Homework # 12 Solutions

set of units used: MKSA

$\boxed{\text{-Problem 1-}}$ Unpolarized light is a superposition of equal amounts of each polarization. "Unpolarized" means the two polarizations don' t interfere. Show that if a beam of unpolarized $X-$rays (intensity I_o) is incident on a single electron, the intensity of the scattered radiation is

$$I_s = I_o \frac{(1+\cos^2\theta)}{2}\frac{R_e^2}{r^2} \quad , \tag{1}$$

where $R_e = e^2/(4\pi\varepsilon_o mc^2) = 2.81\times 10^{-15} m$ is the classical radius of the electron. (This is Reitz and Milford, problem 20-12, p. 544).

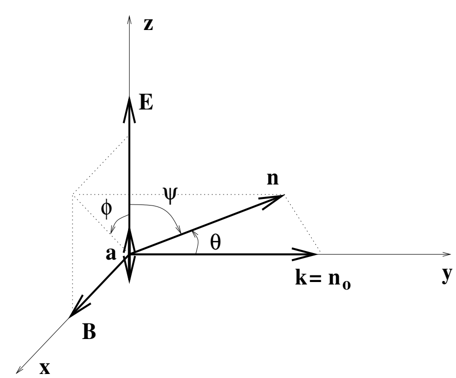

Figure 1: Propagation and polarization vectors for the incident and the diffused radiation.

SOLUTION

When a monocromatic electromagnetic plane wave scatters on a single electron of charge e and mass m, the electron accelerates and radiates. The radiation is emitted in directions different from the one of the incident wave, but assuming *non relativistic* motion of the electron it will have the same frequency of the incident radiation. The whole process can be depicted as a scattering process (the *Thompson scattering*). We know that the power per unit solid angle radiated by a charge in a non relativistic motion is given by

$$\frac{dP}{d\Omega} = \mathbf{S}_s \hat{\mathbf{n}} r^2 = \frac{1}{4\pi\varepsilon_o} \frac{e^2}{4\pi c^3} |\mathbf{a}|^2 \sin^2\psi \quad , \tag{2}$$

where $\mathbf{S}_s$ is the Poynting vector of the scattered radiation and ψ is the angle between the acceleration $\mathbf{a} = e^2\mathbf{E}/m$ and the vector which points towards the observation point from the retarded position of the charge ($\hat{n}$ in fig .1, the propagation direction of the scattered radiation).

It is usefull to express eq. (2) in terms of spherical coordinates with polar axis along the propagation direction of the incident radiation (see figure 1). So we want to express the angle ψ in terms of the polar angle θ and the azimuthal angle ϕ, namely

$$\cos\psi = \sin\theta\cos\phi \quad . \tag{3}$$

Assuming, moreover, that the dimensions of the charge motion during a cycle of oscillation of the electric field is small compared with the wavelength of the radiation, the mean value on time of $|\mathbf{a}|^2$ is $\frac{1}{2}\Re e(\mathbf{a}\mathbf{a}^*)$. and the power radiated per unit solid angle becomes

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{2}\varepsilon_o c|\mathbf{E}|^2 \left(\frac{e^2}{4\pi\varepsilon_o m c^2}\right)^2 (1 - \sin^2\theta\cos^2\phi) \tag{4}$$

$$= \frac{1}{2}\varepsilon_o c|\mathbf{E}|^2 R_e^2 (1 - \sin^2\theta\cos^2\phi) \quad , \tag{5}$$

where R_e is the *classical radius of the electron.* Equation (4) can easily be rewritten in terms of intensities. The intensity of the incident electromagnetic radiation (plane wave with a Poynting vector $\mathbf{S}_o$) is $I_o = \langle|\mathbf{S}_o|\rangle = \varepsilon_o c|\mathbf{E}|^2/2$. The intensity of the scattered radiation along the $\hat{n}$ direction can be written as $I_s = \langle \mathbf{S}_s \hat{\mathbf{n}} \rangle$. From the definition

$$\langle P \rangle = -\oint \langle \mathbf{S}_s \rangle \hat{\mathbf{n}} da \quad , \tag{6}$$

$$\langle dP \rangle = -\langle \mathbf{S}_s \rangle \hat{\mathbf{n}} da \quad , \tag{7}$$

observing that the elementary solid $d\Omega = da/r^2$ we get

$$\left\langle \frac{dP}{d\Omega} \right\rangle = r^2 \langle \mathbf{S}_s \rangle \mathbf{n} = r^2 I_s \quad . \tag{8}$$

Inserting eq (2) into eq. (8) and solving for I_s we get

$$I_s = I_o \frac{(1 - \sin^2\theta \cos^2\phi)}{2} \frac{R_e^2}{r^2} \quad . \tag{9}$$

If the incident beam of $X-$rays in unpolarized then eq. (9) must be averaged on the ϕ angle

$$I_s = I_o R_e^2 (1 - \frac{1}{2}\sin^2\theta) = I_o R_e^2 \frac{1 + \cos^2\theta}{2} \quad . \tag{10}$$

This is called the *Thompson formula* for the diffusion of radiation by a free charge. It is correct for the diffusion of $X-$rays from electrons and for $\gamma-$rays from protons.

-Problem 2- X-rays are attenuated in aluminum by compton scattering. Use the Thompson scattering cross section to calculate the absorption coefficient $\alpha = 2/\delta$ (the fctor of two comes from the fact that absorption is defined in terms of intensity, not amplitude). There are 6.06×10^{28} aluminum atoms per m^3. (This is Reitz and Milford, problem 20-13, p. 544).

SOLUTION

The *differential scattering cross section* in a Thompson scattering (the classical version of Compton scattering) is defined as

$$\frac{d\sigma}{d\Omega} = \frac{\text{radiated energy/unit time/unitsolid angle}}{\text{flux of incident energy (energy/unit area/unit time)}} \quad . \tag{1}$$

As we have learned from problem 2 the numerator is given by

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{2}\varepsilon_o c E_o R_e^2 \frac{1+\cos^2\theta}{2} \quad , \tag{2}$$

where $R_e = e^2/4\pi\varepsilon_o mc$ is the classical radius of the electron, E_o is the amplitude of the incident unpolarized radiation and θ is the scattering angle. The denumerator is given by

$$\langle \mathbf{S}_o \rangle = \frac{1}{2}\varepsilon_o c E_o \quad , \tag{3}$$

where $\mathbf{S}_o$ is the Poynting vector of the incident radiation. Inserting in eq. (1) results (2) and (3) we get

$$\frac{d\sigma}{d\Omega} = R_e^2 \frac{1+\cos^2\theta}{2} \quad . \tag{4}$$

Integrating over the solid angle we get the total cross section in the diffusion of $X-$rays from a free classical charge with non-relativistic motion, namely

$$\sigma = \frac{8\pi}{3} R_e^2 \quad . \tag{5}$$

This is the *Thompson scattering cross section* which for electron equals $0.665 \times 10^{-28} m^2$ and for protons is $\sim (2000)^2$ times smaller (since $m_p \sim 2000 m_e$).

Assume that the main contribution to the power loss of the $X-$ray beam going through the aluminum sample (see figure 2) is due to the Thompson

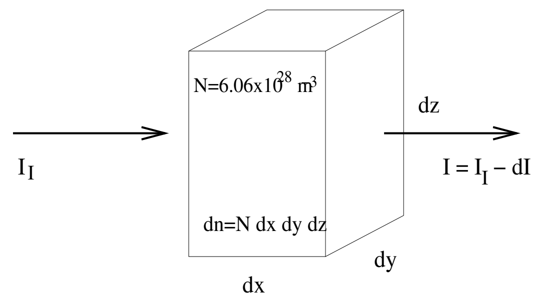

Figure 2: The dn scatterers removes energy from the incident beam I_I, reducing the intensity of a quantity dI.

scattering on the three valence electrons per aluminum atom (which can better be described as free charges). If I_I is the intensity of the incident beam, the intensity after traveling a thickness dz of material of volume $dv = dxdydz$ and containing a number of scatterers $dn = 3Ndv$ ($N = 6.06 \times 10^{28} m^{-3}$) is $I_I - dI$ with [1]

$$dI = \frac{dP}{dxdy} = \frac{dn(\sigma I)}{dxdy} = \sigma 3NIdz \quad , \tag{6}$$

where dP is the power subtracted to the beam by Thompson scattering on the dn scatterers. This means that

$$\frac{dI(z)}{dz} = -\sigma 3NI(z) \quad , \tag{7}$$

i.e. there is an attenuation length $z_o = 1/\alpha = 1/(\sigma 3N)$ and

$$I(z) = I_I \exp(-z/z_o) \quad . \tag{8}$$

So we get for the absorption coefficient the following result

$$\alpha = 3N\sigma = 8\pi N R_e^2 \sim 12 \ m^{-1} \quad . \tag{9}$$

[1] We have made two assumptions: (i.) the scatterers don' t interfere with each other (we add the intensities not the amplitudes) (ii.) the scatterers don' t hide behind one another (never an issue unless the positions are correlated).

$\boxed{\text{-Problem 3-}}$ Show that initially polarized radiation, when scattered by a collection of charged particles, becomes at least partially polarized, and that the maximum polarization occurs when the scattering angle is 90^o. (This is H&M, problem 10-1, p.370).

SOLUTION

As shown in the previous problems the differential scattering cross section for Thompson scattering by linearly polarized radiation on a free classical charge with non-relativistic motion is given by

$$\frac{d\sigma}{d\Omega} = \sigma_{tot}\frac{3\pi}{8}(1 - \sin^2\theta\cos^2\phi) \quad , \tag{1}$$

where as shown in figure 1 θ and ϕ are respectively the polar and azimuthal angle choosing as polar axis the direction of propagation of the incident light, and σ_{tot} is the Thompson scattering cross section.

The *polarization* of the diffused radiation is defined as

$$\Pi(\theta) = \frac{\dfrac{d\sigma_\perp}{d\Omega} - \dfrac{d\sigma_\parallel}{d\Omega}}{\dfrac{d\sigma_\perp}{d\Omega} + \dfrac{d\sigma_\parallel}{d\Omega}} \quad , \tag{2}$$

where the suffixes $\perp$ and $\parallel$ stand for polarization perpendicular ($\phi = \pi/2$) and parallel ($\phi = 0$) to the diffusion plane respectively (in figure 1 the plane determined by the versors $\hat{\mathbf{n}}$ along the direction of propagation of the diffused radiation and $\hat{\mathbf{n}}_o$ along the direction of propagation of the incident radiation). From eq. (1) we get

$$\frac{d\sigma_\perp}{d\Omega} = \sigma_{tot}\frac{3\pi}{8} \quad , \tag{3}$$

$$\frac{d\sigma_\parallel}{d\Omega} = \sigma_{tot}\frac{3\pi}{8}\cos^2\theta \quad . \tag{4}$$

Inserting into eq. (2) we get

$$\Pi(\theta) = \frac{\sin^2\theta}{1 + \cos^2\theta} \quad . \tag{5}$$

This result shows that in the Thompson scattering from unpolarized radiation the diffused radiation from a single scatterer is partially polarized for

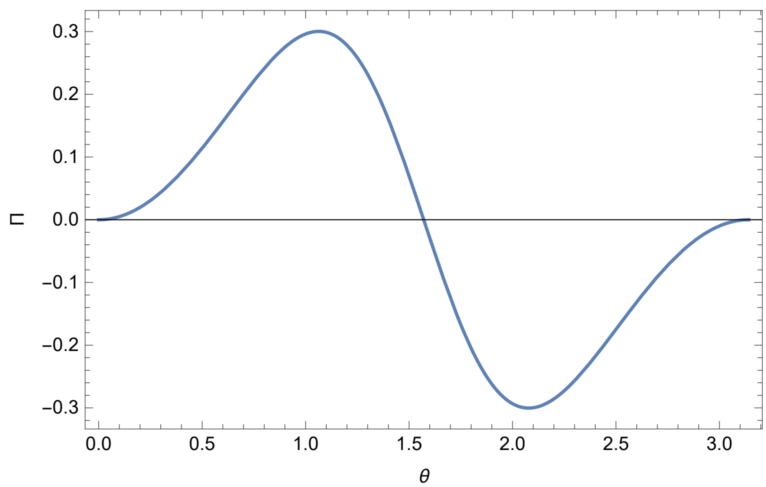

Figure 3: Graph of the polarization of the radiation diffused in Thompson scattering times $\cos(\theta)$ with θ the scattering angle.

scattering angles $\theta \neq 0, \pi/2, \pi$ is unpolarized for $\theta = 0, \pi$ and polarized normally to the scattering plane for $\theta = \pi/2$. In presence of N scatterers *casually* distributed the result for the polarization doesn' t change since the differential scattering cross section of the collection of particles becomes just N times the differential scattering cross section of a single particle [1].

$$\boxed{\text{happy Christmas \& happy new year : --)}}$$

[1]The same is not true when the scatterers are regularly distributed in space. In particular it can be shown that in this case if N is big the differential scattering cross section is zero for pratically all directions except the direction of incidence.