Solution _S2

1)

To solve this problem, we can use MATLAB:

```
k1=15;
k2=14.9;
A=[8 5 2;21 19 16;39 48 53];
b1=[k1;56;140];
b2=[k2;56;140];
x1=inv(A)*b1
x2=inv(A)*b2
cond(A)
```

```
x1 =

    1.0000
    1.0000
    1.0000


x2 =

  -22.9000
   49.9000
  -25.7000


Condition_number =

   6.5887e+04
```

The system is ill-condition since the condition number $\gg 1$.

2)

(Adopted from Applied LA published by K. N. Toosi University of Technology)

To show the vector space, we should check all the nine conditions as follows:

1. $\forall P,Q \in \mathrm{M}_{2\times 2}(\Re) \rightarrow \quad P+Q \in \mathrm{M}_{2\times 2}(\Re)$

$$P+Q=\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}+\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}=\begin{bmatrix} p_{11}+q_{11} & p_{12}+q_{12} \\ p_{21}+q_{21} & p_{22}+q_{22} \end{bmatrix}\in \mathrm{M}_{2\times 2}(\Re)$$

2. $\forall P \in \mathrm{M}_{2\times 2}(\Re), \quad \forall c \in \Re \rightarrow \quad cP \in \mathrm{M}_{2\times 2}(\Re)$

$$cP=c\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}=\begin{bmatrix} cp_{11} & cp_{12} \\ cp_{21} & cp_{22} \end{bmatrix}\in \mathrm{M}_{2\times 2}(\Re)$$

3. $\forall P,Q \in \mathrm{M}_{2\times 2}(\Re) \rightarrow \quad P+Q=Q+P$

$$P+Q=\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}+\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}=\begin{bmatrix} p_{11}+q_{11} & p_{12}+q_{12} \\ p_{21}+q_{21} & p_{22}+q_{22} \end{bmatrix}$$

$$=\begin{bmatrix} q_{11}+p_{11} & q_{12}+p_{12} \\ q_{21}+p_{21} & q_{22}+p_{22} \end{bmatrix}=\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}+\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}=Q+P$$

4. $\forall P, Q, R \in \mathrm{M}_{2\times 2}(\Re) \rightarrow \quad P + (Q + R) = (P + Q) + R$

$$P + (Q + R) = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \left(\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \right)$$

$$= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} q_{11} + r_{11} & q_{12} + r_{12} \\ q_{21} + r_{21} & q_{22} + r_{22} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} + q_{11} + r_{11} & p_{12} + q_{12} + r_{12} \\ p_{21} + q_{21} + r_{21} & p_{22} + q_{22} + r_{22} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} + q_{11} & p_{12} + q_{12} \\ p_{21} + q_{21} & p_{22} + q_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$= \left(\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \right) + \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = (P + Q) + R$$

5. $\forall P \in \mathrm{M}_{2\times 2}(\Re), \quad \exists \mathbf{O} \in \mathrm{M}_{2\times 2}(\Re) \rightarrow \qquad P + \mathbf{O} = \mathbf{O} + P = P$

$$P + \mathbf{O} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_{11} + 0 & p_{12} + 0 \\ p_{21} + 0 & p_{22} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + p_{11} & 0 + p_{12} \\ 0 + p_{21} & 0 + p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \mathbf{O} + P$$

6. $\forall P \in \mathrm{M}_{2\times 2}(\Re), \quad \exists -P \in \mathrm{M}_{2\times 2}(\Re) \rightarrow \quad P + (-P) = (-P) + P = \mathbf{O}$

$$P + (-P) = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} -p_{11} & -p_{12} \\ -p_{21} & -p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} - p_{11} & p_{12} - p_{12} \\ p_{21} - p_{21} & p_{22} - p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -p_{11} + p_{11} & -p_{12} + p_{12} \\ -p_{21} + p_{21} & -p_{22} + p_{22} \end{bmatrix} = \begin{bmatrix} -p_{11} & -p_{12} \\ -p_{21} & -p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$= (-P) + P = \mathbf{O}$$

7. $\forall P, Q \in \mathrm{M}_{2\times 2}(\Re), \quad \forall a, b \in \Re \rightarrow \quad (a+b)P = aP + bP, \quad a(P+Q) = aP + aQ$

$$(a+b)P = (a+b)\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a+b)p_{11} & (a+b)p_{12} \\ (a+b)p_{21} & (a+b)p_{22} \end{bmatrix} = \begin{bmatrix} ap_{11} + bp_{11} & ap_{12} + bp_{12} \\ ap_{21} + bp_{21} & ap_{22} + bp_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ap_{11} & ap_{12} \\ ap_{21} & ap_{22} \end{bmatrix} + \begin{bmatrix} bp_{11} & bp_{12} \\ bp_{21} & bp_{22} \end{bmatrix} = aP + bP$$

$$a(P+Q) = a\left(\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}\right) = a\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + a\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ap_{11} & ap_{12} \\ ap_{21} & ap_{22} \end{bmatrix} + \begin{bmatrix} aq_{11} & aq_{12} \\ aq_{21} & aq_{22} \end{bmatrix} = aP + bQ$$

8. $\forall P \in \mathrm{M}_{2\times 2}(\Re), \quad \forall a, b \in \Re \rightarrow \quad a(bP) = (ab)P$

$$a(bP) = a\left(b\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\right) = a\begin{bmatrix} bp_{11} & bp_{12} \\ bp_{21} & bp_{22} \end{bmatrix}$$

$$= \begin{bmatrix} abp_{11} & abp_{12} \\ abp_{21} & abp_{22} \end{bmatrix} = ab\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = (ab)P$$

9. $\forall P \in \mathrm{M}_{2\times 2}(\Re), \quad \exists 1 \in \Re \rightarrow \quad 1P = P$

$$1P = 1\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1\times p_{11} & 1\times p_{12} \\ 1\times p_{21} & 1\times p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = P$$

3)

To have the linear independent condition for $\boldsymbol{u}$ and $\boldsymbol{v}$, it is only required to obtain the determinant of the following matrix:

$$\begin{vmatrix} 1-\lambda & 2+\lambda \\ 2+\lambda & 1-\lambda \end{vmatrix} \neq 0$$

$$det = -3 - 6\lambda \neq 0 \quad \rightarrow \quad \lambda \neq \frac{1}{2}$$

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4)

To conditions should be checked: **a)** Linear independent and **b)** the Span.

a)

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} c_1 + c_2 + c_3 + c_4 & c_2 + c_3 + c_4 \\ c_3 + c_4 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

b)

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} c_1 + c_2 + c_3 + c_4 & c_2 + c_3 + c_4 \\ c_3 + c_4 & c_4 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$$

$$(reminder: Session\ 6) \rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Thus, two conditions are realized and the mentioned set forms a basis for $M_{2\times2}$.

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5)

$\boldsymbol{R(A) \leq min(m, n)}$

$$A = \begin{bmatrix} 3 & 12 & -1 & -6 \\ 6 & 24 & -2 & -12 \\ -3 & -12 & 1 & 6 \end{bmatrix}$$

$1 \times 1 \quad \rightarrow \quad Yes$

$2 \times 2 \quad \rightarrow \quad No$

Thus, the rank of A is one.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$1 \times 1 \quad \rightarrow \quad Yes$

$2 \times 2 \quad \rightarrow \quad Yes$

$3 \times 3 \quad \rightarrow \quad No$

Thus, the rank of A is two.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & a \\ -a & -1 & 1 \end{pmatrix}, \ where\ a \in \mathbb{R}.$$

$1 \times 1 \quad \rightarrow \quad Yes$

$2 \times 2 \quad \rightarrow \quad ?$

$$\begin{vmatrix} 1 & a \\ -a & 1 \end{vmatrix} = a^2 + 1$$

For any $a \in R$, the determinant of the above matrix is not zero. Thus, the rank of A is two.
