بسم الله الرحمن الرحیم

فصل دوم - تجزیه و تحلیل سیستم های LTI در حوزه زمان

کانولوشن ها:
- گسسته : جمع کانولوشن
- پیوسته : انتگرال کانولوشن

جمع کانولوشن با سیستم های LTI گسسته:

مقدمه اول -

$$x[n]\delta[n] = x[0]\delta[n]$$

$$x[n]\delta[n-1] = x[1]\delta[n-1]$$

$$\vdots \qquad\qquad \vdots$$

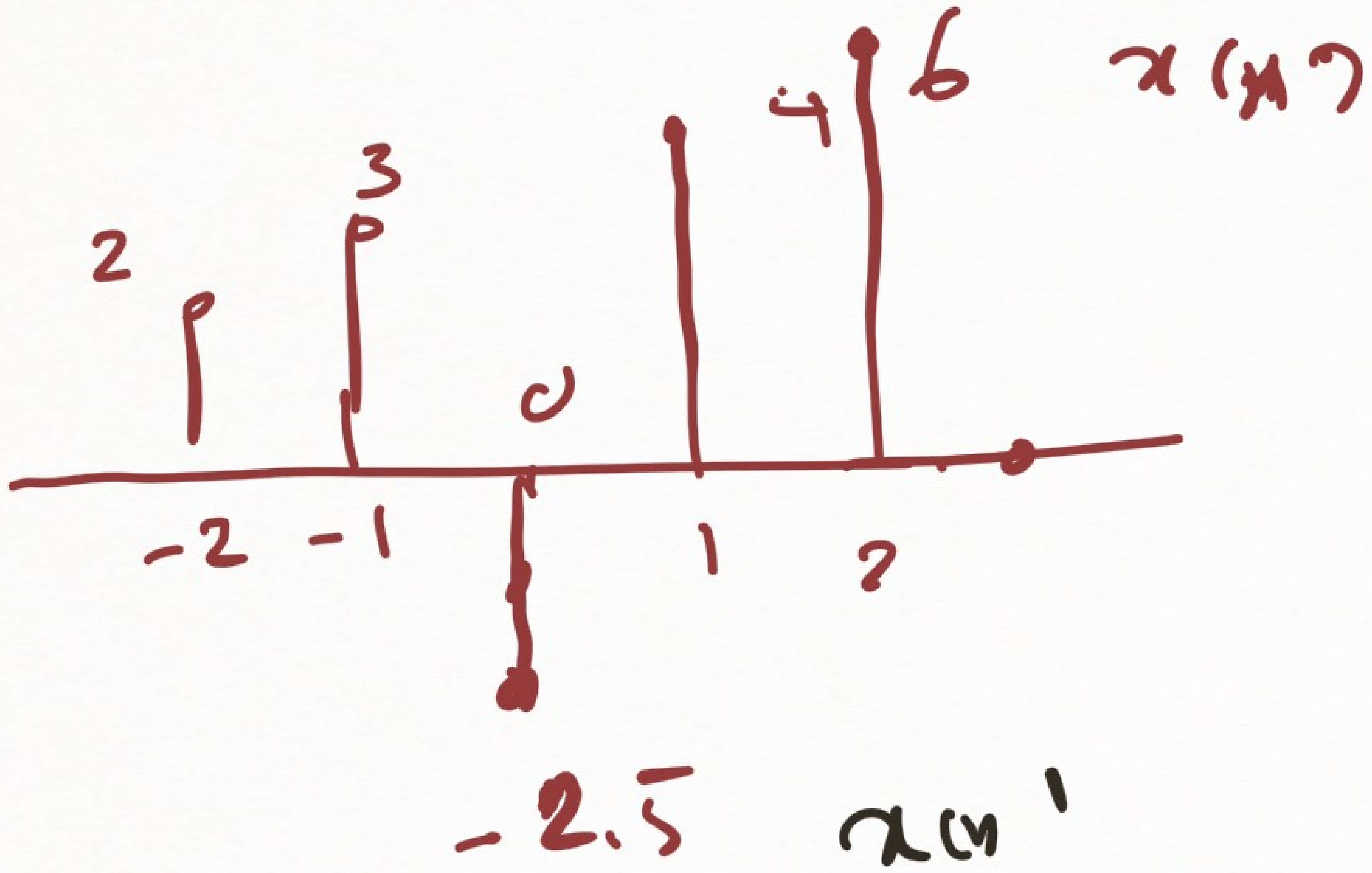
$$\sum_{k=-\infty}^{+\infty} x[n]\delta[n-k] = \sum_{-\infty}^{+\infty} x[k]\delta[n-k]$$

$$x[n]\underbrace{\cancel{\sum_{k=-\infty}^{+\infty}\delta[n-k]}}_{1} = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

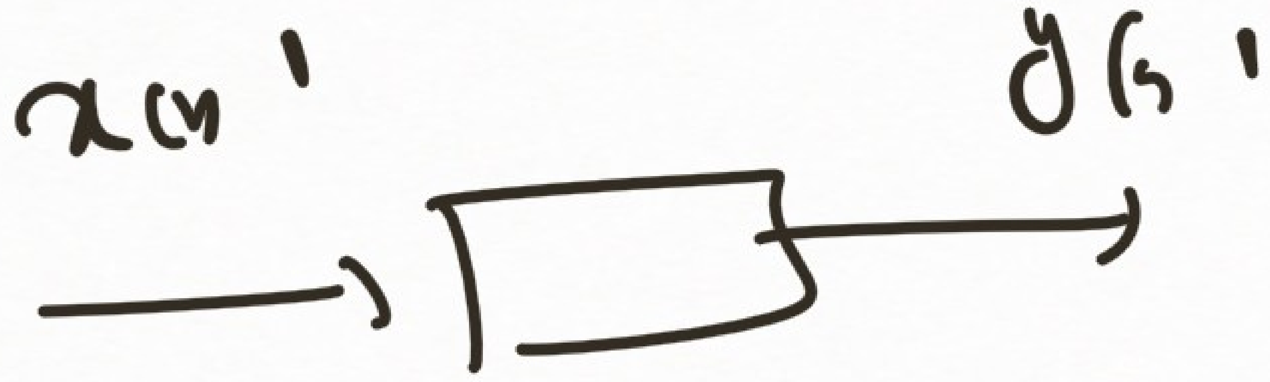
$$x[n] = \cdots x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

هر سیگنال گسسته را میتوان توسط مجموع ضربه ها نشان داد.



$$\Longrightarrow x[n] = 2\delta[n+2] + 3\delta[n+1] - 2.5\,\delta[n] + 4\,\delta[n-1] + 6\,\delta[n-2]$$

تمرین - کتاب سیم فقط صفحی دوشنبه ...



$$y[n] = \mathcal{L}\{x[n]\}$$

$$\delta[n] \longrightarrow h_0[n] = \mathcal{L}\{\delta[n]\}$$

$$\delta[n-1] \longrightarrow h_1[n] = \mathcal{L}\{\delta[n-1]\}$$

$$\delta[n-2] \longrightarrow h_2[n] = \mathcal{L}\{\delta[n-2]\}$$

$$\vdots \qquad\qquad \vdots$$

$$\delta[n-k] \longrightarrow h_k[n] = \mathcal{L}\{\delta[n-k]\}$$

$$y[n] = \mathcal{L}\left\{\sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]\right\}$$

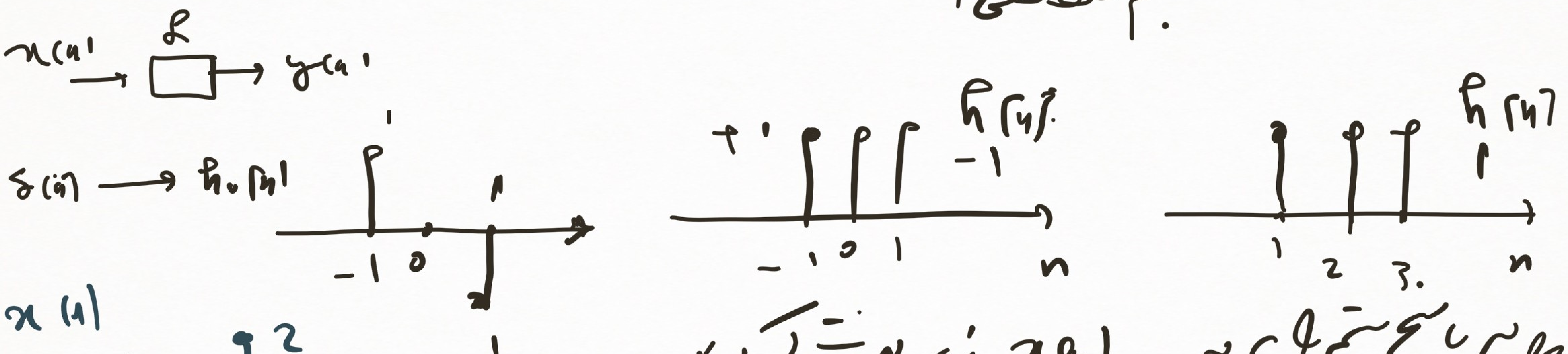
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\underbrace{\mathcal{L}\{\delta[n-k]\}}_{h_k[n]}$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h_k[n]$$

$$y[n] = \cdots x[-2]\,h_{-2}[n] + x[-1]\,h_{-1}[n] + x[0]\,h_0[n] + x[1]\,h_1[n] + x[2]\,h_2[n] + \cdots$$

سیستم فقط خطی است.

مثال -



می‌خواهیم پاسخ سیستم را به سیگنال x[n] زیر بدست آوریم.

$y[n] = ?$

این نمودار رو رسم کنیم.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h_k[n] = \sum_{k=-1}^{1} x[k]\, h_k[n]$$

$$y[n] = x[-1]\, h_{-1}[n] + x[0]\, h_0[n] + x[1]\, h_1[n] = -1\, h_{-1}[n] + h_0[n] + 2\, h_1[n]$$

-1 $\quad$ 1 $\quad$ 2

دارم $\quad$ دارم $\quad$ ندارم

جمع کانولوشن

معادل ورودی خروجی یک سیستم LTI را

$$x(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n)$$

$$\delta[n] \longrightarrow h_0(n) = h[n]$$

$$\delta(n-1) \longrightarrow h_1(n) = h(n-1)$$

$$\delta(n-2) \longrightarrow h_2(n) = h(n-2)$$

$$\vdots$$

$$\delta(n-k) \longrightarrow h_k(n) = h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{+\infty} x(n-k)\, h(k)$$

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, \cancel{h_k(n)} \longrightarrow h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)\, h(\underbrace{n-k}_{n-k=k})$$

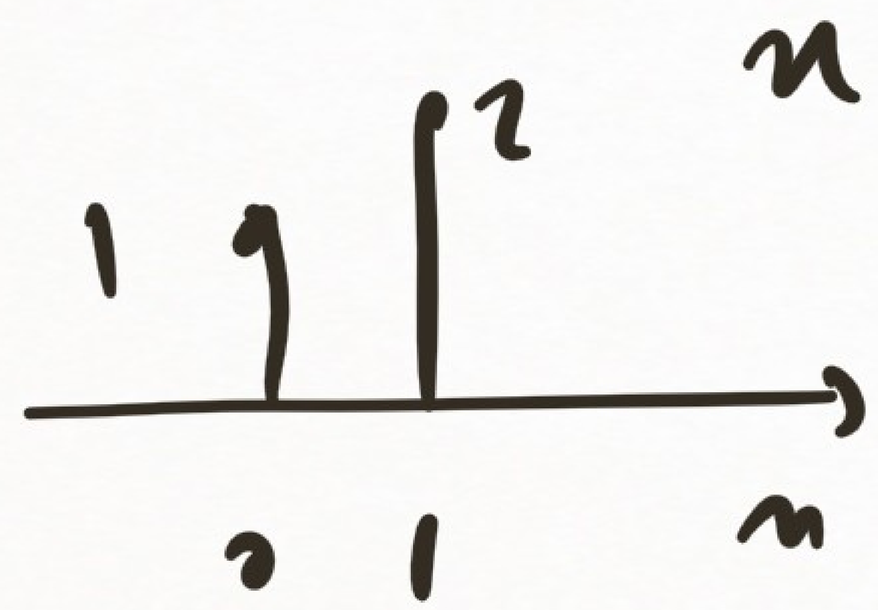
$$y(n) = x(n) * h(n)$$

LTI -سی

$h[n]$

پاسخ چیه؟

پاسخ ضربه را داریم:



$x[n]$

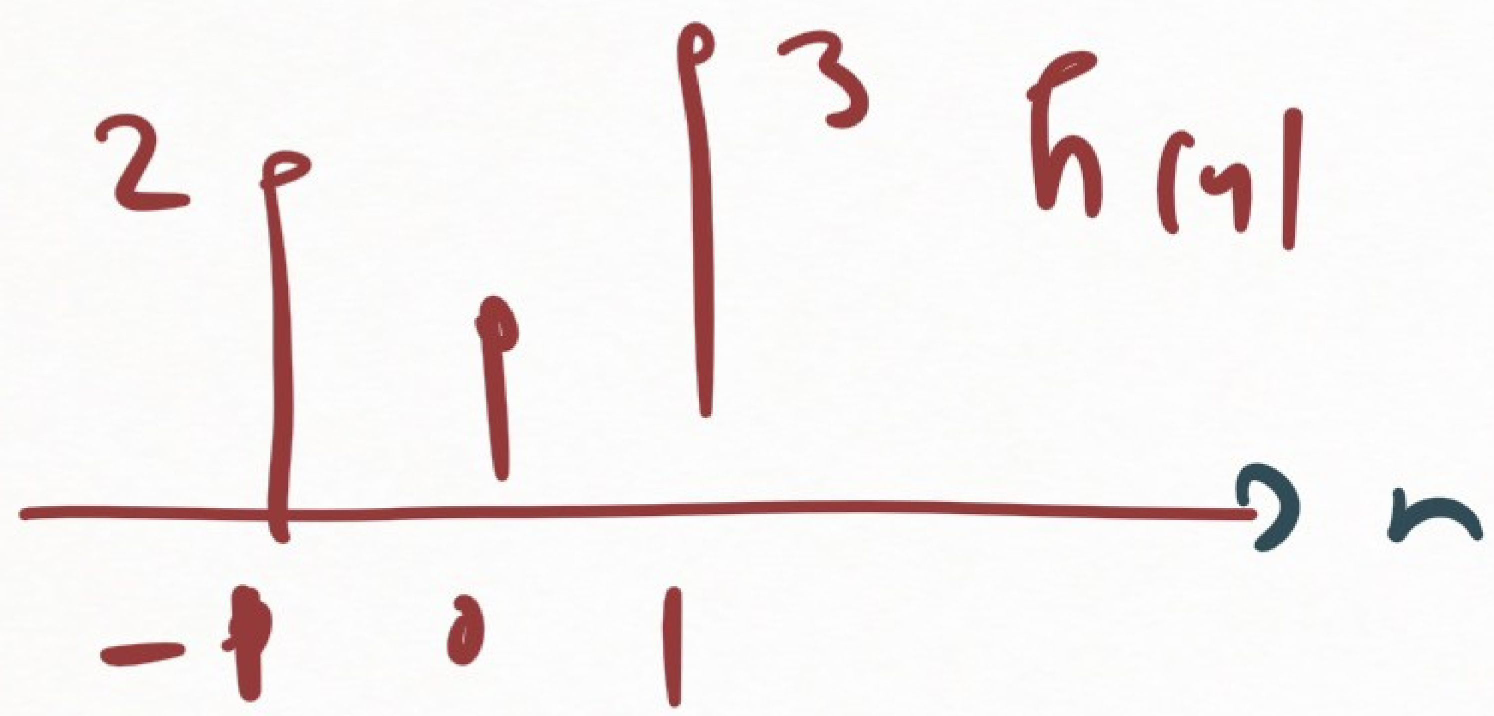
$\longrightarrow \quad y[n] = ?$

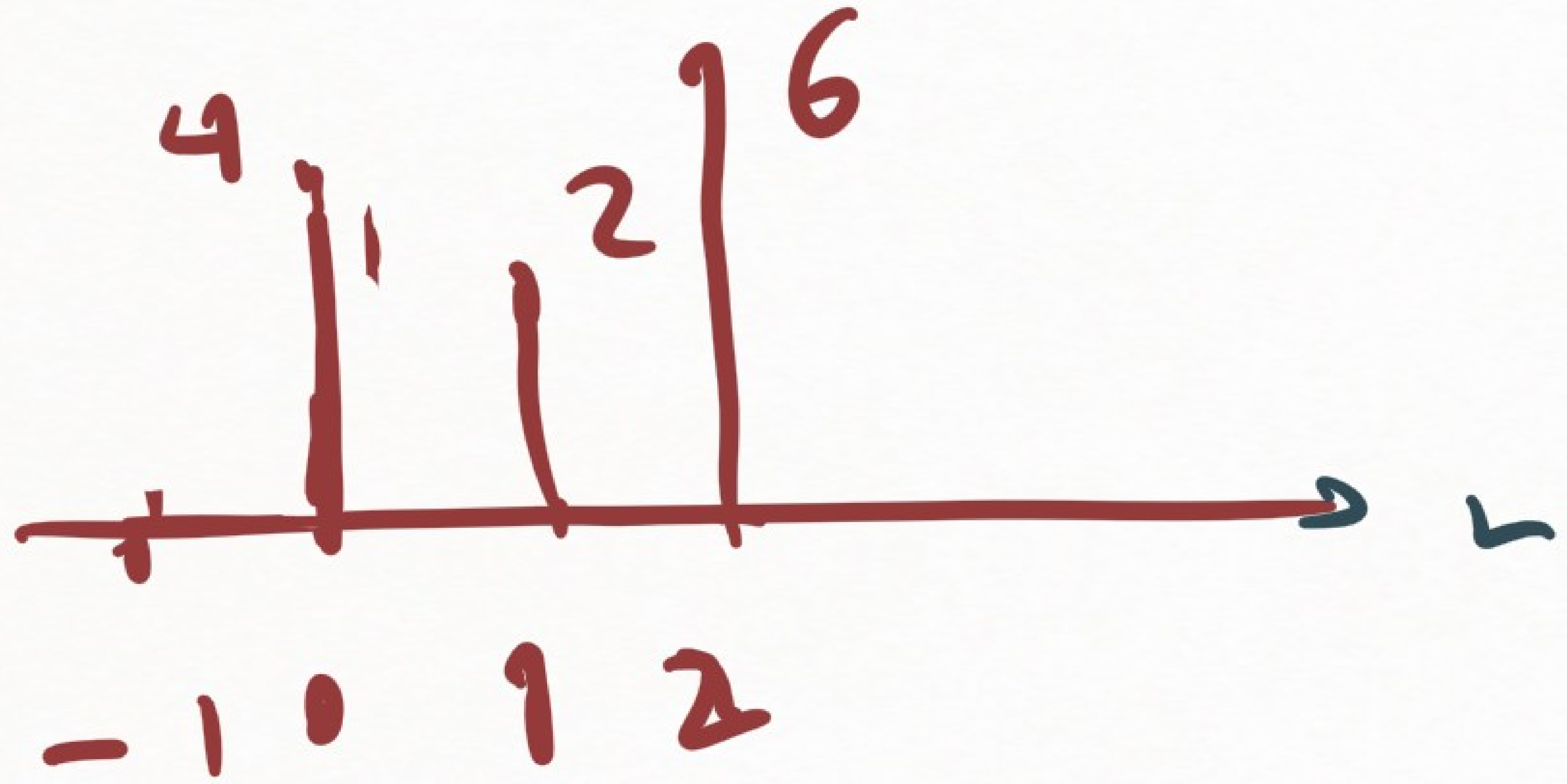
$$y[n] = x[n] * h[n] = \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]}_{1} = \overbrace{\sum_{k=-\infty}^{+\infty} x[n-k]\,h[k]}^{2}$$

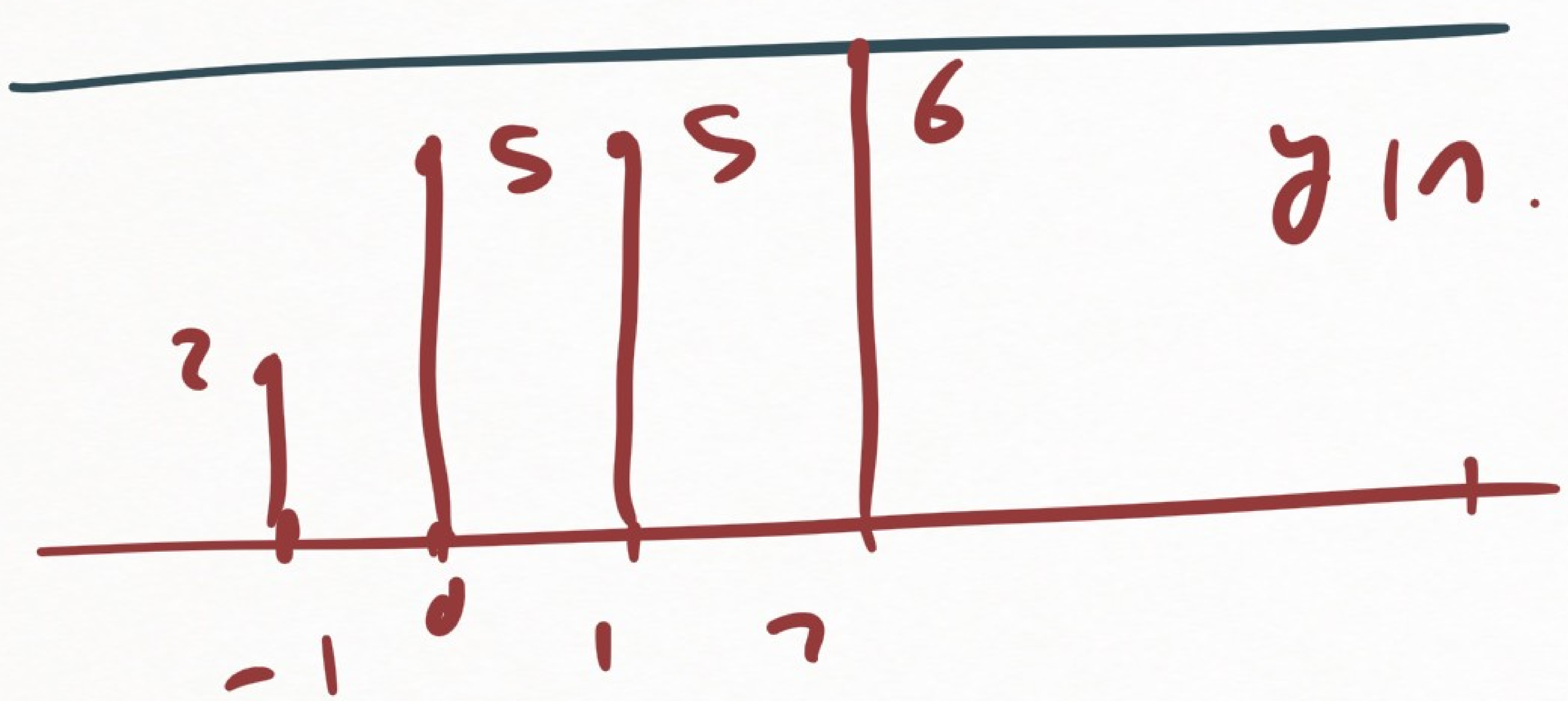
ما باید جمع کانولوشن را بهینه کنیم

$$1 \Rightarrow \quad y[n] = \sum_{k=0}^{1} x[k]\,h[n-k] = x[0]\,h[n] + x[1]\,h[n-1] = \underline{h[n] + 2\,h[n-1]}$$

$$y[n] = h[n] + 2h[n-1]$$







تمرین:

$$② \Longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[n-k]\, h[k],$$

$$y[n] = \sum_{k=-1}^{1} h[k]\, x[n-k]$$

$$y[n] = h[-1]\, x[n+1] + h[0]\, x[n] + h[1]\, x[n-1]$$

$$y[n] = 2\, x[n+1] + x[n] + 3x[n-1]$$

$$y[n]=\overbrace{\sum_{-\infty}^{} x[k]\,h[n-k]}^{1}=\overbrace{\sum_{k=-\infty}^{+\infty} x[n-k]\,h[k]}^{2}$$

روش گرافیکی و یا جاروبی کانولوشن:

1) $x[k]$ و $h[k]$

2) $\underbrace{h[-k]}_{1}$ یا $\underbrace{x[-k]}_{2}$

3) $h[n-k]$ یا $x[n-k]$

4)
$$\left.\begin{array}{l} \sum x[k]\,h[n-k] \\ \sum h[k]\,x[n-k] \end{array}\right\}$$

هر کدام ساده‌تر

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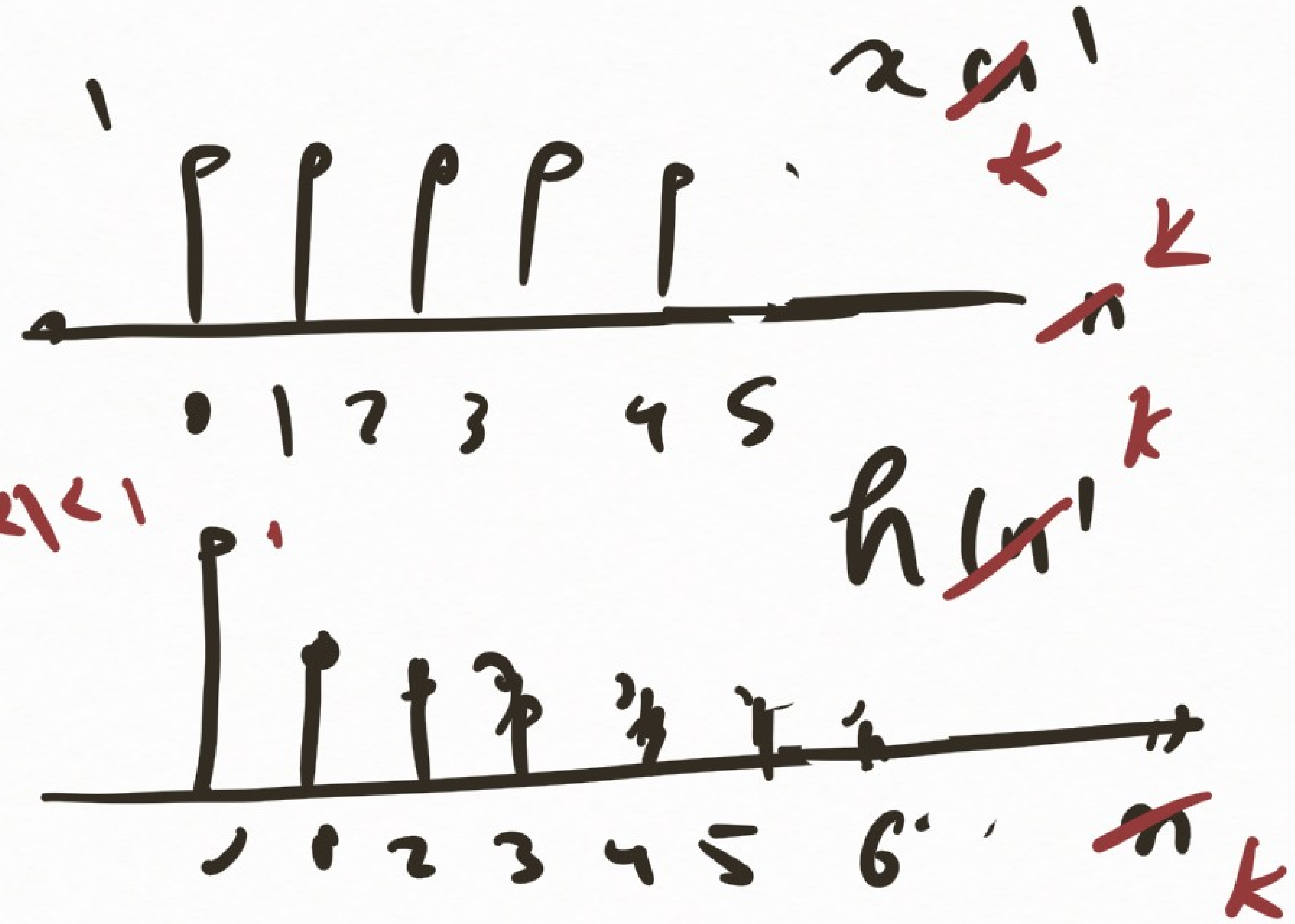
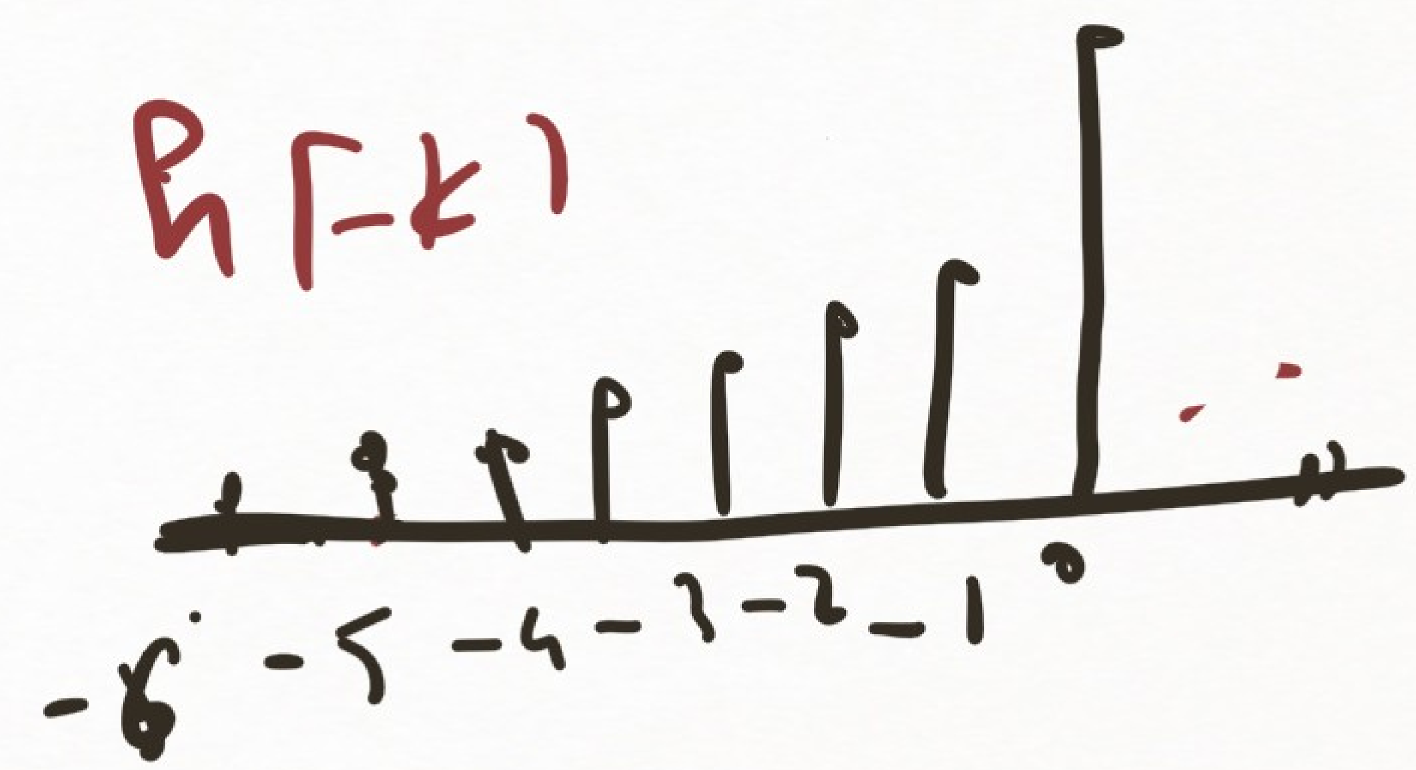
$$x(n)=\begin{cases} 1 ; & 0\le n\le 4 \\ 0 ; & oth \end{cases}$$

$$h(n)=\begin{cases} \alpha^{n} ; & 0\le n\le 6 \\ 0 ; & oth \end{cases}$$

$|\alpha|<1$

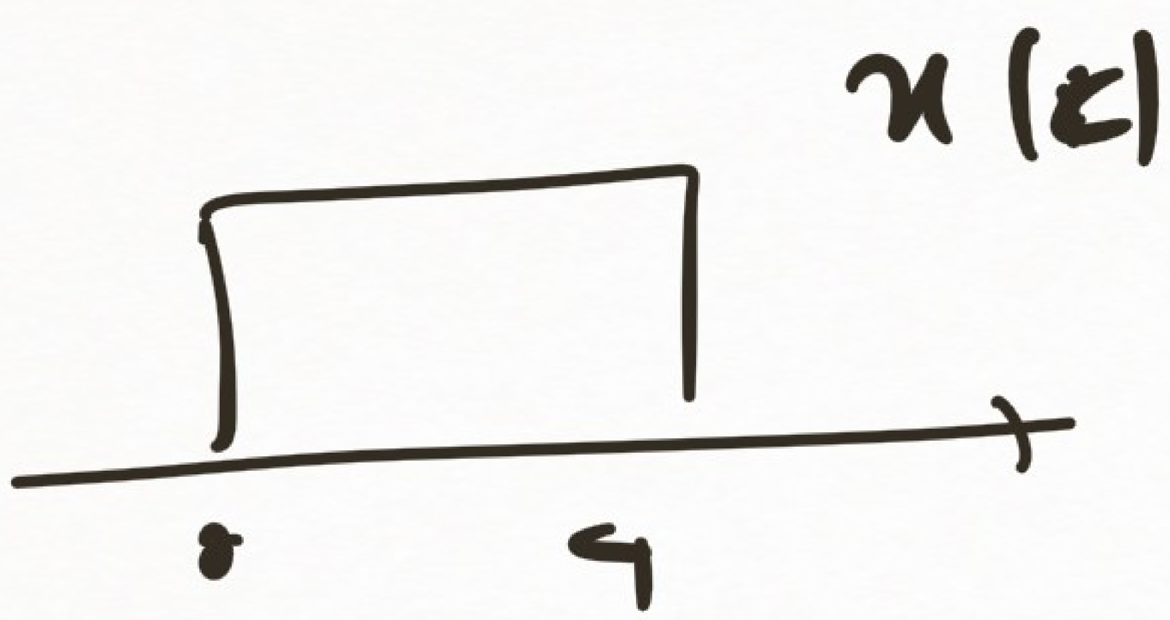


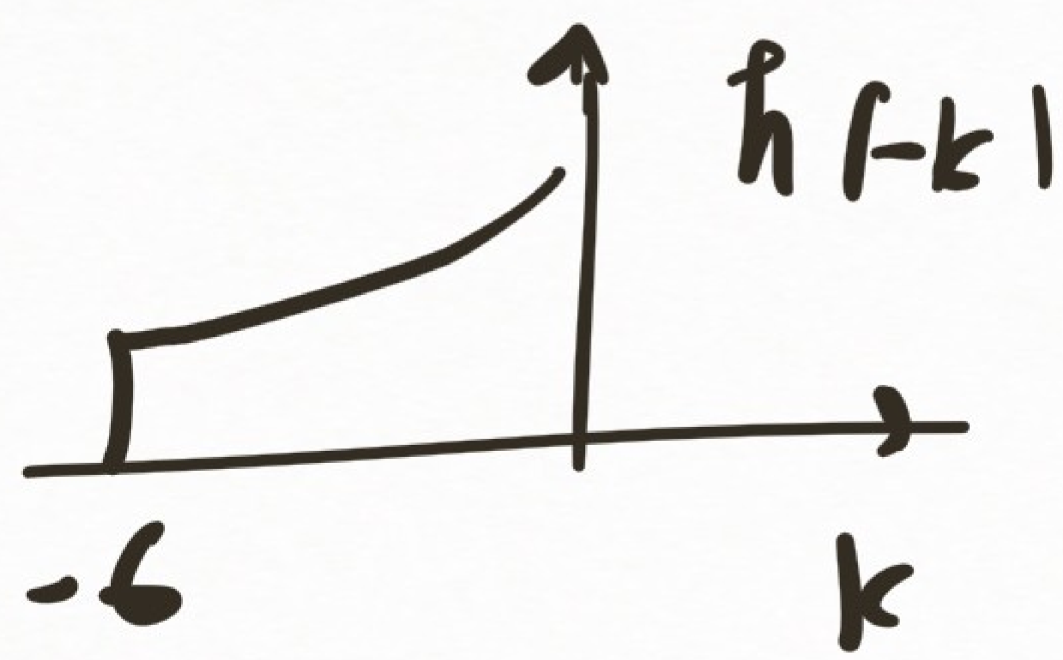
$$y(n)=x(n)*h(n)=?$$

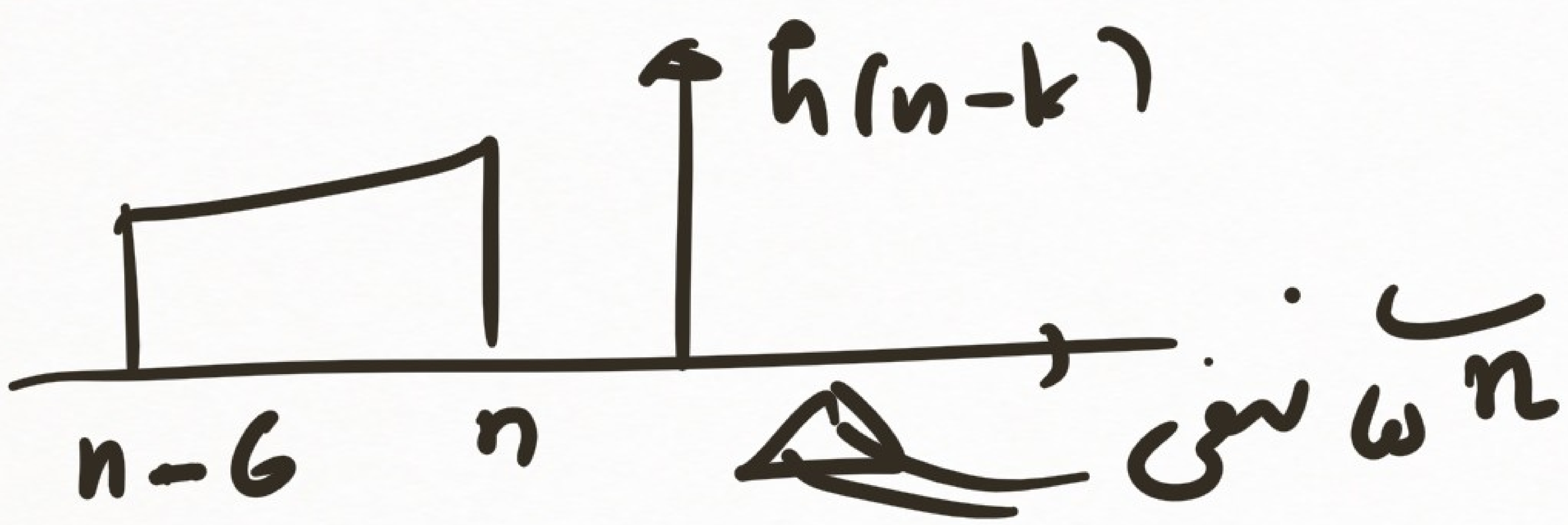


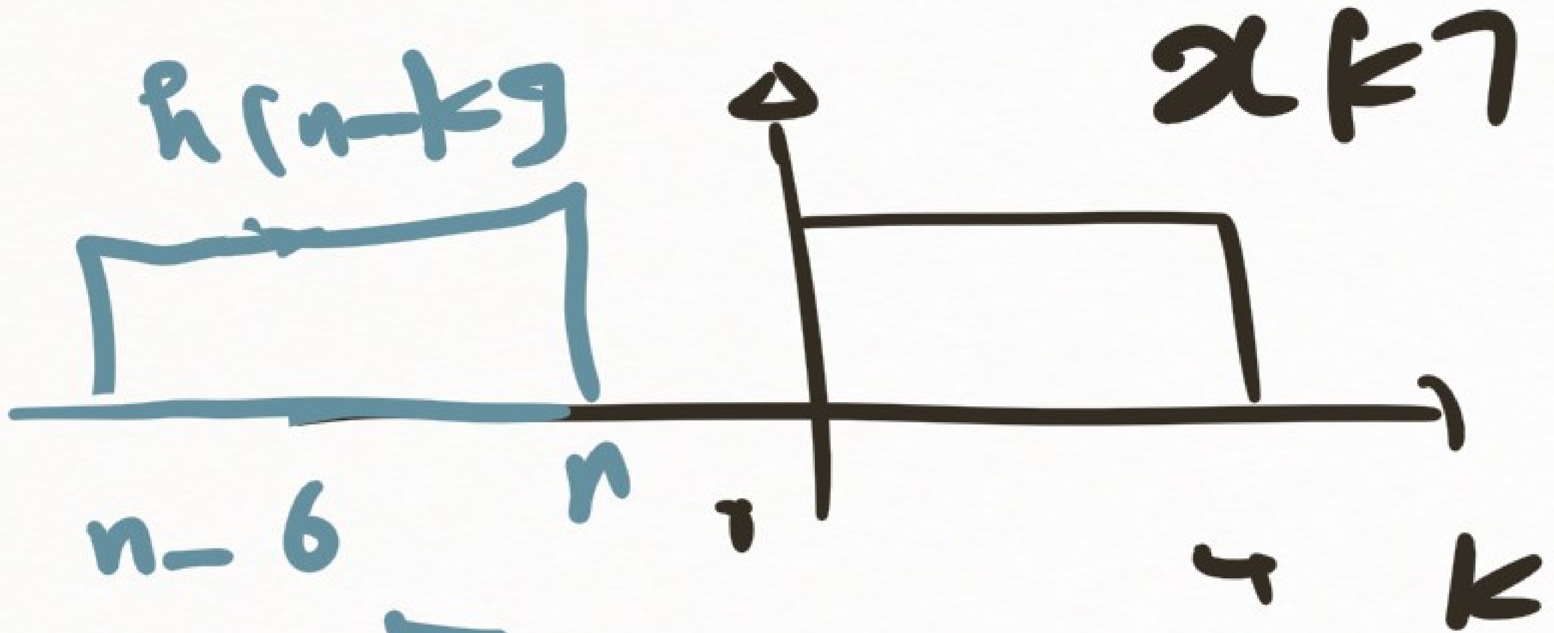
شیفت به راست 2 ← $h[2-k]$

شیفت به چپ 2 ← $h[-2-k]$

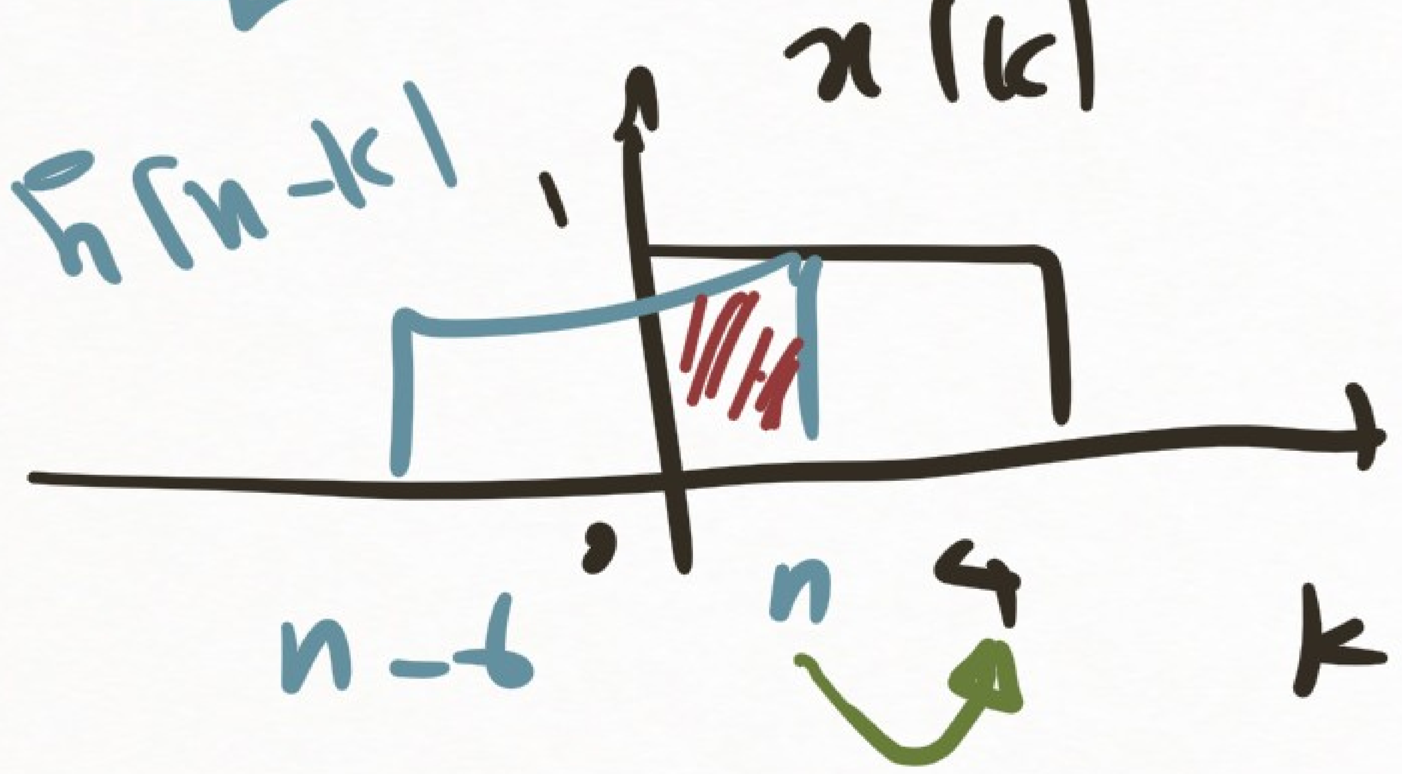




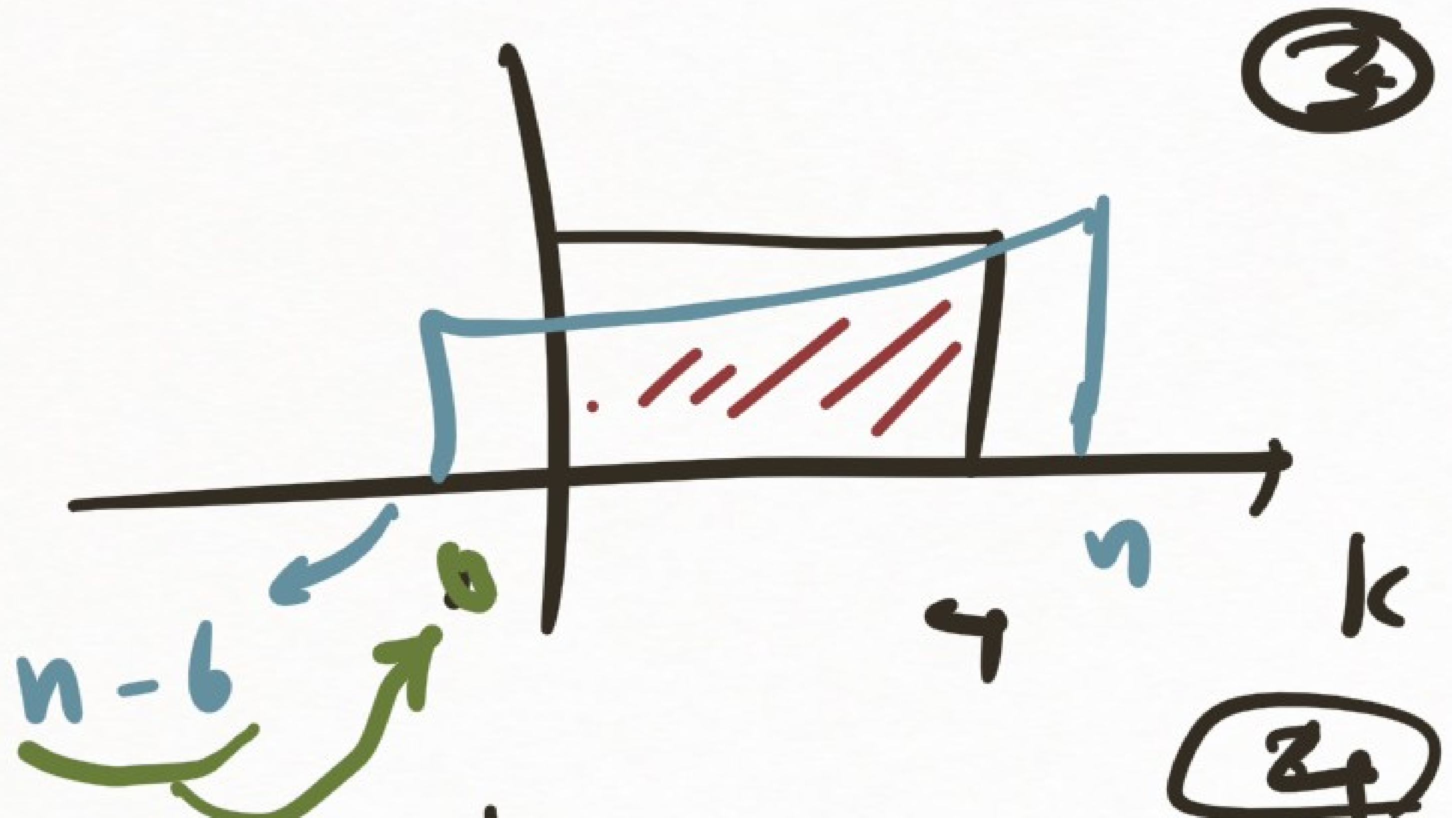




1. $n < 0 \implies y[n] = 0$

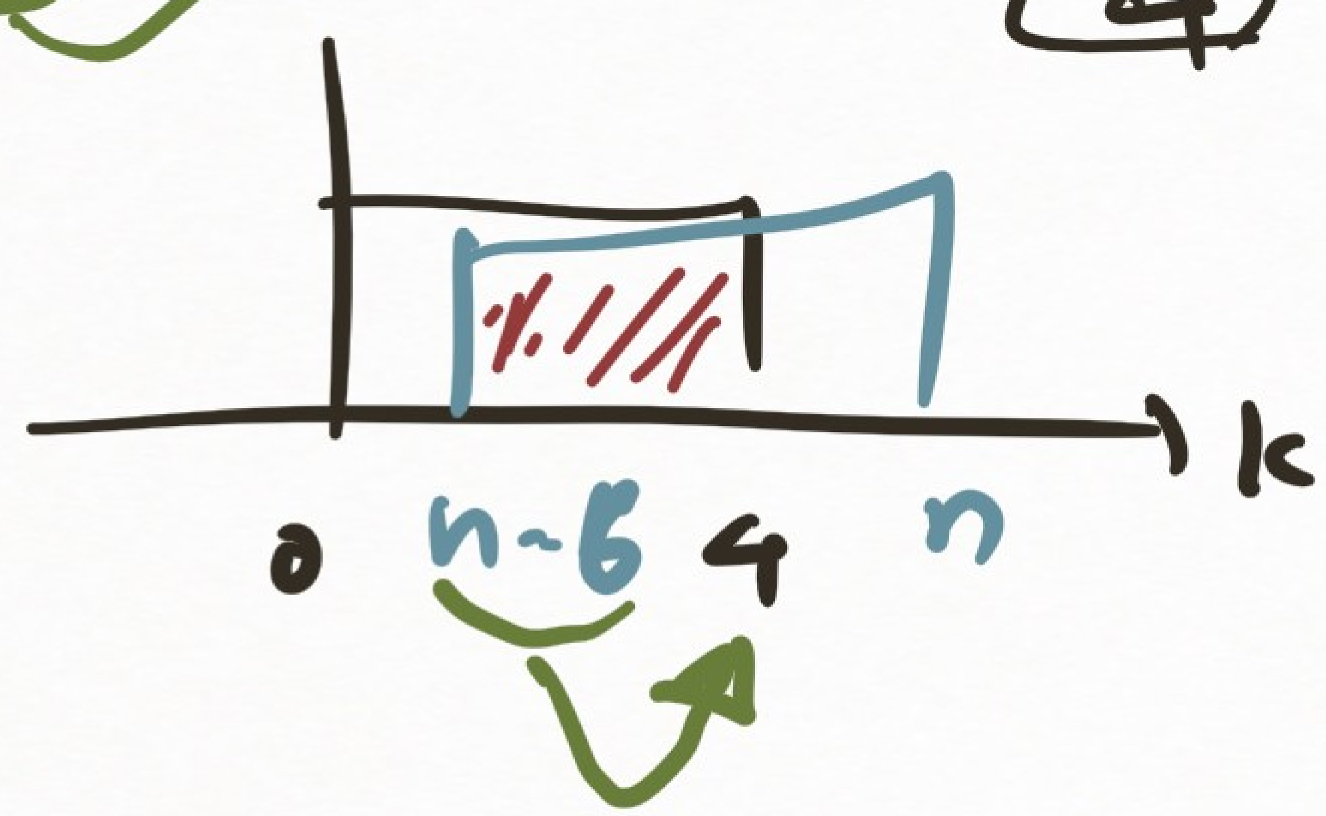


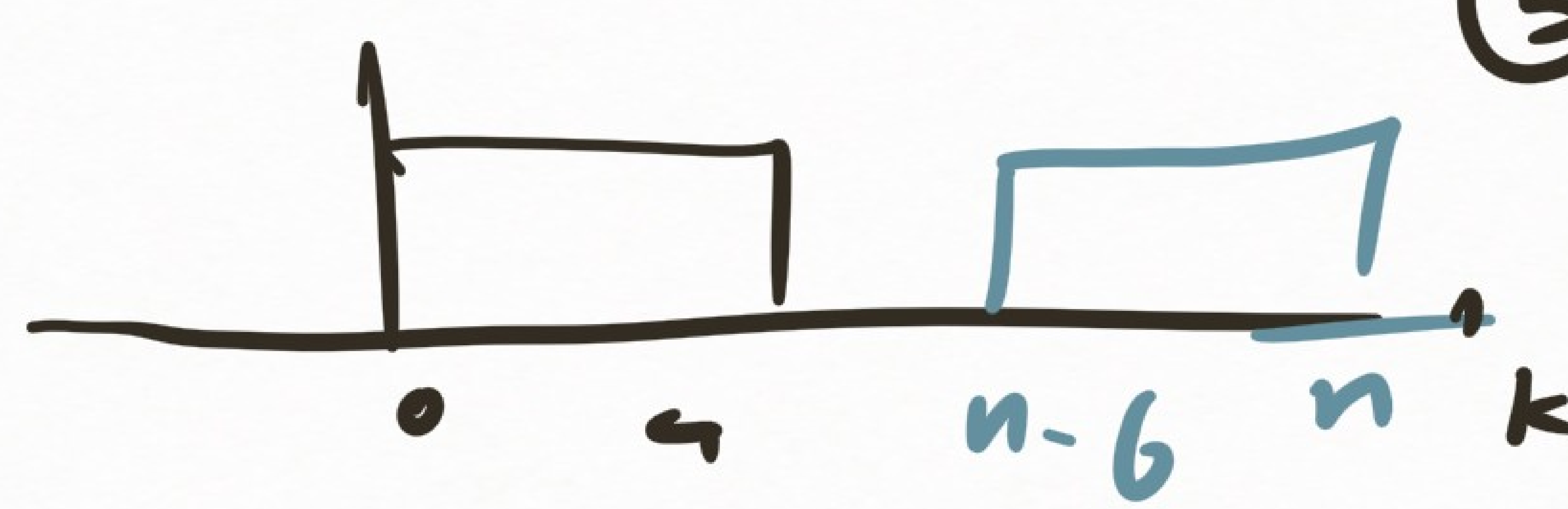
2. $0 \leq n < 4 \implies y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h(n-k) = \sum_{k=0}^{n} 1 \times \alpha^{n-k} = \dots$



3. $4 \leq n < 6 \implies y[n] = \sum_{k=0}^{4} 1 \times \alpha^{n-k} = \sum_{k=0}^{4} \alpha^{n-k} = \dots$

4. $6 \leq n \leq 10 \implies y[n] = \sum_{n-6}^{4} 1 \times \alpha^{n-k} = \sum_{k=n-6}^{4} \alpha^{n-k} = \dots$





5. $n > 10$

$\implies y[n] = 0$

چند فرمول مهم برای Σ:

a/ $$\sum_{n=0}^{N-1} \alpha^{n} = \begin{cases} N, & \alpha = 1 \\ \dfrac{1-\alpha^{N}}{1-\alpha}, & \alpha \neq 1 \end{cases}$$

b/ $$\sum_{n=N_1}^{N_2} \alpha^{n} = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

c/ $$\sum_{n=0}^{\infty} \alpha^{n} = \frac{1}{1-\alpha}, \quad |\alpha| < 1$$

d/ $$\sum_{n=0}^{\infty} n\alpha^{n} = \frac{\alpha}{(1-\alpha)^2}, \quad |\alpha| < 1$$

راه مستقیم: ما برای Σ ها از [illegible] استفاده کنیم.

② $y(n) = \sum_{k=0}^{n} \alpha^{n-k} = \sum_{r=n}^{0} \alpha^{r} = \sum_{r=0}^{n} \alpha^{r} = \boxed{\frac{1-\alpha^{n+1}}{1-\alpha}}$

$n-k=r$

③ $\Rightarrow y(n) = \sum_{k=0}^{4} \alpha^{n-k} = \cdots = \boxed{\frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}}$

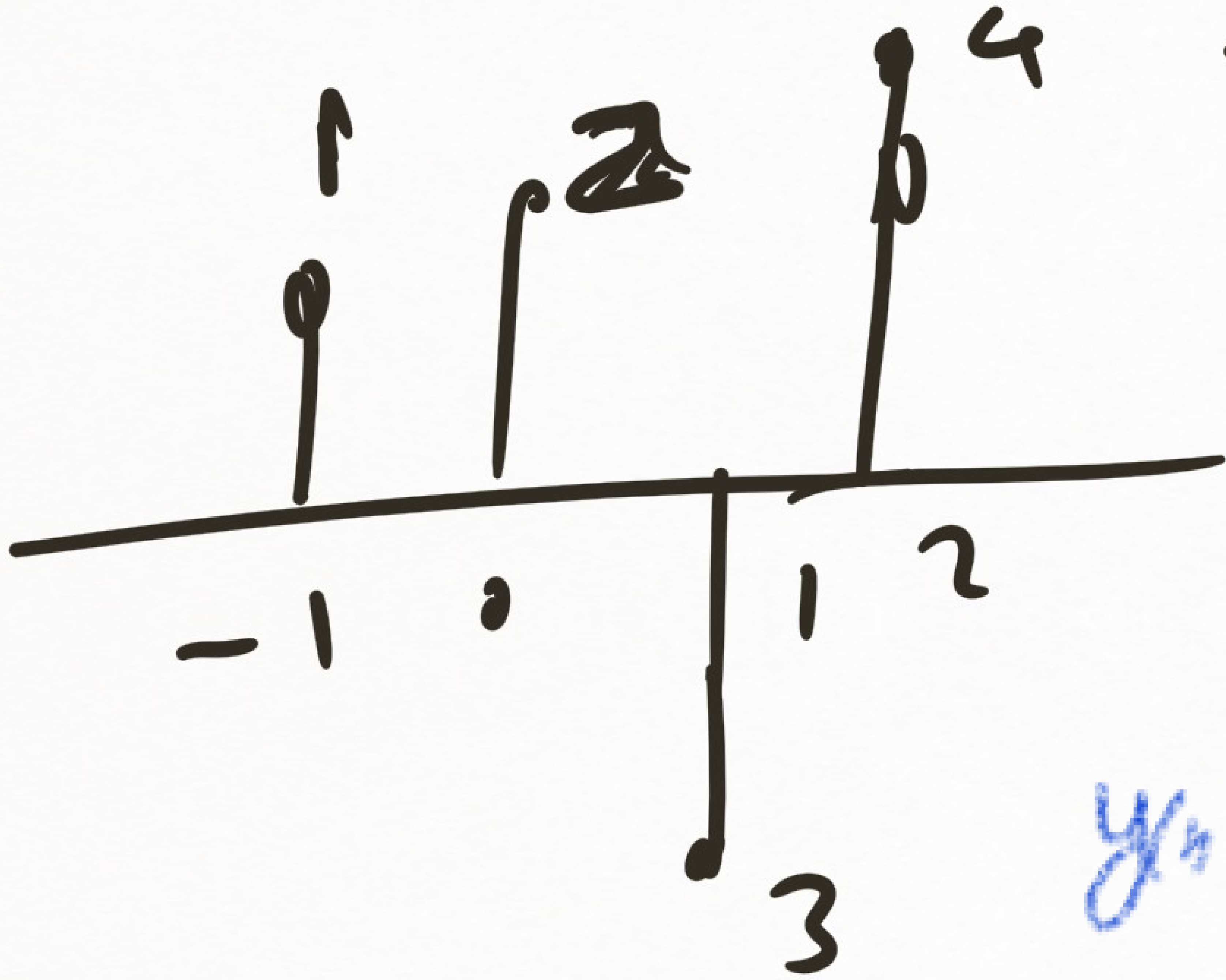
④ $\Rightarrow y(n) = \sum_{k=n-6}^{4} \alpha^{n-k} = \cdots = \boxed{\frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}}$

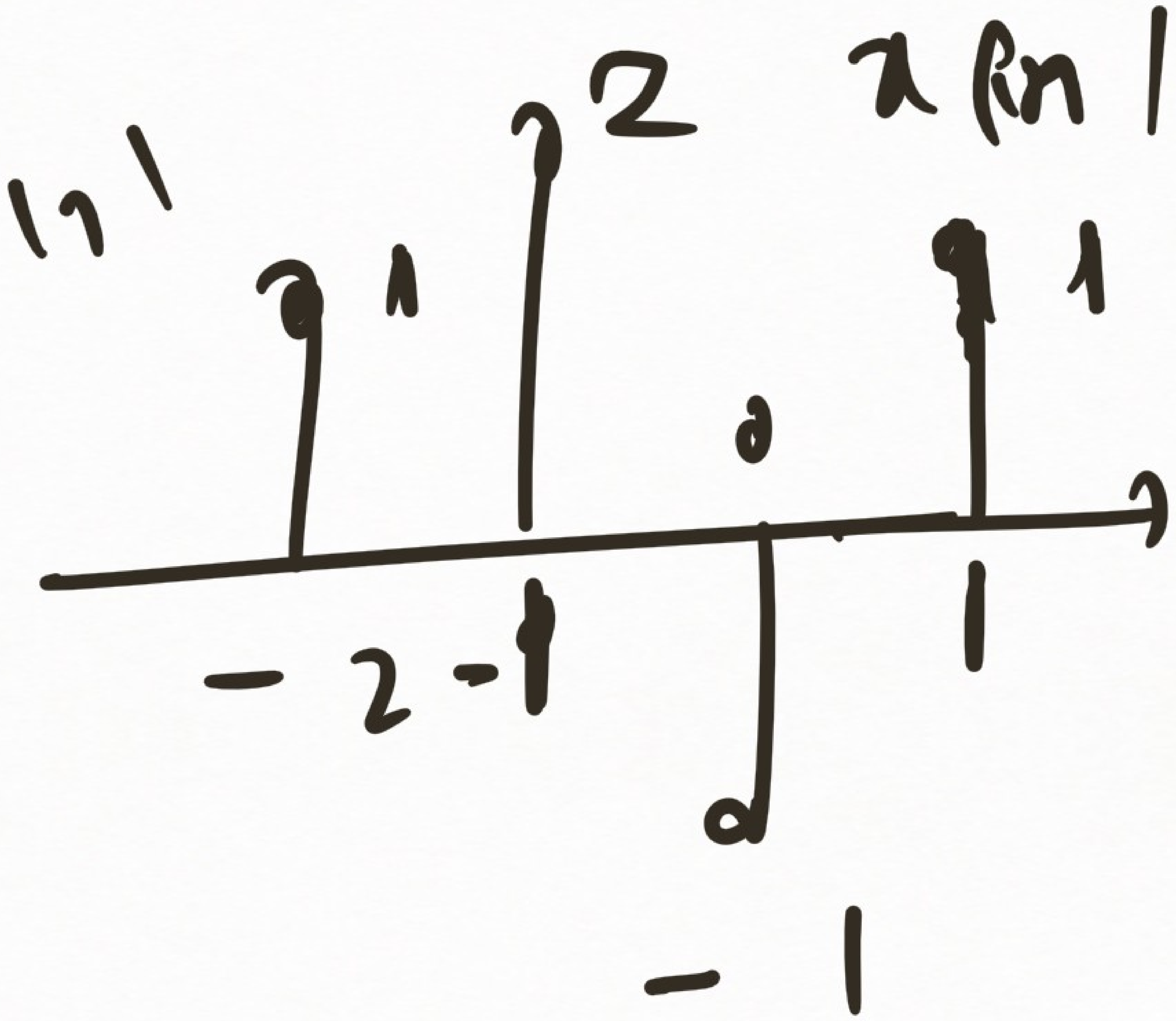
⑤ $\Rightarrow y(n) = 0$


y(n)
0 1 2 3 4 5 6 7 8 9 10
4+6
0+0
x(n)
h(n).
N1 N2
N3 N4
y(n)
N1+N3
N2+N4

تمرین:

$$\begin{cases} x[n] = u[n] \\ h[n] = \alpha^n u[n] \end{cases} \longrightarrow y[n] = ?$$





$\longrightarrow y[n]$

$x[n] * \delta[n] = x[n]$

$y[n] = x[n] * \underbrace{\delta[n-n_0]}_{h[k]} = x[n-n_0]$

تعمیم

$k = n - n_0$

تغییر متغیر

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{+\infty} x[k]\, \delta[n-k-n_0]$$

$$= x[n-n_0] \cancel{\sum_{k=-\infty}^{+\infty} \delta[n-k-n_0]}$$