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1. Perform one-time step for the following methods using h = 0.1:

$$\begin{cases} \frac{dx}{dt} = -tx^2 \\ x(0) = 2 \end{cases}$$

(a) Taylor Method of order 4

(b) Runge-Kutta Method of order 4

(a) pg 277

$f^{(1)}(t_i, w_i)$ = -t * x^2
$f^{(1)}(t_i, w_i)$ = -0*2^2
$f^{(1)}(t_i, w_i)$ = 0
$f^{(2)}(t_i, w_i)$ =2 * t^2 * x^3 - x^2
$f^{(2)}(t_i, w_i)$ = 2 * 0^2 * 2^3 - 2^2
$f^{(2)}(t_i, w_i)$ = -4
$f^{(3)}(t_i, w_i)$ = -6 * t^3 * x^4 + 6 * t * x^3
$f^{(3)}(t_i, w_i)$ = -6 * 0^3 * 2^4 + 6 * 0 * 2^3
$f^{(3)}(t_i, w_i)$ = 0
$f^{(4)}(t_i, w_i)$ = 24 * t^4 * x^5 - 36 * t^2 * x^4 + 6 * x^3
$f^{(4)}(t_i, w_i)$ = 24 * 0^4 * 2^5 - 36 * 0^2 * 2^4 + 6 * 2^3
$f^{(4)}(t_i, w_i)$ = -8

$T^{(4)}(t_i, w_i)$ = x + h*$f^{(1)}(t_i, w_i)$ + (h^2 / 2)*$f^{(2)}(t_i, w_i)$ + (h^3 / 6)*$f^{(3)}(t_i, w_i)$ + (h^4 / 24)*$f^{(4)}(t_i, w_i)$
$T^{(4)}(t_i, w_i)$ = 2 + 0.1*0 + (0.1^2 / 2)*-4 + (0.1^3 / 6)*0 + (0.1^4 / 24)*-8
$T^{(4)}(t_i, w_i)$ = 1.9799

(b) pg 282

k1 = h * f(t,w)
k1 = 0.1 * f(0.0,2.0)
k1 = 0.0

k2 = h * f((t + h/2.0), (w + k1/2.0))
k2 = 0.1 * f((0.0 + 0.1/2.0), (2.0 + -0.0/2.0))
k2 = -0.02

k3 = h * f((t + h/2.0), (w + k2/2.0))
k3 = 0.1 * f((0.0 + 0.1/2.0), (2.0 + -0.02/2.0))
k3 = -0.01980

k4 = h * f((t + h), (w + k3))
k4 = 0.1 * f((0.0 + 0.1), (2.0 + -0.01980))
-0.03921

w = w + (k1 + (2.0 * k2) + (2.0 * k3) + k4)/6.0
w = 2 + (-0.0 + (2.0 * -0.02) + (2.0 * -0.01980) + -0.03921)/6.0

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w = 1.9802

----

2. Reduce this differential equation into a system of first order equations

$$\begin{cases} x''' = 2x' + \log x'' + \cos x \\ x(0) = 1, x'(0) = -3, x''(0) = 5 \end{cases}$$

Solution: Higher order ODEs video. Around 56 minutes in

Old Variable	New Variable	IV	DE
x	x_1	1	$x_1' = x_2$
x'	x_2	3	$x_2' = x_3$
x''	x_3	5	$x_3' = 2x_2 + \log x_3 + \cos x_1$

...

x' =

x_2
x_3
$2x_2 + \log x_3 + \cos x_1$

$x(0) = [1,3,5]^T$

----

3. Given the following differential equations

$$\begin{cases} x''' = t + x + 2x' + 3x'' \\ x(1) = 3, x'(1) = -7, x''(1) = 4 \end{cases}$$

(a) Reduce the differential equation to first order system and perform one time step of Euler Method using h = 0.1

Solution: Higher order ODEs video. Around 56 minutes in

Old Variable	New Variable	IV	DE
x	x_1	3	$x_1' = x_2$
x'	x_2	-7	$x_2' = x_3$
x''	x_3	4	$x_3' = t + x_1 + 2x_2 + 3x_3$

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...

x' =

x_2
x_3
$t + x_1 + 2x_2 + 3x_3$

$x(1) = [3,-7,4]^T$

$w_1 = w_0 + hf(t,w)$
$w_1 = w_0 + hf(t,w)$
$w_1 = 4 + (0.1 * (0.1 + 3 + (2*-7) + (3*4)))$
$w_1 = 4.11$

----

4. Perform one-time step of Forward Euler Method for the first order system of differential equation using h= 0.1

$$\begin{cases} x_1' = x_1^2 + e^t - t^2 \\ x_2' = x_2 - \cos t \\ x_1(0) = 0, x_2(1) = 0 \end{cases}$$

$x'_1 = x_1^2 + e^t - t^2$
$w_1 = w_0 + hf(t,w)$
$w_1 = 0 + (0.1)*f(0.1,0)$
$w_1 = 0 + (0.1)*(0 + e^{(0.1)} - (0.1)^2)$
$w_1 = 0.1095$

$x'_3 = x_2 - \cos(t)$
$w_1 = 0 + (0.1)*f(0.1,0)$
$w_1 = 0 + (0.1)*(0 - \cos(0.1))$
$w_1 = -0.09950$

----

5. Let

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & -2 \\ 0 & -2 & 8 \end{bmatrix}$$

(a) Find $||A||_\infty$.
(b) Find $\rho(A)$.
(c) Find an eigenvector of A corresponding to the eigenvalue for which $|\lambda| = \rho(A)$.

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(d) Is matrix A symmetric? Is matrix A positive definite?

(a) Video “Review of Test 3”

$\max(||A||_\infty)$

Row 1 = |8| + |1| + |0| = 9

Row 2 = |1| + |4| + |-2| = 7

Row 3 = |0| + |-2| + |8| = 10

$||A||_\infty$ = max(9, 7, 10) = 10

(b)

$\rho(A) = \max(\lambda_i)$

Solve (c) first

$\rho(A)$ = max(3, 8, 9) = 9

(c)

Use characteristic polynomial > eigenvalues > eigenvectors

$\rho(\lambda) = \det(A - \lambda I) = 0$

----

det(

8	1	0
1	4	-2
0	-2	8

)

$-\lambda$(

1	0	0
0	1	0
0	0	1

) = 0

----

det(

8	1	0
1	4	-2
0	-2	8

-

λ	0	0
0	λ	0

0	0	λ

) = 0

----

det(

$8 - \lambda$	1	0
1	$4 - \lambda$	-2
0	-2	$8 - \lambda$

) = 0

$-\lambda^3 + 20\lambda^2 - 123\lambda + 216 = 0$

$\lambda = 3, 8, 9$ (Eigenvalues)

$\lambda_1 = 3$

$(A - \lambda I)x = 0$

$(A - 3I)x = 0$

$8 - \lambda$	1	0
1	$4 - \lambda$	-2
0	-2	$8 - \lambda$

5	1	0	x_1	0
1	1	-2	x_2	0
0	-2	5	x_3	0

(d)

All eigenvalues are positive so the matrix is not symmetric but is positive definite.

----

6. Find the first two iterations of Jacobi method for the following linear systems using $x^{(0)} = 0$;

$$4x_1 + x_2 - x_3 + x_4 = -2$$
$$x_1 + 4x_2 - x_3 - x_4 = -1$$
$$-x_1 - x_2 + 5x_3 + x_4 = 0$$
$$x_1 - x_2 + x_3 + 3x_4 = 1$$

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Solution: pg 456

$x^{(0)} = (0,0,0,0)$

$x^{(1)}_1$		$-x^{(0)}_2/4$	$x^{(0)}_3/4$	$-x^{(0)}_4/4$	-2/4
$x^{(1)}_2$	$-x^{(0)}_1/4$		$x^{(0)}_3/4$	$x^{(0)}_4/4$	-1/4
$x^{(1)}_3$	$x^{(0)}_1/5$	$x^{(0)}_2/5$		$-x^{(0)}_4/5$	0
$x^{(1)}_4$	$-x^{(0)}_1/3$	$x^{(0)}_2/3$	$-x^{(0)}_3/3$		1/3

$x^{(1)} = (-1/2,-1/4,0,1/3)$

$x^{(2)}_1$		$-x^{(1)}_2/4$	$x^{(1)}_3/4$	$-x^{(1)}_4/4$	-2/4
$x^{(2)}_2$	$-x^{(1)}_1/4$		$x^{(1)}_3/4$	$x^{(1)}_4/4$	-1/4
$x^{(2)}_3$	$x^{(1)}_1/5$	$x^{(1)}_2/5$		$-x^{(1)}_4/5$	0
$x^{(2)}_4$	$-x^{(1)}_1/3$	$x^{(1)}_2/3$	$-x^{(1)}_3/3$		1/3

$x^{(2)}_1$		-(-1/4) /4	0/4	-(1/3)/4	-2/4
$x^{(2)}_2$	-(-1/2)/4		0/4	(1/3)/4	-1/4
$x^{(2)}_3$	(-1/2)/5	(-1/4)/5		(1/3)/5	0
$x^{(2)}_4$	-(-1/2)/3	(-1/4)/3	-0/3		1/3

$x^{(2)} = (-0.5208, -0.04166, -0.2166, 0.4166)$

----

7. Find the first two iterations of Gauss-Seidel for the following linear systems using $x^{(0)} = 0$;

$$4x_1 + x_2 - x_3 + x_4 = -2$$
$$x_1 + 4x_2 - x_3 - x_4 = -1$$
$$-x_1 - x_2 + 5x_3 + x_4 = 0$$
$$x_1 - x_2 + x_3 + 3x_4 = 1$$

$4x_1 + x_2 - x_3 + x_4 = -2$

$x_1^1 = (-2 - x_2^0 + x_3^0 - x_4^0)/4$

$x_1 + 4x_2 - x_3 - x_4 = -1$

$x_2^1 = (-1 - x_1^1 + x_3^0 + x_4^0)/4$

$-x_1 - x_2 + 5x_3 + x_4 = 0$

$x_3^1 = (x_1^1 + x_2^1 - x_4^0)/5$

$x_1 - x_2 + x_3 + 3x_4 = 1$

$x_4^1 = (1 - x_1^1 + x_2^1 - x_3^1)/3$

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$x_1^{(1)}= (-2 - (-1) + (0) - (1))/4 = -1/2$
$x_2^1 = (-1 - (-1/2) + (0) + (1))/4 = -1/8$
$x_3^1 = ((-1/2) + (1/8)- (1))/5 = -1/8$
$x_4^1 = (1 - (-1/2)+ (-1/8)- (-1/8))/3 = 1/2$

$x_1^2= (-2 - x_2^1 + x_3^1 - x_4^1)/4$
$x_2^2 = (-1 - x_1^2 + x_3^1 + x_4^1)/4$
$x_3^2 = (x_1^2 + x_2^2- x_4^1)/5$
$x_4^2 = (1 - x_1^2 + x_2^2 - x_3^2)/3$

$x_1^2= (-2 - (-1/8) + (-1/8)- (1/2))/4 = -5/8$
$x_2^2 = (-1 - (-5/8) + (-1/8) + (1/2))/4 = 0$
$x_3^2 = ((-5/8) + (0)- (1/2))/5 = -9/40$
$x_4^2 = (1 - (-5/8) + (0) - (-9/40))/3 = 37/60$

$x_{(1)} = (-0.5, -0.125, -0.125, 0.5)$
$x_{(2)} = (-0.625, 0, -0.225, 0.6166)$

----

8. Compute the eigenvalues and associated eigenvectors of the following matrix.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$$

Det(

$3 - \lambda$	2	-1
1	$-\lambda -2$	3
2	0	$4- \lambda$

)
$= -\lambda^3 + 5\lambda^2 + 2\lambda - 24$

$\lambda^1 = 4$
$\lambda^2 = 3$
$\lambda^3 = -2$

$\lambda^1 = 4$

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3 - 4	2	-1
1	-4-2	3
2	0	4-4

=

-1	2	-1
1	-6	3
2	0	0

$\lambda^2 = 3$

3 - 3	2	-1
1	-3-2	3
2	0	4-3

=

0	2	-1
1	-5	3
2	0	1

$\lambda^3 = -2$

3 + 2	2	-1
1	2-2	3
2	0	4+2

=

5	2	-1
1	0	3

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2	0	6

----

9. Find the l_2 norm for the matrix below.

$$B = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{2} & 4 \\ 2 & 2 & -\frac{1}{3} \end{bmatrix}$$

(a) maximum eigenvalue of B^tB (pg 453 example 3).

$B*B^T$ =

1/4	-1/2	1
-1/2	69/4	-7/3
1	-7/3	73/9

¼ - λ	-1/2	1
-1/2	69/4 - λ	-7/3
1	-7/3	73/9 - λ

$-\lambda^3 + (461/18)*\lambda^2 - (2233/16)*\lambda + (2401/144)$

$\lambda_1 = 17.8408$
$\lambda_2 = 0.1221$
$\lambda_3 = 7.6480$

l_2 norm = $\sqrt{max(\lambda)} = \sqrt{17.8408} = 4.2238$

----

10. The following linear system **Ax = b** have **x** as the actual solution and $\bar{\mathbf{x}}$ as an approximate solution. Compute $||x - \bar{x}||_\infty$ and $||A\bar{x} - b||_\infty$.

$0.04x_1 + 0.01x_2 - 0.01x_3 = 0.0478$
$0.4x_1 + 0.1x_2 - 0.2x_3 = 0.413$
$x_1 + 2x_2 + 3x_3 = 0.14$

where

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$x = (1.81, -1.81, 0.65)^t$
$\bar{x} = (2, -2, 1)^t$

(pg 448 Example 7,8d)

$||x_1 - \bar{x}_1|| = ||1.81 - 2||$
$||x_1 - \bar{x}_1|| = ||-0.19||$
$||x_1 - \bar{x}_1|| = 0.19$

$||x_2 - \bar{x}_2|| = ||-1.81 + 2||$
$||x_1 - \bar{x}_1|| = ||0.19||$
$||x_1 - \bar{x}_1|| = 0.19$

$||x_3 - \bar{x}_3|| = ||0.65 - 1||$
$||x_1 - \bar{x}_1|| = ||-0.35||$
$||x_1 - \bar{x}_1|| = 0.35$

max(0.19, 0.19, 0.35) = 0.35

A =

0.04	0.01	-0.01
0.4	0.1	-0.2
1	2	3

$\bar{x}$ =

2
-2
1

b =

0.0478
0.413
0.14

$(A * \bar{x})$ =

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0.05- 0.0478 = 0.0022
0.4 - 0.413 = -0.013
1 - 0.14 = 0.86

max(|0.022|, |-0.013|, |0.86|) = 0.86
