Distribution of the windows

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1 Joint distributions

Let R_1, R_2 be two random variables representing the appearance of a window region; R_1 being the outside of the window, R_2 being the inside. The appearance of these two regions is expected to be different; as a result we are interested in the distribution of such difference. Mathematically, we want to find,

$$f_{R_1 - R_2}(d) \tag{1}$$

or

$$F_{R_1 - R_2}(d) = P(R_1 - R_2 \leq d) \tag{2}$$

in order to do so, we make use of the following theorem.

Theorem 1.1. *If X and Y have joint density function f, then $X + Y$ has density function*

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z - x)dx \tag{3}$$

and if X and Y are independent then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z - x)dx = \int_{-\infty}^{\infty} f_X(z - y)f_Y(y)dy \tag{4}$$

We can use theorem 1.1 to deduce:

$$f_{R_1 - R_2}(z) = \int_{-\infty}^{\infty} f(r_2, z + r_2)dr_2 \tag{5}$$

assuming independence we obtain:

$$f_{R1-R2}(z) = \int_{-\infty}^{\infty} f_{R_2}(r_2)f_{R_1}(z + r_2)dr_2 \tag{6}$$

Furthermore the distribution of the regions is given by the joint distribution of the voxels in that region i.e $= P(R_1 \in WF) = P(X_1 \in WF, ..., X_n \in WF)$ for $X_1, ..., X_n \in R1$ and WF being window frame. Assuming that voxels are independent leads to

$$P(R_1 \in WF) = P(X_1 \in WF)P(X_2 \in WF)...P(X_n \in WF) \tag{7}$$

assuming independent voxels, we also know

$$\begin{aligned} f_{X_1,...X_n}(x_1,...x_n) = &= \frac{\delta^n}{\delta x_1 ....\delta x_n} \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f_{X_1,...X_n}(u_1,...,u_n) du_1...du_n && (8) \\ &= f_{X_1}(x_1) f_{X_2}(x_2) ... f_{X_n}(x_n) && (9) \end{aligned}$$

An analogous analysis can be carried for R_2. Each voxel appearance is modeled by a mixture of three gaussians. That is

$$f_{X_i}(x_i) = \sum_{k=1}^{3} w_i^k G_i^k(x) \quad \text{where } G_i^k \sim N(\mu_{G_i^k}, \sigma_{G_i^k}) \tag{10}$$

$$\begin{aligned} f_{X_1,...X_n}(x_1,...x_n) &= f_{X_1}(x_1) f_{X_2}(x_2) ... f_{X_n}(x_n) && (11) \\ &= \prod_{i=1}^{n} \sum_{k=1}^{3} w_i^k G_i^k(x) && (12) \end{aligned}$$

This is 3^n products of Gaussians to be convolved with 3^n . For $n = 10, 3^n \sim 60000$.

$$\begin{aligned} f_{R1-R2}(z) &= \int_{-\infty}^{\infty} f_{R_2}(r_2) f_{R_1}(z + r_2) dr_2 && (13) \\ &= \int_{-\infty}^{\infty} \prod_{i=1}^{n} \sum_{k=1}^{3} w_i^k G_i^k(r_2) \prod_{i=1}^{n} \sum_{k=1}^{3} v_i^k H_i^k(z + r_2) dr_2 && (14) \end{aligned}$$

2 Reducing the number of mixtures

The above calculations are the ideal solution to our problem setup. However, they pose a computational challenge to our system. A simple way of reducing the number of calculations is reducing the number of mixtures of the appearance distribution. If the most likely mode is persistently dominant across mixtures, then it is reasonable to let this component represent the distribution. If various of the mixtures are representative, then, a combination of the mixtures may represent the distribution in a better way. Merging mixtures is widely used across literature [1]. Suppose that for a mixture in the shape of eq. (10) we wish to merge the $i^t h$ and $j^t h$ component into the $i'th$ component, then the following equations are satisfied.

$$\begin{aligned} w^{i'} &= w^i + w^j && (15) \\ w^{i'} G^{i'}(x) &= w^i G^i(x) + w^j G^j(x) && (16) \end{aligned}$$

We can obtain the mean as follows

$$\begin{aligned} w^{i'}\mu^{i'} = w^{i'} E_{G^{i'}}(x) &= w^{i'} \int x G^{i'}(x) dx \\ &= \int x (w^i G^i(x) + w^j G^j(x)) dx \\ &= w^i \mu^i + w^j \mu^j && (17) \end{aligned}$$

We can obtain the variance using a similar technique. For ease of notation, let $\sigma^{2^i} = v^i$, then,

$$\begin{aligned} v^{i'} &= E_{G^{i'}}[(x - \mu^{i'})(x - \mu^{i'})] \\ &= E_{G^{i'}}(x^2) - (\mu^{i'})^2 \\ &= \frac{w^i}{w^{i'}} E_{G^i}(x^2) + \frac{w^j}{w^{i'}} E_{G^j}(x^2) - (\mu^{i'})^2 \\ &= \frac{1}{w^{i'}} [w^i v^i + w^i (\mu^i)^2 + w^j v^j + w^j (\mu^j)^2] - (\mu^{i'})^2 && (18) \end{aligned}$$

We obtain the final relation

$$w^{i'}(v^{i'} + (\mu^{i'})^2) = w^i (v^i + (\mu^i)^2) + w^j (v^j + (\mu^j)^2) \quad (19)$$

2.1 In particular: Equations from three mixtures to one mixture

We can use the equations derived above to write out the form of merging three-component gaussian mixture (ith, jth, kth) to a single component (i' th). It is easy to check that merging three mixtures give th efollowing equations

$$\begin{aligned} w^{i'} &= w^i + w^j + w^k && (20) \\ w^{i'} G^{i'}(x) &= w^i G^i(x) + w^j G^j(x) + w^k G^k(x) && (21) \\ w^{i'} \mu^{i'} &= w^i \mu^i + w^j \mu^j + w^k \mu^k && (22) \\ v^{i'} &= \frac{1}{w^{i'}} [w^i v^i + w^i (\mu^i)^2 + w^j v^j + w^j (\mu^j)^2 + w^k v^k + w^k (\mu^k)^2] - (\mu^{i'})^2 && (23) \\ w^{i'}(v^{i'} + (\mu^{i'})^2) &= w^i (v^i + (\mu^i)^2) + w^j (v^j + (\mu^j)^2 + w^k (v^k + (\mu^k)^2) && (24) \end{aligned}$$

The distribution of the i'th component is considered to be normal with mean and variance given by equations 23 and 24. Equations 21, 23, 24 are called merging equations [1].

3 Putting it all together

References

[1] Z Zhang. Em algorithms for gaussian mixtures with split-and-merge operation. *Pattern Recognition*, 36(9):1973–1983, Sep 2003.