1 Physical Domain Decomposition

Complex structures are divided into non-overlapping substructures. System of equations for each substructure is formulated and subsequently assembled to form the global system of equations.

1.1 Substructures' System of Equations

This part is from [**craig1968coupling**].
The degree of freedoms associated with shared nodes (between two or more substructures) are referred to as boundary DoFs while other are referred to as internal DoFs. The static system of equations for substructure r is given by the following:

$$\begin{bmatrix} \mathbf{F}_r^I \\ \mathbf{F}_r^B \end{bmatrix} = \begin{bmatrix} \mathbf{K}_r^{II} & \mathbf{K}_r^{IB} \\ \mathbf{K}_r^{BI} & \mathbf{K}_r^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{u}_r^I \\ \mathbf{u}_r^B \end{bmatrix} \tag{1}$$

Constraint modes are defined as the mode shapes of internal freedoms due to successive unit displacement of boundary points, all other boundary point being totally constrained. Setting forces associated with internal DoFs to zero in equation (1):

$$\mathbf{0} = \mathbf{K}_r^{IB}\mathbf{u}_r^B + \mathbf{K}_r^{II}\mathbf{u}_r^I + \quad \Longrightarrow \quad \mathbf{u}_r^I = -\left(\mathbf{K}_r^{II}\right)^{-1}\mathbf{K}_r^{IB}\mathbf{u}_r^B$$

From which an expression for constrained modes is obtained:

$$\bar{\mathbf{\Phi}}_r^C = -\left(\mathbf{K}_r^{II}\right)^{-1}\mathbf{K}_r^{IB} \tag{2}$$

Normal modes are defined as the mode shapes of the substructure with totally constrained boundary. These are obtained from the equations

$$\mathbf{0} = \left[\mathbf{K}_r^{II} - \omega_j^2\mathbf{M}_r^{II}\right]\mathbf{\Phi}_{r,j}^I \tag{3}$$

The eigenvectors of equation (3) form the columns of the normal mode matrix $\mathbf{\Phi}_r^N$. Model order reduction is possible through truncation of this matrix's columns i.e. by retaining only some eigenvectors in the reduced normal mode matrix $\bar{\mathbf{\Phi}}_r^N$. Transformation between freedoms in physical space and modal space is given by the following:

$$\mathbf{u}_r = \mathbf{G}_r\bar{\mathbf{u}}_r \quad \text{or} \quad \begin{bmatrix} \mathbf{u}_r^I \\ \mathbf{u}_r^B \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_r^N \\ \bar{\mathbf{u}}_r^C \end{bmatrix} \tag{4}$$

The modal mass and stiffness matrices are given by the followings respectively

$$\bar{\mathbf{M}}_r = \mathbf{G}_r^T\mathbf{M}_r\mathbf{G}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{M}}_r^{NN} & \bar{\mathbf{M}}_r^{NB} \\ \bar{\mathbf{M}}_r^{BN} & \bar{\mathbf{M}}_r^{BB} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_r^{II} & \mathbf{M}_r^{IB} \\ \mathbf{M}_r^{BI} & \mathbf{M}_r^{BB} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{5}$$

$$\bar{\mathbf{K}}_r = \mathbf{G}_r^T\mathbf{K}_r\mathbf{G}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{K}}_r^{NN} & \bar{\mathbf{K}}_r^{NB} \\ \bar{\mathbf{K}}_r^{BN} & \bar{\mathbf{K}}_r^{BB} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_r^{II} & \mathbf{K}_r^{IB} \\ \mathbf{K}_r^{BI} & \mathbf{K}_r^{BB} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{6}$$

The modal stiffness matrix is block-diagonal because

$$\begin{aligned} \bar{\mathbf{K}}_r^{NB} &= \left[\bar{\mathbf{\Phi}}_r^N\right]^T\mathbf{K}_r^{IB} + \left[\bar{\mathbf{\Phi}}_r^N\right]^T\mathbf{K}_r^{II}\bar{\mathbf{\Phi}}_r^C \\ &= \left[\bar{\mathbf{\Phi}}_r^N\right]^T\mathbf{K}_r^{IB} + \left[\bar{\mathbf{\Phi}}_r^N\right]^T\mathbf{K}_r^{II}\left[-\left(\mathbf{K}_r^{II}\right)^{-1}\mathbf{K}_r^{IB}\right] \\ &= \mathbf{0} \end{aligned}$$

The modal load vector is given by the following equation

$$\bar{\mathbf{F}}_r = \mathbf{G}_r^T \mathbf{F}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{F}}_r^N \\ \bar{\mathbf{F}}_r^C \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{F}_r^I \\ \mathbf{F}_r^B \end{bmatrix} \tag{7}$$

1.2 Assembly into Global System of Equations

Dynamic equation of a structural system with two substructures is given by the following

$$\mathbf{F} = \mathbf{D}\mathbf{u} \quad \text{or} \quad \begin{bmatrix} \mathbf{F}_\alpha^I \\ \mathbf{F}_\beta^I \\ \mathbf{F}^B \end{bmatrix} = \begin{bmatrix} \mathbf{D}_\alpha^{II} & \mathbf{0} & \mathbf{D}_\alpha^{IB} \\ \mathbf{0} & \mathbf{D}_\beta^{II} & \mathbf{D}_\beta^{IB} \\ \mathbf{D}_\alpha^{BI} & \mathbf{D}_\beta^{BI} & \mathbf{D}^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{u}_\alpha^I \\ \mathbf{u}_\beta^I \\ \mathbf{u}^B \end{bmatrix} \tag{8}$$

Using transformations $\bar{\mathbf{F}} = \mathbf{G}^T \mathbf{F}$, $\bar{\mathbf{D}} = \mathbf{G}^T \mathbf{D} \mathbf{G}$, and $\bar{\mathbf{u}} = \mathbf{G}\mathbf{u}$, the equation can be transformed into modal space

$$\bar{\mathbf{F}} = \bar{\mathbf{D}}\bar{\mathbf{u}} \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{F}}_\alpha^N \\ \bar{\mathbf{F}}_\beta^N \\ \bar{\mathbf{F}}^C \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{D}}_\alpha^{NN} & \mathbf{0} & \bar{\mathbf{D}}_\alpha^{NC} \\ \mathbf{0} & \bar{\mathbf{D}}_\beta^{NN} & \bar{\mathbf{D}}_\beta^{NC} \\ \bar{\mathbf{D}}_\alpha^{CN} & \bar{\mathbf{D}}_\beta^{CN} & \bar{\mathbf{D}}^{CC} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_\alpha^N \\ \bar{\mathbf{u}}_\beta^N \\ \bar{\mathbf{u}}^C \end{bmatrix} \tag{9}$$

The transformation matrix is given by the following

$$\mathbf{G} = \begin{bmatrix} \bar{\mathbf{\Phi}}_\alpha^N & \mathbf{0} & \bar{\mathbf{\Phi}}_\alpha^C \\ \mathbf{0} & \bar{\mathbf{\Phi}}_\beta^N & \bar{\mathbf{\Phi}}_\beta^C \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{10}$$

Generalisation for more than two substructures is given in ... (future reading).