GR Light Deflection

Question

Using the small-angle approximation, compare the trajectory of a light ray passing near a massive body under Newtonian mechanics and General Relativity. Assume the closest approach distance is R and define the ratio R_8/R, where R_8 is the Schwarzschild radius of the body. Derive approximate expressions for the light path in both cases and sketch/plot the resulting trajectories for different values of R_8/R (e.g., 0, 0.01, 0.02, 0.03, 0.05). Explain how the deflection changes as R_8/R increases.

Solution

1. Core Formulas (with R =1)

Known,

- $sp = \sin\varphi$
- $cp = \cos\varphi = \sqrt{1 - sp^2}$

Newtonian (no deflection)

- $r_{new} = \frac{1}{sp}$
- $x_{new} = r_{new} \cdot cp = \frac{cp}{sp}$
- $y_{new} = r_{new} \cdot sp = 1$ → Newtonian gives a horizontal line $y = 1$.

Approximate GR

- $u = sp + MR$
- $r_{GR} = \frac{1}{u} = \frac{1}{sp+MR}$
- $y_{GR} = r_{GR} \cdot sp = \frac{sp}{sp+MR}$
- $x_{GR} = r_{GR} \cdot cp = \frac{cp}{sp+MR}$

So, the GR point (x_{GR}, y_{GR}) is just the Newtonian (c_p, s_p) scaled by $1/(s_p + MR)$.

2. Full Step-by-Step Example

Take $sp = 1$ and $MR = 0.01$.

1. Compute cp

$$cp = \sqrt{1 - sp^2} = \sqrt{1 - 0.1^2} = \sqrt{0.99} \approx 0.994987$$

2. Newtonian

- $r_{new} = \frac{1}{0.1} = 10$
- $x_{new} = r_{new} \cdot cp = 10 \times 0.994987 = 9.949874$
- $y_{new} = 1$. (Because $r_{new} \cdot sp = \left(\frac{1}{sp}\right) \cdot sp = 1$)

3. GR ($MR = 0.01$)
   - $u = sp + MR = 0.1 + 0.01 = 0.11$
   - $r_{GR} = \frac{1}{0.11} = \frac{100}{11} \approx 9.090909$
   - $y_{GR} = \frac{sp}{sp+MR} = \frac{0.1}{0.11} = \frac{10}{11} \approx 0.909091$
   - $x_{GR} = \frac{cp}{sp+MR} = \frac{0.994987}{0.11} \approx 9.045340$

3. Quick Results (Rounded) for the Chosen Sample Points

$sp = 0.1$ ($cp = 0.994987$)

- Newtonian: $x = 9.949874$, $y = 1.000000$
- $MR = 0.01$: $x \approx 9.045340$, $y \approx 0.909091$
- $MR = 0.02$: $x \approx 8.291562$, $y \approx 0.833333$
- $MR = 0.05$: $x \approx 6.633250$, $y \approx 0.666667$

$sp = 0.5$ ($cp = 0.866025$)

- Newtonian: $x = 1.732051$, $y = 1.000000$
- $MR = 0.01$: $x \approx 1.698089$, $y \approx 0.980392$
- $MR = 0.02$: $x \approx 1.665433$, $y \approx 0.961538$
- $MR = 0.05$: $x \approx 1.574592$, $y \approx 0.909091$

$sp = 0.9$ ($cp = 0.435890$)

- Newtonian: $x = 0.484322$, $y = 1.000000$
- $MR = 0.01$: $x \approx 0.478999$, $y \approx 0.989011$
- $MR = 0.02$: $x \approx 0.473793$, $y \approx 0.978261$
- $MR = 0.05$: $x \approx 0.458831$, $y \approx 0.947368$

4. Short Interpretation

- Newtonian always sits at $y = 1$ (the horizontal top line in the plot).
- GR points have $y_{GR} = \frac{sp}{sp+MR} < 1$. So for fixed angle sp: $larger\ MR \rightarrow smaller\ y_{GR}$ (more downward deflection).
- For **small sp** (e.g. 0.1) the differences between Newtonian and GR are large → that's why the curve looks "sharper" at the peak. For **sp near 1** the GR and Newtonian values are close → the curves meet near the top.

5. A Short Recipe

- Pick sp (e.g. 0.3). Compute $cp = \sqrt{1 - sp^2}$.
- Newtonian: $x_{new} = cp/sp$, $y_{new} = 1$.
- For each: compute $u = sp + MR$. Then, $x_{GR} = cp/u$, $y_{GR} = sp/u$.
- Plot symmetric points $(\pm x, y)$.

Result

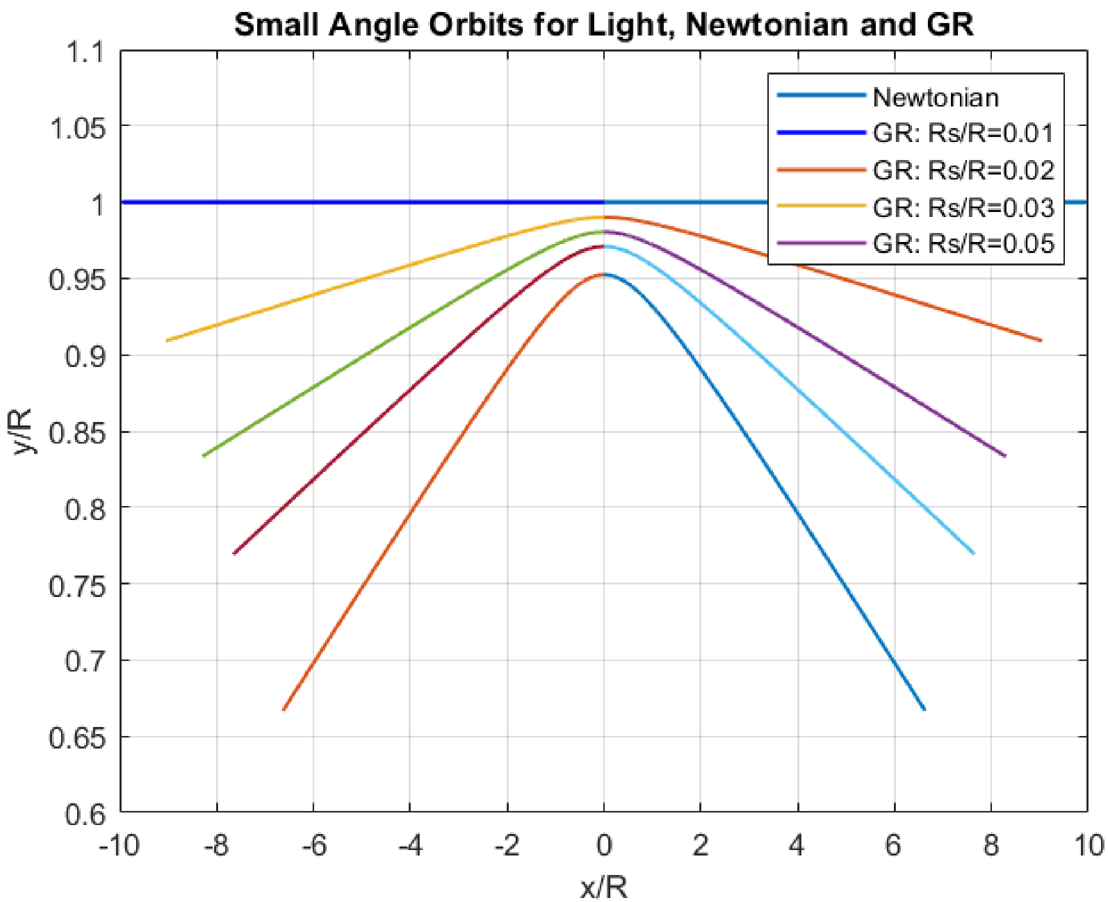

Figure 1. Schematic view of an orbital trajectory for light in GR.