تمرین 1 : ماتریس های زیر را در نظر بگیرید :

$$A = \begin{bmatrix} 6 & -2 \\ -4 & 3 \end{bmatrix} , \quad B = \begin{bmatrix} 8 & 1 \\ -2 & 1 \end{bmatrix} , \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

الف) نشان دهید A و B تحت تبدیل T همانند هستند.

ب) کدام خواص از ماتریس ها تحت تبدیل همانندی تغییر نمی کند ؟

الف) $T^{-1} = \frac{1}{\det(T)} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \frac{1}{2-0} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

باید $\begin{cases} A = TBT^{-1} \\ B = T^{-1}AT \end{cases}$

$$TBT^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 8 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -4 & 3 \end{bmatrix} = A$$

$$T^{-1}AT = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -2 & 1 \end{bmatrix} = B$$

ب) دترمینان ماتریس ، trace ، ترانهاده و ... تغییر نمی کند.

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تمرین 2 : فرم قطری ماتریس زیر را بدست آورید ؟

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix}_{4\times4}$$

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda+1 & 0 & 0 & 0 \\ -1 & \lambda+1 & 0 & 0 \\ 0 & -1 & \lambda & -1 \\ 0 & 0 & 2 & \lambda+2 \end{vmatrix} = 0 \Rightarrow (\lambda+1)(\lambda^3 + 3\lambda^2 + 4\lambda + 2) = 0$$

$$\begin{cases} \lambda_{1,2} = -1 \pm j \\ \lambda_3 = -1 \\ \lambda_4 = -1 \end{cases} \Rightarrow T = \left[ Re\{v_1\} \;\vdots\; Im\{v_1\} \;\vdots\; Re\{v_3\} \;\vdots\; Im\{v_3\} \;\vdots\; \dots v_m \dots v_{m+n} \right]$$

$$\Rightarrow T = \begin{bmatrix} -1 & 1 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \Lambda = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

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تمرین 3 : برای ماتریس های زیر $e^{At}$ را محاسبه کنید ؟

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} , \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

for A : $n=3 \rightarrow R(\lambda) = c^2\lambda^2 + c_1\lambda + c_0 \rightarrow \det(\lambda I - A) = 0 \rightarrow \begin{vmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = 0$

$\lambda_{1,2,3} = 1$

$R(\lambda) = f(\lambda)$

$$f(1) = c^2 + c_1 + c_0 = e^t$$
$$\dot{f}(1) = 2c^2 + c_1 = e^t \Rightarrow \begin{cases} c_2 + c_1 + c_0 = e^t \\ 2c_2 + c_1 + c_0 = e^t \\ 2c_2 = e^t \longrightarrow c_2 = \frac{1}{2}e^t \end{cases}$$
$$\ddot{f}(1) = 2c^2 = e^t$$

$$\begin{cases} \frac{1}{2}e^t + c_1 + c_0 = e^t \\ e^t + c_1 + c_0 = e^t \end{cases} \longrightarrow \begin{cases} -\frac{1}{2}e^t \cancel{-c_1} \cancel{-c_0} = \cancel{-e^t} \\ e^t \cancel{+c_1} \cancel{+c_0} = \cancel{e^t} \end{cases} \Rightarrow e^t - \frac{1}{2}e^t = 0 \quad ??$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \xrightarrow{n=3} \det(\lambda I - A) = 0 \quad \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = 0 \longrightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$\begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2 \\ \lambda_3 = -3 \end{cases} \longrightarrow R(\lambda_i) = f(\lambda_i) \Rightarrow \begin{cases} R(\lambda_1) = f(\lambda_1) \xrightarrow{\lambda_1 = -1} f(-1) = C_2 - C_1 + C_0 = e^{-t} \\ R(\lambda_2) = f(\lambda_2) \xrightarrow{\lambda_2 = -2} f(-2) = 4C_2 - 2C_1 + C_0 = e^{-2t} \\ R(\lambda_3) = f(\lambda_3) \xrightarrow{\lambda = -3} f(-3) = 9C_2 - 3C_1 + C_0 = e^{-3t} \end{cases}$$

$$R(\lambda) = C_2\lambda^2 + C_1\lambda + C_0$$

$$\longrightarrow C_1 = C_2 + C_0 - e^{-t} \Rightarrow \begin{cases} 4C_2 - 2(C_2 + C_0 - e^{-t}) + C_0 = e^{-2t} \\ 9C_2 - 3(C_2 + C_0 - e^{-t}) + C_0 = e^{-3t} \end{cases}$$

$$\Rightarrow \begin{cases} 2C_2 - C_0 = e^{-2t} - 2e^{-t} \\ 6C_2 - 2C_0 = e^{-3t} - 3e^{-t} \end{cases} \xrightarrow{\times -3} \begin{cases} \cancel{-6C_2} + 3C_0 = -3e^{-2t} + 6e^{-t} \\ \cancel{6C_2} - 2C_0 = e^{-3t} - 3e^{-t} \end{cases}$$

$$\Rightarrow \begin{cases} 3C_0 = -3e^{-2t} + 6e^{-t} \\ -2C_0 = e^{-3t} - 3e^{-t} \end{cases} \longrightarrow C_0 = -3e^{-2t} + 6e^{-t} + e^{-3t} - 3e^{-t}$$

$$\Rightarrow \boxed{C_0 = -3e^{-2t} + e^{-3t} + 3e^{-t}}$$

$$\Rightarrow C_2 = \frac{1}{2}\left(-3e^{-2t} + e^{-3t} + 3e^{-t} + e^{-2t} - 2e^{-t}\right) = \boxed{\frac{1}{2}\left(-2e^{-2t} + e^{-3t} + e^{-t}\right) = C_2}$$

$$\Rightarrow C_1 = C_2 + C_0 - e^{-t} \Rightarrow -3e^{-2t} + e^{-3t} + 3e^{-t} - e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t}$$

$$\Rightarrow \boxed{C_1 = \frac{3}{2}e^{-3t} - 4e^{-2t} + \frac{5}{2}e^{-t}}$$

$$\Rightarrow R(\lambda) = f(\lambda) \Rightarrow f(A) = R(A) = e^{At} = C_2A^2 + C_1A + C_0$$

$$= C_2\begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix} + C\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + C_0 = \begin{bmatrix} 3e^{-t} - 3e^{-2t} + e^{-3t} & \frac{11}{2}e^{-t} - 7e^{-2t} + \frac{5}{2}e^{-3t} & \cdots \\ -- & & -- \\ & -- & \\ & & -- \end{bmatrix}$$

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تمرین 4 : به کمک SVD وضعیت ماتریس زیر را مشخص کنید ؟

$$A = \begin{bmatrix} 100 & 0 & -100 \\ 0 & 100 & -100 \\ -100 & -100 & 300 \end{bmatrix}_{3\times 3}$$

$A^TA$ or $AA^T$ فرقی ندارد $\Rightarrow A^T = \begin{bmatrix} 100 & 0 & -100 \\ 0 & 100 & -100 \\ -100 & -100 & 300 \end{bmatrix} \Rightarrow AA^T = \begin{bmatrix} 20000 & 10000 & -40000 \\ 10000 & 20000 & -40000 \\ -40000 & -40000 & 110000 \end{bmatrix}$

$$\Rightarrow \det(\lambda I - AA^T) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 20000 & -10000 & +40000 \\ -10000 & \lambda - 20000 & 40000 \\ 40000 & 40000 & \lambda - 110000 \end{vmatrix} = \lambda^3 - 150000\lambda^2 + 1500000000\lambda - 1000000000000$$

$$\Rightarrow \begin{cases} \lambda_1 = 1.39 \times 10^5 \\ \lambda_2 = 0.1 \times 10^5 \\ \lambda_3 = 0.0072 \times 10^5 \end{cases} \Rightarrow K = \frac{S_{max}}{S_{min}} = \frac{\sqrt{1.39 \times 10^5}}{\sqrt{0.1 \times 10^5}} = 3.72$$

خوش حالت

تمرین 5 : ماتریس روبه رو را به مقادیر منفرد آن تجزیه کنید ؟ $A = \begin{bmatrix} 3 & 0 & 4 \\ -4 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}_{3\times 3}$

$$A_{m\times n} = U_{3\times3} \sum_{3\times3} V^T_{3\times3}$$

$$AA^T = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 9 \end{bmatrix} \rightarrow \det(\lambda I - AA^T) = 0 \Rightarrow \begin{vmatrix} \lambda-25 & 0 & 0 \\ 0 & \lambda-25 & 0 \\ 0 & 0 & \lambda-9 \end{vmatrix} = (\lambda-9)(\lambda-25)^2 = 0$$

$$\begin{cases} \lambda_1 = 25 \\ \lambda_2 = 25 \\ \lambda_3 = 9 \end{cases} \rightarrow (\lambda_i I - AA^T)u_1 \Rightarrow (\lambda_1 I - AA^T)u_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(\lambda_2 I - AA^T)u_2 \Rightarrow u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(\lambda_3 I - AA^T)u_3 \Rightarrow \begin{bmatrix} -16 & 0 & 0 \\ 0 & -16 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = u_3$$

$$\Rightarrow U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^TA = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} \rightarrow \det(\lambda I - A^TA) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-25 & 0 & 0 \\ 0 & \lambda-9 & 0 \\ 0 & 0 & \lambda-25 \end{vmatrix} \Rightarrow \begin{cases} \lambda_1 = 25 \\ \lambda_2 = 25 \\ \lambda_3 = 9 \end{cases} \Rightarrow V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sum_{3\times3} \rightarrow \det(\lambda I - A^TA) = 0 \quad \begin{cases} \lambda_1 = 25 \\ \lambda_2 = 25 \\ \lambda_3 = 9 \end{cases} \rightarrow \begin{cases} \delta_1 = 5 \\ \delta_2 = 5 \\ \delta_3 = 3 \end{cases} \Rightarrow \sum_{3\times3} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow A = U \sum V^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$