Notes and Documentations for Master's Thesis
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Non-Intrusive Surrogate Modelling for Stochastic Linear Structural Systems Based on Collocation Using Substructuring Methods

Rezha Adrian Tanuharja*

Motivation and Reasonings

Why NI-RPCE?

In recent comparison, non - intrusive surrogate model based on rational of polynomial chaos expansions displayed promising efficiency and effectiveness to propagate uncertainties for linear structural systems. The main advantage of the model is its ability to decouple degree of freedoms, which enable parallel model training (once training data is available) and evaluation.

What to improve?

In the aforementioned comparison, majority of computation time is spent on training data generation. In addition, the computation time associated with this process has the highest growth rate with respect to the structural system's complexity (size). There are two possible improvements: reduce the number of required training data or reduce computation time associated with generation and evaluation of the deterministic model (by performing model order reduction and/or domain decomposition).

Why deterministic model order reduction and domain decomposition?

Time to generate training data is proportional to the number of data sets (linear). On the other hand, time to generate and evaluate deterministic model has complexity $O(n^m)$, with $m \geq 3$. Therefore, it is presumed that bigger time reduction can be achieved through model order reduction (size reduction) and domain decomposition (size reduction and parallel computing).

Why Hurty / Craig - Bampton (HCB) domain decomposition method?

This domain decomposition method is a proven and popular method in structural dynamics. Consequently, literatures and resources on the method are plenty and widely available including multiple improvements to tackle disadvantages of the original method e.g. Schur complement for efficient parallelisation, interface mode reduction for further model order reduction, dual method to avoid locking caused by low number of modes, quasi-static component modes to improve mid- and high-frequency performance, etc..

Further potential improvement:

Approximation of random eigenvectors and eigenvalues of the reduced model via perturbation method. HCB use eigenmode truncation to reduce model's order, therefore this approximation integrates naturally with the method. It can save significant time since repeated eigenvectors and eigenvalues computations can be avoided when generating training data; only deterministic ones are computed while the random ones are approximated. However, this potentially reduces accuracy as perturbation method is known to have poor performance in the presence of high uncertainties.

* *Technische Universität München*

1 Physical Domain Decomposition

Complex structures are divided into non-overlapping substructures. System of equations for each substructure is formulated and subsequently assembled to form the global system of equations.

1.1 Substructures' System of Equations

This part is adapted from [1].

The degree of freedoms associated with shared nodes (between two or more substructures) are referred to as boundary DoFs while other are referred to as internal DoFs. The static system of equations for substructure r is given by the following:

$$\begin{bmatrix} \mathbf{F}_r^I \\ \mathbf{F}_r^B \end{bmatrix} = \begin{bmatrix} \mathbf{K}_r^{II} & \mathbf{K}_r^{IB} \\ \mathbf{K}_r^{BI} & \mathbf{K}_r^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{u}_r^I \\ \mathbf{u}_r^B \end{bmatrix} \tag{1}$$

Constraint modes are defined as the mode shapes of internal freedoms due to successive unit displacement of boundary points, all other boundary point being totally constrained. Setting forces associated with internal DoFs to zero in equation (1):

$$\mathbf{0} = \mathbf{K}_r^{IB}\mathbf{u}_r^B + \mathbf{K}_r^{II}\mathbf{u}_r^I + \quad \Longrightarrow \quad \mathbf{u}_r^I = -\left(\mathbf{K}_r^{II}\right)^{-1}\mathbf{K}_r^{IB}\mathbf{u}_r^B$$

From which an expression for constrained modes is obtained:

$$\bar{\mathbf{\Phi}}_r^C = -\left(\mathbf{K}_r^{II}\right)^{-1}\mathbf{K}_r^{IB} \tag{2}$$

Normal modes are defined as the mode shapes of the substructure with totally constrained boundary. These are obtained from the equations

$$\mathbf{0} = \left[\mathbf{K}_r^{II} - \omega_j^2\mathbf{M}_r^{II}\right]\mathbf{\Phi}_{r,j}^I \tag{3}$$

The eigenvectors of equation (3) form the columns of the normal mode matrix $\mathbf{\Phi}_r^N$. Model order reduction is possible through truncation of this matrix's columns i.e. by retaining only some eigenvectors in the reduced normal mode matrix $\bar{\mathbf{\Phi}}_r^N$. Transformation between freedoms in physical space and modal space is given by the following:

$$\mathbf{u}_r = \mathbf{G}_r\bar{\mathbf{u}}_r \quad \text{or} \quad \begin{bmatrix} \mathbf{u}_r^I \\ \mathbf{u}_r^B \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_r^N \\ \bar{\mathbf{u}}_r^C \end{bmatrix} \tag{4}$$

The modal mass and stiffness matrices are given by the followings respectively

$$\bar{\mathbf{M}}_r = \mathbf{G}_r^T\mathbf{M}_r\mathbf{G}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{M}}_r^{NN} & \bar{\mathbf{M}}_r^{NB} \\ \bar{\mathbf{M}}_r^{BN} & \bar{\mathbf{M}}_r^{BB} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_r^{II} & \mathbf{M}_r^{IB} \\ \mathbf{M}_r^{BI} & \mathbf{M}_r^{BB} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{5}$$

$$\bar{\mathbf{K}}_r = \mathbf{G}_r^T\mathbf{K}_r\mathbf{G}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{K}}_r^{NN} & \bar{\mathbf{K}}_r^{NB} \\ \bar{\mathbf{K}}_r^{BN} & \bar{\mathbf{K}}_r^{BB} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_r^{II} & \mathbf{K}_r^{IB} \\ \mathbf{K}_r^{BI} & \mathbf{K}_r^{BB} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{6}$$

By using the definition of constraint mode matrix before, the modal stiffness matrix becomes block-diagonal because

$$\begin{aligned}\bar{\mathbf{K}}_r^{NB} &= \left[\bar{\mathbf{\Phi}}_r^N\right]^T \mathbf{K}_r^{IB} + \left[\bar{\mathbf{\Phi}}_r^N\right]^T \mathbf{K}_r^{II} \bar{\mathbf{\Phi}}_r^C \\ &= \left[\bar{\mathbf{\Phi}}_r^N\right]^T \mathbf{K}_r^{IB} + \left[\bar{\mathbf{\Phi}}_r^N\right]^T \mathbf{K}_r^{II} \left[-\left(\mathbf{K}_r^{II}\right)^{-1} \mathbf{K}_r^{IB}\right] \\ &= \mathbf{0}\end{aligned}$$

The modal load vector is given by the following equation

$$\bar{\mathbf{F}}_r = \mathbf{G}_r^T \mathbf{F}_r \quad \text{or} \quad \begin{bmatrix}\bar{\mathbf{F}}_r^N \\ \bar{\mathbf{F}}_r^C\end{bmatrix} = \begin{bmatrix}\bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I}\end{bmatrix}^T \begin{bmatrix}\mathbf{F}_r^I \\ \mathbf{F}_r^B\end{bmatrix} \tag{7}$$

1.2 Assembly into Global System of Equations

This part is adapted from [2].

First, we look at assembly and formulation for two substructures. Dynamic equation of a structural system with two substructures is given by the following

$$\mathbf{F} = \mathbf{D}\mathbf{u} \quad \text{or} \quad \begin{bmatrix}\mathbf{F}_\alpha^I \\ \mathbf{F}_\beta^I \\ \mathbf{F}_\alpha^B + \mathbf{F}_\beta^B\end{bmatrix} = \begin{bmatrix}\mathbf{D}_\alpha^{II} & \mathbf{0} & \mathbf{D}_\alpha^{IB} \\ \mathbf{0} & \mathbf{D}_\beta^{II} & \mathbf{D}_\beta^{IB} \\ \mathbf{D}_\alpha^{BI} & \mathbf{D}_\beta^{BI} & \mathbf{D}_\alpha^{BB} + \mathbf{D}_\beta^{BB}\end{bmatrix} \begin{bmatrix}\mathbf{u}_\alpha^I \\ \mathbf{u}_\beta^I \\ \mathbf{u}^B\end{bmatrix} \tag{8}$$

Using transformations $\bar{\mathbf{F}} = \mathbf{G}^T\mathbf{F}$, $\bar{\mathbf{D}} = \mathbf{G}^T\mathbf{D}\mathbf{G}$, and $\bar{\mathbf{u}} = \mathbf{G}\mathbf{u}$, the equation can be transformed into modal space

$$\bar{\mathbf{F}} = \bar{\mathbf{D}}\bar{\mathbf{u}} \quad \text{or} \quad \begin{bmatrix}\bar{\mathbf{F}}_\alpha^N \\ \bar{\mathbf{F}}_\beta^N \\ \bar{\mathbf{F}}_\alpha^C + \bar{\mathbf{F}}_\beta^C\end{bmatrix} = \begin{bmatrix}\bar{\mathbf{D}}_\alpha^{NN} & \mathbf{0} & \bar{\mathbf{D}}_\alpha^{NC} \\ \mathbf{0} & \bar{\mathbf{D}}_\beta^{NN} & \bar{\mathbf{D}}_\beta^{NC} \\ \bar{\mathbf{D}}_\alpha^{CN} & \bar{\mathbf{D}}_\beta^{CN} & \bar{\mathbf{D}}_\alpha^{CC} + \bar{\mathbf{D}}_\beta^{CC}\end{bmatrix} \begin{bmatrix}\bar{\mathbf{u}}_\alpha^N \\ \bar{\mathbf{u}}_\beta^N \\ \bar{\mathbf{u}}^C\end{bmatrix} \tag{9}$$

The transformation matrix is given by the following

$$\mathbf{G} = \begin{bmatrix}\bar{\mathbf{\Phi}}_\alpha^N & \mathbf{0} & \bar{\mathbf{\Phi}}_\alpha^C \\ \mathbf{0} & \bar{\mathbf{\Phi}}_\beta^N & \bar{\mathbf{\Phi}}_\beta^C \\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix} \tag{10}$$

The resulting system of equations can be solved efficiently and in parallel using Schur complement which is detailed below. For systems with multiple substructures, solution is obtained via Schur complement without assembling the global matrix.

1.3 Schur Complement

This part is adapted from [2].

The idea of Schur complement is to solve the system of equations in two steps. In the first step, condensation is performed and the resulting equation is solved for the constrained modes. Subsequently, the constrained modes are substituted to the remaining equations to solve for the internal modes.

The formula for internal modes' forces are obtained from equation (9) and are given by the following:

$$\bar{\mathbf{F}}_\alpha^N = \bar{\mathbf{D}}_\alpha^{NN}\bar{\mathbf{u}}_\alpha^N + \bar{\mathbf{D}}_\alpha^{NC}\bar{\mathbf{u}}^C \quad \text{and} \quad \bar{\mathbf{F}}_\beta^N = \bar{\mathbf{D}}_\beta^{NN}\bar{\mathbf{u}}_\beta^N + \bar{\mathbf{D}}_\beta^{NC}\bar{\mathbf{u}}^C$$

To perform condensation, expressions of the internal modes are required. Therefore, the two equations are rearranged and solved for the internal modes respectively:

$$\bar{\mathbf{u}}_{\alpha}^{N} = \left(\bar{\mathbf{D}}_{\alpha}^{NN}\right)^{-1} \bar{\mathbf{F}}_{\alpha}^{N} - \left(\bar{\mathbf{D}}_{\alpha}^{NN}\right)^{-1} \bar{\mathbf{D}}_{\alpha}^{NC} \bar{\mathbf{u}}^{C} \tag{11}$$

$$\bar{\mathbf{u}}_{\beta}^{N} = \left(\bar{\mathbf{D}}_{\beta}^{NN}\right)^{-1} \bar{\mathbf{F}}_{\beta}^{N} - \left(\bar{\mathbf{D}}_{\beta}^{NN}\right)^{-1} \bar{\mathbf{D}}_{\beta}^{NC} \bar{\mathbf{u}}^{C} \tag{12}$$

Equations (11) and (12) are substituted into the formula of constraint mode's force from equation (9) to form the following equation, with each terms detailed in subsequent equations.

$$\left[\mathbf{S}_{\alpha} + \mathbf{S}_{\beta}\right] \bar{\mathbf{u}}^{C} = \mathbf{P}_{\alpha} + \mathbf{P}_{\beta} \tag{13}$$

$$\mathbf{S}_{\alpha} = \bar{\mathbf{D}}_{\alpha}^{CC} - \bar{\mathbf{D}}_{\alpha}^{CN} \left(\bar{\mathbf{D}}_{\alpha}^{NN}\right)^{-1} \bar{\mathbf{D}}_{\alpha}^{NC} \tag{14}$$

$$\mathbf{S}_{\beta} = \bar{\mathbf{D}}_{\beta}^{CC} - \bar{\mathbf{D}}_{\beta}^{CN} \left(\bar{\mathbf{D}}_{\beta}^{NN}\right)^{-1} \bar{\mathbf{D}}_{\beta}^{NC} \tag{15}$$

$$\mathbf{P}_{\alpha} = \bar{\mathbf{F}}_{\alpha}^{C} - \bar{\mathbf{D}}_{\alpha}^{CN} \left(\bar{\mathbf{D}}_{\alpha}^{NN}\right)^{-1} \bar{\mathbf{F}}_{\alpha}^{N} \tag{16}$$

$$\mathbf{P}_{\beta} = \bar{\mathbf{F}}_{\beta}^{C} - \bar{\mathbf{D}}_{\beta}^{CN} \left(\bar{\mathbf{D}}_{\beta}^{NN}\right)^{-1} \bar{\mathbf{F}}_{\beta}^{N} \tag{17}$$

The procedure is parallelised by assigning each substructure to a different computing resource and compute the Schur complement matrix $\mathbf{S}_{\cdot}$ and force $\mathbf{P}_{\cdot}$ independently. After solving for the constraint modes, the result is substituted back and the internal modes for each domain can be computed independently using the following two equations:

$$\bar{\mathbf{D}}_{\alpha}^{NN} \bar{\mathbf{u}}_{\alpha}^{N} = \bar{\mathbf{F}}_{\alpha}^{N} - \bar{\mathbf{D}}_{\alpha}^{NC} \bar{\mathbf{u}}^{C} \tag{18}$$

$$\bar{\mathbf{D}}_{\beta}^{NN} \bar{\mathbf{u}}_{\beta}^{N} = \bar{\mathbf{F}}_{\beta}^{N} - \bar{\mathbf{D}}_{\beta}^{NC} \bar{\mathbf{u}}^{C} \tag{19}$$

The generalisation for more than two substructures is adapted from [3].

While all constraint DoFs belong to two substructures before, this is not the case for more than two substructures. However, by collecting all constraint DoFs into the vector $\bar{\mathbf{u}}^{C}$, the formulation for arbitrary number of substructures is straightforward and given in the following.

$$\left[\sum_{r=1}^{n_{ss}} \mathbf{S}_{r}\right] \bar{\mathbf{u}}^{C} = \sum_{r=1}^{n_{ss}} \mathbf{P}_{r} \tag{20}$$

Just like before, each substructure is assigned to a different computating resource and the Schur complement matrix and force are computed independently, so is the internal DoFs upon substitution of constraint DoFs.

1.4 Interface DoFs Reduction

This part will be adapted from [4] (future reading).

The Hurty/Craig-Bampton method in structural dynamics represents the interior dynamics of each subcomponent in a substructured system with a truncated set of normal modes and retains all of the physical degrees of freedom at the substructure interfaces. This makes the assembly of substructures into a reduced-order system model very simple, but means that the reduced-order assembly will have as many interface degrees of freedom as the full model. When the full-model mesh is highly refined, and/or when the system is divided into many subcomponents, this can lead to an unacceptably large system of equations of motion. To overcome this, interface reduction methods aim to reduce the size of the Hurty/Craig-Bampton model by reducing the number of interface degrees of freedom. This research presents a survey of interface reduction methods for Hurty/Craig-Bampton models, and proposes improvements and generalizations to some of the methods. Some of these interface reductions operate on the assembled system-level matrices while others perform reduction locally by considering the uncoupled substructures. The advantages and disadvantages of these methods are highlighted and assessed through comparisons of results obtained from a variety of representative finite element models.

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