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Brief paper

Unbiased minimum-variance state estimation for linear systems with unknown input☆

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ABSTRACT

The problem of state estimation for a linear system with unknown input, which affects both the system and the output, is discussed in this paper. A recursive optimal filter with global optimality in the sense of unbiased minimum variance over all linear unbiased estimators, is provided. The necessary and sufficient condition for the convergence and stability is also given, which is milder than existing approaches.

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1. Introduction

Problem of state estimation for linear systems with unknown input, has received intensive attention in the past three decades. A traditional way to deal with the problem is augmenting the state vector with the unknown input, which requires some assumption on the unknown input (Friedland, 1969; Hsieh, 2006; Ignani, 1990).

To make the method valid in the case that no prior information about unknown input is available, Kitanidis (1987) proposed an optimal recursive state filter, whose stability and convergence conditions were established by Darouach and Zasadzinski (1997). Kerwin and Prince (2000) further indicated that it is necessary to check whether the global linear minimum-variance unbiased estimation lies within the recursive framework in Darouach and Zasadzinski (1997) and Kitanidis (1987), and proved their optimality. In addition, Hsieh (2000) used a two-stage Kalman filtering technique to estimate both the state and the unknown input, and Gillins and De Moor (2007a) proved that the estimation of the unknown input in Hsieh (2000) is global optimal in a similar way to the one provided in Kerwin and Prince (2000).

Different from the above papers which considered linear systems without direct feedthrough of the unknown input to the output (i.e. without unknown input in the output equation) Darouach, Zasadzinski, and Boutayeb (2003) proposed a recursive state filter which included unknown input in the output equation. However, as pointed out by the author, that filter is a suboptimal one, although it is optimal in the recursive form at each step. Hsieh (2006) also found that Darouacha's result might exhibit a filtering performance degradation problem according to a numerical simulation example. Gillins and De Moor (2007b) proposed a three-step recursive estimator for linear discrete-time systems with unknown input in the output function. However, in Gillins and De Moor (2007b), the distribution matrix before the unknown input in the output equation must be of full column rank, and the global optimality of the method was not mentioned.

In this paper, a state estimator for a system with unknown input in both the system equation and the output equation is proposed, in which the requirement on the distribution matrix of the unknown input is milder than the one in Gillins and De Moor (2007b), i.e. being of full column rank. Then the global optimality of the recursive filter is proved.

2. Main result

2.1. Problem formulation

In this paper, the unbiased minimum variance state estimation of the following linear discrete-time stochastic time-varying system will be given:

$$\begin{aligned} x_{k+1} &= A_k x_k + G_k d_k + w_k \\ y_k &= C_k x_k + H_k d_k + v_k \end{aligned} \tag{1}$$

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where $x_k \in R^n$, $d_k \in R^{n_d}$, $y_k \in R^{n_y}$, $w_k \in R^n$, and $v_k \in R^{n_y}$ are the state vector, deterministic unknown input, output, process noise and measurement noise, respectively. A_k, G_k, C_k, H_k are deterministic known matrices with appropriate dimensions. w_k and v_k are uncorrelated white noises with covariance matrices $Q_k = E[w_k w_k^{\mathrm{T}}] \geq 0$ and $R_k = E[v_k v_k^{\mathrm{T}}] > 0$, respectively. The initial state x_0 is of mean $\hat{x}_0$ and covariance P_0, and is independent of w_k and v_k.

2.2. Filter design

Let $n_{d1,k}$ denote the rank of H_k. Because we do not assume that H_k must be of full column rank, there exists $n_{d1,k} \leq n_d$. Then the singular value decomposition of H_k can be written as

$$H_k = U_k \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} V_k^{\mathrm{T}} = \begin{bmatrix} U_{1,k} & U_{2,k} \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1,k}^{\mathrm{T}} \\ V_{2,k}^{\mathrm{T}} \end{bmatrix} \tag{2}$$

where $\Sigma_k \in R^{n_{d1,k} \times n_{d1,k}}$ is a diagonal matrix of full rank, U_k and V_k are unitary matrices, and $U_{1,k} \in R^{n_y \times n_{d1,k}}$, $U_{2,k} \in R^{n_y \times (n_y - n_{d1,k})}$, $V_{1,k} \in R^{n_d \times n_{d1,k}}$, $V_{2,k} \in R^{n_d \times (n_d - n_{d1,k})}$.

Let

$$\begin{aligned} d_{1,k} &= V_{1,k}^{\mathrm{T}} d_k \\ d_{2,k} &= V_{2,k}^{\mathrm{T}} d_k. \end{aligned} \tag{3}$$

Since V_k is a unitary matrix, it follows that

$$d_k = \begin{bmatrix} V_{1,k} & V_{2,k} \end{bmatrix} \begin{bmatrix} d_{1,k} \\ d_{2,k} \end{bmatrix} = V_{1,k} d_{1,k} + V_{2,k} d_{2,k}. \tag{4}$$

Substituting (4) into (1), we have:

$$\begin{aligned} x_{k+1} &= A_k x_k + G_k V_{1,k} d_{1,k} + G_k V_{2,k} d_{2,k} + w_k \\ y_k &= C_k x_k + H_k V_{1,k} d_{1,k} + H_k V_{2,k} d_{2,k} + v_k. \end{aligned} \tag{5}$$

Similarly, the output y_k can be transformed into $z_{1,k} \in R^{n_{d1,k}}$ and $z_{2,k} \in R^{n_y - n_{d1,k}}$ via a nonsingular transformation matrix T_k, i.e.

$$\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = T_k y_k$$

where

$$T_k = \left(\begin{bmatrix} I_{n_{d1,k} \times n_{d1,k}} & -U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} \right). \tag{6}$$

Since T_k is nonsingular, which can ensure that there is a one-to-one relationship between the two new measurements $z_{1,k}$, $z_{2,k}$ and the output y_k, we can replace the output function of (5) with two new functions obtained by substituting it into (6). Then (5) becomes:

$$\begin{aligned} x_{k+1} &= A_k x_k + G_{1,k} d_{1,k} + G_{2,k} d_{2,k} + w_k \\ z_{1,k} &= C_{1,k} x_k + \Sigma_k d_{1,k} + v_{1,k} \\ z_{2,k} &= C_{2,k} x_k + v_{2,k} \end{aligned} \tag{7}$$

where

$$\begin{aligned} & G_{1,k} = G_k V_{1,k}; \qquad G_{2,k} = G_k V_{2,k} \\ & C_{1,k} = \begin{bmatrix} I & -U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} C_k \\ & C_{2,k} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} C_k = U_{2,k}^{\mathrm{T}} C_k \\ & v_{1,k} = \begin{bmatrix} I & -U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} v_k \\ & v_{2,k} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} v_k = U_{2,k}^{\mathrm{T}} v_k. \end{aligned} \tag{8}$$

Define $R_{1,k}$ and $R_{2,k}$ as the variance of $v_{1,k}$ and $v_{2,k}$, and $R_{12}(k, i)$ as their covariance. Then it follows:

$$\begin{aligned} R_{1,k} &= E\left[v_{1,k} v_{1,k}^{\mathrm{T}} \right] \\ &= U_{1,k}^{\mathrm{T}} R_k U_{1,k} - U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} U_{2,k}^{\mathrm{T}} R_k U_{1,k} \\ R_{2,k} &= E\left[v_{2,k} v_{2,k}^{\mathrm{T}} \right] = U_{2,k}^{\mathrm{T}} R_k U_{2,k} \\ R_{12}(k, k) &= E\left[v_{1,k} v_{2,k}^{\mathrm{T}} \right] \\ &= \begin{bmatrix} I & -U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} R_k \begin{bmatrix} U_{1,k} & U_{2,k} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= U_{1,k}^{\mathrm{T}} R_k U_{2,k} - U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} U_{2,k}^{\mathrm{T}} R_k U_{2,k} \\ &= 0 \\ R_{12}(k, i) &= E\left[v_{1,k} v_{2,i}^{\mathrm{T}} \right] \\ &= \begin{bmatrix} I & -U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} E\left[v_k v_i^{\mathrm{T}} \right] \begin{bmatrix} U_{1,k} & U_{2,k} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = 0 \quad \text{for } k \neq i. \end{aligned} \tag{9}$$

Because process noise and measurement noise are uncorrelated and independent of the initial state, and the measurement noise is white, we have:

$$\begin{aligned} & \operatorname{cov}\left[v_{1,k}, w_i \right] \\ & \quad = \begin{bmatrix} I & -U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} E\left[v_k w_i^{\mathrm{T}} \right] = 0 \\ & \operatorname{cov}\left[v_{2,k}, w_i \right] = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} E\left[v_k w_i^{\mathrm{T}} \right] = 0 \\ & \operatorname{cov}\left[v_{1,k}, v_{1,i} \right] = 0 \quad \text{for } k \neq i \\ & \operatorname{cov}\left[v_{2,k}, v_{2,i} \right] = 0 \quad \text{for } k \neq i \\ & \operatorname{cov}\left[v_{1,k}, x_0 \right] \\ & \quad = \begin{bmatrix} I & -U_{1,k}^{\mathrm{T}} R_k U_{2,k} \left(U_{2,k}^{\mathrm{T}} R_k U_{2,k} \right)^{-1} \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} \operatorname{cov}\left[v_k, x_0 \right] = 0 \\ & \operatorname{cov}\left[v_{2,k}, x_0 \right] = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} U_{1,k}^{\mathrm{T}} \\ U_{2,k}^{\mathrm{T}} \end{bmatrix} \operatorname{cov}\left[v_k, x_0 \right] = 0 \\ & \operatorname{cov}\left[w_k, x_0 \right] = 0. \end{aligned} \tag{10}$$

Since the state vectors of (1) and (7) are the same, and there is a one-to-one relationship between the measurements of the two systems, the minimum-variance unbiased (MVU) estimation of (1), given the measurement y_k, should be the same as that of (7), given the measurements $z_{1,k}$ and $z_{2,k}$.

Compared with (1), in (7) only the measurement $z_{1,k}$ is directly affected by the unknown input, and the distribution matrix of the unknown input in its measurement equation (i.e. Σ_k) is of full rank.

In addition, according to (9) and (10), $v_{1,k}$, $v_{2,k}$ and w_k are uncorrelated. So (7) is more convenient to deal with.

According to Darouach et al. (2003), only the projection of output y_k to the left null space of H_k, which is equal to $z_{2,k}$ in this paper according to (2) and (6), can be used at time instant k in the recursive filter, because the introduction of $z_{1,k}$ at instant k will lead to a biased estimation. In order to take full advantage of available information, we add a term of $z_{1,k-1}$ at instant k, and propose the following recursive linear filter, whose unbiasedness will be proved later:

$$\hat{x}_k = \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + L_k\tilde{z}_{2,k} + G_{1,k-1}\Sigma_{k-1}^{-1}z_{1,k-1} \tag{11}$$

where

$$\tilde{z}_{2,k} = z_{2,k} - C_{2,k}\left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1k-1}\right)\hat{x}_{k-1} - C_{2,k}G_{1,k-1}\Sigma_{k-1}^{-1}z_{1,k-1}. \tag{12}$$

The state estimation error is:

$$\begin{aligned} e_k &= \hat{x}_k - x_k \\ &= \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + L_k\tilde{z}_{2,k} \\ &\quad + G_{1,k-1}\Sigma_{k-1}^{-1}z_{1,k-1} - x_k \\ &= \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\left(\hat{x}_{k-1} - x_{k-1}\right) \\ &\quad + G_{1,k-1}\Sigma_{k-1}^{-1}v_{1,k-1} - w_{k-1} - G_{2,k-1}d_{2,k-1} + L_k\tilde{z}_{2,k} \\ &= \left(I - L_kC_{2,k}\right)\left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)e_{k-1} \\ &\quad + \left(I - L_kC_{2,k}\right)G_{1,k-1}\Sigma_{k-1}^{-1}v_{1,k-1} - \left(I - L_kC_{2,k}\right)w_{k-1} \\ &\quad - \left(G_{2,k-1} - L_kC_{2,k}G_{2,k-1}\right)d_{2,k-1} + L_kv_{2,k}. \end{aligned} \tag{13}$$

In terms of (13), given that $\hat{x}_{k-1}$ is an unbiased estimation of x_{k-1}, it is obvious that (11) is an unbiased estimator of x_k if, and only if,

$$G_{2,k-1} - L_kC_{2,k}G_{2,k-1} = 0 \tag{14}$$

which means that

$$\operatorname{rank}\left(\begin{bmatrix} G_{2,k-1} \\ C_{2,k}G_{2,k-1} \end{bmatrix}\right) = \operatorname{rank}\left(C_{2,k}G_{2,k-1}\right) \tag{15}$$

is a necessary and sufficient condition for the existence of unbiased state estimation.

From (13), e_k can be represented by the combination of x_0, $v_{1,0}, \ldots, v_{1,k-1}, w_0, \ldots, w_{k-1}, v_{2,0}, \ldots, v_{2,k}$. From (9) and (10), it is obvious that

$$\begin{aligned} &E\left[e_{k-1}v_{1,k-1}^{\mathrm{T}}\right] = 0; \qquad E\left[e_{k-1}w_{k-1}^{\mathrm{T}}\right] = 0; \\ &E\left[e_{k-1}v_{2,k}^{\mathrm{T}}\right] = 0; \qquad E\left[v_{1,k-1}w_{k-1}^{\mathrm{T}}\right] = 0; \\ &E\left[w_{k-1}v_{2,k}^{\mathrm{T}}\right] = 0. \end{aligned} \tag{16}$$

Then, by use of (9), (13) and (16), the variance of the estimation error can be written as

$$P_k = \left(I - L_kC_{2,k}\right)\tilde{P}_k\left(I - L_kC_{2,k}\right)^{\mathrm{T}} + L_kR_{2,k}L_k^{\mathrm{T}} \tag{17}$$

where

$$\tilde{P}_k = \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)P_{k-1}\left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)^{\mathrm{T}} + G_{1,k-1}\Sigma_{k-1}^{-1}R_{1,k-1}\left(\Sigma_{k-1}^{-1}\right)^{\mathrm{T}}G_{1,k-1}^{\mathrm{T}} + Q_{k-1}.$$

By minimizing the trace of the variance P_k under the constraint of (14), we can get the matrix L_k for the MVU estimator.

To use the optimization approach proposed by Darouach and Zasadzinski (1997), define the following two parameter matrices:

$$\hat{A}_{k-1} = \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)$$
$$\hat{Q}_{k-1} = G_{1,k-1}\Sigma_{k-1}^{-1}R_{1,k-1}\left(\Sigma_{k-1}^{-1}\right)^{\mathrm{T}}G_{1,k-1}^{\mathrm{T}} + Q_{k-1}.$$

Then (17) becomes:

$$P_k = \left(I - L_kC_{2,k}\right)\tilde{P}_k\left(I - L_kC_{2,k}\right)^{\mathrm{T}} + L_kR_{2,k}L_k^{\mathrm{T}} \tag{18}$$

where

$$\tilde{P}_k = \hat{A}_{k-1}P_{k-1}\hat{A}_{k-1}^{\mathrm{T}} + \hat{Q}_{k-1}.$$

Now the optimization problem for (17) under the constraint of (14) becomes optimizing (18) under the constraint of (14), which is as same as the one in Darouach and Zasadzinski (1997) and Kitanidis (1987). As a result, the solution of L_k can be given as follows, due to Darouach and Zasadzinski (1997):

$$L_k = G_{2,k-1}\Pi_k + \left[\tilde{P}_kC_{2,k}^{\mathrm{T}}N_k^{\mathrm{T}} - G_{2,k-1}\Pi_{k-1}\left(C_{2,k}\tilde{P}_kC_{2,k}^{\mathrm{T}} + R_{2,k}\right)N_k^{\mathrm{T}}\right] \times \left(N_k\left(C_{2,k}\tilde{P}_kC_{2,k}^{\mathrm{T}} + R_{2,k}\right)N_k^{\mathrm{T}}\right)^{-1}N_k \tag{19}$$

where Π_k is the Moore–Penrose inverse matrix of $C_kG_{2,k-1}$, and N_k is the left null matrix of $C_kG_{2,k-1}$:

$$\Pi_k = \left(C_kG_{2,k-1}\right)^{\dagger}, \qquad N_k = \mathrm{Null}\left(\left(C_kG_{2,k-1}\right)^{\mathrm{T}}\right)^{\mathrm{T}}.$$

Substituting (9) into (11), we can get the optimal state estimator in the recursive form of (11).

Remark 1. (15) is the necessary and sufficient condition for the existence of the unbiased filter. Substituting (8) into (15), the condition becomes:

$$\operatorname{rank}\left(\begin{bmatrix} G_kV_{2,k} \\ U_{2,k}^{\mathrm{T}}C_kG_kV_{2,k} \end{bmatrix}\right) = \operatorname{rank}\left(U_{2,k}^{\mathrm{T}}C_kG_kV_{2,k}\right). \tag{20}$$

As a result, if H_k is of full column rank, $V_{2,k}$ will be an empty matrix according to (2). Then all the matrices in (20) will be empty matrices, and (20) will definitely hold. While even if H_k is not of full column rank, as long as (15) or (20) is satisfied, the filter still exists. So our method can deal with a more general class of systems than the one proposed in Gillins and De Moor (2007b).

2.3. Convergence and stability

In this section we consider the time invariant case. As mentioned in the introduction, Darouach and Zasadzinski (1997) established the stability and convergence conditions for the case without unknown input in the output function, which can be directly used to give the convergence and stability condition of filter (11), i.e.:

$$\operatorname{rank}\left(\begin{bmatrix} zI - \hat{A} & -G_2 \\ C_2 & 0 \end{bmatrix}\right) = n + \operatorname{rank}(G_2) \quad \forall z \in C, \|z\| \geq 1 \tag{21}$$

$$\operatorname{rank}\begin{bmatrix} \hat{A} - e^{j\omega}I & G_2 & \hat{Q}^{1/2} & 0 \\ e^{j\omega}C_2 & 0 & 0 & R_2^{1/2} \end{bmatrix} = n + n_y - n_{d1} \quad \forall \omega \in [0, 2\pi]. \tag{22}$$

Remark 2. It can be further proved that the above convergence and stability condition is milder than the one proposed in Darouach et al. (2003). The detailed proof is omitted here due to the page limit. An intuitive interpretation is that since the filter proposed in this paper is the optimal linear filter in the minimum-variance unbiased sense, its variance will definitely be convergent and stable if the system satisfies the convergence and stability condition of another suboptimal linear filter.

2.4. Global optimality

In Kerwin and Prince (2000), it was proved that the optimal solution over the class of all linear unbiased estimates for systems without unknown input in the output function can be written in the form of a linear recursive filter proposed in Kitanidis (1987).

Similarly, we will prove that the state estimation of system (1) with minimum mean square error over the class of all linear

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unbiased estimations based on Y_k (i.e. the collection of y_i up to instant k) can be written in the recursive form (11).

Consider the most general linear combination of measurements and the mean of initial state:

$$\hat{x}_k = \Phi_0(k)\hat{x}_0 + \sum_{i=0}^{k} H_i(k) z_{1,i} + \sum_{i=0}^{k} B_i(k) z_{2,i}. \tag{23}$$

Now we need to seek the unbiased minimum variance estimator. (23) can be rewritten as:

$$\hat{x}_k = \hat{\Phi}_0(k)\hat{x}_0 + \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + \sum_{i=0}^{k} \hat{H}_i(k) z_{1,i} + \sum_{i=0}^{k} \hat{B}_i(k)\tilde{z}_{2,i} \tag{24}$$

where $\tilde{z}_{2,i}$ is defined in (12). Because $\hat{x}_i$ is a linear combination of $\hat{x}_0, z_{1,0}, z_{2,0}, \dots, z_{1,i}$ and $z_{2,i}$, $\tilde{z}_{2,i}$ is a linear combination of $\hat{x}_0$, $z_{1,0}, z_{2,0}, \dots, z_{1,i-1}, z_{2,i-1}$ and $z_{2,i}$, and $z_{2,i}$ can also be recovered as a linear combination of $z_{1,0}, z_{2,0}, \dots, z_{1,i-1}, z_{2,i-1}$ and $\tilde{z}_{2,i}$, the equivalence of the two formulations is obvious. Theorem 1 gives the subset of estimators (24) whose elements are unbiased.

Theorem 1. *(24) is an unbiased state estimator if, and only if:*

1. *The matrices $\hat{H}_i(k)$ for $i = 0, \dots, k-2$ and $i = k$, and $\hat{\Phi}_0(k)$ are zero matrices;*
2. *$\hat{H}_{k-1}(k) = G_{1,k-1}\Sigma_{k-1}^{-1}$;*
3. *$\hat{B}_i(k) C_{2,i} G_{2,i-1}$ are zero matrices for $i = 0, \dots, k-1$; and*
4. *$\hat{B}_k(k) C_{2,k} G_{2,k-1} = G_{2,k-1}$ are satisfied.*

Proof. The estimation error of the estimator (24) is

$$\begin{aligned}\hat{e}_k &= \hat{x}_k - x_k = \left(I - \hat{B}_k(k) C_{2,k}\right)\left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{e}_{k-1} \\ &+ \left(I - \hat{B}_k(k) C_{2,k}\right) G_{1,k-1}\Sigma_{k-1}^{-1} v_{1,k-1} - \left(I - \hat{B}_k(k) C_{2,k}\right) w_{k-1} \\ &+ \hat{B}_k(k) v_{2,k} - \left(G_{2,k-1} - \hat{B}_k(k) C_{2,k} G_{2,k-1}\right) d_{2,k-1} + \hat{\Phi}_0(k)\hat{x}_0 \\ &+ \sum_{i=0}^{k-2} \hat{H}_i(k) z_{1,i} + \hat{H}_k(k) z_{1,k} + \left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right) z_{1,k-1} \\ &+ \sum_{i=0}^{k-1} \hat{B}_i(k)\tilde{z}_{2,i}.\end{aligned}$$

By use of (12) and (7)

$$\tilde{z}_{2,i} = C_{2,i}\left(A_{i-1} - G_{1,i-1}\Sigma_{i-1}^{-1}C_{1,i-1}\right)\left(x_{i-1} - \hat{x}_{i-1}\right) + C_{2,i}G_{2,i-1}d_{2,i-1} + v_{2,i} + C_{2,i}w_{i-1} - C_{2,i}G_{1,i-1}\Sigma_{i-1}^{-1}v_{1,i-1}. \tag{25}$$

Then we have

$$\begin{aligned}E\left(\hat{e}_k\right) &= -\left(G_{2,k-1} - \hat{B}_k(k) C_{2,k} G_{2,k-1}\right) d_{2,k-1} \\ &+ \sum_{i=0}^{k-1} \hat{B}_i(k) C_{2,i} G_{2,i-1} d_{2,i-1} + \hat{\Phi}_0(k)\hat{x}_0 + \sum_{i=0}^{k-2} E\left(\hat{H}_i(k) z_{1,i}\right) \\ &+ E\left(\left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right) z_{1,k-1}\right) + E\left(\hat{H}_k(k) z_{1,k}\right) \\ &+ \left(I - \hat{B}_k(k) C_{2,k}\right)\left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right) E\left(\hat{e}_{k-1}\right) \\ &+ \sum_{i=0}^{k-1} \hat{B}_i(k) C_{2,i}\left(A_{i-1} - G_{1,i-1}\Sigma_{i-1}^{-1}C_{1,i-1}\right) E\left(\hat{e}_{i-1}\right).\end{aligned}$$

Given that the state estimations at the previous instants (i.e. $\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{k-1}$) are unbiased, the expectation of the state estimate error at instant k becomes:

$$\begin{aligned}E\left(\hat{e}_k\right) &= -\left(G_{2,k-1} - \hat{B}_k(k) C_{2,k} G_{2,k-1}\right) d_{2,k-1} \\ &+ \sum_{i=0}^{k-1} \hat{B}_i(k) C_{2,i} G_{2,i-1} d_{2,i-1} + \hat{\Phi}_0(k)\hat{x}_0 \\ &+ \sum_{i=0}^{k-2} E\left(\hat{H}_i(k) z_{1,i}\right) + E\left(\left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right) z_{1,k-1}\right) \\ &+ E\left(\hat{H}_k(k) z_{1,k}\right).\end{aligned} \tag{26}$$

Sufficiency: Obviously, in terms of (26) if all the four conditions are satisfied, $E\left(\hat{e}_k\right)$ will be equal to zero.

Necessity: In order to ensure the estimator unbiased, (26) has to be zero.

With (7), the term $E\left(\hat{H}_k(k) z_{1,k}\right)$ in (26) can be rewritten as

$$E\left(\hat{H}_k(k) z_{1,k}\right) = \hat{H}_k(k) C_{1,k} E\left(x_k\right) + \hat{H}_k(k)\Sigma_k d_{1,k}.$$

Because $d_{1,k}$ is an arbitrary vector and not included in any other terms of (26), a necessary condition to make $E\left(\hat{e}_k\right) \equiv 0$ is $\hat{H}_k(k)\Sigma_k = 0$. In addition, because Σ_k is a full rank diagonal matrix, $E\left(\hat{e}_k\right) \equiv 0$ is equivalent to $\hat{H}_k(k) = 0$.

Similarly, the term $E\left(\left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right) z_{1,k-1}\right)$ can be written as

$$\begin{aligned}&E\left(\left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right) z_{1,k-1}\right) \\ &= \left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right) C_{1,k-1} x_{k-1} \\ &+ \left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right)\Sigma_{k-1} d_{1,k-1}.\end{aligned}$$

Then, when $\hat{H}_k(k) = 0$ holds, since $d_{1,k-1}$ is an arbitrary vector and not included in any other terms of (26), a necessary condition to make $E\left(\hat{e}_k\right) \equiv 0$ is $\left(\hat{H}_{k-1}(k) - G_{1,k-1}\Sigma_{k-1}^{-1}\right)\Sigma_{k-1} = 0$, which is equivalent to $\hat{H}_{k-1}(k) = G_{1,k-1}\Sigma_{k-1}^{-1}$.

Repeating the same procedure, we can conclude that a necessary condition to make $E\left(\hat{e}_k\right) \equiv 0$ is that $\hat{H}_{k-2}(k), \hat{H}_{k-3}(k), \dots, \hat{H}_0(k)$ and $\hat{\Phi}_0(k)$ are all zero matrices, which further leads to

$$E\left(\hat{e}_k\right) = -\left(G_{2,k-1} - \hat{B}_k(k) C_{2,k} G_{2,k-1}\right) d_{2,k-1} + \sum_{i=0}^{k-1} \hat{B}_i(k) C_{2,i} G_{2,i-1} d_{2,i-1}.$$

Because $d_{2,i}$ for $i = 0, \dots, k$ are arbitrary vectors, the matrices before them must be zero matrices to ensure the unbiased estimation.

Which completes the proof. □

Remark 3. From Theorem 1 we can find that (24) is an unbiased estimator if, and only if, it is in the following form:

$$\hat{x}_k = \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + G_{1,k-1}\Sigma_{k-1}^{-1} z_{1,k-1} + \sum_{i=0}^{k} \hat{B}_i(k)\tilde{z}_{2,i}$$

$$= \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + G_{1,k-1}\Sigma_{k-1}^{-1}z_{1,k-1} + \hat{B}_k(k)\,\tilde{z}_{2,k} + \sum_{i=0}^{k-1}\hat{B}_i(k)\,\tilde{z}_{2,i} \tag{27}$$

with the constrains that $\hat{B}_i(k)\,C_{2,i}G_{2,i-1} = 0, i = 0, \ldots, k-1$ and $\hat{B}_k(k)\,C_{2,k}G_{2,k-1} = G_{2,k-1}$.

Let

$$\begin{aligned} a_k &= G_{1,k-1}\Sigma_{k-1}^{-1}z_{1,k-1} \\ \chi_k &= x_k - a_k \\ \bar{z}_{2,k} &= z_{2,k} - C_{2,k}a_k. \end{aligned} \tag{28}$$

Since the only difference between x_k and χ_k is a deterministic vector a_k, the only difference between the estimation of x_k and χ_k should also be this vector, i.e. $\hat{\chi}_k = \hat{x}_k - a_k$.

As a result, by use of (27), the estimator of χ_k should be in the following form:

$$\begin{aligned} \hat{\chi}_k &= \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + G_{1,k-1}\Sigma_{k-1}^{-1}z_{1,k-1} \\ &\quad + \hat{B}_k(k)\,\tilde{z}_{2,k} + \sum_{i=0}^{k-1}\hat{B}_i(k)\,\tilde{z}_{2,i} - a_k \\ &= \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} \\ &\quad + \hat{B}_k(k)\,\tilde{z}_{2,k} + \sum_{i=0}^{k-1}\hat{B}_i(k)\,\tilde{z}_{2,i}. \end{aligned} \tag{29}$$

Substitute (28) into (7) and (12) respectively. Then it follows:

$$\begin{aligned} \chi_{k+1} &= \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)x_k \\ &\quad + G_{2,k}d_{2,k} + \left(w_k - G_{1,k}\Sigma_k^{-1}v_{1,k}\right) \end{aligned} \tag{30}$$

$$\bar{z}_{2,k+1} = C_{2,k}\chi_{k+1} + v_{2,k+1}$$

and

$$\tilde{z}_{2,i} = \bar{z}_{2,i} - C_{2,i}\left(A_{i-1} - G_{1,i-1}\Sigma_{i-1}^{-1}C_{1,i-1}\right)\hat{x}_{i-1}. \tag{31}$$

In addition, according to (10), the system noise and the measurement noise in Eq. (30) are white noises and they are mutually uncorrelated. We then have the system equations (Eq. (30)) and the estimator (Eqs. (29) and (31)) in the form considered by Kerwin and Prince (2000). As a result, we can directly use the result from Kerwin and Prince (2000), i.e. the global optimal (global LMVU) estimator of χ_k should be in the form:

$$\hat{\chi}_k = \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + \hat{B}_k(k)\,\tilde{z}_{2,k}.$$

As a result, the LMVU estimator of $\hat{x}_k$ should be in the form:

$$\hat{x}_k = \left(A_{k-1} - G_{1,k-1}\Sigma_{k-1}^{-1}C_{1,k-1}\right)\hat{x}_{k-1} + \hat{B}_k(k)\,\tilde{z}_{2,k} + a_k$$

which is in the same form of (11).

3. Example

To illustrate the MVU filter, an example similar to that in Keller, Summerer, Boutayeb, and Darouach (1996) is given in this section. Suppose that the matrices of system (1) are given by:

$$A = \begin{bmatrix} 0.5 & 2 & 0 & 0 & 0 \\ 0 & 0.2 & 1 & 0 & 1 \\ 0 & 0 & 0.3 & 0 & 1 \\ 0 & 0 & 0 & 0.7 & 1 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 & -0.3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

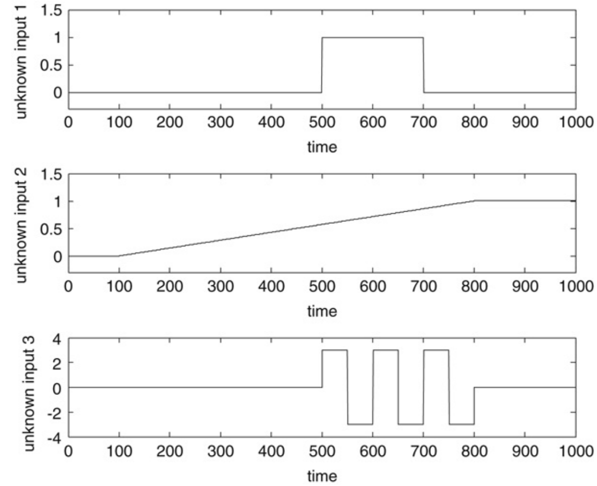

Fig. 1. Unknown input d_k.

$$C = I_5, \qquad H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R = 10^{-2} \times \begin{bmatrix} 1 & & & 0.5 & \\ & 1 & & & 0.3 \\ & & 1 & & \\ 0.5 & & & 1 & \\ & 0.3 & & & 1 \end{bmatrix}$$

$$Q = 10^{-4} \times \begin{bmatrix} 1 & & & & \\ & 1 & 0.5 & & \\ & 0.5 & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

Because H is not of full column rank, the state filter provided in Gillins and De Moor (2007b) cannot be used.

In addition, the filter proposed in Darouach et al. (2003) will not converge for the above system, because

$$n + \operatorname{rank}(G) + \operatorname{rank}(H) = 5 + 2 + 2 = 9$$

$$\operatorname{rank}\begin{bmatrix} zI - A & -G & 0 \\ C & 0 & H \end{bmatrix} \leq 8 \neq 9$$

which means that the convergence condition in Darouach et al. (2003) cannot be satisfied.

Now consider the method proposed in this paper.

First of all, calculate the matrices G_2, C_2 and $\hat{A}$ and R_2:

$$G_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\hat{A} = \begin{bmatrix} 0.8 & 2 & 0 & -0.15 & 0 \\ 0 & 0.2 & 1 & 0 & 1 \\ 0 & 0 & 0.3 & 0 & 1 \\ 0 & 0 & 0 & 0.7 & 1 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 1 & 0 & 0.3 \\ 0 & 1 & 0 \\ 0.3 & 0 & 1 \end{bmatrix}.$$

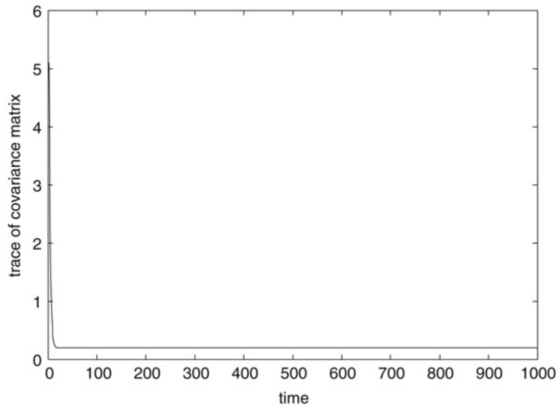

Fig. 2. Trace of the covariance matrix P_k.

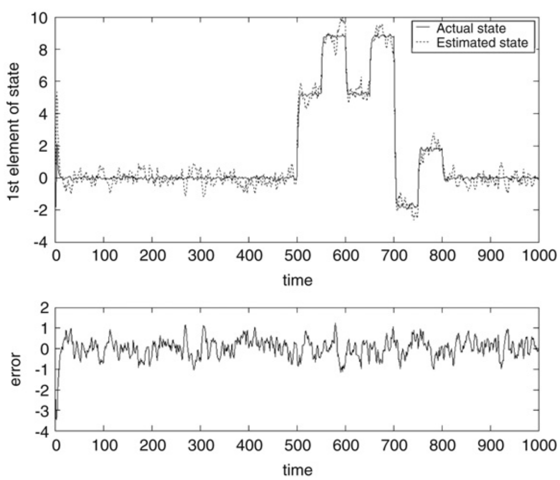

Fig. 3. Actual and estimated value of the first element of state.

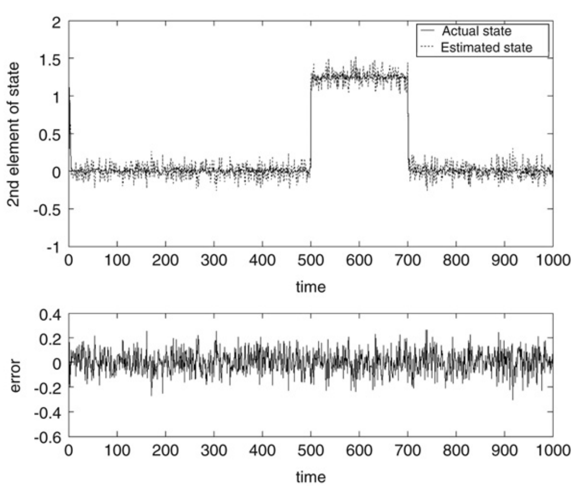

Fig. 4. Actual and estimated value of the second element of state.

Then it can be easily proved that both (21) and (22) are satisfied, which means that our filter is convergent and stable.

Fig. 1 presents the unknown input d_k. The convergence of the trace of covariance matrix P_k is shown in Fig. 2. In Figs. 3 and 4, we plot the actual and the estimated value of the first element and the second element of the state vector x_k respectively.

4. Conclusion

A state estimator for systems with unknown input in both the system and output equations is proposed, its convergence and stability condition is provided, and its optimality is proved.

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