Statistical Gravity and entropy of spacetime

Riccardo Fantoni*
Università di Trieste, Dipartimento di Fisica, strada Costiera 11, 34151 Grignano (Trieste), Italy
(Dated: February 11, 2025)

We discuss the foundations of the statistical gravity theory we proposed in a recent publication [Riccardo Fantoni, Quantum Reports, **6**, 706 (2024)].

Keywords: General Relativity; Einstein-Hilbert action; Statistical Physics; Path Integral; Entropy; Ergodicity

INTRODUCTION

We propose a new horizontal theory which brings together statistical physics and general relativity.

We give statistical physics [1] foundation basis in order to determine the consistency of our theory, already put forward in Ref. [2], for a statistical gravity description.

The key logical point is the connection between thermodynamics and statistical physics made possible by the statistical concept of entropy and its derivative with respect to energy. This defines the temperature. In our statistical gravity theory the energy content is due to matter and electromagnetic fields and the entropy is a count of the quantum states of a quasi closed subregion of spacetime which can be considered closed for a period of time that is long relative to its relaxation time, with energy in a certain interval. Feynman will describe this in chapter 1 of his set of lectures [3] saying "If a system is very weakly coupled to a heat bath at a given 'temperature,' if the coupling is indefinite or not known precisely, if the coupling has been on for a long time, and if all the 'fast' things have happened and all the 'slow' things not, the system is said to be in *thermal equilibrium*".

Our Eq. (2) has long been studied by John Klauder [4] and the form chosen here is just representative and in substitution of the much more rigorous one offered by that author. Other alternative points of view are also present today [5].

This theory based on the mathematical properties of a Wick rotation would open a new sight of the statistical properties of spacetime as a physical entity.

Our theory can be considered a *first step* towards a more sophisticated and dignified description of spacetime.

GENTROPY

Let us define a *subregion* of a macroscopic spacetime region as a part of spacetime that is very small respect to the whole Universe yet macroscopic.

The subregion is not closed. It interacts with the other parts of the Universe. Due to the large number of degrees of freedom of the other parts, the state of the subregion varies in a complex and intricate way.

In order to formulate a statistical theory of gravity we need to determine the statistical distribution of a subregion of a macroscopic spacetime region.

Since different subregions "interact" weekly among themselves then:

1. It is possible to consider them as *statistically independent*, i.e. the state of a subregion does not affect the probability of the states of another subregion. If $\hat{\rho}_{12}$ is the density matrix of the subregion composed by the subregion 1 and by the subregion 2 then
$$\hat{\rho}_{12} = \hat{\rho}_1\hat{\rho}_2, \tag{1}$$
where $\hat{\rho}_i$ is the density matrix of the subregion i.

2. It is possible to consider a subregion as closed for a sufficiently small time interval. The time evolution of the density matrix of the subregion in such an interval of time is
$$\frac{\partial}{\partial t}\hat{\rho}_i = \frac{i}{\hbar}[\hat{\rho}_i, \hat{H}_i], \tag{2}$$
where $\hat{H}_i$ is the Hamiltonian of the quasi closed subregion i.

3. After a sufficiently long period of time the spacetime reaches the state of statistical equilibrium in which the density matrices of the subregions must be stationary. We must then have
$$[\prod_i \hat{\rho}_i, \hat{H}] = 0, \tag{3}$$
where $\hat{H}$ is the Hamiltonian of the closed macroscopic spacetime. This condition is certainly satisfied if
$$[\hat{\rho}_i, \hat{H}] = 0, \tag{4}$$
for all i.

We then find that the logarithm of the density matrix of a subregion is an additive integral of motion of the spacetime.

This is certainly satisfied if
$$\ln\hat{\rho}_i = \alpha_i + \beta_i\hat{H}_i. \tag{5}$$

In the time interval in which the subregion can be considered closed it is possible to diagonalize simultaneously $\hat{\rho}_i$ and $\hat{H}_i$. We then find

$$\ln \rho_n^{(i)} = \alpha_i + \beta_i E_n^{(i)}, \tag{6}$$

where the probabilities $\rho_n^{(i)} = w(E_n^{(i)})$ represent the distribution function in statistical gravity.

If we consider the closed spacetime as composed of many subregions and we neglect the "interactions" among them, each state of the entire spacetime can be described specifying the state of the various subregions. Then the number $d\Gamma$ of quantum states of the closed spacetime corresponding to an infinitesimal interval of his energy must be the product

$$d\Gamma = \prod_i d\Gamma_i, \tag{7}$$

of the numbers $d\Gamma_i$ of the quantum states of the various subregions.

We can then formulate the expression for the *microcanonical distribution function* writing

$$dw \propto \delta(E - E_0) \prod_i d\Gamma_i \tag{8}$$

for the probability to find the closed spacetime in any of the states $d\Gamma$.

Let us consider a spacetime that is closed for a period of time that is long relative to its relaxation time. This implies that the spacetime is in complete statistical equilibrium.

Let us divide the spacetime region in a large number of macroscopic parts and consider one of these. Let $\rho_n = w(E_n)$ be the distribution function for such part. In order to obtain the probability $W(E)dE$ that the subregion has an energy between E and $E + dE$ we must multiply $w(E)$ by the number of quantum states with energies in this interval. Let us call $\Gamma(E)$ the number of quantum states with energies less or equal to E. Then the required number of quantum states with energy between E and $E + dE$ is

$$\frac{d\Gamma(E)}{dE} dE, \tag{9}$$

and the energy probability distribution is

$$W(E) = \frac{d\Gamma(E)}{dE} w(E), \tag{10}$$

with the normalization condition

$$\int W(E) dE = 1. \tag{11}$$

The function $W(E)$ has a well defined maximum in $E = \bar{E}$. We can define the "width" ΔE of the curve $W = W(E)$ through the relation

$$W(\bar{E}) \Delta E = 1. \tag{12}$$

or

$$w(\bar{E}) \Delta\Gamma = 1, \tag{13}$$

where

$$\Delta\Gamma = \frac{d\Gamma(\bar{E})}{dE} \Delta E, \tag{14}$$

is the number of quantum states corresponding to the energy interval ΔE at $\bar{E}$. This is also called the *statistical weight* of the macroscopic state of the subregion, and its logarithm

$$S = \log \Delta\Gamma, \tag{15}$$

is the *entropy* of the subregion. The entropy cannot be negative.

We can also write the definition of entropy in another form, expressing it directly in terms of the distribution function. In fact we can rewrite Eq. (6) as

$$\log w(\bar{E}) = \alpha + \beta \bar{E}, \tag{16}$$

so that

$$\begin{aligned} S = \log \Delta\Gamma &= -\log w(\bar{E}) = -\langle \log w(E_n) \rangle \\ &= -\sum_n \rho_n \log \rho_n = -\mathrm{tr}(\hat{\rho} \log \hat{\rho}), \end{aligned} \tag{17}$$

where 'tr' denotes the trace.

Let us now consider again the closed region and let us suppose that $\Delta\Gamma_1, \Delta\Gamma_2, \ldots$ are the statistical weights of the various subregions, then the statistical weight of the entire region can be written as

$$\Delta\Gamma = \prod_i \Delta\Gamma_i, \tag{18}$$

and

$$S = \sum_i S_i, \tag{19}$$

the entropy is additive.

Let us consider again the microcanonical distribution function for a closed region,

$$\begin{aligned} dw &\propto \delta(E - E_0) \prod_i \frac{d\Gamma_i}{dE_i} dE_i \\ &\propto \delta(E - E_0) e^S \prod_i \frac{dE_i}{\Delta E_i} \\ &\propto \delta(E - E_0) e^S \prod_i dE_i, \end{aligned} \tag{20}$$

where $S = \sum_i S_i(E_i)$ and $E = \sum_i E_i$. Now we know that the most probable values of the energies E_i are the mean values $\bar{E}_i$. This means that the function $S(E_1, E_2, \ldots)$ must have its maximum when $E_i = \bar{E}_i$ for all i. But the

$\bar{E}_i$ are the values of the energies of the subregions that correpond to the complete statistical equilibrium of the region. We then reach the important conclusion that the entropy of a closed region in a state of complete statistical equilibrium has its maximum value (for a given energy of the region E_0).

Let us now consider again the problem to find the distribution function of the subregion, i.e. of any macroscopic region being a small part of a large closed region. We then apply the microcanonical distribution function to the entire region. We will call the "medium" what remains of the spacetime region once the small macroscopic part has been removed. The microcanonical distribution can be written as

$$dw \propto \delta(E + E' - E_0) d\Gamma d\Gamma', \tag{21}$$

where $E, d\Gamma$ and $E', d\Gamma'$ refer to the subregion and to the "medium" respectively, and E_0 is the energy of the closed region that must equal the sum $E + E'$ of the energies of the subregion and of the medium.

We are looking for the probability w_n of one state of the region so that the subregion is in some well defined quantum state (with energy E_n), i.e. a well defined microscopic state. Let us then take $d\Gamma = 1$, set $E = E_n$ and integrate respect to Γ'

$$\begin{aligned}\rho_n &\propto \int \delta(E_n + E' - E_0) d\Gamma' \\ &\propto \int \frac{e^{S'}}{\Delta E'} \delta(E_n + E' - E_0) dE' \\ &\propto \left(\frac{e^{S'}}{\Delta E'} \right)_{E' = E_0 - E_n} .\end{aligned} \tag{22}$$

We use now the fact that, since the subregion is small, its energy E_n will be small respect to E_0

$$S'(E_0 - E_n) \approx S'(E_0) - E_n \frac{dS'(E_0)}{dE_0}. \tag{23}$$

But we know that the derivative of the entropy with respect to the energy is $\beta = 1/k_B T$ where k_B is Boltzmann constant and T is the temperature of the closed spacetime region (that coincides with that of the subregion with which it is in equilibrium). So we finally reach the following result

$$\rho_n \propto e^{-\beta E_n}. \tag{24}$$

which is the *canonical distribution function*.

METRIC REPRESENTATION OF THE DENSITY MATRIX AND PATH INTEGRAL

We then reach to the following expression for the density matrix of spacetime

$$\hat{\rho} \propto e^{-\beta \hat{H}}, \tag{25}$$

where $\hat{H}$ is the spacetime Hamiltonian. In the nonquantum high temperature regime we can let $\beta \to \beta/M$ with M a large integer. Then we can use for the high temperature density matrix the usual classical limit [2, 6–8]

$$\rho(g_{\mu\nu}, g'_{\mu\nu}; \tau) \propto \exp\left[-\tau \int_\Omega \left(\frac{1}{2\kappa} R + \mathcal{L}_F \right) \sqrt{^3 g}\, d^3\mathbf{x} \right] \delta[g_{\mu\nu}(x) - g'_{\mu\nu}(x)], \tag{26}$$

where $g_{\mu\nu}(x)$ is the spacetime metric tensor, $x \equiv (ct, \mathbf{x}) = (x^0, x^1, x^2, x^3)$ is an event in space($\mathbf{x}$)time(t), $\tau = \beta/M$ is a small complex time step, R is the Ricci scalar of the spacetime subregion, $\kappa = 8\pi G c^{-4}$ is Einstein's gravitational constant (G is the gravitational constant and c is the speed of light in vacuum), Ω is the volume of space of the subregion whose spacetime is curved by the matter and electromagnetic fields due to the term $\mathcal{L}_F$, and $^3 g$ is the determinant of the spatial block of the metric tensor. In Eq. (26) the δ is a functional delta [9].

Using then Trotter formula [10] we reach to the path integral expression described in Ref. [2] for the finite temperature case, where the metric tensor path wanders in the spacetime subregion made of the complex time interval $[0, \hbar\beta/c[$ with periodic boundary conditions and the spatial region Ω. The spatial region can be compact in the absence of black holes or not if any are present. In any case it can either include its outermost frontier or not but from a numerical point of view it is convenient to use periodic boundary conditions there in order to simulate a thermodynamic limit so that only the frontiers around eventual black holes matter. The metric tensor 10-dimensional space is an hypertorus with $g_{\mu\nu}(ct + \hbar\beta(\mathbf{x}), \mathbf{x}) = g_{\mu\nu}(ct, \mathbf{x})$ and $g_{\mu\nu}(ct, \mathbf{x} + \boldsymbol{\xi}) = g_{\mu\nu}(ct, \mathbf{x})$. If the periodicities along the imaginary time dimension and along the spatial dimensions are incommensurable, i.e. $\hbar\beta(\mathbf{x})/\xi^i$ cannot be written as rational numbers then the Einstein field equations will let the metric tensor explore its phase space in a quasi-periodic fashion, then one can use either a "molecular-" (or "hydro-") dynamic numerical simulation strategy since the imaginary time averages equal the ensemble averages thanks to ergodicity or a Monte Carlo numerical simulation strategy. Of course from a purely economic numerical point of view the strategy of choice in this case is the Path Integral Monte Carlo one which is born to deal with multidimensional systems.

CONCLUSIONS

We gave logical foundation to the statistical gravity horizontal theory we recently proposed [2, 6]. Our weakness in discussing Eq. (2) does not reflect a weakness in the current knowledge and studies around that equation but is just our lack of deep vertical awareness.

AUTHOR DECLARATIONS

Conflict of interest

The author has no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

* riccardo.fantoni@scuola.istruzione.it

[1] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Course of Theoretical Physics, Vol. 5 (Butterworth Heinemann, 1951) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.

[2] R. Fantoni, Statistical Gravity through Affine Quantization, Quantum Rep. **6**, 706 (2024).

[3] R. P. Feynman, *Statistical Mechanics: A Set of Lectures*, Frontiers in Physics, Vol. 36 (W. A. Benjamin, Inc., 1972) notes taken by R. Kikuchi and H. A. Feiveson, edited by Jacob Shaham.

[4] J. R. Klauder and R. Fantoni, The Magnificent Realm of Affine Quantization: valid results for particles, fields, and gravity, Axioms **12**, 911 (2023).

[5] A. Ashtekar, New variables for classical and quantum gravity, Phys. Rev. Lett. **57**, 2244 (1986).

[6] R. Fantoni, Statistical Gravity, ADM splitting, and AQ, (2024), `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=5098498`.

[7] L. S. Schulman, *Techniques and Applications of Path Integration* (John Wiley & Sons, Technion, Haifa, Israel, 1981).

[8] D. M. Ceperley, Path integrals in the theory of condensed helium, Rev. Mod. Phys. **67**, 279 (1995).

[9] I. M. Gelfand and S. V. Fomin, *Functional Analysis* (Prentice-Hall Inc., Englewoods Cliff, N. J., 1963).

[10] H. F. Trotter, On the Product of Semi-Groups of Operators, Proc. Am. Math. Soc. **10**, 545 (1959).