سوال 5

$x_1 \xleftrightarrow{FS} a_k$ periodic with T_1 $\left(\frac{2\pi}{\omega_1}\right)$

$$x_2(t) = \underbrace{2x_1(1-t)}_{(I)} - \frac{1}{3}x_1(2t-1) \longrightarrow \begin{cases} T = ? \\ b_k = ? \end{cases}$$

$$\text{if } \; x_1(t) \xleftrightarrow{F.S} a_k \Rightarrow x_1(t+1) \xleftrightarrow{F.S} a_k e^{+jk\omega_1 \times 1}$$

$$\text{if } \; x(-t) \xleftrightarrow{F.S} a_{-k} \Rightarrow x_1(-t+1) = x(1-t) \xleftrightarrow{F.S} a_{-k} e^{+jk\omega_1}$$

$$x(t-1) \xleftrightarrow{F.S} a_k e^{-jk\omega_1} \quad , \quad x(2t-1) \xleftrightarrow{F.S} a_k e^{-jk2\omega_1}$$

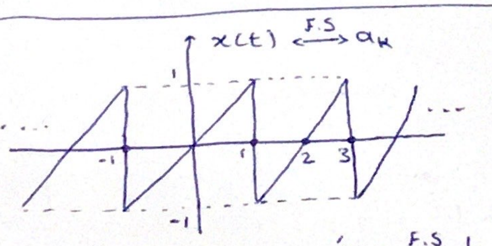
ضرایب تغییر نمی‌کند

فقط ω_1 ، 2 برابر می‌شود ⟵ (T نصف می‌شود)

$$\boxed{\Rightarrow a_k = 2a_{-k} e^{jk\omega_1} - \frac{1}{3} a_k e^{-jk2\omega_1}}$$

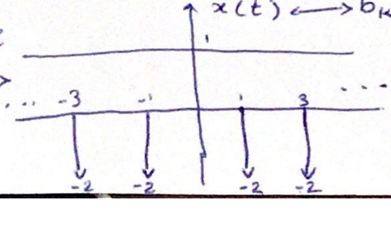
$$, \quad \begin{cases} 2x_1(1-t) \longrightarrow T_1 \\ -\frac{1}{3}x_1(2t-1) \longrightarrow \frac{T_1}{2} \end{cases} \Rightarrow \text{Kmm}\left\{T_1, \frac{T_1}{2}\right\} = T_1$$

سوال 6

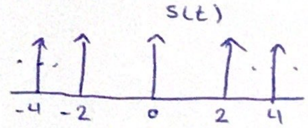


روش اول ⟵ استفاده از خواص

مشتق می‌گیریم



می‌دانیم :



$$S(t) \xleftrightarrow{F.S} d_k = \frac{1}{T} = \frac{1}{2}$$

$$x'(t) = -2\delta(t-1) + 1$$

$$\rightarrow (jk\omega_0)\,a_k = -2e^{-jk\omega_0 \times 1} \times \underbrace{\tfrac{1}{2}}_{=d_k}$$

$$\rightarrow a_k = \frac{-1}{jk\omega_0} e^{-jk\omega_0} \xrightarrow{\omega_0 = \frac{2\pi}{2} = \pi} \cancel{a_k = \frac{-1}{jk\omega_0}}$$

$$\boxed{a_k = \frac{-1}{jk\pi} e^{-jk\pi} = \frac{j}{k\pi}(-1)^k \quad \text{for } k \neq 0}$$

روش دوم $\longrightarrow$ فرمول آنالیز : $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt \xrightarrow{T=2} a_k = \frac{1}{2}\int_{-1}^{+1} \underbrace{t\,e^{-jk\omega_0 t}\,dt}_{!}$

مشتق	انتگرال
t	$e^{-jk\omega_0 t}$
1	$\frac{-1}{jk\omega_0} e^{-jk\omega_0 t}$
0	$\frac{-1}{(jk\omega_0)^2} e^{-jk\omega_0 t}$

$$= \frac{1}{2}\left[\frac{-t}{jk\omega_0} e^{-jk\omega_0 t}\Big|_{-1}^{+1} + \frac{1}{(jk\omega_0)^2} e^{-jk\omega_0 t}\Big|_{-1}^{+1}\right]$$

$$= \frac{1}{2}\left[e^{-jk\omega_0}\left(\frac{1}{(k\omega_0)^2} - \frac{1}{jk\omega_0}\right) - e^{jk\omega_0}\left(\frac{1}{(k\omega_0)^2} + \frac{1}{jk\omega_0}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{(k\omega_0)^2}\left(e^{jk\omega_0} - e^{-jk\omega_0}\right) + \left(e^{jk\omega_0} + e^{-jk\omega_0}\right)\frac{1}{jk\omega_0}\right]$$

سوال 7

$T=3$

a) $a_k = a_{k+2}$

b) $a_k = a_{-k}$

c) $\int_{-0.5}^{0.5} x(t)\,dt = 1$

d) $\int_{0.5}^{1.5} x(t)\,dt = 2$

$x(t) = ?$

طبق خواص سری فوریه می‌دانیم $x(-t) \xleftrightarrow{F.S} a_{-k}$

پس از خاصیت b می‌توان فهمید که : $\boxed{x(t) = x(-t)}$

طبق خاصیت جابه‌جایی در فرکانس : $e^{j\omega_0 M t} x(t) \xleftrightarrow{F.S} a_{k-M}$

از خاصیت a می‌توان فهمید که $\longrightarrow$

$$\underbrace{x(t) = x(t)e^{-j\frac{4\pi}{3}t}}$$

$\rightarrow$ فقط در فرکانس $\Rightarrow t = 0, \pm 1.5, \pm 3, \pm 4.5, \pm 6, \dots$

$t = \pm 1.5t$ مقدار غیرصفر دارد به طوری که شرط a را ارضا کند

$\Leftarrow$ باید $e^{-j\frac{4\pi}{3}t} = 1$ شود t ، a صحیح باشد

از خاصیت c می‌توان فهمید که : $x(t) = \delta(t) \;;\; -\frac{1}{2} < t < \frac{1}{2}$

همچنین از خاصیت d می‌توان فهمید : $x(t) = 2\delta(t-1.5) \;;\; 0.5 < t < 1.5$

بنابراین می‌توان نوشت : $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-3k) + 2\sum_{k=-\infty}^{+\infty} \delta(t-1.5-3k)$

سوال 8

if $x(t) \xleftrightarrow{F.S} a_k$

$$Z(t) = \underbrace{\cos(M\omega_0 t)}_{(III)} \cdot \underbrace{\frac{d}{dt}\Big(\underbrace{x(t-t_0)}_{(I)}\Big)}_{(II)} \xleftrightarrow{F.S} ?$$

I) $x(t-t_0) \xleftrightarrow{F.S} a_k e^{-jk\omega_0 t_0}$

II) $\frac{d}{dt}\left(x(t-t_0)\right) \xleftrightarrow{F.S} jk\omega_0 \left(a_k e^{-jk\omega_0 t_0}\right)$

III) $\cos(M\omega_0 t): \Rightarrow \frac{1}{2}\left(e^{jM\omega_0 t} + e^{-jM\omega_0 t}\right)$

یادآوری: $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$

$$\cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + e^{-j\omega_0 t} \Rightarrow b_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt = \frac{\omega_0}{2\pi} \times \frac{1}{2}\int_0^{\frac{2\pi}{\omega_0}} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-jk\omega_0 t} dt$$

$$= \frac{\omega_0}{4\pi}\int_0^{\frac{2\pi}{\omega_0}} e^{j\omega_0 t} e^{-jk\omega_0 t} dt + \int_0^{\frac{2\pi}{\omega_0}} e^{-j\omega_0 t} \cdot e^{-jk\omega_0 t} dt$$

$$= \frac{\omega_0}{4\pi}\int_0^{\frac{2\pi}{\omega_0}} e^{j(\omega_0 - k\omega_0)t} dt + \int_0^{\frac{2\pi}{\omega_0}} e^{-j(\omega_0 + k\omega_0)t} dt$$

$$= \frac{\omega_0}{4\pi}\left(\frac{2\pi}{\omega_0} e^{j(\omega_0 - k\omega_0)} + \frac{2\pi}{\omega_0} e^{-j(\omega_0 + k\omega_0)}\right) = \underbrace{\frac{1}{2}\left(e^{j(\omega_0 - k\omega_0)} + e^{-j(\omega_0 + k\omega_0)}\right)}_{\text{ضرایب سری فوریه } \cos(\omega_0 t)}$$

$$\Rightarrow \cos(M\omega_0 t) \xleftrightarrow{F.S} \frac{1}{2}\left(e^{j(M\omega_0 - k\omega_0 M)} + e^{-j(M\omega_0 + kM\omega_0)}\right)$$

$$\Rightarrow Z(t) \xleftrightarrow{F.S} C_k = \frac{jk\omega_0}{2}\left(a_k e^{-jk\omega_0 t_0}\right) \cdot \left(e^{jM(\omega_0 - k\omega_0)} + e^{-jM(\omega_0 + k\omega_0)}\right)$$

سوال ۹

$$w(t) = \sum_{k=-\infty}^{+\infty} \left(\delta(t - 0.5 - 2k) - \delta(t + 0.5 - 2k)\right)$$

یادآوری:

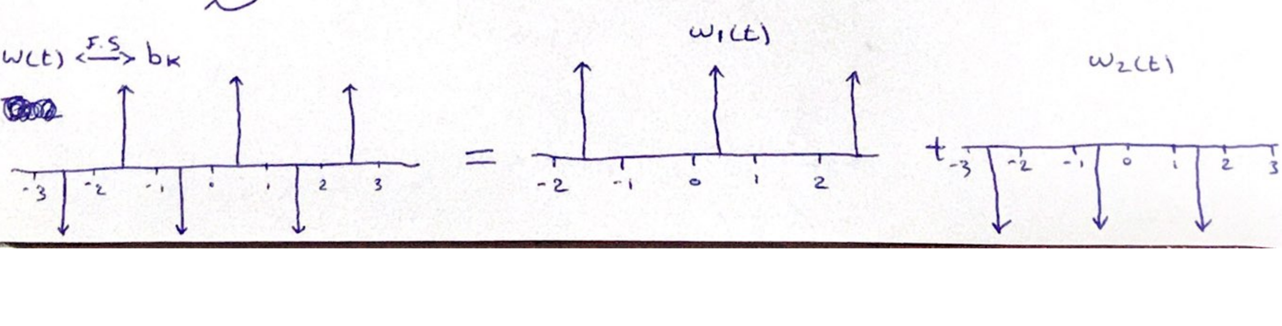
$x(t) \circledast y(t) = \int_T x(\tau) y(t-\tau) d\tau$

$x(t) \circledast y(t) \xleftrightarrow{F.S} T a_k b_k$

$\xleftrightarrow{a_k}$

ضرایب سری فوریه این سیگنال در سوال 6 بدست آورده شد

$w(t) \xleftrightarrow{F.S} b_k$



$$\Rightarrow w(t) = w_1(t) + w_2(t) \xleftrightarrow{F.S} b_k = c_k + d_k$$

$$w_1(t) \xleftrightarrow{F.S} \frac{1}{2} e^{-jk\frac{\pi}{2}}$$
$$w_2(t) \xleftrightarrow{F.S} -\frac{1}{2} e^{jk\frac{\pi}{2}} \quad \Rightarrow b_k = \frac{1}{2}\left(e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right)$$

$$x(t) \xleftrightarrow{F.S} a_k = \frac{-1}{jk\pi} e^{-jk\pi} \Rightarrow T a_k b_k = 2 \times \frac{1}{2}\left(e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right) \times \frac{-1}{jk\pi} e^{-jk\pi}$$

$$= \underbrace{\frac{-1}{jk\pi} e^{-jk\pi}\left(e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right)}_{(I)} \Longrightarrow x(t) \circledast y(t) = FS^{-1}\{I\}$$

روش دوم : $x(t) \circledast y(t) = \int_T x(\tau) y(t-\tau)$

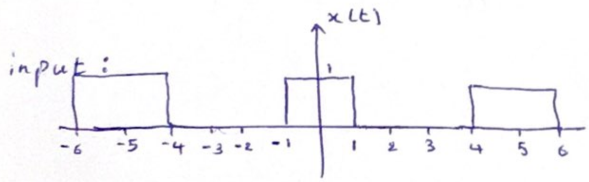
تفاوت کانولوشن پریودیک با کانولوشن معمولی تنها در بازه انتگرال گیری است .

سوال 10

یادآوری :

$$\frac{d^2 y(t)}{dt^2} - 5\frac{dy(t)}{dt} + 6y(t) = 3\frac{dx(t)}{dt}$$

input:



$$x(t) = e^{j\omega t} \longrightarrow \boxed{LTI} \xrightarrow{H(j\omega)e^{j\omega t} = y(t)}$$

$$\Rightarrow \frac{d^2}{dt^2}\left(H(j\omega)e^{j\omega t}\right) - 5\left(H(j\omega)e^{j\omega t}\right) + 6H(j\omega)e^{j\omega t} = 3e^{j\omega t}$$

$$= (j\omega)^2 H(j\omega)e^{j\omega t} - 5(j\omega)H(j\omega)e^{j\omega t} + 6H(j\omega)e^{j\omega t} = 3e^{j\omega t}$$

$$= -\omega^2 H(j\omega)e^{j\omega t} - 5j\omega H(j\omega)e^{j\omega t} + 6H(j\omega)e^{j\omega t} = 3e^{j\omega t}$$

$$\Rightarrow H(j\omega)\left(-\omega^2 e^{j\omega t} - 5j\omega e^{j\omega t} + 6e^{j\omega t}\right) = 3e^{j\omega t}$$

$$\Rightarrow H(j\omega) = \frac{3e^{j\omega t}}{6e^{j\omega t} - 5j\omega e^{j\omega t} - \omega^2 e^{j\omega t}} = \frac{3\cancel{e^{j\omega t}}}{\cancel{e^{j\omega t}}(6 - 5j\omega - \omega^2)} = \frac{3}{-\omega^2 - 5j\omega + 6}$$

$$\Rightarrow y(t) = \sum_{k=-\infty}^{+\infty} \underbrace{a_k H(jk\omega_0)}_{= b_k} e^{jk\omega_0 t}, \quad x(t) \xleftrightarrow{F.S} a_k = \underset{\text{duty cycle}}{d} \cdot \text{Sinc}(kd) = \frac{2}{5}\text{Sinc}\left(\frac{2k}{5}\right)$$

$$\boxed{\text{می دانیم: راه دوم} \quad y(t) = x(t) * h(t) \xleftrightarrow{F.T} Y(j\omega) = X(j\omega)H(j\omega)}$$

$$a_0 = \frac{2}{5} \qquad a_2 = \frac{2}{5}\text{Sinc}\left(\frac{4}{5}\right)$$
$$a_1 = \frac{2}{5}\text{Sinc}\left(\frac{2}{5}\right) \qquad a_3 = \frac{2}{5}\text{Sinc}\left(\frac{6}{5}\right)$$

$$\Rightarrow y(t) = \frac{2}{5}H(0)e^{j0} + \frac{2}{5}\left(\text{Sinc}\left(\frac{2}{5}\right)H(j\omega_0)e^{j\omega_0 t} + \text{Sinc}\left(\frac{4}{5}\right)H(2j\omega_0)e^{j\omega_0 t} + \dots\right)$$

#1

$$\begin{cases} 1)\ x[n] \text{ periodic with } N=10 \\ 2)\ x[n] \xleftrightarrow{F.S} a_k \\ 3)\ x[n] : \text{حقیقی و فرد} \\ 4)\ P = 8 \end{cases}$$

$5)\ 2|a_3| = |a_4|$

$0 < \measuredangle a_3 < \pi$

$0 < \measuredangle a_4 < \pi$

از ③ می دانیم : a_k ها موهومی خالص و فرد هستند

$$P = \frac{1}{N} \cdot \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2 = \sum_{k=0}^{10} |a_k|^2 = |a_0|^2 + |a_1|^2 + \dots + |a_{10}|^2$$

← فقط ضرایب a_3 و a_4 غیرصفر هستند.

$$\Rightarrow |a_3|^2 + |a_4|^2 \xrightarrow{(5)} |a_3|^2 + \left[2|a_3|\right]^2 = 5|a_3|^2 = 8 \Rightarrow |a_3|^2 = \frac{8}{5}$$

$$\begin{cases} |a_3| = 2\sqrt{\frac{2}{5}} \\ |a_4| = 4\sqrt{\frac{2}{5}} \end{cases} \xrightarrow{\text{طبق دیتا ③}} a_k = -a_{-k} \Rightarrow \begin{cases} a_3 = -a_{-3} = 2j\sqrt{\frac{2}{5}} \\ a_4 = -a_{-4} = 4j\sqrt{\frac{2}{5}} \end{cases}$$

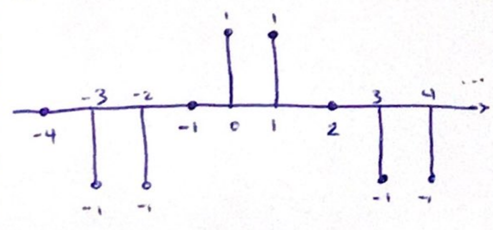
$$\Rightarrow x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{10}n} = \sum_{k=0}^{10} a_k e^{jk\frac{\pi}{5}n} = a_3 e^{j\frac{3\pi}{5}n} + a_4 e^{j\frac{4\pi}{5}n}$$

$$= 2j\sqrt{\frac{2}{5}}\, e^{j\frac{3\pi}{5}n} + 4j\sqrt{\frac{2}{5}}\, e^{j\frac{4\pi}{5}n} = x[n]$$

پریودیک با $N=10$

#2

$x[n] \xleftrightarrow{F.S} a_k = ?$



$$a_k = \frac{1}{N} \cdot \sum_{k=\langle N \rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \xrightarrow{N=6}$$

$$= \frac{1}{6} \cdot \sum_{k=-3}^{3} x[n] e^{-jk\frac{\pi}{3}n} = \frac{1}{6} \cdot \Big[x[-3] e^{+j\pi n} + x[-2] e^{j\frac{2\pi}{3}n}$$

$$+ \cancel{x[-1]}_{0}\, e^{j\frac{\pi}{3}n} + x[0] e^{0} + x[1] e^{-j\frac{\pi}{3}n} + \cancel{x[2]}_{0}\, e^{-j\frac{2\pi}{3}n} + x[3] e^{-j\pi n} \Big]$$

$$\Rightarrow a_k = \frac{1}{6} \cdot \left[-e^{j\pi n} - e^{j\frac{2\pi}{3}n} + 1 + e^{-j\frac{\pi}{3}n} - e^{-j\pi n} \right]$$

#3

$$\begin{cases} \xrightarrow{x[n]} \boxed{\text{علّی}} \xrightarrow{y[n]} \\ y[n] - \frac{1}{4}y[n-1] = x[n] \\ \text{if } \ x[n] = \cos\left(\frac{\pi}{4}n\right) + 3\sin\left(\frac{2\pi}{3}n\right) \Rightarrow y[n] \xleftrightarrow{F.S} a_k = ? \end{cases}$$

با فرض اینکه سیستم LTI است :

$$\xrightarrow{x[n]=e^{j\omega n}} \boxed{LTI} \xrightarrow{H(j\omega)e^{j\omega n}}$$

$\Rightarrow$ طبق معادله دیفرانسیل داده شده، اگر ورودی سیستم را $e^{j\omega n}$ فرض کنیم :

$$H(e^{j\omega})e^{j\omega n} - \frac{1}{4}H(e^{j\omega})e^{-j\omega} \cdot e^{j\omega n} = e^{j\omega n}$$

$$\Rightarrow H(e^{j\omega}) \cdot \left[e^{j\omega n} - \frac{1}{4}e^{-j\omega}e^{j\omega n}\right] = e^{j\omega n}$$

$$\Rightarrow H(e^{j\omega}) = \frac{e^{j\omega n}}{e^{j\omega n} - \frac{1}{4}e^{j\omega(n-1)}} \xrightarrow{\div e^{j\omega n}} H(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

$$y[n] = \sum_{k=\langle N \rangle} a_k H(e^{j\frac{2\pi k}{N}}) e^{jk\left(\frac{2\pi}{N}\right)n} \quad , \quad x[n] = \underbrace{\cos\left(\frac{\pi}{4}n\right)}_{N_1 = 8} + \underbrace{3\sin\left(\frac{2\pi}{3}n\right)}_{N_2 = 3}$$

$$\Rightarrow \text{KMM}\{N_1, N_2\} = \boxed{24}$$

$$x[n] = \underbrace{\frac{1}{2}}_{a_3} e^{j\frac{\pi}{4}n} + \underbrace{\frac{1}{2}}_{a_{-3}} e^{-j\frac{\pi}{4}n} + \underbrace{\frac{3}{2j}}_{a_8} e^{j\frac{2\pi}{3}n} - \underbrace{\frac{3}{2j}}_{a_{-8}} e^{-j\frac{2\pi}{3}n} \longrightarrow \left| \begin{array}{l} a_3 = a_{-3} = \frac{1}{2} \\ a_8 = \frac{3}{2j} \\ a_{-8} = -\frac{3}{2j} \end{array} \right.$$

$$b_3 = a_3 H(e^{j\frac{\pi}{4}}) = \frac{1}{2} \times \frac{1}{1 - \frac{1}{4}e^{-j\frac{\pi}{4}}} \quad , \quad b_{-3} = a_{-3} H(e^{-j\frac{\pi}{4}}) = \frac{1}{2} \times \frac{1}{1 - \frac{1}{4}e^{j\frac{\pi}{4}}}$$

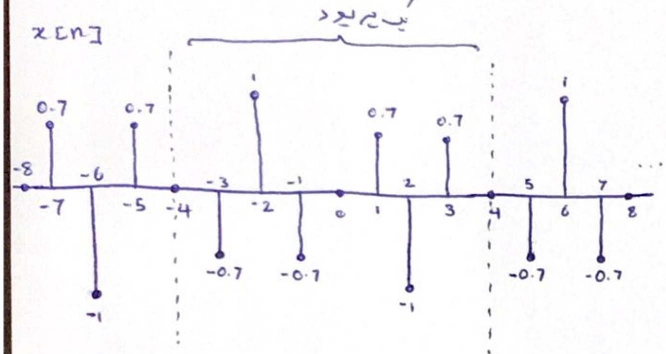
$$b_8 = a_8 H(e^{j\frac{2\pi}{3}}) = \frac{3}{2j} \times \frac{1}{1 - \frac{1}{4}e^{-j\frac{2\pi}{3}}} \quad , \quad b_{-8} = a_{-8} H(e^{j\frac{2\pi}{3}}) = \frac{-3}{2j} \times \frac{1}{1 - \frac{1}{4}e^{j\frac{2\pi}{3}}}$$

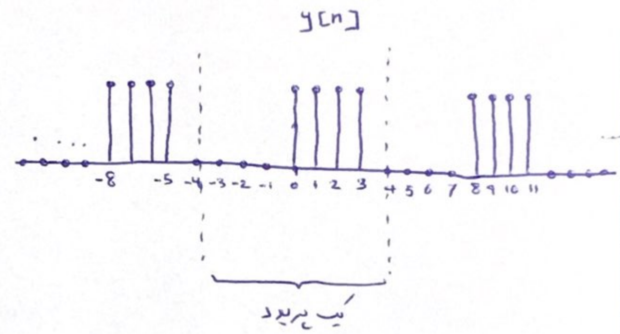
#4

$x[n] = \sin\left(\frac{3\pi}{4} n\right)$

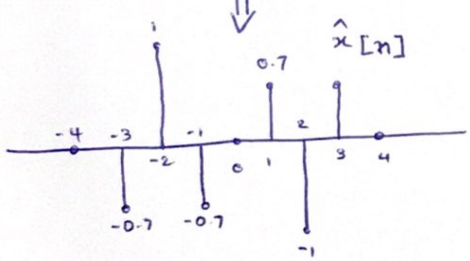
$$y[n] = \begin{cases} 1 & ; \quad 0 \le n \le 3 \\ 0 & ; \quad 4 \le n \le 7 \end{cases} \qquad \Rightarrow \quad x[n] \circledast y[n] = ?$$

$\underline{N = 8}$





$\Downarrow$



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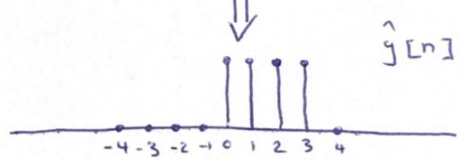
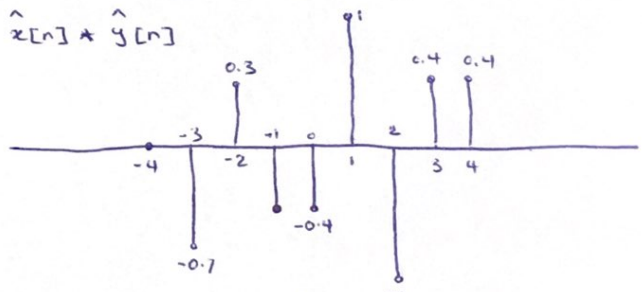
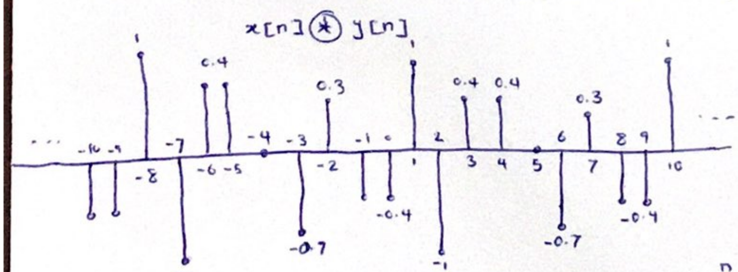


$$\hat{x}[n] * \hat{y}[n] = \sum_{k=-\infty}^{+\infty} \hat{x}[k]\,\hat{y}[n-k] \quad \Rightarrow$$



حاصل کانولوشن معمولی فقط در یک پریود <= حاصل کانولوشن پریودیک این دو سیگنال، متناوب شده سیگنال کانولوشن‌شده با پریود 8 است یعنی:



$$a_k = \frac{1}{8} \cdot \sum_{k=-4}^{4} x[n] e^{-jk\frac{2\pi}{8}n} = \frac{1}{8}\left[x[-4] e^{-jk\frac{\pi}{4}(-4)} + x[-3] e^{-jk\frac{\pi}{4}(-3)} + \cdots + x[3] e^{-jk\frac{\pi}{4}(3)} \right]$$

$$= \frac{1}{8}\left[-0.7 e^{-jk\frac{3\pi}{4}} + 0.3 e^{jk\frac{\pi}{2}} - 0.4 e^{jk\frac{\pi}{4}} - 1 + e^{-jk\frac{\pi}{4}} - e^{-jk\frac{\pi}{2}} + 0.4 e^{-jk\frac{3\pi}{4}} \right]$$