Calculus 1120, Class 19

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Trapezoidal/Simpson's Rule

Trapezoidal Rule.

$$T_n(f) = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Error Estimate: $E_{T,n}(f) \leq \frac{M(b-a)^3}{12n^2}$, where M is the maximum value of $|f^{(2)}(x)|$ for $a \leq x \leq b$.

Key Point: The function f has to have two derivatives, and $f^{(2)}(x)$ must be continuous over the interval $[a, b]$.

Trapezoidal/Simpson's Rule

You can use the trapezoidal rule to estimate $\int_0^1 \sqrt{x}\, dx$. But the error estimate does not apply. The integral exists because the function is continuos, and you can use Riemann sums to approximate it. But the first and second derivatives do not exist at $x = 0$, let alone be continuous.

Trapezoidal/Simpson's Rule

Simpson's Rule.

$$S_n(f) = \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

Error Estimate: $E_{S,n} \leq \frac{(b-a)^5 M}{180n^4}$ where M is the maximum of $|f^{(4)}(x)|$ over the interval $a \leq x \leq b$.

Key Fact: $f^{(4)}$ has to exist and be continuous over the interval $[a, b]$.

Problem 28, Section 8.6

The **error function** is defined to be $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$, and plays an important role in probability, signal processing and heat flow.
(a) Use $L_n(f), R_n(f)$, $T_n(f)$ and $S_n(f)$ to estimate Erf(1) using $n = 10$.
(b) Give error estimates as well.
(c) How large should n be so that $E_{S,n}(f) \leq 10^{-6}$?

Answer: We estimate $\frac{\sqrt{\pi}}{2}\,\mathrm{Erf}(1)$ by $\int_0^1 y(x)\, dx$ with $y = e^{-x^2}$. y is decreasing because $y' = -2xe^{-x^2} < 0$ in the interval $[0, 1]$.

Problem 28, Section 8.6

The derivatives are

$$y' = -2xe^{-x^2}, \qquad y^{(2)} = \left(4x^2 - 2\right) e^{-x^2},$$
$$y^{(3)} = \left(-8x^3 + 12x\right) e^{-x^2}, \quad y^{(4)} = \left(16x^4 - 48x^2 + 12\right) e^{-x^2}.$$

Problem 28, Section 8.6

The first derivative is < 0 so the function is decreasing. The second derivative changes sign at $x = 1/\sqrt{2}$. The function is concave down for $0 \leq x \leq 1/\sqrt{2}$ and concave up for $1/\sqrt{2} \leq x \leq 1$.

$$R_{10}(f) \leq \mathrm{Erf}(1) \leq L_{10}(f) \qquad E_{L,10}, E_{R,10}(f) \leq \frac{1}{10}\left(1 - e^{-1}\right)$$

We cannot use the error estimate $|\frac{L_n(f) - R_n(f)|}{2}$ unless we break up the integral into a sum.
We can use the estimate $E_{T,n}(f) \leq \frac{M(b-a)^3}{12n^2}$.

Problem 28, Section 8.6

$E_{T,10}(f)$

We need to find M, an upper bound for $|y^{(2)}|$ over the interval $0 \le x \le 1$. We can find the maximum and minimum of $y^{(2)}$ over this interval by the 1^{st} derivative test. We need both, because $y^{(2)}$ is both negative and positive in the interval, and we have to find the maximum of $|y^{(2)}|$.

Problem 28, Section 8.6

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The derivative $y^{(3)}(x)$ is 0 at $x = 0, \pm\sqrt{3/2}$, and we can conclude $y^{(3)}(x) > 0$ in the interval, and therefore $y^{(2)}$ is increasing. Since $y^{(2)}(0) = -2$, $y^{(2)}(1) = 2e^{-1}$, $|y^{(2)}(x)| \leq 2$.

This is very time consuming, and we do not need to estimate M this carefully.

It is easier to observe that $4x^2 - 2$ is increasing for $x \geq 0$, so $-2 \leq 4x^2 - 2 \leq 2$. Thus $0 \leq |4x^2 - 2| \leq 2$. Since also $0 \leq e^{-x^2} \leq 1$,

$$|y^{(2)}(x)| \leq |4x^2 - 2|e^{-x^2} \leq 2 \cdot 1 = 2, \qquad 0 \leq x \leq 1.$$

We can take $M = 2$.

An error estimate is

$$E_{T,10} \leq \frac{2(1-0)^3}{12 \cdot 10^2} = \frac{1}{600}.$$

Problem 28, Section 8.6

$E_{S,10}(f)$

For $E_{S,10}$, we need to estimate $|y^{(4)}(x)|$ over the interval $0 \leq x \leq 1$. The problem helps us out by telling us that $|y^{(4)}(x)| \leq 12$ over this interval. So the estimate is

$$E_{S,10} \leq \frac{12(1-0)^5}{180 \cdot 10^4} = \frac{1}{150000}$$

For part c), we need to solve

$$\frac{1}{15n^4} \leq \frac{1}{10^6} \Leftrightarrow 15n^4 \leq 10^6 \Leftrightarrow n \geq \sqrt[4]{10^6/15} \approx 16.086$$

So $n \geq 18$ (an even number!) suffices.

The value of Erf(1) from tables is

$$\mathrm{Erf}(1) \approx 0.842701.$$

Problem 28, Section 8.6

To find an upper bound for $|y^{(4)}(x)|$, use the same shorcut as for $y^{(2)}$.

The derivative of $16x^4 - 48x^2 + 12$ is $64x^3 - 96x = 32x(2x^2 - 3)$ which is < 0 for $0 \le x \le 1$. So $16x^4 - 48x^2 + 12$ is decreasing.

x	0		1
$16x^4 - 48x^2 + 12$	12	$\searrow$	-20

So $|16x^4 - 48x^2 + 12| \le 20$. Combined with $0 \le e^{-x^2} \le 1$, we get $|y^{(4)}(x)| \le 20$.
Using $M = 20$ we get

$$E_{S,10} \le \frac{20(1-0)^5}{180 \cdot 10^4} = \frac{1}{90000}$$

This is worse, but not by that much.
For part c), we need to solve

$$\frac{1}{9n^4} \le \frac{1}{10^6} \Leftrightarrow 9n^4 \le 10^6 \Leftrightarrow n \ge \sqrt[4]{10^6/9} \approx 16.086$$

So $n \ge 18$ (an even number!) suffices.

Problem 28, Section 8.6

Remark: It is true, as the problem states, that in fact $|y^{(4)}(x)| \leq 12$. To show this you need to analyze $y^{(4)}$ more carefully, more involved calculations. A correct solution carries extra bonus points!

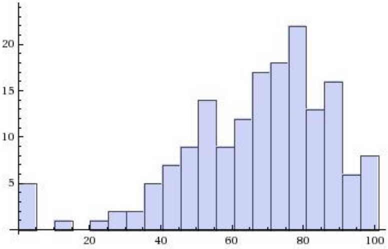
Note: the 1^{st} derivative test says that the maximum/minimum occurs at the endpoints of the interval OR at the critical points. In the examples there were no critical points inside the interval.

Comments on Prelim 1

Mean : 67

Median: 70 (grades above this are typically a B)

Histogram:



A large number of scores were between 75 and 85 which is very good. Students with a score below 63 receive a "warning". Actual grades (IF I were to assign letter grades based on such scores) might be more like

$$F \leq 49 < D \leq 55 < C- \leq 60 < C \leq 65 < C+/B- \leq 69$$

$$69 < B \leq 75 \leq B+ < 85 \leq 89 < A \leq 97 < A+ \leq 100$$

Improper Integrals

Integration via Riemann sums assumes functions are continuous over a bounded interval (or at least piecewise continuous over finitely many bounded intervals).
Deviating from this leads to improper integrals.

- infinite intervals $\displaystyle\int_0^\infty e^{-x}\,dx.$
- unbounded functions, vertical asymptotes $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx.$

The functions in the two examples are > 0, so you can think of the integrals as representing areas below the curves and above the $x-$axis.

Improper Integrals

Improper Integrals

Type I: $y = f(x)$ continuous on $[a, \infty)$ then

$$\int_a^\infty f(x)\, dx = \lim_{A\to\infty} \int_a^A f(x)\, dx.$$

Similarly for $\int_{-\infty}^a f(x)\, dx$.

Example: $\int_0^\infty e^{-x}\, dx := \lim_{R\to\infty} \int_0^R e^{-x}\, dx.$

Improper Integrals

Type I: $y = f(x)$ continuous on $[a, b)$ then

$$\int_a^b f(x)\, dx = \lim_{A \to b^-} \int_a^A f(x)\, dx.$$

Similarly $\int_a^b f(x)\, dx = \lim_{A \to a^+} \int_A^b f(x)\, dx$

Example: $\displaystyle \int_0^1 \frac{1}{\sqrt[3]{x}}\, dx := \lim_{\delta \to 0^+} \int_\delta^1 \frac{1}{\sqrt[3]{x}}\, dx.$

Improper Integrals

Important: If more than one type occurs in the integral, MUST divide into a sum and treat each term separately.

If the interval is infinite in both directions, and there are asymptotes, we need to separate into a sum, and examine each of them.

$$\int_{-\infty}^{\infty} \frac{1}{x}\, dx = \int_{-\infty}^{-a} \frac{1}{x}\, dx + \int_{-a}^{0} \frac{1}{x}\, dx + \int_{0}^{b} \frac{1}{x}\, dx + \int_{b}^{\infty} \frac{1}{x}\, dx.$$

It does not matter what a and b are, except $a < 0$ and $b > 0$.

Improper Integrals

Contrast this with

$$\lim_{R\to 0^+}\left[\int_{-1}^{-R}\frac{1}{x}\,dx\int_{R}^{1}\frac{1}{x}\,dx\right]=0$$

This DOES NOT SAY that $\displaystyle\int_{-1}^{1}\frac{1}{x}\,dx=0$.

The order in which you take the integral and the limit makes a difference!

Improper Integrals

a) $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\, dx,$

b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx?$

Improper Integrals

Similarly if the point of discontinuity is in the middle of the interval, we need to break into two pieces on each side:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\,dx &= \int_{-1}^{0} \frac{1}{\sqrt[3]{x}}\,dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x}}\,dx \\ &= \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{\sqrt[3]{x}}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{\sqrt[3]{x}}\,dx\end{aligned}$$

Improper Integrals

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx.$$

Improper Integrals

$$\int_0^{\infty} \frac{1}{\sqrt{1+x^4}}\, dx$$

Improper Integrals

Important Special Cases:

1. $\int_0^1 \frac{1}{x^p}\, dx$ converges for $0 < p < 1$ and diverges for $p \geq 1$.
2. $\int_1^\infty \frac{1}{x^p}\, dx$ diverges for $0 < p \leq 1$ and converges for $p > 1$.

Improper Integrals

Direct Comparison: If $0 \leq f(x) \leq g(x)$ for $x \geq a$, then

1. $\displaystyle\int_a^\infty g(x)\ dx$ converges implies $\displaystyle\int_a^\infty f(x)\ dx$ converges.
2. $\displaystyle\int_a^\infty f(x)\ dx$ diverges implies $\displaystyle\int_a^\infty g(x)\ dx$ diverges.

Formulas for the Quiz

$$y = (1-x^2)^{-1/2}, \qquad y'(x) = \frac{x}{(1-x^2)^{3/2}}, \qquad y^{(2)}(x) = \frac{1+2x^2}{(1-x^2)^{5/2}}.$$

Trapezoidal Rule.

$$T_n(f) = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

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