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1. The equation $f(x) = 3x - e^x = 0$ has a solution in the interval [1,2].

(a) Using the error formula for the Bisection method find the number of iterations needed for accuracy 0.000001.

(b) Compute p1, p2, p3 to find the root p for the Bisection method.

(a) Verify the formula has a zero in the interval [1,2]

$f(1) = 3(1) - e^1 = 0.2817$

$f(2) = 3(2) - e^2 = -1.3890$

Since 0 is between 0.2817 and -1.3890, there is a point that crosses the x axis.

$\varepsilon > \frac{b_n - a_n}{2^{n+1}}$

$\varepsilon = 10^{-6}$

$10^{-6} > \frac{b_n - a_n}{2^{n+1}}$

$\log(10^{-6}) > \log\left(\frac{1}{2^{n+1}}\right)$

$-6 > -\log(2^{n+1})$

$-6 > -(n+1)\log(2)$

$-\frac{6}{log2} > -\frac{(n+1)(log2)}{log2}$

$19.93 > n+1$

$18.93 > n$

~19 > n

(b) $3x - e^x$

Choosing the half point for p_0; (max) 2 - (min) 1 = 1; Half point = 1.5

$f(p_0) = 3(1.5) - e^{1.5} = 0.01831$

$f(1) * p_0$ = Some positive number

$f(2) * p_0$ = Some negative number

Use f(2) since it passed through axis

$p_1 = \frac{1.5 + 2}{2}$

$p_1 = 1.75$

$f(p_1) = 3x - e^x$

$f(1.75) = 3(1.75) - e^{1.75}$

$f(1.75) = -0.5046$

$f(p_0) * p_1$ = Some negative number

$f(2) * p_1$ = Some positive number

Use p_0 since it passed through axis

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$p_2 = \frac{1.75 + 1.5}{2}$

$p_2 = 1.625$

$f(1.625) = 3(1.625) - e^{1.625}$

$f(1.625) = -0.2034$

$f(p_0) * p_2$ = Some negative number

$f(p_1) * p_2$ = Some positive number

Use p_0 since it passed through axis

$p_3 = \frac{1.625 + 1.5}{2}$

$p_3 = 1.5625$

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2. A natural cubic spline S is define by

$$S(x) = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & if \quad 1 \le x < 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & if \quad 2 \le x \le 3. \end{cases}$$

If S interpolates the data (1,1),(2,1), and (3,0), find B, D, b and d.

$S_0(x) = 1 + B(x - 1) - D(x - 1)^3$

$S'_0(x) = B - 3D(x - 1)^2$

$S''_0(x) = -6D(x - 1)$

$S_1(x) = 1 + b(x - 2) - \frac{3}{4}(x - 2)^2 + d(x - 2)^3$

$S'_1(x) = b - \frac{3}{2}(x - 2) + 3d(x - 2)^2$

$S''_1(x) = -\frac{3}{2} + 6d(x - 2)$

$S_0(1) = 1$

$S_0(2) = 1 + B - D$

$S_1(2) = 1$

$S_1(3) = 1 + b - \frac{3}{4} + d$

$S'_0(1) = B$

$S'_0(2) = B - 3D$

$S'_1(2) = b$

$S'_1(3) = b - \frac{3}{2} + 3d$

$S''_0(1) = 0$

$S''_0(2) = -6D$

$S''_1(2) = -\frac{3}{2}$

$S''_1(3) = -\frac{3}{2} + 6d$

$S''_0(2) = S''_1(2)$

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$-6D = -\frac{3}{2}$
$D = \frac{1}{4}$

$S_0(2) = 1 + B - D = 1$
$B - D = 0$
$B = D$
$B = \frac{1}{4}$

$S''_0(1) = 0 = S''_1(3) = -\frac{3}{2} + 6d$
$0 = -\frac{3}{2} + 6d$
$\frac{3}{2} = 6d$
$d = \frac{1}{4}$

$S'_0(2) = B - 3D = S'_1(2) = b$
$B - 3D = b$
$(\frac{1}{4}) - 3(\frac{1}{4}) = b$
$b = -\frac{1}{2}$

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3. Given the following formula, show that the error term is given $\frac{1}{3}h^2 f'''(\xi)$

$$f' \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$$

$4f(x + h) = 4f(x) + 4hf'(x) + 4\frac{(h)^2}{2} f''(x) + 4\frac{(h)^3}{6} f'''(x)$
$f(x + 2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(x)$
$3f(x) = 3f(x)$

$4f(x + h) - f(x + 2h) = 3f(x) + 2hf'(x) + \frac{h^2}{3} f'''(x)$

$2hf'(x) = 4f(x+h) - 3f(x) - f(x + 2h) + \frac{2h^3}{3} f'''(x)$
$f'(x) = \frac{1}{2h}[4f(x+h) - f(x) - f(x + 2h)] + \frac{h^2}{3} f'''(x)$

The last term is the error so x is ξ.

$f'(x) = \frac{1}{2h}[4f(x+h) - f(x) - f(x + 2h)] + \frac{h^2}{3} f'''(\xi)$

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4. Determine n and h required to approximate

$$\int_0^2 \frac{1}{x+4}dx$$

to within 10^{-5} using

(a) Composite Trapezoidal Rule

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(b) Composite Simpson's Rule

(a) Error = $\left| \frac{2h^2}{12} \, f''(\mu) \right|$

$f'(x) = -\frac{1}{(x+4)^2}$

$f''(x) = \frac{2}{(x+4)^3}$

$\left| \frac{2h^2}{12} \, f''(\mu) \right| = \left| \frac{2h^2}{12} * \frac{2}{(\mu+4)^3} \right|$

$\left| \frac{2h^2}{12} * \frac{2}{(\mu+4)^3} \right|$

$\frac{h^2}{3} * \left| \frac{1}{(\mu+4)^3} \right| < 10^{-5}$

Since 'x' is in the bottom of the function, the bigger it becomes, the smaller the result is. Thus, 0 is max.

$\frac{h^2}{3} * \left| \frac{1}{(0+4)^3} \right| < 10^{-5}$

$\frac{h^2}{3} * \left| \frac{1}{4^3} \right| < 10^{-5}$

$\frac{h^2}{3} * \frac{1}{64} < 10^{-5}$

$h < 0.04381$

$h = \frac{b-a}{n}$

$0.04381 = \frac{2-0}{n}$

$n > 45.64$

$n \geq 46$ (next highest whole number)

(b) Error = $\left| \frac{2h^4}{180} f^{(4)}(\mu) \right|$

$f'(x) = -\frac{1}{(x+4)^2}$

$f''(x) = \frac{2}{(x+4)^3}$

$f'''(x) = -\frac{6}{(x+4)^4}$

$f^{(4)}(x) = \frac{24}{(x+4)^5}$

$\left| \frac{2h^4}{180} f^{(4)}(\mu) \right| = \left| \frac{h^4}{90} * \frac{24}{(\mu+4)^5} \right|$

$\left| \frac{h^4}{90} * \frac{24}{(\mu+4)^5} \right| = \frac{4h^4}{15} \left| \frac{24}{(\mu+4)^5} \right|$

$\frac{4h^4}{15} \left| \frac{24}{(\mu+4)^5} \right| < 10^{-5}$

Still using 0 since it was the max.

$\frac{4h^4}{15} \left| \frac{24}{(0+4)^5} \right| < 10^{-5}$

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$\frac{4h^4}{15} * \frac{24}{4^5} < 10^{-5}$

h = 0.4426

h = $\frac{b-a}{n}$

0.4426 = $\frac{2-0}{n}$

n > 4.52

n $\geq$ 6 (next highest even number)

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5. The Euler Method is given by

$$w_{i+1} = w_i + hf(t_i, w_i) \quad for \quad i = 0, 1, 2, \ldots, N-1$$

$$\text{where} \quad h = \frac{b-a}{N}$$

Given the initial value problem y' = t - y + 2, 0 $\leq$ t $\leq$ 1, y(0) = 3 Let N = 2 and generate w_2 to approximate y(1) using Euler's method.

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y' = t - y + 2

h = $\frac{1-0}{2}$ = 0.5

$w_1 = w_0 + hf(t,w_0)$

$w_1 = 3 + (0.5)f(0.5,3)$

$w_1 = 3 + (0.5)*(-0.5)$

$w_1 = 2.75$

$w_2 = w_1 + hf(t,w_1)$

$w_2 = (2.75) + (0.5)f(1,2.75)$

$w_2 = (2.75) + (0.5)*(0.25)$

$w_2 = 2.875$

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6. Reduce the higher order differential equation to a system of first order differential equations

$$\begin{cases} t^3y''' + t^2y'' - 2ty' + 2y = 8t^3 - 2, & [1,2] \\ y(1) = 2, y'(1) = 8, y''(1) = 6 \end{cases}$$

Old Variable	New Variable	IV	DE
y	y_1	2	$y_1' = y_2$
y'	y_2	8	$y_2' = y_3$

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y''	y_3	6	$y_3' = (8t^3 - 2 - t^2y_3 + 2ty_2 - 2y_1)/t^3$

...
y' =

y_2
y_3
$(8t^3 - 2 - t^2y_3 + 2ty_2 - 2y_1)/t^3$

$y(1) = [2,8,6]^T$

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7. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix}$$

Find the values of α for which

(a) A is strictly diagonally dominant.
(b) A is positive definite.

(a) https://elearn.mtsu.edu/d2l/le/content/7975386/viewContent/68659394/View
(1:25:00)
If strictly diagonally dominant, $|a_{ii}| > \Sigma a_{ik}$
Row 1: |1| > |0| + |-1| = 1
Row 2: |1| > |0| + |1| = 1 (Not true)
This matrix cannot be **strictly** diagonally dominant. The other rows failed.

(b) If symmetric and if $x^tAx > 0$ (pg 419)
α must also be positive (pg 581, Theorem 9.18)
To be symmetric, α must equal -1.
To be positive, α must be greater than 1.
There does not exist an α that meets both conditions.

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8. Find the l_2 norm for the matrix below

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

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maximum eigenvalue of B^tB

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5	-4	1
-4	6	-4
1	-4	5

5 - λ	-4	1
-4	6- λ	-4
1	-4	5- λ

$-\lambda^3 + 16\lambda^2 - 52\lambda + 16$
$\lambda=4$
$\lambda=2(3+2\sqrt{2})= 11.6568$
$\lambda=2(3-2\sqrt{2}) = 0.3431$

l_2 norm = $\sqrt{max(\lambda)} = \sqrt{max(2(3+2\sqrt{2}))}$ = 3.4142

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9. Find the l_∞ norm for the matrix below

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & -2 & -1 \end{bmatrix}$$

$l_\infty = ||A||_\infty = \max(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}, \frac{1}{4} + \frac{1}{2} + \frac{1}{4}, 2 + 2 + 1) = 5$

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10. Use Gaussian elimination and three-digit chopping arithmetic to solve the following linear system and compare the approximation to the actual solution

$$58.9x_1 + 0.03x_2 = 59.2$$
$$-6.10x_1 + 5.31x_2 = 47.0$$

Actual solution is [1,10]

58.9	0.03	59.2
-6.10	5.31	47.0

M = -6.10/58.9 = -0.1035 (3 digit only)

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58.9 * -0.1035 = -6.09	0.03 * -0.1035 = -0.00310	59.2 * -0.1035 = -6.12
-6.10 - (-6.09) = -0.01	5.31 - (-0.00310) = 5.31	47.0 - (-6.12) = 53.1

Zero the first row, first column because of rounding error

-6.09	-0.00310	-6.12
0	5.31	53.1

$(-6.09)x_1 + (-0.00310)x_2 = (-6.12)$
$(5.31)x_2 = (53.1)$

$x_2 = 10.0$
$(-6.09)x_1 + (-0.00310)(1) = (-6.12)$
$x_1 = 1.00$

The results were spot on.
