Tutorial 5

COMP 5361: Discrete Structures and Formal Languages

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Outline

Contents of the section

1. Introduction to Sets
2. Set Operations

Definition

A **set** is an unordered collection of objects, called **elements** or **members** of the set. A set is said to **contain** its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.

Definition

Two sets are **equal** if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x (x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

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- **Singleton Set**: It is a set containing only one element.

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Strategies

- **A Subset of B**: Show that if x belongs to A then x also belongs to B.
- **A Not Subset of B**: Find a single $x \in A$ such that $x \notin B$

Definition

Equality: Two set A and B are equal if and only if A is a subset of B and B is a subset of A.

Definition

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- **Infinite Set**: A set that is not finite, is said to be infinite.

Definition

- **Power Set**: The power set of a set S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$. If a set has n elements, then its power set has 2^n elements.

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Example 1 What is the power set of the set $\{0, 1, 2\}$?

Example 2 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Definition

Cartesian product: The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$, i.e.

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Example 1 What is the Cartesian product $A \times B$?

Example 2 What is the Cartesian product $B \times A$?

Example 3 What is the Cartesian product $A \times B \times C$?

Definition

Relation: A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B. A relation from a set A to itself is called a relation on A.

Definition

Truth Set: The truth set of the statement P is the set of elements x in a domain D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D | P(x)\}$.

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Example

What are the truth sets of the predicates $P(x)$, $Q(x)$, and (x), where the domain is the set of integers and $P(x)$ is "$|x| = 1$," $Q(x)$ is "$x^2 = 2$," and $R(x)$ is "$|x| = x$."

Contents of the section

Set Operations

Definition

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- **Intersection** Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
- **Disjoint**: Two sets are called disjoint if their intersection is the empty set.

Example

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Example 2: What is the intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$?

Example 3: Are two sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$ disjoint?

Definition

- **Difference**: Let A and B be sets. The difference of A and B, denoted by $A - B$ or $A \backslash B$, is the set containing those elements that are in A but not in B.

Set Operations

Definition

- **Difference**: Let A and B be sets. The difference of A and B, denoted by $A - B$ or $A \backslash B$, is the set containing those elements that are in A but not in B.
- **Complement**: Let U be the universal set. The complement of the set A, denoted by $\overline{A}$, is the complement of A with respect to U . Therefore, the complement of the set A is $U - A$.

Example

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Example 2: Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers), what is $\overline{A}$?

Set Identities

$$A \cap U = A \qquad \text{Identity law} \tag{1}$$
$$A \cup \emptyset = A \qquad \text{Identity law} \tag{2}$$
$$A \cup U = U \qquad \text{Domination law} \tag{3}$$
$$A \cap \emptyset = \emptyset \qquad \text{Domination law} \tag{4}$$
$$A \cap A = A \qquad \text{Idempotent law} \tag{5}$$
$$A \cup A = A \qquad \text{Idempotent law} \tag{6}$$
$$\overline{(\overline{A})} = A \qquad \text{Complementation law} \tag{7}$$
$$A \cup B = B \cup A \qquad \text{Commutative law} \tag{8}$$

Set Identities

$$A \cap (B \cap C) = (A \cap B) \cap C \qquad \text{Associative law} \qquad (9)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad \text{Distributive law} \qquad (10)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad \text{Distributive law} \qquad (11)$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B} \qquad \text{De Morgan's law} \qquad (12)$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \text{De Morgan's law} \qquad (13)$$
$$A \cup (A \cap B) = A \qquad \text{Absorption law} \qquad (14)$$
$$A \cap (A \cup B) = A \qquad \text{Absorption law} \qquad (15)$$
$$A \cup \overline{A} = U \qquad \text{Complement law} \qquad (16)$$
$$A \cap \overline{A} = \emptyset \qquad \text{Complement law} \qquad (17)$$

Question 1

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Question 2

Use set builder notation and logical equivalences to prove Question 1.

Question 3

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Question 4

Use a membership table to prove Question 3.

Question 5

Prove the following, using the set identities.

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$