Tutorial 1

COMP 5361: Discrete Structures and Formal Languages

Concordia University

Outline

1. Propositional Equivalences

2. Predicates and Quantifiers

Contents of the section

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Definition

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- A compound proposition that is always false is called a **contradiction**.
- A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.
- Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

Examples

1. Show that $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$ is a tautology.
2. Show that $p \wedge \neg p$ is a contradiction.
3. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent by 2 different methods.
4. Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent using the logical equivalence tables.

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- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.
- If compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**. Unsatisfiable iff the negation is a tautology.

Examples

1. Show that $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is satisfiable, i.e. find a solution.
2. Show that $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is unsatisfiable.

Contents of the section

Predicates

Definition

Statements involving **variables** that their truth assignment depend on the value of the variable are **predicate**, they are a property of the variable. A function is dependent on a variable that verifies the value of the predicate is called **propositional function**. If the variables in a propositional function are assigned values the resulting statement becomes a **proposition**. becomes a proposition

Example

Let $Q(x, y)$ denote the statement $x = y + 3$. Identify:

1. The variables.
2. The predicate.
3. The propositional function.

What are the truth values of the $Q(1, 2)$ and $Q(3, 0)$?

Quantifiers

Definition

Quantification It expresses the extent to which a predicate is true over a range of elements.

- **Universal Quantifier**[a] $\forall$: Indicates that a predicate is true for **all** element **domain of discourse**[b].
- **Existential Quantifier** $\exists$: Indicates that there existsat **at least one** element in **domain of discourse** for which the predicate is true.
- **Uniqueness Quantifier** $\exists!$: Indicates that there exists **exactly one** element in **domain of discourse** for which the predicate is true.

[a] Quantifiers have the highest order of operations

[b] Also called universe of discourse or domain

Universal Quantifier

Key phrases that translate to $\forall$:

- for all
- for every
- all of
- for each
- given any
- for arbitrary
- for any

Existential Quantifier

Key phrases that translate to $\exists$:

- there exists
- for some
- for at least one
- there is

Uniqueness Quantifier

Key phrases that translate to $\exists!$:

- there exists exactly one
- for exactly one
- there is one and only one
- there is exactly one

Watch out...

Empty domains of discourse and universal quantifier

If the domain is empty, then $\forall x P(x)$ is **true** for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.

Empty domains of discourse and existential quantifier

If the domain is empty, then $\exists x Q(x)$ is **false** whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element x in the domain for which $Q(x)$ is true. In this case, by similar logic, $\exists! x Q(x)$ is **false** whenever $Q(x)$ is a propositional function.

Quantifiers with Restricted Domains

Definition

- Domains of quantifiers can be restricted by indicating a condition a variable must satisfy right after the quantifier.
- The restrictions can be replaced by logical statements.
  - In the case of universal quantifier it will be a conditional statement.
  - In the case of existential quantifier it will be conjunction.

Examples

What do the following statements mean, where the domain in each case consists of the real numbers? Rewrite each case so they do not include any restrictions.

1. $\forall x, x < 0(x^2 > 0)$
2. $\forall y, y = 0(y^3 = 0)$
3. $\exists z, z > 0(z^2 = 2)$

- **Bound Variable:** When a quantifier is used on the variable, we say that this occurrence of the variable is bound.
- **Free Variable:** An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.
- **Scope:** The part of a logical expression to which a quantifier is applied is called the scope of this quantifier

Example

Identify the bound variable, free variable, and the scope of the bound variable in the following examples:

1. $\exists x(x + y = 1)$
2. $\exists x(x > 0 \land x \equiv 0(\mathrm{mod} 2)) \lor \forall x(x < 0)$
3. $\forall y(y = 0 \rightarrow y^3 = 0) \land \exists z(z > 0 \land z^2 = 2)$

Negating Quantified Expressions

Rules

- $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- $\neg \exists x Q(x) \equiv \forall x \neg Q(x)$

Examples

Negate the following statements:

1. There is an honest politician.
2. All Canadians love hockey.
3. $\forall x(x2 > x)$
4. $\exists x(x^2 = 2)$

Example

Translate the following sentence into Logical expressions:
Every student in this class has studied calculus.

Translating from English into Logical Expressions

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Step.1: For every student in this class, that student has studied calculus.

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Every student in this class has studied calculus.

Step.1: For every student in this class, that student has studied calculus.

Step.2: For every student x in this class, x has studied calculus.

Example Continues

Caution

At some point you need to decide about the domain of discourse!

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- Domain: All students in the classroom.

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- Domain: All people.

Example Continues

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- Domain: All students in the classroom. We are done!
- Domain: All people. For every person x, if person x is a student in this class then x has studied calculus.

Example Continues

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At some point you need to decide about the domain of discourse!

- Domain: All students in the classroom. We are done!
- Domain: All people. For every person x, if person x is a student in this class then x has studied calculus.

Lets generalize with a two variable quantifier!

Example

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Translate the following sentence into Logical expressions:
"Every student in this class has visited either Canada or Mexico

Step.1: For every x in this class, x has the property that x has visited Mexico or x has visited Canada.

Translating from English into Logical Expressions

Example

Translate the following sentence into Logical expressions:
"Every student in this class has visited either Canada or Mexico

Step.1: For every x in this class, x has the property that x has visited Mexico or x has visited Canada.

Step.2: Decide on the domain.