Applications of Information Bottleneck in DL

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Presentation Overview

Introduction

- Information Theory & Information Transfer
- Machine Learning & Prediction of Parameters
- Information Bottleneck and DNNs

Definitions

- Input Space : $\mathcal{X}$
- Output Space : $\mathcal{Y}$
- random variables $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ with joint distribution $p_{X,Y}$
- x, y represent examples of X, Y respectively

Definitions

Definition

1. Conditional Entropy : $H(X|Y) = \mathbb{E}_{X,Y}[-\log p(X,Y)]$
2. Mutual Information : $I(X;Y) = \mathbb{E}_{X,Y}\left[log \frac{p(X,Y)}{p(X)p(Y)}\right]$

a property of MI:
for every bijective functions f and g: $I(X;Y) = I(f(X), g(Y))$

Relevant Information

a shared concept in Information Theory, Statistics and Machine Learning
a mathmatical formulation of "Relevant Information" is Sufficient Statistics.

Relevant Information

Definition (Sufficient Statistic)

given $S \triangleq f(X)$ for sum function f, S is a sufficient statistic of Y if :

$$\forall x \in \mathcal{X}, y \in \mathcal{Y} : \mathbb{P}\left[X = x \middle| Y = y, S = s\right] = \mathbb{P}\left[X = x \middle| S = s\right]$$

Theorem

S is a sufficient statistic for Y iff:

$$I(S;Y) = I(X;Y)$$

Relevant Information

Definition (Minimal Sufficient Statistic)

S is a MSS if:

$$\forall T;\ T \text{ is sufficient statistic} \Rightarrow \exists g;\ S = g(T)$$

Theorem

given $X \sim P_{Y|X}$, S *is an MSS for* Y *iff:*

$$S \in \underset{T:SS}{argmin}\ I(X;T)$$

subsequently :

Corollary

given $X \sim P_{Y|X}$, S *is an MSS for* Y *iff:*

$$S \in \underset{T:I(Y;T)=I(X;Y)}{argmin}\ I(X;T)$$

Relevant Information

	Markov Chain	Data Processing Inequality
Statistic	Y — X — S	$I(S;Y) \leq I(X;Y)$
Sufficient	Y — SS — X	$I(SS;Y) \geq I(X;Y)$
Minimal	X — SS — MSS	$\forall\, SS : I(MSS;X) \leq I(SS;X)$

Figure: intuition of MSS

The Information Bottleneck framework

let $P_{T|X}$ be a transition kernel from $\mathcal{X}$ to $\mathcal{T}$.

The kernel $P_{T|X}$ can be viewed as transforming $X \sim P_X$ into a representation of $T \sim P_T(\cdot) \triangleq \int P_{T|X}(\cdot|x)dP_X(x)$ in the $\mathcal{T}$ space.

We seek for $P_{T|X}$ s.t extracts information about Y , i.e., high $I(Y;T)$, while maximally compressing X, which is quantified as keeping $I(X;T)$ small.

the IB problem is formulated through the constrained optimization :

$$\inf_{P_{T|X}:I(Y;T)\geq\alpha} I(X;T)$$

α is a parameter which capture the trade-off between $I(X;T)$ and $I(Y;T)$.

Lagrange Dual Form

$\mathcal{L}_\beta(P_{T|X}) \triangleq I(X;T) - \beta I(Y;T)$

minimize $\mathcal{L}_\beta(P_{T|X})$ over all possible $P_{T|X}$ kernels.

Varying $\beta \in [0, \infty)$ regulates the tradeoff between informativeness and compression.

as $\beta \to \infty$, this problem is equivalent to the MSS optimization problem

β and its relation to bias-variance trade-off

Self-Consistent Equations

The Optimal assignment that minimizes $\mathcal{L}_\beta(P_{T|X})$, satisfies the equation:

$$p(t|x) = \frac{p(t)}{Z(x,\beta)} \cdot exp\left(-\beta \sum_y p(y|x) \log \frac{p(y|x)}{p(y|t)}\right)$$

where $Z(\cdot,\cdot)$is the normalization factor; due to the Markov chain condition $Y \to X \to T$,

$$p(y|t) = \frac{1}{p(t)} \sum_x p(y|x) \cdot p(t|x) \cdot p(x)$$

The IB Iterative Algorithm

every iteration (i) consists of three steps:

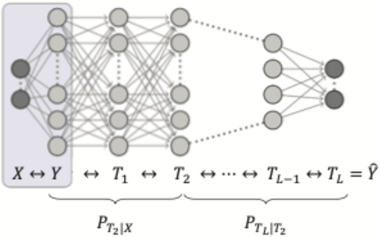
$$p_i(t|x) \leftarrow \frac{p_i(t)}{Z_i(x,\beta)} \cdot exp\left(\beta d(x,t)\right)$$

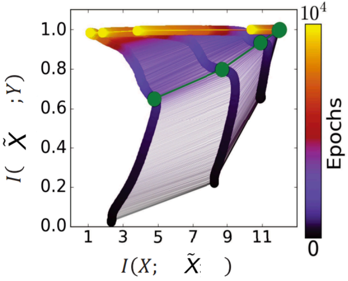
$$p_{i+1}(t) \leftarrow \sum_x p(x) p_i(t|x)$$

$$p_{i+1}(y|t) \leftarrow \sum_y p(y|x) p_i(x|t)$$

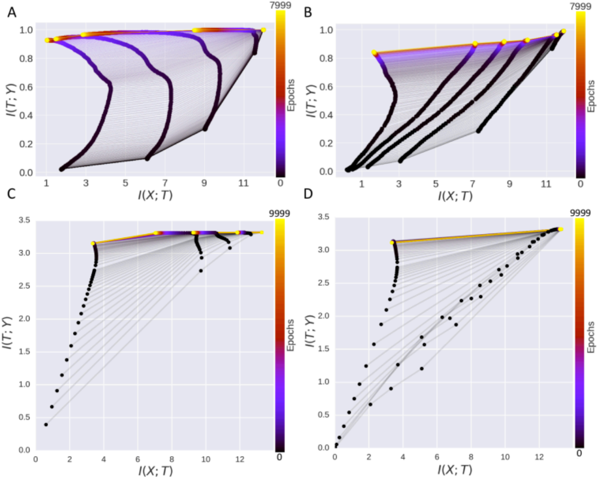
The IB method and Deep Neural Networks

Information Plane

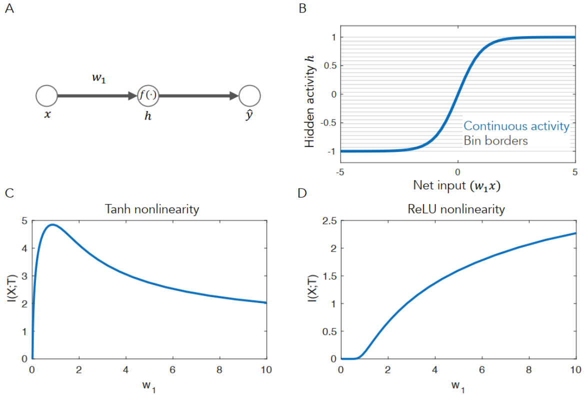




Information Plane with different Activation functions



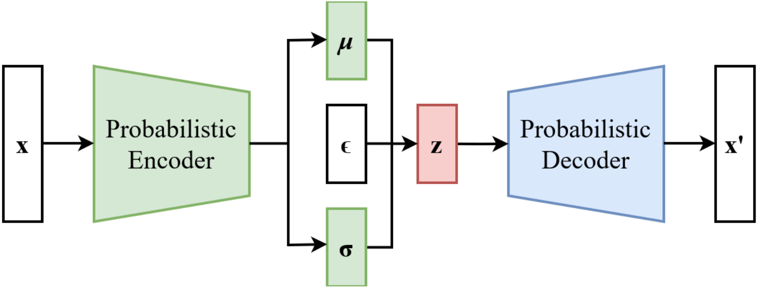
Information Plane with different Activation functions



Information Bottleneck and SGD

- continuous outputs causes infinite values of MI
- the binning method and the Data Processing Principle
- applying binning to the actual activation values
- make the NN stochastic (Amjed et al. 2018, section 5.4)

Variational Auto-Encoders



$Z \sim p_0(Z)$
objective : $\tilde{X} \sim p_\phi(\tilde{x}|x)p(X)$ close to $p_0(z)$ and $X' \sim p_\theta(x'|z)p_0(z)$ close to X

Variational Auto-Encoder

- an alternative representation :

$$\min_{p(\tilde{x}|x)} I(\tilde{X};X) + \beta H(Y|\tilde{X})$$

- approximating the loss function:

$$\mathcal{L}_{approx.} \triangleq \frac{1}{N}\sum_{i=1}^{N} \underbrace{\mathbb{E}_{p(\tilde{x}|x^{(i)})}\left[-\log \mathbb{P}\left[y^{(i)}|\tilde{x}\right]\right]}_{\text{Cross Entropy Loss}} + \beta \underbrace{D_{\mathrm{KL}}(P(\tilde{x}|x^{(i)})||P(\tilde{x}))}_{\textit{regularization}}$$

- a loss function for VAE:

$$\mathcal{L}(\phi,\theta) \triangleq \frac{1}{N}\sum_{i=1}^{N} \underbrace{\mathbb{E}_{p(\tilde{x}|x^{(i)})}\left[-\log \mathbb{P}\left[x^{(i)}|\tilde{x}\right]\right]}_{\text{marginal likelihood of data}} + \underbrace{D_{\mathrm{KL}}(P(\tilde{x}|x^{(i)})||P_0(\tilde{x}))}_{\text{distance to the prior} P_0}$$

differences of VAE and the IB method

- $\beta = 1$
- separate optimization

Beyond the IB method

- $I(X;T)$ vs $I(S;\mathcal{A}(S))$
- a bound for generalization error :
  $\mathbb{P}\left[|err_{test} - err_{train}| > \epsilon\right] < O\left(\frac{I(S;\mathcal{A}(S))}{n\epsilon^2}\right)$

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