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Technical communique

Extension of unbiased minimum-variance input and state estimation for systems with unknown inputs☆

Chien-Shu Hsieh *

Department of Electrical Engineering, Ta Hwa Institute of Technology, Hsinchu 30740, Taiwan, ROC

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ABSTRACT

This paper extends the existing results on joint input and state estimation to systems with arbitrary unknown inputs. The objective is to derive an optimal filter in the general case where not only unknown inputs affect both the system state and the output, but also the direct feedthrough matrix has arbitrary rank. The paper extends both the results of Gillijns and De Moor [Gillijns, S., & De Moor, B. (2007b). Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feedthrough. *Automatica*, *43*, 934–937] and Darouach, Zasadzinski, and Boutayeb [Darouach, M., Zasadzinski, M., & Boutayeb, M. (2003). Extension of minimum variance estimation for systems with unknown inputs. *Automatica*, *39*, 867–876]. The resulting filter is an extension of the recursive three-step filter (ERTSF) and serves as a unified solution to the addressed unknown input filtering problem. The relationship between the ERTSF and the existing literature results is also addressed.

1. Introduction

Unknown input filtering has played a significant role in many applications, e.g., bias compensation (Friedland, 1969; Hsieh & Chen, 1999), geophysical and environmental applications (Kitanidis, 1987), and fault detection and isolation problems (Chen & Patton, 1996). To be more general, it may not be any knowledge concerning the model of the unknown input.

A common approach to solve this unknown input filtering problem is to produce the unknown input decoupled state estimation, which may be achieved by making use of unbiased minimum-variance estimation (Chen & Patton, 1996; Darouach & Zasadzinski, 1997; Gillijns & De Moor, 2007a; Hsieh, 2000; Kitanidis, 1987). In this approach, the filter parameters are first determined to satisfy some algebraic constraints according to the unbiasedness condition of the filter. Next, the parameter matrices are determined in such a way that the state estimation error variance is minimum. It is shown that the results presented by Darouach and Zasadzinski (1997), Hsieh (2000), and Kitanidis (1987) are globally optimal for systems with unknown inputs that only affect the system state (Gillijns & De Moor, 2007a; Kerwin & Prince, 2000). It should be noted that in the above works, only those of Gillijns and De Moor (2007a) and Hsieh (2000) explicitly address the problem of joint input and state estimation.

Trying to extend the aforementioned optimal filters to the case where unknown inputs affect both the system state and the output, Darouach, Zasadzinski, and Boutayeb (2003) have applied the previously proposed parameterizing technique to derive the optimal estimator filter (OEF). More recently, Gillijns and De Moor (2007b) have also proposed a recursive three-step filter (RTSF) to solve the problem of joint input and state estimation for systems with direct feedthrough of the unknown input to the output. It is noted that the RTSF is less restrictive than the OEF since the estimation of the state and the unknown input are interconnected in the former, which may yield better filtering performance. However, the RTSF is applicable only to the case that the direct feedthrough matrix (of the unknown input to the output) has full rank. Some other approaches have also been used to solve the addressed unknown input filtering problem, e.g., the minimax–optimal method (Borisov & Pankov, 1994), the equivalent system description and innovations filtering method (Hou & Patton, 1998), and the time-delayed optimal state estimator design method (Sundaram & Hadjicostis, 2006).

In this technical communique, we intend to present an extension of the RTSF (ERTSF) based on the approaches developed by Darouach et al. (2003) and Gillijns and De Moor (2007b) to estimate both the system state and the unknown input. The result is applicable to the general case where not only unknown inputs affect both the system state and the output, but also the direct feedthrough matrix has arbitrary rank. Moreover, the relationship between the ERTSF and the filters in Borisov and Pankov (1994), Darouach et al. (2003), and Hou and Patton (1998) is also addressed.

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* Corresponding address: Ta Hwa Institute of Technology, 1, Ta Hwa Road, Chiunglin Hsinchu 30740, Taiwan, ROC. Tel.: +886 35927700; fax: +886 35927085.
E-mail address: cshsieh@thit.edu.tw.

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2. Problem formulation

Consider the linear discrete-time stochastic time-varying system with unknown inputs in the form

$$x_{k+1} = A_k x_k + G_k d_k + w_k, \tag{1}$$

$$y_k = C_k x_k + H_k d_k + v_k, \tag{2}$$

where $x_k \in R^n$ is the state vector, $d_k \in R^m$ is an unknown input vector, and $y_k \in R^p$ is the measurement vector. The process noise $w_k \in R^n$ and the measurement noise $v_k \in R^p$ are assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrices, $Q_k = E[w_k w_k^{\mathrm{T}}] \geq 0$ and $R_k = E[v_k v_k^{\mathrm{T}}] > 0$, respectively. The matrices A_k, G_k, C_k, and H_k are known and it is assumed that (A_k, C_k) is observable and that x_0 is independent of v_k and w_k for all k. Moreover, an unbiased estimate $\check{x}_0$ of the initial state x_0 is available with covariance matrix P_0^x.

Recently, to solve the addressed problem, Gillijns and De Moor (2007b) proposed the following RTSF:

Step 1: Estimation of unknown input

$$\tilde{R}_k = C_k P_{k|k-1}^x C_k^{\mathrm{T}} + R_k, \tag{3}$$

$$M_k = (H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k)^{-1} H_k^{\mathrm{T}} \tilde{R}_k^{-1}, \tag{4}$$

$$\hat{d}_{k|k} = M_k (y_k - C_k \hat{x}_{k|k-1}), \tag{5}$$

$$P_{k|k}^d = (H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k)^{-1}. \tag{6}$$

Step 2: Measurement update

$$K_k = P_{k|k-1}^x C_k^{\mathrm{T}} \tilde{R}_k^{-1}, \tag{7}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1} - H_k \hat{d}_{k|k}), \tag{8}$$

$$P_{k|k}^x = P_{k|k-1}^x - K_k (\tilde{R}_k - H_k P_{k|k}^d H_k^{\mathrm{T}}) K_k^{\mathrm{T}}, \tag{9}$$

$$(P_{k|k}^{dx}) = (P_{k|k}^{xd})^{\mathrm{T}} = -(K_k H_k P_{k|k}^d)^{\mathrm{T}}. \tag{10}$$

Step 3: Time update

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + G_k \hat{d}_{k|k}, \tag{11}$$

$$P_{k+1|k}^x = \begin{bmatrix} A_k & G_k \end{bmatrix} \begin{bmatrix} P_{k|k}^x & P_{k|k}^{xd} \\ P_{k|k}^{dx} & P_{k|k}^d \end{bmatrix} \begin{bmatrix} A_k^{\mathrm{T}} \\ G_k^{\mathrm{T}} \end{bmatrix} + Q_k. \tag{12}$$

The initialization of the above RTSF is given as follows:

$$\hat{x}_{0|-1} = \check{x}_0, \qquad P_{0|-1}^x = P_0^x. \tag{13}$$

It is noted that under the following assumption:

$$rank[H_k] = m, \tag{14}$$

the above RTSF can yield the optimal state and input estimates. The necessity of assumption (14) is to promise that (5) yields an unbiased estimate of d_k. It then turns out that the estimates $\hat{x}_{k|k}$ and $\hat{x}_{k+1|k}$ are unbiased for all $k \geq 0$. Since assumption (14) in general may not hold, the unbiasedness of $\hat{d}_{k|k}$ may not be guaranteed, and hence the obtained system state estimator may become a biased one. The main aim of this technical communique is to derive the ERTSF in order to relax the rank condition (14), i.e., in achieving an unbiased system state estimation on the following general rank conditions:

$$0 < rank[H_k] \leq m. \tag{15}$$

Remark 1. A heuristic extension of the above RTSF may be obtained by replacing (4) and (6) with

$$M_k = P_{k|k}^d H_k^{\mathrm{T}} \tilde{R}_k^{-1}, \qquad P_{k|k}^d = (H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k)^{+}, \tag{16}$$

where M^{+} is the Moore–Penrose pseudo-inverse of M. The above modified RTSF is denoted as the RTSF* and its performance will be evaluated in Section 5.

3. Derivation of the ERTSF

We consider the ERTSF as the same form as given in Gillijns and De Moor (2007b), but with different choices of the gain matrices $M_k^* \in R^{m \times p}$ and $L_k^* \in R^{n \times p}$:

$$\hat{d}_{k|k} = M_k^* (y_k - C_k \hat{x}_{k|k-1}), \tag{17}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k^* (y_k - C_k \hat{x}_{k|k-1}), \tag{18}$$

where $\hat{x}_{k|k-1}$ is updated by (11). The basic idea in deriving the ERTSF is to obtain an unbiased system state estimate (18) regardless of the possible inherent bias in the unknown input estimate (17).

First of all, we obtain the errors in the initial estimates $\hat{d}_{0|0}$ and $\hat{x}_{0|0}$ as follows:

$$\tilde{d}_{0|0} = (I - M_0^* H_0) d_0 - M_0^* C_0 \tilde{x}_{0|-1} - M_0^* v_0, \tag{19}$$

$$\tilde{x}_{0|0} = (I - L_0^* C_0) \tilde{x}_{0|-1} - L_0^* v_0 - L_0^* H_0 d_0, \tag{20}$$

where $\tilde{x}_{0|-1} = x_0 - \hat{x}_{0|-1}$, $\tilde{d}_{0|0} = d_0 - \hat{d}_{0|0}$, $\tilde{x}_{0|0} = x_0 - \hat{x}_{0|0}$, $M_0^* = (H_0^{\mathrm{T}} \tilde{R}_0^{-1} H_0)^{+} H_0^{\mathrm{T}} \tilde{R}_0^{-1}$, and $L_0^* = K_0 (I - H_0 M_0^*)$. Using the first condition in (13) and the constraint: $L_0^* H_0 = 0$, we obtain

$$E[\tilde{d}_{0|0}] = (I - M_0^* H_0) d_0, \qquad E[\tilde{x}_{0|0}] = 0. \tag{21}$$

From the first condition in (21), it is clear that if $rank[H_0] < m$ then the unknown input estimator (17) may yield a biased estimate, i.e., $E[\tilde{d}_{0|0}] \neq 0$.

Second, we consider the unbiased state estimation. We obtain the errors in the estimates $\hat{x}_{k|k-1}$, $\hat{d}_{k|k}$, and $\hat{x}_{k|k}$ for $k > 0$ as follows:

$$\tilde{x}_{k|k-1} = A_{k-1} \tilde{x}_{k-1|k-1} + G_{k-1} \tilde{d}_{k-1|k-1} + w_{k-1}, \tag{22}$$

$$\tilde{d}_{k|k} = (I - M_k^* H_k) d_k - M_k^* C_k \tilde{x}_{k|k-1} - M_k^* v_k, \tag{23}$$

$$\tilde{x}_{k|k} = (I - L_k^* C_k) \tilde{x}_{k|k-1} - L_k^* v_k - L_k^* H_k d_k. \tag{24}$$

Introduce the following notations:

$$\Phi_k = M_k^* H_k = I_m - \Pi_k, \qquad \Xi_k = M_k^* C_k G_{k-1}. \tag{25}$$

Substituting (22) into (23) and assuming that $E[\tilde{x}_{k-1|k-1}] = 0$, we obtain

$$\begin{aligned} E[\tilde{d}_{k|k}] &= \Pi_k d_k - (\Xi_k \Pi_{k-1}) d_{k-1} + \Xi_k (\Xi_{k-1} \Pi_{k-2}) d_{k-2} \\ &\quad + \cdots + (-1)^k \Xi_k \times \cdots \times \Xi_2 (\Xi_1 \Pi_0) d_0 \\ &= \Pi_k d_k, \end{aligned} \tag{26}$$

where we assume that $\Xi_i \Pi_{i-1} = 0$ for $i = 1, \ldots, k$, which will be shown later. Using (25) and (26), we obtain

$$M_k^* S_k = \begin{bmatrix} \Phi_k & 0_m \end{bmatrix}, \qquad S_k = \begin{bmatrix} H_k & C_k G_{k-1} \Pi_{k-1} \end{bmatrix}. \tag{27}$$

Note that if $\Phi_k = I_m$ then one can obtain an unbiased unknown input estimator from (17). Substituting (22) into (24) and using (26), we obtain

$$E[\tilde{x}_{k|k}] = (I - L_k^* C_k) G_{k-1} \Pi_{k-1} d_{k-1} - L_k^* H_k d_k,$$

from which we obtain the constraint to achieve an unbiased estimator of x_k as follows:

$$L_k^* S_k = \Gamma_k, \quad \Gamma_k = \begin{bmatrix} 0_{n \times m} & G_{k-1} \Pi_{k-1} \end{bmatrix}. \tag{28}$$

There exist matrices M_k^* and L_k^* satisfying (27) and (28) if and only if

$$rank[S_k] = rank[H_k] + rank[G_{k-1} \Pi_{k-1}],$$

which can be obtained by using the approach of Darouach et al. (2003).

Third, we consider the minimum-variance unbiased state estimation. Using (22)–(24) and (27)–(28), the expressions for (6), (9) and (10) are updated, respectively, as

$$P^d_{k|k} = E[(\tilde{d}_{k|k} - \Pi_k d_k)(\tilde{d}_{k|k} - \Pi_k d_k)^{\mathrm{T}}] = M^*_k \tilde{R}_k (M^*_k)^{\mathrm{T}}, \tag{29}$$

$$P^x_{k|k} = E[\tilde{x}_{k|k}\tilde{x}^{\mathrm{T}}_{k|k}] = P^x_{k|k-1} - L^*_k \tilde{R}_k (L^*_k)^{\mathrm{T}} + L^*_k \Psi_k + (L^*_k \Psi_k)^{\mathrm{T}}, \tag{30}$$

$$P^{dx}_{k|k} = E[(\tilde{d}_{k|k} - \Pi_k d_k)\tilde{x}^{\mathrm{T}}_{k|k}] = M^*_k \Psi_k, \tag{31}$$

where

$$\Psi_k = \tilde{R}_k (L^*_k)^{\mathrm{T}} - C_k P^x_{k|k-1}. \tag{32}$$

We are now in place to derive the gain matrix L^*_k. Applying the approach of Kitanidis (1987) and using (28) and (30), we form the following linear system of equations:

$$\begin{bmatrix} \tilde{R}_k & -S_k \\ S^{\mathrm{T}}_k & 0 \end{bmatrix} \begin{bmatrix} (L^*_k)^{\mathrm{T}} \\ \Lambda^{\mathrm{T}}_k \end{bmatrix} = \begin{bmatrix} C_k P^x_{k|k-1} \\ \Gamma^{\mathrm{T}}_k \end{bmatrix}, \tag{33}$$

where Λ_k is the matrix of Lagrange multipliers. The system (33) has a unique solution if and only if $S^{\mathrm{T}}_k \tilde{R}^{-1}_k S_k$ is nonsingular. In general, this condition does not hold. To remedy this problem, the Moore–Penrose pseudo-inverse M^+ is used instead of the inverse M^{-1}. Thus, the gain matrix L^*_k is obtained as follows:

$$L^*_k = K_k + (\Gamma_k - K_k S_k) S^*_k, \tag{34}$$

where

$$S^*_k = (S^{\mathrm{T}}_k \tilde{R}^{-1}_k S_k)^+ S^{\mathrm{T}}_k \tilde{R}^{-1}_k. \tag{35}$$

It remains to determine the gain matrix M^*_k. According to the well-known matrix equation solution theory (Rao & Mitra, 1971), the general solution of M^*_k to (27) is

$$M^*_k = \begin{bmatrix} \Phi_k & 0_m \end{bmatrix} S^*_k + Z_k (I_p - S_k S^*_k), \tag{36}$$

where Z_k is an arbitrary matrix. Using (36) and solving (29) for Z_k that minimizes the trace of $P^d_{k|k}$, we obtain $Z_k = 0$. Using this optimal Z_k and (27), we obtain

$$\begin{bmatrix} \Phi_k & 0_m \end{bmatrix} (I - S^*_k S_k) = 0. \tag{37}$$

One possible solution of the matrix Φ_k satisfying (37) is given as $\Phi_k = H^+_k H_k$, by which we obtain

$$M^*_k = \begin{bmatrix} H^+_k H_k & 0_m \end{bmatrix} S^*_k, \tag{38}$$

$$S_k = \begin{bmatrix} H_k & C_k G_{k-1}(I_m - H^+_{k-1} H_{k-1}) \end{bmatrix}, \tag{39}$$

$$\Gamma_k = \begin{bmatrix} 0_{n\times m} & G_{k-1}(I_m - H^+_{k-1} H_{k-1}) \end{bmatrix}. \tag{40}$$

Using (25), (27)–(28), (37)–(38), we obtain

$$\Xi_k \Pi_{k-1} = H^+_k C_k \Gamma_k (I - S^*_k S_k) \begin{bmatrix} 0 & I \end{bmatrix}' = 0.$$

Finally, the proposed ERTSF is given by (3), (7), (11)–(12), (17)–(18), (29)–(32), (34)–(35), and (38)–(40).

Remark 2. The above ERTSF can be slightly modified and applied to the following linear dynamic indeterminate-stochastic system (ISS) (Borisov & Pankov, 1994):

$$x_k = A_k x_{k-1} + G_k d_k + w_k,$$
$$y_k = C_k x_k + H_k d_k + v_k.$$

This is achieved by replacing the time update Eqs. (11) and (12), respectively, with

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k}, \qquad P^x_{k+1|k} = A_k P^x_{k|k} A^{\mathrm{T}}_k + Q_k, \tag{41}$$

system matrices S_k (39) and Γ_k (40), respectively, with

$$\tilde{S}_k = \begin{bmatrix} H_k & C_k G_k \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix}^{\mathrm{T}}, \qquad \tilde{\Gamma}_k = \begin{bmatrix} 0 & G_k \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix}^{\mathrm{T}},$$

and by using the following substitutions: $A_k \to A_{k+1}$, $G_k \to G_{k+1}$, and $Q_k \to Q_{k+1}$ in the ERTSF.

4. Relationships with the existing literature results

4.1. Gillijns and De Moor (2007b)

Consider the special case that the matrix H_k is of full column rank, i.e., $rank[H_k] = m$. Then, we have $\Pi_k = 0$ for all $k \geq 0$, and hence one has the following gain relationships:

$$M^*_k = M_k, \qquad L^*_k = K_k(I - H_k M_k).$$

Moreover, it can be easily verified that the matrices (29)–(31) are equivalent to the corresponding covariance matrices given in (6), (9) and (10), respectively. Thus, the ERTSF will be equivalent to the RTSF.

4.2. Darouach et al. (2003)

Consider the following OEF (Darouach et al., 2003):

$$\hat{x}_k = \bar{x}_k + L_k(y_k - C_k \bar{x}_k), \tag{42}$$

$$P_k = \bar{P}_k + L_k \bar{R}_k L^{\mathrm{T}}_k - \bar{P}_k C^{\mathrm{T}}_k L^{\mathrm{T}}_k - L_k C_k \bar{P}_k, \tag{43}$$

where

$$\bar{x}_{k+1} = A_k \hat{x}_k, \qquad \bar{P}_{k+1} = A_k P_k A^{\mathrm{T}}_k + Q_k, \tag{44}$$

$$L_k = \bar{\Gamma}_k \Sigma^+_k + \bar{K}_k \hat{H}_k, \qquad \hat{H}_k = \alpha_k (I - \Sigma_k \Sigma^+_k), \tag{45}$$

$$\bar{K}_k = \left(\bar{P}_k C^{\mathrm{T}}_k - \bar{\Gamma}_k \Sigma^+_k \bar{R}_k\right) \hat{H}^{\mathrm{T}}_k (\hat{H}_k \bar{R}_k \hat{H}^{\mathrm{T}}_k)^{-1}, \tag{46}$$

$$\bar{R}_k = C_k \bar{P}_k C^{\mathrm{T}}_k + R_k, \tag{47}$$

$$\bar{\Gamma}_k = \begin{bmatrix} 0 & G_{k-1} \end{bmatrix}, \qquad \Sigma_k = \begin{bmatrix} H_k & C_k G_{k-1} \end{bmatrix}. \tag{48}$$

In (45), the matrix parameter α_k must be chosen so that $\hat{H}_k$ is of full row rank.

If one intends to make the ERTSF to be invariant with respect to the error $\tilde{d}_{k|k}$ for $k \geq 0$, which corresponds to choosing $\Pi_k = I$, then one has

$$S_k = \Sigma_k, \qquad \Gamma_k = \bar{\Gamma}_k. \tag{49}$$

It will be shown in Theorem 1 that this specific ERTSF can be equivalent to the OEF.

Theorem 1. *If the following initial conditions:*

$$\hat{x}_{0|0} = \hat{x}_0 = \check{x}_0, \qquad P^x_{0|0} = P_0 = P^x_0,$$

are used and the following relationships:

$$\Sigma^+_k = (\Sigma^{\mathrm{T}}_k \tilde{R}^{-1}_k \Sigma_k)^+ \Sigma^{\mathrm{T}}_k \tilde{R}^{-1}_k, \tag{50}$$

$$I - \Sigma_k \Sigma^+_k = \bar{R}_k \hat{H}^{\mathrm{T}}_k (\hat{H}_k \bar{R}_k \hat{H}^{\mathrm{T}}_k)^{-1} \hat{H}_k, \tag{51}$$

hold, the ERTSF, which is given by (3), (7), (11)–(12), (18), (30), (32), (34)–(35), and (49), is equivalent to the OEF, that is referenced by (42)–(48).

Proof. Using (49) in (28) yields

$$(I - L^*_k C_k) G_{k-1} = 0. \tag{52}$$

Next, we note that using (52), (11) and (12) can be simplified as those in (41). Then, it suffices to verify that the gain matrices L_k^* (34) and L_k (45) are equivalent subject to conditions (50) and (51).

Using (7), (35), (49) and (50) in (34) yields

$$L_k^* = \bar{\Gamma}_k \Sigma_k^+ + P_{k|k-1}^x C_k^{\mathrm{T}} \tilde{R}_k^{-1} (I - \Sigma_k \Sigma_k^+). \tag{53}$$

Using (46) and (51) in (45) yields

$$L_k = \bar{\Gamma}_k \Sigma_k^+ + \left(\bar{P}_k C_k^{\mathrm{T}} \bar{R}_k^{-1} - \bar{\Gamma}_k \Sigma_k^+\right) (I - \Sigma_k \Sigma_k^+). \tag{54}$$

Using (53) and (54), one can easily verify by induction that the gain matrices L_k^* and L_k are equivalent. □

4.3. Hou and Patton (1998)

Consider the optimal linear minimum-variance estimator (OLMVE) (Hsieh, 2006), which is obtained by applying the innovations filtering technique (Hou & Patton, 1998) to the considered system (1)–(2), as follows:

$$\hat{x}_k = \bar{x}_k + \bar{K}_k T_k (y_k - C_k \bar{x}_k), \tag{55}$$

$$P_k = \bar{P}_k + \bar{H}_k R_k \bar{H}_k^{\mathrm{T}} - \bar{K}_k \bar{R}_k \bar{K}_k^{\mathrm{T}}, \tag{56}$$

where

$$\begin{aligned} \bar{x}_{k+1} &= D_k (\bar{\Phi}_k \hat{x}_k + G_k H_k^+ y_k) + \bar{H}_{k+1} y_{k+1} \\ &\quad - D_k G_k H_k^+ R_k \bar{T}_k^{\mathrm{T}} \bar{R}_k^+ T_k (y_k - C_k \bar{x}_k), \end{aligned} \tag{57}$$

$$\bar{P}_{k+1} = D_k \breve{P}_{k+1} D_k^{\mathrm{T}}, \qquad \breve{P}_{k+1} = \bar{\Phi}_k P_k \bar{\Phi}_k^{\mathrm{T}} + \bar{Q}_k, \tag{58}$$

$$\bar{K}_k = (\breve{P}_k C_k^{\mathrm{T}} - \bar{H}_k \breve{R}_k) \bar{T}_k^{\mathrm{T}} \bar{R}_k^+, \tag{59}$$

$$\bar{H}_k = G_{k-1} U_{k-1} (T_k C_k G_{k-1} U_{k-1})^+ T_k, \tag{60}$$

$$\bar{R}_k = \bar{T}_k \breve{R}_k \bar{T}_k^{\mathrm{T}}, \qquad \breve{R}_k = C_k \breve{P}_k C_k^{\mathrm{T}} + R_k, \tag{61}$$

$$\begin{aligned} \bar{Q}_k &= Q_k + \bar{\Phi}_k L_k R_k (H_k^+)^{\mathrm{T}} G_k^{\mathrm{T}} + G_k H_k^+ R_k L_k^{\mathrm{T}} \bar{\Phi}_k^{\mathrm{T}} \\ &\quad + G_k H_k^+ (R_k - R_k \bar{T}_k^{\mathrm{T}} \bar{R}_k^+ \bar{T}_k R_k)(H_k^+)^{\mathrm{T}} G_k^{\mathrm{T}}, \end{aligned} \tag{62}$$

$$D_k = I - \bar{H}_{k+1} C_{k+1}, \qquad \bar{\Phi}_k = A_k - G_k H_k^+ C_k, \tag{63}$$

$$T_k = I - H_k H_k^+, \qquad U_k = I - H_k^+ H_k, \tag{64}$$

$$L_k = \bar{H}_k + \bar{K}_k \bar{T}_k, \qquad \bar{T}_k = T_k (I - C_k \bar{H}_k). \tag{65}$$

It will be shown in Theorem 2 that the above OLMVE can be equivalent to the proposed ERTSF. To facilitate the derivation, (55) and (56) are rewritten, respectively, as follows:

$$\hat{x}_k = \bar{x}_{k|k-1} + L_k (y_k - C_k \bar{x}_{k|k-1}),$$

$$P_k = (I - L_k C_k) \breve{P}_k (I - L_k C_k)^{\mathrm{T}} + L_k R_k L_k^{\mathrm{T}}, \tag{66}$$

via using the following substitutions:

$$\bar{x}_k = \bar{x}_{k|k-1} + \bar{H}_k (y_k - C_k \bar{x}_{k|k-1}),$$

$$\bar{x}_{k+1|k} = A_k \hat{x}_k + G_k M_k (y_k - C_k \bar{x}_{k|k-1}),$$

where M_k is given as follows:

$$M_k = H_k^+ (I - C_k \bar{H}_k)(I - \breve{R}_k \bar{T}_k^{\mathrm{T}} \bar{R}_k^+ \bar{T}_k). \tag{67}$$

Theorem 2. *If the following initial conditions:*

$$\hat{x}_{0|-1} = \bar{x}_{0|-1} = \breve{x}_0, \qquad P_{0|-1}^x = \breve{P}_0 = P_0^x$$

are used and the following relationships:

$$\Gamma_k S_k^* = \bar{H}_k, \qquad I - S_k S_k^* = \breve{R}_k \bar{T}_k^{\mathrm{T}} \bar{R}_k^+ \bar{T}_k, \tag{68}$$

hold, the ERTSF, which is given by (3), (7), (11)–(12), (17)–(18), (29)–(32), (34)–(35), *and* (38)–(40), *is equivalent to the OLMVE, that is referenced by* (55)–(65).

Table 1
Performances of the OEF, OLMVE, TDOSE, RTSF*, and ERTSF.

H_k	Filter	$\tilde{x}_{k\|k}^1$	$\tilde{x}_{k\|k}^2$	$\tilde{d}_{k\|k}^1$	$\tilde{d}_{k\|k}^2$
	OEF	0.1000	11.6225	NA	NA
	OLMVE	0.1000	11.6225	NA	NA
H_k^1	TDOSE	0.1000	11.6225	NA	NA
	RTSF*	0.1719	21.9302	3.5707	21.9347
	ERTSF	0.1000	11.6225	3.5707	11.6333
	OEF	0.0662	0.4018	NA	NA
	OLMVE	0.0427	0.3451	NA	NA
H_k^2	TDOSE	0.0603	0.4018	NA	NA
	RTSF*	0.0427	0.3451	0.1085	3.2496
	ERTSF	0.0427	0.3451	0.1085	3.2496
	OEF	0.0839	5.1863	NA	NA
	OLMVE	0.0609	1.0583	NA	NA
H_k^3	TDOSE	0.0609	1.0583	NA	NA
	RTSF*	0.0609	1.0583	0.1171	1.1320
	ERTSF	0.0609	1.0583	0.1171	1.1320

Proof. It suffices to verify that L_k, $\bar{x}_{k+1|k}$, and $\breve{P}_{k+1}$ are equivalent to L_k^*, $\hat{x}_{k+1|k}$, and $P_{k+1|k}^x$, respectively.

Using (59), (65) and (68), we obtain

$$L_k = \Gamma_k S_k^* + (\breve{P}_k C_k^{\mathrm{T}} \breve{R}_k^{-1} - \Gamma_k S_k^*)(I - S_k S_k^*). \tag{69}$$

Using (27)–(28) and (67)–(68), we obtain

$$M_k = H_k^+ (S_k - C_k \Gamma_k) S_k^* = \begin{bmatrix} H_k^+ H_k & 0_m \end{bmatrix} S_k^*. \tag{70}$$

Using (59), (61) and (65)–(67), we have

$$R_k L_k^{\mathrm{T}} - C_k P_k = (I - C_k L_k) \bar{\Psi}_k, \qquad \bar{\Psi}_k = \breve{R}_k L_k^{\mathrm{T}} - C_k \breve{P}_k,$$

$$H_k^+ (I - C_k L_k) = H_k^+ R_k \bar{T}_k^{\mathrm{T}} \bar{R}_k^+ \bar{T}_k + M_k,$$

by which $\breve{P}_{k+1}$ in (58) can be reformulated as

$$\begin{aligned} \breve{P}_{k+1} &= A_k P_k A_k^{\mathrm{T}} + Q_k + G_k M_k \bar{\Psi}_k A_k^{\mathrm{T}} + A_k \bar{\Psi}_k^{\mathrm{T}} M_k^{\mathrm{T}} G_k^{\mathrm{T}} \\ &\quad + G_k M_k \breve{R}_k M_k^{\mathrm{T}} G_k^{\mathrm{T}} + G_k \Delta_k \begin{bmatrix} A_k & G_k \end{bmatrix}^{\mathrm{T}} \\ &\quad + \begin{bmatrix} A_k & G_k \end{bmatrix} \Delta_k^{\mathrm{T}} G_k^{\mathrm{T}}, \end{aligned} \tag{71}$$

where

$$\Delta_k = H_k^+ R_k \bar{T}_k^{\mathrm{T}} \bar{R}_k^+ \bar{T}_k \begin{bmatrix} \bar{\Psi}_k & \breve{R}_k M_k^{\mathrm{T}} \end{bmatrix} = 0. \tag{72}$$

Using (69)–(72), one can easily verify by induction that L_k, $\bar{x}_{k+1|k}$, and $\breve{P}_{k+1}$ are equivalent to L_k^*, $\hat{x}_{k+1|k}$, and $P_{k+1|k}^x$, respectively. □

5. An illustrative example

Consider the numerical example given by Darouach et al. (2003). In this simulation, three cases of H_k, i.e., $H_k = H_k^1 = diag(0, 1)$, $H_k = H_k^2 = diag(1, 0)$, and $H_k = H_k^3 = I_2$, and the filters OEF, OLMVE, RTSF*, ERTSF, and the TDOSE (time-delayed optimal state estimator) given by Sundaram and Hadjicostis (2006) are considered. We list the root-mean-square-errors (rmse) in the state estimates of the aforementioned filters in Table 1, where the estimator delay α of the TDOSE is chosen by 1, 1, and 0 for H_k^1, H_k^2, and H_k^3, respectively.

From Table 1, we may conclude the following results.

(1) The TDOSE and the OEF have similar performances for $\alpha = 1$ and may not give the best performance.
(2) Except for the special case $G_k \Pi_k = G_k$ ($H_k = H_k^1$) the OEF has the worst performance.
(3) The RTSF* may not give the best performance.
(4) The filtering performances of the ERTSF and the OLMVE are the same; both give the best performance. Note that the former also explicitly gives the unknown input estimates.

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References

Borisov, A. V., & Pankov, A. R. (1994). Optimal filtering in stochastic discrete-time systems with unknown inputs. *IEEE Transactions on Automatic Control*, *39*, 2461–2464.

Chen, J., & Patton, R. J. (1996). Optimal filtering and robust fault diagnosis of stochastic systems with unknown disturbances. *IEE Proceedings Control Theory and Applications*, *143*, 31–36.

Darouach, M., & Zasadzinski, M. (1997). Unbiased minimum variance estimation for systems with unknown exogenous inputs. *Automatica*, *33*, 717–719.

Darouach, M., Zasadzinski, M., & Boutayeb, M. (2003). Extension of minimum variance estimation for systems with unknown inputs. *Automatica*, *39*, 867–876.

Friedland, B. (1969). Treatment of bias in recursive filtering. *IEEE Transactions on Automatic Control*, *14*, 359–367.

Gillijns, S., & De Moor, B. (2007a). Unbiased minimum-variance input and state estimation for linear discrete-time systems. *Automatica*, *43*, 111–116.

Gillijns, S., & De Moor, B. (2007b). Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feedthrough. *Automatica*, *43*, 934–937.

Hou, M., & Patton, R. J. (1998). Optimal filtering for systems with unknown inputs. *IEEE Transactions on Automatic Control*, *43*, 445–449.

Hsieh, C.-S. (2000). Robust two-stage Kalman filters for systems with unknown inputs. *IEEE Transactions on Automatic Control*, *45*, 2374–2378.

Hsieh, C.-S. (2006). Optimal filtering for systems with unknown inputs via unbiased minimum-variance estimation. In *Proceedings of IEEE Tencon 2006*.

Hsieh, C.-S., & Chen, F.-C. (1999). Optimal solution of the two-stage Kalman estimator. *IEEE Transactions on Automatic Control*, *44*, 194–199.

Kerwin, W. S., & Prince, J. L. (2000). On the optimality of recursive unbiased state estimation with unknown inputs. *Automatica*, *36*, 1381–1383.

Kitanidis, P. K. (1987). Unbiased minimum-variance linear state estimation. *Automatica*, *23*, 775–778.

Rao, C. R., & Mitra, S. K. (1971). *Generalized inverse of matrices and its applications*. New York: Wiley.

Sundaram, S., & Hadjicostis, C. N. (2006). Optimal state estimators for linear systems with unknown inputs. In *Proceedings of 45th IEEE conference on decision and control* (pp. 4763–4768).