سوال ۴:

inequality constraint $g(x) = Ax - b$ $\qquad g: \mathbb{R}^2 \to \mathbb{R}^3$

Lagrangian function $\mathcal{L}(x, \mu) = f(x) + \mu^T g(x) \;, \quad \mu \in \mathbb{R}^3$

$$= \frac{1}{2} x^T Q x + C^T x + \mu^T (Ax - b)$$

Necessary Conditions:

Stationarity $\nabla_x \mathcal{L}(x^*, \mu) = \partial f(x^*) - Dg(x^*)^T \mu = 0$

$$\Rightarrow Qx^* + C + A^T \mu = 0 \Rightarrow x^* = -Q^{-1}(C + A^T \mu)$$

Primal feasibility $g(x^*) \leq 0$

$$\Rightarrow Ax^* - b \leq 0 \Rightarrow -AQ^{-1}(C + A^T \mu) - b \leq 0$$

Dual feasibility $\mu \geq 0$

Complementary slackness $\mu^T g(x^*) = 0$

$$\Rightarrow \mu^T (A(-Q^{-1}(C + A^T \mu)) - b) = 0$$

$$\mathcal{L}(x^*, \mu) = \frac{1}{2} x^{*T} Q x^* + c^T x^* + \mu^T (A x^* - b)$$

$$= \frac{1}{2}\left(-Q^{-1}(c + A^T\mu)\right)^T Q \left(-Q^{-1}(c + A^T\mu)\right) + c^T\left(-Q^{-1}(c + A^T\mu)\right)$$
$$+ \mu^T\left(A\left(-Q^{-1}(c + A^T\mu)\right) - b\right)$$

$$= \frac{1}{2}(c + A^T\mu)^T Q^{-1}(c + A^T\mu) - c^T Q^{-1}(c + A^T\mu)$$
$$- \mu^T A Q^{-1}(c + A^T\mu) - \mu^T b$$

$$= \frac{1}{2}(c + A^T\mu)^T Q^{-1}(c + A^T\mu) - (c^T + \mu^T A) Q^{-1}(c + A^T\mu) - \mu^T b$$

$$= -\frac{1}{2}(c + A^T\mu)^T Q^{-1}(c + A^T\mu) - \mu^T b$$

Dual problem $\max_{\mu \geq 0} h(\mu) = -\frac{1}{2}(c + A^T\mu)^T Q^{-1}(c + A^T\mu) - \mu^T b$