



Machine Learning

Linear Algebra  
review (optional)

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Matrices and  
vectors

**Matrix:** Rectangular array of numbers:



$$\rightarrow \boxed{\mathbb{R}^{4 \times 2}}$$



$$\boxed{\mathbb{R}^{2 \times 3}}$$

Dimension of matrix: number of rows x number of columns

## Matrix Elements (entries of matrix)

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

$A_{ij}$  = " $i, j$  entry" in the  $i^{th}$  row,  $j^{th}$  column.

$$A_{11} = 1402$$

$$A_{12} = 191$$

$$A_{32} = 1437$$

$$A_{41} = 147$$

$$\cancel{A_{43}} = \text{undefined (error)}$$

Vector: An  $n \times 1$  matrix.

$$\textcircled{y} = \begin{bmatrix} \textcircled{460} \\ \textcircled{232} \\ \textcircled{315} \\ 178 \end{bmatrix}$$

$n = 4$

← 4-dimensional vector.

~~$\mathbb{R}^{3 \times 2}$~~

$\mathbb{R}^4$

$y_i = i^{th}$  element

$$y_1 = 460$$

$$y_2 = 232$$

$$y_3 = 315$$

→ A, B, C, X

a, b, x, y

1-indexed vs 0-indexed:

$y[1]$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

1-indexed

$y[0]$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

0-indexed



Machine Learning

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---

## Addition and scalar multiplication

# Matrix Addition

$$\begin{array}{c} \downarrow \quad \downarrow \\ \rightarrow \begin{bmatrix} \textcircled{1} & 0 \\ \textcircled{2} & 5 \\ \textcircled{3} & 1 \end{bmatrix} + \begin{bmatrix} \textcircled{4} & 0.5 \\ \textcircled{2} & 5 \\ \textcircled{0} & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0.5 \\ 4 & 10 \\ 3 & 2 \end{bmatrix} \\ \underline{\hspace{1.5cm}} \quad \underline{\hspace{1.5cm}} \\ 3 \times 2 \quad 3 \times 2 \\ \text{matrix} \end{array}$$

$$\begin{array}{c} \rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \end{bmatrix} = \\ \underline{\hspace{1.5cm}} \quad \underline{\hspace{1.5cm}} \\ 3 \times 2 \quad 2 \times 2 \end{array}$$

error

# Scalar Multiplication

← real number

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 15 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} \times 3$$

3x2                      3x2

$$\begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} / 4 = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & \frac{3}{4} \end{bmatrix}$$

# Combination of Operands

$$\begin{aligned}
 & \text{Scalar multiplication} \rightarrow 3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} / 3 \quad \text{Scalar division} \\
 & = \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix} \quad \begin{array}{l} \text{matrix subtraction /} \\ \text{vector subtraction} \end{array} \\
 & = \begin{bmatrix} 2 \\ 12 \\ 10 \frac{1}{3} \end{bmatrix} \quad \begin{array}{l} \text{matrix addition /} \\ \text{vector addition} \end{array} \\
 & \quad \begin{array}{l} 3 \times 1 \text{ matrix} \\ 3\text{-dimensional vector} \end{array}
 \end{aligned}$$





Machine Learning

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---

## Matrix-vector multiplication

# Example

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix}_{3 \times 2} \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}_{3 \times 1} \text{ matrix}$$

$$1 \times 1 + 3 \times 5 = 16$$

$$4 \times 1 + 0 \times 5 = 4$$

$$2 \times 1 + 1 \times 5 = 7$$

## Details:

$$\underline{A} \times \underline{x} = \underline{y}$$

$\underline{A}$  is an  $m \times n$  matrix (m rows, n columns).  
 $\underline{x}$  is an  $n \times 1$  matrix (n-dimensional vector).  
 $\underline{y}$  is an  $m$ -dimensional vector.

→ To get  $\underline{y}_i$ , multiply  $\underline{A}$ 's  $i^{th}$  row with elements of vector  $\underline{x}$ , and add them up.

# Example

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 3 & 0 & 4 \\ -1 & -2 & 0 & 0 \end{bmatrix}_{3 \times 4} \begin{matrix} \downarrow \\ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}_{4 \times 1} \end{matrix} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}$$

$$\left. \begin{array}{l} 1 \times 1 + 2 \times 3 + 1 \times 2 + 5 \times 1 = 14 \\ 0 \times 1 + 3 \times 3 + 0 \times 2 + 4 \times 1 = 13 \\ -1 \times 1 + (-2) \times 3 + 0 \times 2 + 0 \times 1 = -7 \end{array} \right\}$$

House sizes:

- 2104
- 1416
- 1534
- 852

Matrix

4x2

1	2104
1	1416
1	1534
1	852

X

$h_0(x)$

2x1

Vector

-40
0.25

=

4x1 matrix

$-40 \times 1 + 0.25 \times 2104$
$-40 \times 1 + 0.25 \times 1416$

$h_0(1416)$

Prediction = Data Matrix  $\otimes$  Parameters

4x1

for  $i = 1:1000$ ,  
prediction(i) = ...

$$h_{\theta}(x) = -40 + 0.25x$$



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## Matrix-matrix multiplication

# Example

$$\begin{array}{l} \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 0 & 1 \\ \hline 5 & 2 \\ \hline \end{array} = \begin{bmatrix} 11 & 10 \\ 9 & 14 \end{bmatrix} \\ \textcircled{2 \times 3} \quad \textcircled{3 \times 2} \\ \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 5 \\ \hline \end{array} = \begin{bmatrix} 11 \\ 9 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \end{array}$$

Handwritten green annotations show the calculation of the 2x2 matrix result. The first row of the result is calculated as  $1 \times 1 + 3 \times 0 + 2 \times 5 = 11$  and  $1 \times 3 + 3 \times 1 + 2 \times 2 = 10$ . The second row is calculated as  $4 \times 1 + 0 \times 0 + 1 \times 5 = 9$  and  $4 \times 3 + 0 \times 1 + 1 \times 2 = 14$ . Arrows point from the intermediate 1x1 and 1x2 results to the corresponding elements in the final 2x2 matrix.

## Details:

$$\underline{A} \times \underline{B} = \underline{C}$$

$m \times n$  matrix  
( $m$  rows,  
 $n$  columns)

$n \times o$  matrix  
( $n$  rows,  
 $o$  columns)

$m \times o$   
matrix

The  $i^{th}$  column of the matrix  $C$  is obtained by multiplying  $A$  with the  $i^{th}$  column of  $B$ . (for  $i = 1, 2, \dots, o$ )



# Example

$$\overset{2 \times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}} \overset{2 \times 2}{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}} =$$

$$\overset{2 \times 2}{\begin{bmatrix} 9 & 7 \\ 15 & 12 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \times 0 + 3 \times 3 \\ 2 \times 0 + 5 \times 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \times 1 + 3 \times 2 \\ 2 \times 1 + 5 \times 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

House sizes:

$$\begin{Bmatrix} \frac{2104}{1416} \\ \frac{1534}{852} \end{Bmatrix}$$

Matrix

$$\begin{bmatrix} 1 & \frac{2104}{1416} \\ 1 & \frac{1534}{852} \\ 1 & \frac{1534}{852} \\ 1 & \frac{1534}{852} \end{bmatrix} \times$$

Matrix

$$\begin{bmatrix} -40 \\ 0.25 \end{bmatrix} \begin{bmatrix} 200 \\ 0.1 \end{bmatrix} \begin{bmatrix} -150 \\ 0.4 \end{bmatrix} =$$

$$\begin{bmatrix} 486 \\ 314 \\ 344 \\ 173 \end{bmatrix} \begin{bmatrix} 410 \\ 342 \\ 353 \\ 285 \end{bmatrix} \begin{bmatrix} 692 \\ 416 \\ 464 \\ 191 \end{bmatrix}$$

Prediction  
of first  
 $h_\theta$

Predictions  
of 2nd  
 $h_\theta$

Have 3 competing hypotheses:

1.  $h_\theta(x) = -40 + 0.25x$

2.  $h_\theta(x) = 200 + 0.1x$

3.  $h_\theta(x) = -150 + 0.4x$



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---

## Matrix multiplication properties

$$3 \times 5 = 5 \times 3$$


"Commutative"

Let  $A$  and  $B$  be matrices. Then in general,  
 $A \times B \neq B \times A$ . (not commutative.)

E.g.


$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$


$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$


$A \times B$   
 $m \times n \quad n \times m$

$A \times B$  is  $m \times m$

$B \times A$  is  $n \times n$



$$\underline{3 \times 5 \times 2}$$

$$3 \times 10 = 30 = 15 \times 2$$

$$3 \times (5 \times 2) = (3 \times 5) \times 2$$

"Associative"

$$\begin{array}{l} A \times (B \times C) \leftarrow \\ \underline{(A \times B)} \times C \leftarrow \end{array}$$

$$A \times B \times C.$$

Let  $D = B \times C$ . Compute  $A \times D$ .

Let  $E = A \times B$ . Compute  $E \times C$ .

$A \times (B \times C)$   
 $(A \times B) \times C$   
 Some  
 answer.

1 is identity

$$1 \times z = z \times 1 = z$$

for any  $z$

# Identity Matrix

Denoted  $I$  (or  $I_{n \times n}$ ).

Examples of identity matrices:

$[1]$   
 $1 \times 1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$3 \times 3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$4 \times 4$

Informally:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

For any matrix  $A$ ,

$$A \cdot I = I \cdot A = A$$

$m \times n$     $n \times n$     $m \times m$     $m \times n$     $m \times n$

$I_{n \times n}$

Note:

$AB \neq BA$  in general

$$AI = IA \checkmark$$



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---

Inverse and  
transpose

$$\underline{1 = \text{"identity"}}$$

$$3 \underbrace{(3^{-1})}_{\frac{1}{3}} = 1$$

$$12 \times \underbrace{(12^{-1})}_{\frac{1}{12}} = 1$$

$$0 \underbrace{(0^{-1})}_{\text{undefined}}$$

Not all numbers have an inverse.

**Matrix inverse:**  $\swarrow$  square matrix  
(#rows = #columns)  $A^{-1}$

If  $A$  is an  $m \times m$  matrix, and if it has an inverse,

$$\rightarrow \underline{A(A^{-1})} = \underline{A^{-1}A} = \underline{I}.$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \swarrow$$

e.g.  $\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 0.4 & -0.1 \\ -0.05 & 0.075 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^{-1}A} = I_{2 \times 2}$

Matrices that don't have an inverse are "singular" or "degenerate"



# Matrix Transpose

Example:

$$\underline{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 9 \end{bmatrix}_{2 \times 3}$$
$$\underline{B} = \underline{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 9 \end{bmatrix}_{3 \times 2}$$

Let  $A$  be an  $m \times n$  matrix, and let  $B = A^T$ .

Then  $B$  is an  $n \times m$  matrix, and

$$\underline{B}_{ij} = \underline{A}_{ji}.$$

$$B_{12} = A_{21} = 2$$

$$B_{32} = 9$$

$$A_{23} = 9.$$