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1. Give linear system

$$x_1 + 4x_2 + \alpha x_3 = 6$$
$$2x_1 - x_2 + \alpha 2x_3 = 3$$
$$\alpha x_1 + 3x_2 + x_3 = 5$$

(a) Find the value(s) of α for which the system has no solutions.
(b) Find the value(s) of α for which the system has an infinite number of solutions.
(c) Assuming a unique solution exists for a given α, find the solution.

Solution: https://elearn.mtsu.edu/d2l/le/content/7975386/viewContent/68580743/View
(About 9 minutes in)

1	4	α		x_1		6
2	-1	α		x_2		3
α	3	1		x_3		5

(a)
There are no solutions when the sum of all variables on the left does not equal to the right.

1	4	α	6
0	-9	$-\alpha$	-9
0	-4a+3	$-a^2+1$	-6a+5

x3 - (x2 * (4a - 3)/9)

1	4	α	6
0	-9	$-\alpha$	-9
0	0	$-a^2 + (1/9)(4a - 3)a + 1$	2 - 2a

$-a^2 + (1/9)(4a - 3)a + 1 = 0$
$a = -(3/10) - 3(\sqrt{21})/10$
and
$a = (3\sqrt{21}/10) - 3/10$

(b)
$-9 - a = -9$
If $a = 0$, $-9 = -9$, infinite solutions

(c)

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$-a^2 + (1/9)(4a - 3)a + 1 = 2 - 2a$
$-(5a^2/9) + (5a/3) - 1 = 0$
$a = (3/2) - (3/2\sqrt{5})$
and
$a = (3/2) + (3/2\sqrt{5})$

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2. Solve the following system using Gaussian elimination with scaled partial pivoting:

1	-1	2		x_1		-2
-2	1	-1		x_2		2
4	-1	2		x_3		-1

Show intermediate matrices at each step.

Solution: Page 380

$s_1 = \max\{|1|,|-1|, |2|\} = 2$
$s_2 = \max\{|-2|,|1|, |-1|\} = 2$
$s_3 = \max\{|4|,|-1|, |2|\} = 4$

$\frac{|a_{11}|}{s_1} = \frac{|1|}{2} = 0.5$
$\frac{|a_{21}|}{s_2} = \frac{|-2|}{2} = 1$
$\frac{|a_{31}|}{s_3} = \frac{|4|}{4} = 1$
s2 = s3 >s1

4	-1	2		x_1		-1
-2	1	-1		x_2		2
1	-1	2		x_3		-2

4	-1	2		x_1		-1
0	0.5	0		x_2		1.5
0	-0.75	1.5		x_3		-1.75

4	-1	2		x_1		-1
0	0.5	0		x_2		1.5

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0	0	1.5	[redacted]	x_3	[redacted]	0.5

$(1.5)x_3 = 0.5$
$x_3 = 0.5 / 1.5$
$x_3 = 0.5 / 1.5$
$x_3 = ⅓$

$(0.5)x_2 = 1.5$
$x_2 = 1.5 / 0.5$
$x_2 = 3$
$(4)x_1 - x_2 + (2)x_3 = -1$
$(4)x_1 - (3) + (2)(1/3) = -1$
$(4)x_1 - (3) + (2/3) = -1$
$(4)x_1 - (7/3) = -1$
$(4)x_1 = -1 + (7/3)$
$(4)x_1 = (4/3)$
$x_1 = (4/3) / 4$
$x_1 = ⅓$

----
3. Find all values of $\alpha > 0$ and $\beta > 0$ so that matrix

A =

3	2	β
α	5	β
2	1	α

A is strictly diagonally dominant.

Solution:https://elearn.mtsu.edu/d2l/le/content/7975386/viewContent/68659394/View
Pg. 417
Around 1:25:10
If strictly diagonally dominant, $|a_{ii}| > \Sigma a_{ik}$

Row 1: $|3| > |2| + |\beta|$
Row 1: β must be between $0 < \beta < 1$

Row 3: $|\alpha| > |1| + |2|$
Row 3: $|\alpha| > 3$

Row 2: $|5| > |\alpha| + |\beta|$

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Row 2: $0 < \beta < 1$; $3 < a < 5 - |\beta|$

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4. Solve the linear system

$$0.211x_1 + 0.811x_2 = 1.52$$
$$1.71x_1 + 1.06x_2 = -0.512$$

using 3 digit chopping arithmetic and Gaussian elimination with partial pivoting.

Solution: pg.381

0.211	0.811	1.52
1.71	1.06	-0.512

|1.71| > |0.211| so we need to switch rows one and two.

1.71	1.06	-0.512
0.211	0.811	1.52

M = 0.211/1.71 = 0.123 (3 digit only)

0.123 * 1.71 = 0.210	0.123 * 1.06 = 0.130	0.123* -0.512 = -0.0629

0.211 - 0.210 = 0.001	0.811 - 0.130 = 0.681	1.52 + 0.0629 = 1.58

1.71	1.06	-0.512
0	0.681	1.58

We turned the first row, first column into zero because of round off error.

$(0.681)x_2 = 1.58$
$x_2 = 1.58/0.681$
$x_2 = 2.32$

$1.71x_1 + 1.06x_2 = -0.512$
$1.71x_1 + 1.06(2.32) = -0.512$
$1.71x_1 + 1.06(2.32) = -0.512$
$1.71x_1 + 2.45 = -0.512$

$1.71x_1 = -2.96$
$x_1 = -2.96/1.71$
$x_1 = -1.73$

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5. Let A = LU where
L =

1	0	0
2	1	0
-1	-2	1

and
U =

4	1	-1
0	2	1
0	0	-1

(a) Solve the linear system Ax = b where b = (0, -2, 2) using Gaussian Elimination
(b) What is the determinant of A?

Solution:pg. 421
(a)
Forward Elimination
Ly = b
$y = \frac{b}{L}$

1	0	0		y_1		0
2	1	0		y_2		-2
-1	-2	1		y_3		2

$y_1 = 0$

$(2)y_1 + (1)y_2 = -2$
$(2)(0) + (1)y_2 = -2$
$(1)y_2 = -2$
$y_2 = -2$

$(-1)y_1 + (-2)y_2 + (1)y_3 = 2$
$(-1)(0) + (-2)(-2) + (1)y_3 = 2$

$-4 + y_3 = 2$
$y_3 = 6$

Backward Elimination
$Ax = b$
$LUx = b$ (and) $Iy = b$
$Ux = y$

4	1	-1		x_1		y_1
0	2	1		x_2		y_2
0	0	-1		x_3		y_3

4	1	-1		x_1		0
0	2	1		x_2		-2
0	0	-1		x_3		6

$-1(x_3) = 6$
$x_3 = -6$

$2(x_2) + (1)(x_3) = -2$
$2(x_2) -6 = -2$
$2(x_2) = 4$
$x_2 = 2$

$(4)(x_1) + (1)(x_2) + (-1)(x_3) = 0$
$(4)(x_1) + (1)(2) + (-1)(-6) = 0$
$(4)(x_1) + 2 + 6 = 0$
$(4)(x_1) + 8 = 0$
$(4)(x_1) = -8$
$x_1 = -2$

(b)
det(A) = det(Lu)
det(A) = det(L) * det(u)
det(A) = (1*1*1) * (4*2*-1)
det(A) = (1) * (-8)
det(A) = -8

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6. Determine values h that will ensure an approximation error of less than 10^{-3} when approximating $\int_0^1 \sin(\pi x^2/2)dx$ and employing **Composite Trapezoid Rule**.

Solution:page 206

$f(x) = \sin(\pi x^2/2)$

$f'(x) = \pi x\cos(\pi x^2/2)$

$f''(x) = \pi x\cos(\pi x^2/2) - \pi x\sin(\pi x^2/2)$

$f''(0) = \pi (0)\cos(\pi (0)^2/2) - \pi (0)\sin(\pi (0)^2/2) = 0$

$f''(1) = \pi (1)\cos(\pi (1)^2/2) - \pi x\sin(\pi (1)^2/2) = 1.2508$

$|f''(0)| < |f''(1)|$

$|E| \leq \frac{(b-a)^3}{12n^2} \; |f''(1)|$

$|E| \leq \frac{(b-a)^3}{12n^2} \; |f''(1)|$

$0.001 = \frac{(1-0)^3}{12n^2} \; 1.2508$

$0.001 = \frac{1.2508}{12n^2}$

$0.001 = \frac{1.2508}{12n^2}$

$0.00079948 = \frac{1}{12n^2}$

$1250.8 = 12n^2$

$104.233 = n^2$

$n > 10.2$

$n > 11$

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7. Find an approximate value of $\int_1^2 x^{-1}dx$ using **Composite Simpson's Rule** with h = 0.25. Give a bound on the error.

Solution:pg 204

(a)

$f(x) = x^{-1}$

$f(x) = \frac{1}{x}$

$f(1) = \frac{1}{1} = 1$

$f(1.25) = \frac{1}{1.25} = \frac{4}{5}$

$f(1.50) = \frac{1}{1.5} = \frac{2}{3}$

$f(1.75) = \frac{1}{1.75} = \frac{4}{7}$

$f(2) = \frac{1}{2}$

$\int_1^2 x^{-1}dx = \frac{0.25}{3}$ [f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)]

$\int_1^2 x^{-1}dx = \frac{0.25}{3}$ [1 + 4($\frac{4}{5}$) + 2($\frac{2}{3}$) + 4($\frac{4}{7}$) + $\frac{1}{2}$]

$\int_1^2 x^{-1}dx = 0.6933$

(b)
$f(x) = x^{-1}$
$f'(x) = \frac{1}{x^2}$
$f''(x) = \frac{2}{x^3}$
$f'''(x) = \frac{6}{x^4}$
$f''''(x) = \frac{24}{x^5}$

$f''''(1) = \frac{24}{(1)^5} = 24$
$f''''(2) = \frac{24}{(2)^5} = 0.75$
$f''''(1) > f''''(2)$

$|24| > |0.75|$

$|E| \leq \frac{(b-a)}{180}\ h^4 * f''''(\zeta)$
$|E| \leq \frac{1}{180}\ (0.25)^4 * 24$
$|E| \leq 0.0005208$

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8. Criticize the following analysis. By Taylor's formula, we have

$$f(x + h) - f(x) = hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6}\ f'''(\zeta_1)$$
$$f(x - h) - f(x) = -hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6}\ f'''(\zeta_2)$$

Therefore, we have

$$\frac{1}{h^2} (f(x + h) - 2f(x) + f(x + h)) = f''(x) + \frac{h}{6} (f'''(\zeta_1) + f'''(\zeta_2))$$

The error in the approximation formula for f'' is thus O(h).

Solution: The two errors are not qualifying for the same criteria so they cannot be summed up. ζ_1 is for x to x + h while ζ_2 is for x to x - h. The combined equation is assuming $\frac{h^3}{6}\ f'''(\zeta_1) = \frac{h^3}{6}\ f'''(\zeta_2)$ which is not true.

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9. Let $f(x) = 3xe^x - \cos x$. **Using the following data and the formula to approximate** $f''(1.3)$ with $h = 0.1$ **and with** $h = 0.01$

$$f''(x_0) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h)) - \frac{h^2}{12} f^{(4)}(\zeta)$$

for some ζ, where $x_0 - h < \zeta < x_0 + h$

x	1.20	1.29	1.30	1.31	1.40
f(x)	11.59006	13.78176	14.04276	14.30741	16.86187

Solution:

$f''(x_0) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h))$

$f''(1.3) = \frac{1}{(0.1)^2}(f(1.3 - 0.1) - 2f(1.3) + f(1.3 + 0.1))$

$f''(1.3) = \frac{1}{(0.1)^2}(f(1.2) - 2f(1.3) + f(1.4))$

$f''(1.3) = \frac{1}{0.01}(11.59006 - 2(14.04276) + 16.86187)$

$f''(1.3) = \frac{1}{0.01}(0.36641)$

$f''(1.3) = 36.641$

$f''(x_0) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h))$

$f''(1.3) = \frac{1}{(0.01)^2}(f(1.3 - 0.01) - 2f(1.3) + f(1.3 + 0.01))$

$f''(1.3) = \frac{1}{(0.01)^2}(f(1.29) - 2f(1.3) + f(1.31))$

$f''(1.3) = \frac{1}{0.0001}(13.78176 - 2(14.04276) + (14.30741))$

$f''(1.3) = 36.5$

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10. In a circuit with impressed voltage $\varepsilon(t)$ and inductance L, Kirchoff's first law gives the relationship

$$\varepsilon(t) = L\frac{di}{dt} + Ri$$

where R is the resistance in the circuit and i is the current. Suppose we measure the current for several values t and obtain:

t	1.00	1.01	1.02	1.03	1.04
i	3.10	3.12	3.14	3.18	3.24

"t" is measured in seconds, "i" is in amperes, the inductance "L" is a constant 0.98 henries, and the resistance is 0.142 ohms. **Approximate the voltage** $\varepsilon(t)$ **when** t = 1.00, 1.01, 1.02, 1.03, 1.04

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Solution: pg 175

$f'(x_0) = \frac{1}{2h}(-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h))$

$f'(1.00) = \frac{1}{2(0.01)}(-3f(1.00) + 4f(1.00 + 0.01) - f(1.00 + 2(0.01)))$

$f'(1.00) = \frac{1}{0.02}(-3f(1.00) + 4f(1.01) - f(1.02))$

$f'(1.00) = \frac{1}{0.02}(-3(3.10) + 4(3.12) - 3.14)$

$f'(1.00) = 2$

$\varepsilon(t) = L\frac{di}{dt} + Ri$

$\varepsilon(1.00) = (0.98)(2) + (0.142)(3.10)$

$\varepsilon(1.00) = 2.4002$

$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h))$

$f'(1.01) = \frac{1}{2(0.01)}(f(1.01 + (0.01)) - f(1.01 - 0.01))$

$f'(1.01) = \frac{1}{0.02}(f(1.02) - f(1.00))$

$f'(1.01) = \frac{1}{0.02}(3.14 - 3.10)$

$f'(1.01) = 2$

$\varepsilon(t) = L\frac{di}{dt} + Ri$

$\varepsilon(1.01) = (0.98)(2) + (0.142)(3.12)$

$\varepsilon(1.01) = 2.40304$

$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h))$

$f'(1.02) = \frac{1}{2(0.01)}(f(1.02 + (0.01)) - f(1.02 - 0.01))$

$f'(1.02) = \frac{1}{0.02}(f(1.03) - f(1.01))$

$f'(1.02) = \frac{1}{0.02}(3.18 - 3.12)$

$f'(1.02) = 3$

$\varepsilon(t) = L\frac{di}{dt} + Ri$

$\varepsilon(1.02) = (0.98)(3) + (0.142)(3.14)$

$\varepsilon(1.02) = 3.38588$

$f'(x_0) = \frac{1}{2h}(f(x_0 + h) - f(x_0 - h))$

$f'(1.03) = \frac{1}{2(0.01)}(f(1.03 + (0.01)) - f(1.03 - 0.01))$

$f'(1.03) = \frac{1}{0.02}(f(1.04) - f(1.02))$

$f'(1.03) = \frac{1}{0.02}(3.24 - 3.14)$

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$f'(1.03) = 5$

$\varepsilon(t) = L\frac{di}{dt} + Ri$
$\varepsilon(1.03) = (0.98)(5) + (0.142)(3.14)$
$\varepsilon(1.03) = 5.34588$

$f'(x_0) = \frac{1}{2h}(-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h))$
$f'(1.04) = \frac{1}{2(-0.01)}(-3f(1.04) + 4f(1.04 - 0.01) - f(1.04 + 2(-0.01)))$
$f'(1.04) = \frac{1}{-0.02}(-3f(1.04) + 4f(1.03) - f(1.02))$
$f'(1.04) = \frac{1}{-0.02}(-3(3.24) + 4(3.18) - 3.14)$
$f'(1.04) = 7$

$\varepsilon(t) = L\frac{di}{dt} + Ri$
$\varepsilon(1.04) = (0.98)(7) + (0.142)(3.14)$
$\varepsilon(1.04) = 7.30588$
