Subject: HW_08 Signal

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۱- تبدیل فوریه

a) $(\frac{1}{2})^{n-1}u[n] \rightarrow 0 \rightarrow +\infty$ $\quad n-1=m$

$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} (\frac{1}{2})^{n-1} e^{-j\omega n} = \sum_{\substack{m+1=0 \\ m=-1}}^{\infty} (\frac{1}{2})^{m} e^{-j\omega(m+1)} = e^{-j\omega} \times \frac{1}{1-(\frac{1}{2})e^{-j\omega}}$

b) $(\frac{1}{2})^{|n-1|} \rightarrow X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} (\frac{1}{2})^{|n-1|} e^{-j\omega n} = \sum_{n=-\infty}^{0} (\frac{1}{2})^{-(n-1)} e^{-j\omega n}$

$+ \underbrace{\sum_{n=1}^{+\infty} (\frac{1}{2})^{n-1} e^{-j\omega n}}_{a \Rightarrow \frac{e^{-j\omega}}{1-(\frac{1}{2})e^{-j\omega}}} = \rightarrow \sum_{n=-\infty}^{0} (\frac{1}{2})^{-(n-1)} e^{-j\omega n} = \sum_{m=0}^{+\infty} (\frac{1}{2})^{m+1} e^{j\omega m} = \frac{1}{2} \times \frac{1}{1-(\frac{1}{2})e^{j\omega}}$

$\hookrightarrow -n=m$

$\Rightarrow \frac{e^{-j\omega}}{1-(\frac{1}{2})e^{-j\omega}} + \frac{1}{2} \times \frac{1}{1-(\frac{1}{2})e^{j\omega}} = \frac{\frac{3}{4}e^{-j\omega}}{\frac{5}{4}-\cos\omega}$ ✓

c) $\delta(n-1)+2\delta(n+2)$

$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \delta[n-1]e^{-j\omega n} + 2\delta[n+2]e^{-j\omega n} =$

$\hookrightarrow \delta[n-n_0]x[n] = \delta[n-n_0]x[n_0]$

$\boxed{1 \times e^{-j\omega} + 2 \times e^{+2j\omega} = e^{-j\omega} + 2e^{2j\omega}}$ ✓

d) $(\frac{1}{2})^{-n}u[-n-1] = X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{-1} (\frac{1}{2})^{-n} e^{-j\omega n} = \sum_{m=1}^{+\infty} (\frac{1}{2})^{m} e^{j\omega m}$ $\quad -n=m$

$\sum_{m=1}^{+\infty} (\frac{1}{2}e^{j\omega})^m = \frac{e^{j\omega}}{2} \times \frac{1}{1-\frac{1}{2}e^{j\omega}}$

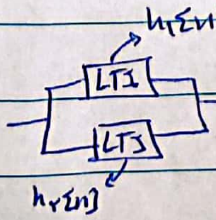
e) $\sin[n\frac{\pi}{2}] + \cos[n] = \frac{1}{2j}(e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}) + \frac{1}{2}(e^{jn} + e^{-jn})$

$X(\omega) = \frac{\pi}{j}(\delta(\omega-\frac{\pi}{2}) - \delta(\omega+\frac{\pi}{2})) + \pi(\delta(\omega-1) - \delta(\omega+1))]$

۲-

$h_1[n] = (\frac{1}{3})^n u[n]$

$h_2[n] = b[n]$



$\rightarrow H(\omega) = \frac{-12+5e^{j\omega}}{12-7e^{-j\omega}+e^{-2j\omega}}$

$b[n] = h_2[n] = ?$

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اگر دو سیستم با هم موازی شوند پاسخ ضربه سیستم معادل برابر با جمع پاسخ ضربه های دو سیستم است

$h_t[n] = h_1[n] + h_2[n]$

$\rightarrow H_t(\omega) = H_1(\omega) + H_2(\omega)$ $\qquad h_1[n] = (\frac{1}{2})^n u[n] \xrightarrow{\mathcal{F}} H_1(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$H_2(\omega) = H_t(\omega) - H_1(\omega) = \frac{-12+5e^{+j\omega}}{-12-7e^{-j\omega}+e^{-2j\omega}} - \frac{1}{1-\frac{1}{2}e^{-j\omega}} =$$

$e^{-j\omega} = s$

$e^{j\omega} = t$

$$\Rightarrow \frac{-12+5t}{-12-7s+s^2} - \frac{1}{1-\frac{1}{2}s} = \frac{-3}{1-\frac{1}{9}e^{-j\omega}}$$

$$\boxed{h_2[n] = b[n] = -3(\frac{1}{9})^n u[n]}$$

۲- سیستم علی و پایدار LTI با معادله ⇐ $y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$

الف) $H(\omega)$ ؟

ب) $h[n]$ ؟

$$\xrightarrow{\mathcal{F}} Y(\omega) - \frac{1}{6}Y(\omega)e^{-j\omega} - \frac{1}{6}Y(\omega)e^{-2j\omega} = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}} = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})} \quad e^{-j\omega} = s$$

$$\frac{1}{(1-\frac{1}{2}s)(1+\frac{1}{3}s)} = \frac{A}{1-\frac{s}{2}} + \frac{B}{1+\frac{s}{3}} \Rightarrow A = (1-\frac{s}{2})H(\omega)\Big|_{s=2} = \frac{1}{\frac{5}{3}} = \boxed{\frac{3}{5}}$$

$$B = (1+\frac{s}{3})H(\omega)\Big|_{s=-3} \Rightarrow \frac{1}{\frac{5}{2}} = \boxed{\frac{2}{5}} \Rightarrow \frac{\frac{3}{5}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{2}{5}}{1+\frac{1}{3}e^{-j\omega}}$$

$$h[n] = \frac{3}{5}(\frac{1}{2})^n u[n] + \frac{2}{5}(-\frac{1}{3})^n u[n]$$

٤- سیستم LTI و پایدار $\left(\frac{4}{5}\right)^n u[n] \to n\left(\frac{4}{5}\right)^n u[n]$

الف) $H(\omega) = ?$

ب) معادله‌ی توصیف کننده سیستم؟

$x[n] \to \boxed{LTI} \to y[n]$

$$x[n] = \left(\frac{4}{5}\right)^n u[n] \to X(\omega) = \frac{1}{1-\frac{4}{5}e^{-j\omega}}$$

$$y[n] = n\left(\frac{4}{5}\right)^n u[n] \qquad Y(\omega) = j\frac{d}{d\omega}X(\omega) = \frac{\left(\frac{4}{5}\right)e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\dfrac{\left(\frac{4}{5}\right)e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2}}{\dfrac{1}{1-\frac{4}{5}e^{-j\omega}}} = \frac{\left(\frac{4}{5}\right)e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}}$$

$$Y(\omega)\left(1-\frac{4}{5}e^{-j\omega}\right) = X(\omega)\left(\frac{4}{5}e^{-j\omega}\right) \to y[n] - \frac{4}{5}y[n-1] = \frac{4}{5}x[n] \checkmark$$

٥- سیستم LTI:

$h[n] = \left(\frac{1}{2}\right)^n u[n]$ با استفاده از خواص پاسخ سیستم به ورودی‌های زیر؟

ابتدا تبدیل فوریه پاسخ ضربه یا همان $H(\omega)$ را پیدا می‌کنیم

1) $x[n] = \left(\frac{3}{4}\right)^n u[n]$ $\qquad H(\omega) = \frac{Y(\omega)}{X(\omega)}$

۲) $(n+1)\left(\frac{1}{4}\right)^n u[n]$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{\mathcal{F}} H(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{Y(\omega)}{X(\omega)} \Rightarrow$$

$$\left.\begin{array}{l} Y(\omega) - \frac{1}{2}Y(\omega)e^{-j\omega} = X(\omega) \to \boxed{y[n] - \frac{1}{2}y[n-1] = x[n]} \end{array}\right\}$$ معادله‌ی سیستم

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{1-\frac{3}{4}e^{-j\omega}} \times \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{\left(1-\frac{3}{4}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$e^{-j\omega} = S \Rightarrow \frac{1}{\left(1-\frac{3}{4}S\right)\left(1-\frac{1}{2}S\right)} \Rightarrow \frac{A}{\left(1-\frac{3}{4}S\right)} + \frac{B}{\left(1-\frac{1}{2}S\right)}$$

$$A = Y(\omega)\left(1-\frac{3}{4}S\right)\Big|_{S=\frac{4}{3}} = \boxed{3} \qquad B = Y(\omega)\left(1-\frac{1}{2}S\right)\Big|_{S=2} = \boxed{-2}$$

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$$Y(\omega) = \frac{3}{1-\frac{3}{4}e^{-j\omega}} - \frac{2}{1-\frac{1}{2}e^{-j\omega}} \xrightarrow{\mathcal{F}^{-1}} y[n] = 3\left(\tfrac{3}{4}\right)^n u[n] - 2\left(\tfrac{1}{2}\right)^n u[n] \quad \checkmark$$

$$x[n] = (n+1)\left(\tfrac{1}{4}\right)^n u[n] \rightarrow n\left(\tfrac{1}{4}\right)^n u[n] + \left(\tfrac{1}{4}\right)^n u[n] \xrightarrow{\mathcal{F}}$$

$$X(\omega) = j\frac{d}{d\omega}\left(\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right) + \frac{1}{1-\frac{1}{4}e^{-j\omega}} = \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

$$Y(\omega) = H(\omega)\cdot X(\omega) \Rightarrow \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} \times \frac{1}{1-\frac{1}{2}e^{-j\omega}} \Rightarrow e^{-j\omega} = S$$

$$\frac{1}{\left(1-\frac{1}{4}S\right)^2\left(1-\frac{1}{2}S\right)} = \frac{A}{\left(1-\frac{1}{4}S\right)} + \frac{B}{\left(1-\frac{1}{4}S\right)^2} + \frac{C}{\left(1-\frac{1}{2}S\right)}$$

$$A = \frac{d}{dS}\left(\left(1-\tfrac{1}{4}S\right)^2 (Y(\omega))\right)\Big|_{S=4} = \boxed{-2}$$

$$C = \left(1-\tfrac{1}{2}S\right)Y(\omega)\Big|_{S=2} \Rightarrow \boxed{4}$$

$$B = \left(1-\tfrac{1}{4}S\right)^2 Y(\omega)\Big|_{S=4} = \boxed{-1}$$

$$Y(\omega) = \frac{-2}{1-\frac{1}{4}e^{-j\omega}} \; \frac{-1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{4}{1-\frac{1}{2}e^{-j\omega}} \xrightarrow{\mathcal{F}^{-1}} y[n] = \left[-2\left(\tfrac{1}{4}\right)^n - (n+1)\left(\tfrac{1}{4}\right)^n + 4\left(\tfrac{1}{2}\right)^n\right]u[n]$$

۴- سیستم LTI علّی:

$$y[n] + \tfrac{1}{2}y[n-1] = x[n]$$

الف) $H(\omega)$؟

ب) پاسخ ورودی های

$$x[n] = \left(-\tfrac{1}{2}\right)^n u[n]$$

$$x[n] = \delta[n] - \tfrac{1}{2}\delta[n-1]$$

$$\xrightarrow{\mathcal{F}} Y(\omega) + \tfrac{1}{2}Y(\omega)e^{-j\omega} = X(\omega) \rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

$$X_1(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \rightarrow Y(\omega) = H(\omega)X(\omega) \rightarrow \frac{1}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{2}e^{-j\omega}\right)} \quad e^{-j\omega} = S$$

$$Y(\omega) = \frac{1}{\left(1+\frac{1}{2}S\right)\left(1-\frac{1}{2}S\right)} \Rightarrow \frac{A}{\left(1+\frac{1}{2}S\right)} + \frac{B}{\left(1-\frac{1}{2}S\right)} \qquad A = \left(1+\tfrac{1}{2}S\right)Y(\omega) = \Big|_{S=-2} = \boxed{\tfrac{1}{2}}$$

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$$B = (1-\tfrac{1}{2}S)\,Y(\omega)\Big|_{S=2} = \tfrac{1}{2} \qquad \frac{\frac{1}{2}}{(1+\frac{1}{2}e^{-j\omega})} + \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\omega}} =$$

$$y[n] = \tfrac{1}{2}(-\tfrac{1}{2})^n u[n] + \tfrac{1}{2}(\tfrac{1}{2})^n u[n]$$

$$Y(\omega) = H(\omega)\,X_2(\omega) \;\rightarrow\; X_2(\omega) = 1-\tfrac{1}{2}e^{-j\omega}$$

$$\frac{1}{1+\frac{1}{2}e^{-j\omega}} \times (1-\tfrac{1}{2}e^{-j\omega}) = \frac{2}{1+\frac{1}{2}e^{-j\omega}} - 1 \xRightarrow{\mathcal{F}^{-1}} y[n] = -\delta[n] + 2(-\tfrac{1}{2})^n u[n] \;\checkmark$$

$\rightarrow \boxed{LTI_1} \rightarrow \boxed{LTI_2} \rightarrow$ -✓

$h_1[n]$ $\quad h_2[n]$ $\quad \rightarrow h_3[n] = ?$ معادله‌ی توصیف کننده

$$H_1(\omega) = \frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$$

$$H_3(\omega) = H_1(\omega)\,H_2(\omega)$$

$$H_2(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-2j\omega}}$$

$$y[n] + \tfrac{1}{8}y[n-3] = 2x[n] - x[n-1] \;\checkmark$$

$$H_3(\omega) = \frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}} \times \frac{1}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-2j\omega}} \Rightarrow e^{-j\omega} = S$$

$$\frac{2-S}{(1+\frac{1}{2}S)(1-\frac{1}{2}S+\frac{1}{4}S^2)} = \frac{2-S}{1-\frac{1}{8}S^3} = \frac{2-e^{-j\omega}}{1-\frac{1}{8}e^{-3j\omega}} \Rightarrow H(\omega) = \frac{2-S}{(1-\frac{1}{8}S^3)}$$

$$\frac{2-S}{\frac{1}{8}S(\frac{8}{S}-S^2)} = \frac{A}{S} + \frac{B}{8-S^3} \Rightarrow A = S\,H(\omega)\Big|_{S=0}$$

$$B = (\tfrac{8}{S}-S^2)\,H(\omega)\Big|_{S=2} = 0 \quad \times$$