Tutorial 4

COMP 5361: Discrete Structures and Formal Languages

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Outline

Contents of the section

- **Theorem**: A theorem is a statement that can be shown to be true and is reserved for a statement that is considered at least somewhat important.
- **Propositions**: Propositions are theorems with less importance.
- **Proof**: Demonstrates whether a statement (theorem) is true.
- **Axioms**: Axioms or postulates are statements we assume to be true.
- **Lemma**: Lemma is a less important theorem that is helpful in the proof of other results.
- **Corollary** A corollary is a theorem that can be established directly from a theorem that has been proved.
- **Conjecture** A conjecture is a statement that is being proposed to be a true statement.

Direct Proofs

Strategy

In a direct proof of a conditional statement $p \rightarrow q$:

1. Assume that p is true.
2. Take advantage of rules of inference.
3. Show that q must also be true.

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Example

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Proof by Contraposition

Strategy

Instead of proving $p \to q$, prove its contrapositive, $\neg q \to \neg p$ (since they are equivalent).

1. Assume $\neg q$ is true.
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Example

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Vacuous Proofs

Caution

While proving $p \rightarrow q$ if you realize p is false then you can conclude the statement.

Caution

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Example

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Trivial Proofs

Caution

While proving $p \rightarrow q$ if we can show q is true then you can conclude the statement.

Trivial Proofs

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While proving $p \to q$ if we can show q is true then you can conclude the statement.

Example

Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$" where the domain consists of all non-negative integers. Show that $P(0)$ is true.

Proofs by Contradiction

Strategy

Suppose we want to prove that a statement p is true.

1. Find a contradiction q such that $\neg p \to q$ is true.
2. Since q is false and $\neg p \to q$ is true, then $\neg p$ is false.
3. Conclude p is true.

Contradiction q is usually of the form $r \wedge \neg r$ for some proposition r.

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Proofs of Equivalence

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Example

Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

Counter Example

Strategy

In order to prove a statement of the form $\forall x P(x)$ is false, you can find a c for which $P(c)$ is false.

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Example

Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Watch Out for Mistakes

One important fallacy

Many incorrect arguments are based on a fallacy called **begging the question**. This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. In other words, this fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called **circular reasoning**.

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Exhaustive Proof and Proof by Cases

Idea

It is based on the following tautology:

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

It let us proceed with the proof with extra assumptions!

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Example

1. Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

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It is based on the following tautology:

$$[(p_1 \lor p_2 \lor \cdots \lor p_n) \to q] \leftrightarrow [(p_1 \to q) \land (p_2 \to q) \land \cdots \land (p_n \to q)]$$

It let us proceed with the proof with extra assumptions!

Example

1. Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.
2. Prove that if n is an integer, then $n^2 \geq n$

Caution

WLOG

We assert that by proving one case of a theorem, no additional argument is required to prove other specified cases. That is, other cases follow by making straightforward changes to the argument, or by filling in some straightforward initial step.

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Danger

Although this will heavily simply the proofs involving proof by cases, an incorrect use of it will lead to an invalid proof. Be very careful with this statement!

Existence Proofs

Definition

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Strategies

- **Constructive proof**: The proof can be given by finding an element (also called and example or a witness) for which the statement is true.
- **Non-constructive proof**: You do not look for an example but rather approach the problem using other methods discussed earlier. One common method of giving non-constrcutive proof is proof by contradiction.

Examples

1. A Constructive Existence Proof:
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1. A Constructive Existence Proof:
   - Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.
   - $1729 = 10^3 + 9^3 = 12^3 + 1^3$
2. A Nonconstructive Existence Proof:
   - Show that there exist irrational numbers x and y such that x^y is rational.
   - *Hint:* $\sqrt{2}$

Uniqueness Proofs

Definition

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Strategies

To prove a statement of this type we need to show that an element with this property exists and that no other element has this property. The two parts of a uniqueness proof are:

- **Existence**: We show that an element x with the desired property exists.
- **Uniqueness**: We show that if $y \neq x$, then y does not have the desired property.

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Examples

Question

Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

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Solution:

- **Existence**: We note that $r = \frac{-b}{a}$ is a solution of $ar + b = 0$ i.e. a real number r exists for which $ar + b = 0$.

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Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Solution:

- **Existence**: We note that $r = \frac{-b}{a}$ is a solution of $ar + b = 0$ i.e. a real number r exists for which $ar + b = 0$.
- **Uniqueness**: suppose that s is a real number such that $as + b = 0$. Then we have:

$$\begin{aligned} ar + b &= as + b \\ ar &= as \\ r &= s \end{aligned}$$

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Practise Questions

Question 1

Use rules of inference to show that if $\forall x(P(x) \to Q(x))$, $\forall x(Q(x) \to R(x))$, and $\exists x(\neg R(x))$ are true, then $\exists x(\neg P(x))$ is true.

Question 2

Give a direct proof of: "If x is an odd integer and y is an even integer, then $x + y$ is odd."

Question 3

Give a proof by contradiction of: "If n is an odd integer, then n^2 is odd."

Question 4

Give an indirect proof of: "If x is an odd integer, then $x + 2$ is odd."

Question 5

Use a proof by cases to show that there are no solutions in positive integers to the equation $x^4 + y^4 = 100$.

Question 6

Prove that given a non-negative integer n, there is a unique non-negative integer m, such that $m^2 \leq n < (m+1)^2$.

Question 7

For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step.

1. Premise: $\{\neg p \vee q \to r, s \vee \neg q, \neg t, p \to t, \neg p \wedge r \to \neg s\}$, conclusion: $\neg q$.

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2. Premise: $\{\neg p \to r \wedge \neg s, t \to s, u \to \neg p, \neg w, u \vee w\}$, conclusion: $\neg t \vee w$.

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2. Premise: $\{\neg p \rightarrow r \wedge \neg s, t \rightarrow s, u \rightarrow \neg p, \neg w, u \vee w\}$, conclusion: $\neg t \vee w$.
3. Premise: $\{p \vee q, q \rightarrow r, p \wedge s \rightarrow t, \neg r, \neg q \rightarrow u \wedge s\}$, conclusion: t.

Question 8

Prove that the following four statements are equivalent:

1. n^2 is odd.
2. $1 - n$ is even.
3. n^3 is odd.
4. $n^2 + 1$ is even.

Question 9

Consider the statement concerning integers "If $m+n$ is even, then $m-n$ is even."

1. Give a direct proof of the statement.

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Consider the statement concerning integers "If $m + n$ is even, then $m - n$ is even."

1. Give a direct proof of the statement.
2. Give a proof by contraposition of the statement.
3. Prove the statement by contradiction.