Tutorial 4

COMP 5361: Discrete Structures and Formal Languages

Mohammad Reza Davari

Concordia University

Outline

1. Introduction to Proofs

2. Proof Methods and Strategy

Contents of the section

1 Introduction to Proofs

2 Proof Methods and Strategy

Definition

- **Theorem**: A theorem is a statement that can be shown to be true and is reserved for a statement that is considered at least somewhat important.
- **Propositions**: Propositions are theorems with less importance.
- **Proof**: Demonstrates whether a statement (theorem) is true.
- **Axioms**: Axioms or postulates are statements we assume to be true.
- **Lemma**: Lemma is a less important theorem that is helpful in the proof of other results.
- **Corollary** A corollary is a theorem that can be established directly from a theorem that has been proved.
- **Conjecture** A conjecture is a statement that is being proposed to be a true statement.

Direct Proofs

Strategy

In a direct proof of a conditional statement $p \rightarrow q$:

1. Assume that p is true.
2. Take advantage of rules of inference.
3. Show that q must also be true.

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Example

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Proof by Contraposition

Strategy

Instead of proving $p \rightarrow q$, prove its contrapositive, $\neg q \rightarrow \neg p$ (since they are equivalent).

1. Assume $\neg q$ is true.
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Example

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Vacuous Proofs

Caution

While proving $p \to q$ if you realize p is false then you can conclude the statement.

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Example

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Trivial Proofs

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Example

Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$" where the domain consists of all non-negative integers. Show that $P(0)$ is true.

Proofs by Contradiction

Strategy

Suppose we want to prove that a statement p is true.

1. Find a contradiction q such that $\neg p \to q$ is true.
2. Since q is false and $\neg p \to q$ is true, then $\neg p$ is false.
3. Conclude p is true.

Contradiction q is usually of the form $r \wedge \neg r$ for some proposition r.

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3. Prove if $3n + 2$ is odd, then n is odd.

Proofs of Equivalence

Strategy

In order to prove a statement of the form $p \leftrightarrow q$ we need to:

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Example

Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Counter Example

Strategy

In order to prove a statement of the form $\forall x P(x)$ is false, you can find a c for which $P(c)$ is false.

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Example

Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Watch Out for Mistakes

One important fallacy

Many incorrect arguments are based on a fallacy called **begging the question**. This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. In other words, this fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called **circular reasoning**.

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Exhaustive Proof and Proof by Cases

Idea

It is based on the following tautology:

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

It let us proceed with the proof with extra assumptions!

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Example

1. Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.
2. Prove that if n is an integer, then $n^2 \geq n$

WLOG

We assert that by proving one case of a theorem, no additional argument is required to prove other specified cases. That is, other cases follow by making straightforward changes to the argument, or by filling in some straightforward initial step.

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Danger

Although this will heavily simply the proofs involving proof by cases, an incorrect use of it will lead to an invalid proof. Be very careful with this statement!

Existence Proofs

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Strategies

- **Constructive proof**: The proof can be given by finding an element (also called and example or a witness) for which the statement is true.
- **Non-constructive proof**: You do not look for an example but rather approach the problem using other methods discussed earlier. One common method of giving non-constrcutive proof is proof by contradiction.

1. A Constructive Existence Proof:
   - Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Examples

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   - $1729 = 10^3 + 9^3 = 12^3 + 1^3$

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1. A Constructive Existence Proof:
   - Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.
   - $1729 = 10^3 + 9^3 = 12^3 + 1^3$
2. A Nonconstructive Existence Proof:
   - Show that there exist irrational numbers x and y such that x^y is rational.
   - *Hint:* $\sqrt{2}$