MATH 242 – Advanced Linear Algebra

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Prof. Johnson

Part (a): Problem 1

Vector Spaces and Subspaces

Let V be a vector space over a field F. Prove the following properties:

1) The zero vector 0_V is unique.
2) For each $v \in V$, the additive inverse $-v$ is unique.
3) If $a \in F$ and $a \cdot v = 0_V$ for some $v \in V$, then either $a = 0$ or $v = 0_V$.

1. Uniqueness of the zero vector:

Theorem 1 *Uniqueness of Zero Vector.* In any vector space V, the zero vector 0_V is unique.

Suppose there exist two zero vectors, 0_V and $0'_V$. By definition of a zero vector: $0_V + 0'_V = 0'_V$ (since 0_V is a zero vector) $0_V + 0'_V = 0_V$ (since $0'_V$ is a zero vector)

Therefore, $0_V = 0'_V$, proving that the zero vector is unique.

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2. Uniqueness of the additive inverse:

Lemma 1 *Uniqueness of Additive Inverse.* For each vector v in a vector space V, the additive inverse $-v$ is unique.

Suppose $v \in V$ has two additive inverses, w and w'. Then: $v + w = 0_V$ and $v + w' = 0_V$

Adding w to both sides of the second equation:

$w + (v + w') = w + 0_V$ $(w + v) + w' = w$ $0_V + w' = w$ $w' = w$

Therefore, the additive inverse is unique.

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Part (b): Problem 2

Comparison of Different Vector Space Properties

Compare and contrast the following vector spaces.

Definition 1 *Real Vector Spaces.* A vector space over the field of real numbers $\mathbb{R}$.

Properties:

- Contains real-valued vectors
- Operations: addition and scalar multiplication by real numbers
- Examples: $\mathbb{R}^n$, continuous functions on an interval

Definition 2 *Complex Vector Spaces.* A vector space over the field of complex numbers $\mathbb{C}$.

Properties:

- Contains complex-valued vectors
- Operations: addition and scalar multiplication by complex numbers
- Examples: $\mathbb{C}^n$, analytic functions

Part (c): Problem 3

Linear Transformations

Explore the properties of linear transformations between vector spaces.

Theorem 2 *Rank-Nullity Theorem.* Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Then:

$\dim(\ker(T)) + \mathrm{di}(\mathrm{im}(T)) = \dim(V)$

Let $K = \ker(T)$ and let $\{v_1, v_2, ..., v_k\}$ be a basis for K.

Extend this to a basis $\{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ for V.

We claim that $\{T(v_{k+1}), ..., T(v_n)\}$ is a basis for $\mathrm{im}(T)$.

For linear independence, suppose $\sum_{i=k+1}^{n} a_i T(v_i) = 0$. Then $T\left(\sum_{i=k+1}^{n} a_i v_i\right) = 0$, which means $\sum_{i=k+1}^{n} a_i v_i in K$.

This implies $\sum_{i=k+1}^{n} a_i v_i = \sum_{j=1}^{k} b_j v_j$ for some scalars b_j.

By the linear independence of the basis of V, all coefficients must be zero. So $\{T(v_{k+1}), ..., T(v_n)\}$ is linearly independent.

For spanning, any $w \in \mathrm{im}(T)$ can be written as $w = T(v)$ for some $v \in V$. We can write $v = \sum_{i=1}^{n} c_i v_i$. Since $T(v_1) = ... = T(v_k) = 0$, we have $w = \sum_{i=k+1}^{n} c_i T(v_i)$.

Thus, $\dim(\mathrm{im}(T)) = n - k = \dim(V) - \dim(\ker(T))$.

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Corollary 1 *Injective Case.* If T is injective, then $\ker(T) = \{0\}$, so $\dim(\mathrm{im}(T)) = \dim(V)$.

This means T preserves dimension.

Corollary 2 *Surjective Case.* If T is surjective, then $\mathrm{im}(T) = W$, so $\dim(\ker(T)) = \dim(V) - \dim(W)$. If $\dim(V) < \dim(W)$, then T cannot be surjective.