Homework # 9 Solutions

set of units used: MKSA

-Problem 1- A rectangular waveguide has a $1cm \times 2cm$ cross section.

(a) What are the two minimum frequencies ?

(b) What are the phase and group velocities of the single propagating mode just below the higher of the two minimum frequencies ?

SOLUTION

Consider a **wavguide** (hollow cylindrical pipe) along the $\hat{\mathbf{z}}$ axis. For a given frequency ω, an electromagnetic wave can propagate inside it only for particular values of the wavevector, namely

$$k_\lambda = \sqrt{\mu_o \varepsilon_o}\sqrt{\omega^2 - \omega_\lambda^2} \quad , \tag{1}$$

with the *cut-off frequencies* $\omega_\lambda = \gamma_\lambda / \sqrt{\mu_o}\varepsilon_o$, where γ_λ for $\lambda = 1, 2, 3, \ldots$ represents the spectrum of eigenvalues of the following eigenvalues problem

$$(\nabla^2 - \frac{\partial^2}{\partial z^2})\psi = \gamma\psi \quad , \tag{2}$$

with the following boundary conditions

$$\begin{cases} \psi|_s = 0 & \text{for TM waves} \quad , \\ \hat{\mathbf{n}} \cdot \nabla\psi|_s = 0 & \text{for TE waves} \quad , \end{cases} \tag{3}$$

where $\psi \exp(\pm ikz)$ represents $E_z(H_z)$ for TM (TE) waves, s is the waveguide surface and $\hat{\mathbf{n}}$ its normal. In corrispondence to γ_λ exists an orthogonal set of solutions ψ_λ called the *modes* of the waveguide.

If $\omega < \omega_\lambda$ the wavenumber runs imaginary and instead of a traveling wave we have exponentially attenuated fields. For this reason ω_λ is called the *cut-off frequency* for the mode in question.

(a) In a waveguide with a rectangular shape with height a and width b (suppose $a > b$) solving eq. (2) one find $\lambda = \{m, n\}$ and

$$\gamma^2_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \quad , \tag{4}$$

$$\omega_{m,n} = \frac{\pi}{\sqrt{\mu_o \varepsilon_o}} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad , \tag{5}$$

For example the $\mathrm{TE}_{m,n}$ mode will be given by

$$\psi_{m,n} = H_o \cos \frac{n\pi}{b} x \ \cos \frac{m\pi}{a} y \quad , \tag{6}$$

$$\mathbf{k} = k_{m,n} \hat{\mathbf{z}} + \frac{n\pi}{b} \hat{\mathbf{x}} + \frac{m\pi}{a} \hat{\mathbf{y}} \quad . \tag{7}$$

The two minimum cut-off frequencies are $\omega_{1,0} = \pi c/a$ and $\omega_{0,1} = \pi c/b$. In the problem $a = 2b = 2cm$ then we find

$$\omega_{0,1} = 2\omega_{1,0} \simeq \left(\frac{\pi \ 3 \times 10^{10}}{1} \right) S^{-1} = 9.4 \times 10^{10} S^{-1} \quad . \tag{8}$$

(b) The mode $\{m, n\}$ travels along the waveguide with a *phase velocity* given by

$$v = \frac{\omega}{k_{m,n}} = \frac{c}{\sqrt{1 - (\omega_{m,n}/\omega)^2}} > c \quad . \tag{9}$$

However the energy carried by the wave travels at the *group velocity*

$$v_g = \frac{1}{dk_{m,n}/d\omega} = c\sqrt{1 - (\omega_{m,n}/\omega)^2} \quad . \tag{10}$$

At ω just below $\omega_{0,1}$ the mode $\{0, 1\}$ cannot propagate and the only mode which travels in the wave guide will be the $\{1, 0\}$. [1] then we get for the phase velocity (9)

$$v = \frac{\omega}{k_{m,n}} = \frac{c}{\sqrt{1 - (\omega_{1,0}/\omega_{0,1})^2}} = \frac{c}{\sqrt{1 - b/a}} \tag{11}$$

$$= \frac{2}{\sqrt{3}} c \simeq 3.5 \times 10^{10} cm/S \quad , \tag{12}$$

[1] for $\omega_{1,0} < \omega < \omega_{0,1}$ the $\{1, 0\}$ is the only mode that can propagate into the waveguide.

and for the group velocity (10)

$$v_g = \frac{1}{dk_{m,n}/d\omega} = c\sqrt{1-(\omega_{1,0}/\omega_{0,1})^2} = c\sqrt{1-b/a} \tag{13}$$

$$= \frac{\sqrt{3}}{2}c \simeq 2.6 \times 10^{10} cm/S \quad . \tag{14}$$

$\boxed{\text{-Problem 2-}}$ Calculate the capacitance and inductance per unit length of a coaxial cable. To calculate C, put $\pm\lambda$ on the two conductors and use $C = \lambda/V$. Similarly, calculate L, let $\pm I$ flow and use $L = \phi/I$. The flux is through the surface shown in figure 1. Show that the speed of a wave, $1/\sqrt{LC} = c$. What ratio b/a is needed for the impedence to equal 75Ω (a typical value) ?

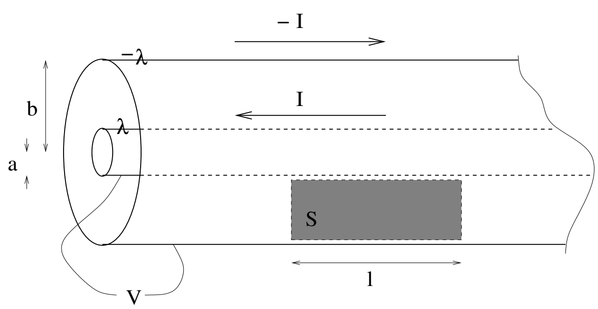

Figure 1: Coaxial cable

SOLUTION

Consider the coaxial cable as a cylindrical capacitor. Immagine that a linear charge density $+\lambda$ is uniformily distributed along the inner cylinder of radius a (and $-\lambda$ along the outer cylinder of radius b). The electric field between the two cylinder is radial and his modulus can be found by Gauss's law

$$E(2\pi r)l = \frac{\lambda l}{\varepsilon_o} \quad \Rightarrow \quad \mathbf{E} = \frac{\lambda}{2\pi\varepsilon_o}\frac{\hat{\mathbf{r}}}{r} \quad , \tag{1}$$

where we have taken as Gauss' s surface a cylinder of height l, radius r such that $a < r < b$ and coaxial with the cable.

The potential difference between the two conducting cylinders is

$$\Delta V = \int \mathbf{E}\, d\mathbf{l} = \frac{\lambda}{2\pi\varepsilon_o}\int_a^b \frac{1}{r}\, dr = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right) \quad . \tag{2}$$

The capacitance per unit length can then be determined as

$$C = \frac{\lambda}{\Delta V} = \frac{2\pi\varepsilon}{\ln(b/a)} \quad . \tag{3}$$

Consider the inner cylinder of the coaxial cable as a wire traveled by a current $+I$ (with a current $-I$ flowing in the outer cylindrical conductor). Then by Ampere' s law we can determine the magnetic field inside the coaxial cable as follows

$$B(2\pi r) = \mu_o I \quad \Rightarrow \quad \mathbf{B} = \frac{\mu_o I}{2\pi r}\hat{\phi} \quad , \tag{4}$$

where we took as Amperian loop a circle of radius r such that $a < r < b$ and coaxial with the cable (note that the current flowing in the outer conductor is not concatenated with the loop).

The flux of the magnetic field through the rectangular shaded region S shown in figure 1 is then

$$\Phi_B = \int \mathbf{B}\ d\mathbf{a} = \int_a^b \left(\frac{\mu_o I}{2\pi r}\right) l\ dr = \frac{\mu_o I l}{2\pi} \ln\left(\frac{a}{b}\right) \quad . \tag{5}$$

The inductance per unit lenght of the coaxial cable is then found by

$$L = \frac{\Phi_B}{lI} = \frac{\mu_o}{2\pi} \ln\left(\frac{a}{b}\right) \quad . \tag{6}$$

Using eqs. (3) and (6) we can find the speed of an electromagnetic wave traveling into the cable as

$$v = \frac{1}{\sqrt{LC}} = (\frac{\mu_o}{2\pi} \ln(a/b) 2\pi\varepsilon_o \frac{1}{\ln(a/b)})^{-1/2} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c \quad . \tag{7}$$

Using eqs. (3) and (6) we can find the impedence of the cable as

$$Z = \sqrt{\frac{L}{C}} = \frac{\ln(a/b)}{2\pi}\sqrt{\frac{\mu_o}{\varepsilon_o}} \quad . \tag{8}$$

Knowing that $Z = 75\Omega$ we can finally determine the ratio b/a from equation (8)

$$\frac{b}{a} = \exp(2\pi Z c \varepsilon_o) \simeq 3.5 \quad . \tag{9}$$

-Problem 3- Consider a two conductor transmission line (TL),

(a) What is the relation between the current and the voltage in it for a wave traveling to the right ($v_x > 0$) ?

(b) What is the relation for a wave traveling to the left ?

(c) Suppose the TL is terminated by a resistor, R, which connects the two conductors. What is the relation between the current and the voltage at the termination point ?

(d) Use the results of parts (a), (b), and (c) to calculate the reflection coefficient when a TL of impedence Z is terminated by a resistance $R_o \neq Z$

(e) Suppose the termination is a capacitor C_o. Redo parts (c) and (d) for this case.

SOLUTION

(a,b) Consider a two conductor transmission line (TL) with a capacitance per unit length C (F/S) and an inductance per unit length L (H/S), lying along the $\hat{\mathbf{x}}$ direction. Call $V(x,t)$ and $Q(x,t)$ respectively the potential drop between the two conductor and the charge, at point x and time t. These quantities must satisfy the following relationships

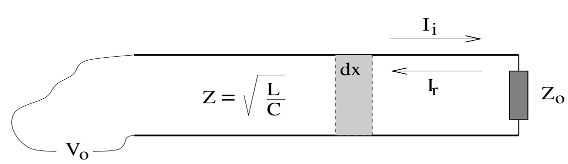

Figure 2: Schematic representation of a two conductor trasmission line (TL) excited by a potential $V_o(t)$ and terminated on an impedence Z_o. the impedence of the TL is $Z = \sqrt{L/C}$ where L and C are respectively the inductance and the capacitance per unit length.

$$V(x,t) = \frac{1}{C}\frac{\partial Q(x,t)}{\partial x} \quad , \tag{1}$$

and if $I(x,t) = \partial Q(x,t)/\partial t$

$$\frac{\partial I(x,t)}{\partial t} = \frac{1}{L}\frac{\partial V(x,t)}{\partial x} \quad , \tag{2}$$

these two eqs. (1) and (2) can be justified observing that an element of length dx of the TL (shaded part in figure 2) have a capacitance $(C\ dx)$ and an inductance $(L\ dx)$.

Deriving eq. (2) two times with respect to time one gets

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{C}\frac{\partial^2 I}{\partial x \partial t} \quad , \tag{3}$$

deriving eq. (2) once with respect to x one gets

$$\frac{\partial^2 I}{\partial t \partial x} = \frac{1}{L}\frac{\partial^2 V}{\partial x^2} \quad . \tag{4}$$

Combining eqs (3) and (4) one gets the following wave equation for $V(x,t)$

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V}{\partial x^2} \quad . \tag{5}$$

In an analogous way one find that $I(x,t)$ satisfies the same equation

$$\frac{\partial^2 I}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 I}{\partial x^2} \quad . \tag{6}$$

The solutions to eqs. (5) and (6) are a "potential wave" and a "current wave" traveling into the TL at a speed $v_x^{\pm} = \pm 1/\sqrt{LC}$. If we excite the TL with a potential $V_o(t) = V_o \exp(i\omega t)$ at $x = 0$ (see figure 2) the potential and current waves generated can be written as

$$V_{\pm}(x,t) \;=\; V_o \exp[i(k_{\pm}x - \omega t)] \quad , \tag{7}$$
$$I_{\pm}(x,t) \;=\; I_o \exp[i(k_{\pm}x - \omega t)] \quad , \tag{8}$$

where $k_{\pm} = \omega/v_x^{\pm}$, V_+, I_+ represents waves traveling to the right and V_+, I_+ represents waves traveling to the left. From eq. (1) one gets

$$-i\omega V_{\pm}(x,t) = \frac{1}{C} i k_{\pm} I_{\pm}(x,t) \quad , \tag{9}$$

from which follows

$$\frac{V_\pm(x,t)}{I_\pm(x,t)} = -\frac{1}{v_x^\pm} = \mp\sqrt{\frac{L}{C}} = \mp Z \quad , \tag{10}$$

where Z is the impedence of the TL.

(c) At the termination point, say $x = l$, one has

$$v(l,t) = Z_o I(l,t) \quad , \tag{11}$$

where Z_o is the impedence on which the TL is terminated (see figure 2).

(d,e) Call V_i, I_i the potential and current waves incideing on Z_o and V_r, I_r the potential and current waves reflected from Z_o. From equation (10) we get

$$V_i = -ZI_i \quad , \tag{12}$$

$$V_r = ZI_r \quad , \tag{13}$$

and from equation (11)

$$V_r + V_i = Z_o(I_r + I_i) \quad . \tag{14}$$

Using eqs. (12) and (13) in eq. (14) one gets

$$Z(I_r - I_i) = Z_o(I_r + I_i) \quad \Rightarrow \tag{15}$$

$$I_r(Z - Z_o) = I_i(Z + Z_o) \quad \Rightarrow \tag{16}$$

$$\frac{V_r}{V_i} = -\frac{I_r}{I_i} = \frac{Z_o - Z}{Z_o + Z} \quad . \tag{17}$$

When the TL is terminated by a resistance R then $Z_o = R$ is real. When the TL is terminated by a capacitor C then $Z_o = 1/(i\omega C)$ is imaginary [1].

[1] In general $C = Q(t)/V(t)$ then $V(t) = (1/C)\int I(t)dt$. If $I(t) \sim \exp(iwt)$ then $Z_o = V(t)/I(t) = 1/(i\omega C)$.