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Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feedthrough☆

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Abstract

This paper extends previous work on joint input and state estimation to systems with direct feedthrough of the unknown input to the output. Using linear minimum-variance unbiased estimation, a recursive filter is derived where the estimation of the state and the input are interconnected. The derivation is based on the assumption that no prior knowledge about the dynamical evolution of the unknown input is available. The resulting filter has the structure of the Kalman filter, except that the true value of the input is replaced by an optimal estimate.

Keywords: Kalman filtering; Recursive state estimation; Unknown input estimation; Minimum-variance estimation

1. Introduction

Systematic measurement errors and model uncertainties such as unknown disturbances or unmodeled dynamics can be represented as unknown inputs. The problem of optimal filtering in the presence of unknown inputs has therefore received a lot of attention.

Friedland (1969) and Park, Kim, Kwon, and Kwon (2000) solved the unknown input filtering problem by augmenting the state vector with an unknown input vector. However, this method is limited to the case where a model for the dynamical evolution of the unknown input is available.

A rigorous and straightforward state estimation method in the presence of unknown inputs is developed by Hou and Müller (1994) and Hou and Patton (1998). The approach consists in first building an equivalent system which is decoupled from the unknown inputs, and then designing a minimum-variance unbiased (MVU) estimator for this equivalent system.

Another approach consists in parameterizing the filter equations and then calculating the optimal parameters by minimizing the trace of the state covariance matrix under an unbiasedness condition. An optimal filter of this type was first developed by Kitanidis (1987). The derivation of Kitanidis (1987) is limited to linear systems without direct feedthrough of the unknown input to the output and yields no estimate of the input. An extension to state estimation for systems with direct feedthrough was developed by Darouach, Zasadzinski, and Boutayeb (2003). Extensions to joint input and state estimation for systems without direct feedthrough are addressed by Hsieh (2000) and Gillijns and De Moor (2007).

In this paper, we combine both extensions of Kitanidis (1987) by addressing the problem of joint input and state estimation for linear discrete-time systems with direct feedthrough of the unknown input to the output. Using linear minimum-variance unbiased estimation, we develop a recursive filter where the estimation of the state and the input are interconnected. The estimation of the input is based on the least-squares (LS) approach developed by Gillijns and De Moor (2007), while the state estimation problem is solved using the method developed by Kitanidis (1987).

This paper is outlined as follows. In Section 2, we formulate the filtering problem and present the recursive three-step structure of the filter. Next, in Sections 3–5, we consider each of

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the three steps separately and derive equations for the optimal input and state estimators. Finally, in Section 6, we summarize the filter equations.

2. Problem formulation

Consider the linear discrete-time system

$$x_{k+1} = A_k x_k + G_k d_k + w_k, \quad (1)$$
$$y_k = C_k x_k + H_k d_k + v_k, \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $d_k \in \mathbb{R}^m$ is an unknown input vector, and $y_k \in \mathbb{R}^p$ is the measurement. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^p$ are assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrices, $Q_k = \mathbb{E}[w_k w_k^{\mathrm{T}}] \geqslant 0$ and $R_k = \mathbb{E}[v_k v_k^{\mathrm{T}}] > 0$, respectively. Results are easily generalized to the case where w_k and v_k are correlated by applying a preliminary transformation to the system (Anderson & Moore, 1979). Also, results are easily generalized to systems with both known and unknown inputs. The matrices A_k, G_k, C_k and H_k are known and it is assumed that rank $H_k = m$. Throughout the paper, we assume that (A_k, C_k) is observable and that x_0 is independent of v_k and w_k for all k. Also, we assume that an unbiased estimate $\hat{x}_0$ of the initial state x_0 is available with covariance matrix P_0^x.

The objective of this paper is to design an optimal recursive filter which estimates both the system state x_k and the input d_k based on the initial estimate $\hat{x}_0$ and the sequence of measurements $\{y_0, y_1, \ldots, y_k\}$. No prior knowledge about the dynamical evolution of d_k is assumed to be available and no prior assumption is made. The unknown input can be any type of signal.

The optimal state estimation problem for a system with direct feedthrough of the unknown input d_k to the output y_k is conceptually not very different from the case where $H_k = 0$. A single filter and a single existence condition, valid for both cases, can be found in Darouach et al. (2003) and Hou and Müller (1994). In contrast, the optimal input estimation problem is conceptually very different in both cases. If $H_k = 0$, the unknown input d_k must be estimated with one step delay because the first measurement containing information on d_k is y_{k+1} (Gillijns & De Moor, 2007). On the other hand, if $H_k \neq 0$, the first measurement containing information on d_k is y_k. Consequently, the structure of the input estimator and the existence conditions are totally different in both cases.

We consider a recursive three-step filter of the form

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + G_{k-1}\hat{d}_{k-1}, \quad (3)$$
$$\hat{d}_k = M_k(y_k - C_k\hat{x}_{k|k-1}), \quad (4)$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - C_k\hat{x}_{k|k-1}), \quad (5)$$

where the matrices $M_k \in \mathbb{R}^{m\times p}$ and $L_k \in \mathbb{R}^{n\times p}$ still have to be determined. The first step, which we call the *time update*, yields an estimate of x_k given measurements up to time $k-1$. This step is addressed in Section 3. The second step yields an estimate of the unknown input. The calculation of the optimal matrix M_k is addressed in Section 4. Finally, the third step, the so-called *measurement update*, yields an estimate of x_k given measurements up to time k. This step is addressed in Section 5, where we calculate the optimal value of L_k.

3. Time update

First, we consider the time update. Let $\hat{x}_{k-1|k-1}$ and $\hat{d}_{k-1}$ denote the optimal unbiased estimates of x_{k-1} and d_{k-1} given measurements up to time $k-1$, then the time update is given by

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + G_{k-1}\hat{d}_{k-1}.$$

The error in the estimate $\hat{x}_{k|k-1}$ is given by

$$\begin{aligned}\tilde{x}_{k|k-1} &:= x_k - \hat{x}_{k|k-1},\\ &= A_{k-1}\tilde{x}_{k-1|k-1} + G_{k-1}\tilde{d}_{k-1} + w_{k-1},\end{aligned}$$

with $\tilde{x}_{k|k} := x_k - \hat{x}_{k|k}$ and $\tilde{d}_k := d_k - \hat{d}_k$. Consequently, the covariance matrix of $\hat{x}_{k|k-1}$ is given by

$$\begin{aligned}P^x_{k|k-1} &:= \mathbb{E}[\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^{\mathrm{T}}],\\ &= [\,A_{k-1} \quad G_{k-1}\,]\begin{bmatrix} P^x_{k-1|k-1} & P^{xd}_{k-1} \\ P^{dx}_{k-1} & P^d_{k-1}\end{bmatrix}\begin{bmatrix} A^{\mathrm{T}}_{k-1} \\ G^{\mathrm{T}}_{k-1}\end{bmatrix} + Q_{k-1},\end{aligned}$$

with $P^x_{k|k} := \mathbb{E}[\tilde{x}_{k|k}\tilde{x}^{\mathrm{T}}_{k|k}]$, $P^d_k := \mathbb{E}[\tilde{d}_k\tilde{d}^{\mathrm{T}}_k]$ and $(P^{xd}_k)^{\mathrm{T}} = P^{dx}_k := \mathbb{E}[\tilde{d}_k\tilde{x}^{\mathrm{T}}_{k|k}]$. Expressions for these covariance matrices will be derived in the next sections.

4. Input estimation

In this section, we consider the estimation of the unknown input. In Section 4.1, we determine the matrix M_k such that (4) yields an unbiased estimate of d_k. In Section 4.2, we extend to MVU input estimation.

4.1. Unbiased input estimation

Defining the *innovation* $\tilde{y}_k := y_k - C_k\hat{x}_{k|k-1}$, it follows from (2) that

$$\tilde{y}_k = H_k d_k + e_k, \quad (6)$$

where e_k is given by

$$e_k = C_k\tilde{x}_{k|k-1} + v_k. \quad (7)$$

Since $\hat{x}_{k|k-1}$ is unbiased, it follows from (7) that $\mathbb{E}[e_k] = 0$ and consequently from (6) that $\mathbb{E}[\tilde{y}_k] = H_k\mathbb{E}[d_k]$. This indicates that an unbiased estimate of the unknown input d_k can be obtained from the innovation $\tilde{y}_k$.

Theorem 1. *Let $\hat{x}_{k|k-1}$ be unbiased, then* (3)–(4) *is an unbiased estimator for all possible d_k if and only if M_k satisfies $M_k H_k = I$.*

Proof. The proof is similar to that of Theorem 1 in Gillijns and De Moor (2007) and is omitted. □

It follows from Theorem 1 that rank $H_k = m$ is a necessary and sufficient condition for the existence of an unbiased input estimator of the form (4). Note that this condition implies $p \geqslant m$. The matrix $M_k = (H_k^{\mathrm{T}} H_k)^{-1} H_k^{\mathrm{T}}$ corresponding to the LS solution of (6) satisfies the condition of Theorem 1. The LS solution is thus unbiased. However, it follows from the Gauss–Markov theorem (Kailath, Sayed, & Hassibi, 2000) that it is not necessarily minimum-variance because in general

$$\tilde{R}_k := \mathbb{E}[e_k e_k^{\mathrm{T}}] = C_k P_{k|k-1}^x C_k^{\mathrm{T}} + R_k \neq cI,$$

where c denotes a positive real number.

4.2. MVU input estimation

An MVU estimate of d_k based on the innovation $\tilde{y}_k$ is obtained by weighted LS estimation with weighting matrix equal to the inverse of $\tilde{R}_k$.

Theorem 2. *Let $\hat{x}_{k|k-1}$ be unbiased and let $\tilde{R}_k$ and $H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k$ be nonsingular, then for M_k given by*

$$M_k^\star = (H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k)^{-1} H_k^{\mathrm{T}} \tilde{R}_k^{-1},$$

(4) *is the MVU estimator of d_k given $\tilde{y}_k$. The variance of the optimal input estimate is given by*

$$P_k^{\star d} = (H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k)^{-1}.$$

Proof. The proof is similar to that of Theorem 2 in Gillijns and De Moor (2007) and is omitted. □

We denote the optimal input estimate corresponding to $M_k^\star$ by $\hat{d}_k^\star$ and derive an equation for $\tilde{d}_k^\star := d_k - \hat{d}_k^\star$. It follows from (4), (6) and the unbiasedness of the input estimator that $\tilde{d}_k^\star$ is given by

$$\tilde{d}_k^\star = (I - M_k^\star H_k) d_k - M_k^\star e_k = -M_k^\star e_k. \tag{8}$$

This equation will be used in the next section, where we consider the measurement update.

5. Measurement update

Finally, we consider the update of $\hat{x}_{k|k-1}$ with the measurement y_k. We calculate the gain matrix L_k which yields the MVU estimator of the form (5). Using (5) and (6), we find that

$$\tilde{x}_{k|k} = (I - L_k C_k)\tilde{x}_{k|k-1} - L_k H_k d_k - L_k v_k. \tag{9}$$

Consequently, (5) is unbiased for all possible d_k if and only if L_k satisfies

$$L_k H_k = 0. \tag{10}$$

Let L_k satisfy (10), then it follows from (9) that $P_{k|k}^x$ is given by

$$P_{k|k}^x = (I - L_k C_k) P_{k|k-1}^x (I - L_k C_k)^{\mathrm{T}} + L_k R_k L_k^{\mathrm{T}}. \tag{11}$$

An MVU state estimator is then obtained by calculating the gain matrix L_k which minimizes the trace of (11) under the unbiasedness condition (10).

Theorem 3. *The gain matrix L_k given by*

$$L_k^\star = K_k^\star (I - H_k M_k^\star), \tag{12}$$

where $K_k^\star = P_{k|k-1}^x C_k^{\mathrm{T}} \tilde{R}_k^{-1}$, minimizes the trace of (11) *under the unbiasedness condition* (10).

Proof. We use the approach of Kitanidis (1987), where a similar optimization problem is solved using Lagrange multipliers. The Lagrangian is given by

$$\begin{aligned} &\mathrm{trace}\{L_k \tilde{R}_k L_k^{\mathrm{T}} - 2 P_{k|k-1}^x C_k^{\mathrm{T}} L_k^{\mathrm{T}} + P_{k|k-1}^x\} \\ &\quad - 2\,\mathrm{trace}\{L_k H_k \Lambda_k^{\mathrm{T}}\}, \end{aligned} \tag{13}$$

where $\Lambda_k \in \mathbb{R}^{p \times n}$ is the matrix of Lagrange multipliers and the factor "2" is introduced for notational convenience. Setting the derivative of (13) with respect to L_k equal to zero, yields

$$\tilde{R}_k L_k^{\mathrm{T}} - C_k P_{k|k-1}^x - H_k \Lambda_k^{\mathrm{T}} = 0. \tag{14}$$

Eqs. (14) and (10) form the linear system of equations

$$\begin{bmatrix} \tilde{R}_k & -H_k \\ H_k^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} L_k^{\mathrm{T}} \\ \Lambda_k^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} C_k P_{k|k-1}^x \\ 0 \end{bmatrix}, \tag{15}$$

which has a unique solution if and only if the coefficient matrix is nonsingular. Let $\tilde{R}_k$ be nonsingular, then the coefficient matrix is nonsingular if and only if $H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k$, the Schur complement of $\tilde{R}_k$, is nonsingular. Finally, premultiplying left- and right-hand side of (15) by the inverse of the coefficient matrix, yields (12). □

We denote the state estimate corresponding to the gain matrix $L_k^\star$ by $\hat{x}_{k|k}^\star$. Substituting (12) in (5), yields the equivalent state updates

$$\begin{aligned} \hat{x}_{k|k}^\star &= \hat{x}_{k|k-1} + K_k^\star (I - H_k M_k^\star)(y_k - C_k \hat{x}_{k|k-1}), \\ &= \hat{x}_{k|k-1} + K_k^\star (y_k - C_k \hat{x}_{k|k-1} - H_k \hat{d}_k^\star), \end{aligned}$$

from which we conclude that the optimal state estimator implicitly estimates the unknown input by weighted LS estimation.

Finally, we derive expressions for the covariance matrices $P_{k|k}^{\star x} := \mathbb{E}[\tilde{x}_{k|k}^\star \tilde{x}_{k|k}^{\star \mathrm{T}}]$ and $P_k^{\star xd} := \mathbb{E}[\tilde{x}_{k|k}^\star \tilde{d}_k^{\star \mathrm{T}}]$ where

$$\begin{aligned} \tilde{x}_{k|k}^\star &:= x_k - \hat{x}_{k|k}^\star, \\ &= (I - L_k^\star C_k)\tilde{x}_{k|k-1} - L_k^\star v_k. \end{aligned} \tag{16}$$

By substituting (12) in (11), we obtain the following expression for $P_{k|k}^{\star x}$,

$$P_{k|k}^{\star x} = P_{k|k-1}^x - K_k^\star (\tilde{R}_k - H_k P_k^{\star d} H_k^{\mathrm{T}}) K_k^{\star \mathrm{T}}.$$

Using (16) and (8), it follows that

$$P_k^{\star xd} = -P_{k|k-1}^x C_k^{\mathrm{T}} M_k^{\star \mathrm{T}} = -K_k^\star H_k P_k^{\star d}.$$

6. Summary of filter equations

In this section, we summarize the filter equations. We assume that $\hat{x}_0$, the estimate of the initial state, is unbiased and has known variance P_0^x. The initialization step of the filter is then given by:

Initialization:

$$\hat{x}_0 = \mathbb{E}[x_0],$$
$$P_0^x = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^{\mathrm{T}}].$$

The recursive part of the filter consists of three steps: the estimation of the unknown input, the measurement update and the time update. These three steps are given by

Estimation of unknown input:

$$\tilde{R}_k = C_k P_{k|k-1}^x C_k^{\mathrm{T}} + R_k,$$
$$M_k = (H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k)^{-1} H_k^{\mathrm{T}} \tilde{R}_k^{-1},$$
$$\hat{d}_k = M_k (y_k - C_k \hat{x}_{k|k-1}),$$
$$P_k^d = (H_k^{\mathrm{T}} \tilde{R}_k^{-1} H_k)^{-1}.$$

Measurement update:

$$K_k = P_{k|k-1}^x C_k^{\mathrm{T}} \tilde{R}_k^{-1},$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - C_k \hat{x}_{k|k-1} - H_k \hat{d}_k),$$
$$P_{k|k}^x = P_{k|k-1}^x - K_k (\tilde{R}_k - H_k P_k^d H_k^{\mathrm{T}}) K_k^{\mathrm{T}},$$
$$P_k^{xd} = (P_k^{dx})^{\mathrm{T}} = -K_k H_k P_k^d.$$

Time update:

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + G_k \hat{d}_k,$$
$$P_{k+1|k}^x = [\, A_k \quad G_k \,] \begin{bmatrix} P_{k|k}^x & P_k^{xd} \\ P_k^{dx} & P_k^d \end{bmatrix} \begin{bmatrix} A_k^{\mathrm{T}} \\ G_k^{\mathrm{T}} \end{bmatrix} + Q_k.$$

Note that the time and measurement update of the state estimate take the form of the Kalman filter, except that the true value of the input is replaced by an optimal estimate. Also, note that in case $H_k = 0$ and $G_k = 0$, the Kalman filter is obtained.

7. Conclusion

This paper has studied the problem of joint input and state estimation for linear discrete-time systems with direct feedthrough of the unknown input to the output. A recursive filter was developed where the update of the state estimate has the structure of the Kalman filter, except that the true value of the input is replaced by an optimal estimate. This input estimate is obtained from the innovation by weighted LS estimation, where the optimal weighting matrix is computed from the covariance matrices of the state estimator.

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References

Anderson, B. D. O., & Moore, J. B. (1979). *Optimal filtering*. Englewood Cliffs, NJ: Prentice-Hall.

Darouach, M., Zasadzinski, M., & Boutayeb, M. (2003). Extension of minimum variance estimation for systems with unknown inputs. *Automatica*, *39*, 867–876.

Friedland, B. (1969). Treatment of bias in recursive filtering. *IEEE Transactions on Automatic Control*, *14*, 359–367.

Gillijns, S., & De Moor, B. (2007). Unbiased minimum-variance input and state estimation for linear discrete-time systems. *Automatica*, *43*(1), 111–116.

Hou, M., & Müller, P. C. (1994). Disturbance decoupled observer design: A unified viewpoint. *IEEE Transactions on Automatic Control*, *39*(6), 1338–1341.

Hou, M., & Patton, R. J. (1998). Optimal filtering for systems with unknown inputs. *IEEE Transactions on Automatic Control*, *43*(3), 445–449.

Hsieh, C. S. (2000). Robust two-stage Kalman filters for systems with unknown inputs. *IEEE Transactions on Automatic Control*, *45*(12), 2374–2378.

Kailath, T., Sayed, A. H., & Hassibi, B. (2000). *Linear estimation*. Upper Saddle River, NJ: Prentice-Hall.

Kitanidis, P. K. (1987). Unbiased-minimum variance linear state estimation. *Automatica*, *23*(6), 775–778.

Park, S. H., Kim, P. S., Kwon, O., & Kwon, W. H. (2000). Estimation and detection of unknown inputs using optimal FIR filter. *Automatica*, *36*, 1481–1488.