Effect of temperature on the equilibrium of a white dwarf in general relativity

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CONTENTS

I. INTRODUCTION

In this work we consider *spacetime* as a smooth manifold $\mathscr{M}$ of dimension d and metric tensor $\boldsymbol{g}$ with covariant components $g_{\alpha\beta}$. We will denote with an arrow over a bold face letter the corresponding 4 vector and with just the bold face symbol the corresponding 3 dimensional vector. Greek indexes run over the d spacetime dimensions. Roman indexes run only over the $d-1$ space dimensions. We use Einstein summation convention of tacitly assuming a sum over repeated indexes. We will assume the speed of light $c = G = 1$ throughout.

In Ref. [1] we determined how the Chandrasekhar argument for the limiting mass of a white dwarf at zero temperature could be modified to take into account the effects of a finite non zero temperature.

In Ref. [2] we showed that in general relativity hydrodynamics a *perfect fluid* with a stress energy tensor, given by $T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}$ in its isotropic frame, one finds in a local Lorentz frame

$$\frac{d\rho}{d\tau} + (\rho + p)\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}} = 0, \tag{1.1}$$

$$(\rho + p)\vec{\boldsymbol{a}} = -\vec{\boldsymbol{\nabla}}p - \vec{\boldsymbol{u}}\frac{dp}{d\tau}, \tag{1.2}$$

where ρ is the mass energy density, p is the pressure, τ is the fluid proper time, $\boldsymbol{u} = (\gamma, \gamma\boldsymbol{v})$ is the fluid 4 velocity, where $u^0 = dt/d\tau = \gamma = (1 - v^2)^{-1/2}$ is Lorentz factor, and $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{u}}/d\tau$ is the 4 acceleration. The first Eq. (1.1) is the *continuity equation* and the second (1.2) is the *Euler equation*.

In §6.10 of the book [3] the effects of general relativity on the Newtonian Chandrasekhar argument are determined. Here we want to carry out the same analysis but taking care of a finite non zero temperature. In particular in their book Shapiro and Teukolsky [3] treat the zero temperature case. They start [4] from the non relativistic version of (1.1) and (1.2),

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}(\rho\boldsymbol{v}) = 0, \tag{1.3}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p, \tag{1.4}$$

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which are obtained from (1.1) and (1.2) in the non relativistic limit $v \ll 1$, $\gamma \approx 1$, and $p \ll \rho$ (since the thermal energy is much smaller than the rest mass of the fluid). And in the left hand side of Euler equation they assume the Newtonian result

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \frac{m}{r^2}\hat{\boldsymbol{r}}, \tag{1.5}$$

where m is the mass of the star inside a radius r. They suggest that working within these approximation to the full general relativity picture is enough to extract the main general relativity effects on the *stability* of the star (see their (5.10.28) equation). The *equilibrium* properties of the white dwarf being unaltered from the Chandrasekhar analysis (see their (6.10.26) equation).

First of all it would be nice to show that in order to take care properly of general relativity effects on the white dwarf equilibrium and stability this limiting approximations are legitimate in the sense that the result of approximating the general Euler equation (1.2) with its non relativistic and Newtonian limits does not appreciably alter the picture that includes general relativity effects.

The equivalence principle allows to rewrite the spatial component of Euler Eq. (1.2) as follows

$$\rho\frac{D\boldsymbol{v}}{d\tau} = -\boldsymbol{\nabla}p - \frac{D(\boldsymbol{v}p)}{d\tau}, \tag{1.6}$$

where $D\ldots/d\tau$ is a covariant derivative with $Dv^i/d\tau = -\Gamma^i{}_{\alpha\beta}u^\alpha u^\beta$ and $D(v^ip)/d\tau = u^\beta\nabla_\beta(v^ip) = -p\Gamma^i{}_{\alpha\beta}u^\alpha u^\beta + v^i dp/d\tau$. Here Γ are the Christoffel symbols and the geodesic equation for the free fall of the particles of the fluid has been used. An important result of Newtonian gravitation is that at any point outside a spherical mass distribution, the gravitational field depends only on the mass interior to that point. Moreover, even if the mass interior is moving spherically symmetrically, the field outside is constant in time. This result is also true in general relativity, where it is known as *Birkhoff theorem*: the only vacuum, spherically symmetric gravitational field is static. It is the Schwarzschild metric solution of Einstein field equations for which the only non zero radial Christoffel symbols are $\Gamma^r{}_{00} = \Gamma^r{}_{rr} = f'(r)/2$, $\Gamma^r{}_{\theta\theta} = rf(r)$, and $\Gamma^r{}_{\phi\phi} = -rf(r)\sin^2\theta$, where $f(r) = 1-2m/r$. Using this solution and recalling that the star is static, so that $dp/d\tau = 0$, one reaches the *Tolman-Oppenheimer-Volkoff* (TOV) equation [5, 6]

$$\frac{dp}{dr} = -\frac{\rho m}{r^2}\left(1+\frac{p}{\rho}\right)\left(1+\frac{4\pi r^3 p}{m}\right)\left(1-\frac{2m}{r}\right)^{-1}, \tag{1.7}$$

where the mass of the star must satisfy

$$\frac{dm}{dr} = 4\pi r^2\rho\left(1-\frac{2m}{r}\right)^{-1/2}, \tag{1.8}$$

with boundary condition $m(r=0)=0$. The Newtonian limit is recovered by taking $p \ll \rho$ and $m \ll r$. A star made of a perfect fluid must also satisfy an adiabatic equation of state

$$p = K\rho^\Gamma, \tag{1.9}$$

known as a *polytrope* of polytropic index $n = 1/(\Gamma-1)$. As shown in Ref. [1] $\Gamma = d\ln p/d\ln\rho \to 4/3$ at high mass density, which corresponds to the extremely relativistic regime $v \gg 1$. Assuming that the white dwarf is made up of a completely ionized plasma then we may write $\rho = \mu_e m_u n$ with n the electrons number density, m_u the atomic mass unit, and $\mu_e = A/Z$ with Z the atomic number and A the mass number, so that for example for a ^{4}He or a ^{12}C white dwarf $\mu_e = 2$ and for a ^{56}Fe white dwarf $\mu_e = 56/26 \approx 2.134$. Then K in the polytrope Eq. (1.9) is only a function of temperature

$$K = \frac{\pi^{2/3}\hbar c}{g^{1/3}(\mu_e m_u)^{4/3}}\frac{f_4(z)}{f_3^{4/3}(z)}, \tag{1.10}$$

where $g=2$ is the electrons spin degeneracy and $f_\mu(z) = -\sum_{\nu=1}^\infty(-z)^\nu/\nu^\mu$ for an ideal Fermi-Dirac gas of electrons in thermal equilibrium at a temperature $T = 1/k_B\beta$ with k_B Boltzmann constant so that $z = \exp(\beta\mu)$ is the activity of the gas with μ the chemical potential of the electrons, as shown in §61 of Ref. [7].

In order to solve the Euler equation of TOV (1.7) one chooses a central mass density $\rho(r=0)=\rho_c$ and pressure $p(r=0)=p_c$ and integrates it with Eqs. (1.8) and (1.9) till $r=R$ where $p(r=R)=0$ and $m(r=R)=M$, with R, M respectively the radius and mass of the star.

We integrated numerically TOV equation and its Newtonian limit. For example choosing a central electron number density $n_c = 10^{36}cm^{-3}$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\rho_c = \mu_e m_u n_c$ and choosing the Chandrasekhar case of degenerate electrons at $T \to 0$, i.e. $z \to \infty$ we found the results of Fig. 1. From the figure we see that the effects of general relativity are non negligible. In the Newtonian case (N) we recover the Chandrasekhar result of a limiting mass independent from the central density, whereas the TOV equation, unlike the Lane-Emden equation does not have this property. Nonetheless we see that in order to be able to observe some general relativity effects we had to have a central mass density $\rho_c \approx 3 \times 10^{12} g/cm^3$ which is one order of magnitude above the neutron drip density of $4 \times 10^{11} g/cm^3$.

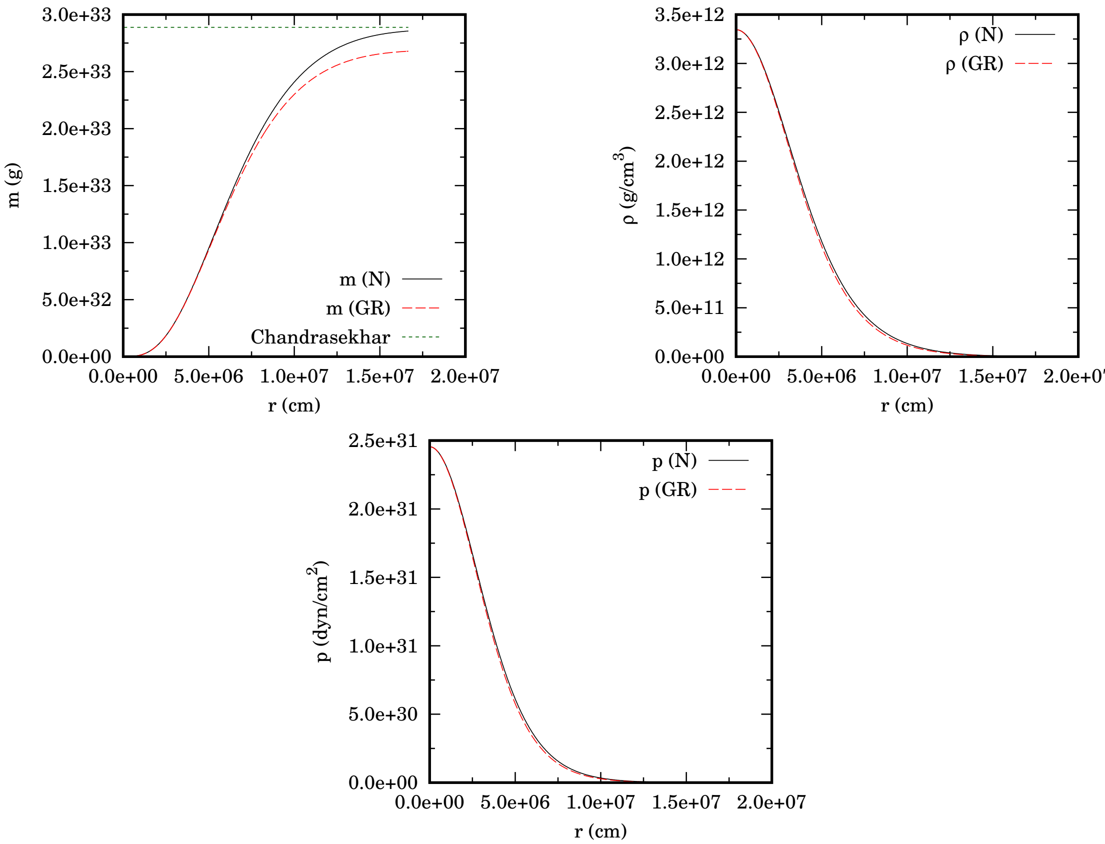

FIG. 1. We compare the numerical integration of the TOV equilibrium equation (GR) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central electron number density $n_c = 10^{36} cm^{-3}$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\rho_c = \mu_e m_u n_c$ at zero temperature $z \to \infty$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. In the first panel we also show the Chandrasekhar limiting value for the star mass. As one can see from the second panel the central mass density is an order of magnitude higher than the density for neutron drip. So the white dwarf would very likely turn into a neutron star starting from in its core.

Nonetheless choosing a central density $\rho_c = 10^{11} g/cm^3$, below neutron drip, with a central pressure $p_c = K\rho_c^{4/3}$ but a finite non zero temperature for example $z = 10^{-3}$ we observe the results of Fig. 2 which shows a relevant effect of general relativity on the white dwarf equilibrium. Of course in the Newtonian limit the Emden-Lane equation still predicts a limiting mass independent from the central mass density in the extreme relativistic $\Gamma = 4/3$ case, as shown in Ref. [1], but again in the TOV case the limiting mass depends on the central mass density.

II. CONCLUSIONS

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

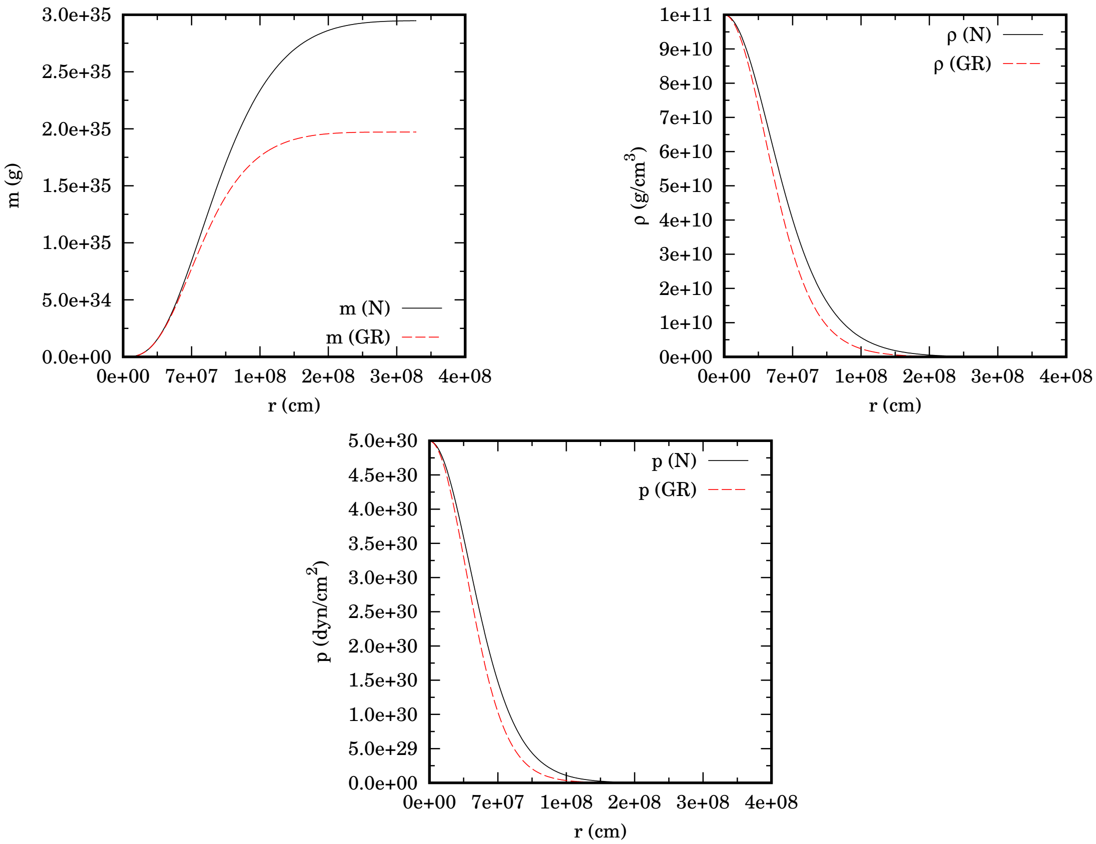

FIG. 2. We compare the numerical integration of the TOV equilibrium equation (GR) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central mass density $\rho_c = 10^{11} g/cm^3$, below neutron drip, and a central pressure $p_c = K\rho_c^{4/3}$ at $z = 10^{-3}$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. As we can see the effect of general relativity is not negligible like in the zero temperature case.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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