Tutorial 6

COMP 5361: Discrete Structures and Formal Languages

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Outline

Defenition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$. In this definition we have:

- A is the domain of f.
- B is codomain of f.
- For $f(a) = b$, we say that b is the image of a and a is a preimage of b.
- **Range/Image**: The range, or image, of f is the set of all images of elements of A.

Defenition

- **One-to-One**: A function f is said to be one-to-one, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be injective if it is one-to-one.
- **Onto**: A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.
- **Bijection**: A function f is a bijection if it is both one-to-one and onto. We also say that such a function is bijective.

Strategies

Let f be $f : A \to B$,

- **Showing f is injective**: Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
- **Showing f is not injective**: Find some elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- **Showing f is surjective**: Take an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- **Strictly Decreasing**: f is increasing if $f(x) > f(y)$ when $x < y$ and $x, y \in$ Domain.

Defenition

Let f be a function whose domain and codomain are subsets of the set of real numbers then:

- **Increasing**: f is increasing if $f(x) \leq f(y)$ when $x < y$ and $x, y \in$ Domain.
- **Strictly Increasing**: f is increasing if $f(x) < f(y)$ when $x < y$ and $x, y \in$ Domain.
- **Decreasing**: f is increasing if $f(x) \geq f(y)$ when $x < y$ and $x, y \in$ Domain.
- **Strictly Decreasing**: f is increasing if $f(x) > f(y)$ when $x < y$ and $x, y \in$ Domain.

Inverse Functions

Defenition

Let f be a bijection from set A to the set B. The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of f is denoted by f^{-1}.

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2. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x) = x^2$. Is f invertible?
3. In the example above what happens if we restrict the domain and codomain to nonnegative real numbers?

Function Composition

Defenition

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$

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Example

f and g be the functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Partial vs. Total Function

Defenition

A **partial function** f from a set A to a set B is an assignment to each element $a \in A$, called the domain of definition of f , of a unique element $b \in B$. The sets A and B are called the domain and codomain of f , respectively. We say that f is undefined for elements in A that are not in the domain of definition of f . When the domain of definition of f equals A, we say that f is a total function.

Contents of the section

Example 1

Determine whether f is a function from $\mathbb{R}$ to $\mathbb{R}$ if:

1. $f(x) = \frac{1}{x}$
2. $f(x) = \sqrt{x}$
3. $f(x) = \pm\sqrt{(x^2+1)}$

Example 2

Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$ if:

1. $f(x) = -3x + 4$
2. $f(x) = -3x^2 + 7$
3. $f(x) = \frac{x+1}{x+2}$
4. $f(x) = \sqrt[5]{x} + 1$

Example 3

Prove the followings:

1. Let $f : R \to R$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = \frac{1}{f(x)}$ is strictly decreasing.
2. Prove that a strictly increasing function from R to itself is one-to-one.