# Tutorial 7

## COMP 5361: Discrete Structures and Formal Languages

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# Outline

# Contents of the section

### Definition

- **Binary Relation**: Let $A$ and $B$ be sets. A binary relation from $A$ to $B$ is a subset of $A \times B$.

### Definition

- **Binary Relation**: Let $A$ and $B$ be sets. A binary relation from $A$ to $B$ is a subset of $A \times B$.
- **Relation on a Set**: A relation on a set $A$ is a relation from $A$ to $A$.

## Example

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# Properties of Relations

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- **Symmetric**: A relation $R$ on a set $A$ is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
- **Antisymmetric**: A relation $R$ on a set $A$ such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.
- **Transitive**: A relation $R$ on a set $A$ is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

## Example

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2. Which of the relations on slide 5 are symmetric?

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4. Which of the relations on slide 5 are transitive?

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## Definition

- Let $R$ be a relation from a set $A$ to a set $B$ and $S$ a relation from $B$ to a set $C$. The composite of $R$ and $S$ is the relation consisting of ordered pairs $(a, c)$, where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.
- Let $R$ be a relation on the set $A$. The powers $R^n$ , $n = 1, 2, 3, \ldots,$ are defined recursively by:

$$R^1 = R \tag{1}$$
$$R^{n+1} = R^n \circ R \tag{2}$$

# Combining Relations

**Theorem**

The relation $R$ on a set $A$ is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$ .

**Example**

1. Let $R = (1,1), (2,1), (3,2), (4,3)$. Find the powers $R^n$ , $n = 2, 3, 4, \dots$.

**Theorem**

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**Example**

1. Let $R = (1,1), (2,1), (3,2), (4,3)$. Find the powers $R^n$ , $n = 2, 3, 4, \ldots$.
2. Let $R$ be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers $n$.

# Contents of the section

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# Closures of Relations

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- **Reflexive Closure**: The reflexive closure of $R$ is the smallest relation that contains $R$ and is reflexive.
- **Symmetric Closure**: The symmetric closure of $R$ is the smallest relation that contains $R$ and is symmetric.
- **Transitive Closure**: The transitive closure of $R$ is the smallest relation that contains $R$ and is transitive.

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## Example

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1. What is the reflexive closure of the relation $R = \{(a, b) | a < b\}$ on the set of integers?
2. What is the symmetric closure of the relation $R = \{(a, b) | a > b\}$ on the set of positive integers?
3. Find the transitive closures of the relation $R$ on a, b, c, d, e, where $R$ is given by:

$$\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$$

## Definition

- Let $R$ be a relation on a set $A$. The connectivity relation $R^*$ is defined as:

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- Let $R$ be a relation on a set $A$. The connectivity relation $R^*$ is defined as:
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- The transitive closure of a relation $R$ equals the connectivity relation $R^*$.

## Example 1

Let $R$ be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), (5, 2)$, and $(5, 4)$. Find:

1. $R^2$
2. $R^3$
3. $R^4$
4. $R^5$
5. $R^6$
6. $R^*$

# Example

## Example 2

Find the transitive closures of these relations on $\{1, 2, 3, 4\}$.

1. $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$
2. $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
3. $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$

# Equivalence Relations

## Definition

- **Equivalence Relation**: A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive.
- **Equivalent**: Two elements $a$ and $b$ that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that $a$ and $b$ are equivalent elements with respect to a particular equivalence relation.

## Example

1. Let $R$ be the relation on the set of integers such that $aRb$ if and only if $a = b$ or $a = -b$. Is $R$ an equivalence relation?

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2. Let $R$ be the relation on the set of real numbers such that $aRb$ if and only if $a - b$ is an integer. Is R an equivalence relation?
3. Let m be an integer with $m > 1$. Show that the relation (Congruence Modulo $m$):

$$R = \{(a, b)|a \equiv b(modm)\}$$

is an equivalence relation on the set of integers.

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3. Let m be an integer with $m > 1$. Show that the relation (Congruence Modulo $m$):
$$R = \{(a, b)|a \equiv b(mod m)\}$$
is an equivalence relation on the set of integers.
4. Show that the "divides" relation is the set of positive integers in not an equivalence relation.

## Definition

- Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element a of $A$ is called the equivalence class of $a$. The equivalence class of $a$ with respect to $R$ is denoted by $[a]_R$ . When only one relation is under consideration, we can delete the subscript $R$ and write $[a]$ for this equivalence class.
- In the above definition $a$ is the representative of this equivalence class.

## Properties

If $a \sim b$ then the followings are equivalent:

1. $aRb$
2. $[a] = [b]$

**Example**

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2. What are the equivalence classes of 0 and 1 for congruence modulo 7?

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1. Let $R$ be the relation on the set of integers such that $aRb$ if and only if $a = b$ or $a = -b$. What is the equivalence class of an integer?
2. What are the equivalence classes of 0 and 1 for congruence modulo 7?
3. What are the equivalence classes of 5 and 2 for congruence modulo 5?

## Definition

- **Partition**: A partition of a set $S$ is a collection of disjoint nonempty subsets of $S$ that have $S$ as their union.

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- **Partition**: A partition of a set $S$ is a collection of disjoint nonempty subsets of $S$ that have $S$ as their union.
- Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i | i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i$ , $i \in I$ , as its equivalence classes.

# Example

## Example 1

Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. Give a partition of set $S$ with 3 classes.

# Example

## Example 2

List the ordered pairs in the equivalence relation $R$ produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$.

**Example 3**

What are the sets in the partition of the integers arising from congruence modulo 7?

# Example

## Example 4

What is the congruence class $[4]_m$ when $m$ is:

1. 2?
2. 3?
3. 6?
4. 8?

## Definition

- A relation $R$ on a set $S$ is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set $S$ together with a partial ordering $R$ is called a partially ordered set, or poset, and is denoted by $(S, R)$. Members of $S$ are called elements of the poset.
- The elements $a$ and b of a poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$. When $a$ and $b$ are elements of $S$ such that neither $a \preceq b$ nor $b \preceq a$, $a$ and $b$ are called incomparable.

# Partial Orderings

## Example

1. Show that the greater than or equal relation is a partial ordering on the set of integers.

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2. Show that the inclusion relation is a partial ordering on the power set of a set $S$.
3. In the poset $(Z^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

## Definition

- **Totally Ordered**: If $(S, \preceq)$ is a poset and every two elements of $S$ are comparable, $S$ is called a totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order. A totally ordered set is also called a chain.
- **Well-ordered Set**: $(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering and every nonempty subset of $S$ has a least element.
- **Lexicographic Ordering**: The lexicographic ordering $\preceq$ on $A_1 \times A_2$ is defined by, $(a_1, a_2) \prec (b_1, b_2)$ when, either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

## Example

1. Determine whether the following posets are totally ordered or not:
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1. Determine whether the following posets are totally ordered or not:
   1. $(Z, \leq)$
   2. $(Z^+, |)$
2. Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbb{Z} \times \mathbb{Z}, \preceq)$, where $\preceq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbb{Z}$.

Definition

- **Covering Relation**: Let $(S, \preceq)$ be a poset. We say that an element $y \in S$ covers an element $x \in S$ *if*$x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs $(x, y)$ such that $y$ covers $x$ is called the covering relation of $(S, \preceq)$.

**Example**

1. Draw the Hasse diagram representing the partial ordering $\{(a, b) | a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.