Homework # 5 Solutions

set of units used: MKSA

-Problem 1- For a transverse wave, $\mathbf{f} = \mathbf{f}_o \exp[i(kx - \omega t)]$, a 0^o phase difference between f_y and f_z gives plane polarization. A 90^o phase difference gives circular polarization. What do other phase differences give ? To be specific, calculate the path traced out by the tip of the real part of $\mathbf{f}$ if $\mathbf{f}_o = \hat{\mathbf{y}} + \hat{\mathbf{z}}(1+i)$. Feel free to use a computer, if you wish.

SOLUTION

$\mathbf{f}(x,t)$ is a *transverse wave* propagating in the positive $\hat{\mathbf{x}}$ direction with a wavevector k (wavelength $2\pi/k$), frequency ω (period $T = 2\pi/\omega$) and velocity $v = \omega/k$ [1]. Since the wave is transverse we must have $\mathbf{f}_o \cdot \hat{\mathbf{x}} = 0$. Then $\mathbf{f}_o$ will be of the form

$$\mathbf{f}_o = \hat{\mathbf{y}} f_{oy} + \hat{\mathbf{z}} f_{oz} \quad , \tag{1}$$

where f_{oy} and f_{oz} are in general complex numbers

$$\begin{cases} f_{oy} = a_y e^{ib_y} & , \\ f_{oz} = a_z e^{ib_z} & . \end{cases} \tag{2}$$

This means that we can have a *phase difference* $(b_y - b_z)$ between the two orthogonal components of the wave.

The polarization vector of the wave $\mathbf{f}(x,t)$ is by definition [2]

$$\begin{aligned} \mathbf{n}(t) &= Re\{\mathbf{f}(x=0,t)\} \\ &= Re\{(\hat{\mathbf{y}} a_y e^{ib_y} + \hat{\mathbf{z}} a_z e^{ib_z})[\cos(kx - \omega t) - i\sin(kx - \omega t)]\} \end{aligned} \tag{3}$$

where $Re(\mathbf{f})$ is the physical observed quantity (for example an electric or a magnetic field).

[1] Too see this just follow a point on the wave. That means: fix a point P at a certain time, for example $x_p = 0$ at time $t = 0$. At that point and that time $\mathbf{f} = \mathbf{f}_o$. Then determine the law of motion $x_p(t)$ of the point P such that $\mathbf{f}(x_p(t), t) = \mathbf{f}_o$. This will be $x_p(t) = (\omega/k)t + 2\pi n/k$ with $n = 0, \pm 1, \pm 2, \ldots$. The velocity of the wave is the velocity of point P, namely ω/k.

[2] We take $x = 0$ just for convenience.

If f_{oz} and f_{oy} have the same phase $b_y = b_z$ then

$$\begin{cases} n_y = a_y \cos(\omega t) \quad , \\ n_z = a_z \cos(\omega t) \quad . \end{cases} \tag{4}$$

This corresponds to a *linear* polarization $n_y = (a_y/a_z)n_z$.

If f_{oz} and f_{oy} have the same modulus $a_y = a_z = a$ and a phase difference $a_y - a_z = \pi/2$ then

$$\begin{cases} n_y = a \cos(\omega t) \quad , \\ n_z = \pm a \sin(\omega t) \quad . \end{cases} \tag{5}$$

This corresponds to a *circular* polarization $n_y^2 + n_z^2 = a^2$.

In all other cases the polarization is *elliptical*. In particular, in the case of the problem we have

$$\begin{cases} f_{oy} = 1 \quad , \\ f_{oz} = \sqrt{2} e^{i\frac{\pi}{4}} \quad . \end{cases} \tag{6}$$

Then the phase difference is $\pi/4$ and the polarization vector becomes (see figure 1)

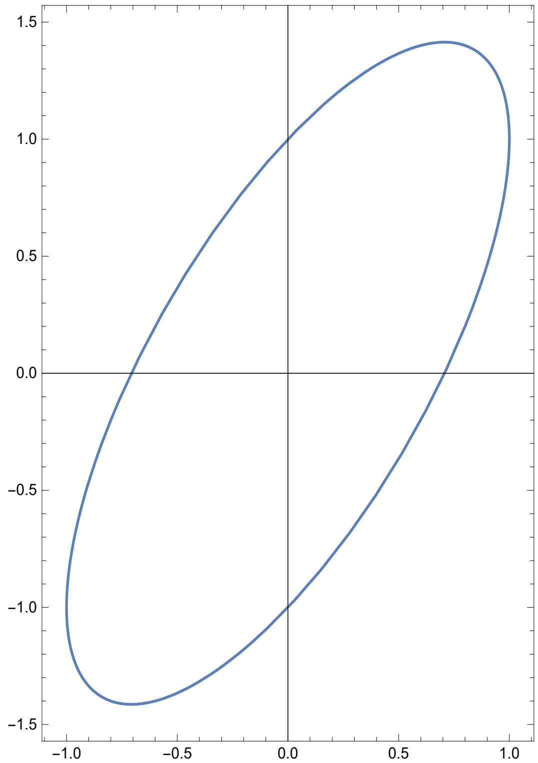

Figure 1: Elliptical polarization. The tip of the real part of $\mathbf{f}(0,t)$ trace the ellipses in a clockwise fashion

$$\left\{ \begin{array}{l} n_y = \cos(\omega t) \quad , \\ n_z = \cos(\omega t) + \sin(\omega t) \quad . \end{array} \right. \tag{7}$$

In figure 2 we show the surface traced out by the tip of the real part of $\mathbf{f}(x,t)$ as it evolves along the positive $\hat{\mathbf{x}}$ axis.

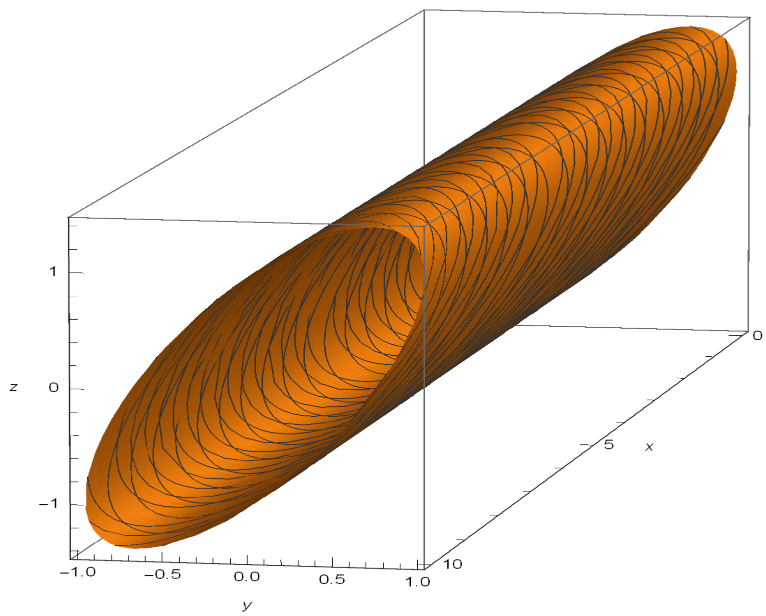

Figure 2: Evolution along the $\hat{\mathbf{x}}$ direction

$\boxed{\text{-Problem 2-}}$ A wave on a string has these values at $t = 0$ for $x \in [-2\pi/k, 2\pi/k]$

$$f(x,0) = a\sin(kx) \quad , \tag{1}$$

$$\dot{f}\,(x,0) = b\cos(kx) \quad . \tag{2}$$

Calculate the functions $g(x - vt)$ and $h(x + vt)$ which describe the left and right going waves. Sketch a picture, similar to Griffiths fig. 8.4, which shows the situation after some time t.

SOLUTION

We have to calculate the two functions $g(x - vt)$ and $h(x + vt)$ describeing the left and right going waves. The function describing the wave traveling on the string will then be $f(x,t) = g(x - vt) + h(x + vt)$. Given the initial conditions (see figure 3)

$$\begin{cases} f(x,0) = g(x) + h(x) = a\sin(kx) \quad , \\ \left.\dfrac{df(x,t)}{dt}\right|_{t=0} = -vg(x) + vh(x) = b\cos(kx) \quad , \end{cases} \tag{3}$$

we get

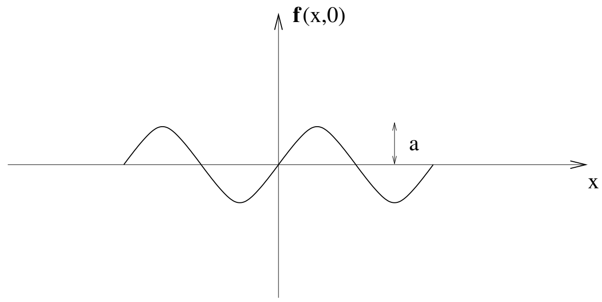

Figure 3: Initial condition.

$$\begin{cases} g(x) = \frac{1}{2}(f(x,0) - \frac{1}{v}\int_0^x \left.\frac{df(y,t)}{dt}\right|_{t=0} dy) \quad , \\ h(x) = \frac{1}{2}(f(x,0) + \frac{1}{v}\int_0^x \left.\frac{df(y,t)}{dt}\right|_{t=0} dy) \quad . \end{cases} \tag{4}$$

Then using the initial conditions (3) we have

$$\begin{cases} g(x) = \frac{1}{2}[a\sin(kx) - \frac{b}{kv}\cos(kx)] \quad , \\ h(x) = \frac{1}{2}[a\sin(kx) + \frac{b}{kv}\cos(kx)] \quad . \end{cases} \tag{5}$$

Calculating $g(x)$ in $(x - vt)$ and $f(x)$ in $(x + vt)$, we get finally

$$\begin{cases} g(x - vt) = \frac{1}{2}[a\sin(kx - \omega t) - \frac{b}{kv}\cos(kx - \omega t)] \quad , \\ h(x + vt) = \frac{1}{2}[a\sin(kx + \omega t) + \frac{b}{kv}\cos(kx + \omega t)] \quad , \end{cases} \tag{6}$$

where $\omega = vk$ is the frequency of the wave. The situation after some time t is sketched in figure 4.

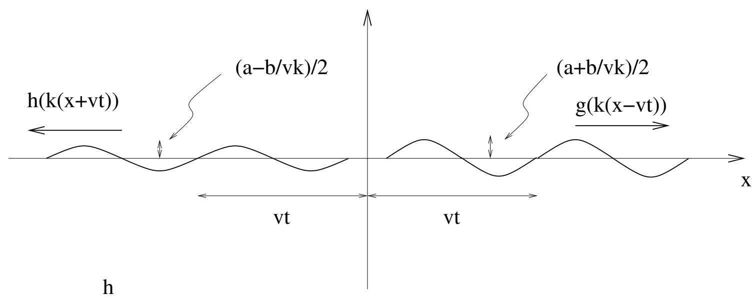

Figure 4: After some time t the two components $g(k(x-vt))$ and $h(k(x+vt))$ are travelling in opposite directions.

-Problem 3- The universe appears to be filled with millimiter wavelength radiation, the cosmic microwave background (CMB). Its energy density is about $4 \times 10^{-14} J/m^3$.

(a) Calculate the peak electric and magnetic field strenghts (in V/m and $Tesla$, respectively).

(b) At what distance from a $1KW$ radio trasmitter is the intensity the same of the CMB ?

SOLUTION

(a) The Cosmic Mirowave Background (CMB) fill uniformly the universe When we measure the average on time of the energy density of CMB, at a given point in space we get

$$\langle U(t)\rangle = \frac{1}{2}\langle \varepsilon_o E^2(x,t) + \frac{1}{\mu_o} B^2(x,t)\rangle \qquad (1)$$

$$= \varepsilon_o \langle E^2(x,t)\rangle = 4 \times 10^{-14} J/m^3 \quad , \qquad (2)$$

where $\mathbf{E}(x,t) = \mathbf{E}_o \cos(\mathbf{kr} - \omega t + \delta)$ and $\mathbf{B}(x,t) = \mathbf{B}_o \cos(\mathbf{kr} - \omega t + \phi)$ are the electric and magnetic field of the CMB radiation. In (1) we used the relation $B(x,t) = E(x,t)/c$. The symbol $\langle \ldots \rangle$ indicate the average over the space. Since the average of $\cos^2(x)$ is $1/2$ [1] then the peak value for $E(x,t)$ is

$$E_o = \sqrt{\frac{2\langle U\rangle}{\varepsilon}} \sim 9.5 \times 10^{-2} V/m \tag{3}$$

and the peak value for $B(x,t)$ is

$$B_o = \frac{E_o}{c} \sim 3 \times 10^{-10} Tesla \tag{4}$$

(b) The intensity of an electromagnetic wave is defined as the average on time of its Poynting vector. For the CMB radiation we get

$$I_{CMB} = \langle S\rangle = c\langle U\rangle = 1.2 \times 10^{-5} W/m^2 \quad . \tag{5}$$

The intensity of the signal from the radio trasmitter at a distance R from it can be written as

$$I_{trasmitter} = \frac{P}{4\pi R^2} \tag{6}$$

where we are assuming spherical symmetry and $P = 1KW$ is the power of the radio transmitter.

From the equality $I_{CMB} = I_{trasmitter}$ follows

$$R = \sqrt{\frac{P}{I_{CMB} 4\pi}} \sim 2.6 Km \tag{7}$$

[1] We have

$$\langle \cos^2(\mathbf{kr} - \omega t + \delta)\rangle = \frac{1}{T}\int_0^T \cos^2(\mathbf{kr} - \omega t + \delta)dt = \frac{1}{2\pi}\int_0^{2\pi} \cos^2(x)dx = \frac{1}{2}$$

where $\mathbf{kr} + \delta$ is a constant and $\omega = 2\pi/T$.