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#1. if A is nonsingular, explane why $\det(A^{-1}) = \frac{1}{\det(A)}$

I have : $\overbrace{I}^{=1} = AA^{-1} \Rightarrow \det(I) = \det(AA^{-1}) = \det(A)\cdot\det(A^{-1}) \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

#2. if A is $n\times n$, explane why $\det(\alpha A) = \alpha^n \det(A)$ for all scaler α.

$$\alpha A = \alpha I A = (\alpha I) A \xrightarrow{\det} \det(\alpha A) = \det(\alpha I A) = \underbrace{\det(\alpha I)}_{=\alpha^n} \det(A) = \alpha^n \det(A)$$

#3. find the all matrix solution of matrix equation $X^2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ where a is any number different from 0.

assume : $X = \begin{bmatrix} b & c \\ d & e \end{bmatrix} \longrightarrow X^2 = X.X = \begin{bmatrix} b & c \\ d & e \end{bmatrix}\begin{bmatrix} b & c \\ d & e \end{bmatrix} = \begin{bmatrix} b^2+cd & bc+ce \\ db+de & dc+e^2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} b^2+cd & c(b+e) \\ d(b+e) & dc+e^2 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} b^2+cd = 1 & (\mathrm{I}) \\ cb+ce = a & (\mathrm{II}) \\ db+de = 0 & (\mathrm{III}) \\ dc+e^2 = 1 & (\mathrm{IV}) \end{cases}$$

from (III) : $d(b+e) = 0$ ↗ $\boxed{d=0}$ ↘ $\boxed{b=-e}$ *

(II) : $c(b+e) = a \xrightarrow{*} c(-e+e) = a \longrightarrow \boxed{c=0}$ **

(I) : $b^2+cd = 1 \xrightarrow[c=0]{d=0} b^2 = 1 \Rightarrow \boxed{b = \pm 1}$ $\longrightarrow$ choice : $\begin{cases} d=0 \\ c=0 \\ b=-1 \\ e=+1 \end{cases}$

IV) : $dc+e^2 = 1 \xrightarrow{d=c=0} e^2 = 1 \Rightarrow \boxed{e = \pm 1}$

#4. ~~xxxx~~ Compute the determinant of the following matrix theoretically. In addition, obtain it by the MATLAB or Python.

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & 7 \\ 2 & 4 & -3 & 2 \\ 3 & 0 & 15 & 3 \end{bmatrix} \longrightarrow \det(A) = ?$$

$$\det_{n\times n}(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\cdot\det(ij) = +1\cdot\underbrace{\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3 & 2 \\ 3 & 15 & 3 \end{vmatrix}}_{(I)} + 2\cdot\underbrace{\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 2 \\ 3 & 0 & 3 \end{vmatrix}}_{(II)} + 7\cdot\underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -3 \\ 3 & 0 & 15 \end{vmatrix}}_{(III)}$$

$$\text{I)}\ \begin{vmatrix} 1 & 3 & -1 \\ 2 & -3 & 2 \\ 3 & 15 & 3 \end{vmatrix} = 1\begin{vmatrix} -3 & 2 \\ 15 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 3 & 15 \end{vmatrix} = (-9-30) - 3\cancel{(6-6)}^{0} - 1(30+9) = -78$$

$$\text{II)}\ \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 2 \\ 3 & 0 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 3(4+4) + 3\cancel{(4-4)}^{0} = 24$$

$$\text{III)}\ \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -3 \\ 3 & 0 & 15 \end{vmatrix} = 3\begin{vmatrix} 2 & 3 \\ 4 & -3 \end{vmatrix} + 15\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 3(-6-12) + 15\cancel{(4-4)}^{0} = -54$$

$$\Rightarrow -1(\text{I}) + 2(\text{II}) - 7(\text{III}) = +1(-78) + 2(24) + 7(-54) = -78 + 48 + 378 = \boxed{-504}$$

#5. given the matrix $\mathbf{A}$, find $\det(A^{-1}A^{T}A)$.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 7 & 0 \\ 4 & 0 & 5 \end{bmatrix} \longrightarrow \text{from Question 1 I know: } \det(A^{-1}) = \frac{1}{\det(A)}$$

$$\Rightarrow \det(A^{-1}A^{T}A) = \cancel{\det(A^{-1})}\cdot\cancel{\det(A)}\cdot\det(A^{T}) \Rightarrow \det(A^{-1}AA^{T}) = \det(A^{T}) = \det(A)$$

$$\det(A) = +7\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = +7(10-12) = \boxed{-14}$$

#6. Pick any number that add to $x+y+z=0$. find the angle betwen your Vector $v=(x,y,z)$ and the Vector $w=(z,x,y)$. Explain why $\frac{v \cdot w}{\|v\|\|w\|}$ is always $-\frac{1}{2}$?

$$\|w\| = \sqrt{x^2+y^2+z^2}$$
$$\|v\| = \sqrt{x^2+y^2+z^2} \quad \Rightarrow \quad \|w\| = \|v\|$$

$$\langle v \cdot w \rangle = (x,y,z)\cdot(z,x,y) = xz + yx + zy$$

I know: $(a+b+c)^2 = a^2+b^2+c^2 + 2(ab+ac+bc)$

$$\Rightarrow \underbrace{(x+y+z)^2}_{=0} = \underbrace{x^2+y^2+z^2}_{=\|v\|\cdot\|w\|} + 2\underbrace{(xy+xz+yz)}_{=\langle v\cdot w\rangle} \quad \Rightarrow \quad 0 = \|v\|\|w\| + 2\langle v,w\rangle$$

$$\Rightarrow \|v\|\cdot\|w\| = -2\langle v,w\rangle \quad \Rightarrow \quad \langle v,w\rangle = \frac{-1}{2}\|v\|\|w\| \; *$$

I know: $\cos\theta = \frac{\langle v,w\rangle}{\|v\|\|w\|} \overset{*}{\Rightarrow} \cos\theta = \frac{-\frac{1}{2}\|v\|\|w\|}{\|v\|\|w\|} = -\frac{1}{2}$

$$\Rightarrow \boxed{\cos\theta = \frac{\langle v,w\rangle}{\|v\|\|w\|} = -\frac{1}{2}}$$