

# GR Light Deflection

## Question

Using the small-angle approximation, compare the trajectory of a light ray passing near a massive body under Newtonian mechanics and General Relativity. Assume the closest approach distance is  $R$  and define the ratio  $R_g/R$ , where  $R_g$  is the Schwarzschild radius of the body. Derive approximate expressions for the light path in both cases and sketch/plot the resulting trajectories for different values of  $R_g/R$  (e.g., 0, 0.01, 0.02, 0.03, 0.05). Explain how the deflection changes as  $R_g/R$  increases.

## Solution

### 1. Core Formulas (with $R=1$ )

Known,

- $sp = \sin \varphi$
- $cp = \cos \varphi = \sqrt{1 - sp^2}$

**Newtonian (no deflection)**

- $r_{new} = \frac{1}{sp}$
- $x_{new} = r_{new} \cdot cp = \frac{cp}{sp}$
- $y_{new} = r_{new} \cdot sp = 1 \rightarrow$  Newtonian gives a horizontal line  $y = 1$ .

**Approximate GR**

- $u = sp + MR$
- $r_{GR} = \frac{1}{u} = \frac{1}{sp+MR}$
- $y_{GR} = r_{GR} \cdot sp = \frac{sp}{sp+MR}$
- $x_{GR} = r_{GR} \cdot cp = \frac{cp}{sp+MR}$

So, the GR point  $(x_{GR}, y_{GR})$  is just the Newtonian  $(c_p, s_p)$  scaled by  $1/(s_p + MR)$ .

### 2. Full Step-by-Step Example

Take  $sp = 1$  and  $MR = 0.01$ .

1. Compute  $cp$

$$cp = \sqrt{1 - sp^2} = \sqrt{1 - 0.1^2} = \sqrt{0.99} \approx 0.994987$$

2. Newtonian

- $r_{new} = \frac{1}{0.1} = 10$
- $x_{new} = r_{new} \cdot cp = 10 \times 0.994987 = 9.949874$
- $y_{new} = 1$ . (Because  $r_{new} \cdot sp = \left(\frac{1}{sp}\right) \cdot sp = 1$ )

### 3. GR ( $MR = 0.01$ )

- $u = sp + MR = 0.1 + 0.01 = 0.11$
- $r_{GR} = \frac{1}{0.11} = \frac{100}{11} \approx 9.090909$
- $y_{GR} = \frac{sp}{sp+MR} = \frac{0.1}{0.11} = \frac{10}{11} \approx 0.909091$
- $x_{GR} = \frac{cp}{sp+MR} = \frac{0.994987}{0.11} \approx 9.045340$

## 3. Quick Results (Rounded) for the Chosen Sample Points

$sp = 0.1$  ( $cp = 0.994987$ )

- Newtonian:  $x = 9.949874, y = 1.000000$
- $MR = 0.01$ :  $x \approx 9.045340, y \approx 0.909091$
- $MR = 0.02$ :  $x \approx 8.291562, y \approx 0.833333$
- $MR = 0.05$ :  $x \approx 6.633250, y \approx 0.666667$

$sp = 0.5$  ( $cp = 0.866025$ )

- Newtonian:  $x = 1.732051, y = 1.000000$
- $MR = 0.01$ :  $x \approx 1.698089, y \approx 0.980392$
- $MR = 0.02$ :  $x \approx 1.665433, y \approx 0.961538$
- $MR = 0.05$ :  $x \approx 1.574592, y \approx 0.909091$

$sp = 0.9$  ( $cp = 0.435890$ )

- Newtonian:  $x = 0.484322, y = 1.000000$
- $MR = 0.01$ :  $x \approx 0.478999, y \approx 0.989011$
- $MR = 0.02$ :  $x \approx 0.473793, y \approx 0.978261$
- $MR = 0.05$ :  $x \approx 0.458831, y \approx 0.947368$

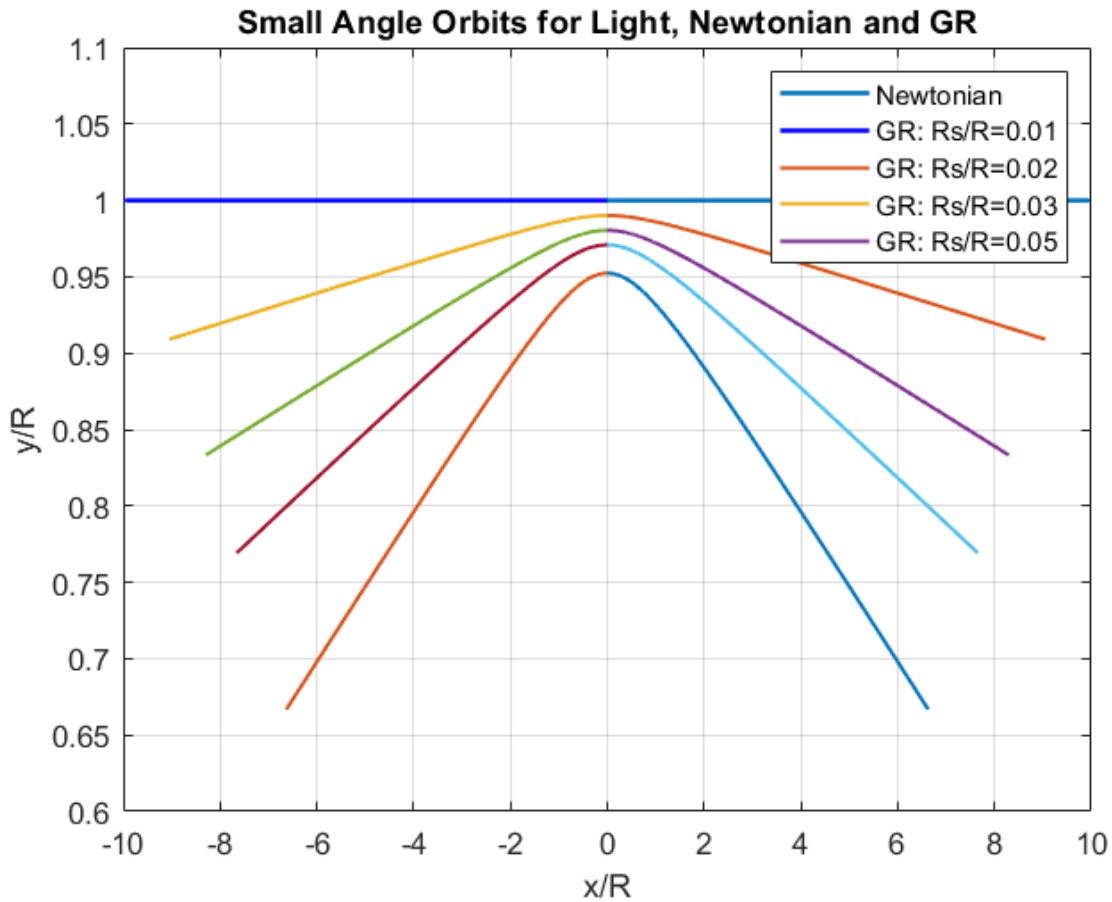
## 4. Short Interpretation

- Newtonian always sits at  $y = 1$  (the horizontal top line in the plot).
- GR points have  $y_{GR} = \frac{sp}{sp+MR} < 1$ . So for fixed angle  $sp$ : *larger*  $MR \rightarrow$  *smaller*  $y_{GR}$  (more downward deflection).
- For **small**  $sp$  (e.g. 0.1) the differences between Newtonian and GR are large  $\rightarrow$  that's why the curve looks "sharper" at the peak. For  **$sp$  near 1** the GR and Newtonian values are close  $\rightarrow$  the curves meet near the top.

## 5. A Short Recipe

- Pick  $sp$  (e.g. 0.3). Compute  $cp = \sqrt{1 - sp^2}$ .
- Newtonian:  $x_{new} = cp/sp$ ,  $y_{new} = 1$ .
- For each: compute  $u = sp + MR$ . Then,  $x_{GR} = cp/u$ ,  $y_{GR} = sp/u$ .
- Plot symmetric points  $(\pm x, y)$ .

## Result



**Figure 1** Schematic view of an orbital trajectory for light in GR.