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ON THE OPTIMALITY OF TWO-STAGE KALMAN FILTERING FOR SYSTEMS WITH UNKNOWN INPUTS

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ABSTRACT

This paper is concerned with the optimal solution of two-stage Kalman filtering for linear discrete-time stochastic time-varying systems with unknown inputs affecting both the system state and the outputs. By means of a newly-presented modified unbiased minimum-variance filter (MUMVF), which appears to be the optimal solution to the addressed problem, the optimality of two-stage Kalman filtering for systems with unknown inputs is defined and explored. Two extended versions of the previously proposed robust two-stage Kalman filter (RTSKF), augmented-unknown-input RTSKF (ARTSKF) and decoupled-unknown-input RTSKF (DRTSKF), are presented to solve the general unknown input filtering problem. It is shown that under less restricted conditions, the proposed ARTSKF and DRTSKF are equivalent to the corresponding MUMVFs. An example is given to illustrate the usefulness of the proposed results.

Key Words: Robust filter, two-stage Kalman filter, unknown input decoupled filter, unbiased estimation, minimum variance estimation.

I. INTRODUCTION

Unknown input filtering has played a significant role in many applications, *e.g.*, bias compensation [1, 2], geophysical and environmental applications [3], and fault detection and isolation problems [4]. When the statistics of the unknown input are known, the optimal state estimator can be represented by the optimal two-stage Kalman estimator (OTSKE) [2]. This can be seen as a special implementation of the well-known Kalman filter. This OTSKE, however, is limited to systems with unknown inputs that have a prescribed statistical model. To be more general, this may not add any knowledge when applied to the unknown input model. This paper considers state estimations for systems with unknown inputs that have arbitrary profiles.

A common approach to solving this problem is to apply unknown input decoupled state estimation, *e.g.*, [3–13]. Four major problem-solving approaches have been used in the literature. The first is unbiased minimum-variance estimation (UMVE) [3–5, 7, 9, 10]. In this approach, the filter parameters are determined first to satisfy some algebraic constraints according to the unbiasedness requirements of the filter. In general, the solutions of the algebraic constraints are parameterized using parameter matrices. Next, the parameter matrices are determined such that the estimation error variance is minimal. The second approach is the equivalent system description (ESD) method [6, 8]. In this approach, an equivalent system description for designing an unknown input decoupled filter for the considered system is first established. Then, the standard derivation of the linear minimum covariance filtering determines the optimal filter by making use of the innovation's filtering technique. The system

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considered in [8] has the most general form where the unknown input affects both the system state and measurements and where the system and measurement noises are correlated. The third is the robust two-stage Kalman filtering (RTSKF) method [11]. In this approach, the OTSKE and a specific unknown input filtering technique are applied to derive the RTSKF; this is independent of the underlying input model. Note that the result in [11] is limited to unknown inputs that only enter into the system state. In the last, the filter is designed based on state estimation techniques for descriptor systems [12, 13].

In recent years, many researchers have devoted attention to this more general unknown input filtering problem, *e.g.*, [14–19]. In [14], a recursive filter was developed where the estimation of the state and the input are interconnected. The input estimate is obtained from the innovation by least-squares estimation and the state estimation is transformed into a standard Kalman filtering problem. An extension of [14] to state estimation for systems with direct feedthrough is addressed in [15], where the state estimation problem is solved using the method developed in [3]. To the best of our knowledge, the filter derived in [15], the recursive three-step filter (RTSF) provides the most recent result in the related literature. Nevertheless, the RTSF is only applicable where the direct feedthrough matrix has full column rank. In [16], the concept of input and state observability is introduced and both the unknown input filter and the state estimator are derived in an unbiased minimum-variance filtering sense. In contrast, the results in [17–19] are devoted to a time-delayed state estimator design; nevertheless, this is not the focus of this paper. A unified solution to the unknown input filtering problem is also explored, *e.g.* [20–22]. In [20], this study proposed the optimal unbiased minimum-variance filter (OUMVF) via the UMVE method, by which the relationship between results in the existing literature (*i.e.* [4, 8, 10]) is further explored. Specifically, the OUMVF can exactly implement the optimal estimator filter (OEF) in [10]. It is, however, also known that the optimal linear minimum-variance estimator (OLMVE) [20], which is derived using the equivalent system description and innovations filtering method in [8], could not be re-derived via the OUMVF. To alleviate this problem, the extended OUMVF (EOUMVF) was proposed in [21] to exactly implement the OLMVE. Moreover, an extension of the RTSKF, named the extended RTSKF (ERTSKF) [22], has been proposed as an alternative to the OUMVF. This provides a direct connection between the two existing filtering methods for systems with unknown inputs which affect both the system state and the outputs: the UMVE method and the RTSKF technique. It should be stressed that the ERTSKF is more compact than the OUMVF in terms of a filter's coding; moreover, the former has a potential advantage over the latter as it explicitly yields optimal estimates for the unknown input (see [14–16] for details).

This paper aims to extend the work of [22] and further explore the optimal solution of the RTSKF in solving the aforementioned general unknown input filtering problem. The obtained filter will be named the optimal RTSKF (ORTSKF). It is noted that the ERTSKF results from the OUMVF via a direct filter transformation and by applying some pre-assumptions; this shows that the former is actually derived using the UMVE method. In this paper, we intend to re-derive the ERTSKF solely by means of the RTSKF technique. Moreover, the OUMVF may not provide optimal filtering performance as compared to that of the EOUMVF (see Section III for details), hence the ERTSKF may not provide the optimal solution to the addressed problem. Via an innovative ESD method, the ORTSKF can be readily derived through the RTSKF simplifying the derivation of the optimal filter. Moreover, the proposed ESD method guarantees that the obtained optimal state estimator will yield an effective unknown input estimator. Two versions of the ORTSKF: the augmented-unknown-input RTSKF (ARTSKF) and the decoupled-unknown-input RTSKF (DRTSKF), are proposed to solve the addressed unknown input filtering problem. In Section III in order to facilitate the discussions of the optimality of the two derived ORTSKFs, this paper presents a compact version of the EOUMVF, referred to as the modified unbiased minimum-variance filter (MUMVF) which will serve as a refined version of the OUMVF.

The remainder of this paper is organized as follows. In Section II, the statement of the problem is addressed. Section III revisits the EOUMVF and facilitates discussion by presenting the MUMVF that optimally solves the addressed problem. In Section IV, the ARTSKF and DRTSKF are derived to optimally solve the addressed problem and their connection is addressed. Via the new derived MUMVF, the optimality issue of the ARTSKF and DRTSKF is explored in Section V. Section VI gives an example to illustrate the usefulness of the proposed results. Section VII presents the conclusions. For easy reference, the acronyms used in this paper and their full names are tabulated in Table I. Moreover, to better understand the relationship between the aforementioned unknown input decoupled filters, Table II lists the corresponding filter form, approach, implications and implicit restrictions.

Table I. List of Acronyms.

Acronym	Full Name
ARTSKF	Augmented-unknown-input robust two-stage Kalman filter
DRTSKF	Decoupled-unknown-input robust two-stage Kalman filter
EOUMVF	Extended optimal unbiased minimum-variance filter
ERTSKF	Extended robust two-stage Kalman filter
ESD	Equivalent system description
MUMVF	Modified unbiased minimum-variance filter
OEF	Optimal estimator filter
OLMVE	Optimal linear minimum-variance estimator
ORTSKF	Optimal robust two-stage Kalman filter
OTSKE	Optimal two-stage Kalman estimator
OUMVF	Optimal unbiased minimum-variance filter
RTSF	Recursive three-step filter
RTSKF	Robust two-stage Kalman filter
UMVE	Unbiased minimum-variance estimation

Table II. The filter forms, approaches, implications, and restrictions of the unknown input decoupled filters.

Filter	Filter Form	Approach	Implications	Restrictions
RTSKF	(3)	RTSKF	Optimal state estimator without direct feedthrough	$G_k=0$ & (18)
ERTSKF	(3)	UMVE	Optimal state estimator with constraint	(18) & (41)
ARTSKF	(3)	RTSKF	Optimal state estimator with explicit input estimate	(18)
DRTSKF	(3)	RTSKF	Optimal state estimator with constraint	$rank[G_k]<p$ & (18)
MUMVF	(5)	UMVE	Optimal state estimator with explicit input estimate	(18)
OEF	(6)	UMVE	Optimal state estimator without delay measurements	$K_k=0$ & (18)
OUMVF	(6)	UMVE	Optimal state estimator with constraint	(18) & (41)
OLMVE	(8)	ESD	Optimal state estimator with implicit input estimate	(18)
EOUMVF	(8)	UMVE	Optimal state estimator with implicit input estimate	(18)
RTSF	(35)	UMVE	Optimal state estimator with constraint	$rank[G_k]=q$

II. STATEMENT OF THE PROBLEM

Consider the linear discrete-time stochastic time-varying system with unknown inputs in the form:

$$x_{k+1}=A_kx_k+B_ku_k+F_kd_k+w_k \tag{1}$$

$$y_k=H_kx_k+G_kd_k+v_k \tag{2}$$

where $x_k\in R^n$ is the system state, $u_k\in R^m$ is the known input, $d_k\in R^q$ is the unknown input, and $y_k\in R^p$ is the measurement. Matrices A_k, B_k, F_k, H_k, and G_k have appropriate dimensions. The process noise w_k and the measurement noise v_k are zero-mean white noise sequences with the following covariances $E\{w_kw_l^T\}=Q_k\delta_{kl}$, $E\{v_kv_l^T\}=R_k\delta_{kl}$, and $E\{w_kv_l^T\}=0$, where δ_{kl} denotes the Kronecker delta function and $R_k>0$. The initial state x_0 is with mean $\hat{x}_0$ and covariance P_0 and is independent of w_k and v_k.

The main aim of this paper is to explore the optimal solution of the following RTSKF:

$$\hat{x}_k=\bar{x}_{k|k}+V_kd_{k|k} \tag{3}$$

$$P_k=P^{\bar{x}}_{k|k}+V_kP^d_{k|k}V_k^T \tag{4}$$

where $\bar{x}_{k|k}$, $d_{k|k}$, $P^{\bar{x}}_{k|k}$, $P^d_{k|k}$, and V_k are to be determined, such that $E[e_k]=0$ and $tr(E[e_ke_k^T])$ is minimized, where $e_k=x_k-\hat{x}_k$. The ORTSKF given by (3) and (4) that optimally achieves the UMVE for the system (1) and (2) is formally defined below.

Definition 1. The optimal filter considered in this paper is defined by the following full-order modified unbiased minimum-variance filter (MUMVF) (see Section III for details):

$$\hat{x}_k=N_k\hat{x}_{k-1}+E_ku_{k-1}+K_k\tilde{y}_{k-1}+L_ky_k \tag{5}$$

where N_k, E_k, K_k, L_k, and $\tilde{y}_{k-1}$ are determined to achieve $E[e_k]=0$ and $min\,tr(E[e_ke_k^T])$.

Definition 2. The ORTSKF given by (3) and (4) is optimal if it is equivalent to the MUMVF given in Definition 1.

It is noted that if $G_k=0$, then the RTSKF in [11] gives the ORTSKF (see [14]). The following section will first revisit the existing EOUMVF and will then derive the MUMVF with the same filtering performance as that of the OLMVE. Next, Section IV will present two ORTSKFs that solve the addressed general unknown input filtering problem via the direct application of the RTSKF. The optimality issue of these two ORTSKFs is addressed in Section V.

III. MODIFIED UNBIASED MINIMUM-VARIANCE FILTER DESIGN

Using the standard unbiased minimum-variance filtering technique and considering the following full-order stochastic Luenberger observer [9, 10]

$$\hat{x}_k = N_k \hat{x}_{k-1} + E_k u_{k-1} + K_k y_{k-1} + L_k y_k \tag{6}$$

the OUMVF that solves the addressed optimal unknown input filtering problem can be obtained in [20]. Nevertheless, it is also known that the OLMVE cannot be obtained via the OUMVF where the former also appears to be an optimal filter which addresses the considered problem. To extend the OUMVF to exactly implement the OLMVE, the EOUMVF was proposed. This, however, is not given in the conventional observer-type filter (6) but in the following two-step filter:

$$\begin{aligned}\bar{x}_k = {} & \bar{A}_{k-1}\hat{x}_{k-1} + \bar{B}_{k-1}u_{k-1} + \bar{F}_{k-1}y_{k-1} + \breve{G}_k y_k \\ & + \tilde{K}_{k-1}T_{k-1}(y_{k-1} - H_{k-1}\bar{x}_{k-1})\end{aligned} \tag{7}$$

$$\hat{x}_k = \bar{x}_k + \breve{K}_k T_k (y_k - H_k \bar{x}_k) \tag{8}$$

where $T_k = I - G_k G_k^+$ and M^+ denotes any one-condition generalized inverse of M, *i.e.*, $MM^+M = M$. Unlike the conventional one-step prediction used in standard Kalman filtering, the determination of $\bar{x}_k$ needs measurement information at time k. This section shows that the above EOUMVF can be reformulated in a more compact form, *i.e.*, the MUMVF, to facilitate the discussions in Section IV. Furthermore, in Section IV we show that the latter has a potential advantage over the former because it explicitly yields the optimal estimates of the unknown input.

First, according to [21], the matrices $\bar{A}_{k-1}$, $\bar{B}_{k-1}$, $\bar{F}_{k-1}$, and $\tilde{K}_{k-1}$ in (7) are updated as follows:

$$\begin{aligned}\bar{A}_k &= D_k A_k - \bar{F}_k H_k, \quad \bar{B}_k = D_k B_k, \\ \bar{F}_k &= D_k F_k G_k^+, \quad \tilde{K}_k = -\bar{F}_k R_k \breve{T}_k{}^T \breve{C}_k{}^+\end{aligned} \tag{9}$$

where

$$\begin{aligned}D_k &= I - \breve{G}_{k+1} H_{k+1}, \quad \breve{T}_k = T_k - \bar{H}_k \breve{G}_k, \\ \breve{C}_k &= \bar{H}_k \breve{P}_k \bar{H}_k^T + \breve{T}_k R_k \breve{T}_k{}^T, \quad \bar{H}_k = T_k H_k\end{aligned} \tag{10}$$

the gain matrix $\breve{K}_k$ in (8) is given as

$$\breve{K}_k = (\breve{P}_k \bar{H}_k^T - \breve{G}_k R_k \breve{T}_k{}^T)\breve{C}_k{}^+ \tag{11}$$

where

$$\begin{aligned}\breve{P}_k &= D_{k-1} P_{k|k-1}^{\bar{x}} D_{k-1}^T, \\ P_{k|k-1}^{\bar{x}} &= \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + \bar{Q}_{k-1}\end{aligned} \tag{12}$$

$$P_k = \breve{P}_k + \breve{G}_k R_k \breve{G}_k{}^T - \breve{K}_k \breve{C}_k \breve{K}_k{}^T \tag{13}$$

$$\Phi_k = A_k - F_k \Lambda_k H_k,$$

$$\Lambda_k = \begin{cases} 0 & \text{for } K_{k+1} = 0 \\ G_k^+ & \text{for } K_{k+1} \neq 0 \end{cases} \tag{14}$$

$$\begin{aligned}\bar{Q}_k = {} & Q_k + \Phi_k L_k R_k \Lambda_k^T F_k^T + F_k \Lambda_k R_k L_k^T \Phi_k^T \\ & + F_k \Lambda_k (R_k - R_k \breve{T}_k{}^T \breve{C}_k{}^+ \breve{T}_k R_k) \Lambda_k^T F_k^T,\end{aligned} \tag{15}$$

and the matrix $\breve{G}_k$ in (7) satisfies the following unbiasedness conditions (see equations (15) and (16) in [21])

$$(I - \breve{G}_k H_k) U_{k-1} = 0, \quad \breve{G}_k G_k = 0 \tag{16}$$

where

$$U_k = F_k (I - \Lambda_k G_k). \tag{17}$$

It is noted that the constraints in (16) always hold if the following rank condition is satisfied (see [10] for details):

$$rank[G_k \;\; H_k U_{k-1}] = rank[G_k] + rank[U_{k-1}]. \tag{18}$$

Next, (7)-(8) are transformed into the filter form (5). Let $\tilde{y}_k$ be given as:

$$\tilde{y}_k = y_k - H_k \bar{x}_k. \tag{19}$$

Then, using (8) and (19), one attains:

$$y_k = H_k \hat{x}_k + (I - H_k \breve{K}_k T_k)\tilde{y}_k. \tag{20}$$

Using (19) and (20) in (7) yields:

$$\begin{aligned}\bar{x}_k = {} & (\bar{A}_{k-1} + \bar{F}_{k-1} H_{k-1})\hat{x}_{k-1} + \bar{B}_{k-1} u_{k-1} \\ & + (\tilde{K}_{k-1} T_{k-1} + \bar{F}_{k-1}(I - H_{k-1}\breve{K}_{k-1} T_{k-1})) \\ & \times \tilde{y}_{k-1} + \breve{G}_k y_k.\end{aligned} \tag{21}$$

From this result and using (9) one can easily reformulate (7)–(8) in the filter form (5), where:

$$N_k = (I - \breve{K}_k \bar{H}_k) D_{k-1} A_{k-1} \tag{22}$$

$$E_k = (I - \breve{K}_k \bar{H}_k) D_{k-1} B_{k-1} \tag{23}$$

$$K_k = (I - \breve{K}_k \bar{H}_k) D_{k-1} F_{k-1} \Pi_{k-1} \tag{24}$$

$$L_k = (I - \breve{K}_k \bar{H}_k) \breve{G}_k + \breve{K}_k T_k \tag{25}$$

$$\tilde{y}_k = (I - H_k \breve{G}_k)(y_k - H_k \bar{x}_{k|k-1}) \tag{26}$$

$$\Pi_k = \Lambda_k (I - H_k \breve{K}_k T_k - R_k \breve{T}_k{}^T \breve{C}_k{}^+ T_k) \tag{27}$$

$$\begin{aligned} \bar{x}_{k|k-1} &= A_{k-1} \hat{x}_{k-1} + B_{k-1} u_{k-1} \\ &\quad + F_{k-1} \Pi_{k-1} \tilde{y}_{k-1}. \end{aligned} \tag{28}$$

Finally, it remains to determine the matrix $\breve{G}_k$ that satisfies (16), from which one may choose the following four different values of $\breve{G}_k$:

1) $MUMVF^1$: $\quad \breve{G}_k = \bar{U}_{k-1} \bar{S}_k^+ \quad (29)$

2) $MUMVF^2$: $\quad \breve{G}_k = \bar{U}_{k-1} (\bar{S}_k^T C_k^{-1} \bar{S}_k)^+ \bar{S}_k^T C_k^{-1} \quad (30)$

3) $MUMVF^3$: $\quad \breve{G}_k = U_{k-1} (\breve{S}_k^T \bar{C}_k^+ \breve{S}_K)^+ \breve{S}_k^T \bar{C}_k^+ = U_{k-1} (S_k^T \bar{C}_k^+ S_k)^+ S_k^T \bar{C}_k^+ \quad (31)$

4) $MUMVF^4$: $\quad \breve{G}_k = U_{k-1} \breve{S}_k^+ \quad (32)$

where

$$\begin{aligned} \bar{U}_k &= [0 \;\; U_k], \quad \bar{S}_k = [G_k \;\; S_k], \\ \breve{S}_k &= T_k S_k, \quad S_k = H_k U_{k-1} \end{aligned} \tag{33}$$

$$\bar{C}_k = T_k C_k T_k^T, \quad C_k = H_k P_{k|k-1}^{\bar{x}} H_k^T + R_k. \tag{34}$$

To facilitate discussion in the following section, Filter (5) can be easily reformulated in the following Kitanidis' filter [3]:

$$\hat{x}_k = \bar{x}_{k|k-1} + L_k^* (y_k - H_k \bar{x}_{k|k-1}) \tag{35}$$

$$P_k = (I - L_k^* H_k) P_{k|k-1}^{\bar{x}} (I - L_k^* H_k)^T + L_k^* R_k (L_k^*)^T \tag{36}$$

$$L_k^* = \breve{G}_k + \breve{K}_k \breve{T}_k \tag{37}$$

where $\breve{K}_k$ given by (11) can be reformulated by using the relationship $\bar{H}_k D_{k-1} = \breve{T}_k H_k$ as follows:

$$\begin{aligned} \breve{K}_k &= (P_{k|k-1}^{\bar{x}} H_k^T - \breve{G}_k C_k) \breve{T}_k^T \breve{C}_k^+, \\ \breve{C}_k &= \breve{T}_k C_k \breve{T}_k^T. \end{aligned} \tag{38}$$

It is noted that (36) is obtained by verifying the following equations:

$$\begin{aligned} P_k &= (D_{k-1} - \breve{K}_k \breve{T}_k H_k) P_{k|k-1}^{\bar{x}} (D_{k-1} - \breve{K}_k \breve{T}_k H_k)^T \\ &\quad + (\breve{G}_k + \breve{K}_k T_k) R_k (\breve{G}_k + \breve{K}_k T_k)^T \\ &= \breve{P}_k + \breve{G}_k R_k \breve{G}_k^T + \breve{K}_k \breve{C}_k \breve{K}_k^T - (P_{k|k-1}^{\bar{x}} H_k^T \\ &\quad - \breve{G}_k C_k) \breve{T}_k^T \breve{K}_k^T - \breve{K}_k \breve{T}_k (P_{k|k-1}^{\bar{x}} H_k^T - \breve{G}_k C_k)^T \\ &= \breve{P}_k + \breve{G}_k R_k \breve{G}_k^T + \breve{K}_k \breve{C}_k \breve{K}_k^T - (P_{k|k-1}^{\bar{x}} H_k^T \\ &\quad - \breve{G}_k C_k) \breve{T}_k^T \breve{C}_k^+ \breve{C}_k \breve{K}_k^T \\ &\quad - \breve{K}_k \breve{C}_k \breve{C}_k^+ \breve{T}_k (P_{k|k-1}^{\bar{x}} H_k^T - \breve{G}_k C_k)^T \\ &= \breve{P}_k + \breve{G}_k R_k \breve{G}_k^T - \breve{K}_k \breve{C}_k \breve{K}_k^T \end{aligned}$$

where (10), (12)-(13), and (37)-(38) are used.

Remark 1. It can be verified that if $\Lambda_k = G_k^+$, *i.e.*, choosing $K_k \neq 0$, the above $MUMVF^4$ will be equivalent to the OLMVE. Moreover, if the following relationship holds

$$\Lambda_k R_k \breve{T}_k^T \breve{C}_k^+ T_k = 0 \tag{39}$$

which is achieved via

$$\Lambda_k R_k \breve{T}_k^T = 0 \tag{40}$$

then from (20) and (27) one has:

$$\Pi_k \tilde{y}_k = \Lambda_k (I - H_k \breve{K}_k T_k) \tilde{y}_k = \Lambda_k (y_k - H_k \hat{x}_k)$$

from which the obtained $MUMVF^1$ (or $MUMVF^4$) will be equivalent to the OUMVF. If this is the case, the OUMVF serves as a compact version of the MUMVF. It is noted that $\breve{T}_k$ takes the following form: $\breve{T}_k = \breve{C}_k \bar{C}_k^+ T_k$ (see (115) and (119) for details). Hence, the following condition will always guarantee (40):

$$\Lambda_k R_k T_k^T = 0 \tag{41}$$

which is satisfied if the covariance matrix R_k takes the specific form $R_k = \sigma^2 I$. Specifically, if $\Lambda_k = 0$, *i.e.*, choosing $K_k = 0$, the $MUMVF^1$ will be equivalent to the OEF.

Remark 2. If $\breve{T}_k$ given by (10) is equal to zero, then all of the above MUMVFs will be given by the following degenerated MUMVF filter form:

$$\hat{x}_k = \bar{x}_{k|k-1} + \breve{G}_k(y_k - H_k\bar{x}_{k|k-1}) \tag{42}$$

$$P_k = (I - \breve{G}_k H_k)P^{\bar{x}}_{k|k-1}(I - \breve{G}_k H_k)^T + \breve{G}_k R_k \breve{G}_k^T. \tag{43}$$

IV. EXTENSIONS OF THE ROBUST TWO-STAGE KALMAN FILTER

4.1 Equivalent system description

In order to derive the ORTSKF using the RTSKF technique, we first rewrite System (1) as follows:

$$x_{k+1} = A_k x_k + B_k u_k + F_k\hat{d}_k + F_k\tilde{d}_k + w_k \tag{44}$$

$$\tilde{d}_k = d_k - \hat{d}_k, \quad \hat{d}_k = \Pi_k\tilde{y}_k \tag{45}$$

where $\hat{d}_k$ can be thought of as an effective unknown input estimator. Then, viewing $\Pi_k\tilde{y}_k$ as a known input vector, (44) can be reformulated as:

$$x_{k+1} = A_k x_k + \bar{B}_k\bar{u}_k + F_k\tilde{d}_k + w_k \tag{46}$$

where

$$\bar{B}_k = [B_k \quad F_k], \quad \bar{u}_k = [u_k^T \quad (\Pi_k\tilde{y}_k)^T].$$

Next, using (9) and (27)-(28) in (21) yields:

$$\bar{x}_k = D_{k-1}\bar{x}_{k|k-1} + \breve{G}_k y_k. \tag{47}$$

Using (10), (19), and (47), we obtain:

$$\tilde{y}_k = (I - H_k\breve{G}_k)(y_k - H_k\bar{x}_{k|k-1}). \tag{48}$$

Using (27) and (48), one obtains:

$$\begin{aligned}\Pi_k\tilde{y}_k &= \Lambda_k(I - H_k\breve{K}_k T_k - R_k\breve{T}_k^T\breve{C}_k^+ T_k)\\ &\quad\times(I - H_k\breve{G}_k)(y_k - H_k\bar{x}_{k|k-1})\\ &= \Lambda_k(I - H_k\breve{G}_k)(I - C_k\breve{T}_k^T\breve{C}_k^+\breve{T}_k)\\ &\quad\times(y_k - H_k\bar{x}_{k|k-1})\\ &= \Xi_k(y_k - H_k\bar{x}_{k|k-1})\end{aligned} \tag{49}$$

from which, along with using (1), (2), (28), (45) and the relationship: $\Xi_k G_k = \Lambda_k G_k$, we obtain:

$$\begin{aligned}\tilde{d}_k &= (I - \Lambda_k G_k)d_k - \Xi_k H_k F_{k-1}\tilde{d}_{k-1}\\ &\quad - \Xi_k(H_k(A_{k-1}e_{k-1} + w_{k-1}) + v_k).\end{aligned} \tag{50}$$

Taking the expectation of (50) and using the facts: $E[e_{k-1}]=0$, $E[w_{k-1}]=0$, and $E[v_k]=0$, one obtains:

$$E[\tilde{d}_k] = (I - \Lambda_k G_k)E[d_k] - \Xi_k H_k F_{k-1}E[\tilde{d}_{k-1}]. \tag{51}$$

Then, we show that the last term of (51), which can be expressed as shown below, equals zero:

$$\begin{aligned}&\Xi_k H_k F_{k-1}E[\tilde{d}_{k-1}]\\ &\quad = (\Xi_k H_k U_{k-1})E[d_{k-1}]\\ &\quad\quad - \Xi_k H_k F_{k-1}(\Xi_{k-1}H_{k-1}U_{k-2})E[d_{k-2}] + \cdots.\end{aligned} \tag{52}$$

Using (10), (16), and (49), we obtain for $i = 1, \ldots, k$:

$$\begin{aligned}\Xi_i H_i U_{i-1} &= \Lambda_i(I - (I - H_i\breve{G}_i)C_i\breve{T}_i^T\breve{C}_i^+ T_i)\\ &\quad\times(I - H_i\breve{G}_i)H_i U_{i-1}\\ &= \Lambda_i(I - (I - H_i\breve{G}_i)C_i\breve{T}_i^T\breve{C}_i^+ T_i)H_i\\ &\quad\times(I - \breve{G}_i H_i)U_{i-1} = 0.\end{aligned} \tag{53}$$

Using (51)-(53), we obtain:

$$E[\tilde{d}_k] = (I - \Lambda_k G_k)E[d_k]. \tag{54}$$

Finally, using (17) and (54), (46) can be alternatively represented as follows:

$$x_{k+1} = A_k x_k + \bar{B}_k\bar{u}_k + U_k\bar{d}_k + w_k \tag{55}$$

where $\bar{d}_k$ is a new unknown input vector such that $E[\bar{d}_k]=0$ for $\Lambda_k G_k = I$ and $E[\bar{d}_k] = E[d_k]$ for $\Lambda_k G_k \neq I$. The implication of (55) is that an explicit unknown input estimator is needed in the assumed system model, hence in the derivation of the following ORTSKF the optimal state as well as unknown input estimates will both be obtained. This joint input and state estimation also holds for the MUMVF since the term $\Pi_k\tilde{y}_k$ is needed in the algorithm. On the other hand, it is not necessary for the OLMVE and EOUMVF to explicitly give the unknown input estimates in order to obtain the optimal state estimates.

4.2 Augmented unknown-input approach

In the first place, it is noted that the system [(55) and (2)] can be represented as follows:

$$x_{k+1} = A_k x_k + \bar{B}_k\bar{u}_k + \bar{U}_k d_k^a + w_k \tag{56}$$

$$y_k = H_k x_k + \bar{G}_k d_k^a + v_k \tag{57}$$

where $\bar{U}_k$ is given in (33) and

$$d_k^a = [d_k^T \;\; \bar{d}_k^T], \quad \bar{G}_k = [G_k \;\; 0].$$

Next, according to [2], if the unknown input vector d_k^a enters into the measurement equation, then the matrix S_k in the blending matrix V_k of the RTSKF corresponding to (56)-(57) should be modified as follows:

$$S_k = H_k \bar{U}_{k-1} \rightarrow H_k \bar{U}_{k-1} + \bar{G}_k = \bar{S}_k.$$

Summarizing the above results, a direct application of the RTSKF with a minor modification in solving the addressed problem is given as the following ARTSKF:

$$\hat{x}_k = \bar{x}_{k|k} + V_k d_{k|k} \tag{58}$$

$$P_k = P_{k|k}^{\bar{x}} + V_k P_{k|k}^d V_k^T \tag{59}$$

where $\bar{x}_{k|k}$ is given by

$$\bar{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + F_{k-1}\Pi_{k-1}\tilde{y}_{k-1} \tag{60}$$

$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K_k^{\bar{x}}(y_k - H_k\bar{x}_{k|k-1}) \tag{61}$$

$$P_{k|k-1}^{\bar{x}} = \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + \bar{Q}_{k-1} \tag{62}$$

$$K_k^{\bar{x}} = P_{k|k-1}^{\bar{x}} H_k^T C_k^{-1} \tag{63}$$

$$P_{k|k}^{\bar{x}} = (I - K_k^{\bar{x}} H_k) P_{k|k-1}^{\bar{x}} \tag{64}$$

$d_{k|k}$ is given by

$$d_{k|k} = K_k^d (y_k - H_k \bar{x}_{k|k-1}) \tag{65}$$

$$K_k^d = P_{k|k}^d \bar{S}_k^T C_k^{-1} \tag{66}$$

$$P_{k|k}^d = (\bar{S}_k^T C_k^{-1} \bar{S}_k)^+ \tag{67}$$

and

$$V_k = \bar{U}_{k-1} - K_k^{\bar{x}} \bar{S}_k. \tag{68}$$

Remark 3. Equations (66) and (67) can be implemented more efficiently without the calculation of the inverse matrix C_k^{-1} as follows:

$$K_k^d = \bar{S}_k^+, \quad P_{k|k}^d = K_k^d C_k (K_k^d)^T.$$

Remark 4. Let $\breve{G}_k$ in $\tilde{y}_k$ be given by (30). Under the following condition

$$\breve{T}_k^T \breve{C}_k^+ \breve{T}_k = C_k^{-1} - C_k^{-1}\bar{S}_k(\bar{S}_k^T C_k^{-1}\bar{S}_k)^+ \bar{S}_k^T C_k^{-1} \tag{69}$$

and using (49) and (66)-(67), $\Pi_k \tilde{y}_k$ and $\bar{Q}_k$ can be further simplified as follows:

$$\begin{aligned}\Pi_k \tilde{y}_k &= \Lambda_k (I - H_k \breve{G}_k)\bar{S}_k(\bar{S}_k^T C_k^{-1}\bar{S}_k)^+ \bar{S}_k^T C_k^{-1} \\ &\quad \times (y_k - H_k \bar{x}_{k|k-1}) \\ &= \Lambda_k [G_k \;\; 0] K_k^d (y_k - H_k \bar{x}_{k|k-1})\end{aligned} \tag{70}$$

$$\begin{aligned}\bar{Q}_k &= Q_k + \Phi_k L_k R_k \Lambda_k^T F_k^T + F_k \Lambda_k R_k L_k^T \Phi_k^T \\ &\quad + F_k \Lambda_k (R_k - R_k C_k^{-1} \\ &\quad \times (I - \bar{S}_k K_k^d) R_k) \Lambda_k^T F_k^T.\end{aligned} \tag{71}$$

Specifically, letting $B_k = 0$, $rank[G_k] = q$, and $\Lambda_k = G_k^+$, from (69)-(71) it can be verified that the obtained ARTSKF will be equivalent to the RTSF.

4.3 Decoupled unknown-input approach

The considered problem may also be solved by decoupling the unknown input from the measurement equation. This is achieved by pre-multiplying (2) by T_k, which leads to:

$$T_k y_k = \bar{H}_k x_k + T_k v_k. \tag{72}$$

It is noted that (72) exists only if $rank[G_k] < p$.

Applying the RTSKF to the system [(55) and (72)], we obtain the following DRTSKF:

$$\hat{x}_k = \bar{x}_{k|k} + \bar{V}_k \bar{d}_{k|k} \tag{73}$$

$$P_k = P_{k|k}^{\bar{x}} + \bar{V}_k P_{k|k}^{\bar{d}} \bar{V}_k^T \tag{74}$$

where $\bar{x}_{k|k}$ is given by

$$\bar{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + F_{k-1}\Pi_{k-1}\tilde{y}_{k-1} \tag{75}$$

$$\bar{x}_{k|k} = \bar{x}_{k|k-1} + K_k^{\bar{x}} T_k (y_k - H_k \bar{x}_{k|k-1}) \tag{76}$$

$$P_{k|k-1}^{\bar{x}} = \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + \bar{Q}_{k-1} \tag{77}$$

$$K_k^{\bar{x}} = P_{k|k-1}^{\bar{x}} \bar{H}_k^T \bar{C}_k^+ \tag{78}$$

$$P_{k|k}^{\bar{x}} = (I - K_k^{\bar{x}} \bar{H}_k) P_{k|k-1}^{\bar{x}} \tag{79}$$

$\bar{d}_{k|k}$ is given by

$$\bar{d}_{k|k} = K_k^{\bar{d}} T_k (y_k - H_k \bar{x}_{k|k-1}) \tag{80}$$

$$K_k^{\bar{d}} = P_{k|k}^{\bar{d}} \breve{S}_k^T \bar{C}_k^+ \tag{81}$$

$$P_{k|k}^{\bar{d}} = (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \tag{82}$$

and

$$\bar{V}_k = U_{k-1} - K_k^{\bar{x}} \breve{S}_k. \tag{83}$$

Remark 5. Let $\breve{G}_k$ in $\tilde{y}_k$ be given by (31). Under the following condition

$$\breve{T}_k^T \breve{C}_k^+ \breve{T}_k = \bar{C}_k^+ - \bar{C}_k^+ S_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+$$

and using (49) and (81)-(82), $\Pi_k \tilde{y}_k$ and $\bar{Q}_k$ can be further simplified as follows:

$$\begin{aligned}\Pi_k \tilde{y}_k &= \Lambda_k (I - S_k K_k^{\bar{d}} - C_k \bar{C}_k^+ (I - S_k K_k^{\bar{d}})) \\ &\quad \times (y_k - H_k \bar{x}_{k|k-1}) \\ &= \Lambda_k (I - C_k \bar{C}_k^+)(I - S_k K_k^{\bar{d}})(y_k - H_k \bar{x}_{k|k-1}) \\ \bar{Q}_k &= Q_k + \Phi_k L_k R_k \Lambda_k^T F_k^T + F_k \Lambda_k R_k L_k^T \Phi_k^T \\ &\quad + F_k \Lambda_k (R_k - R_k \bar{C}_k^+ (I - S_k K_k^{\bar{d}}) R_k) \Lambda_k^T F_k^T.\end{aligned}$$

Furthermore, as illustrated in Remark 1 if the relationship (39) holds, then the obtained DRTSKF will be equivalent to the ERTSKF. If this is the case, the ERTSKF serves as a compact version of the DRTSKF.

4.4 Connection between the ARTSKF and DRTSKF

It can be easily confirmed that under the special case $G_k = 0$, the ARTSKF and the DRTSKF will be both equivalent to the RTSKF. This subsection will further show the conditions under which the ARTSKF is equivalent to the DRTSKF. To achieve this, the ARTSKF and the DRTSKF are each reformulated as the Kitanidis' UMVF form as presented below:

1. ARTSKF-based UMVF:

$$\hat{x}_k = \bar{x}_{k|k-1} + L_k^a (y_k - H_k \bar{x}_{k|k-1}) \tag{84}$$

$$\begin{aligned}P_k &= (I - L_k^a H_k) P_{k|k-1}^{\bar{x}} (I - L_k^a H_k)^T \\ &\quad + L_k^a R_k (L_k^a)^T\end{aligned} \tag{85}$$

where

$$L_k^a = K_k^{\bar{x}} + \Lambda_k^a \bar{S}_k^T C_k^{-1}, \quad \Lambda_k^a = V_k (\bar{S}_k^T C_k^{-1} \bar{S}_k)^+ \tag{86}$$

and $K_k^{\bar{x}}$ is given by (63). The gain matrix L_k^a and Lagrange multipliers Λ_k^a satisfy the following constraint:

$$\begin{bmatrix} C_k & -\bar{S}_k \\ \bar{S}_k^T & 0 \end{bmatrix} \begin{bmatrix} (L_k^a)^T \\ (\Lambda_k^a)^T \end{bmatrix} = \begin{bmatrix} H_k P_{k|k-1}^{\bar{x}} \\ \bar{U}_{k-1}^T \end{bmatrix}. \tag{87}$$

2. DRTSKF-based UMVF:

$$\hat{x}_k = \bar{x}_{k|k-1} + L_k^d (y_k - H_k \bar{x}_{k|k-1}) \tag{88}$$

$$\begin{aligned}P_k &= (I - L_k^d H_k) P_{k|k-1}^{\bar{x}} (I - L_k^d H_k)^T \\ &\quad + L_k^d R_k (L_k^d)^T\end{aligned} \tag{89}$$

where

$$\begin{aligned}L_k^d &= (K_k^{\bar{x}} + \Lambda_k^d \breve{S}_k^T \bar{C}_k^+) T_k, \\ \Lambda_k^d &= \bar{V}_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+\end{aligned} \tag{90}$$

and $K_k^{\bar{x}}$ is given by (78). The gain matrix L_k^d and Lagrange multipliers Λ_k^d satisfy the following constraint:

$$\begin{bmatrix} \bar{C}_k & -\breve{S}_k \\ \breve{S}_k^T & 0 \end{bmatrix} \begin{bmatrix} (L_k^d)^T \\ (\Lambda_k^d)^T \end{bmatrix} = \begin{bmatrix} \bar{H}_k P_{k|k-1}^{\bar{x}} \\ U_{k-1}^T \end{bmatrix}. \tag{91}$$

It is noted that the first equations of (87) and (91) are obtained by solving the minimum-variance property of the obtained filters whereas the second ones are obtained to achieve unbiased conditions.

Next, we rewrite the gain matrices in (86) and (90) respectively, as follows:

$$\begin{aligned}L_k^a &= \bar{U}_{k-1} (\bar{S}_k^T C_k^{-1} \bar{S}_k)^+ \bar{S}_k^T C_k^{-1} + P_{k|k-1}^{\bar{x}} H_k^T C_k^{-1} \\ &\quad \times (I - \bar{S}_k (\bar{S}_k^T C_k^{-1} \bar{S}_k)^+ \bar{S}_k^T C_k^{-1})\end{aligned} \tag{92}$$

$$\begin{aligned}L_k^d &= U_{k-1} (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+ + P_{k|k-1}^{\bar{x}} H_k^T \bar{C}_k^+ \\ &\quad \times (I - \breve{S}_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+).\end{aligned} \tag{93}$$

Note that (93) is obtained by using the relationship $T_k^T \bar{C}_k^+ T_k = \bar{C}_k^+$, which will be verified by (105). Defining the following notations

$$X_k = C_k^{-1/2} \bar{S}_k, \quad \bar{X}_k = (\bar{C}_k^+)^{1/2} \breve{S}_k \tag{94}$$

and using the following relationship

$$M_k^+ = (M_k^T M_k)^+ M_k^T \tag{95}$$

the gain matrices (92) and (93) respectively can be rewritten as follows:

$$\begin{aligned}L_k^a &= \bar{U}_{k-1} X_k^+ C_k^{-1/2} + P_{k|k-1}^{\bar{x}} H_k^T C_k^{-1} \\ &\quad \times (I - \bar{S}_k X_k^+ C_k^{-1/2})\end{aligned} \tag{96}$$

$$\begin{aligned}L_k^d &= U_{k-1} \bar{X}_k^+ (\bar{C}_k^+)^{1/2} + P_{k|k-1}^{\bar{x}} H_k^T \bar{C}_k^+ \\ &\quad \times (I - \breve{S}_k \bar{X}_k^+ (\bar{C}_k^+)^{1/2}).\end{aligned} \tag{97}$$

Using the second constraints of (87) and (91) in (96) and (97) respectively, the unbiasedness conditions of the ARTSKF and the DRTSKF respectively, are determined as follows:

$$rank\begin{bmatrix} X_k \\ \bar{U}_{k-1} \end{bmatrix} = rank[X_k] \tag{98}$$

$$rank\begin{bmatrix} \bar{X}_k \\ U_{k-1} \end{bmatrix} = rank[\bar{X}_k]. \tag{99}$$

Finally, using (96) and (97), the conditions under which the ARTSKF is equivalent to the DRTSKF can be readily obtained:

$$\bar{U}_{k-1}X_k^+ = U_{k-1}\bar{X}_k^+\tilde{C}_k^T \tag{100}$$

$$(I - X_k X_k^+) = \tilde{C}_k(I - \bar{X}_k\bar{X}_k^+)\tilde{C}_k^T \tag{101}$$

where

$$\tilde{C}_k = C_k^{1/2}(\bar{C}_k^+)^{1/2}. \tag{102}$$

The next section will present some primitive conditions which guarantee the conditions (100) and (101).

Remark 6. If the following conditions hold:

$$I - X_k X_k^+ = 0, \quad I - \bar{X}_k\bar{X}_k^+ = 0 \tag{103}$$

the gain matrices of the ARTSKF and the DRTSKF respectively will be given as the following degenerated forms:

$$L_k^a = \bar{U}_{k-1}(\bar{S}_k^T C_k^{-1}\bar{S}_k)^+\bar{S}_k^T C_k^{-1}$$

$$L_k^d = U_{k-1}(\breve{S}_k^T\bar{C}_k^+\breve{S}_k)^+\breve{S}_k^T\bar{C}_k^+.$$

V. ON THE OPTIMALITY OF TWO-STAGE KALMAN FILTERS

This section will demonstrate that the proposed ORTSKFs, *i.e.*, the ARTSKF and the DRTSKF, can both be optimal solutions to the considered unknown input filtering problem in the sense that they are equivalent to the corresponding MUMVFs subject to less restricted conditions. In the discussions below, the degenerated MUMVFs, *i.e.*, the MUMVF with $\breve{T}_k = 0$, are not considered and the following primitive relationship is assumed

$$\bar{C}_k\bar{C}_k^+ = T_k \tag{104}$$

which implies the following identities:

$$T_k^T\bar{C}_k^+T_k = \bar{C}_k^+\bar{C}_k\bar{C}_k^+\bar{C}_k\bar{C}_k^+ = \bar{C}_k^+. \tag{105}$$

To facilitate the following discussion, we first define the notation Y_k as follows

$$Y_k = C_k^{1/2}\breve{T}_k{}^T$$

which yields:

$$\begin{aligned} Y_k Y_k^+ &= C_k^{1/2}\breve{T}_k{}^T(\breve{T}_k C_k\breve{T}_k{}^T)^+\breve{T}_k C_k^{1/2} \\ &= C_k^{1/2}\breve{T}_k{}^T\breve{C}_k{}^+\breve{T}_k C_k^{1/2}. \end{aligned} \tag{106}$$

As a result, we derive the following theorems.

Theorem 1. Let the condition (104) hold. If the following relationship is satisfied

$$\begin{aligned} C_k\bar{C}_k^+M_kC_k\bar{C}_k^+ &= M_k, \\ M_k &= I - C_k^{1/2}X_kX_k^+C_k^{-1/2} \end{aligned} \tag{107}$$

the ARTSKF, which is given by (58)–(68), is equivalent to the MUMVF[2] which is in turn referenced by (5), (10)–(15), (17), (22)–(28), (30), and (33)–(34).

Proof. Comparing (84)–(85) with (35)–(36), it is clear that the theorem is proved by verifying $L_k^* = L_k^a$.

First, we show the following equation:

$$X_kX_k^+ + Y_kY_k^+ = I. \tag{108}$$

Using (30), (94), and (95), we obtain:

$$\breve{G}_k = \bar{U}_{k-1}X_k^+C_k^{-1/2}. \tag{109}$$

Using (10), (33), (94), (107), and (109), we obtain:

$$\begin{aligned} \breve{T}_k &= T_k(I - \bar{S}_kX_k^+C_k^{-1/2}) \\ &= T_kC_k^{1/2}(I - X_kX_k^+)C_k^{-1/2} = T_kM_k. \end{aligned} \tag{110}$$

Using the following fact

$$(I - X_kX_k^+)(I - X_kX_k^+)^T = (I - X_kX_k^+) \tag{111}$$

and (34), (38), (104)–(105), (107), and (110), we have

$$\begin{aligned} \breve{C}_k\bar{C}_k^+ &= T_kC_k^{1/2}(I - X_kX_k^+)C_k^{-1/2}C_k\bar{C}_k^+ \\ &= \bar{C}_k\bar{C}_k^+M_kC_k\bar{C}_k^+ = T_kM_k = \breve{T}_k \end{aligned} \tag{112}$$

from which along with using (106)–(107), we obtain:

$$\begin{aligned} Y_kY_k^+ &= C_k^{1/2}\bar{C}_k^+\breve{C}_k\bar{C}_k^+C_k^{1/2} \\ &= C_k^{1/2}\bar{C}_k^+\bar{C}_k\bar{C}_k^+M_kC_k\bar{C}_k^+C_k^{1/2} \\ &= C_k^{-1/2}M_kC_k^{1/2} = I - X_kX_k^+. \end{aligned}$$

Next, we show the following relationship:

$$\breve{G}_k C_k \breve{T}_k^T = 0 \tag{113}$$

which is verified by using (109)–(111) as follows:

$$\begin{aligned}\breve{G}_k C_k \breve{T}_k^T &= \bar{U}_{k-1} X_k^+ C_k^{-1/2} C_k C_k^{-1/2} \\ &\quad \times (I - X_k X_k^+) C_k^{1/2} T_k^T = 0.\end{aligned}$$

Finally, using (38), (94), (106), (108)–(109), and (113), the gain matrix L_k^* (37) can be rewritten as follows:

$$\begin{aligned}L_k^* &= \bar{U}_{k-1} X_k^+ C_k^{-1/2} \\ &\quad + P_{k|k-1}^{\bar{x}} H_k^T C_k^{-1/2} Y_k Y_k^+ C_k^{-1/2} \\ &= \bar{U}_{k-1} X_k^+ C_k^{-1/2} \\ &\quad + P_{k|k-1}^{\bar{x}} H_k^T C_k^{-1} (I - \bar{S}_k X_k^+ C_k^{-1/2})\end{aligned}$$

which is equivalent to (96). This completes the proof. □

Theorem 2. Let the conditions (104) and (107) hold. If the following relationship is satisfied:

$$\bar{S}_k^+ = X_k^+ C_k^{-1/2} \tag{114}$$

the ARTSKF, which is given by (58)–(68), is equivalent to the MUMVF[1], as referenced by (5), (10)–(15), (17), (22)–(28), (29), and (33).

Proof. We note by (95) that (114) implies the following:

$$\bar{S}_k^+ = (\bar{S}_k^T C_k^{-1} \bar{S}_k)^+ \bar{S}_k^T C_k^{-1}$$

From this relationship and via the same approach given in the proof of Theorem 1, one can easily verify the theorem.

Theorem 3. Let the condition (104) hold. The DRTSKF, which is given by (73)–(83), is equivalent to the MUMVF[3], which is referenced by (5), (10)–(15), (17), (22)–(28), (31), and (33)–(34).

Proof. First, we note by (104) the following relationships:

$$\bar{C}_k^+ = \bar{C}_k^+ \bar{C}_k \bar{C}_k^+ = \bar{C}_k^+ T_k = T_k^T \bar{C}_k^+. \tag{115}$$

Then, using (10), (31), and (33), we have

$$\breve{T}_k = T_k - \breve{S}_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+ \tag{116}$$

from which, along with using (31), (33)–(34), (104), and (115)–(116), we have:

$$\begin{aligned}\breve{G}_k C_k \breve{T}_k^T &= U_{k-1} (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+ C_k \\ &\quad \times (T_k^T - \bar{C}_k^+ \breve{S}_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T) \\ &= U_{k-1} (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+ \bar{C}_k \\ &\quad \times (I - \bar{C}_k^+ \breve{S}_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T) \\ &= U_{k-1} (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ (I - (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k) \\ &\quad \times (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+) \breve{S}_k^T \\ &= 0.\end{aligned} \tag{117}$$

Next, we show the following equation:

$$Y_k Y_k^+ = \tilde{C}_k (I - \bar{X}_k \bar{X}_k^+) \tilde{C}_k^T. \tag{118}$$

Using (33)-(34), (38), (104), and (115)-(116), we have

$$\breve{C}_k \bar{C}_k^+ = \bar{C}_k \bar{C}_k^+ - \breve{S}_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+ = \breve{T}_k \tag{119}$$

from which we obtain:

$$\breve{T}_k{}^T \breve{C}_k{}^+ \breve{T}_k = \bar{C}_k^+ \breve{C}_k \bar{C}_k^+. \tag{120}$$

Using (94)–(95), (102), (106), and (119)–(120), we obtain:

$$\begin{aligned}Y_k Y_k^+ &= C_k^{1/2} \bar{C}_k^+ \breve{C}_k \bar{C}_k^+ C_k^{1/2} \\ &= C_k^{1/2} \bar{C}_k^+ (\bar{C}_k - \breve{S}_k (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T) \bar{C}_k^+ C_k^{1/2} \\ &= C_k^{1/2} (\bar{C}_k^+)^{1/2} (I - \bar{X}_k \bar{X}_k^+) (\bar{C}_k^+)^{1/2} C_k^{1/2} \\ &= \tilde{C}_k (I - \bar{X}_k \bar{X}_k^+) \tilde{C}_k^T.\end{aligned}$$

Finally, using (31), (38), (94)–(95), (102), (106), and (117)–(118) in (37) we find:

$$\begin{aligned}L_k^* &= U_{k-1} \bar{X}_k^+ (\bar{C}_k^+)^{1/2} \\ &\quad + P_{k|k-1}^{\bar{x}} H_k^T C_k^{-1/2} Y_k Y_k^+ C_k^{-1/2} \\ &= U_{k-1} \bar{X}_k^+ (\bar{C}_k^+)^{1/2} \\ &\quad + P_{k|k-1}^{\bar{x}} H_k^T \bar{C}_k^+ (I - \breve{S}_k \bar{X}_k^+ (\bar{C}_k^+)^{1/2})\end{aligned}$$

which is equivalent to (97). This has completed the proof. □

Theorem 4. Let the condition (104) hold. If the following relationship is satisfied:

$$\breve{S}_k^+ = \bar{X}_k^+ (\bar{C}_k^+)^{1/2} \tag{121}$$

the DRTSKF, which is given by (73)-(83), is equivalent to the MUMVF^4, as referenced by (5), (10)-(15), (17), (22)-(28), and (32).

Proof. We note by (95) that (121) implies the following:

$$\breve{S}_k^+ = (\breve{S}_k^T \bar{C}_k^+ \breve{S}_k)^+ \breve{S}_k^T \bar{C}_k^+$$

from which, along with using the same approach given in the proof of Theorem 3, one can easily verify the theorem.

Based on the above, the following facts can be concluded:

1. Conditions (104), (107), and (114) imply that the ARTSKF is equivalent to both the MUMVF^1 and the MUMVF^2.
2. Conditions (104) and (121) imply that the DRTSKF is equivalent to the MUMVF^4 as well as the MUMVF^3.
3. Subject to the conditions (104), (107), (114), and (121) and further assuming the following condition

$$\bar{U}_{k-1} \bar{S}_k^+ = U_{k-1} \breve{S}_k^+ \tag{122}$$

the ARTSKF and the DRTSKF are both optimal solutions to the considered problem. If this is the case, the four MUMVFs presented are all the same. It can be also verified that (104) and (107) guarantee Condition (101) while (114), (121), and (122) guarantee Condition (100).

Remark 7. Conditions (104) and (107) respectively, imply the following relationships

$$T_k \bar{C}_k = \bar{C}_k, \quad \breve{T}_k T_k = \breve{T}_k$$

which are always satisfied in the addressed unknown input filtering problem owing to the facts: $(T_k)^2 = T_k$ and $\breve{G}_k G_k = 0$. This illustrates that Conditions (104) and (107) are less restricted and are reasonable assumptions that demonstrate the optimality of the ARTSKF and DRTSKF.

VI. AN ILLUSTRATIVE EXAMPLE

To show the proposed results, the numerical example given in [20] is considered, where the system parameters of (1) and (2) are given as follows:

$$A_k = \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix},$$

$$H_k = I_{3\times3}, \quad B_k = \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix}, \quad F_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$G_k = G_k^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The covariance matrices are given as $Q_k = diag\{0.1^2, 0.1^2, 0.01^2\}$ and $R_k = 0.1^2 I_{3\times3}$. In the simulation, we set $u_k = 10$, $x_0 = \hat{x}_0 = 0$, $P_0 = 0.1^2 I_{3\times3}$, and

$$d_k = \begin{bmatrix} \Delta a_{11} & \Delta a_{12} & \Delta a_{13} \\ \Delta a_{21} & \Delta a_{22} & \Delta a_{23} \end{bmatrix} x_k + \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} u_k$$

where $\Delta a_{ij} = -0.5 a_{ij}$ and $\Delta b_j = 0.5 b_j$. The simulation time is $N = 100$ time steps.

The root-mean-square-errors (rmse) are listed in the state estimates of the addressed filters, *i.e.*, the OEF, ERTSKF, MUMVFs, ARTSKF, and DRTSKF, in Table III, where both the filter parameters, *i.e.*, $\Lambda_k = 0$ and $\Lambda_k = G_k^+$, are considered and $e^i(rmse)$ stands for $\sqrt{\sum_{k=1}^N (x_k^{(i)} - \hat{x}_k^{(i)})^2/N}$. Note that the filter parameter $\Lambda_k = 0$ implies that the considered filter does not contain delay measurements, *i.e.*, $K_k = 0$, which is the main consideration of the original conference paper [23]. In such a case, the MUMVF will be equivalent to the OUMVF. It can be seen from Table III that for $\Lambda_k = 0$, the filtering performances of the MUMVF^1, MUMVF^2, and ARTSKF are the same; this is also true for the MUMVF^3, MUMVF^4, and DRTSKF, mainly due to the fact that in this simulation case all the relationships (104), (107), (114), and (121) are satisfied. Nevertheless, owing to the fact that the relationship given by (122) does not hold, the performances of the ARTSKF and DRTSKF are not the same, with the performance of the latter much degraded. The performance of 162.1685 indicates that the corresponding state estimates are incorrect. This illustrates that the DRTSKF with $\Lambda_k = 0$, designed using the decoupled unknown-input approach has ignored more useful

Table III. Performance of the OEF, ERTSKF, MUMVFs, ARTSKF, and DRTSKF.

		$\Lambda_k = 0$			$\Lambda_k = G_k^+$		
Case	Filter	e^1(rmse)	e^2(rmse)	e^3(rmse)	e^1(rmse)	e^2(rmse)	e^3(rmse)
G_k^1	OEF	0.2672	0.0949	0.0682	0.2672	0.0949	0.0682
	MUMVF^1	0.2672	0.0949	0.0682	0.1502	0.0949	0.0682
	MUMVF^2	0.2672	0.0949	0.0682	0.1502	0.0949	0.0682
	ARTSKF	0.2672	0.0949	0.0682	0.1502	0.0949	0.0682
	ERTSKF	162.1285	0.0949	0.0682	0.1502	0.0949	0.0682
	MUMVF^3	162.1285	0.0949	0.0682	0.1502	0.0949	0.0682
	MUMVF^4	162.1285	0.0949	0.0682	0.1502	0.0949	0.0682
	DRTSKF	162.1285	0.0949	0.0682	0.1502	0.0949	0.0682
G_k^2	OEF	0.1060	0.0949	0.0782	0.1060	0.0949	0.0782
	MUMVF^1	0.1060	0.0949	0.0782	0.1060	0.0858	0.0657
	MUMVF^2	0.1060	0.0949	0.0782	0.1060	0.0858	0.0657
	ARTSKF	0.1060	0.0949	0.0782	0.1060	0.0858	0.0657
	ERTSKF	0.1060	0.0949	0.0782	0.1060	0.0858	0.0657
	MUMVF^3	0.1060	0.0949	0.0782	0.1060	0.0858	0.0657
	MUMVF^4	0.1060	0.0949	0.0782	0.1060	0.0858	0.0657
	DRTSKF	0.1060	0.0949	0.0782	0.1060	0.0858	0.0657
G_k^3	OEF	0.1060	3.4216	2.7859	0.1060	3.4216	2.7859
	MUMVF^1	0.1060	3.4216	2.7859	0.1034	0.2068	0.1106
	MUMVF^2	0.1060	3.4216	2.7859	0.1034	0.2068	0.1106
	ARTSKF	0.1060	3.4216	2.7859	0.1034	0.2068	0.1106
	ERTSKF	0.1060	2.1774	1.7701	0.1034	0.2068	0.1106
	MUMVF^3	0.1060	2.1774	1.7701	0.1034	0.2068	0.1106
	MUMVF^4	0.1060	2.1774	1.7701	0.1034	0.2068	0.1106
	DRTSKF	0.1060	2.1774	1.7701	0.1034	0.2068	0.1106
G_k^4	OEF	0.2221	0.2041	0.1654	0.2221	0.2041	0.1654
	MUMVF^1	0.5456	3.4239	2.7673	0.118	0.1430	0.0927
	MUMVF^2	0.2221	0.2041	0.1654	0.118	0.1430	0.0927
	ARTSKF	0.2221	0.2041	0.1654	0.118	0.1430	0.0927
	ERTSKF	0.2221	0.2041	0.1654	0.118	0.1430	0.0927
	MUMVF^3	0.2221	0.2041	0.1654	0.118	0.1430	0.0927
	MUMVF^4	0.5456	3.4239	2.7673	0.118	0.1430	0.0927
	DRTSKF	0.2221	0.2041	0.1654	0.118	0.1430	0.0927

information in filtering as compared to that of the ARTSKF with $\Lambda_k = 0$, designed using the augmented unknown-input approach. Moreover, the OEF has the same performance as that of the MUMVF^1 (see Remark 1); the same is true for the ERTSKF and the DRTSKF (see Remark 5). It should be emphasized that the unbiasedness conditions (18), (98), and (99) are not satisfied for the case $\Lambda_k = 0$, which renders all the considered filters as biased estimators. On the other hand, for the case $\Lambda_k = G_k^+$, all the unbiasedness conditions (18), (98), and (99) are satisfied; hence all filters except the OEF will have the best filtering performance. This shows that the filtering performance of the filter designed using $\Lambda_k = G_k^+$ is better than that using $\Lambda_k = 0$ because the obtained filter has fewer restricted unbiased constraints.

Next, we consider the case that the relationship (122) is satisfied, to which we choose:

$$G_k^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The simulation results are also given in Table III, which shows that all filtering performances of the considered filters are the same for $\Lambda_k = 0$. The same result holds for all filters (except the OEF) using $\Lambda_k = G_k^+$. As expected, all the conditions given by (104), (107), (114), and (121)-(122) are satisfied for this simulation case. It is noted that the unbiasedness conditions (18), (98), and (99) are both satisfied for $\Lambda_k = 0$ and $\Lambda_k = G_k^+$, which renders all the considered filters to be unbiased estimators. It is observed that the filter designed using $\Lambda_k = G_k^+$ is slightly better than that using $\Lambda_k = 0$.

Then, we consider a case where the DRTSKF with $\Lambda_k = 0$ has better performance than the ARTSKF with $\Lambda_k = 0$, to which we choose:

$$G_k^3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the case $\Lambda_k = 0$, since the conditions (104), (107), (114), and (121) are satisfied while that of (122) does not hold, the following two observations can be made based on Table III: The filtering performances of the OEF, MUMVF^1, MUMVF^2, and ARTSKF are the same and the filtering performances of the ERTSKF, MUMVF^3, MUMVF^4, and DRTSKF are the same. Unlike the result of the first case, the DRTSKF is shown to have better performance than the ARTSKF. For the case $\Lambda_k = G_k^+$, all filters except the OEF have the same filtering performance, which is much better than results obtained using $\Lambda_k = 0$ since the unbiasedness conditions (18), (98), and (99) are not satisfied for the latter.

It is noted that all the above simulation cases consider that some measurements are not corrupted by the unknown input; hence some state estimates obtained by using $\Lambda_k = 0$ and $\Lambda_k = G_k^+$ are almost the same. Finally, we consider a case where all measurements are corrupted by the unknown input, further illustrating the superiority of the filters designed using $\Lambda_k = G_k^+$ over those using $\Lambda_k = 0$. To achieve this, we choose:

$$G_k^4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

For the case $\Lambda_k = 0$, since Conditions (104), (107), and (122) are satisfied while those of (114) and (121) do not hold, we have the following result: the filtering performances of the OEF, MUMVF^2, ARTSKF, ERTSKF, MUMVF^3, and DRTSKF are the same but different from those of MUMVF^1 and MUMVF^4. In this case, all the unbiasedness conditions (18), (98), and (99) are satisfied. This case also illustrates that the performance of the MUMVF is dependant on the chosen value of the matrix $\breve{G}_k$. For the case $\Lambda_k = G_k^+$, all filters except the OEF have the same filtering performance, which is still better than results obtained by using $\Lambda_k = 0$.

Based on the simulation, the following can be concluded:

1. Under the less restricted conditions (104), (107), and (114), the filtering performance of the MUMVF^1, MUMVF^2, and ARTSKF are all the same.

2. Under the less restricted conditions: (104) and (121), the filtering performances of the MUMVF^3, MUMVF^4, and DRTSKF are all the same.

3. The filtering performance of the OEF is equivalent to that of the ARTSKF for the case $\Lambda_k = 0$.

4. Under Condition (41), the filtering performance of the ERTSKF is equivalent to that of the DRTSKF.

5. For the case $\Lambda_k = 0$, the filtering performance of the MUMVF is dependant on the chosen value of the matrix $\breve{G}_k$.

6. For the case $\Lambda_k = G_k^+$, the filtering performances of the MUMVFs, ARTSKF, and DRTSKF are all the same.

7. The filtering performance of the filter designed by using $\Lambda_k = G_k^+$ is better than that using $\Lambda_k = 0$.

VII. CONCLUSION

In this paper, the ORTSKF design via the RTSKF technique for systems with unknown inputs that have arbitrary profiles was addressed and explored. Moreover, a new MUMVF design whose filtering performance can be made equivalent to that of the OEF (resp. EOUMVF) using the design parameter $\Lambda_k = 0$ (resp. $\Lambda_k = G_k^+$) was also proposed to facilitate discussions of the optimality of the ORTSKF. Specifically, four different implementations of the MUMVF based on the obtained unbiasedness conditions were presented. Using a novel ESD method and the RTSKF, two ORTSKFs, *i.e.*, ARTSKF and DRTSKF, were derived. It was proven that under some less restricted conditions the ORTSKFs were equivalent to the corresponding MUMVFs. Furthermore, the conditions under which the ARTSKF is equivalent to the DRTSKF were derived. The issue of the equivalence of the MUMVF (resp. DRTSKF) and the OUMVF (resp. ERTSKF) was also addressed. Simulation results verified the effectiveness of the proposed results. This research also showed that the presented RTSKF technique serves as an alternative to the UMVE method in deriving optimal state estimators to be used in solving general unknown input filtering problems.

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