Tutorial 1

COMP 355: Introduction to Theoretical Computer Science

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Outline

Contents of the section

General Information

- **Focus of The Tutorials**: The focus of the tutorials will be on problem solving. The problems presented in the tutorials will usually be similar to the ones you will see on your assignments, so you would have an easier time with the assignments.

Contents of the section

Question 1 (Sets)

Problem

Prove the following, using the set identities.

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Question 1 (Sets)

Solution

$$\text{(LHS)}\ \overline{A \cup (B \cap C)} = \qquad (1)$$

Question 1 (Sets)

Solution

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\begin{aligned}
\text{(LHS)} \quad \overline{A \cup (B \cap C)} &= && (1)\\
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\text{(LHS)} \quad & \overline{A \cup (B \cap C)} & = & \quad (1)\\
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\text{Commutative law} \quad & \overline{(B \cap C)} \cap \overline{A} & = & \quad (3)\\
\text{DeMorgan's Laws} \quad & (\overline{B} \cup \overline{C}) \cap \overline{A} & = & \quad (4)\\
\text{Commutative law} \quad & (\overline{C} \cup \overline{B}) \cap \overline{A} & (RHS) & \quad (5)
\end{aligned}$$

Question 2 (Functions)

Problem

Determine whether the followings are total functions or partial functions:

1. $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \frac{1}{x}$
2. $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$
3. $f : \mathbb{N} \rightarrow \mathbb{R}, f(x) = \sqrt{x}$
4. $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sqrt[5]{x} + 1$

Question 2 (Functions)

Solution

1. It is not a total function since it is not define at 0 hence not covering the whole domain.

Question 2 (Functions)

Solution

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Question 3 (Relations)

Problem

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation,

1. $R = \{(a, b) | a \text{ divides } b\}$
2. $R = \{(a, b) | a \leq b\}$
3. $R = \{(a, b) | a > b\}$
4. $R = \{(a, b) | a = b \vee a = -b\}$
5. $R = \{(a, b) | a = b\}$

Question 3 (Relations)

Solution

1. $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

Solution

1. $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$
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Question 3 (Relations)

Solution

1. $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$
2. $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$
3. $\{(4,3),(4,2),(4,1),(3,2),(3,1),(2,1)\}$

Question 3 (Relations)

Solution

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Question 3 (Relations)

Solution

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Question 4 (Relations)

Problem

1. Which of the relations in Question 3 are reflexive?
2. Which of the relations in Question 3 are symmetric?
3. Which of the relations in Question 3 are antisymmetric?
4. Which of the relations in Question 3 are transitive?
5. Which of the relations in Question 3 are equivalence?

Question 4 (Relations)

Solution

1. Relations: $\{1, 2, 4, 5\}$

Question 4 (Relations)

Solution

1. Relations: $\{1, 2, 4, 5\}$
2. Relations: $\{4, 5\}$

Question 4 (Relations)

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Question 4 (Relations)

Solution

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Question 4 (Relations)

Solution

1. Relations: $\{1, 2, 4, 5\}$
2. Relations: $\{4, 5\}$
3. Relations: $\{1, 2, 3, 4, 5\}$
4. Relations: $\{1, 2, 3, 4, 5\}$
5. Relations: $\{4, 5\}$

Problem

Prove that the following four statements are equivalent:

1. n^2 is odd.
2. $1 - n$ is even.
3. n^3 is odd.
4. $n^2 + 1$ is even.

Solution: Question 5

It is enough to show:

$$1^{st} \text{ Statement} \rightarrow 2^{nd} \text{ Statement}$$

$$2^{nd} \text{ Statement} \rightarrow 3^{rd} \text{ Statement}$$

$$3^{rd} \text{ Statement} \rightarrow 4^{th} \text{ Statement}$$

$$4^{th} \text{ Statement} \rightarrow 1^{st} \text{ Statement}$$

In this part we aim to prove, given that n^2 is odd then $1 - n$ is even.
Proof by Contraposition: Assume $1 - n$ is odd, we need to prove n^2 is even. Since $1 - n$ is odd we can write it as $2k + 1$ for some $k \in \mathbb{Z}$.
Therefore we have:

$$\begin{aligned}
1 - n &= 2k + 1 && (1)\\
n &= 2(-k) && (2)\\
n^2 &= 4k^2 && (3)\\
&= 2(2k^2) && (4)
\end{aligned}$$

Equation 4 shows that n^2 is a multiple of 2 and hence even.

Solution: Question 5 Part (2)

In this part we show the proof for: "If $1 - n$ is even then n^3 is odd".
Direct Proof: Assume $1 - n$ is even, hence it can be written as $2k$ for some $k \in \mathbb{Z}$. Therefore, $n = 2(-k) + 1$, i.e. n is odd. For simplicity lets rename $-k$ to q we then have:

$$\begin{aligned} n &= 2q + 1 && (1) \\ n^3 &= (2q+1)^3 && (2) \\ &= 8q^3 + 12q^2 + 6q + 1 && (3) \\ &= 2(4q^3 + 6q^2 + 3q) + 1 && (4) \end{aligned}$$

Equation 4 shows that n^3 is odd, and hence our proof is completed for this part.

Solution: Question 5 Part (3)

In this part we will prove that $n^2 + 1$ is even given that n^3 is odd. We first prove the following lemma which will be helpful to prove the statement.

Lemma: If n^3 is odd then n is odd.

Proof by Contraposition: Assume n is even, we need to show that n^3 is even. Since, n is even, it can be written as $2k$ for some $k \in \mathbb{Z}$. Then $n^3 = 8k^3 = 2(4k^3)$, which shows n^3 is even.

Solution: Question 5 Part (3)

Now that we have the lemma, we will prove the statement as follows:
Direct Proof: Since n^3 is odd, then by the lemma we have n is odd. Hence we can write n as $2t+1$ for some $t \in \mathbb{Z}$. We now have:

$$
\begin{aligned}
n^2+1 &= (2t+1)^2+1 && (1)\\
&= 4t^2+4t+2 && (2)\\
&= 2(2t^2+2t+1) && (3)
\end{aligned}
$$

Equation 3 shows that n^2+1 is even and that the proof for this part is completed.

Solution: Question 5 Part (4)

In this part we need to show n^2 is odd given that $n^2 + 1$ is even.
Direct Proof:Assume $n^2 + 1$ is even, we need to show that n^2 is odd. Since, $n^2 + 1$ is even it can be written as $2k$ for some $k \in \mathbb{Z}$. Hence, $n^2 + 1 = 2k + 1$ i.e. it is odd and therefore the proof is completed.

Now that we have show the proof for all 4 parts, we can conclude that the 4 statements given by the problem are all equivalent.