# Simple Harmonic Motion (SHM) — Algebra-Based Notes

Mass–Spring Oscillator: Position, Velocity, Acceleration

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## 1. What SHM Looks Like (Big Picture)

Simple harmonic motion is periodic motion where the restoring force is proportional to the displacement from equilibrium:

$$F = -ky.$$

For a mass–spring system, this produces a sinusoidal motion in time.

### Key time ideas

- **Period** $T$: time for one full cycle.
- **Frequency** $f = \frac{1}{T}$.
- **Angular frequency** $\omega = 2\pi f = \frac{2\pi}{T}$.

## 2. Position Function

If the mass is at its **maximum height/displacement at** $t = 0$, the cleanest choice is a cosine:

$$y(t) = y_{\max}\cos(\omega t) = y_{\max}\cos\left(\frac{2\pi}{T}\,t\right).$$

### Why cosine here?

Because $\cos(0) = 1$, so at $t = 0$:

$$y(0) = y_{\max}.$$

That matches "starting at maximum."

## 3. Velocity Function

Velocity is the time derivative of position:

$$v(t) = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Differentiate $y(t) = y_{\max}\cos(\omega t)$:

$$v(t) = -y_{\max}\omega\sin(\omega t) = -y_{\max}\left(\frac{2\pi}{T}\right)\sin\left(\frac{2\pi}{T}\,t\right).$$

### Maximum speed

The sine function ranges from $-1$ to $+1$, so the velocity ranges from $-y_{\max}\omega$ to $+y_{\max}\omega$:

$$v_{\max} = y_{\max}\omega = y_{\max}\left(\frac{2\pi}{T}\right).$$

So you can also write:

$$v(t) = -v_{\max}\sin\!\left(\frac{2\pi}{T}\,t\right).$$

## 4. Acceleration Function

Acceleration is the derivative of velocity (or the second derivative of position):

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}.$$

Differentiate $v(t) = -y_{\max}\omega\sin(\omega t)$:

$$a(t) = -y_{\max}\omega^2\cos(\omega t) = -y_{\max}\left(\frac{2\pi}{T}\right)^2\cos\!\left(\frac{2\pi}{T}\,t\right).$$

### Maximum acceleration

$$a_{\max} = y_{\max}\omega^2 = y_{\max}\left(\frac{2\pi}{T}\right)^2.$$

### Important relationship

Because $y(t) = y_{\max}\cos(\omega t)$ and $a(t) = -y_{\max}\omega^2\cos(\omega t)$:

$$\boxed{a(t) = -\omega^2\,y(t)}$$

Acceleration always points toward equilibrium (opposite the displacement).

## 5. Phase Relationships (Who Leads/Lags?)

- $y(t)$ is a cosine wave.
- $v(t)$ is a negative sine wave: shifted by $\frac{T}{4}$ relative to $y(t)$.
- $a(t)$ is a negative cosine wave: exactly opposite $y(t)$.

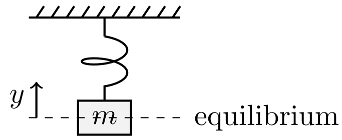
### Checkpoint times (starting at $y = y_{\max}$ at $t = 0$)

Let $\theta = \omega t = \dfrac{2\pi}{T}t$.

| Time | $\theta$ | $y$ | $v$ | $a$ |
|---|---|---|---|---|
| 0 | 0 | $+y_{\max}$ | 0 | $-a_{\max}$ |
| $T/4$ | $\pi/2$ | 0 | $-v_{\max}$ | 0 |
| $T/2$ | $\pi$ | $-y_{\max}$ | 0 | $+a_{\max}$ |
| $3T/4$ | $3\pi/2$ | 0 | $+v_{\max}$ | 0 |
| $T$ | $2\pi$ | $+y_{\max}$ | 0 | $-a_{\max}$ |

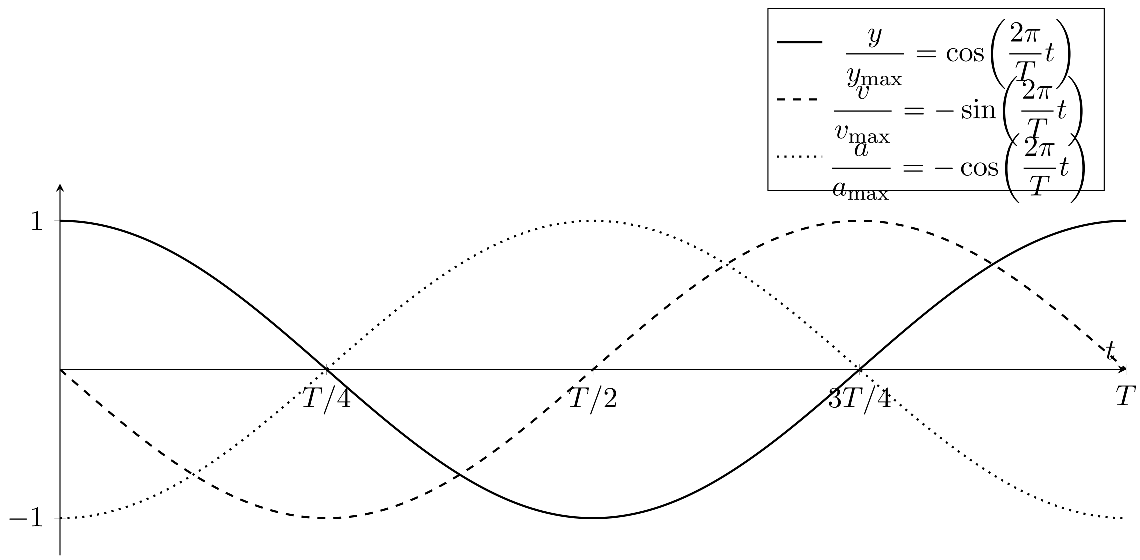
## 6. Quick Sketches (Spring + Graphs)

### Spring–mass diagram (simple)




### Position, velocity, acceleration vs. time (one period)

Below we plot one full period using $\theta = \dfrac{2\pi}{T}t$ on the horizontal axis.




## 7. Example

**Problem.** Write the position equation of a mass attached to a spring with period $T = 2\,\mathrm{s}$ and maximum height (amplitude) $y_{\max} = 2\,\mathrm{cm} = 0.02\,\mathrm{m}$. Assume the mass is at maximum height at $t = 0$. Draw the position diagram. Also write $v(t)$ and $a(t)$.

### Step 1: Find $\omega$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\,\mathrm{s}} = \pi \text{ rad/s}.$$

### Step 2: Position

Starting at maximum means cosine:

$$\boxed{y(t) = 0.02\cos(\pi t)\ \mathrm{m}}$$

**Step 3: Velocity**

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(0.02\cos(\pi t)\big) = -0.02\pi\sin(\pi t).$$

$$\boxed{v(t) = -0.02\pi\ \sin(\pi t)\ \mathrm{m/s}}$$

Maximum speed:

$$v_{\max} = y_{\max}\omega = 0.02(\pi) = 0.02\pi\ \mathrm{m/s}.$$

**Step 4: Acceleration**

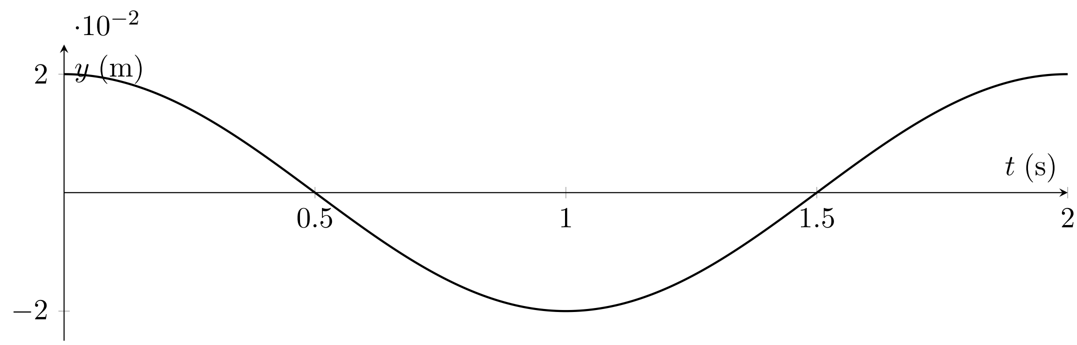
$$a(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(-0.02\pi\sin(\pi t)\big) = -0.02\pi^2\cos(\pi t).$$

$$\boxed{a(t) = -0.02\pi^2\ \cos(\pi t)\ \mathrm{m/s^2}}$$

Maximum acceleration:

$$a_{\max} = y_{\max}\omega^2 = 0.02(\pi^2) = 0.02\pi^2\ \mathrm{m/s^2}.$$

**Position sketch for the example (0 to 2 s)**




## 8. Trig Range Reminder

Because

$$-1 \le \cos\theta \le 1 \qquad \text{and} \qquad -1 \le \sin\theta \le 1,$$

the maximum values come directly from the coefficients:

$$|y| \le y_{\max}, \quad |v| \le v_{\max} = y_{\max}\omega, \quad |a| \le a_{\max} = y_{\max}\omega^2.$$

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**Core SHM Summary (starting at maximum):**

$y = y_{\max}\cos(\omega t),\ v = -y_{\max}\omega\sin(\omega t),\ a = -y_{\max}\omega^2\cos(\omega t) = -\omega^2 y$