Qualitative Representation of Images

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Contents

1 Terminology and Notation

Concept	**Definition**	**Notation**
Topographical surface		$z = f(x, y)$
Gradient	$\nabla f = [f_x, f_y]$	∇f
Hessian	$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$	H
Determinant of Hessian	$\det(H) = f_{xx}f_{yy} - f_{xy}^2$	$\det(H)$
Trace of Hessian	$\mathrm{Tr}(H) = f_{xx} + f_{yy}$	$\mathrm{Tr}(H)$
Principal curvatures	eigenvalues of H	λ_1, λ_2
Principal curvature directions	eigenvectors of H	e_1, e_2
Regular point	Non-critical point where $\lvert\nabla f\rvert > 0$	
Peak (Morse max)	$\lvert\nabla f\rvert = 0$, $\det(H) > 0$, $\mathrm{Tr}(H) < 0$	$\triangle$
Pit (Morse min)	$\lvert\nabla f\rvert = 0$, $\det(H) > 0$, $\mathrm{Tr}(H) > 0$	$\bigcirc$
Pass/bar (Morse saddle)	$\nabla f = 0$, $\det(H) < 0$	$+$
Degenerate critical point	$\det(H) = 0$	
Slope line (integral curve)	$\alpha(0) = p$, $\alpha'(s) = -\nabla f(\alpha(s))$	$\alpha(s)$
Slope line segment	a slope line whose end points are critical points.	
Hill	$\{p \mid \exists$ slope line segment through p that ends at $p_0\}$	$\mathrm{Hill}(p_0)$
Dale	$\{p \mid \exists$ slope line segment through p that ends at $p_0\}$	$\mathrm{Dale}(p_0)$
Watershed line	Special slope lines connecting saddles to peaks. According to Rothe/Reiger [3], this definition may not be true under degenerate cases. Dale boundaries.	
Watercourse line	Special slope lines connecting saddles to pits. According to Rothe/Reiger [3], this definition may not be true under degenerate cases. Hill boundaries.	
Slope district	The monotonic regions obtained by intersecting hills and dales [2].	

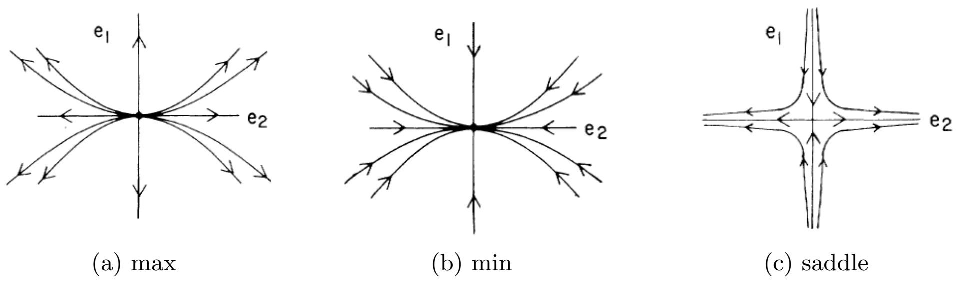

Figure 1: Slope line behaviour near critical points. e_1 and e_2 are the principal curvature directions. Figures were taken from [2].

2 Questions, Propositions and Proofs

Proposition 1 *The slope line ODE, $\alpha'(s) = -\nabla f(\alpha(s))$, with an initial value for $\alpha(0)$ has a unique solution.*

This is an explicit ODE for which there is a unique solution by the existense and uniqueness theorem [3]. Also see [5].

Question 1 *Can we have two slope lines intersecting tangentially or transversely at a regular point?*

Assume two slope lines meet at regular point p. Then, if we solve the slope line ODE with $\alpha(0) = p$, we should get two separate curves. But, we already know that the solution is unique, so there is a contradiction.

Question 2 *Can the curvature of a slope line be multi-valued?*

For a slope line α, we know that $\alpha'(s) = -\nabla f(\alpha(s))$. To analyze curvature, we need to look at $\alpha''(s) = -\frac{d}{ds}[\nabla f(\alpha(s))]$. Since $\alpha(s)$ and ∇f are single-valued, so is $\alpha''(s)$. Thus, the curvature cannot be multi-valued.

Question 3 *How do slope lines behave near min/max points?*

Nackman explains the behaviour in his paper [2]. All slope lines approach to min/max points tangent to one of the principal curvatures. See Figures 1a and 1b. If the point is umbilical, the slope lines approach from all directions. This can be verified using a Monge patch. Let $f(x, y) = ax^2 + by^2$ be the surface of interest where the signs of a and b are the same. Note that $(0, 0)$ is either a max or a min point. Let $p_0 = (\epsilon \cos\theta, \epsilon \sin\theta)$ be a point on a slope line originating from $(0, 0)$. As ϵ goes to 0, $\nabla f(p_0) = (2a\epsilon \cos\theta, 2b\epsilon \sin\theta)$ becomes tangent to $(\cos\theta, \sin\theta)$. This implies that $(a \cos\theta, b \sin\theta) \cdot (\sin\theta, -\cos\theta) = 0$ which reduces to $(a - b) \sin 2\theta = 0$. If the point is not umbilical, $a \neq b$, then $\theta = 0$ or $\theta = \pi/2$ which are the pricincipal curvature directions. On the other hand, if $a = b$, the value of θ is irrelevant.
This proof is incomplete!

Question 4 *How do slope lines behave near saddle points?*

See Figure 1c. We can use the same argument to show that the slope lines approach saddles along e_1 and e_2. But, this is not sufficient!

Question 5 *What is a hill corresponding to a peak?*

Let p_0 be a peak. Then, $\text{Hill}(p_0) = \{p \mid \exists \text{ slope line segment through } p \text{ that ends at } p_0\}$. Similarly, if p_0 is a pit, $\text{Dale}(p_0) = \{p \mid \exists \text{ slope line segment through } p \text{ that ends at } p_0\}$.

Proposition 2 *Both end points of a slope line segment cannot be a peak.*

Let both p_1 and p_2 be peaks. Assume that there exists a slope line connecting these two points. Without loss of generality, assume that the downward flow is from p_1 to p_2. Since all slope lines around a peak must be outgoing, there is a contradiction in p_2 having an incoming slope line. Therefore, we can say that both end points cannot be a peak. A similar proof can be given for the pit case.

Question 6 *Can two saddle points be connected with watershed/watercourse lines in generic images?*

This can happen, but it is an intirinsically unstable event [4], so it is not generic.

Question 7 *Can we have disconnected hills/dales?*

Suppose we have a hill with two disconnected regions. The points in the region without the peak must be connected to the peak with slope lines by definition. Since a region cannot be disconnected from a region containing the peak and have slope lines connected to the peak at the same time, we can say that hills cannot be composed of disconnected regions.

Question 8 *Do hills partion the image domain completely? In other words, is there any point in the domain that does not belong to any hill?*

There is a slope line for every regular point. If we follow slope lines in the upward flow direction starting at a regular point p, we will reach a critical point which is either a peak or a saddle. If it is a peak, then p belongs to the associated hill. But if it is a saddle, then p belongs to a watercourse line. Therefore, the points other than those belonging to watercourse lines belong to hills.

Proposition 3 *Hills are tightly surrounded by watercourse lines. In other words, watercourse lines correspond to hill boundaries.*

TODO

Question 9 *Is there any critical point that does not belong to a watershed/watercourse line?*

TODO

Question 10 *Can there be any watershed/watercourse line without any critical points?*

No. Watershed/watercourse lines are slope line segments, so by definition their end points are always critical points.

Proposition 4 *There is no other max point on a hill boundary.*

Assume that p is a max point on the boundary of hill h. In some neighborhood of p with no other critical points, all the upward flowing slope lines converge to p. Since this neighborhood intersects h, all the points in the intersection must belong to h and the hill corresponding to p at the same time. Since this is not possible, we can say that there cannot be any peaks on a hill boundary.
Another explanation: Hill boundaries are composed of watercourse lines, so there can only be regular, min and saddle points.

Proposition 5 *There is at least one saddle and one min (max) point on a hill (dale) boundary.*

Since hills are bounded by watercourse lines whose end points are always a min and a saddle point, it is clear that there must be at least one min and one saddle point on a hill boundary.

Proposition 6 *Saddles and min points are arranged in an alternating pattern on a hill boundary.*

Since there is no slope line with min-min or saddle-saddle end points, min and saddle points must be in an alternating pattern.

Proposition 7 *The number of saddles and min points is the same on a hill boundary.*

The closed alternating pattern guarantees that the number of saddles and min points is the same.

Question 11 *Can there be another critical point inside a hill (dale) other than the associated peak (pit)?*

There is an example showing a special case in [1] on page 3. A min and a saddle point are inside the centeral hill. This example is important, because it is generic and might affect our definition of slope districts.

References

[1] Peer-Timo Bremer, Herbert Edelsbrunner, Bernd Hamann, and Valerio Pascucci. A topological hierarchy for functions on triangulated surfaces. *IEEE Trans. Vis. Comput. Graph.*, 10(4):385–396, 2004.

[2] Lee R. Nackman. Two-dimensional critical point configuration graphs. *IEEE Trans. Pattern Anal. Mach. Intell.*, 6(4):442–450, 1984.

[3] J. H. Rieger. Topographical properties of generic images. *International Journal of Computer Vision*, 23(1):79–92, 1997.

[4] P.L. Rosin. Early image representation by slope districts. *Journal of Visual Communication and Image Representation*, 6(3):228–243, 1995.

[5] Wikipedia. Picard-Lindelöf Theorem. `http://en.wikipedia.org/wiki/Picard%E2%80%93Lindel%C3%B6f_theorem`.