Notes and Documentations for Master's Thesis
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Non-Intrusive Surrogate Modelling for Stochastic Linear Structural Systems Based on Collocation Using Substructuring Methods

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1 Physical Domain Decomposition

Complex structures are divided into non-overlapping substructures. System of equations for each substructure is formulated and subsequently assembled to form the global system of equations.

1.1 Substructures' System of Equations

This part is adapted from [1].

The degree of freedoms associated with shared nodes (between two or more substructures) are referred to as boundary DoFs while other are referred to as internal DoFs. The static system of equations for substructure r is given by the following:

$$\begin{bmatrix} \mathbf{F}_r^I \\ \mathbf{F}_r^B \end{bmatrix} = \begin{bmatrix} \mathbf{K}_r^{II} & \mathbf{K}_r^{IB} \\ \mathbf{K}_r^{BI} & \mathbf{K}_r^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{u}_r^I \\ \mathbf{u}_r^B \end{bmatrix} \tag{1}$$

Constraint modes are defined as the mode shapes of internal freedoms due to successive unit displacement of boundary points, all other boundary point being totally constrained. Setting forces associated with internal DoFs to zero in equation (1):

$$\mathbf{0} = \mathbf{K}_r^{IB}\mathbf{u}_r^B + \mathbf{K}_r^{II}\mathbf{u}_r^I + \quad \Longrightarrow \quad \mathbf{u}_r^I = -\left(\mathbf{K}_r^{II}\right)^{-1}\mathbf{K}_r^{IB}\mathbf{u}_r^B$$

From which an expression for constrained modes is obtained:

$$\bar{\mathbf{\Phi}}_r^C = -\left(\mathbf{K}_r^{II}\right)^{-1}\mathbf{K}_r^{IB} \tag{2}$$

Normal modes are defined as the mode shapes of the substructure with totally constrained boundary. These are obtained from the equations

$$\mathbf{0} = \left[\mathbf{K}_r^{II} - \omega_j^2\mathbf{M}_r^{II}\right]\mathbf{\Phi}_{r,j}^I \tag{3}$$

The eigenvectors of equation (3) form the columns of the normal mode matrix $\mathbf{\Phi}_r^N$. Model order reduction is possible through truncation of this matrix's columns i.e. by retaining only some eigenvectors in the reduced normal mode matrix $\bar{\mathbf{\Phi}}_r^N$. Transformation between freedoms in physical space and modal space is given by the following:

$$\mathbf{u}_r = \mathbf{G}_r\bar{\mathbf{u}}_r \quad \text{or} \quad \begin{bmatrix} \mathbf{u}_r^I \\ \mathbf{u}_r^B \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_r^N \\ \bar{\mathbf{u}}_r^C \end{bmatrix} \tag{4}$$

The modal mass and stiffness matrices are given by the followings respectively

$$\bar{\mathbf{M}}_r = \mathbf{G}_r^T\mathbf{M}_r\mathbf{G}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{M}}_r^{NN} & \bar{\mathbf{M}}_r^{NB} \\ \bar{\mathbf{M}}_r^{BN} & \bar{\mathbf{M}}_r^{BB} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_r^{II} & \mathbf{M}_r^{IB} \\ \mathbf{M}_r^{BI} & \mathbf{M}_r^{BB} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Phi}}_r^N & \bar{\mathbf{\Phi}}_r^C \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{5}$$

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$$\bar{\mathbf{K}}_r = \mathbf{G}_r^T \mathbf{K}_r \mathbf{G}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{K}}_r^{NN} & \bar{\mathbf{K}}_r^{NB} \\ \bar{\mathbf{K}}_r^{BN} & \bar{\mathbf{K}}_r^{BB} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^{N} & \bar{\mathbf{\Phi}}_r^{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_r^{II} & \mathbf{K}_r^{IB} \\ \mathbf{K}_r^{BI} & \mathbf{K}_r^{BB} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Phi}}_r^{N} & \bar{\mathbf{\Phi}}_r^{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{6}$$

The modal stiffness matrix is block-diagonal because

$$\begin{aligned} \bar{\mathbf{K}}_r^{NB} &= \left[\bar{\mathbf{\Phi}}_r^{N}\right]^T \mathbf{K}_r^{IB} + \left[\bar{\mathbf{\Phi}}_r^{N}\right]^T \mathbf{K}_r^{II} \bar{\mathbf{\Phi}}_r^{C} \\ &= \left[\bar{\mathbf{\Phi}}_r^{N}\right]^T \mathbf{K}_r^{IB} + \left[\bar{\mathbf{\Phi}}_r^{N}\right]^T \mathbf{K}_r^{II} \left[-\left(\mathbf{K}_r^{II}\right)^{-1} \mathbf{K}_r^{IB}\right] \\ &= \mathbf{0} \end{aligned}$$

The modal load vector is given by the following equation

$$\bar{\mathbf{F}}_r = \mathbf{G}_r^T \mathbf{F}_r \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{F}}_r^{N} \\ \bar{\mathbf{F}}_r^{C} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{\Phi}}_r^{N} & \bar{\mathbf{\Phi}}_r^{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{F}_r^{I} \\ \mathbf{F}_r^{B} \end{bmatrix} \tag{7}$$

1.2 Assembly into Global System of Equations

This part is adapted from [2].

Dynamic equation of a structural system with two substructures is given by the following

$$\mathbf{F} = \mathbf{D}\mathbf{u} \quad \text{or} \quad \begin{bmatrix} \mathbf{F}_\alpha^{I} \\ \mathbf{F}_\beta^{I} \\ \mathbf{F}^{B} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_\alpha^{II} & \mathbf{0} & \mathbf{D}_\alpha^{IB} \\ \mathbf{0} & \mathbf{D}_\beta^{II} & \mathbf{D}_\beta^{IB} \\ \mathbf{D}_\alpha^{BI} & \mathbf{D}_\beta^{BI} & \mathbf{D}^{BB} \end{bmatrix} \begin{bmatrix} \mathbf{u}_\alpha^{I} \\ \mathbf{u}_\beta^{I} \\ \mathbf{u}^{B} \end{bmatrix} \tag{8}$$

Using transformations $\bar{\mathbf{F}} = \mathbf{G}^T\mathbf{F}$, $\bar{\mathbf{D}} = \mathbf{G}^T\mathbf{D}\mathbf{G}$, and $\bar{\mathbf{u}} = \mathbf{G}\mathbf{u}$, the equation can be transformed into modal space

$$\bar{\mathbf{F}} = \bar{\mathbf{D}}\bar{\mathbf{u}} \quad \text{or} \quad \begin{bmatrix} \bar{\mathbf{F}}_\alpha^{N} \\ \bar{\mathbf{F}}_\beta^{N} \\ \bar{\mathbf{F}}^{C} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{D}}_\alpha^{NN} & \mathbf{0} & \bar{\mathbf{D}}_\alpha^{NC} \\ \mathbf{0} & \bar{\mathbf{D}}_\beta^{NN} & \bar{\mathbf{D}}_\beta^{NC} \\ \bar{\mathbf{D}}_\alpha^{CN} & \bar{\mathbf{D}}_\beta^{CN} & \bar{\mathbf{D}}^{CC} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}_\alpha^{N} \\ \bar{\mathbf{u}}_\beta^{N} \\ \bar{\mathbf{u}}^{C} \end{bmatrix} \tag{9}$$

The transformation matrix is given by the following

$$\mathbf{G} = \begin{bmatrix} \bar{\mathbf{\Phi}}_\alpha^{N} & \mathbf{0} & \bar{\mathbf{\Phi}}_\alpha^{C} \\ \mathbf{0} & \bar{\mathbf{\Phi}}_\beta^{N} & \bar{\mathbf{\Phi}}_\beta^{C} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{10}$$

Generalisation for more than two substructures is given in [3] (future reading).

1.3 Schur Complement

This part is adapted from [2].

The assembled modal system equation can be solved in two steps. First, the following equation is solved for constrained modes:

$$\left[\bar{\mathbf{D}}^{CC} - \bar{\mathbf{D}}_\alpha^{CN}\left(\bar{\mathbf{D}}_\alpha^{NN}\right)^{-1}\bar{\mathbf{D}}_\alpha^{NC} - \bar{\mathbf{D}}_\beta^{CN}\left(\bar{\mathbf{D}}_\beta^{NN}\right)^{-1}\bar{\mathbf{D}}_\beta^{NC}\right]\bar{\mathbf{u}}^{C} = \bar{\mathbf{F}}^{C} - \bar{\mathbf{D}}_\alpha^{CN}\left(\bar{\mathbf{D}}_\alpha^{NN}\right)^{-1}\bar{\mathbf{F}}_\alpha^{N} - \bar{\mathbf{D}}_\beta^{CN}\left(\bar{\mathbf{D}}_\beta^{NN}\right)^{-1}\bar{\mathbf{F}}_\beta^{N} \tag{11}$$

Second, the solution is substituted to the following equations and they can be solved for the internal modes simultaneously.

$$\bar{\mathbf{D}}_{\alpha}^{NN}\bar{\mathbf{u}}_{\alpha}^{N} = \bar{\mathbf{F}}_{\alpha}^{N} - \bar{\mathbf{D}}_{\alpha}^{NC}\bar{\mathbf{u}}^{C} \tag{12}$$

$$\bar{\mathbf{D}}_{\beta}^{NN}\bar{\mathbf{u}}_{\beta}^{N} = \bar{\mathbf{F}}_{\beta}^{N} - \bar{\mathbf{D}}_{\beta}^{NC}\bar{\mathbf{u}}^{C} \tag{13}$$

Generalisation for more than two substructures is given in [3] (future reading).

1.4 Interface DoFs Reduction

This part will be adapted from [4] (future reading).

The Hurty/Craig-Bampton method in structural dynamics represents the interior dynamics of each subcomponent in a substructured system with a truncated set of normal modes and retains all of the physical degrees of freedom at the substructure interfaces. This makes the assembly of substructures into a reduced-order system model very simple, but means that the reduced-order assembly will have as many interface degrees of freedom as the full model. When the full-model mesh is highly refined, and/or when the system is divided into many subcomponents, this can lead to an unacceptably large system of equations of motion. To overcome this, interface reduction methods aim to reduce the size of the Hurty/Craig-Bampton model by reducing the number of interface degrees of freedom. This research presents a survey of interface reduction methods for Hurty/Craig-Bampton models, and proposes improvements and generalizations to some of the methods. Some of these interface reductions operate on the assembled system-level matrices while others perform reduction locally by considering the uncoupled substructures. The advantages and disadvantages of these methods are highlighted and assessed through comparisons of results obtained from a variety of representative finite element models.

References

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