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COMS 6500 Homework 1

For each question show how you arrived at the answer.

1. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.
    a. $f(x) = \dfrac{2x}{x^2 + 1}$, [0,2] **5 points**
    b. $f(x) = x^3 - 4x + 2$, [1,2] **5 points**
2. Answer the following questions and up to seven (7) decimal places
    a. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x + 1}$ about $x_0 = 0$. **5 points**
    b. Approximate 0.5, 0.75, 1.25, 1.5 using $P_3(x)$ **10 points**
    c. Find the absolute errors and relative errors **5 points**
    d. Find the truncation error **5 points**

    Please use the table below to summarize your results

	0.5	0.75	1.25	1.5
f(x)				
$P_3(x)$				
Absolute error				
Relative error				
Truncation error				

Absolute error = $|p - p^*|$

Relative error = $\left| \dfrac{p - p^*}{p} \right|$

Truncation Error = $\dfrac{1}{(n+1)!} f^{(n+1)}(\eta)(x - x_0)^{(n+1)}$

3. Compute the absolute errors and relative error in the approximation of *p* by *p** put the answer in the form **a.o x 10^n** (**<u>Use three significant figures</u>**.)
    a. $p = e^{10}$, $p^* = 22000$ **5 points**
    b. $p = 10^{\square}$, $p^* = 1400$ **5 points**
    c. $p = 8$, $p^* = 39900$ **5 points**

	Absolute Error	Relative Error
$p = e^{10}$, $p^* = 22000$		
$p = 10^{\square}$, $p^* = 1400$		

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p = 8, p* = 39900		

Absolute error = |p - p*|

Relative error = $\left| \dfrac{p - p*}{p} \right|$

COMS 6500 Homework 1 Walkthrough

1a . f(x) = $\dfrac{2x}{x^2 + 1}$, [0,2]

Self note: The video posted stated we could also just use WolframAlpha:

- https://www.wolframalpha.com/input/?i=find+the+derivative+1%2F%282+sqrt%281+%2B+x%29%29

This is solved by using the quotient rule.

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

The following parts were used:

- f(x) = $\dfrac{2x}{x^2 + 1}$, f'(x) = 2 , g(x) = x^2 + 1 , g'(x) = 2x
- $\dfrac{(x^2 + 1)(2x) - (2x) - (2x)(2x)}{(x^2 + 1)^2}$
- = $\dfrac{(x^2 + 1)(2) - (2x)(2x)}{(x^2 + 1)^2}$
- = $\dfrac{2x + 2 - 4x^2}{(x^2 + 1)^2}$
- = $\dfrac{2(-x^2 + 1)}{(x^2 + 1)^2}$
- The derivative can equal zero when x = 1. Because it has a zero derivative, that point might be a maximum or minimum. We add this to the chart and pug it into the function.
-

x	f(x)
0	$\dfrac{2(0)}{(0)^2 + 1}$ = 0
1	$\dfrac{2(1)}{(1)^2 + 1}$ = 1

2	$\frac{2(2)}{(2)^2+1} = \frac{4}{5}$

- The max given the parameters is "1".

1b. $f(x) = x^3 - 4x + 2$, [1,2]

- First step is to find the derivative of f(x).
- $f'(x) = 3x^2 - 4$
- Factoring $(3x^2 - 4) = \mp\sqrt{1.3333} = \mp 1.155$
- That is in [1,2] so we include it on the chart.
-

x	f(x)
1	$(1)^3 - 4(1) + 2 = -1$
1.155	$(1.155)^3 - 4(1.155) + 2 = -1.079$
2	$(2)^3 - 4(2) + 2 = 2$

- The max given the parameters is "2". The min is "1.155"

2a. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x+1}$ about $x_0 = 0$

- This is the Taylor polynomial formula:

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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- Self note: Here is a video explaining it:
  - https://www.youtube.com/watch?v=8SsC5st4LnI
- Self note: Here is a video with a really close example problem:
  - https://www.youtube.com/watch?v=39s-mgfuuno
- Make a chart for what each part will be:
  -

Part	Formula
Equation	$\sqrt{x+1}$
First Degree Taylor Polynomial Component	$\left(\frac{1}{2\sqrt{x+1}}\right) / 1! * (x - 0)^1$
Second Degree Taylor Polynomial Component	$\left(-\frac{1}{4\sqrt{(x+1)^3}}\right) / 2! * (x - 0)^2$

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Third Degree Taylor Polynomial Component	$(\frac{3}{8\sqrt{(x+1)^5}}) / 3! * (x - 0)^3$

- Note that for some parts above, the answers were simplified from dividing a fraction.
- First part to the first degree taylor polynomial is $\frac{\frac{1}{2\sqrt{x+1}}}{1!}$ without simplification.

* Use the formula from above to calculate the third degree taylor polynomial.
* $\sqrt{x+1} + (\frac{1}{2\sqrt{x+1}}) * 1! * (x - 0)^1 + (- \frac{1}{4\sqrt{(x+1)^3}}) * 2! * (x - 0)^2 + (\frac{3}{8\sqrt{(x+1)^5}}) * 3! * (x - 0)^3$
*

Part	x	Formula	Answer
Equation	0.5	(See below)	1.4146936
First Degree Taylor Polynomial Component	0.75	(See below)	1.5824851
Second Degree Taylor Polynomial Component	1.25	(See below)	1.8748714
Third Degree Taylor Polynomial Component	1.5	(See below)	2.0056739

* For 0.5:
* $\sqrt{0.5+1} + (\frac{1}{2\sqrt{0.5+1}}) / 1! * (0.5 - 0)^1 + (- \frac{1}{4\sqrt{(0.5+1)^3}}) / 2! * (0.5 - 0)^2 + (\frac{3}{8\sqrt{(0.5+1)^5}}) / 3! * (0.5 - 0)^3$
* $\sqrt{1.5} + (\frac{1}{2\sqrt{1.5}}) / 1 * 0.5 + (- \frac{1}{4\sqrt{(1.5)^3}}) / 2 * 0.5^2 + (\frac{3}{8\sqrt{(1.5)^5}}) / 6 * (0.5)^3$
* $1.2247448 + (\frac{1}{2.4494897}) * 0.5 + (- \frac{1}{7.3484692}) / 2 * 0.25 + (\frac{3}{22.0454076}) / 6 * 0.125$
* 1.2247448 + 0.2041241 - 0.0170103 + 0.0028350
  - 1.4146936
* For 0.75:
* $\sqrt{0.75+1} + (\frac{1}{2\sqrt{0.75+1}}) / 1! * (0.75 - 0)^1 + (- \frac{1}{4\sqrt{(0.75+1)^3}}) / 2! * (0.75 - 0)^2 + (\frac{3}{8\sqrt{(0.75+1)^5}}) / 3! * (0.75 - 0)^3$
* $\sqrt{1.75} + (\frac{1}{2\sqrt{1.75}}) / 1 * 0.75 + (- \frac{1}{4\sqrt{(0.75+1)^3}}) / 2 * (0.75 - 0)^2 + (\frac{3}{8\sqrt{(0.75+1)^5}}) / 6 * (0.75 - 0)^3$
* 1.3228756 + 0.2834733 - 0.0303721 + 0.0065083
  - 1.5824851
* For 1.25:
* $\sqrt{x+1} + (\frac{1}{2\sqrt{x+1}}) / 1! * (x - 0)^1 + (- \frac{1}{4\sqrt{(x+1)^3}}) / 2! * (x - 0)^2 + (\frac{3}{8\sqrt{(x+1)^5}}) / 3! * (x - 0)^3$

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- $\sqrt{1.25 + 1}$ + ($\frac{1}{2\sqrt{1.25+1}}$) / 1! * (x - 0)^1 + (- $\frac{1}{4\sqrt{(1.25+1)^3}}$) / 2! * (1.25 - 0)^2 + ($\frac{3}{8\sqrt{(1.25+1)^5}}$) / 3! * (1.25 - 0)^3
- 1.5 + ($\frac{1}{3}$) / 1 * 1.25 + (- $\frac{1}{4\sqrt{(1.25+1)^3}}$) / 2 * (1.25 - 0)^2 + ($\frac{3}{8\sqrt{(1.25+1)^5}}$) / 6 * 1.953125
- 1.5 + 0.4166666 - 0.0578703 + 0.0160751
  - 1.8748714
- For 1.5
- $\sqrt{x + 1}$ + ($\frac{1}{2\sqrt{x+1}}$) / 1! * (x - 0)^1 + (- $\frac{1}{4\sqrt{(x+1)^3}}$) / 2! * (x - 0)^2 + ($\frac{3}{8\sqrt{(x+1)^5}}$) / 3! * (x - 0)^3
- $\sqrt{1.5 + 1}$ + ($\frac{1}{2\sqrt{1.5+1}}$) / 1! * (1.5 - 0)^1 + (- $\frac{1}{4\sqrt{(1.5+1)^3}}$) / 2! * (1.5 - 0)^2 + ($\frac{3}{8\sqrt{(1.5+1)^5}}$) / 3! * (1.5 - 0)^3
- $\sqrt{2.5}$ + ($\frac{1}{2\sqrt{2.5}}$) / 1 * 1.5 + (- $\frac{1}{4\sqrt{(2.5)^3}}$) / 2 * (1.5)^2 + ($\frac{3}{8\sqrt{(2.5)^5}}$) / 6 * (1.5)^3
- 1.5811388 + 0.474341 - 0.0711512 + 0.0213453
  - 2.0056739

Currently, the table from earlier would be:

	0.5	0.75	1.25	1.5
f(x)	1.2247448	1.3228756	1.5	1.5811388
P_3(x)	1.4146936	1.5824851	1.8748714	2.0056739
Absolute error				
Relative error				
Truncation error				

- To find the absolute error, we use: Absolute error = |p - p*|

	0.5	0.75	1.25	1.5
f(x)	1.2247448	1.3228756	1.5	1.5811388
P_3(x)	1.4146936	1.5824851	1.8748714	2.0056739
Absolute error	0.1899488	0.2596095	0.3748714	0.4245351
Relative error				
Truncation error				

- To find the relative error, we use: Relative error = $\left| \frac{p - p*}{p} \right|$

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	0.5	0.75	1.25	1.5
f(x)	1.2247448	1.3228756	1.5	1.5811388
P_3(x)	1.4146936	1.5824851	1.8748714	2.0056739
Absolute error	0.1899488	0.2596095	0.3748714	0.4245351
Relative error	0.1550925	0.1962463	0.2499142	0.2684995
Truncation error				

- To find the truncation error, we use: Truncation Error = $\frac{1}{(n+1)!} f^{(n+1)}(ɳ)(x - x_0)^{(n+1)}$
  - This is also known as Lagrange's formula
- We don't know what "ɳ" is. We only know there is a variable.
- We also can find $f^{(n+1)}$ by taking fourth degree taylor polynomial.
  - $-\frac{15}{16}(x+1)^{(-7/2)} = -\frac{15}{16\sqrt{(x+1)^7}}$
- We can then find $f^{(n+1)}(ɳ)$ by finding the max
- f(0.5)
  - $\frac{1}{(n+1)!} f^{(n+1)}(ɳ)(x - x_0)^{(n+1)}$
  - $\frac{1}{(3+1)!} f^{(3+1)}(ɳ)(x - x_0)^{(3+1)}$
  - $\frac{1}{24} f^{4}(ɳ)(x - x_0)^{4}$
  - $\frac{1}{24}\left(-\frac{15}{16\sqrt{(x+1)^7}}\right)(x - x_0)^{4}$
  - $\frac{1}{24}\left(-\frac{15}{16\sqrt{(0.5+1)^7}}\right)(0.5 - 1.2247448)^{4}$
  - -0.0026072
- f(0.75)
  - $\frac{1}{(n+1)!} f^{(n+1)}(ɳ)(x - x_0)^{(n+1)}$
  - $\frac{1}{(3+1)!} f^{(3+1)}(ɳ)(x - x_0)^{(3+1)}$
  - $\frac{1}{24} f^{4}(ɳ)(x - x_0)^{4}$
  - $\frac{1}{24}\left(-\frac{15}{16\sqrt{(x+1)^7}}\right)(x - x_0)^{4}$
  - $\frac{1}{24}\left(-\frac{15}{16\sqrt{(0.75+1)^7}}\right)(0.75 - 1.3228756)^{4}$

    - -0.0005934
- f(1.25)
    - $\frac{1}{(n+1)!}\ f^{(n+1)}(\eta)(x - x_0)^{(n + 1)}$
    - $\frac{1}{(3+1)!}\ f^{(3+1)}(\eta)(x - x_0)^{(3 + 1)}$
    - $\frac{1}{24}\ f^{4}(\eta)(x - x_0)^{4}$
    - $\frac{1}{24}\ (-\frac{15}{16\sqrt{(x+1)^7}})(x - x_0)^{4}$
    - $\frac{1}{24}\ (-\frac{15}{16\sqrt{(1.25+1)^7}})(1.25 - 1.5)^{4}$
    - -0.000008
- f(1.5)
    - $\frac{1}{(n+1)!}\ f^{(n+1)}(\eta)(x - x_0)^{(n + 1)}$
    - $\frac{1}{(3+1)!}\ f^{(3+1)}(\eta)(x - x_0)^{(3 + 1)}$
    - $\frac{1}{24}\ f^{4}(\eta)(x - x_0)^{4}$
    - $\frac{1}{24}\ (-\frac{15}{16\sqrt{(x+1)^7}})(x - x_0)^{4}$
    - $\frac{1}{24}\ (-\frac{15}{16\sqrt{(1.5+1)^7}})(1.5 - 1.5811388)^{4}$
    - -0.0000001
        - Actual is $-6.86 * 10^{-8}$
- Finally, fill in the last item needed in the chart:

	0.5	0.75	1.25	1.5
f(x)	1.2247448	1.3228756	1.5000000	1.5811388
$P_3(x)$	1.4146936	1.5824851	1.8748714	2.0056739
Absolute error	0.1899488	0.2596095	0.3748714	0.4245351
Relative error	0.1550925	0.1962463	0.2499142	0.2684995
Truncation error	-0.0026072	-0.0005934	-0.0000080	-0.0000001

3. Compute the absolute errors and relative error in the approximation of *p* by *p** put the answer in the form **a.o x 10^n**. (**Use three significant figures**.)

- Still using the same formulas we used before:

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- Absolute error = |p - p*|
- Relative error = $\left| \frac{p - p*}{p} \right|$

3a. p = e^{10}, p* = 22000

- Start with the absolute error:
  - |p - p*|
  - |e^{10} - 22000|
  - 26.4657948
- Then do the relative error:
  - $\left| \frac{p - p*}{p} \right|$
  - $\left| \frac{e^{10} - 22000}{e^{10}} \right|$
  - 0.0012015

3b. p = $10^{\square}$, p* = 1400

- Start with the absolute error:
  - |p - p*|
  - |$10^{\square}$ - 1400|
  - 14.5442686
- Then do the relative error:
  - $\left| \frac{10^{\square} - 1400}{p} \right|$
  - $\left| \frac{10^{\square} - 1400}{10^{\square}} \right|$
  - 0.0104978

3c. p = 8!, p* = 39900

- Start with the absolute error:
  - |p - p*|
  - |8! - 39900|
  - 420
- Then do the relative error:
  - $\left| \frac{p - p*}{p} \right|$
  - $\left| \frac{8! - 39900}{8!} \right|$
  - 0.0104

3. Finally, reformat the answers and fill in the chart:

	Absolute Error	Relative Error

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$p = e^{10}$, $p^* = 22000$	$2.65 * 10^{1}$	$1.20 * 10^{-3}$
$p = 10^{\square}$, $p^* = 1400$	1.45 &* 10^{1}	$1.05 * 10^{-2}$
$p = 8$, $p^* = 39900$	$4.20 * 10^{2}$	$1.04 * 10^{-2}$