

## Brief paper

Unbiased minimum-variance input and state estimation  
for linear discrete-time systems<sup>☆</sup>

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## Abstract

This paper addresses the problem of **simultaneously** estimating the state and the input of a linear discrete-time system. A recursive filter, optimal in the minimum-variance unbiased sense, is developed where the estimation of the state and the input are **interconnected**. The input estimate is obtained from the innovation by least-squares estimation and the state estimation problem is transformed into a standard Kalman filtering problem. Necessary and sufficient conditions for the existence of the filter are given and relations to earlier results are discussed.

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## 1. Introduction

Thanks to its applications in fault detection and in state estimation for geophysical processes with unknown disturbances, the problem of state estimation for linear systems with unknown inputs has received considerable attention during the last decades.

For continuous-time systems, necessary and sufficient conditions for the existence of an optimal state estimator are well-established (Darouach, Zasadzinski, & Xu, 1994; Hou & Müller, 1992; Kudva, Viswanadham, & Ramakrishna, 1980). Furthermore, design procedures for the reconstruction of unknown inputs have received considerable attention (Hou & Patton, 1998; Xiong & Saif, 2003).

For discrete-time systems, earliest approaches were based on augmenting the state vector with an unknown input vector, where a **prescribed** model for the unknown input is assumed. To reduce computation costs of the augmented state filter, Friedland (1969) proposed the two-stage Kalman filter where the estimation of the state and the unknown input are

decoupled. Although successfully used in many applications, both methods are limited to the case where a model for the dynamical **evolution** of the unknown input is available.

Kitanidis (1987), on the other hand, developed an optimal recursive state filter which is based on the assumption that no prior information about the unknown input is available. His result was extended by Darouach and Zasadzinski (1997) who established stability and convergence conditions and developed a new design method for the optimal state filter.

Hsieh (2000) established a connection between the two-stage filter and the Kitanidis filter by showing that Kitanidis' result can be derived by making the two-stage filter independent of the underlying input model. Furthermore, his method yields an estimate of the unknown input. However, the optimality of the input estimate has not been proved.

This paper is an extension of Kitanidis (1987) and Darouach and Zasadzinski (1997) **to joint** minimum-variance unbiased (MVU) input and state estimation. We propose a recursive filter where the estimation of the unknown input and the state are interconnected. We prove that this approach yields the same state update as in Kitanidis (1987) and Darouach and Zasadzinski (1997) and the same input estimate as in Hsieh (2000), thereby also showing that the latter input estimate is indeed optimal.

This paper is organized as follows. In Section 2, the problem is formulated and the structure of the recursive filter is presented. Section 3 deals with optimal reconstruction of the

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unknown input. Next, the state estimation problem is solved in Section 4. Finally, a proof of global optimality is provided in Section 5.

## 2. Problem formulation

Consider the linear discrete-time system

$$x_{k+1} = A_k x_k + G_k d_k + w_k, \quad (1)$$

$$y_k = C_k x_k + v_k, \quad (2)$$

where  $x_k \in \mathbb{R}^n$  is the state vector,  $d_k \in \mathbb{R}^m$  is an unknown input vector, and  $y_k \in \mathbb{R}^p$  is the measurement. The process noise  $w_k \in \mathbb{R}^n$  and the measurement noise  $v_k \in \mathbb{R}^p$  are assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrices,  $Q_k = \mathbb{E}[w_k w_k^T]$  and  $R_k = \mathbb{E}[v_k v_k^T]$ , respectively. Results are easily generalized to the case where  $w_k$  and  $v_k$  are correlated by transforming (1)–(2) into an equivalent system where process and measurement noise are uncorrelated (Anderson & Moore, 1979, Chapter 5.5).

Throughout the paper, we assume that  $(C_k, A_k)$  is observable and that  $x_0$  is independent of  $v_k$  and  $w_k$  for all  $k$ . Also, we assume that the following sufficient condition for the existence of an unbiased state estimator is satisfied.

**Assumption 1** (Darouach & Zasadzinski, 1997; Kitanidis, 1987). *rank  $C_k G_{k-1} = \text{rank } G_{k-1} = m$ , for all  $k$ .*

Note that Assumption 1 implies  $n \geq m$  and  $p \geq m$ .

The objective of this paper is to make MVU estimates of the system state  $x_k$  and the unknown input  $d_{k-1}$ , given the sequence of measurements  $Y_k = \{y_0, y_1, \dots, y_k\}$ . No prior knowledge about  $d_{k-1}$  is assumed to be available and no prior assumption is made. The unknown input  $d_{k-1}$  can be any type of signal.

We consider a recursive filter of the form

$$\hat{x}_{k|k-1} = A_{k-1} \hat{x}_{k-1|k-1}, \quad (3)$$

$$\hat{d}_{k-1} = M_k (y_k - C_k \hat{x}_{k|k-1}), \quad (4)$$

$$\hat{x}_{k|k}^* = \hat{x}_{k|k-1} + G_{k-1} \hat{d}_{k-1}, \quad (5)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k}^* + K_k (y_k - C_k \hat{x}_{k|k}^*), \quad (6)$$

where  $M_k \in \mathbb{R}^{m \times p}$  and  $K_k \in \mathbb{R}^{n \times p}$  still have to be determined. Let  $\hat{x}_{k-1|k-1}$  be an unbiased estimate of  $x_{k-1}$ , then  $\hat{x}_{k|k-1}$  is biased due to the unknown input in the true system. Therefore, an unbiased estimate of the unknown input is calculated from the measurement in (4) and used to obtain an unbiased state estimate  $\hat{x}_{k|k}^*$  in (5). In the final step, the variance of  $\hat{x}_{k|k}^*$  is minimized by using an update similar to the Kalman filter.

Conditions on the matrix  $M_k$  to obtain unbiased and MVU estimates of the unknown input, are derived in Section 3. The gain matrix  $K_k$  minimizing the variance of  $\hat{x}_{k|k}$ , is computed in Section 4.

## 3. Input estimation

In this section, we consider the estimation of the unknown input. In Section 3.1, we determine the matrix  $M_k$  such that (4)

is an unbiased estimator of  $d_{k-1}$ . In Section 3.2, we extend to MVU input estimation.

### 3.1. Unbiased input estimation

Defining the innovation  $\tilde{y}_k$  by

$$\tilde{y}_k \triangleq y_k - C_k \hat{x}_{k|k-1}, \quad (7)$$

it follows from (1) to (3) that

$$\tilde{y}_k = C_k G_{k-1} d_{k-1} + e_k, \quad (8)$$

where  $e_k$  is given by

$$e_k = C_k (A_{k-1} \tilde{x}_{k-1} + w_{k-1}) + v_k, \quad (9)$$

with  $\tilde{x}_k \triangleq x_k - \hat{x}_{k|k}$ .

Let  $\hat{x}_{k-1|k-1}$  be unbiased, then it follows from (9) that  $\mathbb{E}[e_k] = 0$  and consequently from (8) that

$$\mathbb{E}[\tilde{y}_k] = C_k G_{k-1} d_{k-1}. \quad (10)$$

Eq. (10) indicates that an unbiased estimate of the unknown input  $d_{k-1}$  can be obtained from the innovation.

**Theorem 1.** *Let  $\hat{x}_{k-1|k-1}$  be unbiased, then (3)–(4) is an unbiased estimator of  $d_{k-1}$  if and only if  $M_k$  satisfies*

$$M_k C_k G_{k-1} = I_m. \quad (11)$$

**Proof.** Substituting (8) in (4), yields

$$\hat{d}_{k-1} = M_k C_k G_{k-1} d_{k-1} + M_k e_k.$$

Noting that  $\hat{d}_{k-1}$  is unbiased if and only if  $M_k$  satisfies  $M_k C_k G_{k-1} = I_m$ , concludes the proof.  $\square$

The matrix  $M_k$  corresponding to the least-squares (LS) solution of (8), satisfies (11). The LS solution is thus unbiased. However, it does not have minimum-variance because  $e_k$  does not have unit variance and thus (8) does not satisfy the assumptions of the Gauss–Markov theorem (Kailath, Sayed, & Hassibi, 2000, Chapter 3.4.2). Nevertheless, the variance of  $e_k$  can be computed from the covariance matrices of the state estimator. An MVU estimator of  $d_{k-1}$  is then obtained by weighted LS (WLS) estimation with weighting matrix  $(\mathbb{E}[e_k e_k^T])^{-1}$ , as will be shown in the next section.

### 3.2. Minimum-variance unbiased input estimation

Denoting the variance of  $e_k$  by  $\tilde{R}_k$ , a straightforward calculation yields

$$\begin{aligned} \tilde{R}_k &= \mathbb{E}[e_k e_k^T], \\ &= C_k (A_{k-1} P_{k-1|k-1} A_{k-1}^T + Q_k) C_k^T + R_k, \end{aligned} \quad (12)$$

where  $P_{k|k} \triangleq \mathbb{E}[\tilde{x}_k \tilde{x}_k^\top]$ . Furthermore, defining

$$P_{k|k-1} \triangleq A_{k-1} P_{k-1|k-1} A_{k-1}^\top + Q_{k-1},$$

it follows that  $\tilde{R}_k$  can be rewritten as

$$\tilde{R}_k = C_k P_{k|k-1} C_k^\top + R_k.$$

An MVU input estimate is then obtained as follows.

**Theorem 2.** *Let Assumption 1 hold, let  $\hat{x}_{k-1|k-1}$  be unbiased, let  $\tilde{R}_k$  be positive definite and let  $M_k$  be given by*

$$M_k = (F_k^\top \tilde{R}_k^{-1} F_k)^{-1} F_k^\top \tilde{R}_k^{-1}, \quad (13)$$

where  $F_k \triangleq C_k G_{k-1}$ , then (4) is the MVU estimator of  $d_{k-1}$  given the innovation  $\tilde{y}_k$ . The variance of the corresponding input estimate, is given by  $(F_k^\top \tilde{R}_k^{-1} F_k)^{-1}$ .

**Proof.** Under the assumption that  $\tilde{R}_k$  is positive definite, an invertible matrix  $\tilde{S}_k \in \mathbb{R}^{p \times p}$  satisfying  $\tilde{S}_k \tilde{S}_k^\top = \tilde{R}_k$ , can always be found, for example by a Cholesky factorization. We now transform (8) to

$$\tilde{S}_k^{-1} \tilde{y}_k = \tilde{S}_k^{-1} C_k G_{k-1} d_{k-1} + \tilde{S}_k^{-1} e_k. \quad (14)$$

Under the assumption that  $\tilde{S}_k^{-1} C_k G_{k-1}$  has full column rank, the LS solution  $\hat{d}_{k-1}$  of (14) equals

$$\hat{d}_{k-1} = (F_k^\top \tilde{R}_k^{-1} F_k)^{-1} F_k^\top \tilde{R}_k^{-1} \tilde{y}_k, \quad (15)$$

where  $F_k = C_k G_{k-1}$ . Note that solving (14) by LS estimation is equivalent to solving (8) by WLS estimation with weighting matrix  $\tilde{R}_k^{-1}$ . In addition, since the weighting matrix is chosen such that  $\tilde{S}_k^{-1} e_k$  has unit variance, Eq. (14) satisfies the assumptions of the Gauss–Markov theorem. Hence, (15) is the MVU estimate of  $d_{k-1}$  given  $\tilde{y}_k$  (Kailath et al., 2000, Chapter 2.2.3). The variance of the WLS solution (15) is given by  $(F_k^\top \tilde{R}_k^{-1} F_k)^{-1}$ .  $\square$

This input estimator has a strong connection to the filter designed in Hsieh (2000).

**Theorem 3.** *Let  $M_k$  be given by (13), then we obtain the same input estimate as in Hsieh (2000, Section III).*

In Hsieh (2000, Section III), the input estimate follows by making the two-stage Kalman filter independent of the underlying input model. However, the optimality of the input estimate has not been shown. Here, we obtain the same estimate from the innovation in an optimal way, showing that the input estimate of Hsieh is indeed optimal.

## 4. State estimation

Consider a state estimator for system (1)–(2) which takes the recursive form (3)–(6). In Section 4.1, we search for a condition on the gain matrix  $K_k$  such that (6) is an unbiased estimator of  $x_k$ . In Section 4.2, we extend to MVU state estimation.

### 4.1. Unbiased state estimation

Defining  $\tilde{x}_k^* \triangleq x_k - \hat{x}_{k|k}^*$ , it follows from (1) to (3) and (5) that

$$\tilde{x}_k^* = A_{k-1} \tilde{x}_{k-1} + G_{k-1} \tilde{d}_{k-1} + w_{k-1}, \quad (16)$$

where  $\tilde{d}_k \triangleq d_k - \hat{d}_k$ . The following theorem is a direct consequence of (16).

**Theorem 4.** *Let  $\hat{x}_{k-1|k-1}$  and  $\hat{d}_{k-1}$  be unbiased, then (5)–(6) are unbiased estimators of  $x_k$  for any value of  $K_k$ .*

This unbiased state estimator has a strong connection to the filter designed in Kitanidis (1987). Substituting (4) and (5) in (6), yields

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k + (I_n - K_k C_k) G_{k-1} \hat{d}_{k-1}, \quad (17)$$

$$= \hat{x}_{k|k-1} + K_k \tilde{y}_k + (I_n - K_k C_k) G_{k-1} M_k \tilde{y}_k. \quad (18)$$

Defining

$$L_k \triangleq K_k + (I_n - K_k C_k) G_{k-1} M_k,$$

Eq. (18) is rewritten as

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (y_k - C_k \hat{x}_{k|k-1}), \quad (19)$$

which is the kind of update considered in Kitanidis (1987).

**Theorem 5.** *Let  $M_k$  be given by (13) and  $K_k$  by*

$$K_k = P_{k|k-1} C_k \tilde{R}_k^{-1}, \quad (20)$$

then we obtain the state update of Kitanidis (1987).

In Kitanidis (1987) only state estimation is considered. However, we conclude from the equivalence of (17) and (19) that Kitanidis' filter implicitly estimates the unknown input from the innovation by WLS estimation.

### 4.2. Minimum-variance unbiased state estimation

In this section, we calculate the optimal gain matrix  $K_k$  as function of  $M_k$ . The derivation holds for any  $M_k$  satisfying (11) and yields the MVU estimate  $\hat{x}_{k|k}$  of  $x_k$  given the value of  $M_k$  used in (4).

First, we search for an expression of  $\tilde{d}_{k-1}$ . It follows from (4) and (8)–(9) that

$$\tilde{d}_{k-1} = (I_m - M_k C_k G_{k-1}) d_{k-1} - M_k e_k = -M_k e_k, \quad (21)$$

where the last step follows from the unbiasedness of the input estimator. Substituting (21) in (16), yields

$$\tilde{x}_k^* = A_{k-1}^* \tilde{x}_{k-1} + w_{k-1}^*, \quad (22)$$

where

$$A_{k-1}^* = (I_n - G_{k-1} M_k C_k) A_{k-1}, \quad (23)$$

$$w_{k-1}^* = (I_n - G_{k-1} M_k C_k) w_{k-1} - G_{k-1} M_k v_k. \quad (24)$$

An expression for the error covariance matrix  $P_{k|k}^* \triangleq \mathbb{E}[\tilde{x}_k^* \tilde{x}_k^{*\top}]$  follows from (22) to (24),

$$\begin{aligned} P_{k|k}^* &= A_{k-1}^* P_{k-1|k-1} A_{k-1}^{*\top} + Q_{k-1}^* \\ &= (I_n - G_{k-1} M_k C_k) P_{k-1|k-1} (I_n - G_{k-1} M_k C_k)^\top \\ &\quad + G_{k-1} M_k R_k M_k^\top G_{k-1}^\top, \end{aligned} \quad (25)$$

where  $Q_k^* \triangleq \mathbb{E}[w_k^* w_k^{*\top}]$ .

Next, we search for an expression of the error covariance matrix  $P_{k|k}$ . It follows from (6) that

$$\tilde{x}_k = (I_n - K_k C_k) \tilde{x}_{k-1}^* - K_k v_k. \quad (26)$$

Substituting (22) in (26), yields

$$\tilde{x}_k = (I_n - K_k C_k) (A_{k-1}^* \tilde{x}_{k-1}^* + w_{k-1}^*) - K_k v_k, \quad (27)$$

where  $\mathbb{E}[w_{k-1}^* v_k^\top] = -G_{k-1} M_k R_k$ . Note that (27) has a close connection to the Kalman filter. This expression represents the dynamical evolution of the error in the state estimate of a Kalman filter with gain matrix  $K_k$  for the system  $(A_k^*, C_k)$ , where the process noise  $w_{k-1}^*$  is correlated with the measurement noise  $v_k$ . The calculation of the optimal gain matrix  $K_k$  has thus been reduced to a standard Kalman filtering problem.

It follows from (26) and (25) that the error covariance matrix  $P_{k|k}$  is given by

$$P_{k|k} = K_k \tilde{R}_k^* K_k^\top - V_k^* K_k^\top - K_k V_k^{*\top} + P_{k|k}^*, \quad (28)$$

where

$$\begin{aligned} \tilde{R}_k^* &= C_k P_{k|k}^* C_k^\top + R_k + C_k S_k^* + S_k^{*\top} C_k^\top, \\ V_k^* &= P_{k|k}^* C_k^\top + S_k^*, \\ S_k^* &= \mathbb{E}[\tilde{x}_k^* v_k^\top] = -G_{k-1} M_k R_k. \end{aligned} \quad (29)$$

Note that  $\tilde{R}_k^*$  equals the variance of the zero-mean signal  $\tilde{y}_k^*$ ,  $\tilde{R}_k^* = \mathbb{E}[\tilde{y}_k^* \tilde{y}_k^{*\top}]$ , where

$$\tilde{y}_k^* \triangleq y_k - C_k \hat{x}_{k|k}^* = (I_p - C_k G_{k-1} M_k) e_k. \quad (30)$$

Using (30) and (12), (29) can be rewritten as

$$\tilde{R}_k^* = (I_p - C_k G_{k-1} M_k) \tilde{R}_k (I_p - C_k G_{k-1} M_k)^\top.$$

From Kalman filtering theory, we know that uniqueness of the optimal gain matrix  $K_k$  requires invertibility of  $\tilde{R}_k^*$ . However, we now show that  $\tilde{R}_k^*$  is singular by proving that  $I_p - C_k G_{k-1} M_k$  is not of full rank.

**Lemma 6.** Let  $M_k$  satisfy (11), then  $I_p - C_k G_{k-1} M_k$  has rank  $p - m$ .

**Proof.** Because  $M_k$  satisfies (11), it is a left inverse of  $C_k G_{k-1}$ . Consequently,  $C_k G_{k-1} M_k$  and  $I_p - C_k G_{k-1} M_k$  are idempotent (Bernstein, 2005, Facts 3.8.7 and 3.8.9). The rank of  $I_p - C_k G_{k-1} M_k$  is then given by

$$\begin{aligned} \text{rank } I_p - C_k G_{k-1} M_k &= p - \text{rank } C_k G_{k-1} M_k \\ &= p - m, \end{aligned}$$

where the first equality follows from Bernstein (2005, Fact 3.8.6) and the second equality from Bernstein (2005, Proposition 2.6.2).  $\square$

Consequently, the optimal gain matrix  $K_k$  is not unique. Let  $r$  be the rank of  $\tilde{R}_k^*$ , we then propose a gain matrix  $K_k$  of the form

$$K_k = \tilde{K}_k \alpha_k, \quad (31)$$

where  $\alpha_k \in \mathbb{R}^{r \times p}$  is an arbitrary matrix which has to be chosen such that  $\alpha_k \tilde{R}_k^* \alpha_k^\top$  has full rank. The optimal gain matrix  $K_k$  is then given in the following theorem.

**Theorem 7.** Let  $M_k$  satisfy (11) and let  $\alpha_k \in \mathbb{R}^{r \times p}$ , with  $r = \text{rank } \tilde{R}_k^*$ , be an arbitrary matrix, chosen such that  $\alpha_k \tilde{R}_k^* \alpha_k^\top$  has full rank, then the gain matrix  $K_k$  of the form (31) minimizing the variance of  $\hat{x}_{k|k}$ , is given by

$$K_k = (P_{k|k}^* C_k^\top + S_k^*) \alpha_k^\top (\alpha_k \tilde{R}_k^* \alpha_k^\top)^{-1} \alpha_k. \quad (32)$$

**Proof.** Substituting (31) in (28) and minimizing the trace of  $P_{k|k}$  over  $\tilde{K}_k$ , yields (32).  $\square$

Substituting (32) in (28), yields the following update for the error covariance matrix,

$$P_{k|k} = P_{k|k}^* - K_k (P_{k|k}^* C_k^\top + S_k^*)^\top.$$

We now give the relation to Darouach and Zasadzinski (1997).

**Theorem 8.** Let  $M_k$  satisfy (11) and let  $K_k$  be given by (32) with  $r = p - m$ , then we obtain the same state update as Darouach and Zasadzinski (1997). Furthermore, for  $M_k$  given by (13) and  $\alpha_k = [0 \ I_r] U_k^\top \tilde{S}_k^{-1}$ , where  $U_k$  is an orthogonal matrix containing the left singular vectors of  $\tilde{S}_k^{-1} C_k G_{k-1}$  in its columns, the Kitanidis filter is obtained.

By parameterizing the unbiasedness conditions in Kitanidis (1987), Darouach and Zasadzinski (1997) showed that the gain matrix is not unique. Here, the same result is obtained by a procedure which has a closer connection to the Kalman filter.

Note that the expression (32) implicitly depends on the choice of  $M_k$ . Given the value of  $M_k$  used in (4), (32) yields the gain matrix  $K_k$  for which the variance of  $\hat{x}_{k|k}$  is minimal. Our result does not allow to conclude which value(s) of  $M_k$  should optimally be used in (4) to minimize the variance of  $\hat{x}_{k|k}$ .

## 5. Proof of optimality

In Kerwin and Prince (2000), it is proved that a recursive MVU state estimator which can be written in the form (3), (19), minimizes the mean square error of  $\hat{x}_{k|k}$  over the class of all linear unbiased state estimates based on  $Y_k$ . By a similar derivation, we now prove that the estimate of  $d_{k-1}$  minimizing the mean square error over the class of all linear unbiased estimates based on  $Y_k$ , can be written in the form (4). The proof is inspired by the optimality proof in Kerwin and Prince (2000).



We relax the recursivity assumption and consider  $\hat{d}_{k-1}$  to be the most general linear combination of  $\hat{x}_{0|0}$  and  $Y_k$ . As pointed out in Kerwin and Prince (2000), because the innovation  $\tilde{y}_k$  is itself a linear combination of  $\hat{x}_{0|0}$  and  $Y_k$ , the most general estimate of  $d_{k-1}$  can be written in the form

$$\hat{d}_{k-1} = M_k \tilde{y}_k + \sum_{i=0}^{k-1} H_i \tilde{y}_i + N \hat{x}_{0|0}, \quad (33)$$

where we dropped the dependence of  $H_i$  and  $N$  on  $k$  for notational simplicity. A necessary and sufficient condition for (33) to be an unbiased estimator of  $d_{k-1}$ , is given in the following lemma.

**Lemma 9.** *The estimator (33) is unbiased if and only if  $N=0$ ,  $M_k$  satisfies (11) and  $H_i C_i G_{i-1} = 0$  for every  $i < k$ .*

**Proof.** *Sufficiency:* It follows from (10) that if  $H_i C_i G_{i-1} = 0$  for every  $i < k$ , then  $\sum_{i=0}^{k-1} H_i \mathbb{E}[\tilde{y}_i] = 0$ . Furthermore, for  $M_k$  satisfying (11),  $M_k \tilde{y}_k$  and consequently also (33), with  $N=0$ , are unbiased estimators of  $d_{k-1}$ .

*Necessity:* Assume that (33) is an unbiased estimator of  $d_{k-1}$ . Since no prior information about  $d_{k-1}$  is available and since  $y_k$  is the first measurement containing information about  $d_{k-1}$ , we conclude that  $\mathbb{E}[M_k \tilde{y}_k] = d_{k-1}$  and that consequently also (11) must hold. Furthermore, the expected value of the sum of the last two terms in (33) is zero for any unknown input sequence  $d_0, d_1, \dots, d_{k-1}$  if and only if  $H_i C_i G_{i-1} = 0$  for every  $i < k$  and  $N = 0$ .  $\square$

In the remainder of this section, we only consider unbiased input estimators of the form (33). We now prove that the mean square error

$$\sigma_{k-1}^2 \triangleq \mathbb{E}[\|d_{k-1} - \hat{d}_{k-1}\|_2^2] \quad (34)$$

achieves a minimum when  $H_0 = H_1 = \dots = H_{k-1} = 0$ .

**Theorem 10.** *Let  $\hat{d}_{k-1}$  given by (33) be unbiased, then the mean square error (34) achieves a minimum when  $H_0 = H_1 = \dots = H_{k-1} = 0$ .*

In the proof of Theorem 10, we make use of the following lemma, which provides an orthogonality relationship.

**Lemma 11** (see Kerwin & Prince, 2000, Lemma 2). *Let  $\tilde{y}_i$  be defined by (7), then for every  $i < k$  and every  $H_i$  satisfying  $H_i C_i G_{i-1} = 0$ ,  $\mathbb{E}[\tilde{y}_k (H_i \tilde{y}_i)^T] = 0$  and  $\mathbb{E}[d_{k-1} (H_i \tilde{y}_i)^T] = 0$ .*

The proof of Theorem 10 is then given as follows.

**Proof.** Inspired by the proof of Theorem 3 in Kerwin and Prince (2000), we write  $d_{k-1} - \hat{d}_{k-1} = f_M - g_H$ , where  $f_M \triangleq d_{k-1} - M_k \tilde{y}_k$  and  $g_H \triangleq \sum_{i=0}^{k-1} H_i \tilde{y}_i$ . It follows from Lemma 11 that  $\mathbb{E}[f_M g_H^T] = 0$ , so that

$$\begin{aligned} \sigma_{k-1}^2 &= \text{trace}\{(f_M + g_H)(f_M + g_H)^T\} \\ &= \mathbb{E}[\|f_M\|_2^2] + \mathbb{E}[\|g_H\|_2^2]. \end{aligned} \quad (35)$$

The second term in (35) is minimized when  $g_H = 0$ , which occurs for  $H_0 = H_1 = \dots = H_{k-1} = 0$ . That solution also satisfies  $H_i C_i G_{i-1} = 0$ , which completes the proof.  $\square$

It follows from Theorem 10 and (33) that the globally optimal linear estimate of  $d_{k-1}$  based on  $Y_k$  can be written in the recursive form (4). Furthermore, because the matrix  $M_k$  given by (13) minimizes  $\mathbb{E}[\|f_M\|_2^2]$ , it follows that (4) yields the globally optimal linear estimate of  $d_{k-1}$  for this value of  $M_k$ . Combining this result with Theorem 5 and the global optimality of the Kitanidis filter proved in Kerwin and Prince (2000), yields the following theorem.

**Theorem 12.** *Consider a joint input and state estimator of the recursive form (3)–(6). Let  $M_k$  be given by (13) and let  $K_k$  be given by (20), then (4) and (6) are unbiased estimators of  $d_{k-1}$  and  $x_k$  minimizing the mean square error over the class of all linear unbiased estimates based on  $\hat{x}_{0|0}$  and  $Y_k = \{y_0, y_1, \dots, y_k\}$ .*

## 6. Conclusion

An optimal filter is developed which simultaneously estimates the input and the state of a linear discrete-time system. The estimate of the input is obtained from the innovation by least-squares estimation. The state estimation problem is transformed into a standard Kalman filtering problem for a system with correlated process and measurement noise. We prove that this approach yields the same state update as in Kitanidis (1987) and Darouach and Zasadzinski (1997), and the same input estimate as in Hsieh (2000). Finally, a proof is included showing that the optimal input estimate over the class of all linear unbiased estimates may be written in the proposed recursive form.

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## References

- Anderson, B. D. O., & Moore, J. B. (1979). *Optimal filtering*. Englewood Cliffs, NJ: Prentice-Hall.

Bernstein, D. S. (2005). *Matrix mathematics: Theory, facts, and formulas with application to linear systems theory*. Princeton, NJ: Princeton University Press.

- Darouach, M., & Zasadzinski, M. (1997). Unbiased minimum variance estimation for systems with unknown exogenous inputs. *Automatica*, 33(4), 717–719.
- Darouach, M., Zasadzinski, M., & Xu, S. J. (1994). Full-order observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 39(3), 606–609.
- Friedland, B. (1969). Treatment of bias in recursive filtering. *IEEE Transactions on Automatic Control*, 14, 359–367.
- Hou, M., & Patton, R. J. (1998). Input observability and input reconstruction. *Automatica*, 34(6), 789–794.
- Hou, M., & Müller, P. C. (1992). Design of observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 37(6), 871–874.
- Hsieh, C. S. (2000). Robust two-stage Kalman filters for systems with unknown inputs. *IEEE Transactions on Automatic Control*, 45(12), 2374–2378.
- Kailath, T., Sayed, A. H., & Hassibi, B. (2000). *Linear estimation*. Upper Saddle River, NJ: Prentice-Hall.
- Kerwin, W. S., & Prince, J. L. (2000). On the optimality of recursive unbiased state estimation with unknown inputs. *Automatica*, 36, 1381–1383.
- Kitanidis, P. K. (1987). Unbiased minimum-variance linear state estimation. *Automatica*, 23(6), 775–778.
- Kudva, P., Viswanadham, N., & Ramakrishna, A. (1980). Observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 25(1), 113–115.
- Xiong, Y., & Saif, M. (2003). Unknown disturbance inputs estimation based on a state functional observer design. *Automatica*, 39, 1389–1398.



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