Complexity Measures for Time Series Analysis

1 Introduction

This document summarizes various complexity measures used for analyzing real-valued time series, including their definitions, interpretations, and practical considerations. The goal is to use these measures to compare the complexity of different time series.

2 Permutation Entropy

Permutation entropy is a measure of the complexity of a time series based on the order relations between values. It was introduced by Bandt and Pompe in 2002 [1] and has become a widely accepted method for analyzing nonlinear and chaotic systems.

Parameters

- **Order (embedding dimension)**: Number of consecutive points used to create ordinal patterns.
- **Delay (time lag)**: Number of steps between elements in each embedding vector.

The time series is embedded into vectors of the form:

$$[x(t), x(t+\tau), x(t+2\tau), \ldots, x(t+(m-1)\tau)]$$

Each vector is converted to a permutation pattern by ranking its elements, and permutation entropy is computed based on the frequency distribution of these patterns.

Interpretation

- A normalized permutation entropy close to 1 indicates high complexity (random or noisy behavior).
- A value near 0 suggests a highly regular or predictable signal.

3 Lempel-Ziv Complexity (LZC)

Lempel-Ziv Complexity quantifies the complexity of a symbolic sequence by counting the number of distinct substrings needed to reconstruct it. It was originally proposed by Lempel and Ziv in 1976 [2] as a foundation for data compression algorithms.

Computation

Given a symbolic sequence, scan from left to right and identify new substrings as they appear. The count of these substrings is the LZC.

Normalization

To compare sequences of different lengths, LZC is normalized:

$$\mathrm{LZC}_{\mathrm{norm}} = \frac{c(n) \cdot \log_2(n)}{n}$$

where $c(n)$ is the raw LZC and n is the sequence length. This normalization reflects the theoretical upper bound for complexity in random binary sequences.

Application to Real-Valued Time Series

Since LZC requires symbolic input, real-valued time series must be discretized. Common approaches include:

- **Binarization**: Convert values to 0 or 1 based on the median.
- **Multi-symbol discretization**: Partition the range into bins (e.g., quartiles) and assign symbols accordingly.

Interpretation

- Low LZC: The sequence is regular or repetitive.
- High LZC: The sequence is more complex or random.

Example

Given the sequence `010011`, LZC = 4, as the new patterns discovered while scanning are: `0, 1, 00, 11`. A more repetitive sequence like `010101` has LZC = 3.

4 Spectral Entropy

Spectral entropy is a frequency-domain complexity measure that quantifies the spectral diversity of a time series. It is based on the Shannon entropy of the power spectral density (PSD) of the signal.

Computation

Given a real-valued signal, the steps are:

1. Compute the PSD using methods like the Fourier transform or Welch's method.
2. Normalize the PSD so that it sums to 1.
3. Apply the Shannon entropy formula:

$$H = -\sum_i p_i \log_2 p_i$$

   where p_i are the normalized PSD values.

Interpretation

- **Low spectral entropy**: Power is concentrated in few frequencies (e.g., sine waves, periodic signals).
- **High spectral entropy**: Power is spread across many frequencies (e.g., white noise, random signals).

Usage with Real-Valued Time Series

Spectral entropy naturally accepts real-valued input. No discretization is needed. The signal must have a known or assumed sampling frequency (sf), which determines the frequency resolution of the spectral analysis.

Example: Sine Wave vs. White Noise

Let us consider two 10-second time series sampled at 100 Hz:

- A 5 Hz sine wave
- White Gaussian noise

The spectral entropy for these will differ significantly:

- **Sine wave**: $H \approx 0.1$
- **White noise**: $H \approx 0.95$

This illustrates that periodic signals have low spectral entropy due to concentrated frequency content, while noisy or chaotic signals exhibit high entropy from broad spectral content.

Applications

- EEG analysis: Entropy increases with alertness and decreases during seizures.
- Heart rate variability: Higher entropy in healthy subjects.
- Environmental signals (e.g., temperature, wind): Detecting regime shifts or anomalies.

5 SVD Entropy

SVD entropy (Singular Value Decomposition entropy) measures the complexity of a time series by evaluating the diversity of singular values obtained from the time-delay embedded matrix of the signal. It was introduced in the context of physiological signal analysis and has proven useful in detecting the underlying dynamical structure of time series.

Computation

1. Embed the time series into a matrix using time-delay embedding with dimension m and delay τ.
2. Apply singular value decomposition (SVD) to this matrix.
3. Square the singular values and normalize them:
$$p_i = \frac{\sigma_i^2}{\sum_j \sigma_j^2}$$
4. Compute Shannon entropy:
$$H = -\sum_i p_i \log_2 p_i$$

Interpretation

- **Low SVD entropy**: The signal is structured or periodic, dominated by few singular values.
- **High SVD entropy**: The signal is complex or noisy, energy is spread across many singular values.

Applications

- EEG/ECG signal analysis: Identifying seizure activity or cognitive states.
- Physiological monitoring: Heart rate complexity analysis.
- Sensor data streams: Fault detection or novelty detection.

Example

For example, a sine wave will have most of its variance captured by 1 or 2 singular values $\rightarrow$ low entropy. Conversely, white noise will result in a broad spectrum of singular values $\rightarrow$ high entropy.

6 Zero-Crossing Rate (num_zerocross)

Zero-crossing rate (num_zerocross) is a measure of how often a signal changes sign, i.e., how frequently it crosses the zero axis. It is widely used in signal processing to analyze the frequency content, regularity, and complexity of time series.

Computation

Given a time series $x = [x_1, x_2, \dots, x_N]$, the zero-crossing rate is calculated by counting the number of sign changes between consecutive points:

$$\text{num\_zerocross} = \sum_{i=1}^{N-1} \mathbf{1}(x_i \cdot x_{i+1} < 0)$$

where $\mathbf{1}$ is an indicator function that equals 1 when $x_i \cdot x_{i+1} < 0$ (indicating a zero-crossing) and 0 otherwise.

Interpretation

- **Low Zero-Crossing Rate**: The signal is smooth or periodic with few sign changes.
- **High Zero-Crossing Rate**: The signal is complex or noisy with frequent changes in direction.

Applications

- Speech and audio signal analysis: Characterizing sound roughness or periodicity.
- Physiological signals: Evaluating heart rate variability or detecting abnormal rhythms.
- Environmental monitoring: Analyzing fluctuations in temperature, wind speed, or other sensor data.

Example

Consider the time series:

$$x = [1, -2, 3, -4, 5, -6, 7]$$

The number of zero-crossings is 6, as the signal changes sign between every consecutive pair of points.

7 Zero-Crossing Rate

The **zero-crossing rate** is a measure of how many times a signal crosses zero. The **normalized zero-crossing rate** refers to the number of zero-crossings divided by the total number of samples. It can provide insights into the signal's oscillatory nature, randomness, and complexity.

Interpretation of a Normalized Zero-Crossing Rate of 0.5

A **normalized zero-crossing rate** of approximately **0.5** indicates that the signal crosses zero **frequently** relative to its total length, suggesting that the signal is either highly oscillatory or highly random. Here's what it implies:

- **High Oscillatory Behavior**: For periodic signals, such as a high-frequency sine wave, the normalized zero-crossing rate could approach 0.5, because the signal oscillates rapidly and crosses zero frequently within the analyzed time window.
- **Random or Noisy Signals**: In the case of random signals, such as Gaussian white noise, the signal fluctuates randomly and crosses zero many times, which can lead to a normalized zero-crossing rate around 0.5.
- **Complex or Unpredictable Behavior**: A normalized zero-crossing rate of 0.5 suggests a signal with **high complexity** or **high unpredictability**, as it frequently changes direction (positive to negative or vice versa).
- **Comparative Analysis**: In comparison to low-frequency signals, which typically have lower zero-crossing rates (much less than 0.5), signals with a zero-crossing rate of 0.5 indicate rapid changes or fluctuations, either due to high frequency or randomness.

In conclusion, a **normalized zero-crossing rate of 0.5** typically signifies that the signal is **highly oscillatory** or **random**, with frequent sign changes and a complex structure.

Example

For example, a **high-frequency sine wave** with a frequency close to half the sampling rate can have a normalized zero-crossing rate of 0.5. Similarly, a **random signal** like white noise will also exhibit a zero-crossing rate around 0.5 due to its random fluctuations.

7.1 Hjorth Mobility

Hjorth Mobility is a measure used to quantify the **smoothness** or **oscillatory nature** of a signal, particularly in the context of **EEG signals**. It was introduced as part of the Hjorth parameters, which also include **Hjorth Activity** and **Hjorth Complexity**. Hjorth Mobility specifically measures how rapidly a signal changes by computing the ratio of the standard deviation of its first derivative to the standard deviation of the signal itself.

Definition

The Hjorth Mobility M is defined as:

$$M = \frac{\sqrt{\mathrm{Var}(x')}}{\sqrt{\mathrm{Var}(x)}}$$

Where:

- x is the original signal.
- x' is the first derivative of the signal.
- $\mathrm{Var}(x)$ is the variance (or power) of the original signal.
- $\mathrm{Var}(x')$ is the variance of the first derivative of the signal.

Interpretation

The Hjorth Mobility reflects how rapidly the signal changes over time:

- **$M = 1$**: The signal is smooth and exhibits **constant motion**, like a sine wave with regular oscillations.
- **$M > 1$**: The signal is **more oscillatory** or has **higher frequency components**, indicating rapid changes over time.
- **$M < 1$**: The signal exhibits **slower oscillations** or is **stationary**, with fewer fluctuations.

Example

Consider two signals: a **sine wave** and **white noise**.

- **Sine Wave**: For a 1 Hz sine wave, the Hjorth Mobility will be close to 1, as the signal oscillates smoothly at a constant frequency.
- **White Noise**: For white noise, which has random fluctuations, the Hjorth Mobility will be greater than 1, reflecting rapid, irregular changes.

For a **sine wave** sampled at 100 Hz with the equation $x(t) = \sin(2\pi t)$, its first derivative will be $x'(t) = \cos(2\pi t)$. Both the variance of the signal and the variance of the first derivative are constant, and the Hjorth Mobility will be approximately 1.

In contrast, for a **white noise** signal, the first derivative will fluctuate randomly, and the Hjorth Mobility will be greater than 1, reflecting the higher rate of change due to the random variations in the signal.

Applications

Hjorth Mobility is primarily used in the analysis of **EEG** signals to characterize brain activity, but it can also be applied in various other domains where signal dynamics are of interest:

- **EEG Signal Analysis**: Hjorth Mobility helps in distinguishing between different brain states, such as alertness, sleep, and seizure activity.
- **Biomedical Signal Processing**: It can be applied to ECG and EMG signals to assess their dynamics and detect anomalies.
- **Time Series Complexity Analysis**: Hjorth Mobility is useful for quantifying the complexity of any time series, particularly those with rapid or oscillatory changes.

Example Calculation in Python

Here's a simple example of calculating Hjorth Mobility for both a sine wave and a white noise signal in Python:

In this example, the Hjorth Mobility for a sine wave will be close to 1, indicating smooth, periodic oscillations, while the white noise will yield a higher Hjorth Mobility due to its irregular and random changes.

References

[1] Bandt, C., and Pompe, B. (2002). Permutation entropy: a natural complexity measure for time series. *Physical Review Letters*, 88(17), 174102.

[2] Lempel, A., and Ziv, J. (1976). On the complexity of finite sequences. *IEEE Transactions on Information Theory*, 22(1), 75-81.