Solution _S1

1)

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} \quad \Longrightarrow \quad \det(\mathbf{I}) = \det\left(\mathbf{A}^{-1}\mathbf{A}\right) = \det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A})$$
$$\Longrightarrow \quad 1 = \det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A}) \quad \Longrightarrow \quad \det\left(\mathbf{A}^{-1}\right) = 1/\det(\mathbf{A}).$$

2)

$$\alpha\mathbf{A} = (\alpha\mathbf{I})\mathbf{A} \quad \Longrightarrow \quad \det(\alpha\mathbf{A}) = \det(\alpha\mathbf{I})\det(\mathbf{A}) = \alpha^n\det(\mathbf{A}).$$

3)

If we put

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

Then

$$\boldsymbol{X}^2 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}^2 + x_{12}x_{21} & x_{11}x_{12} + x_{12}x_{22} \\ x_{21}x_{11} + x_{22}x_{21} & x_{21}x_{12} + x_{22}^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Thus, we have

$$x_{11}^2 + x_{12}x_{21} = 1 = x_{21}x_{12} + x_{22}^2,$$

Hence $x_{11}^2 = x_{22}^2$.

Furthermore,

$$x_{11}x_{12} + x_{12}x_{22} = x_{12}(x_{11} + x_{22}) = a$$

And $x_{21}x_{11} + x_{22}x_{21} = x_{21}(x_{11} + x_{22}) = 0$.

It follows from $a \neq 0$ that $x_{12} \neq 0$ and $x_{22} \neq -x_{11}$. Since $x_{11}^2 = x_{22}^2$, we must have $x_{22} = x_{11} \neq 0$, and the equations are reduced to

$2x_{11}.x_{12} = a$ and $2x_{11}.x_{21} = 0$,

Hence $x_{21} = 0$. As a result,

$x_{11}^2 + x_{12}x_{21} = x_{11}^2 = 1$, thus $x_{11} = \pm 1$.

For $x_{11} = 1$ we get the solution

$$\boldsymbol{X} = \begin{pmatrix} 1 & \frac{a}{2} \\ 0 & 1 \end{pmatrix}$$

And for $x_{11} = -1$ we get

$$\boldsymbol{X} = \begin{pmatrix} -1 & \frac{-a}{2} \\ 0 & -1 \end{pmatrix}.$$

4)

Easy to compute theoretically.

MATLAB: det(**A**) = -504

5)

$\det(\boldsymbol{A}) = -14$

$\det(\boldsymbol{A}^{-1}\boldsymbol{A}^T\boldsymbol{A}) = \det(\boldsymbol{A})^{-1} * \det(\boldsymbol{A}) * \det(\boldsymbol{A}) = \det(\boldsymbol{A}) = -14$

6)

Try $v = (1,2,-3)$ and $w = (-3,1,2)$ with $\cos(\theta) = -\frac{7}{14}$ and $\theta = 120$.

$$v.w \ = xz + xy + zy = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2).$$

If $x + y + z = 0$ this is $-\frac{1}{2}(x^2+y^2+z^2) = -\frac{1}{2}\|v\|\|w\|$.

Then, $\frac{v.w}{\|v\|\|w\|} = -\frac{1}{2}$.

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7)

A is orthogonal if all the elements are real and $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$.

$$\boldsymbol{A}^T\boldsymbol{A} = \frac{1}{(1+2a^2)^2}\begin{bmatrix} 1 & 2a & 2a^2 \\ -2a & 1-2a^2 & 2a \\ 2a^2 & -2a & 1 \end{bmatrix}\begin{bmatrix} 1 & -2a & 2a^2 \\ 2a & 1-2a^2 & -2a \\ 2a^2 & 2a & 1 \end{bmatrix}$$

$$= \frac{1}{(1+2a^2)^2}\begin{bmatrix} (1+2a^2)^2 & 0 & 0 \\ 0 & (1+2a^2)^2 & 0 \\ 0 & 0 & (1+2a^2)^2 \end{bmatrix} = \boldsymbol{I}$$

For more practice, check the properties of the orthogonal matrix.