Convergence Analysis of GAN training procedures

Reza Pishkoo   Kasra Khoshjoo

Sharif University of Technology

presentation for HDP course
July 8, 2023

Presentation Overview

Generative Adversarial Networks

- latent variable: $Z \sim P_Z(z)$
- generator : $G_\theta(z)$
- discriminator : $D_\psi(x)$
- target function:

$$\mathcal{L}(\theta, \psi) = \mathbb{E}_{P_Z}\left[f(D_\psi(G_\theta(Z)))\right] + \mathbb{E}_{X \sim P_D}\left[f(-D_\psi(X))\right]$$

- zero-sum game

$$\min_\theta \max_\psi \mathcal{L}(\theta, \psi)$$

Iteration Procedures

$$(\theta, \psi)^{(k)} = F(\theta^{(k-1)}, \psi^{(k-1)})$$

- simultaneous gradient descent : $F_h(\theta, \psi) = (\theta, \psi) + hv(\theta, \psi)$ where $v(\theta, \psi) = \begin{bmatrix} -\nabla_\theta \mathcal{L} \\ \nabla_\psi \mathcal{L} \end{bmatrix}$
- alternating gradient descent : $F_h = F_{2,h} \circ F_{1,h}$
- continuous(ODE solver) : $\frac{d(\theta,\psi)}{dt} = v(\theta, \psi)$

Uniqueness of Nash-Equilibrium

Theorem (unique Nash-equilibrium [Goodfellow, 2014])

given that G_θ, D_ψ are powerful enough to approximate any real valued function, there is a unique Nash-equilibrium where :

$$G_\theta(Z) \sim P_D$$

Convergence Theory [Mescheder, 2017]

Theorem (sufficient condition for local convergence [Mescheder, 2017])

given iteration function F at point $\bar{x}$:

- *is stationary, i.e. $F(\bar{x}) = \bar{x}$*
- *has a Jacobian matrix with eigenvalues smaller than. i.e. $|\lambda_i(F'(\bar{x}))| < 1$*

for every initial point x_0 in some neighbourhood of $\bar{x}$, the sequence $F^{(k)}(x_0)$ converges to $\bar{x}$. i.e. $\|F^{(k)}(x_0) - \bar{x}\| \in O(|\lambda_{max}(\bar{x})|^k)$

in practice we have :

$$F'(\theta, \psi) = I + hv'(\theta, \psi)$$

proposition (sufficient condition for convergence [Mescheder, 2017])

there exists $h > 0$ where eigenvalues of $I + hv'(\theta, \psi)$ are within the unit ball if the real part of all eigenvalues of $v'(\theta, \psi)$ are negative.

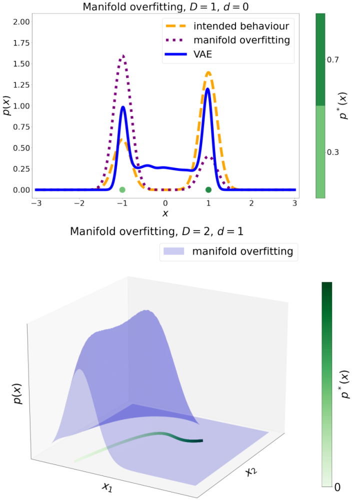
Which Training Methods for GANs do actually Converge?[Mescheder, 2018]

The Manifold Hypothesis

Example (low dimensional dist. [Arjovsky, 2014])

divergance between the distribution of $(0, Z)$ and (θ, Z) where $Z \sim Uni(0,1)$ is :

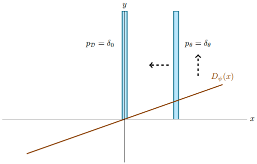
- $W_1(\mathbb{P}_0, \mathbb{P}_\theta) = |\theta|$
- $\mathrm{JS}(\mathbb{P}_0, \mathbb{P}_\theta) = \log 2 \cdot \mathbf{1}_{\{\theta \neq 0\}}$
- $D_{\mathrm{KL}}(\mathbb{P}_0, \mathbb{P}_\theta) = \mathbb{I}_{\{0\}}(\theta)$



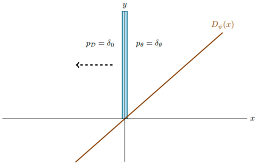
a toy data distribution

Definition (Dirac-GAN [Mescheder, 2018])

The Dirac-GAN consists of a (univariate) generator distribution $p_\theta = \delta_\theta$ and a linear discriminator $D(x) = \psi \cdot x$. The true data distribution P_D is given by a Dirac-distribution concentrated at 0.



(a) $t = t_0$



(b) $t = t_1$

a toy data distribution

the target function of Dirac-GAN is:

$$L(\theta, \psi) = f(\psi \cdot \theta) + f(0) \quad ; \quad f(x) = -\log(1 + \exp(-x))$$

the unique Nash-equilibrium is at $\theta = \psi = 0$.
the eigenvalues of Jacobian matrix of $v(\theta, \psi)$ are $\pm f'(0)i$, real values of which are zero.

a toy data distribution

Lemma (integral curves do not converge)

The integral curves of the gradient vector field $v(\theta, \psi)$ do not converge to the Nash-equilibrium. More specifically, every integral curve $(\theta(t), \psi(t))$ of the gradient vector field $v(\theta, \psi)$ satisfies $\theta(t)^2 + \psi(t)^2 = \text{const}$ for all $t \in [0, \infty)$.

Lemma (simultaneous gradient does not converge)

For simultaneous gradient descent, the Jacobian of the update operator $F_h(\theta, \psi)$ has eigenvalues $\lambda_{\{1,2\}} = 1 \pm hf'(0)i$ with absolute values $\sqrt{1 + h^2 f'(0)^2}$ at the Nash-equilibrium. Independently of the learning rate, simultaneous gradient descent is therefore not stable near the equilibrium.

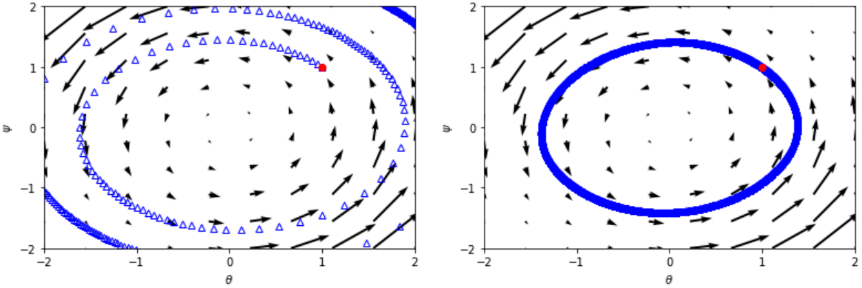
Lemma (alternating gradient descent does not converge)

For alternating gradient descent with n_g generator and n_d discriminator updates, the Jacobian of the update operator $F_h(\theta, \psi)$ has eigenvalues

$$\lambda_{1,2} = 1 - \frac{\alpha^2}{2} \pm \sqrt{\left(1 - \frac{\alpha^2}{2}\right)^2 - 1} \quad ; \quad \alpha = \sqrt{n_g n_d} h f'(0)$$

which implies that eigenvalues cannot be strictly inside the unit circle.

a toy data distribution



(a) SimGD (b) AltGD

Theorem (Convergence for continuous distributions[Nagarajan, 2017])

under some suitable assumptions - gradient descent based GAN optimization is locally convergent for absolutely continuous distributions

Creating convergent procedures

Additive Noise

- adding Independent gaussian noise to data points will make both distributions continuous, thus, making the gradient descent convergent.
- in the Dirac-GAN case:

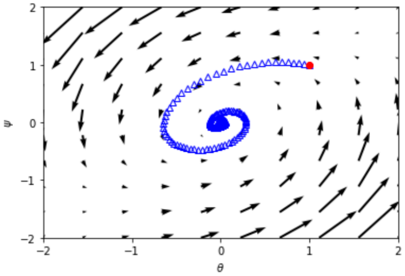
Lemma

When using Gaussian instance noise with standard deviation σ, the eigenvalues of the Jacobian of thegradient vector field are given by

$$\lambda_{1,2} = f^{''}(0)\sigma^2 \pm \sqrt{f''(0)^2\sigma^4 - f'(0)^2}$$

- if $f^{''}(0) < 0$ both eigenvalues have negative real parts which means additive noise leads to convergence in the Dirac-GAN.

Additive Noise



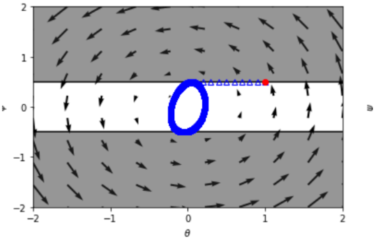
(f) Instance noise

Wasserstein GAN

$$\mathcal{L}(\theta, \psi) = \mathbb{E}_{P_Z}\left[D_\psi(G_\theta(Z))\right] - \mathbb{E}_{X \sim P_D}\left[D_\psi(X)\right]; \quad D_\psi \text{ is 1-Lipschitz}$$

- To make the divergence continuous with respect to the parameters of the generator, Wasserstein GANs (WGANs) replace the Jensen-Shannon divergence used in the original derivation of GANs with the Wasserstein-divergence
- alternating and simultaneous gradient descent for Wasserstein divergance does not converge in the Dirac-GAN problem.

A WGAN trained with simultaneous or alternating gradient descent with a fixed number of discriminator updates per generator update and a fixed learning rate $h > 0$ does generally not converge to the Nash equilibrium for the Dirac-GAN.



(c) WGAN ($n_d = 5$)

regularizing the target function for Wasserstein GAN can make it convergent(Wasserstein GAN with Quadratic Transport Cost [Liu, 2019]).

Regularizing Wasserstein GAN [Liu, 2019]

- consider the linear program:

$$\begin{cases} \max\limits_{H_i, H_j} & \frac{1}{m}\sum H_j - \frac{1}{m}\sum H_i \\ \text{subject to} & |H_i - H_j| \leq \frac{1}{2}\|x_i - y_j\|^2 \end{cases}$$

- consider the optimal transport problem:

$$\sigma(j) = \operatorname*{argmin}_i \ \frac{1}{2}\|x_i - y_j\|^2 + H_i^* - H_j^*$$

- define losses:

$$\min_\psi \frac{1}{2}\left(\frac{1}{m}\sum D_\psi(y_j) - \frac{1}{n}H_j^*\right)^2 + \frac{1}{2}\cdot\frac{1}{n}\sum\left(D_\psi(x_i) - H_i^*\right)^2$$
$$\underbrace{+\frac{\lambda}{2}\mathbb{E}_{X\sim G_\theta(Z)}\left[\left(\|\nabla_x D_\psi(X)\| - \|y_{\sigma^*(X)} - X\|\right)^2\right]}_{\text{gradient regulrization}}$$

$$\min_\theta \mathcal{L}(\theta) = -\frac{1}{n}\sum D_\psi(G_\theta(Z_i))$$

Regularizing Wasserstein GAN [Liu, 2019]

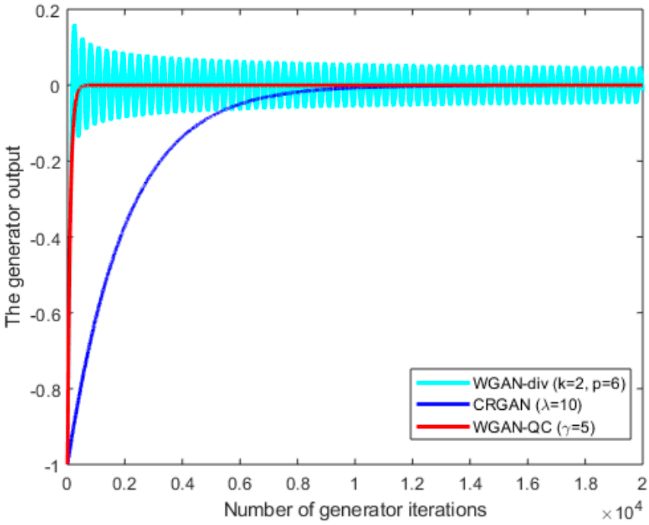
nframe it can be shown that :

$$-v^{'}(\theta^*, \psi^*) = \begin{bmatrix} M_{DD} + M_R & M_{GD} \\ 0 & 0 \end{bmatrix}$$

where :

$$\begin{aligned} M_R &= \lambda \cdot \mathbb{E}_{P_Z}\left[\nabla_{\theta,\psi} D_{\psi^*}(G_\theta(Z)) \nabla_{\theta,\psi} D_{\psi^*}(G_\theta(Z))^T\right] \\ M_{DD} &= \mathbb{E}_{y \sim P_D}\left[\nabla_\psi D_{\psi^*}(y)\right] \mathbb{E}_{y \sim P_D}\left[\nabla_\psi D_{\psi^*}(y)\right]^T \\ &\quad + \mathbb{E}_{y \sim P_D}\left[\nabla_\psi D_{\psi^*}(y) \nabla_\psi D_{\psi^*}(y)^T\right] \\ M_{GD} &= -\mathbb{E}_{Z \sim P_Z}\left[\nabla^2 D_{\psi^*}(G_\theta(Z)) \nabla_\theta G_{\theta^*}(Z)^T\right] \end{aligned}$$

Convergence WGAN-QC in Dirac-GAN



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