Consensus Problems in Networks of Agents with Switching Topology and Time-Delays

Reza Olfati-Saber and Richard M. Murray

Abstract— **In this paper, we discuss consensus problems for networks of dynamic agents with fixed and switching topologies. We analyze three cases: i) directed networks with fixed topology, ii) directed networks with switching topology, and iii) undirected networks with communication time-delays and fixed topology. We introduce two consensus protocols for networks with and without time-delays and provide a convergence analysis in all three cases. We establish a direct connection between the algebraic connectivity (or Fiedler eigenvalue) of the network and the performance (or negotiation speed) of a linear consensus protocol. This required the generalization of the notion of algebraic connectivity of undirected graphs to digraphs. It turns out that balanced digraphs play a key role in addressing average-consensus problems. We introduce disagreement functions for convergence analysis of consensus protocols. A disagreement function is a Lyapunov function for the disagreement network dynamics. We proposed a simple disagreement function that is a common Lyapunov function for the disagreement dynamics of a directed network with switching topology. A distinctive feature of this work is to address consensus problems for networks with directed information flow. We provide analytical tools that rely on algebraic graph theory, matrix theory, and control theory. Simulations are provided that demonstrate the effectiveness of our theoretical results.**

Keywords—**consensus problems, networks of autonomous agents, switching systems, graph Laplacians, networks with time-delays, algebraic graph theory, digraph theory.**

I. Introduction

DISTRIBUTED coordination of networks of dynamic agents has attracted several researchers in recent years. This is partly due to broad applications of multi-agent systems in many areas including cooperative control of unmanned air vehicles (UAVs), formation control [1], [2], [3], [4], [5], flocking [6], [7], [8], distributed sensor networks [9], attitude alignment of clusters of satellites, and congestion control in communication networks [10].

Consensus problems have a long history in the field of computer science, particularly in automata theory and distributed computation [11]. In many applications involving multi-agent/multi-vehicle systems, groups of agents need to agree upon certain quantities of interest. Such quantities might or might not be related to the motion of the individual agents. As a result, it is important to address agreement problems in their general form for networks of dynamic agents with directed information flow under link failure and creation (i.e. switching network topology).

Our main contribution in this paper is to pose and address consensus problems under a variety of assumptions on the network topology (being fixed or switching), presence or lack of communication time-delays, and directed or undirected network information flow. In each case, we provide a convergence analysis. Moreover, we establish a connection between algebraic connectivity of the network and the performance of reaching an agreement. Furthermore, we demonstrate that the maximum time-delay that can be tolerated by a network of integrators applying a linear consensus protocol is inversely proportional to the largest eigenvalue of the network topology or the maximum degree of the nodes of the network. This naturally led to the realization that there exists a fundamental *trade-off* between performance of reaching a consensus and robustness to time-delays.

R. Olfati-Saber is a Postdoctoral Scholar at the Department of Control and Dynamical Systems, at California Institute of Technology, Pasadena, CA 91125, E-mail:olfati@cds.caltech.edu.

R. M. Murray is a Professor of Mechanical Engineering and Control and Dynamical Systems at California Institute of Technology, Pasadena, CA, 91125, E-mail: murray@caltech.edu.

In the past, a number of researchers have worked in problems that are essentially different forms of agreement problems with differences regarding the types of agent dynamics, the properties of the graphs, and the names of the tasks of interest. In [1], [12], *graph Laplacians* are used for the task of *formation stabilization* for groups of agents with linear dynamics. This particular method for formation stabilization has not yet been extended to systems with nonlinear dynamics that are not feedback linearizable. A special case of this approach is known as the *leader-follower architecture* and has been widely used by numerous researchers [13], [14], [15]. In [16], graph Laplacians are used as an essential part of a dynamic theory of graphs.

The problem of *synchronization of coupled oscillators* is closely related to consensus problems on graphs. This is a broad field that is of great interest to researchers in physics, biophysics, neurobiology, and systems biology [17],[18], [19]. In synchronization of coupled oscillators, a consensus is reached regarding the frequency of oscillation of all agents.

In recent years, there has been a tremendous amount of renewed interest in *flocking/swarming* [20], [21], [22], [23], [24], [25], [26], [27] that has been primarily originated from the pioneering work of Reynolds. In [7], *alignment* of heading angles for multiple particles is analyzed from the point of view of statistical mechanics. Moreover, a phase transition phenomenon is observed that occurs when the network topology becomes connected by increasing the density of agents in a bounded region. The work in [28] focuses on attitude alignment on undirected graphs in which the agents have simple dynamics motivated by the model used in [7]. It is shown that the connectivity of the graph on average is sufficient for convergence of the heading angles of the agents. In [29], the authors provide a convergence analysis of linear and nonlinear protocols for undirected networks in presence or lack of communication time-delays. Theoretically, the convergence analysis of consensus protocols on digraphs (or directed graphs) is more challenging than the case of undirected graphs. This is partly due to the fact that the properties of graph Laplacians are mostly known for undirected graphs and, as a result, an algebraic theory of digraphs is practically a nonexistent theory. Here, our main focus is analysis of consensus protocols on directed networks with fixed/switching topology.

In this paper, our analysis relies on several tools from algebraic graph theory [30], [31], matrix theory [32], and control theory. We establish a connection between the performance of a linear consensus protocol on a directed network and the Fiedler eigenvalue of the mirror graph of the information flow (obtained via a mirror operation).

It turns out that a class of directed graphs called balanced graphs have a crucial role in derivation of an invariant quantity and a Lyapunov function for convergence analysis of average-consensus problems on directed graphs. This Lyapunov function, called the disagreement function, is a measure of group disagreement in a network. We show that a directed graph solves the average-consensus problem using a linear protocol if and only if it is balanced. Furthermore, we use properties of balanced networks to analyze the convergence of an agreement protocol for networks with switching topology.

The variation of the network topology is usually due to link failures or creations in networks with mobile nodes. We introduce a common Lyapunov function that guarantees asymptotic convergence to a group decision value in networks with switching topology. Finally, we analyze the effects of communication time-delays in undirected networks with fixed topology. We provide a direct connection between the robustness margin to time-delays and the maximum eigenvalue of the network topology.

An outline of this paper is as follows. In Section II, we define consensus problems on graphs. In Section III, we give two protocols. In Section IV, the network dynamics is given for the cases of fixed and switching topologies and the relation to graph Laplacians is explained. Some background on algebraic graph theory and matrix theory related to the properties of graph Laplacians are provided in Section V. A counterexample is given in Section VI that shows there exists a strongly connected digraph that does not solve an average-consensus problem. In Section VII, balanced graphs are defined and our results on directed networks with fixed topology are stated. In Section VIII, mirror graphs are defined and used to determine the performance (or speed of convergence) of a consensus protocol on digraphs and define the algebraic connectivity of digraphs. In Section IX, our main results on networks with switching topology are presented. Average-consensus problems for networks with communication time-delays is discussed in Section X. The simulation results are presented in Section XI. Finally, in Section XII, concluding remarks are stated.

II. Consensus Problems on Graphs

Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph (or directed graph) of order n with the set of nodes $\mathcal{V} = \{v_1, \ldots, v_n\}$, set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a wighted adjacency matrix $\mathcal{A} = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. The node indices belong to a finite index set $\mathcal{I} = \{1, 2, \ldots, n\}$. An edge of G is denoted by $e_{ij} = (v_i, v_j)$. The adjacency elements associated with the edges of the graph are positive, i.e. $e_{ij} \in \mathcal{E} \iff a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of *neighbors* of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. A *cluster* is any subset $J \subseteq \mathcal{V}$ of the nodes of the graph. The set of neighbors of a cluster N_J is defined by

$$N_J := \bigcup_{v_i \in J} N_i = \{v_j \in \mathcal{V} : v_i \in J, (v_i, v_j) \in \mathcal{E}\} \tag{1}$$

Let $x_i \in \mathbb{R}$ denote the value of node v_i. We refer to $G_x = (G, x)$ with $x = (x_1, \ldots, x_n)^T$ as a *network* (or *algebraic graph*) with value $x \in \mathbb{R}^n$ and *topology* (or *information flow*) G. The value of a node might represent physical quantities including attitude, position, temperature, voltage, and so on. We say nodes v_i and v_j *agree* in a network if and only if $x_i = x_j$. We say the nodes of a network have reached a *consensus* if and only if $x_i = x_j$ for all $i, j \in \mathcal{I}, i \neq j$. Whenever the nodes of a network are all in agreement, the common value of all nodes is called the *group decision value*.

Suppose each node of a graph is a *dynamic agent* with dynamics

$$\dot{x}_i = f(x_i, u_i), \quad i \in \mathcal{I}. \tag{2}$$

A *dynamic graph* (or *dynamic network*) is a dynamical system with a state (G, x) in which the value x evolves according to the *network dynamics* $\dot{x} = F(x, u)$. Here, $F(x, u)$ is the column-wise concatenation of the elements $f(x_i, u_i)$ for $i = 1, \ldots, n$. In a dynamic network with switching topology, the information flow G is a discrete-state of the system that changes in time.

Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables $x_1, \ldots, x_n$ and $a = x(0)$ denote the initial state of the system. The *χ-consensus problem* in a dynamic graph is a distributed way to calculate $\chi(a)$ by applying inputs u_i that only depend on the states of node v_i and its neighbors. We say a state feedback

$$u_i = k_i(x_{j_1}, \ldots, x_{j_{m_i}}) \tag{A}$$

is a *protocol* with topology G if the cluster $J_i = \{v_{j_1}, \ldots, v_{j_m}\}$ of nodes with indices $j_1, \ldots, j_{m_i} \in \mathcal{I}$ satisfies the property $J_i \subseteq \{v_i\} \cup N_i$. In addition, if $|J_i| < n$ for all $i \in \mathcal{I}$, (A) is called a *distributed protocol*.

We say protocol (A) asymptotically solves the χ-consensus problem if and only if there exists an asymptotically stable equilibrium x^* of $\dot{x} = F(x, k(x))$ satisfying $x_i^* = \chi(x(0))$ for all $i \in \mathcal{I}$. We are interested in distributed solutions of the χ-consensus problem in which no node is connected to all other nodes. The special cases with $\chi(x) = \text{Ave}(x) = \frac{1}{n}(\sum_{i=1}^n x_i)$, $\chi(x) = \max_i x_i$, and $\chi(x) = \min_i x_i$ are called *average-consensus*, *max-consensus*, and *min-consensus*, respectively, due to their broad applications in distributed decision-making for multi-agent systems.

Solving the average-consensus problem is an example of *distributed computation* of a linear function $\chi(a) = \text{Ave}(a)$ using a network of dynamic systems (or integrators). This is a more challenging task than reaching a consensus with initial state a. Since an extra condition $x_i^* = \chi(a), \forall i \in \mathcal{I}$ has to be satisfied which relates the limiting state x^* of the system to the initial state a.

III. Consensus Protocols

In this section, we present two consensus protocols that solve agreement problems in a network of continuous-time (CT) integrator agents with dynamics

$$\dot{x}_i(t) = u_i(t) \tag{3}$$

or agents with discrete-time (DT) model

$$x_i(k+1) = x_i(k) + \epsilon u_i(k) \tag{4}$$

and step-size $\epsilon > 0$. We consider two scenarios:

i) Fixed or switching topology and zero communication time-delay: The following linear consensus protocol is used:

$$u_i = \sum_{v_j \in N_i} a_{ij}(x_j - x_i) \tag{A1}$$

where the set of neighbors $N_i = N_i(G)$ of node v_i is variable in networks with switching topology.

ii) Fixed topology $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and communication time-delay $\tau_{ij} > 0$ corresponding to the edge $e_{ij} \in \mathcal{E}$: We use the following linear time-delayed consensus protocol:

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}[x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})] \tag{A2}$$

The primary objective in this paper is *analysis* of protocols (A1) and (A2) for the aforementioned scenarios. We show that in each case consensus is asymptotically reached. We also characterize the class of digraphs that solve the average-consensus problem using protocol (A1). Furthermore, we provide results that directly relate performance and algorithmic robustness of these consensus protocols to the eigenvalues of the network topology.

Remark 1. In [29], the authors have introduced a Lyapunov-based method for convergence analysis of the following nonlinear consensus protocol

$$u_i = \sum_{v_j \in N_i} \phi_{ij}(x_j - x_i), \tag{A3}$$

for undirected networks. Here, ϕ_{ij}'s are continuous $\mathbb{R} \to \mathbb{R}$ mappings with $\phi_{ij}(z) = \phi_{ji}(z)$ for all $e_{ij} \in \mathcal{E}$ which satisfy the following properties: i) ϕ_{ij} is locally Lipschitz, ii) $\phi_{ij}(z) = 0 \iff z = 0$, iii) $z\phi_{ij}(z) > 0, \forall z \neq 0$. The convergence analysis of protocol (A3) is very similar to the proof of Theorem 8 and is omitted from this paper due to the limitation of space.

The reader might wonder whether protocol (A1) is an *ad-hoc* protocol, or it can be analytically derived. For undirected networks, there exists a derivation of this protocol that can be summarized as follows. Define the *Laplacian potential* associated with the undirected graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ as

$$\Phi_G(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_j - x_i)^2 \tag{5}$$

and notice that the gradient-based feedback $u = -\frac{1}{2}\nabla\Phi_G(x)$ is identical to protocol (A1). As a result, the network dynamics for integrator agents applying protocol (A1) is in the form

$$\dot{x} = -\frac{1}{2}\nabla\Phi_G(x) \tag{6}$$

that is a *gradient system* (up to a fixed time-scaling) that is induced by graph G. The same argument is not applicable to the case of digraphs. This is a reason that the analysis in the case of directed networks is more challenging. For graphs with 0-1 adjacency elements, the potential function in (5) is the same as the Laplacian potential introduced in [29] (up to a positive factor) as a measure of group disagreement.

A. Communication/Sensing Cost of Protocols

An important aspect of performing coordinated tasks in a distributed fashion in multi-agent systems is to keep communication and inter-agent sensing costs limited. We define the *communication/sensing cost* C of the topology $(\mathcal{V}, \mathcal{E})$ of a protocol as $\mathcal{E}$, or the total number of the directed edges of the graph $(\mathcal{V}, \mathcal{E})$. In the work of Klavins [33], C is called "communication complexity" of performing a task. For weighted digraphs, the communication/sensing cost can be defined as a function of the adjacency elements by

$$C = \sum_{i,j=1}^{n} \mathrm{sgn}(a_{ij}) \tag{7}$$

where $\mathrm{sgn}(\cdot)$ is the sign function (i.e. $\mathrm{sgn}(z) = 0$ for $z = 0$ and $\mathrm{sgn}(z) = z/|z|$, otherwise). According to this definition, C is the same as $|\mathcal{E}|$ for a digraph.

Apparently, the communication/sensing cost of protocols with directed information flow is smaller than the communication/sensing cost of their undirected counterparts. This is our primary reason for the analysis of consensus protocols for digraphs.

An alternative reason for considering consensus problems on digraphs is multi-agent flocking. In [6], the information flow in a flock is directed and the topology of the network of agents goes through changes that are discrete-event type in nature.

Remark 2. Given a bounded communication cost C, the problem of choosing the weights a_{ij} in protocol (A1) such that a certain performance index is maximized (or minimized) is an optimization problem that falls within the category of *network design problems*. We refer the reader to the work of Xiao and Boyd [34] for a network design problem for reaching average-consensus using a semi-definite programming approach. The framework presented in [34] partially relies on the work in [29] that introduced average-consensus for networks of integrators.

IV. Network Dynamics

Given Protocol (A1), the state of a network of continuous-time integrator agents evolves according to the following linear system

$$\dot{x}(t) = -Lx(t) \tag{8}$$

where L is called the *graph Laplacian* induced by the information flow G and is defined by

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases} \tag{9}$$

Apparently, the stability properties of system (8) depends on the location of the eigenvalues of the graph Laplacian L. Spectral properties of graphs is among the main topics of interest in algebraic graph theory [30], [31]. The basic properties of graph Laplacians that are used here are discussed in Section V.

In a network with *switching topology*, convergence analysis of Protocol (A1) is equivalent to stability analysis for a *hybrid system*

$$\dot{x}(t) = -L_k x(t), \quad k = s(t) \tag{10}$$

where $L_k = \mathcal{L}(G_k)$ is the Laplacian of graph G_k that belongs to a set Γ. The set Γ is a finite collection of digraphs of order n with an index set $\mathcal{I}_\Gamma \subset \mathbb{Z}$. The map $s(t) : \mathbb{R} \to \mathcal{I}_\Gamma$ is a *switching signal* that determines the network topology.

In Section IX, we will see that Γ is a relatively large set for $n \gg 1$. The task of stability analysis for the hybrid system in (10) is rather challenging. One of the reasons is that the product of two Laplacian matrices do not commute in general.

For agents with discrete-time models, applying protocol (A1) gives the following discrete-time network dynamics

$$x(k+1) = P_\epsilon x(k) \tag{11}$$

with

$$P_\epsilon = I - \epsilon L. \tag{12}$$

Let $d_{max} = \max_i l_{ii}$ denote the maximum node out-degree of digraph G. Then, P_ϵ is a nonnegative and stochastic matrix for all $\epsilon \in (0, 1/d_{max})$. We refer to P_ϵ as the *Perron matrix* induced by G.

The convergence analysis of Protocol (A1) for discrete-time agents heavily relies on the theory of nonnegative matrices [35], [32] and will be discussed in a separate paper. Our approach presents a Lyapunov-based convergence analysis for agreement in networks with discrete-time models. This is different than the approach pursued in the work of Jadbabaie *et al.* which strongly relies on matrix theoretic properties and infinite right-convergent products (RCP) of stochastic matrices [36].

V. Algebraic Graph Theory and Matrix Theory

In this section, we introduce some basic concepts and notation in graph theory that will be used throughout the paper. More information is available in [31], [37]. A comprehensive survey on properties of Laplacians of undirected graphs can be found in [38]. However, we need to use some basic properties of Laplacians of digraphs. These properties cannot be found in the graph theory literature and will be stated here.

Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph (or digraph) with n nodes. The in-degree and out-degree of node v_i are, respectively, defined as follows:

$$\deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji},\ \deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}. \tag{13}$$

For a graph with 0-1 adjacency elements, $\deg_{out}(v_i) = |N_i|$. The *degree matrix* of the digraph G is a diagonal matrix $\Delta = [\Delta_{ij}]$ where $\Delta_{ij} = 0$ for all $i \neq j$ and $\Delta_{ii} = \deg_{out}(v_i)$. The *graph Laplacian* associated with the digraph G is defined as

$$\mathcal{L}(\mathcal{G}) = L = \Delta - \mathcal{A}. \tag{14}$$

This definition is consistent with the definition of L in (9).

Remark 3. The graph Laplacian L does not depend on the diagonal elements a_{ii} of the adjacency matrix of G. These diagonal elements correspond to the weights of *loops* (v_i, v_i) (i.e. cycles of length one) in a graph. We assume $a_{ii} = 0$ for all i, unless stated otherwise.

For undirected graphs, the Laplacian potential defined in (5) can be expressed as a quadratic form with a kernel L, or

$$\Phi_G(x) = x^T L x = \frac{1}{2} \sum_{i,j} a_{ij}(x_j - x_i)^2. \tag{15}$$

This shows that the Laplacian of an undirected graph is positive semidefinite. This positive definiteness of L does not necessarily hold for digraphs. As an example, consider a digraph G_2 with two nodes and an adjacency matrix and graph Laplacian given by

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ L = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{16}$$

We have $\Phi_{G_2}(x) = x^T L x = x_1^2 - x_1 x_2$ that is a sign-indefinite quadratic form.

By definition, every row sum of the Laplacian matrix is zero. Therefore, the Laplacian matrix always has a zero eigenvalue corresponding to a right eigenvector

$$w_r = \mathbf{1} = (\mathbf{1}, \mathbf{1}, \dots, \mathbf{1})^{\mathbf{T}}$$

with identical nonzero elements. This means that $rank(L) \leq n - 1$.

A digraph is called *strongly connected (SC)* if and only if any two distinct nodes of the graph can be connected via a path that follows the direction of the edges of the digraph. The following theorem establishes a direct relation between the SC property of a digraph and the rank of its Laplacian. According to the following theorem, the Laplacian of a strongly connected digraph has an isolated eigenvalue at zero.

Theorem 1. *Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph with Laplacian L. If G is strongly connected, then rank$(L) = n - 1$.*

Proof: See the Appendix. □

Remark 4. For an undirected graph G, Theorem 1 can be stated as follows: G is connected if and only if rank$(L) = n - 1$. The proof for the *undirected case* is available in the literature [30], [31]. The opposite side of Theorem 1 does not hold. A counterexample is the digraph G_2 specified in equation (16). Clearly, G_2 is not strongly connected because there is no path connecting node v_2 to node v_1. But rank$(L) = 1 = n - 1$.

For a connected graph G that is *undirected*, the following well-known property holds [31]:

$$\min_{\substack{x \neq 0 \\ \mathbf{1^T x = 0}}} \frac{x^T L x}{\|x\|^2} = \lambda_2(L) \tag{17}$$

The proof follows from a special case of Courant–Fischer Theorem in [32]. We will later establish a connection between $\lambda_2(\hat{L})$ with $\hat{L} = (L + L^T)/2$, called the *Fiedler eigenvalue* of $\hat{L}$ [39] and the *performance* (i.e. worst-case speed of convergence) of protocol (A1) on digraphs.

Remark 5. The notion of *algebraic connectivity* (or λ_2) of graphs was originally defined by M. Fiedler for undirected graphs [39]. We extend this notion to *algebraic connectivity of digraphs* by defining the mirror operation on digraphs that produces an undirected graph $\hat{G}$ from a digraph G (See Definition 2).

The key in the stability analysis of system (8) is in the spectral properties of graph Laplacian. The following result is well-known for undirected graphs (e.g. see [38]). Here, we state the result for digraphs and prove it using Geršgorin disk theorem [32].

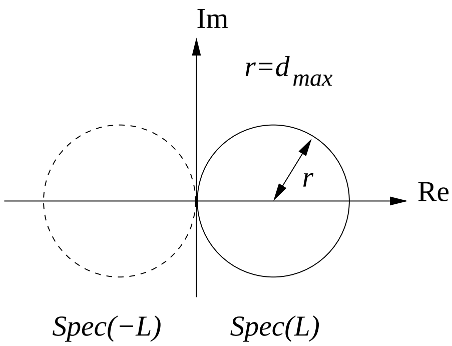

Fig. 1. A demonstration of Geršgorin Theorem applied to graph Laplacian.

Theorem 2. *(spectral localization) Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a digraph with the Laplacian L. Denote the maximum node out-degree of the digraph G by $d_{max}(G) = \max_i \deg_{out}(v_i)$. Then, all the eigenvalues of $L = \mathcal{L}(G)$ are located in the following disk*

$$D(G) = \{z \in \mathbb{C} : |z - d_{max}(G)| \leq d_{max}(G)\} \tag{18}$$

centered at $z = d_{max}(G) + 0j$ in the complex plane (see Figure 1).

Proof: Based on the Geršgorin disk theorem, all the eigenvalues of $L = [l_{ij}]$ are located in the union of the following n disks

$$D_i = \{z \in \mathbb{C} : |z - l_{ii}| \leq \sum_{j \in \mathcal{I}, j \neq i} |l_{ij}|\}. \tag{19}$$

But for the digraph G, $l_{ii} = \Delta_{ii}$ and

$$\sum_{j \in \mathcal{I}, j \neq i} |l_{ij}| = \deg_{out}(v_i) = \Delta_{ii}.$$

Thus, $D_i = \{z \in \mathbb{C} : |z - \Delta_{ii}| \leq \Delta_{ii}\}$. On the other hand, all these n disks are contained in the largest disk $D(G)$ with radius $d_{max}(G)$. Clearly, all the eigenvalues of $-L$ are located in the disk $D'(G) = \{z \in \mathbb{C} : |z + d_{max}(G)| \leq d_{max}(G)\}$ that is the mirror image of $D(G)$ with respect to the imaginary axis. □

Here is an immediate corollary and the first convergence proof for protocol (A1) for a directed network with fixed topology G.

Corollary 1. *Consider a network of integrators $\dot{x}_i = u_i$ where each node applies protocol* (A1). *Assume G is a strongly connected digraph. Then, protocol* (A1) *globally asymptotically solves a consensus problem.*

Proof: Since G is strongly connected, rank$(L) = n - 1$ and L has a simple eigenvalue at zero. Based on Theorem 2, the rest of the eigenvalues of $-L$ have negative real-parts and therefore the linear system in (8) is stable. On the other hand, any equilibrium x^* of (8) is a right eigenvector of L associated with $\lambda = 0$. Since the eigenspace associated with the zero eigenvalue is one-dimensional, there exists an $\alpha \in \mathbb{R}$ such that $x^* = \alpha\mathbf{1}$, i.e. $x_i^* = \alpha$ for all i. □

Keep in mind that Corollary 1 does not guarantee whether the group decision value α is equal to Ave$(x(0))$, or not. In other words, Corollary 1 does not necessarily address the average-consensus problem.

We need to provide a limit theorem for exponential matrices of the form $\exp(-Lt)$. Considering that the solution of (8) with fixed topology is given by

$$x(t) = \exp(-Lt)x(0), \tag{20}$$

by explicit calculation of $\exp(-Lt)$, one can obtain the group decision value for a general digraph. The following theorem is closely related to a famous limit theorem in the theory of nonnegative matrices known as the *Perron-Frobenius Theorem* [32]. We will use this theorem for characterization of the class of digraphs that solve average-consensus problems using protocol (A1).

Notation. Following the notation in [32], we denote the set of $m \times n$ real matrices by $M_{m,n}$ and the set of square $n \times n$ matrices by M_n. Furthermore, throughout this paper, the right and left eigenvectors of the Laplacian L associated with $\lambda_1 = 0$ are denoted by w_r and w_l, respectively.

Theorem 3. *Assume G is a strongly connected digraph with Laplacian L satisfying $Lw_r = 0$, $w_l^T L = 0$, and $w_l^T w_r = 1$. Then*

$$R = \lim_{t \to +\infty} \exp(-Lt) = w_r w_l^T \in M_n \tag{21}$$

Proof: Let $A = -L$ and let J be the Jordan form associated with A, i.e. $A = SJS^{-1}$. We have $\exp(At) = S\exp(Jt)S^{-1}$ and as $t \to +\infty$, $\exp(Jt)$ converges to a matrix $Q = [q_{ij}]$ with a single nonzero element $q_{11} = 1$. The fact that other blocks in the diagonal of $\exp(Jt)$ vanish is due to the property that Re$(\lambda_k(A)) < 0$ for all $k \geq 2$ where $\lambda_k(A)$ is the kth largest eigenvalue of A in terms of magnitude $|\lambda_k|$. Notice that $R = SQS^{-1}$. Since $AS = SJ$ the first column of S is w_r. Similarly, $S^{-1}A = JS^{-1}$ that means the first row of S^{-1} is w_l^T. Due to the fact that $S^{-1}S = I$, w_l satisfies the property $w_l^T w_r = 1$ as stated in the question. A straightforward calculation shows that $R = w_r w_l^T \in M_n$. □

VI. A Counterexample for Average-Consensus

A sufficient condition for the decision value α of each node in the proof of Corollary 1 to be equal to Ave$(x(0))$ is that $\sum_{i=1}^n u_i \equiv 0$. If G is undirected (i.e. $a_{ij} = a_{ji} > 0, \forall i, j : a_{ij} \neq 0$), automatically the condition $\sum_{i=1}^n u_i = 0, \forall x$ holds and Ave$(x(t))$ is an invariant quantity [29]. However, this property does not hold for a general digraph.

A simple counterexample is a digraph of order $n = 3$ with

$$\mathcal{V} = \{v_1, v_2, v_3\}, \mathcal{E} = \{e_{12}, e_{23}, e_{31}, e_{13}\}$$

as shown in Figure 2. Assume the graph has 0-1 weights. Notice that G is a strongly connected digraph. Given $G = (\mathcal{V}, \mathcal{E})$, we

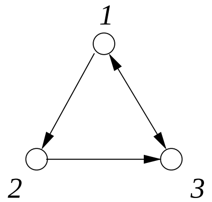

Fig. 2. A connected digraph of order 3 that does not solve the average-consensus problem using Protocol (A1).

have $\sum_{i=1}^{3} u_i = x_3 - x_1$. Thus, if nodes v_1 and v_3 disagree, the property $\sum_{i=1}^{3} u_i = 0$ does not hold for all x. On the other hand, the reader can verify that for this example

$$L = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Using Theorem 3, one obtains the limit $x_i^* = [x_1(0) + x_2(0) + 2x_3(0)]/4$ for $i = 1, 2, 3$. This group decision value is different from Ave$(x(0))$ if and only if $x_1(0) + x_2(0) \neq 2x_3(0)$. As a result, for all initial conditions satisfying $x_1(0) + x_2(0) \neq 2x_3(0)$, Protocol (A1) does not solve the average-consensus problem, but all nodes asymptotically reach a consensus. This motivates us to *characterize the class of all digraphs that solve the average-consensus problem*.

VII. Networks with Fixed Topology and Balanced Graphs

The following class of digraphs turns out to be instrumental in solving average-consensus problems for networks with both fixed and switching topologies:

Definition 1. (balanced graphs) We say the node v_i of a digraph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is *balanced* if and only if its in-degree and out-degree are equal, i.e. $\deg_{out}(v_i) = \deg_{in}(v_i)$. A graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is called *balanced* if and only if all of its nodes are balanced, or

$$\sum_j a_{ij} = \sum_j a_{ji}, \forall i. \tag{22}$$

Any undirected graph is balanced. Furthermore, the digraphs shown in Figure 3 are all balanced. Here is our first main result:

Theorem 4. *Consider a network of integrators with a fixed topology $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that is a strongly connected digraph. Then, protocol* (A1) *globally asymptotically solves the average-consensus problem if and only if G is balanced.*

Proof: The proof follows from Theorems 5 and 6, below. □

Remark 6. According to Theorem 4, if a graph is not balanced, then protocol (A1) does not (globally) solve the average-consensus problem for all initial conditions. This assertion is consistent with the counterexample given in Figure 2.

Theorem 5. *Consider a network of integrator agents with a fixed topology $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that is a strongly connected digraph. Then, protocol* (A1) *globally asymptotically solves the average-consensus problem if and only if $\mathbf{1^T L} = \mathbf{0}$.*

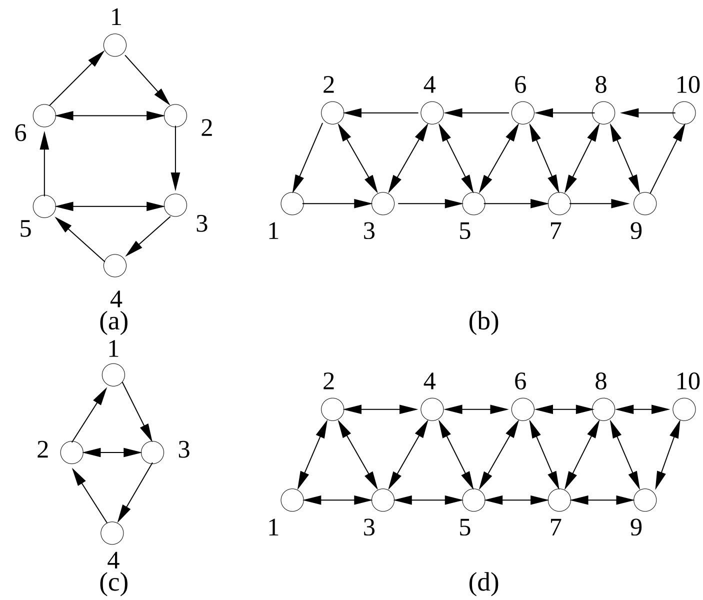

Fig. 3. Four examples of balanced graphs.

Proof: From Theorem 3, with $w_r = \frac{1}{\sqrt{n}}\mathbf{1}$ we obtain

$$x^* = \lim_{t \to +\infty} x(t) = Rx_0 = w_r(w_l^T x_0) = \frac{1}{\sqrt{n}}(w_l^T x_0)\mathbf{1}.$$

This implies Protocol 1 globally exponentially solves a consensus problem with the decision value $\frac{1}{\sqrt{n}}(w_l^T x_0)$ for each node. If this decision value is equal to Ave$(x_0), \forall x_0 \in \mathbb{R}^n$, then necessarily $\frac{1}{\sqrt{n}} w_l = \frac{1}{\sqrt{n}}$, i.e. $w_l = w_r = \frac{1}{\sqrt{n}}\mathbf{1}$. This implies that $\mathbf{1}$ is the left eigenvector of L. To prove the converse, assume that $\mathbf{1^T L} = \mathbf{0}$. Let us take $w_r = \frac{1}{\sqrt{n}}\mathbf{1}$, $w_l = \beta\mathbf{1}$ with $\beta \in \mathbb{R}, \beta \neq 0$. From condition $w_l^T w_r = 1$, we get $\beta = \frac{1}{\sqrt{n}}$ and $w_l = \frac{1}{\sqrt{n}}\mathbf{1}$. This means that the decision value for every node is $\frac{1}{\sqrt{n}}(w_l^T x_0) = \frac{1}{n}\mathbf{1^T x_0} = \text{Ave}(\mathbf{x_0})$. □

The following result provides the group decision value for arbitrary digraphs including the ones that are unbalanced.

Corollary 2. *Assume all the conditions in Theorem 5 hold. Suppose L has a nonnegative left eigenvector $\gamma = (\gamma_1, \ldots, \gamma_n)^T$ associated with $\lambda = 0$ that satisfies $\sum_i \gamma_i > 0$. Then, after reaching a consensus, the group decision value is*

$$\alpha = \frac{\sum_i \gamma_i x_i(0)}{\sum_i \gamma_i}, \tag{23}$$

i.e. the decision value belongs to the convex hull of the initial values.

Proof: Due to $\gamma^T L = 0$, we get $\gamma^T u \equiv 0$ (because $u = -Lx$). Hence, $\beta = \gamma^T x$ is an invariant quantity. Suppose the digraph G is not balanced. Then, an agreement is asymptotically reached. Let α be the decision value of all nodes after reaching a consensus. We have $\gamma^T x^* = \gamma^T x(0)$ because of the invariance of $\gamma^T x(t)$. But $x^* = \alpha\mathbf{1}$, thus we obtain

$$\left(\sum_i \gamma_i\right)\alpha = \gamma^T x(0)$$

and the result follows. □

The following result shows that if one of the agents uses a relatively small update rate (or step-size), then the group decision value will be relatively close to x_i^*. In other words, the agent v_{i^*} plays the role of a leader in a *leader-follower architecture*.

Corollary 3. *(multi-rate integrators) Consider a network of multi-rate integrators with the node dynamics*

$$\gamma_i \dot{x}_i = u_i, \quad \gamma_i > 0, \forall i \in \mathcal{I} \tag{24}$$

Assume the network has a fixed topology $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and each node applies Protocol (A1)*. Then, an agreement is globally asymptotically reached and the group decision value will be*

$$\alpha = \frac{\sum_i \gamma_i x_i(0)}{\sum_i \gamma_i} \tag{25}$$

Proof: The dynamics of the network evolves according to

$$D\dot{x} = -Lx$$

where $D = \text{diag}(\gamma)$ is a diagonal matrix with the ith diagonal element $\gamma_i > 0$. The last equation can be rewritten as

$$\dot{x} = -\tilde{L}x$$

where $\tilde{L} = D^{-1}L = \text{diag}(1/\gamma_1, \ldots, 1/\gamma_n)L$. Note that $\tilde{L}$ is a valid Laplacian matrix for a digraph $\tilde{G}$ with the adjacency matrix $\tilde{A} = D^{-1}\mathcal{A}$. To obtain $\tilde{G}$ from G, one needs to divide the weights of the edges leaving node v_i by γ_i. Clearly, γ is a vector with positive elements that is the left eigenvector of $\tilde{L}$ and based on Corollary 2 the decision value is in the weighted average of $x_i(0)$'s with the weights that are specified by γ. □

Remark 7. The discrete-time model and attitude alignment protocol discussed in Jadbabaie *et al.* [28] correspond to the first-order Euler approximation of equation (24) with protocol (A1) and the special choice of $\gamma_i = \deg_{out}(v_i) + 1$ in Corollary 3. In [1], a Laplacian matrix is defined as $I - D^{-1}\mathcal{A}$ which in the context of this paper is equivalent to a multi-rate network of integrators with $\gamma_i = \deg_{out}(v_i) \geq 0$. The singularity of D that is caused by the choice of $\gamma_i = \deg_{out}(v_i)$ is avoided in [28] by properly adding a positive constant to $\deg_{out}(v_i)$.

Theorem 6. *Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a digraph with an adjacency matrix $\mathcal{A} = [a_{ij}]$. Then, all the following statements are equivalent:*
i) G is balanced,
ii) $w_l = \mathbf{1}$ is the left eigenvector of the Laplacian of G associates with the zero eigenvalue, i.e. $\mathbf{1^T L} = \mathbf{0}$.
iii) $\sum_{i=1}^n u_i = 0, \forall x \in \mathbb{R}^n$ with $u_i = \sum_{v_j \in N_i} a_{ij}(x_j - x_i)$.

Proof: We show $i) \iff ii)$ and $ii) \iff iii)$.

Proof of $i) \iff ii)$: We have $\Delta_{ii} = \deg_{out}(v_i)$ and $\deg_{in}(v_i) = \sum_{j, j \neq i} a_{ji}$, thus the ith column sum of L is equal to zero, or

$$\sum_i l_{ji} = \sum_{i,j \neq i} l_{ji} + l_{ii} = -\deg_{in}(v_i) + \deg_{out}(v_i) = 0,$$

if and only if node v_i of G is balanced. Noting that the i column sum of L is the same as the ith element of the row vector $\mathbf{1^T L}$, one concludes that $\mathbf{1^T L} = \mathbf{0}$ iff all the nodes of G are balanced, i.e. G is balanced.

Proof of $ii) \iff iii)$: Since $u = -Lx$, $(\sum_i u_i = 0, \forall x) \Leftrightarrow (\mathbf{1^T u} = -(\mathbf{1^T L})\mathbf{x} = \mathbf{0}, \forall \mathbf{x}) \iff \mathbf{1^T L} = \mathbf{0}$. □

Notice that in Theorem 6, graph G does not need to be connected.

VIII. Performance of Protocols and Mirror Graphs

In this section, we discuss performance issues of Protocol (A1) with balanced graphs. An important consequence of Theorem 6 is that *for networks with balanced information flow, $\alpha = Ave(x)$ is an invariant quantity*. This is certainly not true for an arbitrary digraph. The invariance of Ave(x) allows decomposition of x according to the following equation:

$$x = \alpha\mathbf{1} + \delta \tag{26}$$

where $\alpha = \text{Ave}(x)$ and $\delta \in \mathbb{R}^n$ satisfies $\sum_i \delta_i = 0$. We refer to δ as the (group) *disagreement vector*. The vector δ is orthogonal to $\mathbf{1}$ and belongs to an $(n-1)$-dimensional subspace called the *disagreement eigenspace* of L provided that G is strongly connected. Moreover, δ evolves according to the (group) *disagreement dynamics* given by

$$\dot{\delta} = -L\delta. \tag{27}$$

Define the *Laplacian disagreement function* of a digraph G as

$$\Phi_G(x) = x^T Lx \tag{28}$$

with $L = \mathcal{L}(G)$. The Laplacian disagreement for digraphs is not necessarily nonnegative. An example of a digraph with a Laplacian disagreement that is sign-indefinite is given in equation (16).

In the following, we show that for any balanced digraph G, there exists an undirected graph $\hat{G}$ with a Laplacian disagreement function that is identical to the Laplacian disagreement of G. This proves that the Laplacian of balanced graphs is positive-semidefinite. Here is the definition of this induced undirected graph:

Definition 2. (mirror graph/operation) Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be weighted digraph. Let $\tilde{\mathcal{E}}$ be the set of *reverse edges* of G obtained by reversing the order of nodes of all the pairs in $\mathcal{E}$. The *mirror* of G denoted by $\hat{G} = \mathcal{M}(\mathcal{G})$ is an undirected graph in the form $\hat{G} = (\mathcal{V}, \hat{\mathcal{E}}, \hat{\mathcal{A}})$ with the same set of nodes as G, the set of edges $\hat{\mathcal{E}} = \mathcal{E} \cup \tilde{\mathcal{E}}$, and the symmetric adjacency matrix $\hat{\mathcal{A}} = [\hat{a}_{ij}]$ with elements

$$\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij} + a_{ji}}{2} \geq 0 \tag{29}$$

The following result shows that the operations of $\mathcal{L}$ and Sym on a weighted adjacency matrix $\mathcal{A}$ commute.

Theorem 7. *Let G be a digraph with adjacency matrix $\mathcal{A} = adj(G)$ and Laplacian $L = \mathcal{L}(G)$. Then $L_s = Sym(L) = (L + L^T)/2$ is a valid Laplacian matrix for $\hat{G} = \mathcal{M}(\mathcal{G})$ if and only if*

G is balanced, or equivalently the following diagram commutes if and only if G is balanced

$$\begin{array}{ccccc} G & \xrightarrow{adj} & \mathcal{A} & \xrightarrow{\mathcal{L}} & L \\ \mathcal{M}\big\downarrow & & Sym\big\downarrow & & Sym\big\downarrow \\ \hat{G} & \xrightarrow[adj]{} & \hat{\mathcal{A}} & \xrightarrow[\mathcal{L}]{} & \hat{L} \end{array} \tag{30}$$

Moreover, if G is balanced, the Laplacian disagreement functions of G and $\hat{G}$ are equal.

Proof: We know that G is balanced iff $\mathbf{1^T L} = \mathbf{0}$. Since $L\mathbf{1} = \mathbf{0}$, we have $\mathbf{1^T L} = \mathbf{0} \iff \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{L} + \mathbf{L^T})\mathbf{1} = \mathbf{0}$. Thus, G is balanced iff L_s has a right eigenvector of $\mathbf{1}$ associated with $\lambda = 0$, i.e. L_s is a valid Laplacian matrix. Now, we prove that $L_s = \mathcal{L}(\hat{G})$. For doing so, let us calculate $\hat{\Delta}$ element-wise, we get

$$\begin{aligned} \hat{\Delta}_{ii} &= \textstyle\sum_j \dfrac{a_{ij} + a_{ji}}{2} = \dfrac{1}{2}(\deg_{out}(v_i) + \deg_{in}(v_i)) \\ &= \deg_{out}(v_i) = \Delta_{ii} \end{aligned}$$

Thus, $\hat{\Delta} = \Delta$. On the other hand, we have

$$L_s = \frac{1}{2}(L + L^T) = \Delta - \frac{A + A^T}{2} = \hat{\Delta} - \hat{A} = \hat{L} = \mathcal{L}(\hat{G})$$

The last part simply follows from the fact that $\hat{L}$ is equal to the symmetric part of L and $x^T(L - L^T)x \equiv 0$. □

Notation. For simplicity of notation, in the context of algebraic graph theory, $\lambda_k(G)$ is used to denote $\lambda_k(\mathcal{L}(G))$.

Now, we are ready to present our main result on the performance of protocol (A1) in terms of the worst-case speed of reaching an agreement.

Theorem 8. *(performance of agreement) Consider a network of integrators with a fixed topology G that is a strongly connected digraph. Given protocol* (A1), *the following statements hold:*
i) the group disagreement (vector) δ, as the solution of the disagreement dynamics in (27), *globally asymptotically vanishes with a speed equal to $\kappa = \lambda_2(\hat{G})$, or the Fiedler eigenvalue of the mirror graph induced by G, i.e.*

$$\|\delta(t)\| \leq \|\delta(0)\| \exp(-\kappa t), \tag{31}$$

ii) the following smooth, positive definite, and proper function

$$V(\delta) = \frac{1}{2}\|\delta\|^2 \tag{32}$$

is a valid Lyapunov function for the disagreement dynamics.

Proof: We have

$$\begin{aligned} \dot{V} &= -\delta^T L\delta = -\delta^T L_s\delta = -\delta^T \hat{L}\delta \\ &\leq -\lambda_2(\hat{G})\|\delta\|^2 = -2\kappa V(\delta) < 0, \forall\delta \neq 0 \end{aligned} \tag{33}$$

This proves that $V(\delta)$ is a valid Lyapunov function for the group disagreement dynamics. Moreover, $\delta(t)$ vanishes globally exponentially fast with a speed of κ as $t \to +\infty$.. The fact that $L_s = \hat{L}$ is a valid Laplacian matrix of the undirected graph $\hat{G}$ (i.e. the mirror of G) is based on Theorem 7. In addition, the inequality

$$\delta^T \hat{L}\delta \geq \lambda_2(\hat{G})\|\delta\|^2, \quad \forall\delta : \mathbf{1^T}\delta = \mathbf{0} \tag{34}$$

follows from equation (17). □

A well-known observation regarding the Fiedler eigenvalue of an undirected graph is that for dense graphs λ_2 is relatively large and for sparse graphs λ_2 is relatively small [31]. This is why λ_2 is called the *algebraic connectivity* of the graph. According to this observation, from Theorem 8, one can conclude that a network with dense interconnections solves an agreement problem faster than a connected but sparse network. As a special case, a cycle of length n that creates a balanced digraph on n nodes solves an agreement problem. However, this is a relatively slow way to solve such a consensus problem.

IX. Networks with Switching Topology

Consider a network of mobile agents that communicate with each other and need to agree upon a certain objective of interest or perform synchronization. Since, the nodes of the network are moving, it is not hard to imagine that some of the existing communication links can fail simply due to the existence of an obstacle between two agents. The opposite situation can arise where new links between nearby agents are created because the agents come to an effective range of detection with respect to each other. In terms of the network topology G, this means that certain number of edges are added or removed from the graph. Here, we are interested to investigate that in case of a *network with switching topology* whether it is still possible to reach a consensus, or not.

Consider a *hybrid system* with a continuous-state $x \in \mathbb{R}^n$ and a discrete-state G that belongs to a finite collection of digraphs $\Gamma_n = \{G\}$ such that G is a digraph of order n that is *strongly connected* and *balanced*. This set can be analytically expressed as

$$\Gamma_n = \{G = (\mathcal{V}, \mathcal{E}, \mathcal{A}) : \text{rank}(\mathcal{L}(G)) = n - 1, \mathbf{1^T}\mathcal{L}(\mathbf{G}) = \mathbf{0}\}.$$

Given protocol (A1), the continuous-state of the system evolves according to the following dynamics

$$\dot{x}(t) = -\mathcal{L}(G_k)x(t), \quad k = s(t), G_k \in \Gamma_n \tag{35}$$

where $s(t) : \mathbb{R}_{\geq 0} \to \mathcal{I}_{\Gamma_n}$ is a *switching signal* and $\mathcal{I}_{\Gamma_n} \subset \mathbb{N}$ is the index set associated with the elements of Γ_n. The set Γ_n is finite because at most a graph of order n is complete and has $n(n-1)$ directed edges.

The key in our analysis for reaching an average-consensus in mobile networks with directed switching topology is a basic property of the disagreement function in (32). This disagreement function does not depend on the network topology G. Moreover, for all $k \in \mathcal{I}_{\Gamma_n}$, the Laplacian of the digraph G_k is positive semi-definite because G_k is balanced. Thus, $V(\delta)$ is non-increasing along the solutions of the switching system. This property of $V(\delta)$ makes it an appropriate candidate as a *common Lyapunov function* for stability analysis of the switching system (35).

Theorem 9. *For any arbitrary switching signal $s(\cdot)$, the solution of the switching system* (35) *globally asymptotically converges to Ave$(x(0))$ (i.e. average-consensus is reached). Moreover, the following smooth, positive definite, and proper function*

$$V(\delta) = \frac{1}{2}\|\delta\|^2 \tag{36}$$

is a valid common Lyapunov function for the disagreement dynamics given by

$$\dot{\delta}(t) = -\mathcal{L}(G_k)\delta(t), \quad k = s(t), G_k \in \Gamma_n. \tag{37}$$

Furthermore, the inequality $\|\delta(t)\| \leq \|\delta(0)\| \exp(-\kappa^* t)$ holds, i.e. the disagreement vector δ vanishes exponentially fast with the least rate of

$$\kappa^* = \min_{G \in \Gamma_n} \lambda_2(\mathcal{L}(\hat{G}))). \tag{38}$$

Proof: Due to the fact that G_k is balanced for all k and $u = -\mathcal{L}(G_k)x$, we have $\mathbf{1^T u} = -(\mathbf{1^T}\mathcal{L}(\mathbf{G_k}))\mathbf{x} \equiv \mathbf{0}$. Thus, $\alpha = \text{Ave}(x)$ is an invariant quantity. This allows the decomposition of x in the form $x = \alpha\mathbf{1} + \delta$. Therefore, the disagreement switching system induced by (35) takes the form (37). Calculating $\dot{V}$, we get

$$\begin{array}{rcl}\dot{V} & = & -\delta^T\mathcal{L}(G_k)\delta = -\delta^T\mathcal{L}(\hat{G}_k)\delta \leq -\lambda_2(\mathcal{L}(\hat{G}_k))\|\delta\|^2 \\ & \leq & -\kappa^*\|\delta\|^2 = -2\kappa^* V(\delta) < 0, \forall \delta \neq 0\end{array} \tag{39}$$

This guarantees that $V(\delta)$ is a valid common Lyapunov function for the disagreement switching system in equation (37). Moreover, we have

$$V(\delta(t)) \leq V(\delta(0))\exp(-2\kappa^* t) \Rightarrow \|\delta(t)\| \leq \|\delta(0)\|\exp(-\kappa^* t)$$

and the disagreement vector $\delta(t)$ globally exponentially vanishes with a speed of $\kappa^* > 0$ as $t \to +\infty$. The minimum in (38) always exists and is achieved because Γ_n is a finite set. □

X. Networks with Communication Time-Delays

Consider a network of continuous-time integrators with a fixed topology $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ in which the state of node v_i passes through a communication channel e_{ij} with time-delay $\tau_{ij} > 0$ before getting to node v_j. The transfer function associated with the edge e_{ij} can be expressed as

$$h_{ij}(s) = e^{-\tau_{ij}s}$$

in the Laplace domain. Given protocol (A2), the network dynamics can be written as

$$\dot{x}_i(t) = \sum_{v_j \in N_i} a_{ij}[x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})]. \tag{40}$$

After taking the Laplace transform of both sides of equation (40), we get

$$sX_i(s) - x_i(0) = \sum_{j \in N_i} a_{ij}h_{ij}(s)(X_j(s) - X_i(s)) \tag{41}$$

where $X_i(s)$ denotes the Laplace transform of $x_i(t)$ for all $i \in \mathcal{I}$. The last set of equations can be rewritten in a compact form as

$$X(s) = (sI + L(s))^{-1}x(0) \tag{42}$$

where $L(s)$ is the Laplacian matrix of a graph with adjacency matrix $\mathcal{A}(s) = [a_{ij}h_{ij}(s)]$. Any linear filtering effects of channel e_{ij} can be incorporated in the transfer function $h_{ij}(s)$ of the link. The convergence analysis of protocol (A2) for a network of integrator agents with communication time-delays reduces to stability analysis for a MIMO transfer function

$$G(s) = (sI + L(s))^{-1}. \tag{43}$$

To gain further insight in the relation between the graph Laplacian and the convergence properties of consensus protocol (A2), we focus on the simplest possible case where the time-delays in all channels are equal to $\tau > 0$ and the network topology is fixed and undirected. Immediately, it follows that $\sum_i u_i \equiv 0$ and thus $\alpha = \text{Ave}(x(t))$ is an invariant quantity. In addition, we have

$$L(s) = e^{-\tau s}L$$

where $L = \mathcal{L}(G)$. Here is our main result for average-consensus in a network with communication time-delays and fixed topology [29]:

Theorem 10. *Consider a network of integrator agents with equal communication time-delay $\tau > 0$ in all links. Assume the network topology G is fixed, undirected, and connected. Then, protocol* (A2) *with $\tau_{ij} = \tau$ globally asymptotically solves the average-consensus problem if and only if either of the following equivalent conditions are satisfied:*
i) $\tau \in (0, \tau^*)$ with $\tau^* = \pi/2\lambda_n, \lambda_n = \lambda_{max}(L)$.
ii) The Nyquist plot of $\Gamma(s) = e^{-\tau s}/s$ has a zero encirclement around $-1/\lambda_k, \forall k > 1$.
Moreover, for $\tau = \tau^*$ the system has a globally asymptotically stable oscillatory solution with frequency $\omega = \lambda_n$.

Proof: See the Appendix. □

A. Trade-Off Between Performance and Robustness

Based on part i) of Theorem 10, one concludes that the upper bound on the admissible channel time-delay in the network is inversely proportional to λ_n, i.e. the largest eigenvalue of the Laplacian of the information flow.

From Geršgorin theorem, we know that $\lambda_n \leq 2d_{max}(G)$ where $d_{max}(G)$ is the maximum out-degree of the nodes of G. Therefore, a sufficient condition for convergence of protocol (A2) is

$$\tau \leq \frac{\pi}{4d_{max}(G)} \tag{44}$$

This means that networks with nodes that have relatively high out-degrees cannot tolerate relatively high communication time-delays. On the other hand, let $\tilde{\mathcal{A}} = k\mathcal{A}$ with $k > 0$ be the adjacency matrix of $\tilde{G}$. Denote the Laplacian of $\tilde{G}$ by $\tilde{L}$ and notice that $\lambda_n(\tilde{L}) = k\lambda_n(L)$. Thus, for any arbitrary delay $\tau > 0$, there exists a sufficiently small $k > 0$ such that $\tau < \pi/(2k\lambda_n)$. As a result, by scaling down the weights of a digraph, an arbitrary large time-delay $\tau > 0$ can be tolerated. The trade-off is that the negotiation speed, or λ_2, degrades by a factor of $1/k > 0$. In other words, there is a *trade-off* between robustness of a protocol to time-delays and its performance.

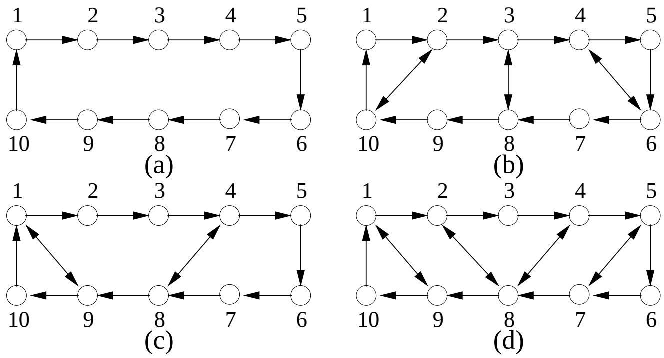

Fig. 4. Four examples of balanced and strongly connected digraphs: (a) G_a, (b) G_b, (c) G_c, and (d) G_d.

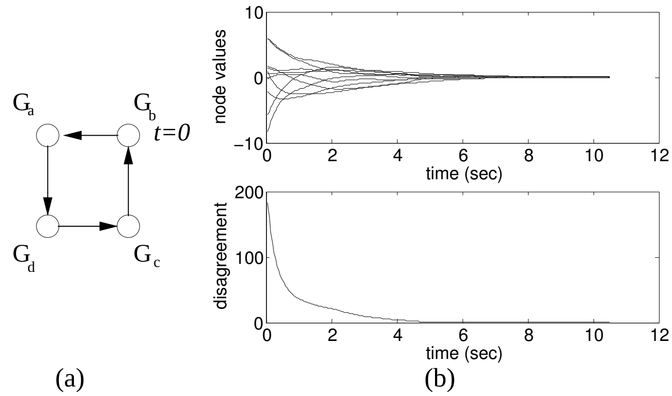

Fig. 5. (a) A finite automaton with four states representing the discrete-states of a network with switching topology and (b) trajectory of the node values and the group disagreement for a network with a switching information flow.

B. *Trade-Off Between High Performance and Low Communication Cost*

For undirected graphs with 0-1 weights, a graph with a relatively high communication cost C is expected to have a relatively high algebraic connectivity λ_2 (e.g. a complete graph). In contrast, a graph with a relatively low communication cost is expected to have a relatively low λ_2 (e.g. a cycle). This implies that there is another *trade-off* between performance and communication cost. This second trade-off is between achieving a high performance and maintaining a low communication cost.

The existence of the aforementioned two trade-offs suggests posing and addressing a *network design problem* that attempts to find an adjacency matrix $\mathcal{A}$ with a bounded communication cost C that attempts to achieve a balanced interplay between performance and robustness (see Remark 2).

XI. Simulation Results

Figure 4 shows four different networks each with $n = 10$ nodes. All digraphs in this figure have 0-1 weights. Moreover, they are all strongly connected and balanced. In Figure 5(a), a finite automaton is shown with the set of states $\{G_a, G_b, G_c, G_d\}$ representing the discrete-states of a network with switching topology as a hybrid system. The hybrid system starts at the discrete-state G_b and switches every $T = 1$ second to the next state according to the state machine in Figure 5(a). The continuous-time state trajectories and the group disagreement (i.e. $\|\delta\|^2$) of the network are shown in Figure 5(b). Clearly, the group disagreement is monotonically decreasing. One can observe that an average-consensus is reached asymptotically. Moreover, the group disagreement vanishes exponentially fast.

For a random initial state satisfying $\text{Ave}(x(0)) = 0$, the state trajectories of the system and the disagreement function $\|\delta\|^2$ in time are shown in Figure 6 for four digraphs. It is clear that as the number of the edges of the graph increases, algebraic connectivity (or λ_2) increases, and the settling time of the state trajectories decreases.

The case of a directed cycle of length 10, or G_a, has the highest over-shoot. In all four cases, a consensus is asymptotically reached and the performance is improved as a function of $\lambda_2(\hat{G}_k)$ for $k \in \{a, b, c, d\}$.

Next, we present simulation results for the average-consensus problem with communication time-delay for a network with a topology shown in Figure 3(d). Figure 7 shows the state trajectories of this network with communication time-delay τ for $\tau = 0, 0.5\tau_{max}, 0.7\tau_{max}, \tau_{max}$ with $\tau_{max} = \pi/2\lambda_{max}(G_e) = 0.269$. Here, the initial state is a random set of numbers with zero-mean. Clearly, the agreement is achieved for the cases with $\tau < \tau_{max}$ in Figures 7(a), (b), and (c). For the case with $\tau = \tau_{max}$, synchronous oscillations are illustrated in Figure 7(d). A second-order Pade approximation is used to model the time-delay as a finite-order LTI system.

XII. Conclusions

We provided the convergence analysis of a consensus protocol for a network of integrators with directed information flow and fixed/switching topology. Our analysis relies on several tools from algebraic graph theory, matrix theory, and control theory. We established a connection between the performance of a linear consensus protocol and the Fiedler eigenvalue of the mirror graph of a balanced digraph. This provides an extension of the notion of algebraic connectivity of graphs to algebraic connectivity of balanced digraphs. A simple disagreement function was introduced as a Lyapunov function for the group disagreement dynamics. This was later used to provide a common Lyapunov function that allowed convergence analysis of an agreement protocol for a network with switching topology. A commutative diagram was given that shows the operations of taking Laplacian and symmetric part of a matrix commute for adjacency matrix of balanced graphs. Balanced graphs turned out to be instrumental in solving average-consensus problems.

For undirected networks with fixed topology, we gave sufficient and necessary conditions for reaching an average-consensus in presence of communication time-delays. It was shown that there is a trade-off between robustness to time-delays and the performance of a linear consensus protocol. Moreover, a second trade-off exists between maintaining a low communication cost and achieving a high performance in reaching a consensus. Extensive simulation results are provided that demonstrate the effectiveness of our theoretical results and analytical tools.

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APPENDIX

I. PROOFS

This section contains the proofs of some of the theorems of the paper.

A. Proof of Theorem 1

Proof: To establish this result, we show that if a digraph of order n is strongly connected, then the null space of its Laplacian is a one-dimensional subspace of $\mathbb{R}^n$.

Define $\phi_{ij}(z) = a_{ij}z$ for all $e_{ij} \in \mathcal{E}$. It is trivial that if $x_i = x_j$ for all $e_{ij} \in \mathcal{E}$, then $u = 0$. Thus, we prove the converse: $u = 0$ implies that all nodes are in agreement. If the values of all nodes are equal, the result follows. Thus, assume there exists a node v_{i^*}, called the *max-leader*, such that $x_{i^*} \geq x_j$ for all $j \neq i^*$, i.e. $i^* = \arg\max_{j\in\mathcal{I}} x_j$ (if i^* is not unique, choose one arbitrarily).

Define the initial cluster $J^{(0)} = \{v_{i^*}\}$ and denote the indices of all the *first-neighbors* of v_{i^*} by $J^{(1)} = N_{i^*}$. Then, $u_{i^*} = 0$ implies that

$$\sum_{j\in N_{i^*}} \phi_{i^*j}(x_j - x_{i^*}) = 0 \tag{45}$$

Since $x_j \leq x_{i^*}$ for all $j \in N_{i^*}$ and $\phi_{ij}(z) \leq 0$ for $z \leq 0$ (i.e. all weights are nonnegative), we get $x_{i^*} = x_j$ for all the first-neighbors $v_j \in J^{(1)}$, (i.e. the max-leader and all of its firsneighbors are in agreement). Next, we define the kth-neighbors of v_{i^*} and show that the max-leader is in agreement with all of its kth-neighbors for $k = 1, \ldots, n-1$. The set of kth-neighbors of v_{i^*} is defined by the following recursive equation

$$J^{(k)} = J^{(k-1)} \cup N_{J^{(k-1)}}, \quad k \geq 1, J^{(0)} = \{i^*\} \tag{46}$$

where N_J denotes the set of neighbors of cluster $J \subseteq \mathcal{V}$ (see equation (1)). By definition, $\{v_{i^*}\} \subset J^{(k)} \subseteq \mathcal{I}$ for $k \geq 1$ and $J^{(k)}$ is a monotonically increasing sequence of clusters (in terms of inclusion).

Notice that in a strongly connected digraph, the maximum length of the minimum path connecting any node $v_j \neq v_{i^*}$ to node v_{i^*} is $n-1$. Thus, $J^{(n-1)} = \mathcal{V}$. By induction, we prove that all the nodes in $J^{(k)}$ are in agreement for $k \geq 1$. The statement holds for $k = 1$ (i.e. the set of first-neighbors of the max-leader). Assume all the nodes in $J^{(k)}$ are in agreement with v_{i^*}, we show that all the nodes in $J^{(k+1)}$ are in agreement with v_{i^*} as well. It is sufficient to show this for an arbitrary node $v_i \in J^{(k)}$ with $N_i \cap (J^{(k+1)} \setminus J^{(k)}) \neq \emptyset$. This is because in a strongly connected digraph, $N_i \neq \emptyset$ for all $v_i \in \mathcal{V}$. Thus, if $N_i \cap (J^{(k+1)} \setminus J^{(k)}) = \emptyset$ for all v_i, we get $J^{(k+1)} = J^{(k)}$ and the statement holds. For node v_i, we have

$$u_i = \sum_{v_j\in N_i} \phi_{ij}(x_j - x_i) = 0. \tag{47}$$

But $N_i = (N_i \cap J^{(k)}) \cup (N_i \cap (\mathcal{V} \setminus J^{(k)}))$ and $\mathcal{V} \setminus J^{(k)} = \mathcal{V} \setminus J^{(k+1)} \cup (J^{(k+1)} \setminus J^{(k)})$. Keeping in mind that $J^{(k)} \subseteq \mathcal{V}$ for all k and $J^{(k+1)}$ contains the set of first-neighbors of node v_i, or $N_i \subseteq J^{(k+1)}$, we have

$$N_i \cap (\mathcal{I} \setminus J^{(k)}) = N_i \cap (J^{(k+1)} \setminus J^{(k)}) \tag{48}$$

and

$$\begin{aligned} u_i &= \textstyle\sum_{v_j\in N_i\cap J^{(k)}} \phi_{ij}(x_j - x_i) \\ &+ \textstyle\sum_{v_j\in N_i\cap(J^{(k+1)}\setminus J^{(k)})} \phi_{ij}(x_j - x_i) = 0 \end{aligned} \tag{49}$$

The first summation is equal to zero because $x_j = x_i$ for all nodes $v_j \in N_i \cap J^{(k)} \subseteq J^{(k)}$. Hence, the second summation must be zero. But $x_{i^*} = x_i \geq x_j$ for all $v_i \in J^{(k)}$ and $v_j \in \mathcal{V} \setminus J^{(k)}$ which implies all nodes in $N_i \cap (J^{(k+1)} \setminus J^{(k)})$ are in agreement with i^*. This means that all nodes v_i in the cluster

$$\begin{aligned} \textstyle\bigcup_{v_i\in J^{(k)}} N_i \cap (J^{(k+1)} \setminus J^{(k)}) &= J^{(k+1)} \cap (J^{(k+1)} \setminus J^{(k)}) \\ &= J^{(k+1)} \setminus J^{(k)} \end{aligned} \tag{50}$$

are in agreement with the max-leader v_{i^*}, i.e. all the nodes in $J^{(k+1)}$ are in agreement. Combining this result with the fact that $J^{(n-1)} = \mathcal{V}$, one concludes that all the nodes in $\mathcal{V}$ are in agreement. □

B. Proof of Theorem 10

Notice that despite the existence of a nonzero delay τ, $\sum_{i=1}^n u_i = 0$. Thus, $\alpha = \text{Ave}(x)$ is an invariant quantity. Given that the solutions of (40) globally asymptotically converge to a limit x^*, due to the invariance of α, $x_i^* = \text{Ave}(x(0)), \forall i \in \mathcal{I}$ and the average-consensus will be reached. To establish the stability of (40), we use a frequency domain analysis. We have $X(s) = G_\tau(s)x(0)$ where

$$G_\tau(s) = (sI_n + e^{-\tau s}L)^{-1}. \tag{51}$$

Define $Z_\tau(s) = G_\tau^{-1}(s) = (sI_n + e^{-\tau s}L)$. We need to find sufficient conditions such that all the zeros of $Z_\tau(s)$ are on the open LHP or $s = 0$. Let w_k be the kth normalized eigenvector of L associated with the eigenvalue λ_k in an increasing order. For a connected graph G, $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n = \lambda_{max}(L)$. Clearly, $s = 0$ in the direction w_1 is a zero of the MIMO transfer function $Z_\tau(s)$, because $Z_\tau(0)w_1 = Lw_1 = 0$. Furthermore, any eigenvector of $Z_\tau(s)$ is an eigenvector of L and vice verse. Let (s, w_k) with $k > 1$ be a right MIMO transmission zero of $Z_\tau(s)$ at frequency s in the direction w_k, i.e. $Z_\tau(s)w_k = 0$. Then, $s \neq 0$ satisfies the following equation

$$s + e^{-\tau s}\lambda_k = 0, \tag{52}$$

or

$$\frac{1}{\lambda_k} + \frac{e^{-\tau s}}{s} = 0, \tag{53}$$

where λ_k is the kth eigenvalue of L corresponding to w_k. This is due to the fact that

$$Z_\tau(s)w_k = sw_k + e^{-\tau s}Lw_k = (s + e^{-\tau s_k}\lambda_k)w_k = 0, \tag{54}$$

but $w_k \neq 0$, thus $s + e^{-\tau}\lambda_k = 0$. Equation (53) provides *a Nyquist criterion for convergence of protocol* (A2). If the net encirclement of the Nyquist plot of $\Gamma(s) = e^{-\tau s}/s$ around $-1/\lambda_k$ for $k > 1$ is zero, then all the zeros of $Z_\tau(s)$ (or poles of $G_\tau(s)$) other than $s = 0$ are stable. For the special case where L is symmetric, all the eigenvalues are real and the Nyquist stability criterion reduces to zero net encirclement of the Nyquist plot of $\Gamma(s)$ around $-1/\lambda_n$ (note that $\lambda_n = \lambda_{max}(L)$). This is because the plot of $\Gamma(j\omega)$ in the s-plane remains on the right hand side of $-\tau$. Since

$$\Gamma(j\omega) = \frac{e^{-j\omega\tau}}{j\omega} = -\frac{\sin(\omega\tau)}{\omega} - j\frac{\cos(\omega\tau)}{\omega}, \qquad (55)$$

and clearly $\mathrm{Re}(\Gamma(j\omega))$ is a sinc function satisfying $\mathrm{Re}(\Gamma(j\omega)) \geq -\tau$. A conservative upper bound on τ can be obtained according to the property $\mathrm{Re}(\Gamma(j\omega)) \geq -\tau$ of the Nyquist plot of $\Gamma(s)$ by setting $-1/\lambda_n > -\tau$ which gives the convergence condition $\tau < 1/\lambda_n$. As a by-product, for $\tau = 0$, the protocol always converges regardless of the value of λ_k for $k > 1$.

A better upper bound on the time-delay τ can be calculated as follows. Let us find the smallest value of the time-delay $\tau > 0$ such that $Z_\tau(s)$ has a zero on the imaginary axis. To do so, set $s = j\omega$ in (52), we have

$$\begin{aligned} j\omega + e^{-j\omega\tau}\lambda_k &= 0, \\ -j\omega + e^{j\omega\tau}\lambda_k &= 0, \end{aligned} \qquad (56)$$

multiplying both sides of the last two equations gives

$$\omega^2 + \lambda_k^2 + j\omega\lambda_k(e^{j\omega\tau} - e^{-j\omega\tau}) = 0, \qquad (57)$$

or

$$\omega^2 + \lambda_k^2 - 2\omega\lambda_k \sin(\omega\tau) = 0, \qquad (58)$$

Assuming $\omega > 0$ (due to $s \neq 0$), from (58), we get

$$(\omega - \lambda_k)^2 + 2\omega\lambda_k(1 - \sin(\omega\tau)) = 0. \qquad (59)$$

Since both terms in the left hand side of the last equation are positive semi-definite, the equality holds if and only if both terms are zero, i.e.

$$\begin{aligned} \omega &= \lambda_k, \\ \sin(\omega\tau) &= 1, \end{aligned} \qquad (60)$$

This implies $\tau\lambda_k = 2l\pi + \pi/2$ for $l = 0, 1, 2, \ldots$, thus the smallest $\tau > 0$ satisfies $\tau\lambda_k = \pi/2$. Therefore, we have

$$\tau^* = \min_{\substack{\tau\lambda_k = \pi/2 \\ k > 1}} \{\tau\} = \min_{k>1} \frac{\pi}{2\lambda_k} = \frac{\pi}{2\lambda_n} \qquad (61)$$

Due to the continuous dependence of the roots of equation (52) in τ and the fact that all the zeros of this equation other than $s = 0$ for $\tau = 0$ are located on the open LHP, for all $\tau \in (0, \tau^*)$, the roots of (52) with $k > 1$ are on the open LHP and therefore the poles of $G_\tau(s)$ (except for $s = 0$) are all stable. One can repeat a similar argument for the assumption that $\omega < 0$ and get the equation

$$(\omega + \lambda_k)^2 - 2\omega\lambda_k(1 + \sin(\omega\tau)) = 0, \qquad (62)$$

which leads to $\omega = -\lambda_k$ and $\tau\lambda_k = 2l\pi + \pi/2$.

For $\tau = \tau^*$, $G_\tau(s)$ has three poles on the imaginary axis given by

$$s = 0, s = \pm j\lambda_n \qquad (63)$$

All other poles of $G_\tau(s)$ are stable and in the steady-state the values of each node takes the following form:

$$x_i^{ss}(t) = a_i + b_i \sin(\lambda_n t + \varphi_i), \quad i \in \mathcal{I} \qquad (64)$$

where a_i, b_i, φ_i are constants that depend on the initial conditions. □

REFERENCES

[1] A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *The IEEE Trans. on Automatic Control*, (to appear) 2004.

[2] R. Olfati-Saber and R. M. Murray, "Distibuted cooperative control of multiple vehicle formations using structural potential functions," *The 15th IFAC World Congress*, June 2002.

[3] R. Olfati-Saber and R. M. Murray, "Graph Rigidity and Distributed Formation Stabilization of Multi-Vehicle Systems," *Proceedings of the IEEE Int. Conference on Decision and Control*, vol. 3, pp. 2965–2971, Dec. 2002.

[4] T. Eren, P. N. Belhumeur, and A. S. Morse, "Closing ranks in vehicle formations based on rigidity," *Proc. of the IEEE Conference on Decision and Control*, Dec. 2002.

[5] R. Vidal, O. Shakernia, and S. Sastry, "Formation control of nonholonomic mobile robots omnidirectional visual servoing and motion segmentation," *Proc. of the IEEE Conference on Robotics and Automation*, 2003.

[6] C. W. Reynolds, "Flocks, herds, and schools: a distributed behavioral model," *Computer Graphics (ACM SIGGRAPH '87 Conference Proceedings)*, vol. 21, no. 4, pp. 25–34, July 1987.

[7] T. Vicsek, A. Czirook, E. Ben-Jacob, and O. Cohen, I. Shochet, "Novel type of phase transition in a system of self-deriven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, August, 1995.

[8] J. Toner and Y. Tu, "Flocks, herds, and schools: A quantitative theory of flocking," *Physical Review E*, vol. 58, no. 4, pp. 4828–4858, October 1998.

[9] J. Cortes and F. Bullo, "Coordination and geometric optimization via distributed dynamical systems," *submitted to the SIAM Journal on Control and Optimization*, May 2003.

[10] F. Paganini, J. Doyle, and S Low, "Scalable laws for stable network congestion control," *Proc. of the Int. Conf. on Decision and Control*, Orlando, FL, Dec. 2001.

[11] N. A. Lynch, *Distributed Algorithms*, Morgan Kaufmann Publishers, Inc., 1997.

[12] H. Yamaguchi, T. Arai, and G. Beni, "A distributed control scheme for multiple robotic vehicles to make group formations," *Robotics and Autonomous Systems*, vol. 36, pp. 125–147, 2001.

[13] M. Mesbahi and F. Y. Hadegh, "Formation flying of multiple spacecraft via graphs, matrix inequalities, and switching," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 24, no. 2, pp. 369–377, March 2000.

[14] J. P. Desai, J. P. Ostrowski, and V. Kumar, "Modeling and control of formations of nonholonomic mobile robots," *IEEE Trans. on Robotics and Automation*, vol. 17, no. 6, December 2002.

[15] R. W. Lawton, J. R. T. Beard and B. J. Young, "A Decentralized Approach to Formation Maneuvers," *IEEE Trans. on Robotics and Automation (to appear)*, 2002.

[16] M. Mesbahi, "On a dynamic extension of the theory of graphs," *Proc. of the American Control Conference*, Anchorange, AL, May 2002.

[17] J. Graver and H. Servatius, B. Servatius, *Chemical oscillators, waves, and turbulance*, Springer-Verlag, Berlin, 1984.

[18] S. H. Strogatz, "Exploring complex networks," *Nature*, vol. 410, pp. 268–276, 2001.

[19] G. B. Ermentrout, "Stable periodic solutions to discrete and continuum arrays of weakly coupled oscillators," *SIAM Journal on Applied Mathematics*, vol. 52, no. 6, pp. 1665–1687, Dec. 1992.

[20] D. Helbing, I. Farkas, and T. Vicsek, "Simulating dynamical features of escape panic," *Nature*, vol. 407, pp. 487–490, 2000.

[21] Y. Liu, K.M. Passino, and M. M. Polycarpou, "Stability analysis of M-dimensional asynchronous swarms with a fixed communication topology," *IEEE Trans. On Automatic Control*, vol. 48, no. 1, pp. 76–95, January 2003.

[22] V. Gazi and K.M. Passino, "Stability analysis of swarms," *IEEE Trans. on Automatic Control (to appear)*, May 2003.

[23] R. Olfati-Saber and R. M. Murray, "Flocking with obstacle avoidance: cooperation with limited communication in mobile networks," *Proc. of the IEEE Conf. on Decision and Control*, Dec. 2003.

[24] D. E. Chang, S. Shadden, J. Marsden, and R. Olfati Saber, "Collision Avoidance for Multiple Agent Systems," *Proc. of the IEEE Conf. on Decision and Control*, Feb. 2003.

[25] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stability of flocking motion," Tech. Rep. MS-CIS-03-03, the GRASP Laboratory, University of Pennsylvania, Philadelphia, PA, May 2003.

[26] R. Olfati-Saber, "Flocking with obstacle avoidance," Tech. Rep. CIT-CDS 03-006, California Institute of Technology, Control and Dynamical Systems, Pasadena, California, Feb. 2003, `http://resolver.caltech.edu/CaltechCDSTR:2003.006`.

[27] R. Olfati-Saber, "A unified look at Reynolds flocking rules," Tech. Rep. CIT-CDS 03-014, California Institute of Technology, Con-

trol and Dynamical Systems, Pasadena, California, Sep. 2003, http://resolver.caltech.edu/CaltechCDSTR:2003.014.

[28] A. Jadbabaie, J. Lin, and S. A. Morse, "Coordination of groups of mobile agents using nearest neighbor rules," *IEEE Trans. on Automatic Control*, vol. 48, no. 6, pp. 988–1001, June 2003.

[29] R. Olfati-Saber and R. M. Murray, "Consensus Protocols for Networks of Dynamic Agents," *Proc. of the American Control Conference*, June 2003.

[30] N. Biggs, *Algebraic Graph Theory*, Cambridge Tracks in Mathematics. Cambridge University Press, 1974.

[31] C. Godsil and G. Royle, *Algebraic Graph Theory*, vol. 207 of *Graduate Texts in Mathematics*, Springer, 2001.

[32] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1987.

[33] E. Klavins, "Communication Complexity of Multi-Robot Systems," *The Fifth International Workshop on the Algorithmic Foundations of Robotics*, Dec. 2002.

[34] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Proc. of the Conference on Decision and Control (also, submitted to Systems and Control Letters)*, December 2003.

[35] F. R. Gantmacher, *Matrix Theory*, vol. II, Chelsea Publishing Company, 1959.

[36] I. Daubechies and J. C. Lagarias, "Sets of matrices all infinite products of which converge," *Linear Algebra and Its Applications*, vol. 161, pp. 227–263, 1992.

[37] R. Diestel, *Graph Theory*, vol. 173 of *Graduate Texts in Mathematics*, Springer-Verlag, 2000.

[38] R. Merris, "Laplacian matrices of a graph: a survey," *Linear Algebra and its Applications*, vol. 197, pp. 143–176, 1994.

[39] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 98, pp. 298–305, 1973.

Reza Olfati-Saber received the B.S. degree in Electrical Engineering from Sharif University in 1994 and the S.M. and Ph.D. degrees in Electrical Engineering and Computer Science from Massachusetts Institute of Technology in 1997 and 2001, respectively. He has been a Postdoctoral Scholar at the Department of Control and Dynamical Systems at California Institute of Technology since 2001. His research interests include distributed control of multi-agent systems, formation control, consensus and synchronization problems, flocking/swarming, particle-based modeling and simulation, self-assembly, cooperative robotics, mobile sensor networks, self-organizing networks, folding and unfolding, biomolecular systems, dynamic graph theory, graph rigidity, computational geometry, nonlinear control theory, and control of aerospace vehicles and UAVs. Dr. Olfati-Saber won the Lotfi Zadeh best student paper award in 1997 and is a member of IEEE, AIAA, and Sigma Xi.

Richard M. Murray received the B.S. degree in Electrical Engineering from California Institute of Technology in 1985 and the M.S. and Ph.D. degrees in Electrical Engineering and Computer Sciences from the University of California, Berkeley, in 1988 and 1991, respectively. He is currently a Professor of Mechanical Engineering and the Chair of the Division of Engineering and Applied Science at the California Institute of Technology, Pasadena. Professor Murray's research is in the application of feedback and control to mechanical, information, and biological systems.

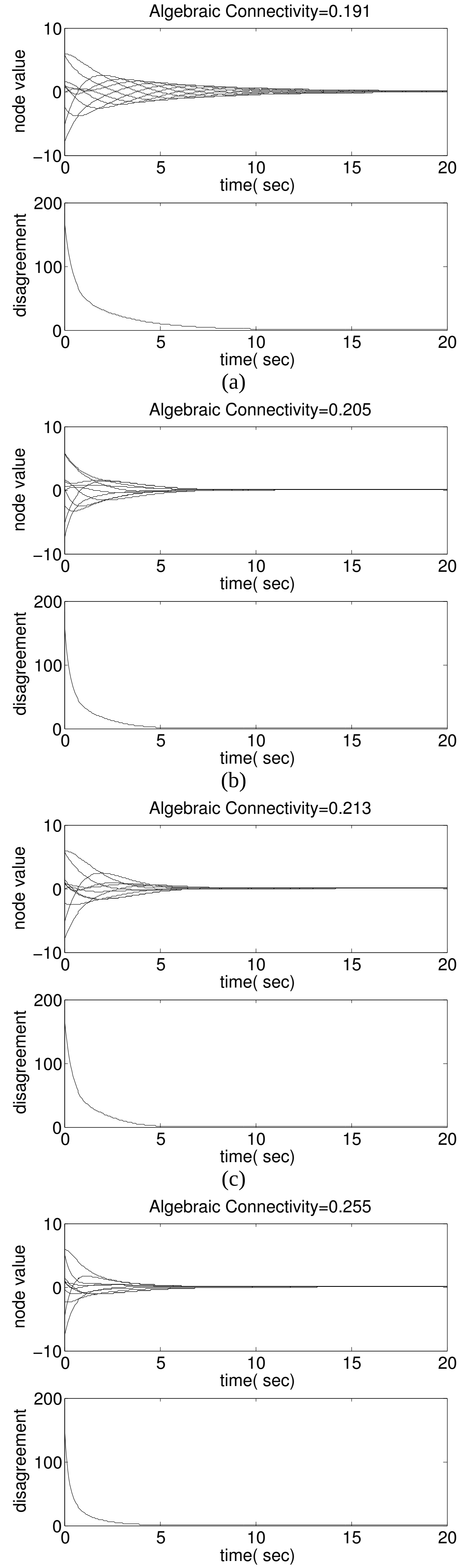

Fig. 6. State trajectories of all nodes corresponding to networks with topologies

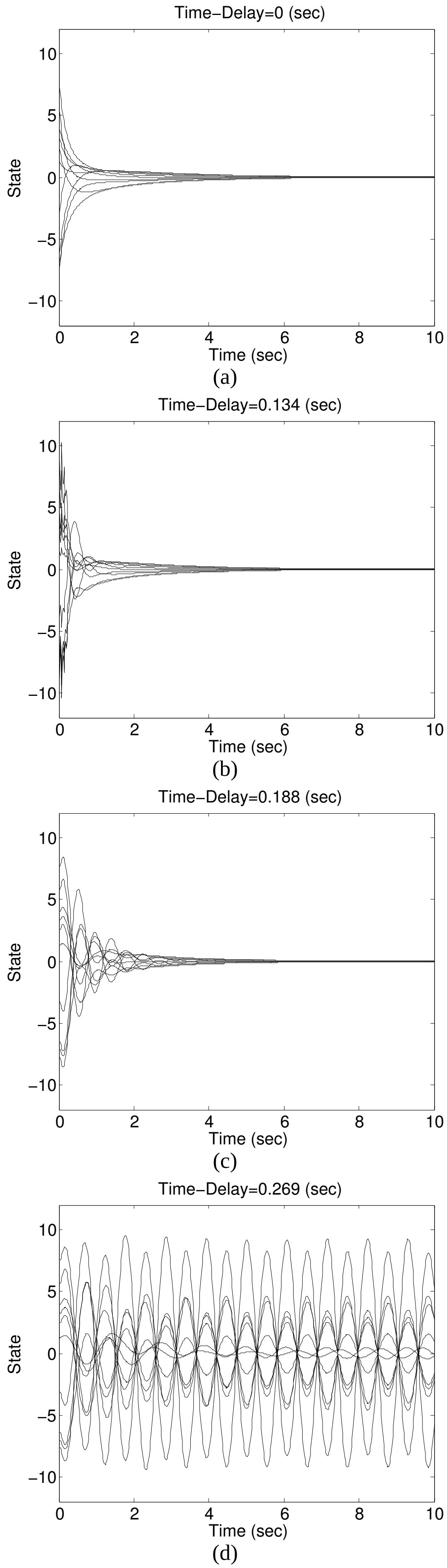

Fig. 7. Consensus problem with communication time-delays on graph G_e given in Figure 3(d): (a) $\tau = 0$, (b) $\tau = 0.5\tau_{max}$,(c) $\tau = 0.7\tau_{max}$, and (d) $\tau =$