OPTION PRICING ON ASSETS
SSM long project

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Abstract

On this project I examine the option-pricing problem for an European call assuming different stochastic evolution for the stock price. First I explore the analytical solution for the Black-Scholes equation and compare it with a numerical solution, using a Monte Carlo method. I apply it using the parameters extracted from the real evolution of the assets of IBEX35 of the last three months. Finally I make further considerations on the SDE proposed by Black and Scholes, so to make it more realistic, thus including stochastic volatilities and jumps, and then compare the numerical solutions of these models.

1 Background theory

Although i am not going to give specific details about the underlying economy concepts and implications, it is important to briefly mention the basics in order to follow the results that I present in the next section.

- An underlying asset can be a currency, market index or the stock itself.
- A derivative is a finantial security (i.e a negotiable finantial instrument that holds some type of monetary value) with a value that rely on an underlying asset. An option is one example of derivative. There are two types: a call option gives its owner a right to buy the underlying asset, while a put option gives its owner a right to sell the underlying asset. The one analysed here are an European option, which only can be exercised at the end of its life, at its madurity T.

1.1 Black-Scholes model

It supposed one of the most significant advances in the option valuation problem. Although its over-simplified and not correct, it is still used as a reference benchmark for stock analysis. It makes possible to obtain European option with a portfolio constituted by the underlying asset and a risk free bonus. It assumes that the price of the underlying asset follow a geometric Brownian motion, i.e the instantaneous log return of the stock price is an infinitessimal random walk with drift. It is described then by the following SDE:

$$dS = \mu S dt + \sigma S dW \tag{1}$$

Where μ represents the expected return, related to the risk-free rate r and σ is the volatility. Both values are assumed to be constant in the model. W is the Wiener process.

So discretizing the time interval, in order to perform simulations for the trajectories, we can write the path for $S(t)$ as:

$$S(t_i) = S(t_{i-1}) e^{(\mu - \sigma^2/2)\Delta t + \sigma w_i \sqrt{\Delta t}} \tag{2}$$

where w_i are stochastic variables normally distributed.

We denote by $C(S,t)$ the price of a call option relying on the underlying asset whose cost is S(t). Applying Itô calculus, we can write that

$$dC = \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \tag{3}$$

Constructing a portfolio with a quantity Δ of the underlying asset S and a loan B with an interest r, such that it has a value that varies as $d\theta = \Delta S - dB = \Delta(\mu S dt + \sigma S dW) - rBdt$. The assumption for the model is that the change on the value of the portfolio is the same as the change on the value of the option. With that and using $\Delta = \frac{\partial C}{\partial S}$, we can arrive to the Black-Scholes equation

$$\frac{\partial C}{\partial t} = rC - rS\frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \tag{4}$$

The boundary condition for an European call option is given by

$$C(S, t = T) = max(S(T) - K, 0) \tag{5}$$

being T the madurity time and K the strike price, which is the price for which the option's owner can buy the underlying security.
By the change of variables, and handling correctly the Black-Scholes equation, we get the diffusion equation. Its solution is proven to be

$$C(S,t) = SN(d_1) - ke^{-r(T-t)}N(d_2) \tag{6}$$

where

$$d_1 = \frac{ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

and $N(d_n)$ the normal distribution

$$N(d_n) = \frac{1}{2\pi}\int_{-\infty}^{d_n} e^{s^2/2} ds$$

1.2 Generalizations

Further considerations can be done so to modify the SDE given on (1). I am going to compare the results for the price of call options given (1) as well as from generalised SDEs, one that include stochastic volatilities, and another one that include stochastic volatilities as well as jumps or discontinuities.

- The stochastic volatility problem
  I follow the assumptions on [2]. Thus consider the same equation given by (1) for the underlying asset evolution, but now we let σ not to be constant. Instead, it is assumed that it obeys the following stochastic process:
  $$dV = \phi V dt + \xi V dW' \tag{7}$$
  Being $V = \sigma^2$. It is assumed to be independent of S, thus θ and ξ are parameters that can depend just on σ and t.
  W' can be, in principle, correlated with the Wiener proces W of the equation for the evolution of the underlying asset. But for the numerical results I assume that there is no such correlation,

thus $\rho = 0$. In order to perform a Monte Carlo simulation on V we discretize the time interval $T - t$ so we can sample the variance evolution as:

$$V_i = V_{i-1} e^{[(\phi - \xi^2/2)\Delta t + \nu_i \xi \sqrt{\Delta t}]} \tag{8}$$

where ν_i are n independent normal variates.
It is assumed that V follows a mean-reverting process, so we can take $\phi = a(\sigma^* - \sigma)$.

- Jumps at random times
  In real markets, the arrival of important new information about the stock has an important effect on price. This arrival is modeled by a Poisson-distributed event, so we have that the probability that an event occurre during a time interval of lenght h would be equal to λh. Hence we can write the following PDE:

  $$dS = (\mu - \lambda k)Sdt + \sigma SdW + Sdq \tag{9}$$

  being $q(t)$ the Poisson process and $k = \epsilon(Y-1)$. $(Y-1)$ is the random variable percentage change in the stock price when an event occur.
  Thus it modifies the solution given by (2) in the following way:

  $$S(t_i) = S(t_{i-1}) e^{(\mu - \sigma^2/2 - \lambda k)\Delta t + \sigma w_i \sqrt{\Delta t}} Y(n) \tag{10}$$

  where $Y(n) = \prod_{j=1}^{n} Y_j$, being Y_j independent and identically distributed and n is Poisson distributed with parameter λt.
  In my simulations I include that the jumps are negative in 70% of the times and the other 30% are positive, but I could explore different scenarios.

1.3 Monte Carlo method

In order to get numerical solutions I perform a Monte Carlo method. Its convergence relies on the Theorem of the Central Limit. First I generate M stochastic trajectories with probability density given by the lognormal processes described above, then I obtain the expectated value for C by averaging all results C_i, $i = 1, ..., M$. The trajectories are sampled until $t = T$, thus the price of the call option is:

$$C_i = e^{-r(T-t)} max(S_T - K, 0) \tag{11}$$

2 Results

After testing generally the algorithms and consistency of the Monte Carlo method, I extracted data from *Yahoo! Finance* of IBEX35 Stock Index. I analyse the closing values of the Index of the last three months, which are 63 trading days, so to get a real trajectory in order to infer the parameters for my simulations of the next three months.

For the SDE in (1), I infer on the value of the volatility taking a constant value. First I calculate the daily returns, as $r_i = log\left(\frac{S_i}{S_{i-1}}\right)$, and the standard deviation for contiguous averaged 10 days, i.e to take the sets $\{\ r_2, ..., r_{11}\}, \{\ r_3, ..., r_{12}\}, ..., \{\ r_{54}, ..., r_{63}\}$, we can assign its averaged value to be the value of the volatility that will be used for a period of $(T - t) = 3$ months.

Thus from the three months analysed, I obtain $\sigma = 0.00824$. I take (just as a reference) for the drift component the risk-free rate given by the *euribor* index, which is actually near to zero, $r = -0.0012$.

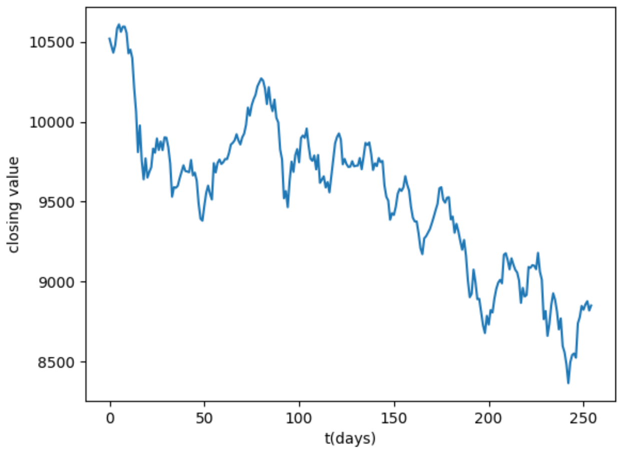

Figure 1. Real trajectory of IBEX35 Stock Index at the closing time from the last year, that corresponds to 262 trading days.

For calculating the parameters needed for the simulation of th SDE that includes a stochastic volatility, which in turn follow the process of (7), I do the same previous procedure but now for the volatilities exctracted for defining the previous averaged value. Thus, for my data, as σ^* is taken to be the averaged one used before, I isolate a_i from each μ_i and I obtain as an average $a = 82.3$. Moreover, $\xi = 0.2727$. So we have that it it has much more dispersion than the stock price, but also a strong drift given a.

For the SDE with stochastic jumps, (9), I rather choose an arbitrary value for the probability of occuring a jump between time steps. It has to be very small, because it represent rare events in markets. I try $\lambda = 0.0005$ which is still pretty high but I get it so to observe significant changes in the 3 months trajectories.

The first step to calculate the price of the derivative is to generate as much trajectories as I can. The next image shows one example for each evolution described in the three PDEs described above.

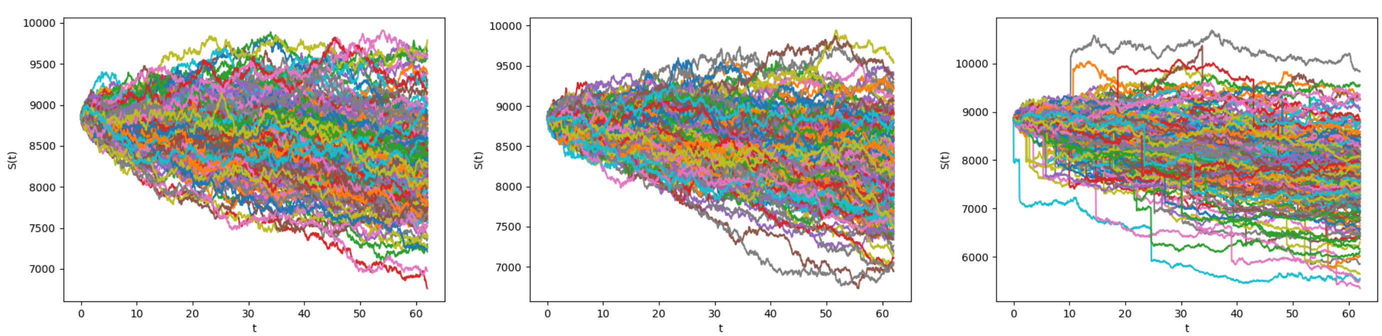

Figure 2. From left to right, 200 trajectories from the basic PDE (1), the one with stochastic volatility using (1) and (7) and the one with stochastic volatility plus jumps (9).

Using the Monte Carlo method now, I use equation (11) on the each price obtained for each trajectory at the madurity time, $S_i(t = T)$. In the next table I summarize the prices of a call European option with time madurity $T = 262$ and $K = 8850$, obtained with the B-S exact solution and with the numerical estimations considering the three different SDE explained, comparing them.

MODEL	BS exact	BS numerical	SV numerical	SVJ numerical
OPTION PRICE	37.21	37.5 ± 0.4	32.2 ± 0.6	30.1 ± 0.9
change			0.13	0.19

Table 1. Results obtained by averaging 10^5 Monte Carlo simulations for a European call option assuming the Black-Scholes (BS) model, the one that incorporates stochastic volatility (SV), and the one that includes also jumps (SVJ). For the last two ones, I show the relative change with respect to the BS model.

3 Conclusions

We can observe that numerical solution of the Black-Scholes model approaches with little error to the exact result. So Monte Carlo method has a good convergence to the analitical solution. Moreover, considering a stochastic volatility in the model, in this case the final price is modified, lowering its value for the option almost a 13%, when introducing stochastic volatilities with the parameters trying to reproduce the same real data, and a 19% when introducing discontinuous random jumps, although it is expected to observe less or no jumps in the period time considered, so the probability given in λ is higher than realistic values. Nevertheless, it is interesting as a case of study and it could be extended as exploring the parameter space λ.

[1] Black, Fischer; Scholes, Myron. "The Pricing of Options and Corporate Liabilities". Journal of Political Economy. 81 (3): 637–654.

[2] Robert C. Merton. "Option pricing when underlying stock returns are discontinuous". Journal of Financial Economics 3 (1976) 125-144.

[3] John Hull; Alan White. "The pricing options on assets with stochastic volatilities". The Journal of Finance, Volume 42, Issue 2 (Jun., 1987), 281-300

[4] Ren-Raw Chen; Shih-Kuo Yeh: "Analytical Upper Bounds for American Option Prices". The Journal of Financial and Quantitative Analysis, Vol. 37, No. 1 (Mar., 2002), pp. 117-135

[5] Mark Broadie; Özgür Kaya. "Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes". OPERATIONS RESEARCH, Vol. 54, No. 2, March–April 2006, pp. 217–231

[6] Notes on "Econofisica" from Pere Colet.