1. The equation $f(x) = x^2 - 2e^x = 0$ has a solution in the interval [-1,1].
   (a) With $p_0 = -1$ and $p_1 = 1$ calculate p_2 using the Secant method.
   (b) With p_2 from part (a) calculate p_3 using Newton's method.

Solution:

(a)

$$p_i = p_{i-1} - \frac{f(p_{i-1})(p_{i-1} - p_{i-2})}{f(p_{i-1}) - f(p_{i-2})}$$

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 1 - \frac{f(1)(1-(-1))}{f(1) - f(-1)} = 1 - \frac{f(1)(1-(-1))}{f(1) - f(-1)}$$

$$p_2 = 1 - \frac{(-4.4365)(1-(-1))}{(-4.4365) - (0.2642)} = \mathbf{-0.8875}$$

(b)

$f'(x) = 2x - 2e^x$

$$p_3 = p2 - \frac{f(p_2)}{f'(p_2)} = p_2 - \frac{f(p_2)}{f'(p_2)} = -0.8875 - \frac{-0.0357}{-2.5983} = \mathbf{-0.9012}$$

----

2. The equation $f(x) = 2 - x^2\sin x = 0$ has a solution in the interval [-1,2].
   (a) Verify that the Bisection method can be applied to the function f(x) on [-1,2].
   (b) Using the error formula for the Bisection method find the number of iterations needed for accuracy 0.000001. Do not do the Bisection calculations.
   (c) Compute p_3 for the Bisection method.

Solution:

(a) Verify the formula has a zero in the interval [-1,2]

$f(x) = 2 - x^2\sin x = 0$
$f(-1) = 2 - (-1)^2\sin(-1)$
$f(-1) = 2.841$

$f(x) = 2 - x^2\sin x = 0$
$f(2) = 2 - (2)^2\sin(2)$
$f(2) = 2 - (2)^2\sin(2)$
$f(2) = -1.637$

Since 0 is between 2.841 and -1.637, there is a point that crosses the x axis.

(b)

$\varepsilon > \frac{b_n - a_n}{2^{n+1}}$

$\varepsilon = 10^{-6}$

$10^{-6} > \frac{b_n - a_n}{2^{n+1}}$

$\log(10^{-6}) > \log\left(\frac{1}{2^{n+1}}\right)$

$-6 > -\log(2^{n+1})$

$-6 > -(n+1)\log(2)$

$-\frac{6}{log2} > -\frac{(n+1)(log2)}{log2}$

$19.93 > n+1$

$18.93 > n$

~19 > n

(c)

$p_0 = 2 - x^2\sin x$

Choosing the half point for p_0; (max) 2 - (min) -1 = 3; 3/2 = 1.5; -1 + 1.5 and 2-1.5 = 0.5

$p_0 = 2 - (0.5)^2\sin(0.5)$

$p_0 = 1.8801$

f(-1) = 2.841

$p_0 = 1.8801$

f(-1) * p_0 = Some positive number

f(2) = -1.637

$p_0 = 1.8801$

f(2) * p_0 = Some negative number

Use f(2) as the marker since it is negative (somewhere in the middle must pass through zero)

$p_1 = \frac{0.5 + 2}{2}$

$p_1 = 1.25$

$f(p_1) = 2 - (1.25)^2\sin(1.25)$

$f(p_1) = 0.5172$

f(1.25) is still positive. Keep using f(2)

$p_2 = \frac{1.25 + 2}{2} = 1.625$

$f(p_2) = 2 - (1.625)^2 \sin(1.625)$
$f(p_2) = -0.6367$

f(1.625) is negative so use p_1 instead of 2.

$p_3 = \frac{p_2 + p_1}{2} = \frac{1.625 + 1.25}{2}$ = **1.4375**

----

3. Suppose the function f(x) has a unique zero p in the interval [a,b]. Further, suppose f''(x) exists and is continuous on the interval [a,b].
   (a) Under what conditions will Newton's method give a quadratically convergent sequence to p?
   (b) Define quadratic convergence.

Solution:

(a) **Under the condition that "a sufficiently accurate initial approximation is chosen"**.(page 70)

(b) **Quadratic convergence means "the speed of convergence of the method decreases to 0 as the procedure continues."** (Page 70)

----

4. Let $f(x) = x^3 - e^{-x}$, $x_0 = 0.5$, $x_1 = 0.7$, $x_2 - 1.0$.
   (a) Find the Lagrange Polynomial, $P_2(x)$, of degree at most 2 for f(x) using x_0, x_1, x_2.
   (b) Evaluate $P_2(0.8)$ and compute the actual error $|f(0.8) - P_2(0.8)|$

Solution: page 108

(a)

f(0.5) = -0.4815
f(0.7) = -0.1535
f(1.0) = 0.6321

$L_0(x) = \frac{(x-0.7)(x-1)}{(0.5-0.7)(0.5-1)}$

$L_1(x) = \frac{(x-0.5)(x-1)}{(0.7-0.5)(0.7-1)}$

$L_2(x) = \frac{(x-0.5)(x-0.7)}{(1-0.5)(1-0.7)}$

$P_2(x) = (-0.4815)L_0(x) + (-0.1535)L_1(x) + (0.6321)L_2(x)$

(b)

$f(0.8) = 0.0626$

$P_2(0.8) = (-0.4815)L_0(0.8) + (-0.1535)L_1(0.8) + (0.6321)L_2(0.8)$
$P_2(0.8) = (-0.4815)(-0.2) + (-0.1535)(1) + (0.6321)(0.2)$
$P_2(0.8) = 0.06922$

$|f(0.8) - P_2(0.8)| = |0.0626 - 0.06922|$
$|f(0.8) - P_2(0.8)| =$ **0.00662**

----

5. Let $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$. Given

$f(x) = -2e^{-x} + 1/4x^4 - \frac{1}{120}x^5 + 2x$

$f'(x) = 2e^{-x} + x^3 - \frac{1}{24}x^4 + 2$

$f''(x) = -2e^{-x} + 3x^2 - \frac{1}{6}x^3$

$f'''(x)= 2e^{-x} + 6x - \frac{1}{2}x^2$

$f^{(4)}(x) = -2e^{-x} + 6 - x$

$f^{(5)}(x) = 2e^{-x} - 1$

$f^{(6)}(x) = -2e^{-x}$

(a) Find the Lagrange Interpolating Polynomial, $P_2(x)$, of degree at most 2 for $f(x)$ using x_0, x_1, x_2.
(b) Give the general error formula for $f(x) - P_2(x)$.
(c) Use the formula from (b) to find a bound for the absolute error at 0.65 assuming $f'''(x)$ has no relevant critical points.

Solution: (Same references as question 4)

(a)

$f(0) = -2$
$f(0.5) = -0.1976$
$f(1) = 1.5059$

$L_0(x) = \dfrac{(x-0.5)(x-1)}{(0-0.5)(0-1)}$

$L_1(x) = \dfrac{(x-0)(x-1)}{(0.5-0)(0.5-1)}$

$L_2(x) = \dfrac{(x-0)(x-0.5)}{(1-0)(1-0.5)}$

$P_2(x) = (-2)L_0(x) + (-0.1976)L_1(x) + (1.5059)L_2(x)$

(b)

$\dfrac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$

$\dfrac{f'''(\xi(x))}{6}$ **$x(x-0.5)(x-1)$**

(c)
f'''(0) = 2
f'''(0.5) = 4.0880
f'''(1) = 6.2357

f'''(1) is the max error; Use in formula below for f'''(ξ(x))

$\frac{f'''(\xi(x))}{6}$ x(x-0.5)(x-1)
$\frac{f'''(\xi(x))}{6}$ (0.65)(0.65-0.5)(0.65-1)
$\frac{f'''(\xi(x))}{6}$ (0.65)(0.65-0.5)(0.65-1)
$\frac{6.2357}{6}$ (0.65)(0.65-0.5)(0.65-1)
=**-0.0354 General Error?**

|f(0.8) - P_2(0.8)|
P_2(0.8) = (-2)(-0.12) + (-0.1976)(0.64) + (1.5059)(0.48)
|0.8010 - 0.8363|
= **0.0353 Absolute Error?**
----
6. Let f(x) = $x^4 - 2x^3 + x^2 - 3$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$.
   (a) Compute the interpolating polynomial, $P_3(x)$, of degree at most 3 for f(x) using the given nodes.
   (b) Find the maximum error in using $P_3(x)$ to approximate f(x) on the interval [0,2].

Solution:
(a)
f(0) = -3
f(0.5) = -2.9375
f(1) = -3
f(1.5) = -2.4375

$L_0(x) = \frac{(x-0.5)(x-1.0)(x-1.5)}{(0-0.5)(0-1.0)(0-1.5)}$
$L_1(x) = \frac{(x-0)(x-1.0)(x-1.5)}{(0.5-0)(0.5-1.0)(0.5-1.5)}$
$L_2(x) = \frac{(x-0)(x-0.5)(x-1.5)}{(1.0-0)(1.0-0.5)(1.0-1.5)}$
$L_3(x) = \frac{(x-0)(x-0.5)(x-1.5)}{(1.5-0)(1.5-0.5)(1.5-1)}$

$P_3(x) = (-3)L_0(x) + (-2.9375)L_1(x) + (-3)L_2(x) + (-2.4375)L_3(x)$

(b)
$f(x) = x^4 - 2x^3 + x^2 - 3$
$f'(x) = 4x^3 - 6x^2 + 2x$
$f''(x) = 12x^2 - 6x + 2$
$f'''(x) = 24x - 6$
$f''''(x) = 24$

$$\frac{f''''(\xi(x))}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$
$$\frac{f''''(\xi(x))}{24}(x-0)(x-0.5)(x-1.0)(x-1.5)$$
$$\frac{24}{24}(x-0)(x-0.5)(x-1.0)(x-1.5)$$

Since $f''''(x) = 24$, there is no maximum.
----

7. Let $f(x) = x\sin(2x) - x^2$, $x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.7$.
   (a) Find Newton's Divided-Difference form of the interpolating polynomial P_2 for f(x) using the three given nodes.
   (b) Add a fourth node $x_3 = 0.9$ and compute the next interpolating polynomial P_3.

Solution: (page 124)
(a) and (b)

x	**f(x)**	**First Divided Difference**	**Second Divided Difference**	**Third Divided Difference**
x_0	$f[x_0]$			
		$f[x_0,x_1] = \dfrac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2] = \dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1,x_2] = \dfrac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3] = \dfrac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1,x_2,x_3] = \dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	

		f[x_2,x_3] = $\frac{f[x_3]-f[x_2]}{x_3-x_2}$		
x_3	f[x_3]			

Filled in:

x_0 = 0	f[x_0] = 0			
		f[x_0,x_1] = $\frac{f[x_1]-f[x_0]}{x_1-x_0}$ = $\frac{0.0793-0}{0.3-0}$ = 0.2643		
x_1 = 0.3	f[x_1] = 0.0793		f[x_0,x_1,x_2] = $\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$ = $\frac{0.30125-0.2643}{0.7-0}$ = 0.0527	
		f[x_1,x_2] = $\frac{f[x_2]-f[x_1]}{x_2-x_1}$ = $\frac{0.1998-0.0793}{0.7-0.3}$ = 0.30125		f[x_0,x_1,x_2,x_3] = $\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$ = $\frac{-0.5020-0.0527}{0.9-0}$ = -1.8516
x_2 = 0.7	f[x_2] = 0.1998		f[x_1,x_2,x_3] = $\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$ = $\frac{-0.667-0.30125}{0.9-0.3}$ = -1.61375	
		f[x_2,x_3] = $\frac{f[x_3]-f[x_2]}{x_3-x_2}$ = $\frac{0.0664-0.1998}{0.9-0.7}$ = -0.667		
x_3 = 0.9	f[x_3] = 0.0664			

P_2 = 0.2643(x)+0.0527(x)(x-0.3)

P_3 = 0.2643(x)+0.0527(x)(x-0.3) -1.8516(x)(x-0.3)(x-0.7)

----

8. Given the partition $x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.5$ of [0,0.5] and f(x) = sin3x;
   (a) Find the cubic spline s with clamped boundary conditions that interpolates f.
   (b) Find an approximation for $\int_0^{0.5} \sin 3x dx$ with $\int_0^{0.5} \sin(x)dx$ and compare the results to the actual value.

Solution: pg 154, Slides CH03_5B

(a)

Solution: (page 148/149, Slides CH03_5B)

$x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.5$

$a_0 = f(x_0) = 0$, $a_1 = f(x_1) = 0.7833$, $a_2 = f(x_2) = 0.9974$

$h_j = x_{j+i} - x_j$

$h_0 = x_1 - x_0 = 0.3 - 0 = 0.3$

$h_1 = x_2 - x_1 = 0.5 - 0.3 = 0.2$

$S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$

A =

$2h_0$	h_0	0
h_0	$2(h_0 + h_1)$	h_1
0	h_1	$2h_1$

A =

0.6	0.3	0
0.3	1	0.2
0	0.2	0.4

B =

0
$3(\frac{1}{h1}(y2 - y1) - \frac{1}{h_0}(y1 - y0))$

0

B =

0
$3(\frac{1}{0.3}(0.9974 - 0.7833) - \frac{1}{0.2}(0.7833 - 0)) = -1.7755$
0

x =

c_0
c_1
c_2

x = (matrix multiplication) A*B

-0.53265
-1.7755
-0.3551

$c_1 = -1.7755$

--Solve for b

$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0)$

$b_0 = \frac{1}{0.3}(0.7833) - \frac{0.3}{3}(-1.7755)$

$b_0 = 2.78855$

$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 + 2c_1)$

$b_1 = \frac{1}{0.2}(0.9974 - 0.7833) - \frac{0.2}{3}(2(-1.7755))$

$b_1 = \frac{1}{0.2}(0.9974 - 0.7833) - \frac{0.2}{3}(2(-1.7755))$

$b_1 = 1.3072$

--Solve for d

$d_0 = \frac{1}{3h_0}(c_1 - c_0)$

$d_0 = \frac{1}{3(0.3)}(-1.7755 - 0)$

$d_0 = -1.9727$

$d_1 = \frac{1}{3h_1}(c_2 - c_1)$

$d_1 = \frac{1}{3(0.2)}(0 + 1.7755)$

$d_1 = 2.9591$

$S(x) = 2.78855(x)-0.53265(x)^2 -1.9727(x)^3$ $\quad\in [0,0.3]$

$S(x) = 0.7833 +1.3072(x-0.3) -1.7755(x-0.3)^2 + 2.9591(x-0.3)^3$ $\quad\in [0.3,0.5]$

(b) Solution on pg 154

$\int_0^{0.5} \sin(x)dx = (a_0+a_1) + \frac{1}{2}(b_0+b_1) + \frac{1}{3}(c_0+c_1) + \frac{1}{4}(d_0+d_1)$

$= (0+0.7883) + \frac{1}{2}(2.78855+1.3072) + \frac{1}{3}(-0.53265+-1.7755) + \frac{1}{4}(-1.9727+2.9591)$

$=2.3133$

$\int_0^{0.5} \sin(x)dx = 0.122417$

----

9. Determine the natural cubic spline that interpolates the function $f(x) = x^6$ over the interval [0,2] using knots 0,1,2.

Solution: (page 148/149, Slides CH03_5B)

$x_0 = 0,\ x_1 = 1,\ x_2 = 2$

$a_0 = f(x_0) = 0, a_1 = f(x_1) = 1,\ a_2 = f(x_2) = 64$

$h_j = x_{j+i} - x_j$

$h_0 = x_1 - x_0 = 1- 0 = 1$

$h_1 = x_2 - x_1 = 2 - 1 = 1$

$S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$

A =

1	0	0

h_0	$2(h_1 + h_0)$	h1
0	0	1

A =

1	0	0
1	2	1
0	0	1

B =

0
$3(\frac{1}{h1}(y2 - y1) - \frac{1}{h_0}(y1 - y0))$
0

B =

0
3((1)(64 - 1) - (1)(1 - 0)) = 188
0

x =

c_0
c_1
c_2

x = (matrix multiplication) A*B

0
376

$\boxed{0}$

$c_1 = 376$

--Solve for b

$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0)$

$b_0 = (1 - 0) - \frac{1}{3}(376 + 0)$

$b_0 = -124.333$

$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 + 2c_1)$

$b_1 = (64 - 1) - \frac{1}{3}(0 + 752)$

$b_1 = -187.6666$

--Solve for d

$d_0 = \frac{1}{3h_0}(c_1 - c_0)$

$d_0 = \frac{1}{3}(376 - 0)$

$d_0 = \frac{1}{3}(376 - 0)$

$d_0 = 120$

$d_1 = \frac{1}{3h_1}(c_2 - c_1)$

$d_1 = \frac{1}{3}(0 - 376)$

$d_1 = -120$

$S(x) = -124.333(x_j) + 120(x_j)^3 \qquad \in [0,1]$

$S(x) = 1 - 187.6666(x-1) + 376(x-1)^2 + -120(x-1)^3 \qquad \in [1,2]$

----

10. Do there exist a, b, c, d such that the function

$$S(x) = \begin{cases} -x & (-10 < x \leq -1) \\ ax^3 + bx^2 + cx + d & (-1 \leq x \leq 1) \\ x & (1 \leq x \leq 10) \end{cases}$$

is a natural cubic spline function?

Solution:

--Natural cubic spline definition (page 146, Theorem 3.11; Last sentence)
$s_0''(x_1) = 0$
$s_2''(x_3) = 0$

--Find points:

$s_0(x_0) = f(-10) = 10$
$s_1(x_1) = f(-1) = -a +b -c +d$
$s_2(x_2) = f(1) = 1$
$s_2(x_3) = f(10) = 10$

$s_0(x_1) = s_1(x_1)$ --Used later in finding b, d
$s_0(-1) = s_1(-1)$
$1 = -a +b -c +d$

$s_1(x_2) = s_2(x_2)$ --Used later in finding b, d
$s_1(1) = s_2(1)$
$a +b +c +d = 1$

$s_0'(x_1) = s_1'(x_1)$ --Used later in finding a, c
$-1 = 3a - 2b + c$
$-1 = 3a - 2b + c$

$s_1'(x_2) = s_2'(x_2)$ --Used later in finding a, c
$1 = 3a + 2b + c$

$s_0''(x_1) = s_1''(x_1)$ --Used later in finding a,b
$0 = -6a + 2b$
$0 = -6a + 2b$

$s_1''(x_2) = s_2''(x_2)$ --Used later in finding a,b
$6a + 2b = 0$ --Used later in finding a

--Find b,d
$1 = -a +b -c +d$
$1 = a +b +c +d$
(Add them)
$2 = 2b + 2d$
$1 = b + d$ --Used later in finding d

--Find a,c

$-1 = 3a - 2b + c$
$1 = 3a + 2b + c$
(Add them)
$0 = 6a + 2c$
$0 = 3a + c$ --Used later in finding c

--Find a,b
$0 = -6a + 2b$
$0 = 6a + 2b$
(Add them)
$0 = 4b$
$0 = b$

--Finding d
$1 = b + d$
$1 = 0 + d$
$1 = d$

--Finding a
$6a + 2b = 0$
$6a + 2(0) = 0$
$a = 0$

--Finding c
$0 = 3a + c$
$0 = 3(0) + c$
$0 = c$
