Tutorial 7

COMP 5361: Discrete Structures and Formal Languages

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Outline

Definition

- **Binary Relation**: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

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- **Binary Relation**: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.
- **Relation on a Set**: A relation on a set A is a relation from A to A.

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- **Symmetric**: A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
- **Antisymmetric**: A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.
- **Transitive**: A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Example

1. Which of the relations on slide 5 are reflexive?

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2. Which of the relations on slide 5 are symmetric?

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4. Which of the relations on slide 5 are transitive?

Example

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Combining Relations

Definition

- Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.
- Let R be a relation on the set A. The powers R^n , $n = 1, 2, 3, \ldots,$ are defined recursively by:

$$R^1 = R \tag{1}$$
$$R^{n+1} = R^n \circ R \tag{2}$$

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Example

1. Let $R = (1,1), (2,1), (3,2), (4,3)$. Find the powers R^n , $n = 2, 3, 4, \ldots$.

Theorem

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Example

1. Let $R = (1,1), (2,1), (3,2), (4,3)$. Find the powers R^n , $n = 2, 3, 4, \ldots$.
2. Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

Contents of the section

Definition

Let R be a relation on set A, then:

- **Reflexive Closure**: The reflexive closure of R is the smallest relation that contains R and is reflexive.

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- **Transitive Closure**: The transitive closure of R is the smallest relation that contains R and is transitive.

Example

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1. What is the reflexive closure of the relation $R = \{(a,b) | a < b\}$ on the set of integers?
2. What is the symmetric closure of the relation $R = \{(a,b) | a > b\}$ on the set of positive integers?
3. Find the transitive closures of the relation R on a, b, c, d, e, where R is given by:
$$\{(a,c),(b,d),(c,a),(d,b),(e,d)\}$$

Definition

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- The transitive closure of a relation R equals the connectivity relation R^*.

Example

Example 1

Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), (5, 2)$, and $(5, 4)$. Find:

1. R^2
2. R^3
3. R^4
4. R^5
5. R^6
6. R^*

Example 2

Find the transitive closures of these relations on $\{1, 2, 3, 4\}$.

1. $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$
2. $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
3. $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$

Contents of the section

Definition

- **Equivalence Relation**: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.
- **Equivalent**: Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

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2. Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?
3. Let m be an integer with $m > 1$. Show that the relation (Congruence Modulo m):

$$R = \{(a, b)|a \equiv b(modm)\}$$

is an equivalence relation on the set of integers.

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3. Let m be an integer with $m > 1$. Show that the relation (Congruence Modulo m):
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is an equivalence relation on the set of integers.
4. Show that the "divides" relation is the set of positive integers in not an equivalence relation.

Definition

- Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.
- In the above definition a is the representative of this equivalence class.

Properties

If $a \sim b$ then the followings are equivalent:

1. aRb
2. $[a] = [b]$

Example

1. Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. What is the equivalence class of an integer?

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Example

1. Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. What is the equivalence class of an integer?
2. What are the equivalence classes of 0 and 1 for congruence modulo 7?
3. What are the equivalence classes of 5 and 2 for congruence modulo 5?

Definition

- **Partition**: A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

Equivalence Relations

Definition

- **Partition**: A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union.
- Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i | i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Example

Example 1

Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. Give a partition of set S with 3 classes.

Example 2

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$.

Example

Example 3

What are the sets in the partition of the integers arising from congruence modulo 7?

Example

Example 4

What is the congruence class $[4]_m$ when m is:

1. 2?
2. 3?
3. 6?
4. 8?

Contents of the section

Partial Orderings

Definition

- A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.
- The elements a and b of a poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$. When a and b are elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called incomparable.

Example

1. Show that the greater than or equal relation is a partial ordering on the set of integers.

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2. Show that the inclusion relation is a partial ordering on the power set of a set S.

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2. Show that the inclusion relation is a partial ordering on the power set of a set S.
3. In the poset $(Z^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Definition

- **Totally Ordered**: If $(S, \preceq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order. A totally ordered set is also called a chain.
- **Well-ordered Set**: $(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering and every nonempty subset of S has a least element.
- **Lexicographic Ordering**: The lexicographic ordering $\preceq$ on $A_1 \times A_2$ is defined by, $(a_1, a_2) \prec (b_1, b_2)$ when, either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

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   2. $(Z^+, |)$
2. Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbb{Z} \times \mathbb{Z}, \preceq)$, where $\preceq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbb{Z}$.

Definition

- **Covering Relation**: Let $(S, \preceq)$ be a poset. We say that an element $y \in S$ covers an element $x \in S$ *if*$x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the covering relation of $(S, \preceq)$.

Example

1. Draw the Hasse diagram representing the partial ordering $\{(a, b)|a divides b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.