Implementation of MNIST Classification Using Multilayer Perceptron and Convolutional Neural Networks

Introduction to Deep Learning CS4851/6851 - Dr. Yue Wang

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1. Introduction

1.1 Background and Motivation

Deep learning has revolutionized fields such as computer vision, natural language processing, and signal processing. Among its many architectures, **Multilayer Perceptron (MLP)** and **Convolutional Neural Network (CNN)** are foundational. MLPs serve as universal function approximators, while CNNs excel in spatial data processing, especially image classification.

This project explores the construction of both MLP and CNN models **from scratch using only NumPy**. By re-implementing these architectures without relying on high-level libraries like PyTorch or TensorFlow, we gain a deeper understanding of how data flows through layers and how gradients are propagated during learning.

1.2 Problem Definition

Our task is to classify handwritten digits from the MNIST dataset using two approaches: - A fully connected MLP with one hidden layer. - A CNN incorporating one convolutional layer.

1.3 System Overview

The system includes the following components: 1. **Data preprocessing** using TorchVision, normalization, and batching. 2. **MLP implementation:** one hidden layer, and an output softmax layer. 3. **CNN implementation:** one convolution layer (with one kernel), and an output softmax layer. 4. **Manual backpropagation** for both architectures, with weight updates using gradient descent. 5. **Evaluation** of model performance on the MNIST test subset.

2. Methodology

2.1 MLP

2.1.1 Architecture The MLP processes a flattened 784-dimensional input (28×28 image) through:

- One hidden layer with a variable number of neurons,
- An output layer with 10 neurons,
- Sigmoid activation in the hidden layer and softmax activation in the output layer.

MLP Forward Pass:

```
def forward(self, x):
    self.hidden_layer_input = np.dot(x, self.weights1) + self.bias1
    self.hidden_layer_output = sigmoid(self.hidden_layer_input)
    self.output_layer_input = np.dot(self.hidden_layer_output, self.weights2) + self.bias2
    outputs = softmax(self.output_layer_input)
    return outputs
```

MLP Backward Pass:

```
def backward(self, x, y, pred):
    one_hot_y = np.eye(self.output_size)[y]
    d_output = pred - one_hot_y
    d_weights2 = np.dot(self.hidden_layer_output.T, d_output)
    d_bias2 = np.sum(d_output, axis=0)
    d_hidden = np.dot(d_output, self.weights2.T)
                * (self.hidden_layer_output
                    * (1 - self.hidden_layer_output))
    d_weights1 = np.dot(x.T, d_hidden)
    d_bias1 = np.sum(d_hidden, axis=0)

    self.weights2 -= self.lr * d_weights2
    self.bias2 -= self.lr * d_bias2
    self.weights1 -= self.lr * d_weights1
    self.bias1 -= self.lr * d_bias1
```

Example `MLP_3.py` output with hidden size = 256:

```
The number of training data: 60000
The number of testing data: 10000
Epoch 1/100, Train Loss: 0.8850 Train Accuracy: 0.7027
Epoch 2/100, Train Loss: 0.2825 Train Accuracy: 0.9169
Epoch 3/100, Train Loss: 0.2201 Train Accuracy: 0.9349
Epoch 4/100, Train Loss: 0.1861 Train Accuracy: 0.9439
Epoch 5/100, Train Loss: 0.1630 Train Accuracy: 0.9516
...
Epoch 96/100, Train Loss: 0.0044 Train Accuracy: 0.9997
Epoch 97/100, Train Loss: 0.0042 Train Accuracy: 0.9997
Epoch 98/100, Train Loss: 0.0042 Train Accuracy: 0.9997
Epoch 99/100, Train Loss: 0.0041 Train Accuracy: 0.9997
Epoch 100/100, Train Loss: 0.0041 Train Accuracy: 0.9997
Test Accuracy: 0.9701
```

2.1.2 Gradient computaion We consider a standard Multilayer Perceptron (MLP) with: - Input: $\mathbf{x} \in \mathbb{R}^d$ (flattened image vector of size 784) - Hidden layer: weights $\mathbf{W}_1 \in \mathbb{R}^{d \times h}$, bias $\mathbf{b}_1 \in \mathbb{R}^h$ - Activation: sigmoid function $\sigma(z) = \frac{1}{1+e^{-z}}$ - Output layer: weights $\mathbf{W}_2 \in \mathbb{R}^{h \times 10}$, bias $\mathbf{b}_2 \in \mathbb{R}^{10}$ - Output activation: softmax

The model computes:

$$\mathbf{z}_1 = \mathbf{x}\mathbf{W}_1 + \mathbf{b}_1$$

$$\mathbf{h} = \sigma(\mathbf{z}_1)$$

$$\mathbf{z}_2 = \mathbf{h}\mathbf{W}_2 + \mathbf{b}_2$$

$$\hat{\mathbf{y}} = \text{softmax}(\mathbf{z}_2)$$

where $\hat{\mathbf{y}}$ is the predicted probability distribution over 10 classes.

The loss is cross-entropy:

$$L = -\sum_{i=1}^{10} y_i \log(\hat{y}_i)$$

where $\mathbf{y}$ is the true one-hot encoded label vector.

Step 1: Derivative of Loss w.r.t. Output Logits For softmax + cross-entropy, the derivative simplifies to:

$$\frac{\partial L}{\partial \mathbf{z}_2} = \hat{\mathbf{y}} - \mathbf{y}$$

This gives the error at the output layer directly.

Step 2: Gradients for Output Layer Parameters Compute gradients of the loss with respect to the second layer weights$\mathbf{W}_2$and bias$\mathbf{b}_2$:

- Gradient w.r.t. $\mathbf{W}_2$:
$$\frac{\partial L}{\partial \mathbf{W}_2} = \mathbf{h}^\top (\hat{\mathbf{y}} - \mathbf{y})$$
- Gradient w.r.t. $\mathbf{b}_2$:
$$\frac{\partial L}{\partial \mathbf{b}_2} = \hat{\mathbf{y}} - \mathbf{y}$$

  where $\mathbf{h}$ is the hidden layer output (after sigmoid activation).

Step 3: Backpropagate Through Hidden Layer

- Backprop through $\mathbf{z}_2$:
$$\frac{\partial L}{\partial \mathbf{h}} = (\hat{\mathbf{y}} - \mathbf{y})\mathbf{W}_2^\top$$
- Sigmoid derivative:
$$\sigma'(z) = \sigma(z)(1 - \sigma(z))$$

  thus:
$$\frac{\partial L}{\partial \mathbf{z}_1} = \frac{\partial L}{\partial \mathbf{h}} \odot \sigma(\mathbf{z}_1)(1 - \sigma(\mathbf{z}_1))$$

  where $\odot$ denotes element-wise (Hadamard) product.

Step 4: Gradients for Hidden Layer Parameters

- Gradient w.r.t. $\mathbf{W}_1$:
$$\frac{\partial L}{\partial \mathbf{W}_1} = \mathbf{x}^\top \left(\frac{\partial L}{\partial \mathbf{z}_1}\right)$$
- Gradient w.r.t. $\mathbf{b}_1$:
$$\frac{\partial L}{\partial \mathbf{b}_1} = \frac{\partial L}{\partial \mathbf{z}_1}$$

Step 5: Update Rules After computing all gradients, the parameters are updated using gradient descent:

- Update weights and biases:

$$\mathbf{W}_2 \leftarrow \mathbf{W}_2 - \eta \frac{\partial L}{\partial \mathbf{W}_2}$$
$$\mathbf{b}_2 \leftarrow \mathbf{b}_2 - \eta \frac{\partial L}{\partial \mathbf{b}_2}$$
$$\mathbf{W}_1 \leftarrow \mathbf{W}_1 - \eta \frac{\partial L}{\partial \mathbf{W}_1}$$
$$\mathbf{b}_1 \leftarrow \mathbf{b}_1 - \eta \frac{\partial L}{\partial \mathbf{b}_1}$$

where η is the learning rate.

Remarks:

- The softmax + cross-entropy derivative simplifies the first step: no need to compute softmax derivative separately.
- Proper handling of element-wise derivatives of the sigmoid activation is critical to avoid incorrect gradient shapes.
- All computations assume batch mode: $\mathbf{x}$ can represent multiple samples at once.

2.2 CNN

2.2.1 Architecture The CNN model processes a 28×28 grayscale image through:

- A single convolutional layer with multiple filters,
- ReLU activation after convolution,
- Flattening,
- A fully connected layer with 10 outputs.

Weight Initialization

```
self.filters = np.random.randn(num_filters,
                               1,
                               kernel_size,
                               kernel_size) *
               np.sqrt(2.0 / (kernel_size * kernel_size))
```

According to *"Delving Deep into Rectifiers: Surpassing Human-Level Performance on ImageNet Classification" – He et al., 2015.*, this rescales the weights to preserve the variance of activations across layers during forward propagation.

CNN Forward Pass:

```
def forward(self, x):
    self.x = x
    self.conv = self.convolve(x, self.filters)
    self.relu = relu(self.conv)
    self.flat = self.relu.reshape(x.shape[0], -1)
    self.logits = np.dot(self.flat, self.fc_weights) + self.fc_bias
    self.probs = softmax(self.logits)
    return self.probs
```

Vectorized CNN Convolution (Efficient Implementation):

```python
def convolve(self, x, filters):
    B, _, H, W = x.shape
    F, _, KH, KW = filters.shape
    OH, OW = H - KH + 1, W - KW + 1
    patches = np.lib.stride_tricks.as_strided(
        x,
        shape=(B, OH, OW, KH, KW),
        strides=(x.strides[0], x.strides[2], x.strides[3], x.strides[2], x.strides[3]),
        writeable=False
    ).reshape(B, OH * OW, KH * KW)
    filters_flat = filters.reshape(F, KH * KW)
    out = np.matmul(patches, filters_flat.T)
    return out.transpose(0, 2, 1).reshape(B, F, OH, OW)
```

Remarks:

- The convolution operation is transformed into a batch matrix multiplication by unfolding local patches into rows ("im2col" style) and reshaping filters into rows. Each patch becomes a vector.
- Mathematically, if `x_patch` has shape `(B, OH*OW, KH*KW)` and `filters_flat` has shape `(F, KH*KW)`, then:

$$\text{output}[b, :, i] = \text{filters} \times x\_\text{patch}[b, i]^T$$

where $x\_\text{patch}[b, i]$ is the flattened receptive field.

CNN Backward Pass:

```python
def backward(self, x, y, pred):
    B = y.shape[0]
    y_onehot = np.eye(self.fc_output_size)[y]
    d_logits = (pred - y_onehot) / B

    d_fc_weights = np.dot(self.flat.T, d_logits)
    d_fc_bias = np.sum(d_logits, axis=0)
    d_flat = np.dot(d_logits, self.fc_weights.T)
    d_relu = d_flat.reshape(self.relu.shape)
    d_conv = d_relu * (self.conv > 0)

    B, F, OH, OW = d_conv.shape
    patches = np.lib.stride_tricks.as_strided(
        self.x,
        shape=(B, OH, OW, self.kernel_size, self.kernel_size),
        strides=(
            self.x.strides[0],
            self.x.strides[2],
            self.x.strides[3],
            self.x.strides[2],
            self.x.strides[3]),
        writeable=False
    ).reshape(B, OH * OW, self.kernel_size * self.kernel_size)

    d_out_flat = d_conv.reshape(B, F, OH * OW)
    d_filters = np.zeros_like(self.filters)
    for b in range(B):
        d_filters += np.matmul(d_out_flat[b], patches[b])
                       .reshape(F, 1, self.kernel_size, self.kernel_size)

    self.fc_weights -= self.lr * d_fc_weights
    self.fc_bias -= self.lr * d_fc_bias
    self.filters -= self.lr * d_filters
```

Remarks:

- During backpropagation, output gradients `d_conv` are reshaped to align with the patch matrix.
- The filter gradient is efficiently computed using:

$\Delta W = \sum_b d\_out[b] \times x\_patch[b]$

where `d_out[b]` is the output gradient for batch `b` reshaped as `(F, OH*OW)`, and `x_patch[b]` is the local receptive fields. - This matrix multiplication accumulates gradients for all patches and batches simultaneously, maintaining full batch parallelism.

Example `CNN_3.py` output with kernel size = 7:

```
Training samples: 60000
Testing samples: 10000
Epoch 1/5, Train Loss: 0.6728 Train Accuracy: 0.7876
Epoch 2/5, Train Loss: 0.3537 Train Accuracy: 0.8979
Epoch 3/5, Train Loss: 0.3120 Train Accuracy: 0.9103
Epoch 4/5, Train Loss: 0.2893 Train Accuracy: 0.9170
Epoch 5/5, Train Loss: 0.2736 Train Accuracy: 0.9219
Test Accuracy: 0.9252
```

2.2.2 Gradient Computation We consider a simple CNN with: - Input: $\mathbf{x} \in \mathbb{R}^{B\times 1\times H\times W}$ (batch of grayscale images) - Convolutional layer: filters $\mathbf{W}_{\text{conv}} \in \mathbb{R}^{F\times 1\times K\times K}$ (F filters) - Activation: ReLU - Fully connected layer: weights $\mathbf{W}_{\text{fc}} \in \mathbb{R}^{d\times 10}$ - Output activation: softmax

The model computes:

1. **Convolution**:
$$\mathbf{z}_{\text{conv}} = \text{conv2d}(\mathbf{x}, \mathbf{W}_{\text{conv}})$$
2. **ReLU Activation**:
$$\mathbf{h}_{\text{conv}} = \text{ReLU}(\mathbf{z}_{\text{conv}})$$
3. **Flattening**:
$$\mathbf{h}_{\text{flat}} = \text{flatten}(\mathbf{h}_{\text{conv}})$$
4. **Fully Connected Layer**:
$$\mathbf{z}_{\text{fc}} = \mathbf{h}_{\text{flat}}\mathbf{W}_{\text{fc}} + \mathbf{b}_{\text{fc}}$$
5. **Softmax Output**:
$$\hat{\mathbf{y}} = \text{softmax}(\mathbf{z}_{\text{fc}})$$

The loss is cross-entropy:

$$L = -\sum_{i=1}^{10} y_i \log(\hat{y}_i)$$

Step 1: Derivative of Loss w.r.t. FC Output

$$\frac{\partial L}{\partial \mathbf{z}_{\text{fc}}} = \hat{\mathbf{y}} - \mathbf{y}$$

Step 2: Gradients for Fully Connected Layer

- Gradient w.r.t. FC weights:
$$\frac{\partial L}{\partial \mathbf{W}_{\text{fc}}} = \mathbf{h}_{\text{flat}}^{\top}(\hat{\mathbf{y}} - \mathbf{y})$$
- Gradient w.r.t. FC bias:
$$\frac{\partial L}{\partial \mathbf{b}_{\text{fc}}} = \hat{\mathbf{y}} - \mathbf{y}$$
- Backpropagate into flattened feature map:
$$\frac{\partial L}{\partial \mathbf{h}_{\text{flat}}} = (\hat{\mathbf{y}} - \mathbf{y})\mathbf{W}_{\text{fc}}^{\top}$$

Step 3: Reshape and Backpropagate Through ReLU

- Reshape:
$$\frac{\partial L}{\partial \mathbf{h}_{\text{conv}}} = \text{reshape}\left(\frac{\partial L}{\partial \mathbf{h}_{\text{flat}}}\right)$$
- ReLU derivative:
$$\frac{\partial L}{\partial \mathbf{z}_{\text{conv}}} = \frac{\partial L}{\partial \mathbf{h}_{\text{conv}}} \odot \mathbf{1}_{\mathbf{z}_{\text{conv}}>0}$$

where $\mathbf{1}_{\mathbf{z}_{\text{conv}}>0}$ is an indicator function that is 1 where $\mathbf{z}_{\text{conv}}$ was positive and 0 elsewhere.

Step 4: Gradients for Convolutional Filters At this point, we need to compute the gradient of the loss with respect to the convolution filters.

Using vectorized notation:

Let: - $\mathbf{P}$ be the matrix of extracted patches from $\mathbf{x}$, with shape $(B, OH \times OW, K \times K)$ -$\delta_{\text{conv}} = \frac{\partial L}{\partial \mathbf{z}_{\text{conv}}}$, reshaped as $(B, F, OH \times OW)$

Then, for each batch$b$, the gradient for convolutional filters is:

$$\Delta \mathbf{W}_{\text{conv}} = \sum_{b=1}^{B} \delta_{\text{conv}}[b] \times \mathbf{P}[b]$$

Specifically: - Multiply each $(F, OH \times OW)$ gradient slice with the corresponding $(OH \times OW, K \times K)$ patch slice. - Resulting in a(F, K, K)tensor.

Summed over the batch.

Step 5: Summary of Update Rules The parameters are updated using gradient descent:

- Update FC layer:

$$\mathbf{W}_{\text{fc}} \leftarrow \mathbf{W}_{\text{fc}} - \eta \frac{\partial L}{\partial \mathbf{W}_{\text{fc}}}$$

$$\mathbf{b}_{\text{fc}} \leftarrow \mathbf{b}_{\text{fc}} - \eta \frac{\partial L}{\partial \mathbf{b}_{\text{fc}}}$$

- Update convolutional filters:

$$\mathbf{W}_{\text{conv}} \leftarrow \mathbf{W}_{\text{conv}} - \eta \Delta \mathbf{W}_{\text{conv}}$$

where η is the learning rate.

Key Remarks:

- Convolution backpropagation is efficiently computed by reshaping the input into patches, and using matrix multiplication to compute gradients.
- ReLU activation masks the gradient, ensuring only the active neurons propagate gradients backward.
- The spatial dimensions shrink in the forward convolution (valid convolution), so the backpropagated gradients must align carefully with reduced sizes.
- Efficient batch matrix multiplications allow computing all filter updates across the batch without explicit nested loops.

3. Evaluation

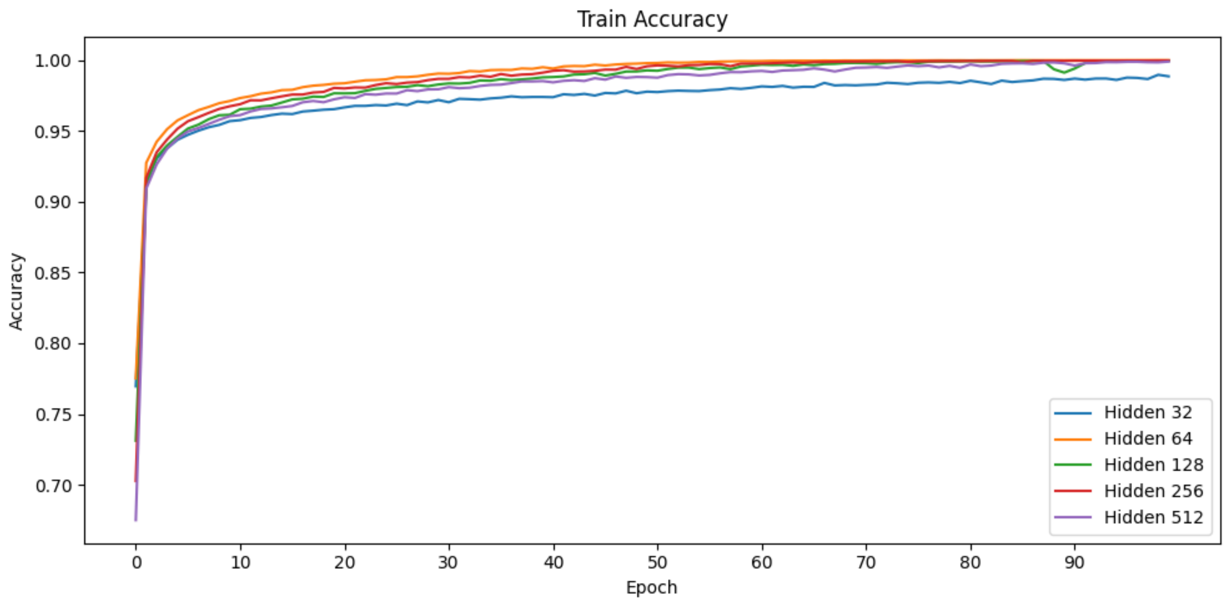
3.1 Experiment Settings

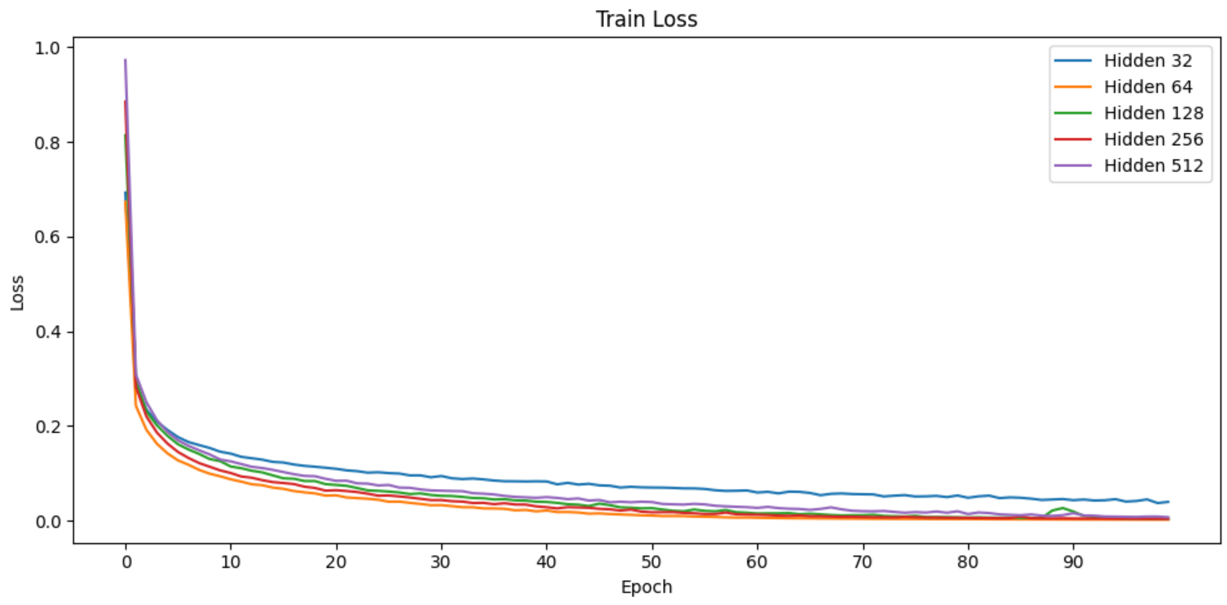
- **Data**: 60,000 training and 10,000 testing images from MNIST
- **MLP**: 784-input, one hidden layer with varying sizes, 10-output, softmax
- **Learning rate**: 0.01
- **Batch size**: 128
- **CNN**: 28x28-input, one convolution kernel with varying sizes, ReLU, FC(10)
- **Epochs**:
  - **MLP**: 100
  - **CNN**: 5

The five hidden sizes tested for MLP were **32, 64, 128, 256, 512**.

3.2 MLP Learning Curves

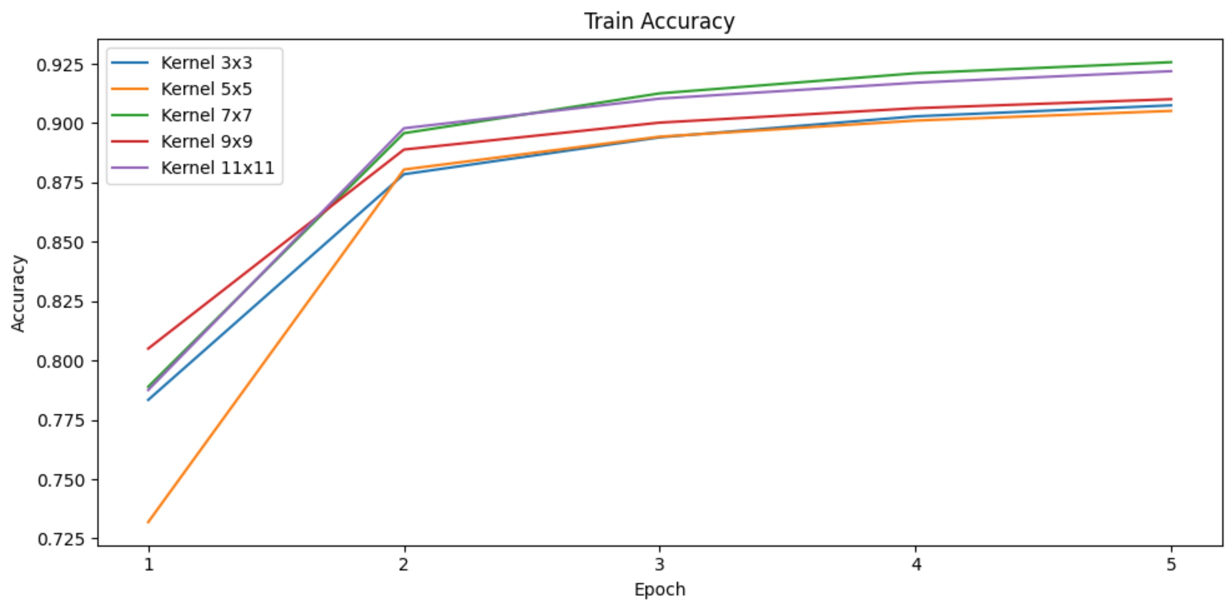
Below are the training loss and accuracy plots for each hidden size:

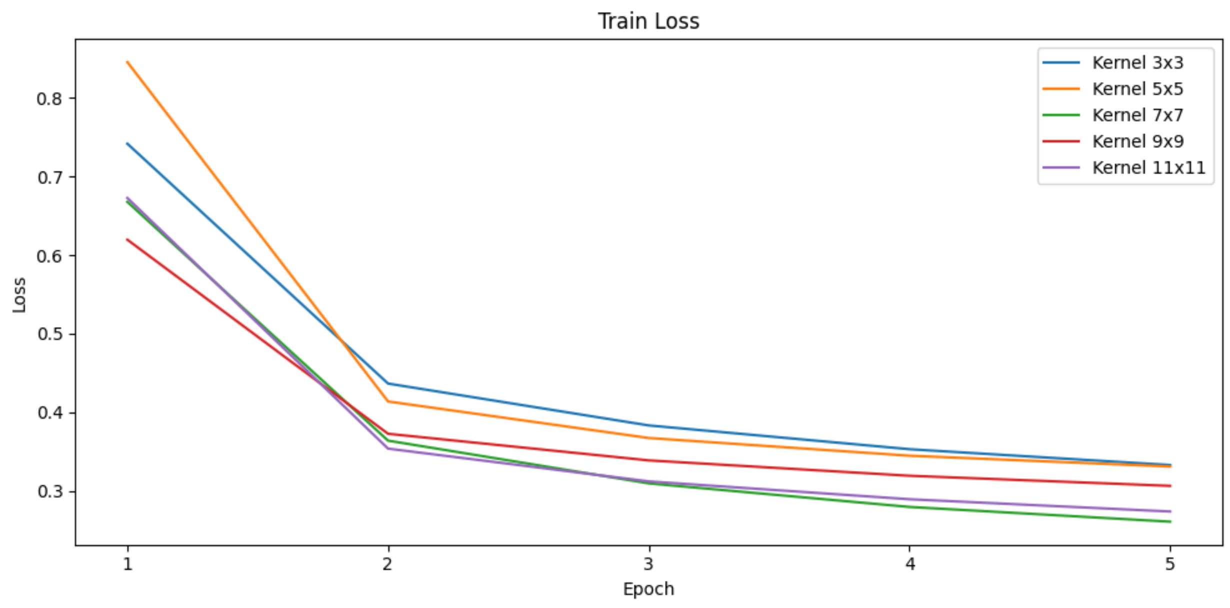




3.3 CNN Learning Curves

Below are the training loss and accuracy plots for each kernel size:





3.4 MLP Accuracy Results

The table below shows test set accuracy for each `hidden_size` value:

Hidden Layer Size	Test Accuracy
32	95.69%
64	96.94%
128	96.79%
256	**97.01%**
512	96.40%

We found that increasing the hidden layer size generally improved accuracy, but performance plateaued beyond 256. The best result was obtained with `hidden_size = 256`.

3.5 CNN Accuracy Results

The table below shows test set accuracy for each `kernel_size` value:

Kernel Size	Test Accuracy
3 x 3	91.16%
5 x 5	90.47%
7 x 7	***92.77%***
9 x 9	91.29%
11 x 11	92.52%

As can be seen, the best result was obtained with `kernel_size = 7`.

4. Learning Outcomes

4.1 Tasks Accomplished

- Implemented both MLP and CNN architectures from scratch in NumPy.
- Derived and implemented full forward and backward propagation manually.
- Designed a full training and evaluation pipeline.
- Found best hyperparameters (`hidden_size` for MLP and `kernel_size` for CNN) during evaluation of our models.
- Gained practical experience debugging gradient flow in neural nets.

4.2 Team Contributions

This project was a team effort by three members:

- **Reza Mansouri**: Implemented the CNN architecture, developed the vectorized convolution and backpropagation, and trained CNN models.
- **Pranjal Patil**: Implemented the MLP architecture, designed the MLP training loop, and handled saving/loading parameters.
- **Ven Mendelssohn**: Integrated training/evaluation pipelines, generated plots, and wrote and organized the final project report.

All members contributed to testing, debugging, and system integration.

4.3 Challenges and Lessons Learned

- Implementing convolutional backpropagation without autograd was complex.
- Proper memory management and vectorization were crucial for speed.
- Manual implementation helped us understand how modern frameworks work under the hood.
- More parameters does not always mean better performance, since overfitting is always possible.

4.4 Conclusion

Rebuilding MLP and CNN architectures from scratch allowed us to deeply understand the fundamental building blocks of deep learning. From activation functions to convolution operations, every detail required precise implementation and validation. Manually constructing and optimizing both models also highlighted the critical roles of proper initialization, gradient flow, and efficient computation. This project not only enhanced our confidence in the theory and application of deep learning models but also provided a strong foundation for further exploration into modern deep learning frameworks.