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For each question show how you arrived at the answer. These questions are on **Numerical differentiation and integration**.

1. Let f(x) = $\dfrac{cosx}{1+x^3}$. Approximate f'(0.9) using the three point centered difference formula with h = 0.2.

Solution: Page 175

f'(x) = $\dfrac{f(x_0+h)-f(x_0-h)}{2h}$

f'(x) = $\dfrac{f(0.9+0.2)-f(0.9-0.2)}{2(0.2)} = \dfrac{f(1.1)-f(0.7)}{0.4}$

f'(x) = $\dfrac{0.1945-0.5695}{0.4}$ = **-0.9372**

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2. Let h = 0.2. Given

f(x) = $-2e^{-x} + 1/4x^4 - \frac{1}{120}x^5 + 2x$

f'(x) = $2e^{-x} + x^3 - \frac{1}{24}x^4 + 2$

f''(x) = $-2e^{-x} + 3x^2 - \frac{1}{6}x^3$

f'''(x) = $2e^{-x} + 6x - \frac{1}{2}x^2$

$f^{(4)}(x) = -2e^{-x} + 6 - x$

$f^{(5)}(x) = 2e^{-x} - 1$

$f^{(6)}(x) = -2e^{-x}$

(i) Approximate f'(0.65) using the three point centered difference formula
(ii) Give the general form of the error formula for the five point centered difference formula.
(iii) Give the error formula for part (i).

Solution: Page 175

(i)

f'(x) = $\dfrac{f(x_0+h)-f(x_0-h)}{2h}$

f'(x) = $\dfrac{f(0.65+0.2)-f(0.65-0.2)}{2(0.2)} = \dfrac{f(0.85)-f(0.45)}{0.4}$

f'(x) = $\dfrac{f(0.85)-f(0.45)}{0.4} = \dfrac{f(0.9719)-f(-0.3651)}{0.4}$

f'(x) = **3.3428**

(ii) Page 176

$\dfrac{h^4}{30}$ **$f^{(4)}(\xi)$ where ξ lies between x_0 and x_0 + 4h**

(iii) Page 175

$\dfrac{h^2}{6}$ **$f^{(3)}(\xi_1)$ where ξ_1 lies between x_0 - h and x_0 + h**

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3. Use the forward-difference formulas and backward-difference formulas to determine the missing row of the table below.

x	f(x)	f'(x)
1.0	1.0000	
1.2	1.2625	
1.4	1.6595	

(i) Compute the actual errors.
(ii) Find the error bounds using the error formulas.

Solution: Page 173

(i)

f'(x) = $\frac{f(x_0+h)-f(x_0)}{h}$ (forward); $\frac{f(x_0)-f(x_0-h)}{h}$ (backward); h = x_1 - x_0 = 0.2

f'(1.0) = $\frac{f(x+h)-f(x)}{h}$ = $\frac{f(1.0+0.2)-f(1.0)}{0.2}$ = $\frac{1.2625-1.0000}{0.2}$ = **1.3125**

f'(1.2) = $\frac{f(x+h)-f(x)}{h}$ = $\frac{f(1.2+0.2)-f(1.2)}{0.2}$ = $\frac{1.6595-1.2625}{0.2}$ = **1.9850**

f'(1.4) = $\frac{f(x)-f(x-h)}{h}$ = $\frac{f(1.4)-f(1.4-0.2)}{0.2}$ = $\frac{1.6595-1.2625}{0.2}$ = **1.9850**

(ii)

f(x) = x^2ln(x) + 1

f'(x) = 2xln(x) + x

f'(1.0) = 2(1.0)ln(1.0) + (1.0) = 1

f'(1.2) = 2(1.2)ln(1.2) + (1.2) = 1.6375

f'(1.4) = 2(1.4)ln(1.4) + (1.4) = 2.3421

E f'(1.0) = |1 - 1.3125| = **0.3125**

E f'(1.2) = |1.6375 - 1.9850| = **0.3474**

E f'(1.4) = |2.3421 - 1.9850| = **0.3571**

(iii)

f''(x) = 2xln(x) + 3

f''(x) = 2xln(x) + 3

f''(1.0) < ξ < f''(1.2) --Use 1.2 since f''(1.2) is greater

$\frac{h}{2}$ f''(ξ) ≤ $\frac{0.2}{2}$ f''(1.2) = **0.3365**

f''(1.2) < ξ < f''(1.4) --Use 1.4 since f''(1.4) is greater

$\frac{h}{2}$ f''(ξ) ≤ $\frac{0.2}{2}$ f''(1.4) = **0.3673**

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4. The Composite Trapezoidal Rule applied to the integral I = $\int_a^b$ f(x)dx gives the error E = - $\frac{b-a}{12}$ $h^3 f''(\mu)$. Suppose f''(x) = $\frac{2 + 2x - e^x}{3}$, a = 0.51, b = 1.0. What values of n and h should be used to approximate I to within 0.00001?

Solution: Page 205

E = - $\frac{b-a}{12}$ $h^3 f''(\mu)$ < 0.00001

f''(0.51) = $\frac{2 + 2x - e^x}{3}$ = $\frac{2 + 2(0.51) - e^{0.51}}{3}$ = 0.45156

f''(1.0) = $\frac{2 + 2x - e^x}{3}$ = $\frac{2 + 2(1.0) - e^{1.0}}{3}$ = 0.4272

Find the max or min

f'''(x) = $\frac{1}{3}$ (2 - e^x)

$\frac{1}{3}$ (2 - e^x) = 0

2 - e^x = 0

e^x = 2

x = ln2 = 0.6931

f''(0.6931) = $\frac{2 + 2x - e^x}{3}$ = $\frac{2 + 2(0.6931) - e^{0.6931}}{3}$ = 0.4620 = μ

E = $\left| \frac{b-a}{12} h^3 f''(\mu) \right|$< 0.00001

h = $\frac{(b-a)}{n}$ = $\frac{(1-0.51)^3}{n^3}$ = $\frac{(0.49)^3}{n^3}$

E = $\left| \frac{(1-0.51)}{12} \frac{(0.49)^3}{n^3} (0.4620) \right|$< 0.00001

E = $\left| \frac{(0.49)^4}{12n^3} (0.4620) \right|$< 0.00001

E = $\left| \frac{(0.49)^4(0.4620)}{n^3} \right|$< (0.00001)*12

E = 0.0266 = 0.00012 * n^3

E = 221.9448 = n^3

E = 6.0545 = n

n = 7 --This is the minimum since it is the closest int without going under

h = $\frac{(b-a)}{n}$ = $\frac{(1-0.51)}{7}$ = **0.07**

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5. Let f(x) = xlnx + x^4

(i) Approximate I = $\int_1^3$ f(x)dx using Composite Simpson's rule with n = 4.

(ii) Find the smallest upper bound for the absolute error using the error formula.

(iii) Find the values of n and h required for an error of at most 0.0001?

Solution: Page 204

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(i)

$I = \int_1^3 f(x)dx = \int_1^3 (x\ln x + x^4)dx$

$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$

$x_i = \{1, 1.5, 2, 2.5, 3\}$

$f(x) = x\ln x + x^4$

$f(x_0) = (1)\ln(1) + (1)^4 = 1$

$f(x_1) = (1.5)\ln(1.5) + (1.5)^4 = 5.6706$

$f(x_2) = (2)\ln(2) + (2)^4 = 17.3862$

$f(x_3) = (2.5)\ln(2.5) + (2.5)^4 = 41.3532$

$f(x_4) = (3)\ln(3) + (3)^4 = 84.2958$

$\frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$

$\frac{0.5}{3}$ [1 + 4(5.6706) + 2(17.3862) + 4(41.3532) + 84.2958] = **51.3605**

(ii)

$f(x) = x\ln x + x^4$

$f'(x) = 4x^3 + \log(x) + 1$

$f''(x) = 12x^2 + \frac{1}{x}$

$f^{(3)}(x) = 24x - \frac{1}{x^2}$

$f^{(4)}(x) = 24 + \frac{2}{x^2}$ -- As "x" gets bigger, this gets smaller. Smallest "x" means the greatest value. Use "x" = 1 = M

$E \leq \frac{M(b-a)^5}{180n^4}$

$E \leq \frac{26(3-1)^5}{180(4)^4}$

E $\leq$ 0.0180

(iii)

$E \leq \frac{M(b-a)^5}{180n^4}$

$0.0001 \leq \frac{26(3-1)^5}{180n^4}$

n = 26
