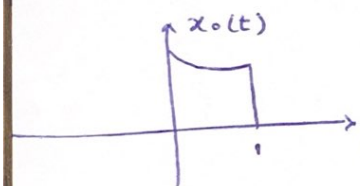
یا لطیف

تمرین سری ۶ سیگنال

1400/2/21

#1 $x_0(t) = \begin{cases} e^{-t} & ; \ 0 \le t \le 1 \\ 0 & ; \ \text{other} \end{cases} \Longrightarrow X(j\omega)$:سیگنال‌های زیر

ابتدا تبدیل فوریه سیگنال $x_0(t)$ را بدست می‌آوریم سپس با استفاده از خواص تبدیل فوریه، تبدیل فوریه سایر سیگنال‌ها را بدست می‌آوریم.



یادآوری: زوج تبدیل فوریه

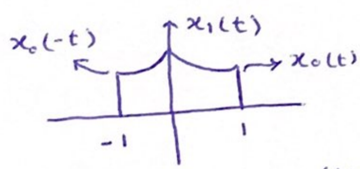
رابطه سنتز: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$

آنالیز: $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$

$$x_0(t) \xleftrightarrow{F} X_0(j\omega) = \int_{-\infty}^{+\infty} x_0(t)e^{-j\omega t}\,dt = \int_0^1 e^{-t}\cdot e^{-j\omega t}\,dt = \int_0^1 e^{-(1+j\omega)t}\,dt$$

$$= \frac{-1}{1+j\omega}\cdot e^{-(1+j\omega)t}\Big|_0^1 = \frac{1}{1+j\omega}\left(1-e^{-(1+j\omega)}\right)$$

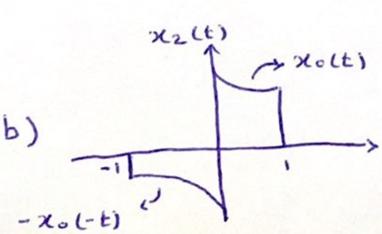
a)



$: \ x_1(t) = x_0(t) + x_0(-t) \xleftrightarrow{F} X_0(j\omega) + X_0(-j\omega)$

$$\Rightarrow X_1(j\omega) = \frac{1}{1+j\omega}\left(1-e^{-(1+j\omega)}\right) + \frac{1}{1-j\omega}\left(1-e^{-(1-j\omega)}\right)$$

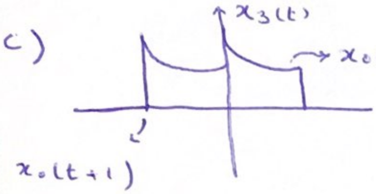
b)



$$x_2(t) = x_0(t) - x_0(-t) \xleftrightarrow{F} X_2(j\omega) = X_0(j\omega) - X_0(-j\omega)$$

$$\Rightarrow X_2(j\omega) = \frac{1}{1+j\omega}\left(1-e^{-(1+j\omega)}\right) - \frac{1}{1-j\omega}\left(1-e^{-(1-j\omega)}\right)$$

c)



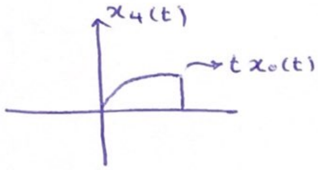
$$x_3(t) = x_0(t) + x_0(t+1) \xleftrightarrow{F} X_0(j\omega) + X_0(j\omega)e^{j\omega}$$

$$\Rightarrow X_3(j\omega) = \frac{1}{1+j\omega}\left(1-e^{-(1+j\omega)}\right) + \frac{1}{1+j\omega}\left(1-e^{-(1+j\omega)}\right)\cdot e^{j\omega}$$

$$= \frac{1}{j\omega+1}\left(1-e^{-(1+j\omega)}\right)\cdot\left[1+e^{j\omega}\right]$$

یادآوری : خاصیت شیفت زمانی : $x(t-t_0) \xleftrightarrow{F} X(j\omega)e^{-j\omega t_0}$

d)



$$x_4(t) = t\,x_0(t) \xleftrightarrow{F} X_4(j\omega) = j\frac{d}{d\omega}X_0(j\omega)$$

$$\Rightarrow X_4(j\omega) = j\frac{d}{d\omega}\left[\frac{1}{1+j\omega}\left(1-e^{-(1+j\omega)}\right)\right] = \frac{1-2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1+j\omega)^2}$$

یادآوری : خاصیت مشتق در فرکانس : $-jt\,x(t) \xleftrightarrow{F} \frac{d}{d\omega}X(j\omega) \xrightarrow{\times j} t\,x(t) \xleftrightarrow{F} j\frac{d}{d\omega}X(j\omega)$

#2 نشان دهید که : $x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \xleftrightarrow[\text{به ازای } a>0]{F} X(j\omega) = \frac{1}{(a+j\omega)^n}$

$$n=1 : \frac{t^{1-1}}{(1-1)!}e^{-at}u(t) = \underbrace{1\,e^{-at}u(t)}_{\substack{\text{سیگنال سوال یک} \\ x_0(t)}} \xleftrightarrow{F} X_0(j\omega) = \frac{1}{a+j\omega}$$

$$n=2 : \frac{t^{2-1}}{(2-1)!}e^{-at}u(t) = \underbrace{t\,\underbrace{e^{-at}u(t)}_{x_0(t)}}_{x_1(t)} \xleftrightarrow[\text{طبق خاصیت مشتق فرکانسی}]{F} j\frac{d}{d\omega}X_0(j\omega) = \frac{1}{(a+j\omega)^2}$$

$$n=3 : \frac{t^2}{2}e^{-at}u(t) = \frac{t}{2}x_1(t) \xleftrightarrow{F} \frac{j}{2}\cdot\frac{d}{d\omega}X_1(j\omega) = \frac{j}{2}\left[\frac{-2j(a+j\omega)}{(a+j\omega)^4}\right] = \frac{1}{(a+j\omega)^3}$$

$$\Rightarrow \frac{t^{n-1}}{(n-1)!}e^{-at}u(t) = \frac{t}{(n-1)}x_{n-1} \xleftrightarrow{F} \frac{j}{(n-1)}\cdot\frac{d}{d\omega}X_{n-1}(j\omega) = j\left[\frac{-(n-1)j\cdot(a+j\omega)^{n-2}}{(n-1)\cdot(a+j\omega)^{2n-2}}\right]$$

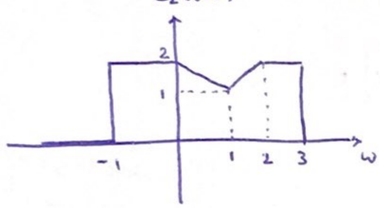
$$= \frac{e^{-at}u(t)}{(n-2)!} = x_{n-1}$$

3 عکس تبدیل فوریه ?

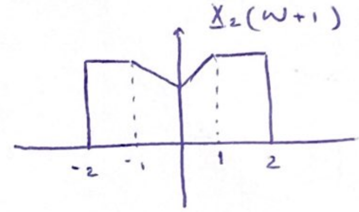
a) $\bar{X}_1(j\omega) = \dfrac{2\sin(3\overbrace{(\omega-2\pi)}^{=\alpha})}{\underbrace{\omega-2\pi}_{=\alpha}} \xleftrightarrow{F^{-1}} x_1(t) = ?$

$\Rightarrow \bar{X}_1(j\omega) = \dfrac{2\sin 3\alpha}{\alpha} \xleftrightarrow{F^{-1}} x_1(t) = \begin{cases} 1 & ; |t|<3 \\ 0 & ; \text{other} \end{cases} \xrightarrow{\alpha=\omega-2\pi} x_1(t) = \begin{cases} e^{j2\pi t} & ; |t|<3 \\ 0 & ; 0 \end{cases}$

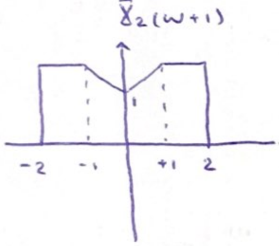
b)



یک واحد به چپ شیفت می دهیم



Real and even



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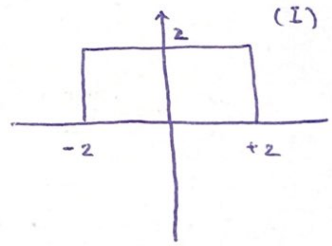
(I)



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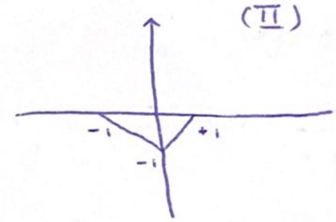
(II)



$\bar{X}_2(\omega+1) = (\mathrm{I}) - (\mathrm{II})$

$(\mathrm{I}) \xleftrightarrow{F^{-1}} \dfrac{4}{2\pi}\,\mathrm{Sinc}\left(\dfrac{4}{2\pi}t\right) = \dfrac{2}{\pi}\,\mathrm{Sinc}\left(\dfrac{2}{\pi}t\right)$

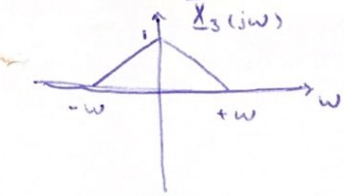
$(\mathrm{II}) \xleftrightarrow{F^{-1}} \dfrac{-2\sin^2(t/2)}{\pi t^2}$

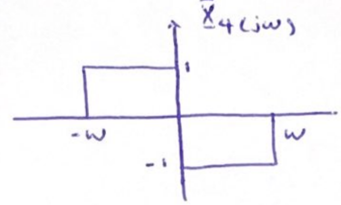
$\Rightarrow \bar{X}_2(\omega+1) \xleftrightarrow{F^{-1}} \dfrac{2}{\pi}\,\mathrm{Sinc}\left(\dfrac{2t}{\pi}\right) + \dfrac{2\sin^2\left(\frac{t}{2}\right)}{\pi t^2}$

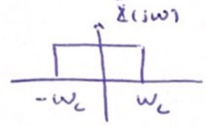
$\Rightarrow \bar{X}_2(j\omega) = \left[\dfrac{2}{\pi}\,\mathrm{sinc}\left(\dfrac{2t}{\pi}\right) + \dfrac{2\sin^2\left(\frac{t}{2}\right)}{\pi t^2}\right] e^{j\omega_0 t}$

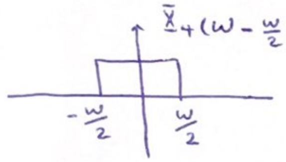
روش دوم : if $x(t)$: Real $\Longleftrightarrow$ $\begin{cases} x_{even}(t) \xleftrightarrow{F} Re\{\bar{X}(j\omega)\} \\ x_{odd}(t) \xleftrightarrow{F} j\,Im\{\bar{X}(j\omega)\} \end{cases}$

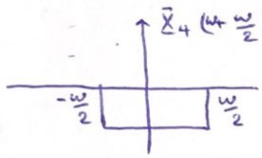
$\Rightarrow x(t) = x_{even}(t) + x_{odd}(t)$

c) 

$\Longrightarrow$ 

یادآوری :  $\overset{F}{\longleftrightarrow} x(t) = \frac{W_c}{\pi}\,\text{sinc}\left(\frac{W_c t}{\pi}\right)$

$\Rightarrow$  $\longleftrightarrow \quad x_4(t) = \frac{W}{2\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\cdot e^{j\frac{W}{2}t}$

 $\longleftrightarrow \quad x_4(t) = \frac{-W}{2\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\cdot e^{-j\frac{W}{2}t}$

$$\Longrightarrow x_4(t) = \frac{W}{2\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\cdot e^{j\frac{W}{2}t} - \frac{W}{2\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\cdot e^{-j\frac{W}{2}t} = \frac{W}{2\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\overbrace{\left[e^{j\frac{W}{2}t} - e^{-j\frac{W}{2}t}\right]}^{=2\cos\left(\frac{W}{2}t\right)}$$

$$= \frac{W}{\cancel{2}\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\cdot\cancel{2}\cos\left(\frac{W}{2}t\right) = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\cdot\cos\left(\frac{Wt}{2}\right)$$

$$\Longrightarrow \frac{d}{dw}\bar{X}_3(jw) = \bar{X}_4(jw) \overset{F}{\longleftrightarrow} -jt\,x_4(t) \Rightarrow \boxed{x_3(t) = -jt\,\frac{W}{\pi}\,\text{sinc}\left(\frac{Wt}{2\pi}\right)\cdot\cos\left(\frac{Wt}{2}\right)}$$

راه دوم : همین روش دوم قسمت b

#4   a)

تبدیل فوریه و کانولوشن

$$\begin{cases} x(t) = te^{-2t}u(t) \overset{F}{\longleftrightarrow} \bar{X}(jw) = \frac{1}{(2+jw)^2} \\ h(t) = e^{-4t}u(t) \overset{F}{\longleftrightarrow} H(jw) = \frac{1}{4+jw} \end{cases}$$

یادآوری :
$$\begin{cases} x(t) = e^{-at}u(t) \overset{F}{\longleftrightarrow} \bar{X}(jw) = \frac{1}{a+jw} \\ x(t) = te^{-at}u(t) \overset{F}{\longleftrightarrow} \bar{X}(jw) = \frac{1}{(a+jw)^2} \end{cases}$$

$$\Rightarrow y(t) = x(t) * h(t) \overset{F}{\longleftrightarrow} Y(jw) = \bar{X}(jw)\,H(jw) = \frac{1}{(2+jw)^2}\times\frac{1}{4+jw} = \frac{1}{(2+jw)^2\cdot(4+jw)}$$

$$\Rightarrow y(t) = F^{-1}\left\{Y(jw)\right\}$$

b) $\begin{cases} x(t) = e^{-t}u(t) \xleftrightarrow{\mathcal{F}} X(j\omega) = \dfrac{1}{1+j\omega} \\ h(t) = e^{t}u(-t) \xleftrightarrow{\mathcal{F}} H(j\omega) = \dfrac{1}{1-j\omega} \end{cases}$

$\rightarrow h(t) = x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$

$$\Rightarrow y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{1+j\omega} \times \frac{1}{1-j\omega} = \frac{1}{1+\omega^2}$$

$$\Rightarrow y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}$$

5

$\mathcal{L}$ $x(t) = \cos(t) \longrightarrow \begin{cases} a)\ h_1(t) = u(t) \\ b)\ h_2(t) = -2\delta(t) + 5e^{-2t}u(t) \\ c)\ h_3(t) = 2te^{-t}u(t) \end{cases} \Rightarrow y_1(t) = y_2(t) = y_3(t)$

$$x(t) = \cos(t) \xleftrightarrow{\mathcal{F}} X(j\omega) = \pi\delta(\omega-1) + \pi\delta(\omega+1)$$

a) $h_1(t) = u(t) \xleftrightarrow{\mathcal{F}} H_1(j\omega) = \dfrac{1}{j\omega} + \pi\delta(\omega)$

$$\Rightarrow Y_1(j\omega) = X(j\omega)H_1(j\omega) = \left(\pi\delta(\omega-1) + \pi\delta(\omega+1)\right)\cdot\left(\frac{1}{j\omega} + \pi\delta(\omega)\right)$$

b) $h_2(t) = -2\delta(t) + 5e^{-2t}u(t) \xleftrightarrow{\mathcal{F}} H_2(j\omega) = -2 + \dfrac{5}{2+j\omega} = \dfrac{1-2j\omega}{2+j\omega}$

$$\Rightarrow Y_2(j\omega) = X(j\omega)H_2(j\omega) = \left(\pi\delta(\omega-1) + \pi\delta(\omega+1)\right)\cdot\left(\frac{1-2j\omega}{2+j\omega}\right)$$

c) $h_3(t) = 2te^{-t}u(t) \xleftrightarrow{\mathcal{F}} H_3(j\omega) = \dfrac{2}{(1+j\omega)^2}$

$$\Rightarrow Y_3(j\omega) = X(j\omega)H_3(j\omega) = \left(\pi\delta(\omega-1) + \pi\delta(\omega+1)\right)\cdot\left(\frac{2}{(1+j\omega)^2}\right)$$

ب) هر ترکیب خطی از h_1، h_2، h_3 خروجی مشابه دارد.

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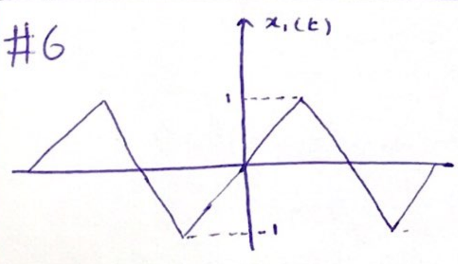


الف) $x_1(t)$: Real and odd $\Rightarrow X_1(j\omega)$ موهومی خالص و فرد

$\Rightarrow \mathrm{Re}\{X(j\omega)\} = 0 \quad , \quad \mathrm{Im}\{X(j\omega)\} \neq 0$

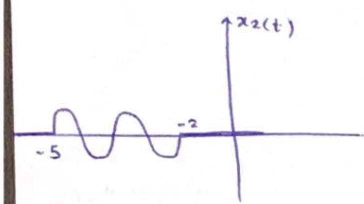
ب) طبق قسمت الف قسمت موهومی $X(j\omega)$ صفر نیست

ج) $\underbrace{e^{j\alpha\omega} X(j\omega)}_{\text{موهومی خالص}} \Rightarrow e^{j\alpha\omega} = j$ به ازای αهایی که شود سیگنال حقیقی می‌شود

ت) $\int_{-\infty}^{+\infty} X(j\omega)\, d\omega \overset{?}{=} 0 \Longrightarrow \cancel{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega}$

$\cancel{x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X}$ سطح زیر نمودار <== چون سیگنال فرد است این فرض درست است .

ث) $\int_{-\infty}^{+\infty} \underbrace{\omega X(j\omega)}_{\text{مشتق زمانی}}\, d\omega \overset{?}{=} 0$ برقرار نیست



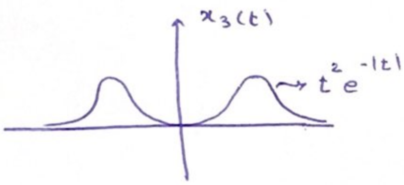
الف) ✗

ب) ✗

ج) ✓ : $x_2(t-3.5) \Rightarrow$ حقیقی و زوج $\alpha = -3.5$

ت) ✓

ث) ✓



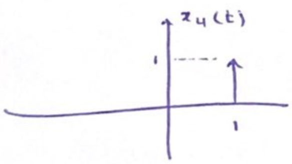
الف) ✗

ب) ✓

ج) $\alpha = 0$ به ازای

ت) ✓

ث) ✗



الف) ✗

ب) ✗

ج) $x_4(t+1) \Rightarrow \alpha = -1$

ت) ✓

ث) ✓) -> چون تبدیل فوریه $x_4(t)$ کمپلکس می شود

$|H(j\omega)| = \begin{cases} \frac{1}{2000\pi}|\omega| & ; |\omega| < 2000\pi \\ 4000\pi & ; 2000\pi \le |\omega| \le 3000\pi \\ 0 & ; \text{other} \end{cases}$

$\measuredangle H(j\omega) = -\measuredangle H(-j\omega)$

$\measuredangle H(j\omega) = \begin{cases} \frac{\pi}{2} & ; 0 < \omega < 2000\pi \\ \frac{\omega}{6000} & ; 2000\pi \le \omega \le 3000\pi \\ 0 & ; \omega > 3000\pi \end{cases}$

$x(t) = \frac{1}{2} + 2\sin(1000\pi t + \frac{\pi}{4}) - 3\cos(2500\pi t - \frac{\pi}{4}) + 4\sin(4000\pi t)$

$X(j\omega) = \left(\frac{1}{2} \times 2\pi\delta(\omega)\right) + \left[\frac{2\pi}{j}\delta(\omega - 1000\pi) + \frac{2\pi}{j}\delta(\omega + 1000\pi)\right] e^{j\frac{\pi}{4}t}$

$+ \left[-3\pi\left(\delta(\omega - 2500\pi) + \delta(\omega + 2500\pi)\right) e^{-j\frac{\pi}{4}t}\right] + \left[4\pi\left(\delta(\omega - 4000\pi) - \delta(\omega + 4000\pi)\right)\right]$

$Y(j\omega) = X(j\omega) H(j\omega) \Rightarrow |Y(j\omega)| = \begin{cases} \frac{1}{2000\pi}|\omega| \cdot X(j\omega) & ; |\omega| < 2000\pi \\ 4000\pi\, X(j\omega) & ; 2000\pi \le |\omega| \le 3000\pi \\ 0 & ; \text{other} \end{cases}$