MATH 242: Advanced Linear Algebra – Problem Set 3

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Part (a): Problem 1

Vector Spaces

Let V be a vector space over a field F. Prove the following properties:

1) The zero vector 0_V is unique.
2) For each $v \in V$, the additive inverse $-v$ is unique.
3) If $a \in F$ and $a \cdot v = 0_V$ for some $v \in V$, then either $a = 0$ or $v = 0_V$.

1. Uniqueness of the zero vector:

Suppose there exist two zero vectors, 0_V and $0'_V$. By definition of a zero vector, we have: $0_V + 0'_V = 0'_V$ (since 0_V is a zero vector) $0_V + 0'_V = 0_V$ (since $0'_V$ is a zero vector)

Therefore, $0_V = 0'_V$, proving that the zero vector is unique.

2. Uniqueness of the additive inverse:

Suppose $v \in V$ has two additive inverses, w and w'. Then: $v + w = 0_V$ and $v + w' = 0_V$

Adding w to both sides of the second equation,

$$\begin{aligned} w + (v + w') &= w + 0_V \\ (w + v) + w' &= w \\ 0_V + w' &= w \\ w' &= w \end{aligned} \tag{1}$$

Therefore, the additive inverse is unique.

3. Zero product property:

Suppose $a \in F$, $v \in V$, and $a \cdot v = 0_V$.

If $a = 0$, we're done. Otherwise, $a \neq 0$, which means a has a multiplicative inverse a^{-1} in F.

Multiplying both sides by a^{-1},

$$\begin{aligned} a^{-1} \cdot (a \cdot v) &= a^{-1} \cdot 0_V \\ (a^{-1} \cdot a) \cdot v &= 0_V \\ 1 \cdot v &= 0_V \\ v &= 0_V \end{aligned} \tag{2}$$

Thus, either $a = 0$ or $v = 0_V$.

Part (b): Problem 2

Linear Transformations

Let $T : R^3 \to R^2$ be a linear transformation defined by: $T(x, y, z) = (2x - y + 3z, 4x + 2y - z)$

1) Find the standard matrix representation of T.
2) Determine $\ker(T)$ and $\text{im}(T)$.
3) Verify that $\dim(\ker(T)) + \dim(\text{im}(T)) = \dim(R^3)$.

1. Standard matrix representation:

For a linear transformation $T : R^3 \to R^2$, the standard matrix is found by applying T to each standard basis vector of R^3 and using the results as columns.

$$\begin{aligned} T(1,0,0) &= (2(1) - 0 + 3(0), 4(1) + 2(0) - 0) = (2,4) \\ T(0,1,0) &= (2(0) - 1 + 3(0), 4(0) + 2(1) - 0) = (-1,2) \\ T(0,0,1) &= (2(0) - 0 + 3(1), 4(0) + 2(0) - 1) = (3,-1) \end{aligned} \tag{3}$$

Therefore, the standard matrix representation is:

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \end{pmatrix} \tag{4}$$

```python
def matrix_multiply(A, v):
    """
    Multiply matrix A by vector v
    """
    result = []
    for row in A:
        product = sum(a_i * v_i for a_i, v_i in zip(row, v))
        result.append(product)
    return result

# Example usage
A = [[2, -1, 3], [4, 2, -1]]
v = [1, 2, 3]
print(matrix_multiply(A, v))  # Should output [7, 6]
```

2. Determining $\ker(T)$ and $\text{im}(T)$:

For $\ker(T)$, we solve $T(x, y, z) = (0, 0)$,

$$\begin{aligned} 2x - y + 3z &= 0 \\ 4x + 2y - z &= 0 \end{aligned} \tag{5}$$

From the second equation: $z = 4x + 2y$ Substituting into the first equation,

$$\begin{aligned} 2x - y + 3(4x + 2y) &= 0 \\ 2x - y + 12x + 6y &= 0 \\ 14x + 5y &= 0 \\ y &= -\frac{14x}{5} \end{aligned} \tag{6}$$

So, $\ker(T) = \{\left(t, -\frac{14t}{5}, \frac{-8t}{5}\right) \mid t \in \mathbb{R}\}$

For $\operatorname{im}(T)$, we analyze the column space of the matrix,

$$\operatorname{span}((2,4),(-1,2),(3,-1)) \tag{7}$$

Since we can find two linearly independent columns, $\operatorname{im}(T) = R^2$.

3. Verification of dimension equation:

$$\begin{aligned} \dim(\ker(T)) &= 1(\text{it's a line in } \mathbb{R}^3) \\ \dim(\operatorname{im}(T)) &= 2 \text{ (it's } \mathbb{R}^2) \\ \dim(\ker(T)) + \dim(\operatorname{im}(T)) &= 1 + 2 = 3 = \dim(R^3) \end{aligned} \tag{8}$$

This verifies the rank-nullity theorem.