Continuity, Bernoulli's Equation, and Hydrostatics

1 Hydrostatic Pressure and the U-Tube

For a fluid at rest, pressure varies with depth according to

$$P = P_0 + \rho g h, \tag{1}$$

where P_0 is the surface pressure and h is the depth below the surface.

U-Tube with Two Fluids

At equilibrium, pressures at the same horizontal level must be equal:

$$P_0 + \rho_{\text{oil}} g h_{\text{oil}} = P_0 + \rho_{\text{water}} g h_{\text{water}}. \tag{2}$$

Canceling P_0,

$$\boxed{\rho_{\text{oil}} h_{\text{oil}} = \rho_{\text{water}} h_{\text{water}}} \tag{3}$$

Key Idea: In static fluids, pressure depends only on depth and density.

Example: U-Shaped Tube with Oil and Water

A U-shaped tube is open to the atmosphere on both sides. One side contains oil ($\rho_{\text{oil}} = 0.5\,\text{g/cm}^3$), and the other contains water ($\rho_{\text{water}} = 1.0\,\text{g/cm}^3$).

If the oil column has height $h_{\text{oil}} = 20\,\text{cm}$, find the height h_{water} of the water column.

Solution At the same horizontal level, pressures are equal:

$$\rho_{\text{oil}} g h_{\text{oil}} = \rho_{\text{water}} g h_{\text{water}}.$$

Thus,

$$h_{\text{water}} = \frac{\rho_{\text{oil}}}{\rho_{\text{water}}} h_{\text{oil}} = \frac{0.5}{1.0}(20\,\text{cm}) = 10\,\text{cm}.$$

2 Continuity Equation

Mass Conservation

For steady flow, mass is conserved:

$$\boxed{\frac{\Delta m_{\text{in}}}{\Delta t} = \frac{\Delta m_{\text{out}}}{\Delta t}} \tag{4}$$

Using $\Delta m = \rho \Delta V$,

$$\rho \frac{\Delta V_{\text{in}}}{\Delta t} = \rho \frac{\Delta V_{\text{out}}}{\Delta t}. \tag{5}$$

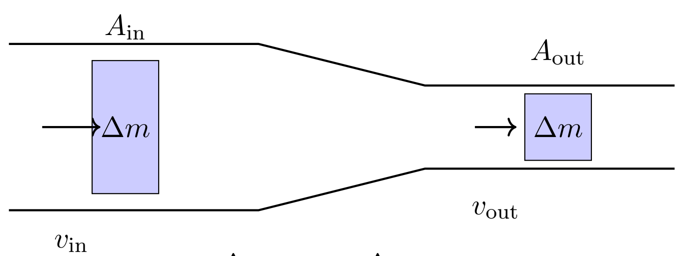
Since $\Delta V = A \Delta x$ and $v = \Delta x / \Delta t$,

$$\rho A_{\text{in}} v_{\text{in}} = \rho A_{\text{out}} v_{\text{out}}. \tag{6}$$

For an incompressible fluid ($\rho =$ constant),

$$\boxed{A_{\text{in}} v_{\text{in}} = A_{\text{out}} v_{\text{out}}} \tag{7}$$

Consequence: If a pipe narrows, the fluid speed increases.



$$\frac{\Delta m_{\text{in}}}{\Delta t} = \frac{\Delta m_{\text{out}}}{\Delta t}$$

Figure 1: Mass conservation for steady incompressible flow. The same mass of fluid enters and exits the pipe in equal time intervals, even though the speed changes.

Example: Filling a Bucket

A hose fills a 10 L bucket in 20 s. The hose opening has cross-sectional area $A = 10\,\text{cm}^2$.

Find the speed of the water exiting the hose.

Solution Convert volume:

$$10\,\text{L} = 10 \times 10^{-3}\,\text{m}^3.$$

Volume flow rate:

$$Q = \frac{\Delta V}{\Delta t} = \frac{10 \times 10^{-3}}{20} = 5 \times 10^{-4}\,\text{m}^3/\text{s}.$$

Using $Q = Av$:

$$v = \frac{Q}{A} = \frac{5 \times 10^{-4}}{10 \times 10^{-4}} = 0.5\,\mathrm{m/s}.$$

3 Bernoulli's Equation

Work done by pressure forces on a fluid element:

$$W = P_1 \Delta V - P_2 \Delta V. \tag{8}$$

Change in kinetic energy:

$$\Delta K = \frac{1}{2}\rho \Delta V (v_2^2 - v_1^2). \tag{9}$$

Change in gravitational potential energy:

$$\Delta U = \rho \Delta V g (y_2 - y_1). \tag{10}$$

Using $W = \Delta K + \Delta U$ and dividing by ΔV:

$$\boxed{P + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}} \tag{11}$$

Assumptions

- Steady flow
- Incompressible fluid
- Negligible viscosity
- Flow along a streamline

Example: Bernoulli Without Gravity

Water flows horizontally through a pipe that narrows from $A_1 = 4\,\mathrm{cm}^2$ to $A_2 = 1\,\mathrm{cm}^2$. The speed at the wide section is $v_1 = 1\,\mathrm{m/s}$.

Find the pressure difference $P_1 - P_2$.

Solution From continuity:

$$v_2 = \frac{A_1}{A_2} v_1 = 4\,\mathrm{m/s}.$$

Bernoulli (same height):

$$P_1 - P_2 = \frac{1}{2}\rho (v_2^2 - v_1^2).$$

Using $\rho = 1000\,\mathrm{kg/m^3}$:

$$P_1 - P_2 = \frac{1}{2}(1000)(16 - 1) = 7500\,\mathrm{Pa}.$$

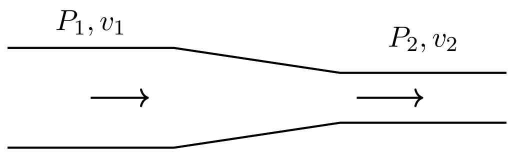


Figure 2: Horizontal flow: pressure decreases as speed increases.

Example: Bernoulli With Gravity

Water flows upward in a pipe from point 1 to point 2. The speed is the same at both points. Point 2 is 5 m higher than point 1.

Find the pressure difference $P_1 - P_2$.

Solution Since $v_1 = v_2$, kinetic terms cancel:

$$P_1 - P_2 = \rho g(y_2 - y_1).$$

Using $\rho = 1000\,\mathrm{kg/m^3}$:

$$P_1 - P_2 = (1000)(9.8)(5) = 4.9 \times 10^4\,\mathrm{Pa}.$$

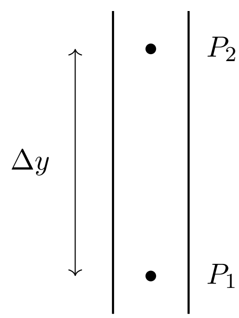


Figure 3: Pressure decreases with height in steady flow.

Example: Bernoulli With Gravity *and* Area Change

Water flows steadily upward through a pipe that both **narrows** and **rises**. At point 1 (lower, wide section): $A_1 = 4\,\mathrm{cm}^2$, $v_1 = 2.0\,\mathrm{m/s}$, $y_1 = 0$. At point 2 (higher, narrow section): $A_2 = 1\,\mathrm{cm}^2$, $y_2 = 3.0\,\mathrm{m}$.

Assume incompressible, nonviscous flow. Find $P_1 - P_2$.

Solution **Step 1: Continuity to get v_2.**

$$A_1 v_1 = A_2 v_2 \quad \Rightarrow \quad v_2 = \frac{A_1}{A_2} v_1 = 4(2.0) = 8.0\,\mathrm{m/s}.$$

Step 2: Bernoulli between 1 and 2.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

so

$$P_1 - P_2 = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) + \rho g(y_2 - y_1).$$

With $\rho = 1000\,\mathrm{kg/m^3}$ and $g = 9.8\,\mathrm{m/s^2}$:

$$P_1 - P_2 = \frac{1}{2}(1000)\left(8.0^2 - 2.0^2\right) + (1000)(9.8)(3.0)$$

$$= 500(64-4) + 29400 = 500(60) + 29400 = 30000 + 29400 = 5.94 \times 10^4\,\mathrm{Pa}.$$

$$\boxed{P_1 - P_2 \approx 5.9 \times 10^4\ \mathrm{Pa}}$$

Flow rises and narrows: speed increases (continuity) and pressure drops further due to both higher speed and higher elevation (Bernoulli).