

The Importance of the Public Global Parameter On Ring-LWE problem-based Key Encapsulation Mechanisms

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Abstract

There are cryptographic systems that are secure against attacks by both quantum and classical computers. Some of these cryptographic systems are the Key Encapsulation Mechanisms (KEM) based on Ring-LWE problem. Some Ring-LWE problem-based KEMs include a public global parameter that is random and uniformly chosen. This parameter is used to generate a public key using in the process one secret key.

In this work, we analyze some values of the public global parameter that leak information about the secret key.

1. Introduction

The Ring-LWE problem was used as the basis of public key encryption schemes, digital signatures, key encapsulation mechanisms (KEM), homomorphic encryptions, and more. The Ring-LWE based approaches are promising, due to the provable security and high efficiency. The Ring-LWE problem is assumed as hard because the best known algorithms for Ring-LWE problem run in exponential time. Quantum computers don't seem to help.

In some Ring-LWE problem-based KEMs, a polynomial a is selected randomly and it is called by public global parameter. The polynomial b is calculated ($b = a.s + e$ where s, e are small polynomials). The polynomials b and a are public and the secret s is hard to find. But there are some values of a that do not offer the necessary security and leak information about the secret s .

2. Preliminaries

2.1 Mathematical Notations

For an integer $q \geq 1$, let \mathbb{Z}_q be the residue class ring modulo q and $\mathbb{Z}_q = \{0, \dots, q-1\}$. Let $\mathcal{R}_q = \mathbb{Z}_q[x]/(x^n+1)$ denote the polynomial ring modulo x^n+1 where the coefficients are in \mathbb{Z}_q . The operations (addition and multiplication) of the elements in \mathcal{R}_q are according to those of polynomials.

The integer $[x]$ is defined as $[x + \frac{1}{2}] \in \mathbb{Z}$.

Centered Binomial Distribution We define centered binomial distribution ψ_η as follows: sample $(a_1, \dots, a_\eta, b_1, \dots, b_\eta) \leftarrow \{0, 1\}^{2\eta}$ and output $\sum_{i=1}^\eta (a_i - b_i)$.

The samples are in the interval $[-\eta, \eta]$.

Discrete Gaussian Distribution It is defined by (χ_σ) where is assigned a weight proportional $\exp\left(-\frac{x^2}{2\sigma^2}\right)$ to all integer x where $\sigma \in \mathbb{R}$ is a standard deviation.

2.2 The Ring-LWE Problem

The schemes based on Ring-LWE have some advantages since there is a quantum reduction that solves a hard problem in ideal lattices in the worst case to solving Ring-LWE problem in the average-case [Regev 2009].

The Ring-LWE problem fixes a power of two n and modulus q . For $s \in \mathcal{R}_q$ called as secret, the Ring-LWE distribution $A_{s,\chi}$ over $\mathcal{R}_q \times \mathcal{R}_q$ is sampled by choosing $a \in \mathcal{R}_q$ uniformly at random, choosing $e \rightarrow \chi_\sigma^n$, and outputting $(a, a.s + e)$. One version of the Ring-LWE problem is the Search Ring-LWE.

Search Ring-LWE $_{q,\chi,k}$: Given k independent samples $(a_i, b_i) \in \mathcal{R}_q \times \mathcal{R}_q$ drawn from $A_{s,\chi}$ for a uniformly random $s \in \mathcal{R}_q$ (fixed for all samples), find s .

Next, we explain two key encapsulation mechanisms to understand the importance of the public global parameter.

2.3 Key Encapsulation Mechanisms based on Ring-LWE

We describe the most relevant concepts and definitions in Ring-LWE KEM. The Number Theoretic Transform (NTT) is used to speed up the polynomial multiplication, and it is not absolutely related in any way to security. To simplify our analysis, we use ordinary multiplication instead of NTT.

2.3.1 Ring-LWE KEM

The public global parameter a is selected uniformly at random from \mathcal{R}_q . The Ring-LWE KEM is shown in Figure 1. Since s_A, s_B, e_A, e_B and e'_B are small, Alice and Bob get the same shared key $s_{K_A} = s_{K_B}$ with high probability.

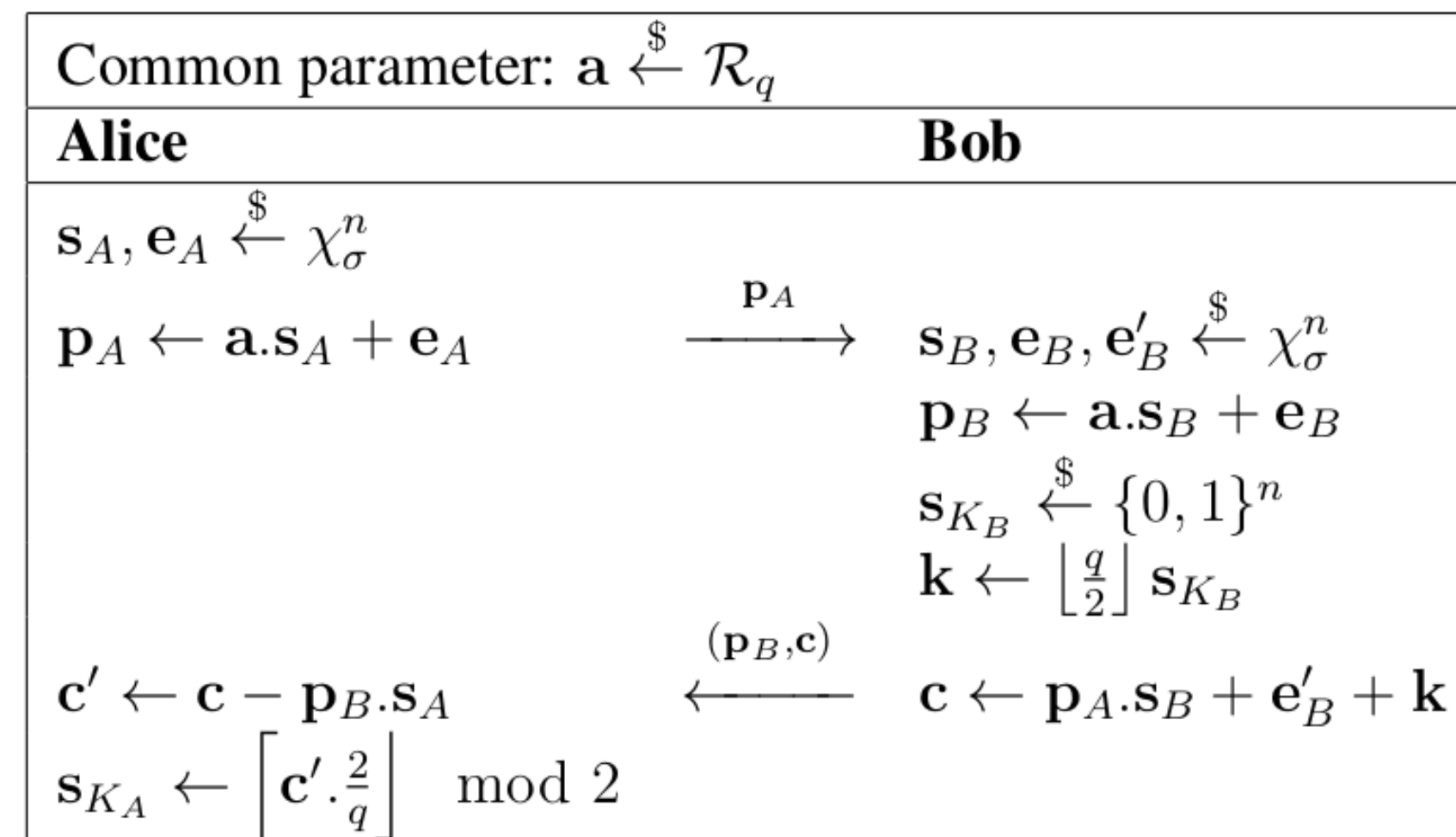


Figure 1: Ring-LWE KEM

3. Bad Values for the Public Global Parameter

Firstly, we explain a trivial case to understand how a third party (Eve) can recover some coefficients of secret keys of both participants.

Case 1: (Trivial Case) Let a be a polynomial with degree 0 (an integer) with value m . This case can happen but with a negligible probability.

Alice generates her public key $p_A = s_A.a + e_A$. The integer m multiplies each coefficient of s_A and its respective error e_A is added. We have:

$$p_A[i] = s_A[i].m + e_A[i] \quad \text{for } 0 \leq i \leq n-1$$

Eve can notice the integer value of $a = m$ and recover s_A applying $[\cdot]$ on $\frac{p_A}{m}$.

$$\left\lceil \frac{p_A[i]}{m} \right\rceil = \left\lceil \frac{s_A[i] + \frac{e_A[i]}{m}}{m} \right\rceil = s_A[i] + \left\lceil \frac{e_A[i]}{m} \right\rceil$$

Where the i -th coefficient of s_A can be retrieved with no error if $-\frac{1}{2} < \frac{e_A[i]}{m} < \frac{1}{2}$.

On NewHope, the value $e_A \in \psi_8^n$ ($-8 \leq e_A[i] \leq 8$), therefore m should be greater and equal to 17 ($m \geq 17$) because $-\frac{1}{2} < \frac{e_A[i]}{m} < \frac{1}{2}$.

Note: The value of a is a polynomial where each coefficient is selected uniformly at random in \mathbb{Z}_q , therefore the value of a being an integer would be suspicious for the participants. Alice and Bob can deny to share a secret using this suspect value of a .

Case 2: Let a be the public global parameter and we define a polynomial $c \in \mathcal{R}_q$ such that $a.c = m$ where m is the integer mentioned before.

Alice generates her public key $p_A = s_A.a + e_A$ and sends it to Bob. Eve takes p_A and multiplies by c . The integer m multiplies each coefficient of s_A .

$$p_A.c = s_A.a.c + e_A.c = s_A.m + e_A.c \quad \text{because } a.c = m$$

Eve applies $[\cdot]$ to $\frac{p_A.c}{m}$

$$\left\lceil \frac{(p_A.c)[i]}{m} \right\rceil = s_A[i] + \left\lceil \frac{(e_A.c)[i]}{m} \right\rceil \quad \text{for } 0 \leq i \leq n-1$$

and can retrieve the i -th coefficient of s_A if $-\frac{1}{2} < \frac{(e_A.c)[i]}{m} < \frac{1}{2}$. The result $e_A.c$ should be a small polynomial where each coefficient of $e_A.c$ divided by m should be between -0.5 and 0.5. One way to ensure this is to make the polynomial c belong to Gaussian or Centered Binomial Distribution. Because a multiplication between two polynomials which belong to Gaussian or Centered Binomial Distribution results in a small polynomial where its coefficients have an expected value equal to 0.

On NewHope, the value $e_A \in \psi_8^n$ and $e_A.c$ should be small therefore c should belong to ψ_μ^n where μ is a small integer. Therefore the parameter a that leaks information about secret keys can be generated using the formula $a = m(\psi_\mu^n)^{-1}$.

Note: Alice and Bob can calculate a^{-1} and multiply by all integers in \mathbb{Z}_q (brute force to determine the value of m). If the result is a small polynomial then it is possible that a leaks information because $c \in \psi_\mu^n$ and $c = a^{-1}.m$.

Case 3: (Adding error to Case 2)

Let a be the public global parameter and we define a polynomial $c \in \mathcal{R}_q$ such that $a.c = m + \psi_\nu^n$ where ψ_ν^n is a small polynomial and ν is a small integer.

Alice generates her public key $p_A = s_A.a + e_A$ and sends it to Bob. Eve takes p_A and multiplies by c . The integer m multiplies each coefficient of s_A .

$$p_A.c = s_A.(m + \psi_\nu^n) + e_A.c \quad \text{because } a.c = m + \psi_\nu^n \\ = s_A.m + s_A.\psi_\nu^n + e_A.c$$

Eve applies $[\cdot]$ to $\frac{p_A.c}{m}$

$$\left\lceil \frac{(p_A.c)[i]}{m} \right\rceil = s_A[i] + \left\lceil \frac{(s_A.\psi_\nu^n + e_A.c)[i]}{m} \right\rceil \quad \text{for } 0 \leq i \leq n-1$$

and can retrieve the i -th coefficient of s_A if $-\frac{1}{2} < \frac{(s_A.\psi_\nu^n + e_A.c)[i]}{m} < \frac{1}{2}$.

Note that the result $s_A.\psi_\nu^n + e_A.c$ should be a small polynomial therefore the polynomial c should belong to Gaussian or Centered Binomial Distribution.

On NewHope, the values $s_A, e_A \in \psi_8^n$. By definition $a.c = m + \psi_\nu^n$, a public global parameter a can be generated using the formula $a = (m + \psi_\nu^n)c^{-1}$ where c should belong to the Centered Binomial Distribution ($c \in \psi_\mu^n$).

Note: In this case, we have $m = a.c - \psi_\nu^n$ where a and m are public (for m we can use brute force). The value ψ_ν^n is an error (small) polynomial and the value c is unknown. It looks like the Search Ring-LWE problem where c is the secret. Therefore Alice and Bob do not have knowledge about a (being generated by c) and its possibility of leaking information.

The same process can be applied to retrieve Bob's secret s_B too in all cases.

4. Experiments

For our experiments, we work with parameters $n = 1024$, $q = 12289$ (parameters of NewHope) and set $m = \frac{q}{17} = 722$ that allows a maximum margin or error. For cases 2 and 3, it was generated 100 values of $a = m.(\psi_\mu^n)^{-1}$ for $\mu = \{1, 2, 4, 8, 16\}$ and $a = (m + \psi_\nu^n).(\psi_\mu^n)^{-1}$ where $\mu = \nu$ for $\mu = \{1, 2, 4, 8, 16\}$, respectively. And for each value a , 100 public keys were generated making a total of 100000 experiments. Each public key was multiplied by its respective c and divided by m . Applying rounding function $[\cdot]$ to this result, we retrieve some coefficients of the secret key. Each experiment takes at most 0.3 seconds.

Value of μ	Case 2					Case 3				
	1	2	4	8	16	1	2	4	8	16
Recovered complete keys	10000	10000	9284	144	0	10000	9372	9	0	0
Avg. recovered coefficients (%)	100.0	100.0	99.9	99.5	95.5	100.0	99.9	99.5	95.4	84.2
Max. # of wrong coefficients	0	0	2	19	90	0	3	17	88	218

Table 1: Results of experiments

In both cases for $\mu = 1, 2$ we retrieve 39372 of 40000 secret keys with zero coefficient errors (giving a success of 98.4 %). And that in the remaining 628 experiments there was an error in at most 3 of 1024 coefficients of the secret key.

5. Concluding remarks

We exposed some values of the public global parameter a that leak information about secret keys. Thus, there is a big responsibility how the public global parameter a is generated. If a has a value that leaks information (selected deliberately or not), then the secrets are exposed. Therefore the great and open question is "How to know when the value of the public global parameter a may or may not leak information about the secret?". In our experiments for cases 2 and 3, the generated values a that leak information always have at least 10 repeated coefficients. Therefore, it is recommended to be careful with values of parameter a that have repeated coefficients.

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References

- [Alkim et al. 2017] Alkim, E., Avanzi, R. M., Bos, J. W., Ducas, L., de la Piedra, A., Pöppelmann, T., and Schwabe, P. (2017). Newhope algorithm specifications and supporting documentation.
- [Barreto et al. 2016] Barreto, P. S., Longa, P., Naehrig, M., Ricardini, J. E., and Zanon, G. (2016). Sharper ring-lwe signatures. *Cryptology ePrint Archive*.

- [Bos et al. 2015] Bos, J. W., Costello, C., Naehrig, M., and Stebila, D. (2015). Post-quantum key exchange for the tls protocol from the ring learning with errors problem. In *2015 IEEE Symposium on Security and Privacy*, pages 553–570. IEEE.
- [de Clercq et al. 2015] de Clercq, R., Roy, S. S., Vercauteran, F., and Verbauwhe, I. (2015). Efficient software implementation of ring-lwe encryption. In *Proceedings of the 2015 Design, Automation & Test in Europe Conference & Exhibition, DATE '15*, page 339–344, San Jose, CA, USA. EDA Consortium.
- [Fan and Vercauteran 2012] Fan, J. and Vercauteran, F. (2012). Somewhat practical fully homomorphic encryption. *Cryptology ePrint Archive*.
- [Lindner and Peikert 2011] Lindner, R. and Peikert, C. (2011). Better key sizes (and attacks) for lwe-based encryption. In Kiayias, A., editor, *Topics in Cryptology – CT-RSA 2011*, pages 319–339, Berlin, Heidelberg. Springer Berlin Heidelberg.
- [Lyubashevsky et al. 2013] Lyubashevsky, V., Peikert, C., and Regev, O. (2013). On ideal lattices and learning with errors over rings. *J. ACM*, 60(6).
- [Peikert 2009] Peikert, C. (2009). Public-key cryptosystems from the worst-case shortest vector problem: Extended abstract. In *Proceedings of the Forty-First Annual ACM Symposium on Theory of Computing, STOC '09*, page 333–342, New York, NY, USA. Association for Computing Machinery.
- [Regev 2009] Regev, O. (2009). On lattices, learning with errors, random linear codes, and cryptography. volume 56, New York, NY, USA. Association for Computing Machinery.
- [Roy et al. 2016] Roy, S. S., Karmakar, A., and Verbauwhe, I. (2016). Ring-lwe: applications to cryptography and their efficient realization. In *International conference on security, privacy, and applied cryptography engineering*, pages 323–331. Springer.
- [Wu et al. 2012] Wu, Y., Huang, Z., Zhang, J., and Wen, Q. (2012). A lattice-based digital signature from the ring-lwe. In *2012 3rd IEEE International Conference on Network Infrastructure and Digital Content*, pages 646–651.