Decision Making Using Linguistic OWA Operators, *Fuzzy Sets and Systems* **79** (1996) 175-190.

[13] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, Choice Processes for Non-Homogeneous Group Decision Making in Linguistic Setting, *Fuzzy Sets and Systems* (1996). To appear.

[14] E.H. Mamdani and S. Assilian, An experiment in linguistic synthesis with a fuzzy logic controller, *Inter. Journal of Man Machine Studies* **7** (1975) 1-13.

[15] J. Montero, "Aggregation of Fuzzy Opinions in a Non-Homogeneous Group," *Fuzzy Sets and Systems* **25** (1988) 15-20.

[16] M. Tong and P. P. Bonissone, A Linguistic Approach to Decision Making with Fuzzy Sets, *IEEE Transactions on Systems, Man and Cybernetics* **11** (1980) 716-723.

[17] V. Torra, The Weighted OWA Operator, *Int. Journal of Intelligent Systems* (1996). To appear.

[18] V. Torra, Weighted OWA Operators for Synthesis of Information, Proc. 5th IEEE Inter. Conference on Fuzzy Systems, New Orleans, USA, 1996, 966-971.

[19] R.R. Yager, A New Methodology for Ordinal Multiple Aspect Decision Based on Fuzzy Sets, *Decision Sciences* **12** (1981) 589-600.

[20] R.R. Yager, A Note on Weighted Queries in Information Retrieval Systems, *J. of the American Society of Information Sciences* **38** (1987) 23-24.

[21] R.R. Yager, On Ordered Weighted Averaging Aggregation Operators in Multicriteria Decision Making, *IEEE Transactions on Systems, Man, and Cybernetics* **18** (1988) 183-190.

[22] R.R. Yager, Applications and Extension of OWA Aggregation, *Int. Journal of Man-Machine Studies* **37** (1992) 103-132.

[23] R.R. Yager, Families of OWA Opertors, *Fuzzy Sets and Systems* **59** (1993) 125-148.

[24] R.R. Yager, Fuzzy Screening Systems, in *Fuzzy Logic: State of the Art*, R. Lowen, Ed. Kluwer Academic Publishers, 1993, pp. 251-261.

[25] R.R. Yager, On Weighted Median Aggregation, *Int. J. Uncertainty, Fuzzines and Knowledge-Based Systems* **2** (1994) 101-113.

[26] R.R. Yager, An Approach to Ordinal Decision Making, *Int. Journal of Approximate Reasoning* **12** (1993) 237-261.

[27] R.R. Yager, Fusion of Ordinal Information Using Weighted Median Aggregation, Machine Intelligence Institute, Technical Report #MII-1520.

[28] L. A. Zadeh, The Concept of a Linguistic Variable and its Applications to Approximate Reasoning. Part I, *Information Sciences* **8** (1975) 199-249, Part II, *Information Sciences* **8** (1975) 301-357, Part III, *Information Sciences* **9** (1975) 43-80.

[29] L. A. Zadeh, A Computational Approach to Fuzzy Quantifiers in Natural Languages, *Computers and Mathematics with Applications* **9** (1983) 149-184.