# Tutorial 6

## COMP 5361: Discrete Structures and Formal Languages

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# Outline

# Contents of the section

## Defenition

Let $A$ and $B$ be nonempty sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$. If $f$ is a function from $A$ to $B$, we write $f : A \to B$. In this definition we have:

- $A$ is the domain of $f$.
- $B$ is codomain of $f$.
- For $f(a) = b$, we say that $b$ is the image of $a$ and $a$ is a preimage of $b$.
- **Range/Image**: The range, or image, of $f$ is the set of all images of elements of $A$.

## Defenition

- **One-to-One**: A function $f$ is said to be one-to-one, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$. A function is said to be injective if it is one-to-one.
- **Onto**: A function $f$ from $A$ to $B$ is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function $f$ is called surjective if it is onto.
- **Bijection**: A function $f$ is a bijection if it is both one-to-one and onto. We also say that such a function is bijective.

## Strategies

Let $f$ be $f : A \to B$,

- **Showing $f$ is injective**: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
- **Showing $f$ is not injective**: Find some elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- **Showing $f$ is surjective**: Take an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- **Showing $f$ is not surjective**: Find some element element $y \in B$ such that there does not exist any $x \in A$ with $f(x) = y$.

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## Defenition

Let $f$ be a function whose domain and codomain are subsets of the set of real numbers then:

- **Increasing**: $f$ is increasing if $f(x) \leq f(y)$ when $x < y$ and $x, y \in$ Domain.
- **Strictly Increasing**: $f$ is increasing if $f(x) < f(y)$ when $x < y$ and $x, y \in$ Domain.
- **Decreasing**: $f$ is increasing if $f(x) \geq f(y)$ when $x < y$ and $x, y \in$ Domain.
- **Strictly Decreasing**: $f$ is increasing if $f(x) > f(y)$ when $x < y$ and $x, y \in$ Domain.

## Defenition

Let $f$ be a bijection from set $A$ to the set $B$. The inverse function of $f$ is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of $f$ is denoted by $f^{-1}$.

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## Example

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

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### Example

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2. Let $f$ be the function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x) = x^2$. Is f invertible?

# Inverse Functions

## Defenition

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## Example

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?
2. Let $f$ be the function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x) = x^2$. Is f invertible?
3. In the example above what happens if we restrict the domain and codomain to nonnegative real numbers?

## Defenition

Let $g$ be a function from the set $A$ to the set $B$ and let $f$ be a function from the set $B$ to the set $C$. The composition of the functions $f$ and $g$, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$

# Function Composition

## Defenition

Let $g$ be a function from the set $A$ to the set $B$ and let $f$ be a function from the set $B$ to the set $C$. The composition of the functions $f$ and $g$, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$

## Example

$f$ and $g$ be the functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of $f$ and $g$? What is the composition of $g$ and $f$ ?

# Partial vs. Total Function

## Defenition

A **partial function** $f$ from a set $A$ to a set $B$ is an assignment to each element $a \in A$, called the domain of definition of $f$ , of a unique element $b \in B$. The sets $A$ and $B$ are called the domain and codomain of $f$ , respectively. We say that $f$ is undefined for elements in $A$ that are not in the domain of definition of $f$ . When the domain of definition of $f$ equals $A$, we say that $f$ is a total function.

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# Examples

## Example 1

Determine whether $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$ if:

1. $f(x) = \frac{1}{x}$
2. $f(x) = \sqrt{x}$
3. $f(x) = \pm\sqrt{(x^2 + 1)}$

## Example 2

Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$ if:

1. $f(x) = -3x + 4$
2. $f(x) = -3x^2 + 7$
3. $f(x) = \frac{x+1}{x+2}$
4. $f(x) = \sqrt[5]{x} + 1$

## Example 3

Prove the followings:

1. Let $f : R \to R$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = \frac{1}{f(x)}$ is strictly decreasing.
2. Prove that a strictly increasing function from R to itself is one-to-one.