

SDE 101 – Stochastic Differential Equations and (Deep) Generative Models

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Definition 1 *Definition of SDEs.*

A stochastic differential equation (SDE) is an equation that describes the evolution of a random process over time.

The general form of an SDE can be written as,

$$dx(t) = \underbrace{(x(t), t) dt}_{\text{drift}} + \underbrace{L(x(t), t) d\beta(t)}_{\text{diffusion}}, \quad (1)$$

where,

- $x(t)$ is the random process,
- $f(x(t), t)$ is the drift term (deterministic part),
- $L(x(t), t)$ is the diffusion term (stochastic part),
- $d\beta(t)$ is standard Brownian motion (Wiener process),

Part (a): Problem 1**Vector Spaces and Subspaces**

Let V be a vector space over a field F . Prove the following properties:

- 1) The zero vector 0_V is unique.
- 2) For each $v \in V$, the additive inverse $-v$ is unique.
- 3) If $a \in F$ and $a \cdot v = 0_V$ for some $v \in V$, then either $a = 0$ or $v = 0_V$.

1. Uniqueness of the zero vector:**Theorem 1** *Uniqueness of Zero Vector.*

In any vector space V , the zero vector 0_V is unique.

Suppose there exist two zero vectors, 0_V and $0'_V$. By definition of a zero vector: $0_V + 0'_V = 0'_V$ (since 0_V is a zero vector) $0_V + 0'_V = 0_V$ (since $0'_V$ is a zero vector)

Therefore, $0_V = 0'_V$, proving that the zero vector is unique. ■

2. Uniqueness of the additive inverse:**Lemma 1** *Uniqueness of Additive Inverse.*

For each vector v in a vector space V , the additive inverse $-v$ is unique.

Suppose $v \in V$ has two additive inverses, w and w' . Then: $v + w = 0_V$ and $v + w' = 0_V$

Adding w to both sides of the second equation:

$$w + (v + w') = w + 0_V \quad (w + v) + w' = w \quad 0_V + w' = w \quad w' = w$$

Therefore, the additive inverse is unique. ■

Part (b): Problem 2**Comparison of Different Vector Space Properties**

Compare and contrast the following vector spaces.

Definition 2 *Real Vector Spaces.*

A vector space over the field of real numbers \mathbb{R} .

Properties:

- Contains real-valued vectors
- Operations: addition and scalar multiplication by real numbers
- Examples: \mathbb{R}^n , continuous functions on an interval

Definition 3 *Complex Vector Spaces.*

A vector space over the field of complex numbers \mathbb{C} .

Properties:

- Contains complex-valued vectors
- Operations: addition and scalar multiplication by complex numbers
- Examples: \mathbb{C}^n , analytic functions

Part (c): Problem 3**Linear Transformations**

Explore the properties of linear transformations between vector spaces.

Theorem 2 Rank-Nullity Theorem.

Let $T : V \rightarrow W$ be a linear transformation between finite-dimensional vector spaces. Then:
 $\dim(\ker(T)) + \dim(\text{im}(T)) = \dim(V)$

Let $K = \ker(T)$ and let $\{v_1, v_2, \dots, v_k\}$ be a basis for K .

Extend this to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V .

We claim that $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $\text{im}(T)$.

For linear independence, suppose $\sum_{i=k+1}^n a_i T(v_i) = 0$. Then $T(\sum_{i=k+1}^n a_i v_i) = 0$, which means $\sum_{i=k+1}^n a_i v_i \in K$.

This implies $\sum_{i=k+1}^n a_i v_i = \sum_{j=1}^k b_j v_j$ for some scalars b_j .

By the linear independence of the basis of V , all coefficients must be zero. So $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly independent.

For spanning, any $w \in \text{im}(T)$ can be written as $w = T(v)$ for some $v \in V$. We can write $v = \sum_{i=1}^n c_i v_i$. Since $T(v_1) = \dots = T(v_k) = 0$, we have $w = \sum_{i=k+1}^n c_i T(v_i)$.

Thus, $\dim(\text{im}(T)) = n - k = \dim(V) - \dim(\ker(T))$.

■

Corollary 1 Injective Case.

If T is injective, then $\ker(T) = \{0\}$, so $\dim(\text{im}(T)) = \dim(V)$.

This means T preserves dimension.

Corollary 2 Surjective Case.

If T is surjective, then $\text{im}(T) = W$, so $\dim(\ker(T)) = \dim(V) - \dim(W)$. If $\dim(V) < \dim(W)$, then T cannot be surjective.