Linear Algebra

Home Work : 2

Reza Adinepour

S.t ID: 9814303

① Find the response of the following equation for $K = 15$ and $K = 14.9$. Is the system ill-condition? Find the condition number

$$AX = b \longrightarrow \begin{bmatrix} 8 & 5 & 2 \\ 21 & 19 & 16 \\ 39 & 48 & 53 \end{bmatrix} X = \begin{bmatrix} K \\ 56 \\ 140 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 5 & 2 \\ 21 & 19 & 16 \\ 39 & 48 & 53 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 56 \\ 140 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 5 & 2 \\ 21 & 19 & 16 \\ 39 & 48 & 53 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 14.9 \\ 56 \\ 140 \end{bmatrix}$$

$$X = A^{-1} \cdot b = \frac{1}{\det(A)} \cdot adj(A) \cdot b$$

$$\det(A) = \begin{vmatrix} 8 & 5 & 2 \\ 21 & 19 & 16 \\ 39 & 48 & 53 \end{vmatrix} = 8 \begin{vmatrix} 19 & 16 \\ 48 & 53 \end{vmatrix} - 5 \begin{vmatrix} 21 & 16 \\ 39 & 53 \end{vmatrix} + 2 \begin{vmatrix} 21 & 19 \\ 39 & 48 \end{vmatrix} = 8(1007 - 768) - 5(1113 - 624)$$

$$+ 2(1008 - 741) = 1$$

$adj(A)$:

$$a_{11} = 8 = \begin{vmatrix} 19 & 16 \\ 48 & 53 \end{vmatrix} = 239 \quad , \quad a_{12} = 5 = -\begin{vmatrix} 21 & 16 \\ 39 & 53 \end{vmatrix} = -489$$

$$a_{13} = 2 = \begin{vmatrix} 21 & 19 \\ 39 & 48 \end{vmatrix} = 267 \quad , \quad a_{21} = 21 = -\begin{vmatrix} 5 & 2 \\ 48 & 53 \end{vmatrix} = -169$$

$$a_{22} = 19 = \begin{vmatrix} 8 & 2 \\ 39 & 53 \end{vmatrix} = 346 \quad , \quad a_{23} = 16 = -\begin{vmatrix} 8 & 5 \\ 39 & 48 \end{vmatrix} = -189$$

$$a_{31} = 39 = \begin{vmatrix} 5 & 2 \\ 19 & 16 \end{vmatrix} = 42 \quad , \quad a_{32} = 48 = -\begin{vmatrix} 8 & 2 \\ 21 & 16 \end{vmatrix} = -86$$

$$a_{33} = 53 = \begin{vmatrix} 8 & 5 \\ 21 & 19 \end{vmatrix} = 47 \implies adj(A) = \begin{bmatrix} 239 & -169 & 42 \\ -489 & 346 & -86 \\ 267 & -189 & 47 \end{bmatrix}$$

$$\implies X = A^{-1} \cdot b = \frac{1}{1} \begin{bmatrix} 239 & -169 & 42 \\ -489 & 346 & -86 \\ 267 & -189 & 47 \end{bmatrix} \times \begin{bmatrix} 14.9 \\ 56 \\ 140 \end{bmatrix} = \begin{bmatrix} -22.9 \\ 49.9 \\ -25.7 \end{bmatrix}$$

if $k = 15$: $\begin{bmatrix} 8 & 5 & 2 \\ 21 & 19 & 16 \\ 39 & 48 & 53 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 56 \\ 140 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \cdot b = \begin{bmatrix} 239 & -169 & 42 \\ -489 & 346 & -86 \\ 267 & -189 & 47 \end{bmatrix} \begin{bmatrix} 15 \\ 56 \\ 140 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

X → for $k = 14.9$: $\begin{bmatrix} -22.9 \\ 49.9 \\ -25.7 \end{bmatrix}$

X → for $k = 15$: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ Large change in X for every 0.1 change in b

So The system is ill-condition

$$K = \|A\| \cdot \|A^{-1}\| \begin{cases} \|A\| = \sqrt{\lambda_{max}} = \sqrt{max(\det(A^TA - \lambda I)) = 0} \\ \|A^{-1}\| = \sqrt{\lambda_{min}} = \sqrt{min(\det(A^TA - \lambda I) = 0)} \end{cases}$$

$$A^T = \begin{bmatrix} 8 & 21 & 39 \\ 5 & 19 & 48 \\ 2 & 16 & 53 \end{bmatrix} \Rightarrow A^TA = \begin{bmatrix} 8 & 5 & 2 \\ 21 & 19 & 16 \\ 39 & 48 & 53 \end{bmatrix} \times \begin{bmatrix} 8 & 21 & 39 \\ 5 & 19 & 48 \\ 2 & 16 & 53 \end{bmatrix} = \begin{bmatrix} 2026 & 2311 & 2419 \\ 2311 & 2690 & 2858 \\ 2419 & 2856 & 3069 \end{bmatrix}$$

$$A^TA - \lambda I = \begin{bmatrix} 2026 & 2311 & 2419 \\ 2311 & 2690 & 2858 \\ 2419 & 2856 & 3069 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2026-\lambda & 2311 & 2419 \\ 2311 & 2690-\lambda & 2858 \\ 2419 & 2856 & 3069-\lambda \end{bmatrix}$$

$$\det(A^TA - \lambda I) = 0 : (2026-\lambda) \begin{vmatrix} 2690-\lambda & 2858 \\ 2856 & 3069-\lambda \end{vmatrix} - 2311 \begin{vmatrix} 2311 & 2858 \\ 2419 & 3069-\lambda \end{vmatrix} + 2419 \begin{vmatrix} 2311 & 2690-\lambda \\ 2419 & 2856 \end{vmatrix}$$

$$(2026-\lambda) \cdot (\lambda^2 - 5759\lambda + 93162) - 2311(-2311\lambda + 178957) + 2419(93106 + 2419\lambda)$$

$$= -\lambda^3 + 7785\lambda^2 - 11760896\lambda + 188746212 + 11192282\lambda - 188346213$$

$$= -\lambda^3 + 7785\lambda^2 - 568614\lambda + 399999 \longrightarrow \text{roots}([-1, 7785, -568614, 399999])$$

$$\begin{cases} \lambda_1 = 0.710 \\ \lambda_2 = 73.02 \\ \lambda_3 = 7711.26 \end{cases} \Rightarrow K = \|A\| \cdot \|A^{-1}\| = \sqrt{7711.26} \times \sqrt{0.710} = \boxed{74}$$

(2) Consider $M_{2\times2}(R)$ is a set of all 2x2 matrix with real elements. show this set form a vector space on R (real number).

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

1) $\forall\ A,B \in \mathbb{R} \Rightarrow A+B \in \mathbb{R}$:

$$A+B=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\in \mathbb{R} \ \checkmark$$

2) $\forall\ A\in\mathbb{R},\ \forall c \in F \Rightarrow cA \in \mathbb{R}$:

$$cA=c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} \checkmark$$

3) $\forall\ A,B \in \mathbb{R} \Rightarrow A+B=B+a$:

$$A+B=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}\, b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}=B+a=\begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{bmatrix} \checkmark$$

4) $\forall\ A,B,C \in \mathbb{R} \Rightarrow A+(B+C)=(A+B)+C$:

$$A+(B+C)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} \end{bmatrix}$$

$$(A+B)+C=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}+\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} \end{bmatrix} = \checkmark$$

5) $\forall\ A\in\mathbb{R},\ \exists\ \vec{o}\in\mathbb{R} \Rightarrow A+\vec{o}=\vec{o}+A$:

$$A+\vec{o}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \checkmark$$

6) $\forall\ A\in\mathbb{R},\ \exists\ -A\in\mathbb{R} \Rightarrow A+(-A)=(-A)+A=\vec{o}$:

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ -A=\begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} \Rightarrow A+(-A)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(-A)+A=\begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}+\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

7) $\forall\ A,B\in\mathbb{R},\ \forall\ a,b\in F \Rightarrow (a+b)\vec{A}=a\vec{A}+b\vec{A},\ a(A+B)=aA+aB$:

$$(a+b)\cdot\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=(a+b)\cdot\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}=\begin{bmatrix} (a+b)a_{11}+b_{11} & (a+b)a_{12}+b_{12} \\ (a+b)a_{21}+b_{21} & (a+b)a_{22}+b_{22} \end{bmatrix}$$

$$=\begin{bmatrix} a(a_{11}+b_{11})+b(a_{11}+b_{11}) & a(a_{12}+b_{12})+b(a_{12}+b_{12}) \\ a(a_{21}+b_{21})+b(a_{21}+b_{21}) & a(a_{22}+b_{22})+b(a_{22}+b_{22}) \end{bmatrix}=\begin{bmatrix} a(a_{11}+b_{11}) & a(a_{12}+b_{12}) \\ a(a_{21}+b_{21}) & a(a_{22}+b_{22}) \end{bmatrix}+$$

$$\begin{bmatrix} b(a_{11}+b_{11}) & b(a_{12}+b_{12}) \\ b(a_{21}+b_{21}) & b(a_{22}+b_{22}) \end{bmatrix} = a\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + b\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \checkmark$$

8) $\forall A \in \mathbb{R},\ \forall a,b \in F \Rightarrow a(b\vec{A}) = (ab)\vec{A}$:

$$a(b\vec{A}) = a\begin{bmatrix} ba_{11} & ba_{12} \\ ba_{21} & ba_{22} \end{bmatrix} = \begin{bmatrix} aba_{11} & aba_{12} \\ aba_{21} & aba_{22} \end{bmatrix},$$

$$= \checkmark$$

$$(ab)\vec{A} = ab\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} aba_{11} & aba_{12} \\ aba_{21} & aba_{22} \end{bmatrix}$$

9) $\forall A \in \mathbb{R},\ \exists 1 \in F \Rightarrow 1A = A$:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow 1A = 1 \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1a_{11} & 1a_{12} \\ 1a_{21} & 1a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \checkmark$$

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(3) Find the linear independent condition for the following vectors:

$$u = \begin{bmatrix} 1-\lambda \\ 2+\lambda \end{bmatrix}, \quad \bar{V} = \begin{bmatrix} 2+\lambda \\ 1-\lambda \end{bmatrix} \Rightarrow C_1 u + C_2 \bar{V} = 0$$

$$C_1 u = C_1\begin{bmatrix} 1-\lambda \\ 2+\lambda \end{bmatrix} = \begin{bmatrix} C_1(1-\lambda) \\ C_1(2+\lambda) \end{bmatrix}, \quad C_2\bar{V} = C_2\begin{bmatrix} 2+\lambda \\ 1-\lambda \end{bmatrix} = \begin{bmatrix} C_2(2+\lambda) \\ C_2(1-\lambda) \end{bmatrix}$$

$$\Rightarrow C_1 u + C_2\bar{V} = \begin{bmatrix} C_1 - C_1\lambda \\ 2C_1 + C_1\lambda \end{bmatrix} + \begin{bmatrix} 2C_2 + C_2\lambda \\ C_2 - C_2\lambda \end{bmatrix} = \begin{bmatrix} C_1 - C_1\lambda + 2C_2 + C_2\lambda \\ 2C_1 + C_1\lambda + C_2 - C_2\lambda \end{bmatrix} = 0$$

$$\begin{cases} C_1 - C_1\lambda + 2C_2 + C_2\lambda = 0 \\ 2C_1 + C_1\lambda + C_2 - C_2\lambda = 0 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 1-\lambda & 2+\lambda \\ 2+\lambda & 1-\lambda \end{bmatrix}}_{=A} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \det(A) \neq 0 : \begin{vmatrix} 1-\lambda & 2+\lambda \\ 2+\lambda & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - (2+\lambda)^2 = 1 - \cancel{\lambda^2} - 2\lambda - 4 - 2\lambda - \cancel{\lambda^2} \neq 0$$

$$\Rightarrow -6\lambda - 3 \neq 0 \Rightarrow \boxed{\lambda \neq -0.5}$$

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(4) check the basis condition for the following set. (for the vector space $M_{2\times2}$)

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\begin{cases} \text{a) Linear independent} \\ \text{b) span} \end{cases}$$

a) $c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}1&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}1&1\\1&0\end{bmatrix}+c_4\begin{bmatrix}1&1\\1&1\end{bmatrix}=0$

$\Rightarrow \begin{bmatrix}c_1&0\\0&0\end{bmatrix}+\begin{bmatrix}c_2&c_2\\0&0\end{bmatrix}+\begin{bmatrix}c_3&c_3\\c_3&0\end{bmatrix}+\begin{bmatrix}c_4&c_4\\c_4&c_4\end{bmatrix}=0$

$\Rightarrow \begin{bmatrix}c_1+c_2+c_3+c_4 & c_2+c_3+c_4\\ c_3+c_4 & c_4\end{bmatrix}=0$

$\Rightarrow \underbrace{\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}}_{=A}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \det(A)=1\neq 0 \Rightarrow$ Linear independent ✓

b) $c_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_2\begin{bmatrix}1&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}1&1\\1&0\end{bmatrix}+c_4\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}r_1&r_2\\r_3&r_4\end{bmatrix}$

$\Rightarrow \begin{bmatrix}c_1&0\\0&0\end{bmatrix}+\begin{bmatrix}c_2&c_2\\0&0\end{bmatrix}+\begin{bmatrix}c_3&c_3\\c_3&0\end{bmatrix}+\begin{bmatrix}c_4&c_4\\c_4&c_4\end{bmatrix}=\begin{bmatrix}r_1&r_2\\r_3&r_4\end{bmatrix}$

$\Rightarrow \begin{bmatrix}c_1+c_2+c_3+c_4 & c_2+c_3+c_4\\ c_3+c_4 & c_4\end{bmatrix}=\begin{bmatrix}r_1&r_2\\r_3&r_4\end{bmatrix}$

$\Rightarrow \underbrace{\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}}_{=A}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}r_1&r_2\\r_3&r_4\end{bmatrix} \Rightarrow \det(A)=1\neq 0 \Rightarrow$ Span ✓

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(5) Find The rank of the following Matrix:

$A=\begin{bmatrix}3&12&-1&-6\\6&24&-2&-12\\-3&-12&1&6\end{bmatrix}$, $A=\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\0&0&1&1\end{bmatrix}$, $A=\begin{bmatrix}1&1&a\\-a&-1&1\end{bmatrix}$, where $a\in\mathbb{R}$

a) $A=\begin{bmatrix}3&12&-1&-6\\6&24&-2&-12\\-3&-12&1&6\end{bmatrix} \Rightarrow R(A)\leq \min(3,4)=3 \Rightarrow$

$\begin{cases} 1\times1 \text{ کهاد} : ✓ \\ 2\times2 \text{ کهاد} : ✗ \\ \Rightarrow \boxed{R(A)=1} \end{cases}$

b) $A=\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\0&0&1&1\end{bmatrix} \Rightarrow R(A)\leq\min(3,4)=3 \Rightarrow$

$\begin{cases} 1\times1 \text{ کهاد} : ✓ \\ 2\times2 \text{ کهاد} : ✓ \Rightarrow \begin{bmatrix}1&1\\0&1\end{bmatrix} \\ 3\times3 \text{ کهاد} : ✗ \\ \Rightarrow \boxed{R(A)=2} \end{cases}$

c) $A=\begin{bmatrix}1&1&a\\-a&-1&1\end{bmatrix} \Rightarrow R(A)\leq\min(2,3)=2 \Rightarrow$

$\begin{cases} 1\times1 \text{ کهاد} : ✓ \\ 2\times2 \text{ کهاد} : \longrightarrow \begin{cases} 1+a>0 \Rightarrow a>-1 \\ -1+a>0 \Rightarrow a>+1 \\ 1+a^2>0 \Rightarrow a\in\mathbb{R} \end{cases} \end{cases}$

$\Rightarrow a>1 \Rightarrow \boxed{R(A)=2}$