# Effect of temperature on the equilibrium of a white dwarf in general relativity

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## I. INTRODUCTION

## II. DISCUSSION

In this work we consider *spacetime* as a smooth manifold $\mathscr{M}$ of dimension $d = 4$ and metric tensor $\boldsymbol{g}$ with covariant components $g_{\alpha\beta}$. We will denote with an arrow over a bold face letter the corresponding 4 vector and with just the bold face symbol the corresponding 3 dimensional vector. Greek indexes run over the $d$ spacetime dimensions. Roman indexes run only over the $d-1$ space dimensions. We use Einstein summation convention of tacitly assuming a sum over repeated indexes. We will use geometrized units $c = G = 1$ throughout.

In Ref. [1] we determined how the Chandrasekhar argument for the limiting mass of a white dwarf at zero temperature could be modified to take into account the effects of a finite non zero temperature.

In Ref. [2] we showed that in general relativity hydrodynamics a *perfect fluid* with a stress energy tensor, given by $T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}$ in its isotropic frame, one finds in a local Lorentz frame

$$\frac{d\rho}{d\tau} + (\rho + p)\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}} = 0, \tag{2.1}$$

$$(\rho + p)\vec{\boldsymbol{a}} = -\vec{\boldsymbol{\nabla}}p - \vec{\boldsymbol{u}}\frac{dp}{d\tau}, \tag{2.2}$$

where $\rho$ is the mass energy density, $p$ is the pressure, $\tau$ is the fluid proper time, $\vec{\boldsymbol{u}} = (\gamma, \gamma\boldsymbol{v})$ is the fluid 4 velocity, where $u^0 = dt/d\tau = \gamma = (1 - v^2)^{-1/2}$ is Lorentz factor, and $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{u}}/d\tau$ is the 4 acceleration. The first Eq. (2.1) is the *continuity equation* and the second (2.2) is the *Euler equation*.

In §6.10 of the book [3] the effects of general relativity on the Newtonian Chandrasekhar argument are determined. They show that for white dwarfs it is often enough to approximate the general Euler equation (2.2) with its non relativistic and Newtonian limits. This does not appreciably alter the picture that includes general relativity effects on the star *equilibrium*. They find that general relativity affects only the *stability* of a white dwarf (see their (6.10.28) equation). Here we will show that albeit very small the general relativity effect on dense white dwarfs equilibrium

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is still appreciable. Moreover we will discuss the effect of taking care of a finite non zero temperature which may in general be considered a function of the distance $r$ from the star center.

In their book Shapiro and Teukolsky [3] treat the uniform zero temperature case. They start [4] from the non relativistic version of (2.1) and (2.2),

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}(\rho\boldsymbol{v})=0, \tag{2.3}$$

$$\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=-\frac{1}{\rho}\boldsymbol{\nabla}p, \tag{2.4}$$

which are obtained from (2.1) and (2.2) in the non relativistic limit $v\ll 1$, $\gamma\approx 1$, and $p\ll\rho$ (since the thermal energy is much smaller than the rest mass of the fluid). And in the left hand side of Euler equation they assume the Newtonian result

$$\frac{d\boldsymbol{v}}{dt}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=\frac{m}{r^2}\hat{\boldsymbol{r}}, \tag{2.5}$$

where $m$ is the mass of the star inside a radius $r$. They suggest that working within these approximation to the full general relativity picture is enough to extract the main general relativity effects on the stability of the star (see their (5.10.28) equation). The equilibrium properties of the white dwarf being unaltered from the Chandrasekhar analysis (see their (6.10.26) equation).

The equivalence principle allows to rewrite the spatial component of Euler Eq. (2.2) as follows

$$\rho\frac{D\boldsymbol{v}}{d\tau}=-\boldsymbol{\nabla}p-\frac{D(\boldsymbol{v}p)}{d\tau}, \tag{2.6}$$

where $D\ldots/d\tau$ is a covariant derivative with $Dv^i/d\tau=-\Gamma^i{}_{\alpha\beta}u^\alpha u^\beta$ and $D(v^ip)/d\tau=u^\beta\nabla_\beta(v^ip)=-p\Gamma^i{}_{\alpha\beta}u^\alpha u^\beta+v^idp/d\tau$. Here $\Gamma$ are the Christoffel symbols and the geodesic equation for the free fall of the particles of the fluid has been used. An important result of Newtonian gravitation is that at any point outside a spherical mass distribution, the gravitational field depends only on the mass interior to that point. Moreover, even if the mass interior is moving spherically symmetrically, the field outside is constant in time. This result is also true in general relativity, where it is known as *Birkhoff theorem*: the only vacuum, spherically symmetric gravitational field is static. It is the Schwarzschild metric solution of Einstein field equations for which the only non zero radial Christoffel symbols are $\Gamma^r{}_{00}=\Gamma^r{}_{rr}=f'(r)/2$, $\Gamma^r{}_{\theta\theta}=rf(r)$, and $\Gamma^r{}_{\phi\phi}=-rf(r)\sin^2\theta$, where $f(r)=1-2m/r$. Using this solution and recalling that the star is static, so that $dp/d\tau=0$, one reaches the *Tolman-Oppenheimer-Volkoff* (TOV) equation [5, 6]

$$\frac{dp}{dr}=-\frac{\rho m}{r^2}\left(1+\frac{p}{\rho}\right)\left(1+\frac{4\pi r^3p}{m}\right)\left(1-\frac{2m}{r}\right)^{-1}, \tag{2.7}$$

where the mass of the star must satisfy

$$\frac{dm}{dr}=4\pi r^2\rho\left(1-\frac{2m}{r}\right)^{-1/2}, \tag{2.8}$$

with boundary condition $m(r=0)=0$. The Newtonian limit is recovered by taking $p\ll\rho$ and $m\ll r$. A star made of a perfect fluid must also satisfy an adiabatic equation of state

$$p=K\rho^\Gamma, \tag{2.9}$$

known as a *polytrope* of polytropic index $1/(\Gamma-1)$. As shown in Ref. [1] $\Gamma=d\ln p/d\ln\rho\to 4/3$ at high electron mass density, which corresponds to the *extremely relativistic regime* $v\gg 1$, i.e. when we can use a dispersion relation $\epsilon(k)=\hbar ck$ for the electrons. Assuming that the white dwarf is made up of a completely ionized plasma then we may write $\rho=\mu_em_un$ with $n$ the electrons number density, $m_u$ the atomic mass unit, and $\mu_e=A/Z$ with $Z$ the atomic number and $A$ the mass number, so that for example for a $^4$He or a $^{12}$C white dwarf $\mu_e=2$ and for a $^{56}$Fe white dwarf $\mu_e=56/26\approx 2.134$. Then $K$ in the polytrope Eq. (2.9) is only a function of temperature

$$K=\frac{\pi^{2/3}\hbar}{g^{1/3}(\mu_em_u)^{4/3}}\frac{f_4(z)}{f_3^{4/3}(z)}, \tag{2.10}$$

where $g=2$ is the electrons spin degeneracy and $f_\mu(z)=-\sum_{\nu=1}^\infty(-z)^\nu/\nu^\mu$ for an ideal Fermi-Dirac gas of extremely relativistic electrons in thermal equilibrium at a temperature $T=1/k_B\beta$ with $k_B$ Boltzmann constant so that

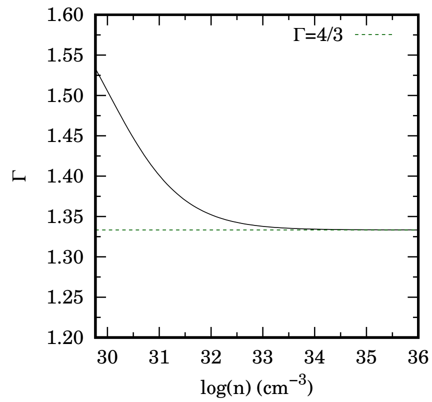


FIG. 1. We show $\Gamma = d\ln p/d\ln n$ for the equation of state of the extremely relativistic electron gas as a function of density at zero temperature $z \to \infty$. $g = 2$ and $n$ is in cm$^{-3}$.

$z = \exp(\beta\mu)$ is the activity of the gas with $\mu$ the chemical potential of the electrons, as shown in §61 of Ref. [7]. Note that at $T \to 0$, $z \to \infty$ and in Eq. (2.10) we find $\lim_{z\to\infty} f_4(z)/f_3^{4/3}(z) = 3^{1/3}/2^{5/3}$. In Fig. 1 we show the polytrope exponent $\Gamma = d\ln p/d\ln n$ for the equation of state of the extremely relativistic electron gas as a function of density at zero temperature. This should be compared with Fig. 1 of Ref. [1].

In order to solve the Euler equation of TOV (2.7) one chooses a central mass density $\rho(r = 0) = \rho_c$ and pressure $p(r = 0) = p_c$ and integrates it with Eqs. (2.8) and (2.9) till $r = R$ where $p(r = R) = 0$ and $m(r = R) = M$, with $R, M$ respectively the radius and mass of the star.

We integrated numerically TOV equation and its Newtonian limit (N) in order to understand when the general relativity effects become important.

At First we chose a high enough central electron number density $n_c = 10^{36}\mathrm{cm}^{-3}$ with $\rho_c = \mu_e m_u n_c$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\Gamma = 4/3$. Moreover, following Chandrasekhar, we chose the case of degenerate electrons at $T \to 0$, i.e. $z \to \infty$ in Eq. 2.10. At this high central number density the electron gas adiabatic equation of state is, to a good approximation, a polytrope in the extreme relativistic $\Gamma \to 4/3$ limit. Our results are shown in Fig. 2. From the figure we see that the effects of general relativity are indeed non negligible. In the Newtonian case (N) we recover the Chandrasekhar result of a limiting mass independent from the central density, whereas the TOV equation, unlike the Lane-Emden equation, still has a limiting mass but it depends on the central density. For the existence of the limiting mass in both the TOV and its N limit it is crucial to choose a polytrope equation of state with index less than 5, i.e. $\Gamma > 6/5$. Nonetheless we see from our figure that in order to be able to observe some general relativity effects we had to have a central mass density $\rho_c \approx 3 \times 10^{12}\mathrm{g/cm}^3$ which is one order of magnitude above the neutron drip density of $4 \times 10^{11}\mathrm{g/cm}^3$.

We therefore chose a central density $\rho_c = 10^{11}\mathrm{g/cm}^3$, below neutron drip, with a central electron number density $n_c = \rho_c/(\mu_e m_u) \approx 10^{34}\mathrm{cm}^{-3}$ for which we have a central Fermi temperature of $T_f = E_f/k_B = (\hbar^2/2m_e k_B)(6\pi^2 n_c/g)^{2/3} \approx 10^{12}\mathrm{K}$, where $m_e$ is the mass of the electron. From Fig. 1 of Ref. [1] we see that in this conditions we are well within the extremely relativistic $\Gamma \to 4/3$ regime. Choosing then again $z \to \infty$ in Eq. (2.10) and a central pressure $p_c = K\rho_c^{4/3}$ we observe the results of Fig. 3 which still shows a non negligible effect of general relativity on the white dwarf equilibrium.

As discussed in Ref. [1], at finite non zero temperature the adiabatic equation of state of the ideal Fermi gas of electrons will enter the extremely relativistic regime at high electron number density. In Fig. 1 of that reference it was shown for example that at a temperature of $T = 20000\mathrm{K}$ the adiabatic equation of state becomes a polytrope with $\Gamma \approx 4/3$ already for $n \gtrsim 10^{25}\mathrm{cm}^{-3}$. In order to take properly into account the effect of temperature on the star equilibrium we would need to introduce a function $T = T(r)$ for the temperature at a radius $r$ within the star. The assumption of Chandrasekhar was to consider negligible the effect of temperature since the average temperatures of a white dwarf $100000\mathrm{K} - 3000\mathrm{K}$ are much smaller than the Fermi temperature so that the Fermi-Dirac distribution function can be considered as the one of a degenerate gas. The electron gas will be in its polytrope extremely relativistic regime in the inner star shells but it will gradually exit this regime in the outer shells where the number density approaches zero, and with it the Fermi temperature, at the star surface.

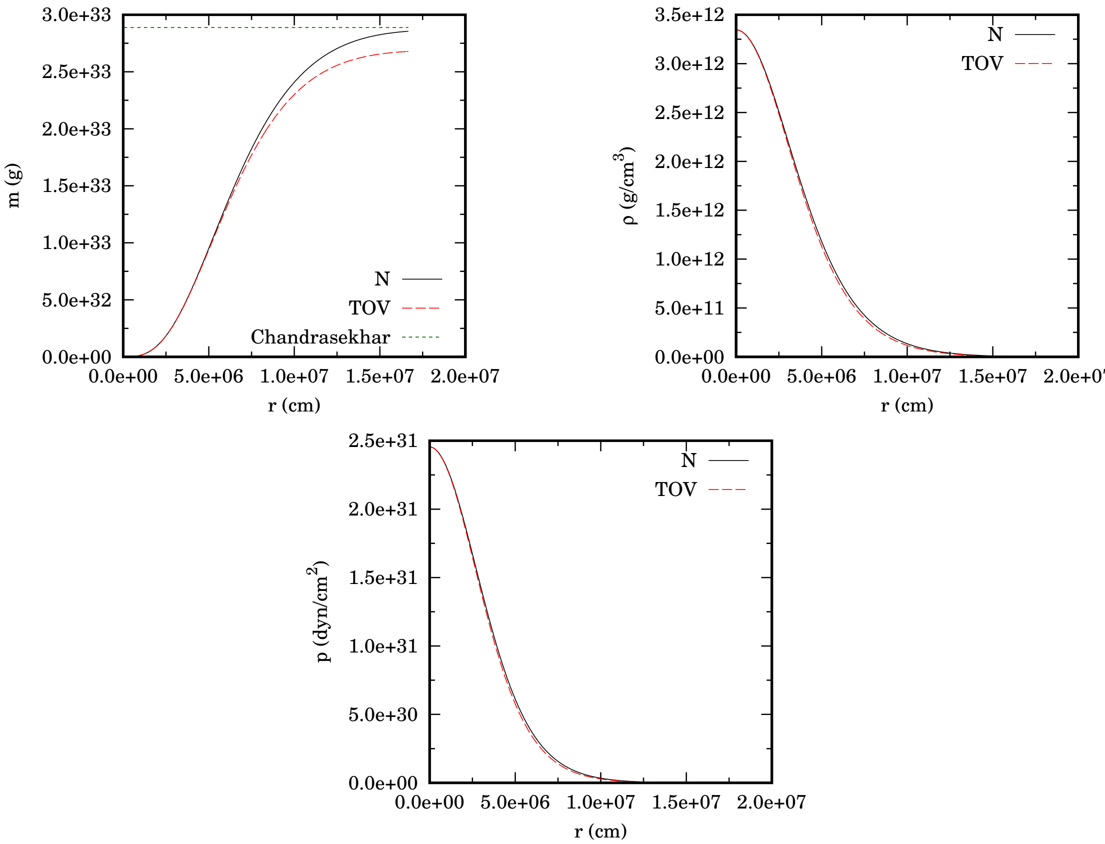


FIG. 2. We compare the numerical integration of the TOV equilibrium equation (TOV) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central electron number density $n_c = 10^{36}\mathrm{cm}^{-3}$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\rho_c = \mu_e m_u n_c$ at zero temperature $z \to \infty$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. In the first panel we also show the Chandrasekhar limiting value for the star mass. As one can see from the second panel the central mass density is an order of magnitude higher than the density for neutron drip. So the white dwarf would very likely turn into a neutron star starting from in its core.

## III. CONCLUSIONS

## AUTHOR DECLARATIONS

### Conflicts of interest

None declared.

### Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

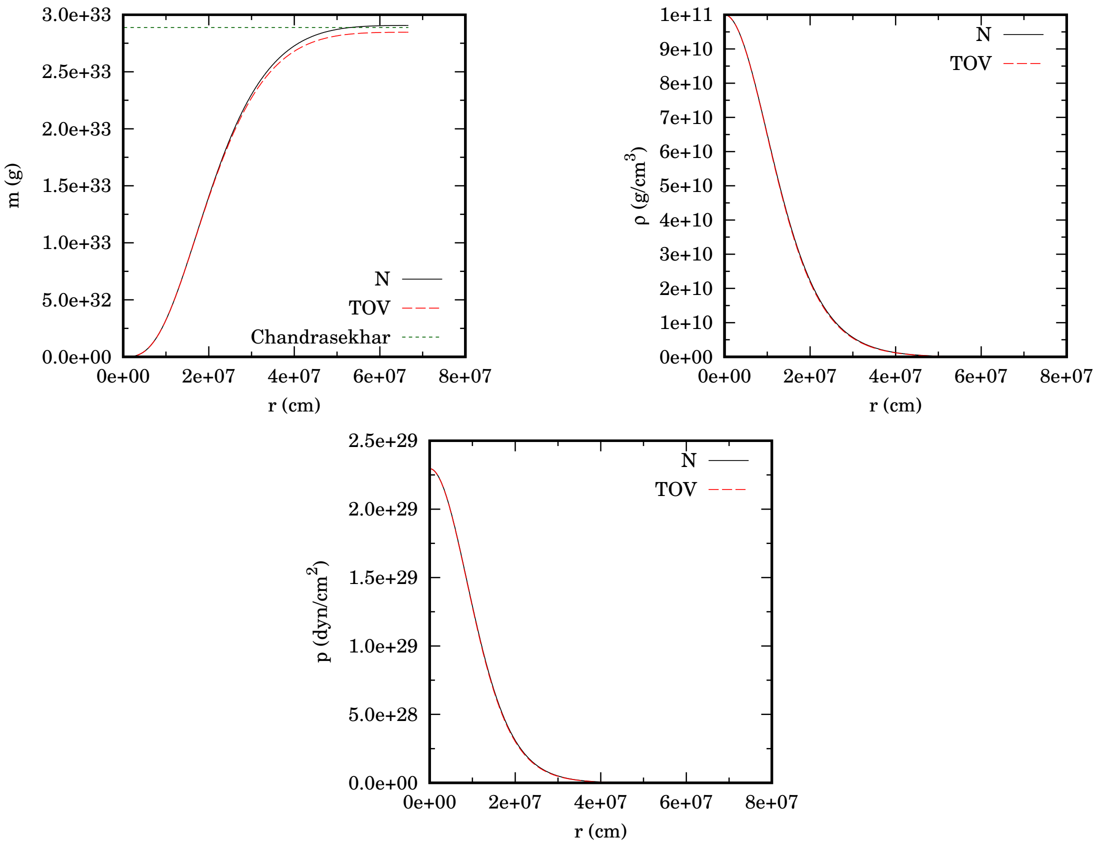


FIG. 3. We compare the numerical integration of the TOV equilibrium equation (TOV) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central mass density $\rho_c = 10^{11}\mathrm{g/cm}^3$, below neutron drip, and a central pressure $p_c = K\rho_c^{4/3}$ at $z \to \infty$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. In the first panel we also show the Chandrasekhar limiting value for the star mass. As we can see from this first panel the effect of general relativity is still not negligible.

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