

# Tutorial 4

## COMP 5361: Discrete Structures and Formal Languages

Mohammad Reza Davari

Concordia University

# Outline

- 1 Introduction to Proofs
- 2 Proof Methods and Strategy
- 3 Practise Questions

# Contents of the section

- 1 Introduction to Proofs
- 2 Proof Methods and Strategy
- 3 Practise Questions

- **Theorem:** A theorem is a statement that can be shown to be true and is reserved for a statement that is considered at least somewhat important.
- **Propositions:** Propositions are theorems with less importance.
- **Proof:** Demonstrates whether a statement (theorem) is true.
- **Axioms:** Axioms or postulates are statements we assume to be true.
- **Lemma:** Lemma is a less important theorem that is helpful in the proof of other results.
- **Corollary** A corollary is a theorem that can be established directly from a theorem that has been proved.
- **Conjecture** A conjecture is a statement that is being proposed to be a true statement.

## Strategy

In a direct proof of a conditional statement  $p \rightarrow q$ :

- 1 Assume that  $p$  is true.
- 2 Take advantage of rules of inference.
- 3 Show that  $q$  must also be true.

## Strategy

In a direct proof of a conditional statement  $p \rightarrow q$ :

- 1 Assume that  $p$  is true.
- 2 Take advantage of rules of inference.
- 3 Show that  $q$  must also be true.

## Example

Give a direct proof of the theorem “If  $n$  is an odd integer, then  $n^2$  is odd.”

# Proof by Contraposition

## Strategy

Instead of proving  $p \rightarrow q$ , prove its contrapositive,  $\neg q \rightarrow \neg p$  (since they are equivalent).

- 1 Assume  $\neg q$  is true.
- 2 Take advantage of rules of inference.
- 3 Show that  $\neg p$  must also be true.

# Proof by Contraposition

## Strategy

Instead of proving  $p \rightarrow q$ , prove its contrapositive,  $\neg q \rightarrow \neg p$  (since they are equivalent).

- 1 Assume  $\neg q$  is true.
- 2 Take advantage of rules of inference.
- 3 Show that  $\neg p$  must also be true.

## Example

Prove that if  $n$  is an integer and  $3n + 2$  is odd, then  $n$  is odd.



## Caution

While proving  $p \rightarrow q$  if you realize  $p$  is false then you can conclude the statement.

## Caution

While proving  $p \rightarrow q$  if you realize  $p$  is false then you can conclude the statement.

## Example

Show that the proposition  $P(0)$  is true, where  $P(n)$  is “If  $n > 1$ , then  $n^2 > n$ ” and the domain consists of all integers.

## Caution

While proving  $p \rightarrow q$  if we can show  $q$  is true then you can conclude the statement.

## Caution

While proving  $p \rightarrow q$  if we can show  $q$  is true then you can conclude the statement.

## Example

Let  $P(n)$  be “If  $a$  and  $b$  are positive integers with  $a \geq b$ , then  $a^n \geq b^n$ ” where the domain consists of all non-negative integers. Show that  $P(0)$  is true.

# Proofs by Contradiction

## Strategy

Suppose we want to prove that a statement  $p$  is true.

- 1 Find a contradiction  $q$  such that  $\neg p \rightarrow q$  is true.
- 2 Since  $q$  is false and  $\neg p \rightarrow q$  is true, then  $\neg p$  is false.
- 3 Conclude  $p$  is true.

Contradiction  $q$  is usually of the form  $r \wedge \neg r$  for some proposition  $r$ .

# Proofs by Contradiction

## Strategy

Suppose we want to prove that a statement  $p$  is true.

- 1 Find a contradiction  $q$  such that  $\neg p \rightarrow q$  is true.
- 2 Since  $q$  is false and  $\neg p \rightarrow q$  is true, then  $\neg p$  is false.
- 3 Conclude  $p$  is true.

Contradiction  $q$  is usually of the form  $r \wedge \neg r$  for some proposition  $r$ .

## Example

- 1 Show that at least four of any 22 days must fall on the same day of the week.

# Proofs by Contradiction

## Strategy

Suppose we want to prove that a statement  $p$  is true.

- 1 Find a contradiction  $q$  such that  $\neg p \rightarrow q$  is true.
- 2 Since  $q$  is false and  $\neg p \rightarrow q$  is true, then  $\neg p$  is false.
- 3 Conclude  $p$  is true.

Contradiction  $q$  is usually of the form  $r \wedge \neg r$  for some proposition  $r$ .

## Example

- 1 Show that at least four of any 22 days must fall on the same day of the week.
- 2 Prove that  $\sqrt{2}$  is irrational.

# Proofs by Contradiction

## Strategy

Suppose we want to prove that a statement  $p$  is true.

- 1 Find a contradiction  $q$  such that  $\neg p \rightarrow q$  is true.
- 2 Since  $q$  is false and  $\neg p \rightarrow q$  is true, then  $\neg p$  is false.
- 3 Conclude  $p$  is true.

Contradiction  $q$  is usually of the form  $r \wedge \neg r$  for some proposition  $r$ .

## Example

- 1 Show that at least four of any 22 days must fall on the same day of the week.
- 2 Prove that  $\sqrt{2}$  is irrational.
- 3 Prove if  $3n + 2$  is odd, then  $n$  is odd.



# Proofs of Equivalence

## Strategy

In order to prove a statement of the form  $p \leftrightarrow q$  we need to:

- 1 Show  $p \rightarrow q$  is true.
- 2 Show  $q \rightarrow p$  is true.

# Proofs of Equivalence

## Strategy

In order to prove a statement of the form  $p \leftrightarrow q$  we need to:

- 1 Show  $p \rightarrow q$  is true.
- 2 Show  $q \rightarrow p$  is true.

## Example

Prove the theorem “If  $n$  is an integer, then  $n$  is odd if and only if  $n^2$  is odd.”

# Counter Example

## Strategy

In order to prove a statement of the form  $\forall x P(x)$  is false, you can find a  $c$  for which  $P(c)$  is false.

# Counter Example

## Strategy

In order to prove a statement of the form  $\forall x P(x)$  is false, you can find a  $c$  for which  $P(c)$  is false.

## Example

Show that the statement “Every positive integer is the sum of the squares of two integers” is false.

# Watch Out for Mistakes

## One important fallacy

Many incorrect arguments are based on a fallacy called **begging the question**. This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. In other words, this fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called **circular reasoning**.

# Contents of the section

- 1 Introduction to Proofs
- 2 Proof Methods and Strategy
- 3 Practise Questions

# Exhaustive Proof and Proof by Cases

## Idea

It is based on the following tautology:

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

It let us proceed with the proof with extra assumptions!

# Exhaustive Proof and Proof by Cases

## Idea

It is based on the following tautology:

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

Let us proceed with the proof with extra assumptions!

## Example

- 1 Prove that  $(n + 1)^3 \geq 3^n$  if  $n$  is a positive integer with  $n \leq 4$ .



# Exhaustive Proof and Proof by Cases

## Idea

It is based on the following tautology:

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

Let us proceed with the proof with extra assumptions!

## Example

- 1 Prove that  $(n+1)^3 \geq 3^n$  if  $n$  is a positive integer with  $n \leq 4$ .
- 2 Prove that if  $n$  is an integer, then  $n^2 \geq n$

## WLOG

We assert that by proving one case of a theorem, no additional argument is required to prove other specified cases. That is, other cases follow by making straightforward changes to the argument, or by filling in some straightforward initial step.

# Caution

## WLOG

We assert that by proving one case of a theorem, no additional argument is required to prove other specified cases. That is, other cases follow by making straightforward changes to the argument, or by filling in some straightforward initial step.

## Danger

Although this will heavily simplify the proofs involving proof by cases, an incorrect use of it will lead to an invalid proof. Be very careful with this statement!

## Definition

A proof of the form  $\exists xP(x)$  is called an existence proof.

## Definition

A proof of the form  $\exists xP(x)$  is called an existence proof.

## Strategies

- **Constructive proof:** The proof can be given by finding an element (also called an example or a witness) for which the statement is true.
- **Non-constructive proof:** You do not look for an example but rather approach the problem using other methods discussed earlier. One common method of giving non-constructive proof is proof by contradiction.

## ① A Constructive Existence Proof:

- Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

## ① A Constructive Existence Proof:

- Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.
- $1729 = 10^3 + 9^3 = 12^3 + 1^3$

## ① A Constructive Existence Proof:

- Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.
- $1729 = 10^3 + 9^3 = 12^3 + 1^3$

## ② A Nonconstructive Existence Proof:

- Show that there exist irrational numbers  $x$  and  $y$  such that  $x^y$  is rational.
- *Hint:*  $\sqrt{2}$



# Uniqueness Proofs

## Definition

A proof of the form  $\exists x(P(x) \wedge \forall y(y = x \rightarrow \neg P(y)))$  is called a uniqueness proof.

# Uniqueness Proofs

## Definition

A proof of the form  $\exists x(P(x) \wedge \forall y(y = x \rightarrow \neg P(y)))$  is called a uniqueness proof.

## Strategies

To prove a statement of this type we need to show that an element with this property exists and that no other element has this property. The two parts of a uniqueness proof are:

- **Existence:** We show that an element  $x$  with the desired property exists.
- **Uniqueness:** We show that if  $y \neq x$ , then  $y$  does not have the desired property.

# Examples

## Question

Show that if  $a$  and  $b$  are real numbers and  $a \neq 0$ , then there is a unique real number  $r$  such that  $ar + b = 0$ .

# Examples

## Question

Show that if  $a$  and  $b$  are real numbers and  $a \neq 0$ , then there is a unique real number  $r$  such that  $ar + b = 0$ .

## Solution:

- **Existence:** We note that  $r = \frac{-b}{a}$  is a solution of  $ar + b = 0$  i.e. a real number  $r$  exists for which  $ar + b = 0$ .

## Question

Show that if  $a$  and  $b$  are real numbers and  $a \neq 0$ , then there is a unique real number  $r$  such that  $ar + b = 0$ .

## Solution:

- **Existence:** We note that  $r = \frac{-b}{a}$  is a solution of  $ar + b = 0$  i.e. a real number  $r$  exists for which  $ar + b = 0$ .
- **Uniqueness:** suppose that  $s$  is a real number such that  $as + b = 0$ . Then we have:

$$ar + b = as + b$$

$$ar = as$$

$$r = s$$

# Contents of the section

- 1 Introduction to Proofs
- 2 Proof Methods and Strategy
- 3 Practise Questions

## Question 1

Use rules of inference to show that if  $\forall x(P(x) \rightarrow Q(x))$ ,  $\forall x(Q(x) \rightarrow R(x))$ , and  $\exists x(\neg R(x))$  are true, then  $\exists x(\neg P(x))$  is true.

# Solution: Question 1

$\forall x(P(x) \rightarrow Q(x))$		Given by the problem	(1)
$\forall x(Q(x) \rightarrow R(x))$		Given by the problem	(2)
$\exists x(\neg R(x))$		Given by the problem	(3)
$Q(c) \rightarrow R(c)$	for an arbitrary $c$	Universal Instantiation: 2	(4)
$\neg(R(c))$	for some $c$	Existential Instantiation: 3	(5)
$\neg(Q(c))$	for some $c$	Modus Tollens: 5,4	(6)
$P(c) \rightarrow Q(c)$	for an arbitrary $c$	Universal Instantiation: 1	(7)
$\neg(P(c))$	for some $c$	Modus Tollens: 7,6	(8)
$\exists x(\neg P(x))$		Existential Generalization: 8	(9)
$\therefore \exists x(\neg P(x))$		By: 9	(10)



## Question 2

Give a direct proof of: “If  $x$  is an odd integer and  $y$  is an even integer, then  $x + y$  is odd.”

## Solution: Question 2

**Direct Proof:** Assume  $x$  is an odd integer and  $y$  is an even integer, we need to show that  $x + y$  is odd. By our assumption we have:

$$x = 2k + 1 \text{ for some } k \in \mathbb{Z} \quad (1)$$

$$y = 2e \text{ for some } e \in \mathbb{Z} \quad (2)$$

$$x + y = 2k + 1 + 2e \quad (3)$$

$$= 2(k + e) + 1 \quad (4)$$

$$\therefore x + y \text{ is odd} \quad (5)$$

Note that in Equation 4  $k + e$  is just an integer and can be renamed to some  $k'$ , hence showing that  $x + y$  has the form of an odd integer.

## Question 3

Give a proof by contradiction of: “If  $n$  is an odd integer, then  $n^2$  is odd.”

## Solution: Question 3

**Proof by Contradiction:** Given that  $n$  is an odd integer, assume  $n^2$  is not odd i.e. it is even. We have:

$$n = 2k + 1 \text{ for some } k \in \mathbb{Z} \quad (1)$$

$$n^2 = 2e \text{ for some } e \in \mathbb{Z} \quad (2)$$

Now using Equation 1, we have:

$$n^2 = (2k + 1)^2 \quad (3)$$

$$= 4k^2 + 4k + 1 \quad (4)$$

$$= 2(2k^2 + 2k) + 1 \quad (5)$$

Equation 5 shows  $n^2$  is odd which is in contradiction with Equation 2. Hence, our assumption was false, i.e  $n^2$  must be odd.

## Question 4

Give an indirect proof of: “If  $x$  is an odd integer, then  $x + 2$  is odd.”

## Solution: Question 4

**Proof by Contraposition** Assume  $x + 2$  is not odd (it is even) we need to show that  $x$  is even. Since  $x + 2$  is even, we can write it as  $2k$  for some  $k \in \mathbb{Z}$ . Therefore we have:

$$x + 2 = 2k \quad (1)$$

$$x = 2(k - 1) \quad (2)$$

The second equation shows that  $x$  is even, hence we reached our desired conclusion.

## Question 5

Use a proof by cases to show that there are no solutions in positive integers to the equation  $x^4 + y^4 = 100$ .

## Solution: Question 5

**Proof by Cases:** Note that neither  $x$  or  $y$  can be greater than 3 (since  $4^4 = 256 > 100$ ). Hence, we only need to search for solutions among  $\{0, 1, 2, 3\}$ .

**Case 1:**  $x = 0$ , in this case  $y$  needs to be as large as possible,  $y = 3$  provides  $81 \neq 100$  hence, there is no solution in this case.

**Case 2:**  $x = 1$ , even when  $y$  attains the largest possible value, the left side of the equation is still smaller than the right side ( $82 \neq 100$ ) hence, there is no solution in this case.

**Case 3:**  $x = 2$ , again with the same reasoning as above cases we can see that the left side of the equation will always be smaller than the right side. Hence, there is no solution in this case.

**Case 4:**  $x = 3$  this case has been covered by  $y = 3$  in the previous cases, hence WLOG we can conclude that there is no solution in this case.





## Question 6

Prove that given a non-negative integer  $n$ , there is a unique non-negative integer  $m$ , such that  $m^2 \leq n < (m+1)^2$ .

## Solution: Question 6

Since this is a uniqueness proof, we need to show the followings:

- Existence
- Uniqueness

We first show the existence of such  $m$ .

**Existence:** Let  $m$  be  $\lfloor \sqrt{n} \rfloor$ , then:

$$m \leq \sqrt{n} \leq m + 1 \quad (1)$$

$$\therefore m^2 \leq n \leq (m + 1)^2 \quad (2)$$

## Solution: Question 6

Now we attempt to show the uniqueness of the solution.

**Uniqueness:** Assume  $\exists k \neq m$ , WLOG we can also assume  $m \geq k$ , such that:

$$k^2 \leq n \leq (k+1)^2 \quad (3)$$

$$k \leq n \leq k+1 \quad (4)$$

Now if we subtract Equations 1 and 4 we would have:

$$m - k \leq 0 \quad (5)$$

Since  $m \geq k$  then  $m - k \geq 0$ , considering the Equation 5, we reach to the conclusion:

$$m - k = 0$$

$$m = k$$

Hence, the solution is unique.

## Question 7

For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step.

- ① Premise:  $\{\neg p \vee q \rightarrow r, s \vee \neg q, \neg t, p \rightarrow t, \neg p \wedge r \rightarrow \neg s\}$ , conclusion:  $\neg q$ .

## Question 7

For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step.

- ① Premise:  $\{\neg p \vee q \rightarrow r, s \vee \neg q, \neg t, p \rightarrow t, \neg p \wedge r \rightarrow \neg s\}$ , conclusion:  $\neg q$ .
- ② Premise:  $\{\neg p \rightarrow r \wedge \neg s, t \rightarrow s, u \rightarrow \neg p, \neg w, u \vee w\}$ , conclusion:  $\neg t \vee w$ .

# Practise Questions

## Question 7

For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step.

- ➊ Premise:  $\{\neg p \vee q \rightarrow r, s \vee \neg q, \neg t, p \rightarrow t, \neg p \wedge r \rightarrow \neg s\}$ , conclusion:  $\neg q$ .
- ➋ Premise:  $\{\neg p \rightarrow r \wedge \neg s, t \rightarrow s, u \rightarrow \neg p, \neg w, u \vee w\}$ , conclusion:  $\neg t \vee w$ .
- ➌ Premise:  $\{p \vee q, q \rightarrow r, p \wedge s \rightarrow t, \neg r, \neg q \rightarrow u \wedge s\}$ , conclusion:  $t$ .

## Solution: Question 7.1

$\neg p \vee q \rightarrow r$	Given by the problem	(1)
$s \vee \neg q$	Given by the problem	(2)
$\neg t$	Given by the problem	(3)
$p \rightarrow t$	Given by the problem	(4)
$\neg p \wedge r \rightarrow \neg s$	Given by the problem	(5)
$\neg p$	Modus Tollens: 3, 4	(6)
$\neg p \vee$	Addition: 6	(7)
$r$	Modus Ponens: 1, 7	(8)
$r \wedge \neg p$	Conjunction: 8, 6	(9)
$\neg s$	Modus Ponens: 9, 5	(10)
$\neg q$	Disjunctive Syllogism: 10, 2	(11)
$\therefore \neg q$	By : 11	(12)

## Solution: Question 7.2

$\neg p \rightarrow r \wedge \neg s$  Given by the problem (1)

$t \rightarrow s$  Given by the problem (2)

$u \rightarrow \neg p$  Given by the problem (3)

$\neg w$  Given by the problem (4)

$u \vee w$  Given by the problem (5)

$u$  Disjunctive Syllogism: 4, 5 (6)

$\neg p$  Modus Ponens: 3, 6 (7)

$r \wedge \neg s$  Modus Ponens: 7, 1 (8)

$\neg s$  Simplification: 8 (9)

$\neg t$  Modus Tollens: 9, 2 (10)

$\neg t \vee w$  Addition: 10 (11)

$\therefore \neg t \vee w$  By: 11 (12)





## Solution: Question 7.3

$p \vee q$	Given by the problem	(1)
$q \rightarrow r$	Given by the problem	(2)
$p \wedge s \rightarrow t$	Given by the problem	(3)
$\neg r$	Given by the problem	(4)
$\neg q \rightarrow u \wedge s$	Given by the problem	(5)
$\neg q$	Modus Tollens: 2, 4	(6)
$p$	Disjunctive Syllogism: 6, 1	(7)
$u \wedge s$	Modus Ponens: 5, 6	(8)
$s$	Simplification: 8	(9)
$p \wedge s$	Conjunction: 7, 8	(10)
$t$	Modus Ponens: 10, 3	(11)
$\therefore t$	By: 11	(12)

## Question 8

Prove that the following four statements are equivalent:

- ①  $n^2$  is odd.
- ②  $1 - n$  is even.
- ③  $n^3$  is odd.
- ④  $n^2 + 1$  is even.

# Solution: Question 8

It is enough to show:

$1^{st} \text{ Statement} \rightarrow 2^{nd} \text{ Statement}$

$2^{nd} \text{ Statement} \rightarrow 3^{rd} \text{ Statement}$

$3^{rd} \text{ Statement} \rightarrow 4^{th} \text{ Statement}$

$4^{th} \text{ Statement} \rightarrow 1^{st} \text{ Statement}$

## Solution: Question 8 Part (1)

In this part we aim to prove, given that  $n^2$  is odd then  $1 - n$  is even.

**Proof by Contraposition:** Assume  $1 - n$  is odd, we need to prove  $n^2$  is even. Since  $1 - n$  is odd we can write it as  $2k + 1$  for some  $k \in \mathbb{Z}$ .

Therefore we have:

$$1 - n = 2k + 1 \quad (1)$$

$$n = 2(-k) \quad (2)$$

$$n^2 = 4k^2 \quad (3)$$

$$= 2(2k^2) \quad (4)$$

Equation 4 shows that  $n^2$  is a multiple of 2 and hence even.

## Solution: Question 8 Part (2)

In this part we show the proof for: “If  $1 - n$  is even then  $n^3$  is odd”.

**Direct Proof:** Assume  $1 - n$  is even, hence it can be written as  $2k$  for some  $k \in \mathbb{Z}$ . Therefore,  $n = 2(-k) + 1$ , i.e.  $n$  is odd. For simplicity let's rename  $-k$  to  $q$  we then have:

$$n = 2q + 1 \quad (1)$$

$$n^3 = (2q + 1)^3 \quad (2)$$

$$= 8q^3 + 12q^2 + 6q + 1 \quad (3)$$

$$= 2(4q^3 + 6q^2 + 3q) + 1 \quad (4)$$

Equation 4 shows that  $n^3$  is odd, and hence our proof is completed for this part.

## Solution: Question 8 Part (3)

In this part we will prove that  $n^2 + 1$  is even given that  $n^3$  is odd. We first prove the following lemma which will be helpful to prove the statement.

**Lemma:** If  $n^3$  is odd then  $n$  is odd.

**Proof by Contraposition:** Assume  $n$  is even, we need to show that  $n^3$  is even. Since,  $n$  is even, it can be written as  $2k$  for some  $k \in \mathbb{Z}$ . Then  $n^3 = 8k^3 = 2(4k^3)$ , which shows  $n^3$  is even.

## Solution: Question 8 Part (3)

Now that we have the lemma, we will prove the statement as follows:

**Direct Proof:** Since  $n^3$  is odd, then by the lemma we have  $n$  is odd.

Hence we can write  $n$  as  $2t + 1$  for some  $t \in \mathbb{Z}$ . We now have:

$$n^2 + 1 = (2t + 1)^2 + 1 \quad (1)$$

$$= 4t^2 + 4t + 2 \quad (2)$$

$$= 2(2t^2 + 2t + 1) \quad (3)$$

Equation 3 shows that  $n^2 + 1$  is even and that the proof for this part is completed.

## Solution: Question 8 Part (4)

In this part we need to show  $n^2$  is odd given that  $n^2 + 1$  is even.

**Direct Proof:** Assume  $n^2 + 1$  is even, we need to show that  $n^2$  is odd. Since,  $n^2 + 1$  is even it can be written as  $2k$  for some  $k \in \mathbb{Z}$ . Hence,  $n^2 + 1 = 2k + 1$  i.e. it is odd and therefore the proof is completed.

Now that we have show the proof for all 4 parts, we can conclude that the 4 statements given by the problem are all equivalent.



## Question 9

Consider the statement concerning integers “If  $m + n$  is even, then  $m - n$  is even.”

- 1 Give a direct proof of the statement.

## Question 9

Consider the statement concerning integers “If  $m + n$  is even, then  $m - n$  is even.”

- 1 Give a direct proof of the statement.
- 2 Give a proof by contraposition of the statement.

## Question 9

Consider the statement concerning integers “If  $m + n$  is even, then  $m - n$  is even.”

- 1 Give a direct proof of the statement.
- 2 Give a proof by contraposition of the statement.
- 3 Prove the statement by contradiction.

# Solution: Question 9 Part (1)

**Direct Proof:** Assume  $m + n$  is even, we need to show  $m - n$  is even. Since  $m + n$  is even we have the following cases:

**Case 1:** Both  $m$  and  $n$  are even: Then in this case,  $m - n$  would also be even as the subtraction of two even numbers is even.

**Case 2:** Both  $m$  and  $n$  are odd: Then in this case,  $m - n$  would be even as the subtraction of two odd numbers is even.

We have proved the desired result for all cases, hence the proof is completed.

## Solution: Question 9 Part (2)

**Proof by Contraposition:** Assume  $m - n$  is odd, we need to show that  $m + n$  is also odd. Since  $m - n$  is odd we have that  $m$  and  $n$  are a pair of odd and even, i.e. one is odd and the other is even. The addition of an odd and even integer is odd, i.e.  $m + n$  is odd. Hence, we reached the desired result and this concludes the proof for this case.

## Solution: Question 9 Part (3)

**Proof by Contradiction:** Assume  $m + n$  is even and  $m - n$  is odd. Sum of an odd and an even integer is odd. Hence if we add  $m + n$  and  $m - n$  we should have an odd integer. However,  $m + n + m - n = 2m$ , i.e. an even integer. Hence we reached a contradiction in this case, which means our original assumption was false, i.e.  $m - n$  must be even.