Differential Geometry

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(Dated: December 1, 2025)

Starting from the lecture notes for the graduate course PHYSICS 498 GR1/ASTRONOMY 469 GR1 held by by Stuart Shapiro on Spring 1997 at Loomis in Urbana/Champaign, IL, USA.

We give a brief description of differential Geometry and exterior calculus ending with their role in the formulation of Einstein theory of Gravitation [1, 2] and Maxwell theory of Electromagnetism [3] respectively. We conclude with some other possible extensions not supported by experiment [4].

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I. PREAMBLE ON DIFFERENTIAL TOPOLOGY

Consider a manifold $\mathscr{M}$ of dimension n.

A *curve* on $\mathscr{M}$

$$C(\lambda) \quad \lambda \text{ is an affine parameter.} \tag{1.1}$$

The tangent *vector* to C

$$\vec{\boldsymbol{u}} = \frac{dC(\lambda)}{d\lambda} = \partial_{\vec{\boldsymbol{u}}}. \tag{1.2}$$

A *basis of vectors* $\{\vec{\boldsymbol{e}}_\alpha\}$ with

$$\vec{\boldsymbol{e}}_\alpha = \partial_{\vec{\boldsymbol{e}}_\alpha} = \partial_\alpha. \tag{1.3}$$

A change of basis vectors is realized as follows

$$\vec{\boldsymbol{e}}_{\alpha'} = L^\alpha{}_{\alpha'}\vec{\boldsymbol{e}}_\alpha, \tag{1.4}$$

where the primed indexes are for the vectors in the new basis and a summation over the repeated index is tacitly assumed here and everywhere else in these manuscript.

In a *coordinate basis*

$$\vec{\boldsymbol{e}}_\alpha = \partial_{\vec{\boldsymbol{e}}_\alpha} = \partial_\alpha = \frac{\partial}{\partial x^\alpha}. \tag{1.5}$$

For a transformation of coordinates $x^{\alpha'} = x^{\alpha'}(x^\beta)$

$$\vec{\boldsymbol{e}}_\alpha = \frac{\partial}{\partial x^\alpha} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\vec{\boldsymbol{e}}_{\alpha'} = L^{\alpha'}{}_\alpha\vec{\boldsymbol{e}}_{\alpha'}. \tag{1.6}$$

with

$$L^{\alpha'}{}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha}, \quad L^\alpha{}_\beta L^\beta{}_\gamma = \delta^\alpha{}_\gamma, \tag{1.7}$$

where $||\delta^\alpha{}_\gamma|| = \mathrm{diag}(1,1,\ldots,1)$ is the identity matrix and $||L^\alpha{}_\beta|| = ||L^\beta{}_\alpha||^{-1}$.

A transformation of coordinates of a vector

$$\vec{\boldsymbol{u}} = u^{\alpha'}\vec{\boldsymbol{e}}_{\alpha'} = u^\alpha\vec{\boldsymbol{e}}_\alpha = u^\alpha L^{\alpha'}{}_\alpha\vec{\boldsymbol{e}}_{\alpha'}, \tag{1.8}$$

with

$$u^{\alpha'} = L^{\alpha'}{}_\alpha u^\alpha. \tag{1.9}$$

The *1-form* $\widetilde{\boldsymbol{\sigma}}$ in the dual space of the tangent vector space

$$\langle \widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{u}} \rangle = \text{a real number}, \tag{1.10}$$

where $\langle \cdot, \cdot \rangle$ is a bilinear two slots machine such that

$$\langle \widetilde{\boldsymbol{\omega}}^\beta, \vec{\boldsymbol{e}}_\alpha \rangle = \delta^\beta{}_\alpha \tag{1.11}$$

with $\{\widetilde{\boldsymbol{\omega}}^\beta\}$ a *basis of 1-forms*.
So if

$$\vec{\boldsymbol{u}} = u^\alpha \vec{\boldsymbol{e}}_\alpha, \tag{1.12}$$

$$\widetilde{\boldsymbol{\sigma}} = \sigma_\beta \widetilde{\boldsymbol{\omega}}^\beta, \tag{1.13}$$

we will have

$$u^\alpha = \langle \widetilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{u}} \rangle, \tag{1.14}$$

$$\sigma_\beta = \langle \widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta \rangle, \tag{1.15}$$

$$\sigma_\alpha u^\alpha = \langle \widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{u}} \rangle. \tag{1.16}$$

A change of basis 1-forms is realized as follows

$$\widetilde{\boldsymbol{\omega}}^{\alpha'} = L^{\alpha'}_\alpha \widetilde{\boldsymbol{\omega}}^\alpha, \tag{1.17}$$

and for the 1-form components

$$\sigma_{\alpha'} = L^\alpha{}_{\alpha'} \sigma_\alpha. \tag{1.18}$$

A particularly important 1-form is the *gradient*, $\widetilde{\boldsymbol{d}}f$, with f a scalar (a function), defined like so

$$\langle \widetilde{\boldsymbol{d}}f, \vec{\boldsymbol{u}} \rangle = \partial_{\vec{\boldsymbol{u}}} f = u^\alpha \partial_\alpha f = u^\alpha f_{,\alpha}, \tag{1.19}$$

where we use the comma to denote a partial derivative

$$f_{,\alpha} = \langle \widetilde{\boldsymbol{d}}f, \vec{\boldsymbol{e}}_\alpha \rangle = \partial_{\vec{\boldsymbol{e}}_\alpha} f = \partial_\alpha f. \tag{1.20}$$

So

$$\widetilde{\boldsymbol{d}}f = f_{,\alpha} \widetilde{\boldsymbol{\omega}}^\alpha. \tag{1.21}$$

In a coordinate basis

$$f_{,\alpha} = \frac{\partial f}{\partial x^\alpha}, \tag{1.22}$$

and $\{\widetilde{\boldsymbol{d}}x^\alpha\}$ is dual to $\{\partial/\partial x^\alpha\}$

$$\langle \widetilde{\boldsymbol{d}}x^\alpha, \partial/\partial x^\beta \rangle = \partial_\beta x^\alpha = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta. \tag{1.23}$$

A tensor $\mathbb{H}$ of rank $\binom{n}{m}$ is a linear machine with n input slots for 1-forms, $\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\lambda}}, \ldots, \widetilde{\boldsymbol{\beta}}$, and m input slots for vectors, $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \ldots, \vec{\boldsymbol{w}}$, which returns a real number

$$H(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\lambda}}, \ldots, \widetilde{\boldsymbol{\beta}}, \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \ldots, \vec{\boldsymbol{w}}) = \text{real number}, \tag{1.24}$$

Up to here we did not use a metric at all so we woked in *differential topology*. We will introduce a metric only later. For the time being let us take a detour on *exterior calculus*.

II. EXTERIOR CALCULUS IN BRIEF

(Chapter 4 in Ref. [1]) We may define a p-form as a completely antisymmetric tensor with all indexes "downstairs". More formally, in our n-dimensional manifold $\mathscr{M}$, we define it like so

$$\begin{aligned}\widetilde{\boldsymbol{\alpha}} &= \frac{1}{p!}\alpha_{\mu_1\mu_2\cdots\mu_p}\widetilde{\boldsymbol{\omega}}^{\mu_1}\wedge\widetilde{\boldsymbol{\omega}}^{\mu_2}\wedge\ldots\wedge\widetilde{\boldsymbol{\omega}}^{\mu_p}\\ &= \alpha_{|\mu_1\mu_2\cdots\mu_p|}\widetilde{\boldsymbol{\omega}}^{\mu_1}\wedge\widetilde{\boldsymbol{\omega}}^{\mu_2}\wedge\ldots\wedge\widetilde{\boldsymbol{\omega}}^{\mu_p},\end{aligned} \tag{2.1}$$

where the vertical bars around the indexes means that the summation extends only over $\mu_1 < \mu_2 < \ldots < \mu_p$ and $\wedge$ is the *wedge product* which is defined by its action on any two 1-forms, $\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}$ (or on any two vectors), as

$$\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}} = \widetilde{\boldsymbol{\alpha}}\otimes\widetilde{\boldsymbol{\beta}} - \widetilde{\boldsymbol{\beta}}\otimes\widetilde{\boldsymbol{\alpha}}, \tag{2.2}$$

where $\otimes$ denotes a *direct product*. So that $\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}} = -\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\alpha}}$ and $\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\alpha}} = 0$. Given any three 1-forms, $\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\boldsymbol{\gamma}}$ (or any three vectors), the wedge product has the following properties

$$(a\widetilde{\boldsymbol{\alpha}} + b\widetilde{\boldsymbol{\beta}})\wedge\widetilde{\boldsymbol{\gamma}} = a\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\gamma}} + b\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\gamma}}, \tag{2.3a}$$

$$(\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}})\wedge\widetilde{\boldsymbol{\gamma}} = \widetilde{\boldsymbol{\alpha}}\wedge(\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\gamma}}) = \widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\gamma}}, \tag{2.3b}$$

$$\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}} = \alpha_\mu\beta_\nu\widetilde{\boldsymbol{\omega}}^\mu\wedge\widetilde{\boldsymbol{\omega}}^\nu = \frac{1}{2}(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)\widetilde{\boldsymbol{\omega}}^\mu\wedge\widetilde{\boldsymbol{\omega}}^\nu. \tag{2.3c}$$

and if $\widetilde{\boldsymbol{\alpha}}$ is a p-form and $\widetilde{\boldsymbol{\beta}}$ is a q-form with p and q greater than 1, then $\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}} = (-1)^{pq}\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\alpha}}$.

Analogously for a p-vector we will have

$$\vec{\boldsymbol{a}} = \frac{1}{p!}a_{\mu_1\mu_2\cdots\mu_p}\vec{\boldsymbol{e}}^{\mu_1}\wedge\vec{\boldsymbol{e}}^{\mu_2}\wedge\ldots\wedge\vec{\boldsymbol{e}}^{\mu_p}. \tag{2.4}$$

A contraction of the p-form $\widetilde{\boldsymbol{\alpha}}$ of Eq. (2.1) and the p-vector $\vec{\boldsymbol{a}}$ of Eq. (2.4) is

$$\langle\widetilde{\boldsymbol{\alpha}}, \vec{\boldsymbol{a}}\rangle = \alpha_{|\mu_1\mu_2\cdots\mu_p|}a^{\mu_1\mu_2\cdots\mu_p}. \tag{2.5}$$

For example the jacobian determinant of a set of p functions $f^k(x^1, x^2, \ldots, x^n)$ with respect to p of their arguments is

$$\left\langle\widetilde{\boldsymbol{d}}f^1\wedge\widetilde{\boldsymbol{d}}f^2\wedge\ldots\wedge\widetilde{\boldsymbol{d}}f^p, \frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\ldots\wedge\frac{\partial}{\partial x^p}\right\rangle = \det\left|\left|\left(\frac{\partial f^\mu}{\partial x^\nu}\right)\right|\right| = \frac{\partial(f^1, f^2, \ldots, f^p)}{\partial(x^1, x^2, \ldots, x^p)}. \tag{2.6}$$

A. Exterior derivative

The exterior derivative is defined by induction:

i. if $\widetilde{\boldsymbol{\sigma}}$ is a p-form $\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\sigma}}$ is a $(p+1)$-form;

ii. a function f is a 0-form and $\widetilde{\boldsymbol{d}}f = f_{,\alpha}\widetilde{\boldsymbol{\omega}}^\alpha$;

iii. if $\widetilde{\boldsymbol{\alpha}}$ is a p-form and $\widetilde{\boldsymbol{\beta}}$ is a q-form then $\widetilde{\boldsymbol{d}}(\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}}) = \widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}} + (-1)^p\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\beta}}$.

It can easily be verified that $\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{d}} = \widetilde{\boldsymbol{d}}^2 = 0$.

B. Integration

We just require a "differentiable manifold" *calm* with or without a metric. In order to integrate a p-form in an n-dimensional manifold one may go through the following steps:

i. consider in a coordinate basis

$$\widetilde{\boldsymbol{\sigma}} = \sigma_{|\mu_1\mu_2\cdots\mu_p|}(x^1, x^2, \ldots, x^n)\widetilde{\boldsymbol{d}}x^{\mu_1}\wedge\widetilde{\boldsymbol{d}}x^{\mu_2}\wedge\ldots\wedge\widetilde{\boldsymbol{d}}x^{\mu_p}; \tag{2.7}$$

ii. substitute a parameterization of the p-dimensional surface of the form, $x^\mu(\lambda^1, \lambda^2, \ldots, \lambda^p)$, so that

$$\tilde{\boldsymbol{\sigma}} = \sigma(\lambda^1, \lambda^2, \ldots, \lambda^p)\tilde{\boldsymbol{d}}\lambda^1 \wedge \tilde{\boldsymbol{d}}\lambda^2 \wedge \ldots \wedge \tilde{\boldsymbol{d}}\lambda^p; \tag{2.8}$$

iii. integrate

$$\begin{aligned}\int \tilde{\boldsymbol{\sigma}} &= \int \left\langle \tilde{\boldsymbol{\sigma}}, \frac{\partial}{\partial\lambda^1} \wedge \frac{\partial}{\partial\lambda^2} \wedge \ldots \wedge \frac{\partial}{\partial\lambda^p} \right\rangle d\lambda^1 d\lambda^2 \ldots d\lambda^p, \\ &= \int \sigma(\lambda^1, \lambda^2, \ldots, \lambda^p) d\lambda^1 d\lambda^2 \ldots d\lambda^p, \end{aligned} \tag{2.9}$$

using the elementary definition of integration.

iv. Stokes theorem

$$\int_\Omega \tilde{\boldsymbol{d}}\tilde{\boldsymbol{\sigma}} = \int_{\partial\Omega} \tilde{\boldsymbol{\sigma}}, \tag{2.10}$$

and Gauss theorem

$$\int_\Omega \tilde{\boldsymbol{d}} \star \tilde{\boldsymbol{\sigma}} = \int_{\partial\Omega} \star \tilde{\boldsymbol{\sigma}}, \tag{2.11}$$

where $\partial\Omega$ is the closed p-dimensional boundary of the $(p+1)$-dimensional surface Ω and $\star$ stands for the *dual form* described in the next Section II C.

C. Dual of a p-form

In an n-dimensional manifold $\mathscr{M}$, the dual of a p-form $\tilde{\boldsymbol{\sigma}}$ is an $(n-p)$-form $\star\tilde{\boldsymbol{\sigma}}$ with components

$$\star\sigma_{\mu_1\mu_2\cdots\mu_{n-p}} = \sigma^{|\nu_1\nu 2\cdots\nu_p|}\varepsilon_{\nu_1\cdots\nu_p\mu_1\cdots\mu_{n-p}}, \tag{2.12}$$

where ε is the *Levi-Civita tensor*, the completely antisymmetric rank n tensor. On a positively oriented basis $\{\vec{\boldsymbol{e}}_\mu\}$, $\varepsilon_{12\cdots n} = \varepsilon(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \ldots, \vec{\boldsymbol{e}}_n) = +1$ and [1]

$$\varepsilon_{\mu_1\mu_2\cdots\mu_n} = [\mu_1, \mu_2, \ldots, \mu_n] = \begin{cases} 0 & \text{unless } \mu_1, \mu_2, \ldots, \mu_n \text{ are all different} \\ +1 & \text{for even permutations of } 1, 2, \ldots, n \\ -1 & \text{for odd permutations of } 1, 2, \ldots, n \end{cases}, \tag{2.13}$$

so that given any matrix Λ

$$\varepsilon_{\mu_1\mu_2\cdots\mu_n}\Lambda^{\mu_1}_{\bar{1}}\Lambda^{\mu_2}_{\bar{2}}\cdots\Lambda^{\mu_n}_{\bar{n}} = \det||\Lambda^\mu{}_{\bar{\nu}}||\,. \tag{2.14}$$

The dual has the following property

$$\tilde{\boldsymbol{\sigma}} \wedge \star\tilde{\boldsymbol{\sigma}} = ||\sigma||^2\varepsilon, \tag{2.15}$$

where

$$||\sigma||^2 = \sigma_{|\mu_1\mu_2\cdots\mu_p|}\sigma^{\mu_1\mu_2\cdots\mu_p}, \tag{2.16}$$

is the norm of the p-form.

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[1] In a manifold with a metric, that will be introduced in the next Section III, ε should be corrected as in Eq. (3.66) (Ex. 8.3 of Ref [1] and Exs. 3.20, 3.21 of Ref. [2])

III. THE METRIC TENSOR

Now we will introduce a *metric* and dwell into differential *geometry* or more specifically into *Riemannian geometry*. [2] We will then work on a *smooth* manifold $\mathscr{M}$, i.e. a *Riemannian manifold*. The metric $\boldsymbol{g}$ is a rank 2 symmetric tensor. In its $\binom{0}{2}$ form

$$g(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta) = \vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = g_{\alpha\beta}, \tag{3.1}$$

$$\boldsymbol{g} = \boldsymbol{ds}^2 = g_{\alpha\beta}\widetilde{\boldsymbol{\omega}}^\alpha \otimes \widetilde{\boldsymbol{\omega}}^\beta, \tag{3.2}$$

where in a coordinate basis $\widetilde{\boldsymbol{\omega}}^\alpha = \widetilde{\boldsymbol{d}}x^\alpha$. If $\vec{\boldsymbol{\xi}} = dx^\alpha \vec{\boldsymbol{e}}_\alpha$ is a displacement vector then

$$\begin{aligned} g(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}}) &= \vec{\boldsymbol{\xi}} \cdot \vec{\boldsymbol{\xi}} = g_{\alpha\beta}\widetilde{\boldsymbol{\omega}}^\alpha \otimes \widetilde{\boldsymbol{\omega}}^\beta(dx^\gamma \vec{\boldsymbol{e}}_\gamma, dx^\delta \vec{\boldsymbol{e}}_\delta) \\ &= g_{\alpha\beta}\langle \widetilde{\boldsymbol{\omega}}^\alpha, dx^\gamma \vec{\boldsymbol{e}}_\gamma\rangle\langle \widetilde{\boldsymbol{\omega}}^\beta, dx^\delta \vec{\boldsymbol{e}}_\delta\rangle \\ &= g_{\alpha\beta} dx^\gamma dx^\delta \langle \widetilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{e}}_\gamma\rangle\langle \widetilde{\boldsymbol{\omega}}^\beta, \vec{\boldsymbol{e}}_\delta\rangle \\ &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= ds^2. \end{aligned} \tag{3.3}$$

In other words

i. Interval between two unspecified displacements $\boldsymbol{ds}^2 = \boldsymbol{g}$;

ii. Interval between two unspecified displacements $ds^2 = g(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}})$;

as for

i. Unspecified direction $\widetilde{\boldsymbol{d}}f$;

ii. Specified direction $df = \langle \widetilde{\boldsymbol{d}}f, \vec{\boldsymbol{v}}\rangle = \partial_{\vec{\boldsymbol{v}}} f = v^\alpha f_{,\alpha}$.

We use $\boldsymbol{g}$ to establish a correspondence between 1-forms and vectors

$$\widetilde{\boldsymbol{u}} \leftrightarrow \vec{\boldsymbol{u}} \quad \text{if and only if} \quad \langle \widetilde{\boldsymbol{u}}, \vec{\boldsymbol{a}}\rangle = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{a}} = g(\vec{\boldsymbol{u}}, \vec{\boldsymbol{a}}) \quad \forall\ \vec{\boldsymbol{a}}. \tag{3.4}$$

In components $\widetilde{\boldsymbol{u}} = u_\beta \widetilde{\boldsymbol{\omega}}^\beta$ and

$$u_\beta = \langle \widetilde{\boldsymbol{u}}, \vec{\boldsymbol{e}}_\beta\rangle = g(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_\beta) = g(u^\alpha \vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta) = u^\alpha g_{\alpha\beta}, \tag{3.5}$$

so we use $g_{\alpha\beta}$ to *lower indexes*.

Also $\widetilde{\boldsymbol{\omega}}^\alpha$ is dual to $\vec{\boldsymbol{e}}_\alpha$. Call $\widetilde{\boldsymbol{e}}^\alpha$ the 1-form corresponding to $\vec{\boldsymbol{e}}_\alpha$, then

$$\langle \widetilde{\boldsymbol{e}}^\alpha, \vec{\boldsymbol{e}}_\beta\rangle = \vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = g_{\alpha\beta} = \langle g_{\alpha\gamma}\widetilde{\boldsymbol{\omega}}^\gamma, \vec{\boldsymbol{e}}_\beta\rangle, \tag{3.6}$$

so $\widetilde{\boldsymbol{e}}^\alpha = g_{\alpha\gamma}\widetilde{\boldsymbol{\omega}}^\gamma = \widetilde{\boldsymbol{\omega}}^\alpha$.

$\boldsymbol{g}$ in its $\binom{1}{1}$ form

$$g^\alpha{}_\beta = g(\widetilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{e}}_\beta) = \langle \widetilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{e}}_\beta\rangle = \delta^\alpha{}_\beta. \tag{3.7}$$

$\boldsymbol{g}$ in its $\binom{2}{0}$ form

$$g^{\alpha\beta} = g(\widetilde{\boldsymbol{\omega}}^\alpha, \widetilde{\boldsymbol{\omega}}^\beta), \tag{3.8}$$

and

$$g^\alpha{}_\beta = g^{\alpha\mu}g_{\mu\beta} = \delta^\alpha{}_\beta \tag{3.9}$$

or $||g^{\alpha\beta}|| = ||g_{\alpha\beta}||^{-1}$.

Consider for example a tensor $\mathbb{H}$ of rank $\binom{1}{2}$, then

$$H(\widetilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{e}}_\beta, \vec{\boldsymbol{e}}_\gamma) = H^\alpha{}_{\beta\gamma}, \tag{3.10}$$

$$\mathbb{H} = H^\alpha{}_{\beta\gamma}\vec{\boldsymbol{e}}_\alpha \otimes \widetilde{\boldsymbol{\omega}}^\beta \otimes \widetilde{\boldsymbol{\omega}}^\gamma, \tag{3.11}$$

$$H_{\alpha\beta\gamma} = H^\delta{}_{\beta\gamma}g_{\delta\alpha} \tag{3.12}$$

$$H_{\alpha'\beta'\gamma'} = L^\alpha{}_{\alpha'}L^\beta{}_{\beta'}L^\gamma{}_{\gamma'}H_{\alpha\beta\gamma}, \tag{3.13}$$

where the last equation is the change of basis.

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[2] Riemannian geometry originated with the vision of Georg Friedrich Bernhard Riemann (Breselenz, 17 September 1826 – Selasca, 20 July 1866) expressed in his inaugural lecture "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" ("On the Hypotheses on which Geometry is Based").

A. The global (non coordinate) orthonormal frame

Through a change of basis $L^{\hat{\mu}}{}_{\mu}$ we can always diagonalize the symmetric metric tensor globally, thanks to the spectral theorem, so to realize an orthogonal frame

$$L^{\mu}{}_{\hat{\mu}}L^{\nu}{}_{\hat{\nu}}g_{\mu\nu} = \widetilde{g}_{\hat{\mu}\hat{\nu}}, \tag{3.14}$$

with $||\widetilde{g}_{\hat{\mu}\hat{\nu}}|| = \mathrm{diag}\{\lambda_{\hat{1}}, \lambda_{\hat{2}}, \ldots, \lambda_{\hat{n}}\}$. Furthermore, it is always possible to rescale each vector (or 1-form) of the orthogonal basis to get $||\widetilde{g}_{\hat{\mu}\hat{\nu}}|| \to ||\eta_{\hat{\mu}\hat{\nu}}|| = \mathrm{diag}\{1, 1, \ldots, 1\}$ so that $ds^2 = \eta_{\hat{\mu}\hat{\nu}}\widetilde{\boldsymbol{\omega}}^{\hat{\mu}} \otimes \widetilde{\boldsymbol{\omega}}^{\hat{\nu}}$ with $\widetilde{\boldsymbol{\omega}}^{\hat{\mu}} = \widetilde{\boldsymbol{\omega}}^{\mu}\sqrt{\lambda_{\hat{\mu}}}$. This at the price of having a non coordinate basis. We will call this the *global Lorentz frame* (LF) [3] or the *global (non coordinate) orthonormal frame*.

B. Commutators

Consider two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. We want to prove that

$$[\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}]f = \partial_{\vec{\boldsymbol{u}}}\partial_{\vec{\boldsymbol{v}}}f - \partial_{\vec{\boldsymbol{v}}}\partial_{\vec{\boldsymbol{u}}}f = \text{vector}, \tag{3.15}$$

for example on a scalar f.

We will prove this in a coordinate basis and then extend the result in a general non coordinate basis:

i. In a coordinate basis $\vec{\boldsymbol{e}}_\alpha = \partial_\alpha = \partial/\partial x^\alpha$ and

$$\begin{aligned}
[\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}] &= u^\alpha\partial_\alpha(v^\beta\partial_\beta) - v^\beta\partial_\beta(u^\alpha\partial_\alpha) \\
&= u^\alpha v^\beta\frac{\partial^2}{\partial x^\alpha\partial x^\beta} + u^\alpha v^\beta{}_{,\alpha}\frac{\partial}{\partial x^\beta} - \\
&\quad v^\beta u^\alpha\frac{\partial^2}{\partial x^\beta\partial x^\alpha} - v^\beta u^\alpha{}_{,\beta}\frac{\partial}{\partial x^\alpha} \\
&= (u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta})\frac{\partial}{\partial x^\alpha},
\end{aligned} \tag{3.16}$$

where we used the commutation of the partial derivatives. For basis vectors $[\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta] = [\partial_\alpha, \partial_\beta] = 0$;

ii. In a non coordinate basis we will have instead

$$[\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta] = c_{\alpha\beta}{}^{\gamma}\vec{\boldsymbol{e}}_\gamma, \tag{3.17}$$

so that

$$\begin{aligned}
[\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}] &= [u^\alpha\vec{\boldsymbol{e}}_\alpha, v^\beta\vec{\boldsymbol{e}}_\beta] \\
&= (u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta} + u^\gamma v^\beta c_{\gamma\beta}{}^{\alpha})\vec{\boldsymbol{e}}_\alpha.
\end{aligned} \tag{3.18}$$

C. Covariant derivative

When taking a derivative on $\mathscr{M}$ we need to take care also of how the basis vectors and 1-forms change. [4] Such a derivative is called a *covariant derivative* for which we will use interchangeably the following three symbols

$$\frac{D\ldots}{\partial\lambda}\ , \quad u^\alpha\nabla_\alpha\ldots\ , \quad u^\alpha(\ldots)_{;\alpha}. \tag{3.19}$$

Let us distinguish four cases:

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[3] Hendrik Antoon Lorentz (Arnhem, 18 July 1853 - Haarlem, 4 February 1928)

[4] In other terms, when taking a derivative of a vector on $\mathscr{M}$ we come across the problem of comparing two vectors at two different points of $\mathscr{M}$. This is solved with the procedure of *parallel transport* where we simply compare the two vectors either at the initial or at the final point after having "copied" the components of the vector respect to the basis at one point on its components respect to the basis at the other point.

i. On a scalar f

$$\nabla_\alpha f = \partial_\alpha f \quad \text{or} \quad f_{;\alpha} = f_{,\alpha}. \tag{3.20}$$

ii. On a vector $\vec{\boldsymbol{v}} v^\alpha \vec{\boldsymbol{e}}_\alpha$. We will prove later that

$$\nabla_\alpha \vec{\boldsymbol{e}}_\beta = \partial_\alpha \vec{\boldsymbol{e}}_\beta = \Gamma^\gamma{}_{\alpha\beta} \vec{\boldsymbol{e}}_\gamma, \tag{3.21}$$

where the Γ are some coefficients called *connection coefficients* for a non coordinate (*anholonomic*) basis and *Christoffel symbols* for a coordinate basis (*holonomic*). [5]

Then

$$\begin{aligned}\nabla_\alpha \vec{\boldsymbol{v}} &= \nabla_\alpha (v^\beta \vec{\boldsymbol{e}}_\beta) \\ &= (\partial_\alpha v^\beta) \vec{\boldsymbol{e}}_\beta + v^\beta \partial_\alpha \vec{\boldsymbol{e}}_\beta \\ &= v^\beta{}_{,\alpha} \vec{\boldsymbol{e}}_\beta + v^\beta \Gamma^\gamma{}_{\alpha\beta} \vec{\boldsymbol{e}}_\gamma \\ &= \left(v^\beta{}_{,\alpha} + v^\gamma \Gamma^\beta{}_{\alpha\gamma} \right) \vec{\boldsymbol{e}}_\beta, \end{aligned} \tag{3.22}$$

or

$$(\nabla_\alpha \vec{\boldsymbol{v}})^\beta = v^\beta{}_{;\alpha} = v^\beta{}_{,\alpha} + \Gamma^\beta{}_{\alpha\gamma} v^\gamma. \tag{3.23}$$

iii. On a 1-form $\tilde{\boldsymbol{\sigma}} = \sigma_\alpha \tilde{\boldsymbol{\omega}}^\alpha$

$$\langle \tilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\alpha \rangle = \sigma_\beta \langle \tilde{\boldsymbol{\omega}}^\beta, \vec{\boldsymbol{e}}_\alpha \rangle = \sigma_\beta \delta^\beta{}_\alpha = \sigma_\alpha, \tag{3.24}$$

taking the covariant derivative of this expression

$$\nabla_\alpha \langle \tilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta \rangle = \sigma_{\beta,\alpha} \tag{3.25}$$

$$\langle \nabla_\alpha \tilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta \rangle + \langle \tilde{\boldsymbol{\sigma}}, \partial_\alpha \vec{\boldsymbol{e}}_\beta \rangle = \sigma_{\beta,\alpha} \tag{3.26}$$

$$\langle \nabla_\alpha \tilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta \rangle = \sigma_{\beta,\alpha} - \langle \tilde{\boldsymbol{\sigma}}, \Gamma^\gamma{}_{\alpha\beta} \vec{\boldsymbol{e}}_\gamma \rangle \tag{3.27}$$

$$(\nabla_\alpha \tilde{\boldsymbol{\sigma}})_\beta = \sigma_{\beta;\alpha} = \sigma_{\beta,\alpha} - \Gamma^\gamma{}_{\alpha\beta} \sigma_\gamma, \tag{3.28}$$

where we see how the correction due to the change of the basis vector enters with a minus sign.

iv. On a tensor $\mathbb{H}$ of rank $\binom{r}{s}$

$$\begin{aligned}(\nabla_\alpha H)^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s} = {} & H^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s;\alpha} = H^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s,\alpha} \\ & + \Gamma^{\mu_1}{}_{\gamma\alpha} H^{\gamma\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s} + \ldots + \Gamma^{\mu_r}{}_{\gamma\alpha} H^{\mu_1\mu_2\cdots\gamma}{}_{\nu_1\nu_2\cdots\nu_s} \\ & - \Gamma^\gamma{}_{\nu_1\alpha} H^{\mu_1\mu_2\cdots\mu_r}{}_{\gamma\nu_2\cdots\nu_s} + \ldots - \Gamma^\gamma{}_{\nu_s\alpha} H^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\gamma}. \end{aligned} \tag{3.29}$$

The connection coefficients

We now want to prove Eq. (3.21) and determine the expression of the connection coefficients in terms of the metric tensor. Start again from the definition (3.21)

$$g(\nabla_\alpha \vec{\boldsymbol{e}}_\beta, \vec{\boldsymbol{e}}_\gamma) = g\left(\Gamma^\delta{}_{\alpha\beta} \vec{\boldsymbol{e}}_\delta, \vec{\boldsymbol{e}}_\gamma \right) = \Gamma^\delta{}_{\alpha\beta}\, g_{\delta\gamma}. \tag{3.30}$$

Then consider the partial derivative of the metric tensor

$$g_{\beta\gamma,\alpha} = g(\nabla_\alpha \vec{\boldsymbol{e}}_\beta, \vec{\boldsymbol{e}}_\gamma) + g(\vec{\boldsymbol{e}}_\beta, \nabla_\alpha \vec{\boldsymbol{e}}_\gamma). \tag{3.31}$$

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[5] Our definition for the connection coefficients is different from the one of Ref. [1] where $\nabla_\alpha \vec{\boldsymbol{e}}_\beta = \Gamma^\gamma{}_{\beta\alpha} \vec{\boldsymbol{e}}_\gamma$. This difference is only relevant for a non coordinate basis.

Rewrite Eq. (3.31) in the following 3 equivalent ways

$$g_{\beta\gamma,\alpha} = g(\vec{e}_\beta, \nabla_\gamma \vec{e}_\alpha) + g(\vec{e}_\gamma, \nabla_\alpha \vec{e}_\beta) - g(\vec{e}_\beta, [\vec{e}_\gamma, \vec{e}_\alpha]), \tag{3.32a}$$
$$g_{\gamma\alpha,\beta} = g(\vec{e}_\alpha, \nabla_\beta \vec{e}_\gamma) + g(\vec{e}_\gamma, \nabla_\alpha \vec{e}_\beta) - g(\vec{e}_\gamma, [\vec{e}_\alpha, \vec{e}_\beta]), \tag{3.32b}$$
$$g_{\alpha\beta,\gamma} = g(\vec{e}_\alpha, \nabla_\beta \vec{e}_\gamma) + g(\vec{e}_\beta, \nabla_\gamma \vec{e}_\alpha) - g(\vec{e}_\alpha, [\vec{e}_\beta, \vec{e}_\gamma]), \tag{3.32c}$$

where we used the symmetry of the metric tensor $\boldsymbol{g}$ and the definition of the commutator $[\cdot,\cdot]$. Adding (3.32a) and (3.32b) and subtracting (3.32c), and using the definitions (3.21) and (3.17) for the Γ and c coefficients respectively, we find

$$2g(\vec{e}_\gamma, \nabla_\alpha \vec{e}_\beta) = g_{\beta\gamma,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} + c_{\gamma\alpha\beta} + c_{\alpha\beta\gamma} - c_{\beta\gamma\alpha}. \tag{3.33}$$

Using Eqs. (3.30) and (3.17) we find for the connection coefficients

$$\Gamma_{\gamma\alpha\beta} = \frac{1}{2}\{g_{\beta\gamma,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} + c_{\gamma\alpha\beta} + c_{\alpha\beta\gamma} - c_{\beta\gamma\alpha}\}. \tag{3.34}$$

Note that the indexes of the c commutation coefficients enter in the 3 terms in cyclical order moving from one term to the next. In a coordinate basis all c are zero and we find the so called Christoffel symbols

$$\Gamma_{\gamma\alpha\beta} = \frac{1}{2}\{g_{\beta\gamma,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}\}, \tag{3.35}$$

which is clearly symmetric in its last two indexes.

An important property of the metric tensor is to be covariantly constant, i.e. $\nabla\boldsymbol{g} = 0$. In fact in an orthonormal frame $g_{\hat\alpha\hat\beta} = \eta_{\hat\alpha\hat\beta}$ and in the next Section III D we will see that it is also always possible to choose a local coordinate orthonormal frame on $\mathscr{M}$ such that $g_{\hat\alpha\hat\beta,\hat\gamma} = 0$ (and of course $c_{\hat\alpha\hat\beta\hat\gamma} = 0$ on the coordinate frame), then $\Gamma_{\hat\alpha\hat\beta\hat\gamma} = 0$ so that $g_{\hat\alpha\hat\beta;\hat\gamma} = g_{\hat\alpha\hat\beta,\hat\gamma} = 0$.

We will now prove 3 properties of Γ:

i. Since the metric is covariantly constant

$$\begin{aligned} 0 = g_{\alpha\beta;\gamma} &= g_{\alpha\beta,\gamma} - \Gamma^\mu{}_{\alpha\gamma} g_{\mu\beta} - \Gamma^\mu{}_{\beta\gamma} g_{\alpha\mu} \\ &= g_{\alpha\beta,\gamma} - \Gamma_{\beta\alpha\gamma} - \Gamma_{\alpha\beta\gamma}, \end{aligned} \tag{3.36}$$

so that

$$\frac{1}{2} g_{\alpha\beta,\gamma} = \Gamma_{(\alpha\beta)\gamma}, \tag{3.37}$$

where the round parenthesis contain indexes on which one symmetrizes. So Γ is antisymmetric on its first two indexes in the local coordinate orthonormal frame described in the next Section III D or in a global (non coordinate) orthonormal frame described in Section III A for which, in both cases, $g_{\hat\alpha\hat\beta,\hat\gamma} = 0$.

ii. From the definition of the commutator (3.17) and the connection coefficient (3.21) follows

$$\begin{aligned} [\vec{e}_\alpha, \vec{e}_\beta] &= \nabla_\alpha \vec{e}_\beta - \nabla_\beta \vec{e}_\alpha \\ &= (\Gamma^\gamma{}_{\alpha\beta} - \Gamma^\gamma{}_{\beta\alpha})\vec{e}_\gamma \\ &= c_{\alpha\beta}{}^\gamma \vec{e}_\gamma, \end{aligned} \tag{3.38}$$

so that

$$\frac{1}{2} c_{\alpha\beta\gamma} = \Gamma_{\gamma[\alpha\beta]}, \tag{3.39}$$

where the square parenthesis contain indexes on which one antisymmetrizes. So Γ is symmetric on its last two indexes in a coordinate reference frame where $c_{\alpha\beta\gamma} = 0$.

iii. Γ is not a tensor. In fact let's see how Γ transforms

$$\begin{aligned}\nabla_{\alpha'}\vec{e}_{\beta'} = \Gamma^{\gamma'}{}_{\alpha'\beta'}\vec{e}_{\gamma'} &= \nabla_{L^{\alpha}{}_{\alpha'}\vec{e}_{\alpha}}(L^{\beta}{}_{\beta'}\vec{e}_{\beta}) = L^{\alpha}{}_{\alpha'}\nabla_{\alpha}(L^{\beta}{}_{\beta'}\vec{e}_{\beta}) \\ &= L^{\alpha}{}_{\alpha'}L^{\beta}{}_{\beta'}\nabla_{\alpha}\vec{e}_{\beta} + L^{\alpha}{}_{\alpha'}L^{\beta}{}_{\beta',\alpha}\vec{e}_{\beta} \\ &= L^{\alpha}{}_{\alpha'}L^{\beta}{}_{\beta'}\Gamma^{\gamma}{}_{\alpha\beta}\vec{e}_{\gamma} + L^{\alpha}{}_{\alpha'}L^{\beta}{}_{\beta',\alpha}L^{\gamma'}{}_{\beta}\vec{e}_{\gamma'},\end{aligned} \tag{3.40}$$

so that

$$\Gamma^{\gamma'}{}_{\alpha'\beta'} = L^{\alpha}{}_{\alpha'}L^{\beta}{}_{\beta'}L^{\gamma'}{}_{\gamma}\Gamma^{\gamma}{}_{\alpha\beta} + L^{\alpha}{}_{\alpha'}L^{\gamma'}{}_{\beta}L^{\beta}{}_{\beta',\alpha}, \tag{3.41}$$

where the last term is in general different from zero.

Useful identities

(Ex. 7.7 in Ref. [1]) We will here enunciate and prove 7 useful identities:

i. From the definition (3.65) follows

$$g_{,\alpha} = g g^{\mu\nu} g_{\mu\nu,\alpha} = -g g_{\mu\nu} g^{\mu\nu}{}_{,\alpha}. \tag{3.42}$$

To prove this identity we first note that for any diagonalizable matrix A the following identity holds

$$\det A = e^{\mathrm{tr}(\ln A)}, \tag{3.43}$$

which clearly holds when A is in its diagonal form. So

$$\begin{aligned} g_{,\alpha} &= \left[e^{\mathrm{tr}(\ln ||g_{\mu\nu}||)}\right]_{,\alpha} \\ &= g[\mathrm{tr}(\ln ||g_{\mu\nu}||)]_{,\alpha} \\ &= g\,\mathrm{tr}[(\ln ||g_{\mu\nu}||)_{,\alpha}] \\ &= g\,\mathrm{tr}(||g_{\mu\nu}||^{-1}||g_{\mu\nu,\alpha}||) \\ &= g g^{\mu\nu} g_{\mu\nu,\alpha} \\ &= -g g_{\mu\nu} g^{\mu\nu}{}_{,\alpha} \end{aligned}$$

where in the last equality we used Eq. (3.9).

All the remaining identities require a coordinate basis.

ii. Contraction of first two indexes of Christoffel symbol

$$\Gamma^{\alpha}{}_{\beta\alpha} = \Gamma^{\alpha}{}_{\alpha\beta} = \left(\ln\sqrt{|g|}\right)_{\beta}. \tag{3.44}$$

From Eq. (3.35) and identity [i.] follows

$$\Gamma^{\alpha}{}_{\beta\alpha} = \frac{1}{2}g^{\alpha\nu}g_{\alpha\nu,\beta} = \frac{1}{2}g_{,\beta}/g = \left(\ln\sqrt{|g|}\right)_{,\beta}. \tag{3.45}$$

iii. Contraction of last two indexes of Christoffel symbol

$$g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = -\frac{1}{\sqrt{|g|}}\left(g^{\alpha\nu}\sqrt{|g|}\right)_{\nu}. \tag{3.46}$$

From Eq. (3.35) follows

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\{g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\}, \tag{3.47}$$

using property [i.]

$$g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\{2g_{\beta\mu},{}^{\mu} - g_{,\beta}/g\}. \tag{3.48}$$

On the other hand sine the contracted index is mute

$$\frac{1}{\sqrt{|g|}}\left(g^{\alpha\nu}\sqrt{|g|}\right)_{\nu} = \frac{1}{\sqrt{|g|}}\left(g^{\alpha\nu}{}_{,\nu}\sqrt{|g|} + g^{\alpha\nu}g_{,\nu}/2\sqrt{|g|}\right) = g^{\alpha\nu}{}_{,\nu} + g^{\alpha\beta}g_{,\beta}/2g, \tag{3.49}$$

and using Eq. (3.9)

$$0 = (g^{\alpha\beta}g_{\beta\mu}),{}^{\mu} = g^{\alpha\beta}g_{\beta\mu},{}^{\mu} + g^{\alpha\nu}{}_{,\nu}. \tag{3.50}$$

Putting together (3.48), (3.49), and (3.50) gives identity [iii.].

iv. Divergence of a vector

$$A^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha}\right)_{,\alpha}. \tag{3.51}$$

From the definition of covariant derivative (3.22) and identity [ii.] follows

$$\begin{aligned}A^{\alpha}{}_{;\alpha} &= A^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}{}_{\beta\alpha}A^{\beta}\\ &= A^{\alpha}{}_{,\alpha} + \frac{\left(\sqrt{|g|}\right)_{,\alpha}}{\sqrt{|g|}}A^{\alpha}\\ &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha}\right)_{,\alpha}.\end{aligned} \tag{3.52}$$

v. Divergence of a rank $\binom{2}{0}$ antisymmetric tensor

$$A^{\alpha\beta}{}_{;\beta} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha\beta}\right)_{,\beta}. \tag{3.53}$$

From the definition of covariant derivative (3.29) and identity [ii.] follows

$$\begin{aligned}A^{\alpha\beta}{}_{;\beta} &= A^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\gamma\beta}A^{\gamma\beta}\Gamma^{\beta}{}_{\gamma\beta}A^{\alpha\gamma}\\ &= A^{\alpha\beta}{}_{,\beta} + \Gamma^{\beta}{}_{\gamma\beta}A^{\alpha\gamma}\\ &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha\beta}\right)_{,\beta},\end{aligned} \tag{3.54}$$

where in the first equality we used the symmetry of the Christoffel symbol respect to its last two indexes and in the last equality we used identity [ii.].

vi. Divergence of a rank $\binom{1}{1}$ tensor

$$A_{\alpha}{}^{\beta}{}_{;\beta} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A_{\alpha}{}^{\beta}\right)_{,\beta} - \Gamma^{\lambda}{}_{\alpha\mu}A_{\lambda}{}^{\mu}. \tag{3.55}$$

From the definition of covariant derivative (3.29) and identity [ii.] follows

$$\begin{aligned}A_{\alpha}{}^{\beta}{}_{;\beta} &= A_{\alpha}{}^{\beta}{}_{,\beta} + \Gamma^{\beta}{}_{\mu\beta}A_{\alpha}{}^{\mu} - \Gamma^{\lambda}{}_{\alpha\mu}A_{\lambda}{}^{\mu}\\ &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A_{\alpha}{}^{\beta}\right)_{,\beta} - \Gamma^{\lambda}{}_{\alpha\mu}A_{\lambda}{}^{\mu}.\end{aligned} \tag{3.56}$$

where again in the last equality we used identity [ii.].

vii. Laplacian

$$\Box S = S_{;\alpha}{}^{\alpha} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}S_{,}{}^{\beta}\right)_{,\beta}, \tag{3.57}$$

where S is a scalar. Since the metric is covariantly constant

$$S_{;\alpha}{}^{\alpha} = (S_{,\alpha})_{;\beta}\, g^{\beta\alpha} = \left(S_{,\alpha}g^{\beta\alpha}\right)_{;\beta} = \left(S_{,}{}^{\beta}\right)_{;\beta} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}S_{,}{}^{\beta}\right)_{,\beta}. \tag{3.58}$$

where in the last equality we used identity [iv.].

D. The local (coordinate) orthonormal frame

(Ex. 13.3 in Ref. [1]) A *local (coordinate) orthonormal frame* is "tangent" to the manifold $\mathscr{M}$ on its point $\mathscr{P}_0$. We will call this a *Local Lorentz Frame* (LLF). It is the closest thing to a global (non coordinate) orthonormal frame "near" $\mathscr{P}_0$. It satisfies the following recipes: [6]

i. $g_{\alpha\beta}(\mathscr{P}_0) = \eta_{\alpha\beta}$;

ii. $g_{\alpha\beta,\gamma}(\mathscr{P}_0) = 0$;

iii. $g_{\alpha\beta,\gamma\delta}(\mathscr{P}_0) \neq 0$ in general;

with $||\eta_{\alpha\beta}|| = \mathrm{diag}\{1, 1, \ldots, 1\}$. It is customary to denote the indexes in a LLF with a hat, like so $\eta_{\hat{\alpha}\hat{\beta}}$, but we will not adopt this convention in this section and simply use unprimed indexes to denote the LLF.

We will now prove that it is always possible to choose a coordinate system such that [i.] and [ii.] hold at an arbitrary point.

Let's first count the number of independent components in a symmetric tensor of dimension n and rank r. For $r = 2$ we have $\binom{n}{2} + \binom{n}{1} = \frac{n(n+1)}{2}$ independent components. For $r = 3$ we have $\binom{n}{3} + 2\binom{n}{2} + \binom{n}{1} = \frac{n(n+1)(n+2)}{6}$. independent components. So for example in GR $n = 4$ and we find 10 for $r = 2$ and 20 for $r = 3$.

Consider now an arbitrary change of coordinates $x^{\alpha'} = f^{\alpha'}(x^{\alpha})$. Taylor expand around $\mathscr{P}_0$ at the origin

$$x^{\alpha'} = f^{\alpha'}{}_{,\mu}x^{\mu} + \frac{1}{2}f^{\alpha'}{}_{,\mu\nu}x^{\mu}x^{\nu} + \frac{1}{6}f^{\alpha'}{}_{,\mu\nu\lambda}x^{\mu}x^{\nu}x^{\lambda} + \ldots. \tag{3.59}$$

Then we can count the independent components of the various coefficients. For example in $n = 4$ the linear term $M^{\alpha'}{}_{\mu} = f^{\alpha'}{}_{,\mu}$ has $4 \times 4 = 16$ components, the quadratic term $N^{\alpha'}{}_{\mu\nu} = f^{\alpha'}{}_{,\mu\nu}$ has $4 \times 10 = 40$ components, and the cubic term $P^{\alpha'}{}_{\mu\nu\lambda} = f^{\alpha'}{}_{,\mu\nu\lambda}$ has $4 \times 20 = 80$ components. Recall that

$$L^{\alpha'}{}_{\mu} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}} = M^{\alpha'}{}_{\mu} + N^{\alpha'}{}_{\mu\nu}x^{\nu} + \frac{1}{2}P^{\alpha'}{}_{\mu\nu\lambda}x^{\nu}x^{\lambda} + \ldots. \tag{3.60}$$

At the origin we want $g_{\mu\nu}(\mathscr{P}_0) = \eta_{\mu\nu}$, but in general

$$\begin{aligned} g_{\mu\nu}(\mathscr{P}_0) = &\left[M^{\alpha'}{}_{\mu} + N^{\alpha'}{}_{\mu\nu}x^{\nu} + \frac{1}{2}P^{\alpha'}{}_{\mu\nu\lambda}x^{\nu}x^{\lambda} + \ldots\right] \times \\ &\left[M^{\beta'}{}_{\nu} + N^{\beta'}{}_{\nu\lambda}x^{\lambda} + \frac{1}{2}P^{\beta'}{}_{\nu\mu\lambda}x^{\mu}x^{\lambda} + \ldots\right] g_{\alpha'\beta'}. \end{aligned} \tag{3.61}$$

Then we conclude the following:

i. The condition on the metric requires

$$g_{\mu\nu}(\mathscr{P}_0) = \eta_{\mu\nu} = M^{\alpha'}{}_{\mu}M^{\beta'}{}_{\nu}g_{\alpha'\beta'}, \tag{3.62}$$

which can always be accommodated and for example in GR we have 10 independent components in $\eta_{\mu\nu}$ and $4 \times 4 = 16$ in $M^{\alpha}{}_{\beta}$. So we have 6 degrees of freedom left over for a Lorentz transformation (3 boosts and 3 rotations) to determine $M^{\nu}{}_{\mu}$.

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[6] In this context $\eta_{\alpha\beta}$ is also known as the *Minkowski tensor* (Aleksotas, 22 June 1864 - Gottinga, 12 January 1909).

ii. The condition on the first derivative of the metric requires

$$0 = g_{\mu\nu,\lambda}(\mathscr{P}_0) = M^{\alpha'}{}_{\mu} M^{\beta'}{}_{\nu} g_{\alpha'\beta',\lambda} + (N^{\alpha'}{}_{\mu\lambda} M^{\beta'}{}_{\nu} + N^{\beta'}{}_{\nu\lambda} M^{\alpha'}{}_{\mu}) g_{\alpha'\beta'}, \tag{3.63}$$

which can also be always accommodated with no degrees of freedom left. For example in GR $g_{\mu\nu,\lambda}$ has $4\times 10 = 40$ independent components then we will always be able to find the exactly 40 components of $N^{\alpha}{}_{\beta\gamma}$.

iii. The condition on the second derivative of the metric $g_{\mu\nu,\lambda\rho}(\mathscr{P}_0) = 0$ cannot in general be satisfied. For example in GR $g_{\mu\nu,\lambda\rho}$ has $10\times 10 = 100$ independent components but $P^{\alpha}{}_{\beta\gamma\delta}$ has only 80, so 20 degrees of freedom cannot be specified. We will see in Section III F that these are exactly the degrees of freedom of the Riemann curvature tensor.

So we can say that a LLF is the closest thing possible to a global orthonormal frame at a particular point $\mathscr{P}_0$ of the Riemannian manifold $\mathscr{M}$, being the tangent space to $\mathscr{M}$ at $\mathscr{P}_0$.

Upon taking the determinant of $L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} g_{\mu'\nu'} = \eta_{\mu\nu}$ we find

$$\det \left|\left| L^{\mu'}{}_{\mu} \right|\right|^2 \det ||g_{\mu'\nu'}|| = 1. \tag{3.64}$$

We will denote

$$g = \det||g_{\mu'\nu'}||. \tag{3.65}$$

In Section IV B we will see that in General Relativity (GR) $\mathscr{M}$ is a pseudo-Riemannian 4-dimensional manifold with $||\eta_{\mu\nu}|| = \mathrm{diag}\{-1, 1, 1, 1\}$ and $\det||L^{\mu'}{}_{\mu}|| = 1/\sqrt{-g}$.

The Levi-Civita tensor in a general basis becomes

$$\begin{aligned} \varepsilon_{\mu_1'\mu_2'\cdots\mu_n'} &= L^{\mu_1}{}_{\mu_1'} L^{\mu_2}{}_{\mu_2'} \cdots L^{\mu_n}{}_{\mu_n'} \varepsilon_{\mu_1\mu_2\cdots\mu_n} \\ &= \det||L^{\mu}{}_{\mu'}|| \varepsilon_{\mu_1\mu_2\cdots\mu_n} \\ &= \sqrt{|g|}\, \varepsilon_{\mu_1\mu_2\cdots\mu_n}, \end{aligned} \tag{3.66}$$

where in the first equality we used the fact that the Levi-Civita tensor is defined as the completely antisymmetric tensor of Eq. (2.13), in the second equality we used property (2.14), and in the last equality we used properties (3.64), (3.65), and (1.7).

E. Geodesics

A *geodesic* is a curve on the manifold $\mathscr{M}$ that parallel transports its tangent vector along itself

$$\nabla_{\vec{\boldsymbol{u}}} \vec{\boldsymbol{u}} = 0, \tag{3.67}$$

i.e. the tangent vector $\vec{\boldsymbol{u}}$ is covariantly constant along the curve

$$u^{\alpha} u^{\beta}{}_{;\beta} = 0, \tag{3.68}$$

$$u^{\alpha} \left(u^{\beta}{}_{,\beta} + \Gamma^{\beta}{}_{\alpha\gamma} u^{\gamma}\right) = 0, \tag{3.69}$$

$$\frac{d^2 x^{\beta}}{d\lambda^2} + \Gamma^{\beta}{}_{\alpha\gamma} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\gamma}}{d\lambda} = 0, \tag{3.70}$$

where x^{α} is a coordinate system on $\mathscr{M}$ and $u^{\alpha} = dx^{\alpha}/d\lambda$ with $\lambda = a\bar{\lambda} + b$ is an affine parameter (the proper time in GR) with a and b two real numbers giving the units (of time) and the origin (of time) respectively.

The geodesic equation (3.70) is a second order differential equation. For a solution it is then necessary to give the initial conditions $x^{\alpha}(0)$ and $\dot{x}^{\alpha}(0)$, where the dot stands fro a derivative respect to λ . Through each point of $\mathscr{M}$ exists a unique geodesic in each direction.

All affine parameters are related by a linear transformation. In fact, let $\bar{\lambda} = \bar{\lambda}(\lambda)$, then $d/d\lambda = \dot{\bar{\lambda}} d/d\bar{\lambda}$ and $d^2/d\lambda^2 = \ddot{\bar{\lambda}} d/d\bar{\lambda} + (\dot{\bar{\lambda}})^2 d^2/d\bar{\lambda}^2$. So Eq. (3.70) becomes

$$\frac{d^2 x^{\beta}}{d\bar{\lambda}^2} + \frac{\ddot{\bar{\lambda}}}{(\dot{\bar{\lambda}})^2} \frac{dx^{\beta}}{d\bar{\lambda}} + \Gamma^{\beta}{}_{\alpha\gamma} \frac{dx^{\alpha}}{d\bar{\lambda}} \frac{dx^{\gamma}}{d\bar{\lambda}} = 0. \tag{3.71}$$

Since the change in the affine parameter must not change the geodetic equation then the second term in Eq. (3.71) must cancel. This occur if $\ddot{\bar{\lambda}} = 0$ or $\bar{\lambda} = a\lambda + b$.

From a variational principle

Alternatively we can define a geodesic as a curve of extremal length. The length of a curve $C(\lambda)$ is given by

$$\mathscr{S} = \int_C ds = \int_C \sqrt{g_{\alpha\beta}(x^\gamma)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}\, d\lambda = \int \mathscr{L}_C(x^\gamma, \dot{x}^\gamma)\, d\lambda, \tag{3.72}$$

where λ is any parameter along the curve. The curve of extremal length is the one obtained through the stationary variational principle $\delta\mathscr{S}[x^\gamma, \dot{x}^\gamma] = 0$. We then find

$$\frac{d}{d\lambda}\left(\frac{\partial \mathscr{L}_{\mathscr{C}}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathscr{L}_{\mathscr{C}}}{\partial x^\alpha} = 0, \tag{3.73}$$

where $\mathscr{C}$ is the geodesic. We will from now on forget about this subscript. Since $\partial\mathscr{L}/\partial\dot{x}^\alpha = g_{\alpha\beta}\dot{x}^\beta/\mathscr{L}$ then Eq. (3.73) becomes

$$\frac{d}{d\lambda}\left(\frac{1}{\mathscr{L}}g_{\alpha\beta}\dot{x}^\beta\right) - \frac{1}{2\mathscr{L}}g_{\gamma\delta,\alpha}\dot{x}^\gamma\dot{x}^\delta = 0, \tag{3.74}$$

or

$$-\frac{1}{\mathscr{L}^2}\frac{d\mathscr{L}}{d\lambda}g_{\alpha\beta}\dot{x}^\beta + \frac{1}{\mathscr{L}}g_{\alpha\beta,\gamma}\dot{x}^\gamma\dot{x}^\beta + \frac{1}{\mathscr{L}}g_{\alpha\beta}\ddot{x}^\beta - \frac{1}{2\mathscr{L}}g_{\gamma\delta,\alpha}\dot{x}^\gamma\dot{x}^\delta = 0, \tag{3.75}$$

or, reordering terms,

$$g_{\alpha\beta}\ddot{x}^\beta + \frac{1}{2}\{g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\gamma\beta,\alpha}\}\dot{x}^\gamma\dot{x}^\beta = \frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{d\lambda}g_{\alpha\beta}\dot{x}^\beta, \tag{3.76}$$

recalling the definition (3.35) for the Christoffel symbol

$$g_{\alpha\beta}\ddot{x}^\beta + \Gamma_{\alpha\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = \frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{d\lambda}\dot{x}_\alpha, \tag{3.77}$$

On the extremal curve $\mathscr{L} = d\mathscr{S}/d\lambda$ =constant so $d\mathscr{L}/d\lambda = 0$ and we recover the geodesic Eq. (3.70). [7]

F. Curvature

(Chapter 11 in Ref. [1]) We will use a geometric introduction.

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[7] Einstein believed that the geodesic equation of motion can be derived from the field equations for empty space, i.e. from the fact that the Ricci curvature vanishes. He wrote [5]:

It has been shown that this law of motion — generalized to the case of arbitrarily large gravitating masses — can be derived from the field equations of empty space alone. According to this derivation the law of motion is implied by the condition that the field be singular nowhere outside its generating mass points.

and [6]

One of the imperfections of the original relativistic theory of gravitation was that as a field theory it was not complete; it introduced the independent postulate that the law of motion of a particle is given by the equation of the geodesic.

A complete field theory knows only fields and not the concepts of particle and motion. For these must not exist independently from the field but are to be treated as part of it.

On the basis of the description of a particle without singularity, one has the possibility of a logically more satisfactory treatment of the combined problem: The problem of the field and that of the motion coincide.

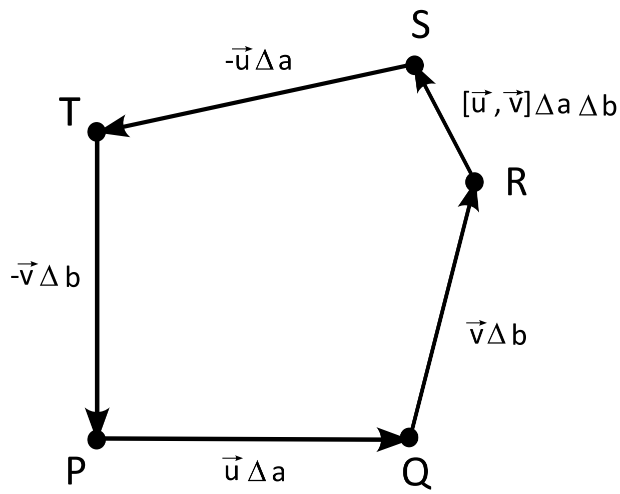

FIG. 1. Closed curve on $\mathscr{M}$ of infinitesimal area.

Consider a closed curve on $\mathscr{M}$ of infinitesimal area as in Fig. 1. We start at P then move to Q to end in R. We then move from P to T to S. The two paths are then closed by moving from R to S. We then consider the change $\delta\vec{\boldsymbol{A}}$ in a vector field $\vec{\boldsymbol{A}}^{\text{field}}$ from parallel transporting itself along the path P→T→S or parallel transporting itself along the path P→Q→R and closing the gap to reach S. At P we will have a vector $\vec{\boldsymbol{A}}_P = \vec{\boldsymbol{A}}_P^{\text{field}}$ where $\vec{\boldsymbol{A}}_P^{\text{field}}$ is our vector field at P. When we move from P to Q we can compare the vector field at Q with the parallel transported vector at Q, $\delta_{P\to Q}\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_Q^{\text{field}} - \vec{\boldsymbol{A}}_{P \overset{||}{\to} Q} = \Delta a \nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{A}}$. We then move to R to find $\delta_{Q\to R}\delta_{P\to Q}\vec{\boldsymbol{A}} = \Delta a \Delta b \nabla_{\vec{\boldsymbol{v}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{A}}$. Going from P to T to S we find $\delta_{T\to S}\delta_{P\to T}\vec{\boldsymbol{A}} = \Delta a \Delta b \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{v}}}\vec{\boldsymbol{A}}$. We then go from R to S to close the curve and we find $\delta_{R\to S}\vec{\boldsymbol{A}} = \Delta a \Delta b \nabla_{[\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}]}\vec{\boldsymbol{A}}$. So the change of the vector field around the curve is

$$\begin{aligned}
\delta\vec{\boldsymbol{A}} &= \delta_{Q\to R}\delta_{P\to Q}\vec{\boldsymbol{A}} + \delta_{R\to S}\vec{\boldsymbol{A}} - \delta_{P\to T}\delta_{T\to S}\vec{\boldsymbol{A}} \\
&= \Delta a \Delta b \left(\nabla_{\vec{\boldsymbol{v}}}\nabla_{\vec{\boldsymbol{u}}} - \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{v}}} - \nabla_{[\vec{\boldsymbol{v}},\vec{\boldsymbol{u}}]}\right)\vec{\boldsymbol{A}} \qquad (3.78)\\
&= \Delta a \Delta b \mathscr{R}(\vec{\boldsymbol{v}},\vec{\boldsymbol{u}})\vec{\boldsymbol{A}}, \qquad (3.79)
\end{aligned}$$

where

$$\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) = \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{v}}} - \nabla_{\vec{\boldsymbol{v}}}\nabla_{\vec{\boldsymbol{u}}} - \nabla_{[\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}]}, \qquad (3.80)$$

is the *curvature (local) operator*.

We will now give 3 properties of this operator:

i. For any 3 vectors $\vec{\boldsymbol{u}},\vec{\boldsymbol{v}},\vec{\boldsymbol{w}}$, and a scalar f

$$\begin{aligned}
\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})f\vec{\boldsymbol{w}} &= f\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}, \qquad (3.81)\\
\mathscr{R}(f\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} &= f\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}, \qquad (3.82)\\
\mathscr{R}(\vec{\boldsymbol{u}},f\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} &= f\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}. \qquad (3.83)
\end{aligned}$$

ii. $\mathscr{R}$ is linear

$$\begin{aligned}
\mathscr{R}(\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} &= \mathscr{R}(\vec{\boldsymbol{a}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} + \mathscr{R}(\vec{\boldsymbol{b}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}, \qquad (3.84)\\
\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}})\vec{\boldsymbol{w}} &= \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{a}})\vec{\boldsymbol{w}} + \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{b}})\vec{\boldsymbol{w}}, \qquad (3.85)\\
\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})(\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}}) &= \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{a}} + \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{b}}. \qquad (3.86)
\end{aligned}$$

iii. $\mathscr{R}$ is local

$$\mathscr{R}(\vec{\boldsymbol{u}}+\vec{\boldsymbol{a}},\vec{\boldsymbol{v}}+\vec{\boldsymbol{b}})(\vec{\boldsymbol{w}}+\vec{\boldsymbol{c}}) \xrightarrow[]{\substack{\vec{a}\to\vec{0}\\ \vec{b}\to\vec{0}\\ \vec{c}\to\vec{0}}} \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}. \qquad (3.87)$$

These 3 properties imply that $\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}$ is a tensor.

G. The Riemann tensor

The *Riemann tensor* $\mathbb{R}$ is defined in terms of the curvature tensor as follows

$$\mathbb{R}(\widetilde{\boldsymbol{\sigma}},\vec{\boldsymbol{c}},\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})=\langle\widetilde{\boldsymbol{\sigma}},\mathscr{R}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\vec{\boldsymbol{c}}\rangle. \tag{3.88}$$

The components of Riemann are as follows

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} &= \mathbb{R}(\widetilde{\boldsymbol{\omega}}^{\alpha},\vec{\boldsymbol{e}}_{\beta},\vec{\boldsymbol{e}}_{\gamma},\vec{\boldsymbol{e}}_{\delta}) && (3.89)\\
&= \langle\widetilde{\boldsymbol{\omega}}^{\alpha},\mathscr{R}(\vec{\boldsymbol{e}}_{\gamma},\vec{\boldsymbol{e}}_{\delta})\vec{\boldsymbol{e}}_{\beta}\rangle && (3.90)\\
&= \langle\widetilde{\boldsymbol{\omega}}^{\alpha},\nabla_{\gamma}\nabla_{\delta}\vec{\boldsymbol{e}}_{\beta}-\nabla_{\delta}\nabla_{\gamma}\vec{\boldsymbol{e}}_{\beta}-\nabla_{[\vec{\boldsymbol{e}}_{\gamma},\vec{\boldsymbol{e}}_{\delta}]}\vec{\boldsymbol{e}}_{\beta}\rangle && (3.91)\\
&= \langle\widetilde{\boldsymbol{\omega}}^{\alpha},\nabla_{\gamma}\left(\Gamma^{\mu}{}_{\delta\beta}\vec{\boldsymbol{e}}_{\mu}\right)-\nabla_{\delta}\left(\Gamma^{\mu}{}_{\gamma\beta}\vec{\boldsymbol{e}}_{\mu}\right)-\nabla_{(c_{\gamma\delta}{}^{\mu}\vec{\boldsymbol{e}}_{\mu})}\vec{\boldsymbol{e}}_{\beta}\rangle && (3.92)\\
&= \langle\widetilde{\boldsymbol{\omega}}^{\alpha},\Gamma^{\mu}{}_{\delta\beta,\gamma}\vec{\boldsymbol{e}}_{\mu}+\Gamma^{\mu}{}_{\delta\beta}\Gamma^{\sigma}{}_{\gamma\mu}\vec{\boldsymbol{e}}_{\sigma}-\Gamma^{\mu}{}_{\gamma\beta,\delta}\vec{\boldsymbol{e}}_{\mu}-\Gamma^{\mu}{}_{\gamma\beta}\Gamma^{\sigma}{}_{\delta\mu}\vec{\boldsymbol{e}}_{\sigma}-c_{\gamma\delta}{}^{\mu}\Gamma^{\sigma}{}_{\mu\beta}\vec{\boldsymbol{e}}_{\sigma}\rangle, && (3.93)
\end{aligned}$$

so

$$R^{\alpha}{}_{\beta\gamma\delta}=\Gamma^{\alpha}{}_{\delta\beta,\gamma}+\Gamma^{\mu}{}_{\delta\beta}\Gamma^{\alpha}{}_{\gamma\mu}-\Gamma^{\alpha}{}_{\gamma\beta,\delta}-\Gamma^{\mu}{}_{\gamma\beta}\Gamma^{\alpha}{}_{\delta\mu}-c_{\gamma\delta}{}^{\mu}\Gamma^{\alpha}{}_{\mu\beta}, \tag{3.94}$$

which for GR are $4\times4\times4\times4=256$ components and we expect a reduction to 20 (see Section III D).

Note that in a flat space $g_{\alpha\beta}=\eta_{\alpha\beta}$ globally, so all $R^{\alpha}{}_{\beta\gamma\delta}=0$.

We give now 4 symmetry properties of Riemann. We will prove these working in a LLF (see Section III D). Since Riemann is local these properties will hold globally. In a LLF $g_{\alpha\beta,\gamma}=0$ and the Christoffel symbols vanish, so we find

$$R_{\alpha\beta\gamma\delta}=g_{\alpha\mu}R^{\mu}{}_{\beta\gamma\delta}=g_{\alpha\mu}(\Gamma^{\mu}{}_{\delta\beta,\gamma}-\Gamma^{\mu}{}_{\gamma\beta,\delta})=\Gamma_{\alpha\delta\beta,\gamma}-\Gamma_{\alpha\gamma\beta,\delta}. \tag{3.95}$$

Using the symmetry of the Christoffel symbol (see Section III C) we easily prove the following:

i. Antisymmetry in the last two indexes

$$R_{\alpha\beta\gamma\delta}=-R_{\alpha\beta\delta\gamma}. \tag{3.96}$$

ii. Cyclic identity

$$R_{\alpha[\beta\gamma\delta]}=R_{\alpha\beta\gamma\delta}+R_{\alpha\gamma\delta\beta}+R_{\alpha\delta\beta\gamma}=0. \tag{3.97}$$

iii. Antisymmetry in the first two indexes

$$R_{\alpha\beta\gamma\delta}=-R_{\beta\alpha\gamma\delta}. \tag{3.98}$$

iv. Pair symmetry

$$R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta}. \tag{3.99}$$

We can then count the number of independent components of Riemann in an n-dimensional manifold $\mathscr{M}$. Due to properties [i.] and [iii.] the number of independent components on these pair of indexes is $M=n(n-1)/2$; due to property [iv.] the number of independent components reduces to $M(M+1)/2$; and we yet have to subtract $\binom{n}{4}$ to the counting since due to properties [i.], [ii.], and [iii.] the 4 indexes cannot be all different. We are then left with $n^2(n^2-1)/12$ independent components. For example for $n=2$ (sphere, see Section III J) we have only 1 component, for $n=2$ we have 6, and for $n=4$ (GR, see Section IV B) we have 20.

Commutation of covariant derivatives

(Ex. 16.3 in Ref. [1]) Covariant derivatives do not generally commute. For any vector $\vec{\boldsymbol{B}}$ we will prove that [8]

$$B^\mu{}_{;\alpha\beta} - B^\mu{}_{;\beta\alpha} = -R^\mu{}_{\gamma\alpha\beta}B^\gamma. \tag{3.100}$$

To prove this we work in a LLF

$$B^\mu{}_{;\alpha\beta} = (B^\mu{}_{;\alpha})_{,\beta} = (B^\mu{}_{,\alpha} + \Gamma^\mu{}_{\gamma\alpha}B^\gamma)_{,\beta} = B^\mu{}_{,\alpha\beta} + \Gamma^\mu{}_{\gamma\alpha,\beta}B^\gamma, \tag{3.101}$$

where in the last equality we used the fact that the Christoffel symbol vanish in a LLF. Then in a LLF

$$B^\mu{}_{;\alpha\beta} - B^\mu{}_{;\beta\alpha} = (\Gamma^\mu{}_{\gamma\alpha,\beta} - \Gamma^\mu{}_{\gamma\beta,\alpha})B^\gamma = R^\mu{}_{\gamma\beta\alpha}B^\gamma = -R^\mu{}_{\gamma\alpha\beta}B^\gamma. \tag{3.102}$$

See (Ex. 9.8 of Ref. [2]) for the extension to tensors.

Bianchi identities

The following *Bianchi identities* hold

$$R^\alpha{}_{\beta[\gamma\delta;\epsilon]} = 0. \tag{3.103}$$

These can be proven working in a LLF where

$$R^\alpha{}_{\beta\gamma\delta;\epsilon} = R^\alpha{}_{\beta\gamma\delta,\epsilon} = \Gamma^\alpha{}_{\delta\beta,\gamma\epsilon} - \Gamma^\alpha{}_{\gamma\beta,\delta\epsilon}, \tag{3.104}$$

and using the fact that partial derivatives commute.

The Ricci tensor

The *Ricci curvature tensor* is defined as

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}. \tag{3.105}$$

It is a symmetric tensor

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta} = g^{\gamma\epsilon}R_{\epsilon\alpha\gamma\beta} = g^{\epsilon\gamma}R_{\gamma\beta\epsilon\alpha} = R_{\beta\alpha}, \tag{3.106}$$

where we used the pair symmetry of Riemann (property [iv.] in Section III G).

The scalar curvature

The *scalar curvature* is the trace of Ricci

$$R = R^\alpha{}_\alpha \tag{3.107}$$

The Einstein tensor

The *Einstein tensor* $\mathbb{G}$ has the following components

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R. \tag{3.108}$$

The Einstein tensor is covariantly constant $\nabla\mathbf{G} = 0$ or

$$G^{\alpha\beta}{}_{;\beta} = 0, \tag{3.109}$$

---

[8] Note that $\mathscr{R}f = 0$ for a scalar f as can be easily proven from the properties (3.20), (3.22) of the covariant derivative, the definition (3.17), and the property (3.39) of the connection coefficients .

which are also known as the *contracted Bianchi identities*. These can be proven using the Bianchi identities

$$R^\alpha{}_{\beta\gamma\delta;\epsilon} + R^\alpha{}_{\beta\epsilon\gamma;\delta} + R^\alpha{}_{\beta\delta\epsilon;\gamma} = 0. \tag{3.110}$$

Contract α and γ and use antisymmetry of Riemann in last two indexes (property [i.] of Section III G) in the second term

$$R_{\beta\delta;\epsilon} - R_{\beta\epsilon;\delta} + R^\alpha{}_{\beta\delta\epsilon;\alpha} = 0. \tag{3.111}$$

Contract β and δ and use antisymmetry of Riemann in first two indexes (property [iii.] of Section III G) in the third term

$$R_{;\epsilon} - R^\beta{}_{\gamma;\beta} - R_{\alpha\epsilon;}{}^\alpha = R_{;\epsilon} - 2R_{\alpha\epsilon;}{}^\alpha = 0, \tag{3.112}$$

so that

$$G_{\alpha\epsilon;}{}^\alpha = R_{\alpha\epsilon;}{}^\alpha - \frac{1}{2} g_{\alpha\epsilon} R_;{}^\alpha = 0. \tag{3.113}$$

The Weyl conformal tensor

(Ex. 13.131 in Ref. [1] and Chapter 9 in Ref. [2]) The *Weyl conformal tensor* is defined as follows

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\delta_{[\gamma}{}^{[\alpha} R_{\delta]}{}^{\beta]} + \frac{1}{3}\delta_{[\gamma}{}^{[\alpha}\delta_{\delta]}{}^{\beta]} R. \tag{3.114}$$

The Weyl conformal tensor has the following properties:

i. Has same symmetries of Riemann;

ii. Is completely trace-free, i.e. contraction of $C_{\alpha\beta\gamma\delta}$ on any two indexes vanishes. It can be considered as the trace-free part of Riemann.

iii. In a manifold $\mathscr{M}$ of dimension n, its number of independent components can be inferred by the two properties above. Recalling the counting for Riemann of Section III G and noticing that property [ii.] above requires that contracting any two indexes we are left with only other two indexes with the proper symmetry constraints we conclude that the number of independent components of the Weyl tensor is given by $\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2}$ for $n \geq 3$ so it must be 0 for $n \leq 3$. Thus for $n \leq 3$ we may assume that the Weyl tensor is identically zero and the Riemann tensor is completely determined by its trace, the Ricci tensor,

iv. $C^{\alpha\beta}{}_{\gamma\delta} = 0$ if and only if $\mathscr{M}$ is *conformally flat*, i.e. if and only if it is reducible to Minkowski space by a *conformal transformation*, i.e. if and only if it exists a coordinate frame where

$$ds^2 = e^{2\phi(x^\alpha)} \eta_{\alpha\beta} dx^\alpha dx^\beta, \tag{3.115}$$

with ϕ a scalar. The function e^ϕ is called the *conformal factor*.

H. Geodesics deviation

Consider a congruence of geodesics $x^\alpha = x^\alpha(\lambda, m)$ with $\vec{\boldsymbol{u}} = \partial/\partial\lambda$, $\vec{\boldsymbol{m}} = \partial/\partial m$, and $\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{u}} = 0$. This is pictorially shown in Fig. 2.

By definition of a connecting vector $[\vec{\boldsymbol{n}}, \vec{\boldsymbol{u}}] = 0$, so $\vec{\boldsymbol{n}}$ and $\vec{\boldsymbol{u}}$ form a coordinate basis with coordinates n and λ respectively. Then $\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}} = \nabla_{\vec{\boldsymbol{n}}}\vec{\boldsymbol{u}}$ and $\mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}}) = \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{n}}} - \nabla_{\vec{\boldsymbol{n}}}\nabla_{\vec{\boldsymbol{u}}}$. So

$$\nabla_{\vec{\boldsymbol{u}}}(\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}}) = \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{n}}}\vec{\boldsymbol{u}} = \nabla_{\vec{\boldsymbol{n}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{u}} + \mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}})\vec{\boldsymbol{u}} = \mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}})\vec{\boldsymbol{u}}, \tag{3.116}$$

where in the last equality we used the geodesic equation $\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{u}} = 0$. We then reached the equation for the geodesics deviation

$$\nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}} = \mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}})\vec{\boldsymbol{u}}, \tag{3.117}$$

which in components $\langle\tilde{\boldsymbol{\omega}}^\alpha, \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}}\rangle + \langle\tilde{\boldsymbol{\omega}}^\alpha, \mathscr{R}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{u}})\vec{\boldsymbol{u}}\rangle = 0$ becomes

$$\frac{D^2 n^\alpha}{d\lambda^2} = u^\gamma u^\beta n^\alpha{}_{;\beta\gamma} = R^\alpha{}_{\beta\delta\gamma} u^\beta u^\delta n^\gamma. \tag{3.118}$$

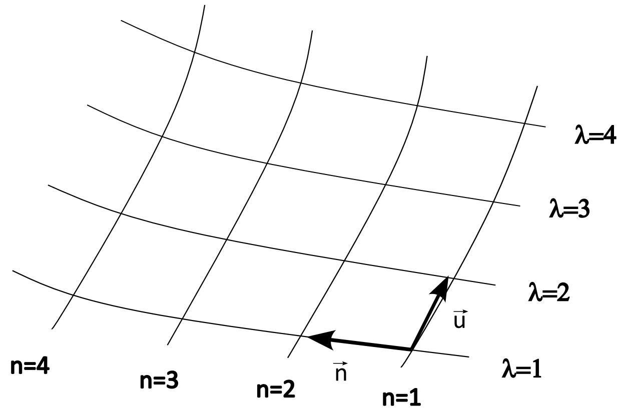

FIG. 2. Congruence of geodesics. λ =affine parameter (proper time in GR), $\vec{\boldsymbol{n}}$ =connecting vector which connects points of equal λ on different geodesics.

I. Cartan structure equations

Cartan, taking profit from the forms language (see Section II), devised a very useful way to calculate the component of the Riemann tensor in a simple way. Cartan structure equations are 3, they need a metric and are true in any frame (coordinate or non coordinate). We will first enunciate them and then proceed to their proof.

i. Introduce the *connection 1-form*

$$\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} = \langle \widetilde{\boldsymbol{\omega}}^{\alpha}, \nabla \vec{\boldsymbol{e}}_{\beta} \rangle = \Gamma^{\alpha}{}_{\beta\gamma} \widetilde{\boldsymbol{\omega}}^{\gamma}, \tag{3.119}$$

where the covariant derivative symbol ∇ has an empty index. Also the Γ we use here is the one of Ref. [1] with the last two indexes interchanged respect to our (this affects only a non coordinate basis, see footnote 5).

Then the first Cartan structure equation is

$$\widetilde{\boldsymbol{d}} \widetilde{\boldsymbol{\omega}}^{\alpha} = -\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} \wedge \widetilde{\boldsymbol{\omega}}^{\beta}. \tag{3.120}$$

We will now outline the proof. Let $\widetilde{\boldsymbol{\omega}}^{\alpha} = L^{\alpha}{}_{\bar{\beta}} \widetilde{\boldsymbol{\omega}}^{\bar{\beta}}$. Then, taking the exterior derivative,

$$\begin{aligned}
\widetilde{\boldsymbol{d}} \widetilde{\boldsymbol{\omega}}^{\alpha} &= L^{\alpha}{}_{\bar{\beta},\bar{\gamma}} \widetilde{\boldsymbol{\omega}}^{\bar{\gamma}} \wedge \widetilde{\boldsymbol{\omega}}^{\bar{\beta}} \\
&= L^{\alpha}{}_{\bar{\beta},\bar{\gamma}} L^{\bar{\gamma}}{}_{\beta} L^{\bar{\beta}}{}_{\gamma} \widetilde{\boldsymbol{\omega}}^{\beta} \wedge \widetilde{\boldsymbol{\omega}}^{\gamma} \\
&= L^{\alpha}{}_{\bar{\beta},\bar{\gamma}} (L^{\bar{\gamma}}{}_{\beta} L^{\bar{\beta}}{}_{\gamma} - L^{\bar{\gamma}}{}_{\gamma} L^{\bar{\beta}}{}_{\beta}) \widetilde{\boldsymbol{\omega}}^{\beta} \otimes \widetilde{\boldsymbol{\omega}}^{\gamma}.
\end{aligned} \tag{3.121}$$

Now

$$\begin{aligned}
-\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} \wedge \widetilde{\boldsymbol{\omega}}^{\beta} &= -\Gamma^{\alpha}{}_{\beta\gamma} \widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\beta} \\
&= (\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta}) \widetilde{\boldsymbol{\omega}}^{\beta} \otimes \widetilde{\boldsymbol{\omega}}^{\gamma} \\
&= -c_{\beta\gamma}{}^{\alpha} \widetilde{\boldsymbol{\omega}}^{\beta} \otimes \widetilde{\boldsymbol{\omega}}^{\gamma},
\end{aligned} \tag{3.122}$$

where in the last equality we used Eq. (3.39).

But

$$[\vec{\boldsymbol{e}}_{\beta}, \vec{\boldsymbol{e}}_{\gamma}] = c_{\beta\gamma}{}^{\alpha} \vec{\boldsymbol{e}}_{\alpha} \tag{3.123}$$

$$\begin{aligned}
&= [L^{\bar{\beta}}{}_{\beta} \partial_{\bar{\beta}}, L^{\bar{\gamma}}{}_{\gamma} \partial_{\bar{\gamma}}] \\
&= L^{\bar{\beta}}{}_{\beta} L^{\bar{\gamma}}{}_{\gamma,\bar{\beta}} \partial_{\bar{\gamma}} - L^{\bar{\gamma}}{}_{\gamma} L^{\bar{\beta}}{}_{\beta,\bar{\gamma}} \partial_{\bar{\beta}} \\
&= (L^{\bar{\gamma}}{}_{\beta} L^{\bar{\beta}}{}_{\gamma,\bar{\gamma}} - L^{\bar{\gamma}}{}_{\gamma} L^{\bar{\beta}}{}_{\beta,\bar{\gamma}}) L^{\alpha}{}_{\bar{\beta}} \vec{\boldsymbol{e}}_{\alpha},
\end{aligned} \tag{3.124}$$

and since $L^{\alpha}{}_{\bar{\beta}}L^{\bar{\beta}}{}_{\gamma} = \delta^{\alpha}{}_{\gamma}$ we have $L^{\alpha}{}_{\bar{\beta}}L^{\bar{\beta}}{}_{\gamma,\bar{\gamma}} = -L^{\alpha}{}_{\bar{\beta},\bar{\gamma}}L^{\bar{\beta}}{}_{\gamma}$ therefore

$$c_{\beta\gamma}{}^{\alpha} = L^{\alpha}{}_{\bar{\beta},\bar{\gamma}}(L^{\bar{\gamma}}{}_{\gamma}L^{\bar{\beta}}{}_{\beta} - L^{\bar{\gamma}}{}_{\beta}L^{\bar{\beta}}{}_{\gamma}). \tag{3.125}$$

Substituting (3.125) in (3.121) and using (3.122) gives the desired equation (3.120).

ii. The second Cartan structure equation is

$$\widetilde{\boldsymbol{d}}g_{\alpha\beta} = \widetilde{\boldsymbol{\omega}}_{\alpha\beta} + \widetilde{\boldsymbol{\omega}}_{\beta\alpha}. \tag{3.126}$$

This can easily be proven as follows

$$\widetilde{\boldsymbol{\omega}}_{\alpha\beta} + \widetilde{\boldsymbol{\omega}}_{\beta\alpha} = (\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma})\widetilde{\boldsymbol{\omega}}^{\gamma} = g_{\alpha\beta,\gamma}\widetilde{\boldsymbol{\omega}}^{\gamma} = \widetilde{\boldsymbol{d}}g_{\alpha\beta}, \tag{3.127}$$

where in the second last equality we used Eq. (3.37). Note that even if we proved the Cartan Eq. (3.126) for a coordinate basis it holds generally.

iii. The *curvature 2-form* is defined as follows

$$\mathscr{R}^{\alpha}{}_{\beta} = \widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} + \widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\sigma} \wedge \widetilde{\boldsymbol{\omega}}^{\sigma}{}_{\beta}. \tag{3.128}$$

The third Cartan structure equation is

$$\mathscr{R}^{\alpha}{}_{\beta} = \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\delta}. \tag{3.129}$$

We will now outline the proof of this equation. Let us develop the two terms on the right hand side of Eq. (3.128). The first

$$\begin{aligned}
\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} &= \widetilde{\boldsymbol{d}}\left(\Gamma^{\alpha}{}_{\beta\delta}\widetilde{\boldsymbol{\omega}}^{\delta}\right) \\
&= \Gamma^{\alpha}{}_{\beta\delta,\gamma}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\delta} \\
&= (\Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta})\widetilde{\boldsymbol{\omega}}^{\gamma} \otimes \widetilde{\boldsymbol{\omega}}^{\delta}.
\end{aligned} \tag{3.130}$$

The second

$$\begin{aligned}
\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\sigma} \wedge \widetilde{\boldsymbol{\omega}}^{\sigma}{}_{\beta} &= \Gamma^{\alpha}{}_{\sigma\gamma}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \Gamma^{\sigma}{}_{\beta\delta}\widetilde{\boldsymbol{\omega}}^{\delta} \\
&= (\Gamma^{\alpha}{}_{\sigma\gamma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma})\widetilde{\boldsymbol{\omega}}^{\gamma} \otimes \widetilde{\boldsymbol{\omega}}^{\delta}.
\end{aligned} \tag{3.131}$$

The right hand side in the Cartan Eq. (3.129)

$$\frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\delta} = R^{\alpha}{}_{\beta\gamma\delta}\widetilde{\boldsymbol{\omega}}^{\gamma} \otimes \widetilde{\boldsymbol{\omega}}^{\delta}, \tag{3.132}$$

due to the antisymmetry of Riemann respect to its last two indexes.

Putting together Eqs. (3.130), (3.131), and (3.132) and recalling the expression (3.94) for the components of the Riemann tensor in a coordinate basis proves Cartan equation (3.129).

J. The sphere

A *sphere* is *the* surface of constant positive curvature.

Spherical coordinates in flat space

Spherical coordinates in flat space are as follows

$$\begin{cases} \vec{\boldsymbol{e}}_r = \frac{\partial}{\partial r} & , \quad r \quad \text{radius} \\ \vec{\boldsymbol{e}}_\theta = \frac{\partial}{\partial \theta} & , \quad \theta \quad \text{polar angle} \\ \vec{\boldsymbol{e}}_\varphi = \frac{\partial}{\partial \varphi} & , \quad \varphi \quad \text{azimuthal angle} \end{cases} \tag{3.133}$$

This is a coordinate basis and $[\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta] = 0$ for any choice of α and β. The metric already diagonal and is given by

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \tag{3.134}$$

so that

$$||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{3.135}$$

with $g = r^4 \sin^2\theta$.

With a (non coordinate) orthonormal basis

(Ex. 8.6 in Ref. [1]) We find the orthonormal (non coordinate) frame by rescaling the basis vector (3.133) like so

$$\begin{cases} \vec{\boldsymbol{e}}_{\hat{r}} = \frac{\partial}{\partial r} \\ \vec{\boldsymbol{e}}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta} \\ \vec{\boldsymbol{e}}_{\hat{\varphi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{cases} \tag{3.136}$$

so that $g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$ but $[\vec{\boldsymbol{e}}_{\hat{\alpha}}, \vec{\boldsymbol{e}}_{\hat{\beta}}] \neq 0$ whenever $\hat{\alpha} \neq \hat{\beta}$. It can easily be verified that

$$\begin{cases} [\vec{\boldsymbol{e}}_{\hat{r}}, \vec{\boldsymbol{e}}_{\hat{\theta}}] = -\frac{1}{r}\vec{\boldsymbol{e}}_{\hat{\theta}} = c_{\hat{r}\hat{\theta}}{}^{\hat{\theta}} \\ [\vec{\boldsymbol{e}}_{\hat{r}}, \vec{\boldsymbol{e}}_{\hat{\varphi}}] = -\frac{1}{r}\vec{\boldsymbol{e}}_{\hat{\varphi}} = c_{\hat{r}\hat{\varphi}}{}^{\hat{\varphi}} \\ [\vec{\boldsymbol{e}}_{\hat{\theta}}, \vec{\boldsymbol{e}}_{\hat{\varphi}}] = -\frac{1}{r\tan\theta}\vec{\boldsymbol{e}}_{\hat{\varphi}} = c_{\hat{\theta}\hat{\varphi}}{}^{\hat{\varphi}} \end{cases} \tag{3.137}$$

Note that J. D. Jackson book of "Classical Electrodynamics" [3] uses the orthonormal basis. For example the gradient of a scalar ψ

$$\begin{aligned} \boldsymbol{\nabla}\psi &= \vec{\boldsymbol{e}}_\alpha \psi_,{}^\alpha = \vec{\boldsymbol{e}}_\alpha g^{\alpha\beta}\psi_{,\beta} \\ &= \vec{\boldsymbol{e}}_{\hat{r}}\frac{\partial\psi}{\partial r} + \vec{\boldsymbol{e}}_{\hat{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \vec{\boldsymbol{e}}_{\hat{\varphi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}. \end{aligned} \tag{3.138}$$

where we used the fact that $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}$.

For the divergence of a vector $\vec{\boldsymbol{A}} = A^{\hat{\alpha}}\vec{\boldsymbol{e}}_{\hat{\alpha}} = A^\alpha \vec{\boldsymbol{e}}_\alpha$ with

$$\begin{cases} A^{\hat{r}} = A^r \\ A^{\hat{\theta}} = rA^\theta \\ A^{\hat{\varphi}} = r\sin\theta A^\varphi \end{cases} \tag{3.139}$$

we find

$$\begin{aligned} \boldsymbol{\nabla}\vec{\boldsymbol{A}} = A^\alpha{}_{;\alpha} &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^\alpha\right)_{,\alpha} \\ &= \frac{1}{r^2}\left(r^2 A^{\hat{r}}\right)_{,r} + \frac{1}{r\sin\theta}\left(\sin\theta A^{\hat{\theta}}\right)_{,\theta} + \frac{1}{r\sin\theta}A^{\hat{\varphi}}{}_{,\varphi}, \end{aligned} \tag{3.140}$$

where we used identity [iv.] in the subsection "Useful identities" of Section III C.

For the Laplacian of a scalar ψ

$$\begin{aligned} \boldsymbol{\nabla}^2\psi = \psi_{;\alpha}{}^\alpha &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\psi_,{}^\alpha\right)_{,\alpha} \\ &= \frac{1}{r^2}\left(r^2\psi_{,r}\right)_{,r} + \frac{1}{r\sin\theta}\left(\sin\theta\psi_{,\theta}\right)_{,\theta} + \frac{1}{r^2\sin^2\theta}\psi_{,\varphi\varphi}. \end{aligned} \tag{3.141}$$

where we used identity [vii.] in the subsection "Useful identities" of Section III C.

Cartesian coordinates

A sphere can be embedded in the 3 dimensional flat space. Here we can as usual choose a Cartesian coordinate system

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\sqrt{x^2+y^2}/z\right) \\ \varphi = \arctan\left(y/x\right) \end{cases} \tag{3.142}$$

From Eq. (1.6) we can determine the basis vectors

$$\begin{cases} \vec{\boldsymbol{e}}_r = \vec{\boldsymbol{e}}_x \frac{x}{r} + \vec{\boldsymbol{e}}_y \frac{y}{r} + \vec{\boldsymbol{e}}_z \frac{z}{r} \\ \vec{\boldsymbol{e}}_\theta = \vec{\boldsymbol{e}}_x \frac{xz}{r^2\sqrt{x^2+y^2}} + \vec{\boldsymbol{e}}_y \frac{yz}{r^2\sqrt{x^2+y^2}} + \vec{\boldsymbol{e}}_z \left(-\frac{\sqrt{x^2+y^2}}{r^2}\right) \\ \vec{\boldsymbol{e}}_\varphi = \vec{\boldsymbol{e}}_x \left(-\frac{y}{x^2+y^2}\right) + \vec{\boldsymbol{e}}_y \frac{x}{x^2+y^2} \end{cases} \tag{3.143}$$

and for the versors

$$\begin{cases} \hat{r} = \frac{\vec{\boldsymbol{e}}_r}{|\vec{\boldsymbol{e}}_r|} \quad |\vec{\boldsymbol{e}}_r| = \sqrt{\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}} = 1 \\ \hat{\theta} = \frac{\vec{\boldsymbol{e}}_\theta}{|\vec{\boldsymbol{e}}_\theta|} \quad |\vec{\boldsymbol{e}}_\theta| = \sqrt{\frac{x^2z^2}{r^4(x^2+y^2)} + \frac{y^2z^2}{r^4(x^2+y^2)} + \frac{x^2+y^2}{r^4}} = \frac{1}{r} \\ \hat{\varphi} = \frac{\vec{\boldsymbol{e}}_\varphi}{|\vec{\boldsymbol{e}}_\varphi|} \quad |\vec{\boldsymbol{e}}_\varphi| = \sqrt{\frac{y^2}{(x^2+y^2)^2} + \frac{x^2}{(x^2+y^2)^2}} = \frac{1}{r\sin\theta} \end{cases} \tag{3.144}$$

Curvature of the sphere in a simple way

We will here use the results of Section III I to determine with the Cartan structure equations, in a rapid and simple way, the 1 independent component of the Riemann tensor for the sphere, the surface for which r is constant.

The 1-form orthonormal basis

$$\begin{cases} \widetilde{\boldsymbol{\omega}}^{\hat{\theta}} = r\widetilde{\boldsymbol{d}}\theta \\ \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}} = r\sin\theta\widetilde{\boldsymbol{d}}\varphi \end{cases} \tag{3.145}$$

so that $ds^2 = \eta_{\hat{\mu},\hat{\nu}}\widetilde{\boldsymbol{\omega}}^{\hat{\mu}} \otimes \widetilde{\boldsymbol{\omega}}^{\hat{\nu}}$. From Cartan structure equation (3.126) it must be $\widetilde{\boldsymbol{\omega}}_{\hat{\mu},\hat{\nu}} + \widetilde{\boldsymbol{\omega}}_{\hat{\nu},\hat{\mu}} = 0$ or $\widetilde{\boldsymbol{\omega}}^{\hat{\mu}}{}_{\hat{\nu}} + \widetilde{\boldsymbol{\omega}}_{\hat{\nu}}{}^{\hat{\mu}} = 0$ so that

$$\begin{cases} \widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\theta}} = \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\varphi}} = 0 \\ \widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} = -\widetilde{\boldsymbol{\omega}}_{\hat{\varphi}}{}^{\hat{\theta}} = -\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\theta}} = 0 \end{cases} \tag{3.146}$$

From the properties of the external derivative and from Cartan structure equation (3.120) it must be $\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\hat{\theta}} = \widetilde{\boldsymbol{d}}(r\widetilde{\boldsymbol{d}}\theta) = r\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{d}}\theta = 0 = -\widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} \wedge \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}$. So it must be either $\widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} = 0$ or $\widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} \propto \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}$. The other basis 1-form gives

$$\begin{aligned} \widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}} &= \widetilde{\boldsymbol{d}}(r\sin\theta\widetilde{\boldsymbol{d}}\varphi) \\ &= r\cos\theta\widetilde{\boldsymbol{d}}\theta \wedge \widetilde{\boldsymbol{d}}\varphi \\ &= \frac{\cot\theta}{r}\widetilde{\boldsymbol{\omega}}^{\hat{\theta}} \wedge \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}} \\ &= -\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\theta}} \propto \widetilde{\boldsymbol{\omega}}^{\hat{\theta}}. \end{aligned} \tag{3.147}$$

So we find that

$$\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\theta}} = \frac{\cot\theta}{r}\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}. \tag{3.148}$$

From Cartan structure equation (3.129) and using the result of Eq. (3.146) then follows

$$
\begin{aligned}
\mathscr{R}^{\hat\theta}{}_{\hat\varphi} &= \widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\hat\varphi}{}_{\hat\theta} \\
&= \widetilde{\boldsymbol{d}}\left(-\frac{\cot\theta}{r}\widetilde{\boldsymbol{\omega}}^{\hat\varphi}\right) \\
&= \widetilde{\boldsymbol{d}}(-\cos\theta\,\widetilde{\boldsymbol{d}}\varphi) \\
&= \sin\theta\,\widetilde{\boldsymbol{d}}\theta\wedge\widetilde{\boldsymbol{d}}\varphi \\
&= \frac{1}{r^2}\widetilde{\boldsymbol{\omega}}^{\hat\theta}\wedge\widetilde{\boldsymbol{\omega}}^{\hat\varphi} \\
&= \frac{1}{2}R^{\hat\theta}{}_{\hat\varphi\hat\alpha\hat\beta}\widetilde{\boldsymbol{\omega}}^{\hat\alpha}\wedge\widetilde{\boldsymbol{\omega}}^{\hat\beta}.
\end{aligned}
\tag{3.149}
$$

So we reach the result that the only independent Riemann component is

$$
R^{\hat\theta}{}_{\hat\varphi\hat\theta\hat\varphi} = 1/r^2. \tag{3.150}
$$

The scalar curvature is then

$$
R = R^{\hat\alpha\hat\beta}{}_{\hat\alpha\hat\beta} = 2R^{\hat\theta\hat\varphi}{}_{\hat\theta\hat\varphi} = 2/r^2. \tag{3.151}
$$

Curvature of the sphere in a hard way

We work in the global (non coordinate) frame (3.136). Due to the antisymmetry of Riemann $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ respect to its first two indexes (3.98) and respect to its last two indexes (3.96), for $\hat\alpha=\hat\beta$ or for $\hat\gamma=\hat\delta$ Riemann vanishes. Recall that Riemann is given by Eq. (3.94)

$$
R_{\hat\alpha\hat\beta\hat\gamma\hat\delta} = \bar{R}_{\hat\alpha\hat\beta\hat\gamma\hat\delta} + \Gamma^{\hat\mu}{}_{\hat\delta\hat\beta}\Gamma_{\hat\alpha\hat\gamma\hat\mu} - \Gamma^{\hat\mu}{}_{\hat\gamma\hat\beta}\Gamma_{\hat\alpha\hat\delta\hat\mu} - c_{\hat\gamma\hat\delta}{}^{\hat\mu}\Gamma_{\hat\alpha\hat\mu\hat\beta}, \tag{3.152}
$$

$$
\bar{R}_{\hat\alpha\hat\beta\hat\gamma\hat\delta} = \Gamma_{\hat\alpha\hat\delta\hat\beta,\hat\gamma} - \Gamma_{\hat\alpha\hat\gamma\hat\beta,\hat\delta}, \tag{3.153}
$$

where by Eq. (3.34)

$$
\Gamma_{\hat\alpha\hat\beta\hat\gamma} = \frac{1}{2}\{c_{\hat\alpha\hat\beta\hat\gamma} + c_{\hat\beta\hat\gamma\hat\alpha} - c_{\hat\gamma\hat\alpha\hat\beta}\}. \tag{3.154}
$$

Remember that we can freely bring up or down indexes since $g_{\hat\alpha\hat\beta}=\eta_{\hat\alpha\hat\beta}$ and note that the 3 indexes of the c commutation coefficients in Γ appear cyclically. Moreover $\bar{R}$ must have the same symmetry properties of Riemann.

From the commutation coefficients of Eq. (3.137) the only relevant commutation coefficients c is $c_{\hat\theta\hat\varphi\hat\varphi}=-\cot\theta/r$ where we have antisymmetry in the first two indexes. Due to the above mentioned symmetry properties of Riemann its only non-zero component is $R_{\hat\theta\hat\varphi\hat\theta\hat\varphi}$.

For the connection coefficients (3.154) we find

$$
\begin{cases}
\Gamma_{\hat\theta\hat\varphi\hat\varphi} = c_{\hat\theta\hat\varphi\hat\varphi}, \\
\Gamma_{\hat\varphi\hat\theta\hat\varphi} = 0, \\
\Gamma_{\hat\varphi\hat\varphi\hat\theta} = -c_{\hat\theta\hat\varphi\hat\varphi},
\end{cases}
\tag{3.155}
$$

where the components with 2 or all 3 indexes equal to $\hat\theta$ or with all 3 indexes equal to $\hat\varphi$ must vanish. From the results of Eq. (3.155). We can easily verify that $R_{\hat\theta\hat\varphi\hat\theta\hat\varphi}-\bar{R}_{\hat\theta\hat\varphi\hat\theta\hat\varphi}=0$.

In order to determine $R_{\hat\theta\hat\varphi\hat\theta\hat\varphi}=\bar{R}_{\hat\theta\hat\varphi\hat\theta\hat\varphi}=\Gamma_{\hat\theta\hat\varphi\hat\varphi,\hat\theta}-\Gamma_{\hat\theta\hat\theta\hat\varphi,\hat\varphi}$ we need to calculate

$$
R_{\hat\theta\hat\varphi\hat\theta\hat\varphi} = c_{\hat\theta\hat\varphi\hat\varphi,\hat\theta} = \frac{\partial\left[\left(-\frac{\cot\theta}{r}\right)\,r\sin\theta\right]}{r\partial\theta}\frac{1}{r\sin\theta} = 1/r^2, \tag{3.156}
$$

where in order to carry out the derivative respect to $\hat\theta$ we had to use the change of basis of Eq. (3.145). An even harder route is the calculation in the usual polar coordinates basis.

IV. PHYSICS

According to Einstein (Ulma, 14 March 1879 - Princeton, 18 April 1955) the arena for Physics is a pseudo Riemannian 4-dimensional manifold $\mathscr{M}$ where a point $\mathscr{P}=(x^0,x^1,x^2,x^4)$ describes an **event** at a given time $t=x^0/c$ in a given place in space $\boldsymbol{r}=(x^1,x^2,x^3)$, where c is the speed of light constant. On the tangent LLF, at a $\mathscr{P}$, *Special Relativity* (SR) holds with Minkowski metric $\eta_{\alpha\beta}=\mathrm{diag}(-1,1,1,1)$. On a global frame *General Relativity* (GR) holds with $g_{\alpha\beta}(x^\gamma)$ a metric field determined by Einstein field equations described in Section IV B. In this section bold face letters without any other decoration describe 3-dimensional vectors.

A. Electromagnetism

According to Einstein **strong** *equivalence principle* (SEP) **all** laws of Physics should be written in the same **form** in a LLF or in a global frame on the manifold $\mathscr{M}$.

Electrostatics

Charles-Augustin de Coulomb (Angoulême, 14 June 1736 - Paris, 23 August 1806) discovered the mathematical law of interaction between two charges of electrical charge q_1 nd q_2 separated by a distance r. *Coulomb force* (in Gauss units)

$$\boldsymbol{F}_{12}=\boldsymbol{r}\frac{q_1q_2}{r^3}, \tag{4.1}$$

gives rise to an *electric field* around charge one $\boldsymbol{E}(r)=\boldsymbol{F}_{12}/q_2$. The electric field is generated by an electric potential $\boldsymbol{E}(r)=-\boldsymbol{\nabla}\varphi(r)$ with $\varphi(r)=q/r$, the Coulomb potential. The Coulomb potential satisfies to the equation of Baron Simón Denis Poisson (Pithiviers, 21 June 1781 - Paris, 25 April 1840) [9]

$$\boldsymbol{\nabla}^2\varphi(r)=-4\pi q\delta^3(\boldsymbol{r}), \tag{4.4}$$

where δ^3 is a Dirac delta function in 3 dimensions. Poisson equation is the equation of Pierre-Simon, Marquis de Laplace (Beaumont-en-Auge, 23 March 1749 - Paris, 5 March 1827) with a source term due to the charge q. Later Johann Carl Friedrich Gauss (Braunschweig, 30 April 1777 - Gottinga, 23 February 1855) discovered that

$$4\pi q=-\int_\Omega\boldsymbol{\nabla}^2\varphi(r)\,d\boldsymbol{r}=-\int_{\partial\Omega}\boldsymbol{n}\cdot\boldsymbol{\nabla}\varphi(r)\,dS=\int_{\partial\Omega}\boldsymbol{n}\cdot\boldsymbol{E}(r)\,dS=\Phi_E, \tag{4.5}$$

which states the important mathematical result that the flux Φ_E of the electric field through *any* closed surface containing charge q is fixed. In Eq. (4.5) $d\boldsymbol{r}$ is the infinitesimal volume integral, $\boldsymbol{n}dS$ is the infinitesimal surface element with $\boldsymbol{n}$ its outward normal versor, Ω is the volume region considered in the volume integral, and $\partial\Omega$ is its bounding surface.

The representation of the electron as a pointwise particle poses the problem of an infinite self-energy diverging as $1/r$. On the other side the electrostatic energy for assembling a system on N point charges of charge q_i is that required to bring them close together from infinity

$$\mathscr{E}=\frac{1}{2}\sum_{i\neq j=1}^N\frac{q_iq_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}, \tag{4.6}$$

But again a divergence problem arises as soon as one introduces a charge density $\rho(\boldsymbol{r})$ to rewrite this energy with a continuous expression

$$\mathscr{E}=\frac{1}{2}\int d\boldsymbol{r}\int d\boldsymbol{r}'\,\frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{4.7}$$

---

[9] For charges *living* [7] in n-dimensions we have

$$\varphi(r)=q\begin{cases}1/r & n=3\\ -\ln(r/\ell) & n=2\\ -r & n=1\end{cases}, \tag{4.2}$$

where ℓ is a length. And the Poisson equation becomes

$$\boldsymbol{\nabla}^2\varphi(r)=-q\delta^n(\boldsymbol{r})\begin{cases}4\pi & n=3\\ 2\pi & n=2\\ 2 & n=1\end{cases}, \tag{4.3}$$

where δ^n is a Dirac delta function in n dimensions.

where one readily recognize a divergence for a linear, planar, or spatial charge density. Also in these cases is necessary to deal with infinities.

Maxwell equations

From the first discoveries of electrostatics soon enough James Clerk Maxwell (Edimburgh, 13 June 1831 – Cambridge, 5 November 1879) wrote his equations for electrodynamics. The most synthetic way to write these important equations describing electromagnetism is through the geometric language of the differential forms (here we use Gauss units and set additionally the speed of light $c = 1$)

$$\tilde{\boldsymbol{d}}\, \tilde{\boldsymbol{F}} = \boldsymbol{0}, \tag{4.8}$$

$$\tilde{\boldsymbol{d}}\, \star \tilde{\boldsymbol{F}} = 4\pi\, \star \tilde{\boldsymbol{J}}. \tag{4.9}$$

Here $\tilde{\boldsymbol{d}}$ stands for an exterior derivative (see Section II A), $\star$ is the Hodge star that stands for the dual, $\boldsymbol{F} = \tilde{\boldsymbol{d}}\boldsymbol{A}$ is the **Faraday** two form that subtend the electromagnetic antisymmetric tensor $F_{\mu\nu}$ containing the electric and magnetic fields (6 components, 6 basis 2-forms)

$$\tilde{\boldsymbol{F}} = \frac{1}{2} F_{\alpha\beta} \tilde{\boldsymbol{d}}x^\alpha \wedge \tilde{\boldsymbol{d}}x^\beta \tag{4.10}$$

$$= E_x \tilde{\boldsymbol{d}}x \wedge \tilde{\boldsymbol{d}}t + E_y \tilde{\boldsymbol{d}}y \wedge \tilde{\boldsymbol{d}}t + E_z \tilde{\boldsymbol{d}}z \wedge \tilde{\boldsymbol{d}}t + B_x \tilde{\boldsymbol{d}}y \wedge \tilde{\boldsymbol{d}}z + B_y \tilde{\boldsymbol{d}}z \wedge \tilde{\boldsymbol{d}}x + B_z \tilde{\boldsymbol{d}}x \wedge \tilde{\boldsymbol{d}}y, \tag{4.11}$$

$\vec{\boldsymbol{A}} = (\varphi, \boldsymbol{A})$ is the electromagnetic 4-potential one form where φ is the electric scalar potential for the electric field $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \partial \boldsymbol{A}/\partial t$ and $\boldsymbol{A}$ the magnetic vector potential for the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, $\star\tilde{\boldsymbol{F}}$ is **Maxwell** two form dual to Faraday, and $\star\tilde{\boldsymbol{J}}$ is the **charge** three form with $\tilde{\boldsymbol{J}} = (\rho, \boldsymbol{J})$ the 4-current density one form with ρ the electric charge density and $\boldsymbol{J}$ the electric current density. So that the total charge Q inside a three dimensional hypersurface region $\mathscr{S}$ is $Q = \int_{\mathscr{S}} \star\tilde{\boldsymbol{J}}$. Also from $\tilde{\boldsymbol{d}}\tilde{\boldsymbol{d}}\star\tilde{\boldsymbol{F}} = \boldsymbol{0}$ follows $\tilde{\boldsymbol{d}}\star\tilde{\boldsymbol{J}} = \boldsymbol{0}$ which is the law of conservation of charge. Eq. (4.8) summarizes Faraday's law and the non-existence of magnetic monopoles and it is a consequence of the general result that $\tilde{\boldsymbol{d}}\tilde{\boldsymbol{d}} = \boldsymbol{0}$. Eq. (4.9) summarizes Ampere's law with Maxwell's correction to take into account of the displacement current and Gauss's law. The importance of the formulation of Eqs. (4.8)-(4.9) lies in the fact that written in the differential form language, Maxwell equations have the same form in Special Relativity or in General Relativity thanks to the strong equivalence principle. This is tantamount to assume that such formulation is appropriate also in any riemannian manifold.

The Maxwell equations are invariant under the *gauge transformation* $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\psi$ and $\varphi \to \varphi - \partial\psi/\partial t$ with the gauge function $\psi(t, \boldsymbol{r})$ any scalar. Which means that electromagnetism has $U(1)$ *gauge freedom*.

Now, start with the scalar φ. Its gradient $\tilde{\boldsymbol{d}}\varphi$ is a one form. Take its dual to get the three form $\star\tilde{\boldsymbol{d}}\varphi$. Take its exterior derivative to get the four form $\tilde{\boldsymbol{d}}\star\tilde{\boldsymbol{d}}\varphi$. Take its dual, to get the scalar $-\star\tilde{\boldsymbol{d}}\star\tilde{\boldsymbol{d}}\varphi = \Box\,\varphi = -(\partial^2\varphi/\partial t^2) + \boldsymbol{\nabla}^2\varphi$. This is the Jean-Baptiste le Rond d'Alembert (16 November 1717 - 29 October 1783) wave operator.

Start with the one form $\tilde{\boldsymbol{A}}$. Get the two form $\tilde{\boldsymbol{d}}\tilde{\boldsymbol{A}}$. Take its dual to get the two form $\star\tilde{\boldsymbol{d}}\tilde{\boldsymbol{A}}$. Take its exterior derivative to get the three form $\tilde{\boldsymbol{d}}\star\tilde{\boldsymbol{d}}\tilde{\boldsymbol{A}}$. Take its dual, to get the one form $4\pi\tilde{\boldsymbol{J}} = \star\tilde{\boldsymbol{d}}\star\tilde{\boldsymbol{d}}\tilde{\boldsymbol{A}}$. This is the wave equation for the electromagnetic 4-potential. And from here follow the electromagnetic waves. For example for the zero component in vacuum in absence of charges one finds $\Box\,\varphi = 0$ whose solution with forward and backward propagation along the direction $\boldsymbol{k}$ is of the form $\varphi(t, \boldsymbol{r}) = f(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) + g(\boldsymbol{k}\cdot\boldsymbol{r} + \omega t)$, where $\omega = 2\pi/T$ is the angular frequency of the wave of period T, $k = 2\pi/\lambda$ is the wave vector for a wavelength λ, the speed of the wave is $\omega/k = \lambda/T = c = 1$, and f, g are arbitrary functions. In spherical symmetry [10] one would otherwise have a spherical wave solution of the following kind, $\varphi(t, r) = [F(kr - \omega t) + G(kr + \omega t)]/r$ with F, G arbitrary functions. Or a Green function $G(t, r) = \delta(kr - \omega t)/r$ forward solution of $\Box\, G = -4\pi\delta^{(4)}(\vec{\boldsymbol{x}})$.

In flat spacetime, express the coordinates of one electron as a function of his proper time as $a^\mu(\tau)$. The density-current 4-vector for this electron is then

$$J^\mu(\vec{\boldsymbol{x}}) = e \int \delta^{(4)}[\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}(\tau)]\, \dot{a}^\mu d\tau, \tag{4.12}$$

[10] Note that in spherical symmetry Eq. (3.141) for the Laplacian reduces to $\boldsymbol{\nabla}^2\psi = (r\psi)_{,rr}/r$.

where $\vec{\boldsymbol{x}} = (t, \boldsymbol{r})$ and as usual we denote with the dot a partial time derivative. This density-current drives the electromagnetic field or $\widetilde{\boldsymbol{F}}$. Then Maxwell equation (4.9) becomes $F_\mu{}^\nu{}_{,\nu} = 4\pi J_\mu$ where as usual the comma stands for a partial derivative. Or $A^\nu{}_{,\nu\mu} - \eta^{\nu\alpha} A_{\mu,\alpha\nu} = 4\pi J_\mu$ where $\eta_{\mu\nu}$ is the metric of the Lorentz coordinate system of the flat spacetime. Make use of the gauge freedom to set *Lorentz gauge*, $A^\nu{}_{,\nu} = 0$, to get

$$\Box A_\mu = -4\pi J_\mu. \tag{4.13}$$

This can be solved through the Green's function method rewriting $A^\mu(\vec{\boldsymbol{x}}) = e \int G[\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}(\tau)]\, \dot{a}^\mu d\tau$. The causal solution Eq. (4.13) is then given in terms of the retarded potential

$$A_\mu(t, \boldsymbol{r}) = \int G(t - t', |\boldsymbol{r} - \boldsymbol{r}'|) J_\mu(t', \boldsymbol{r}')\, d\boldsymbol{r}' dt' = \int \frac{J_\mu(t_{\text{retarded}}, \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d\boldsymbol{r}', \tag{4.14}$$

$$t_{\text{retarded}} = t - |\boldsymbol{r} - \boldsymbol{r}'|, \tag{4.15}$$

where remember that we chose the speed of light $c = 1$ and we carried out the integration over t'.

B. Gravitation

According to Einstein **weak** *equivalence principle* (WEP) the **laws of motion** should be written in the same **form** in a LLF or in a global frame on the manifold $\mathscr{M}$.

(Chapter 16 of Ref. [1]) Consider a 4-dimensional pseudo Riemannian manifold $\mathscr{M}$ describing our "spacetime". Let the "geometry of spacetime" describe the "gravitational field": $g_{\alpha\beta}, \Gamma^\alpha{}_{\beta\gamma}, R^\alpha{}_{\beta\gamma\delta}, \ldots$. And consider the Riemann tensor as the "true gravitational field".

A measure of $g_{\alpha\beta}$ then is a measure of the gravitational field. Measure $\Gamma^\alpha{}_{\beta\gamma}$ from geodesics (see Section III E) and $R^\alpha{}_{\beta\gamma\delta}$ from geodesics deviation (see Section III H). A point of the manifold describes an event at a given time t in a point of space $\boldsymbol{r}$, $\mathscr{P} = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{r})$. At a point $\mathscr{P}$ we can always choose a LLF (see Section III D) where $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, $\Gamma^{\hat{\mu}}{}_{\hat{\nu}\hat{\gamma}} = 0$ and *Special Relativity* (SR) holds.

In the spirit of a field theory we regard a "test particle" as structureless and moving on a unique straight line in a LLF or a geodesic globally, i.e. "freely falling". Spacetime is filled with a congruence of test particles geodesics. One and only one at each point in each direction.

The stress-energy tensor

(Chapter 5 of Ref. [2]) A proper description of the energy, momentum, and stress of a relativistic fluid or field uses the symmetric tensor $\mathbb{T}$, the *stress-energy tensor* or *energy-momentum* tensor. It describes the momentum *density* and energy *"flux"* at each point in spacetime. The contravariant components of this tensor in a LLF of an observer are related by the measurements of that observer as follows

$$\begin{cases} T^{00} = \rho = \text{mass-energy density} \\ T^{0j} = j\text{-component of energy flux} \quad (=) \quad T^{j0} = j\text{-component of momentum density} \\ T^{ij} = \text{components of the ordinary stress tensor} \end{cases} \tag{4.16}$$

where we denote with a roman index just the three spatial components.

Newton (Colsterworth, 25 December 1642 - London, 20 March 1727) equations of motion $\boldsymbol{F} = m\boldsymbol{a}$ in an infinitesimal cubic box [11] around a point $\mathscr{P}_0$ gives $T^{j\alpha}{}_{,\alpha}(\mathscr{P}_0) = 0$. And the rate of change of energy in the box $T^{00}{}_{,0}(\mathscr{P}_0)$ has to equal the energy flux through the box $-T^{0j}{}_{,j}(\mathscr{P}_0)$. Summarizing we conclude that if $\mathbb{T}$ describes all particles, fluids, fields, ... the interrelation of momentum and energy change is summarized by the following *equation of motion*

$$T^{\mu\nu}{}_{,\nu} = 0. \tag{4.17}$$

For example ($c = 1$):

---

[11] For example the component T^{jj} measures the pressure (force per unit area) on the face of the box orthogonal to the direction j.

i. For an *isolated particle* with rest mass m, on a curve $\vec{z}(\tau)$, a 4-velocity $\vec{u} = d\vec{z}/d\tau$, and trajectory $\boldsymbol{q}(t)$

$$T^{\mu\nu}(t,\boldsymbol{r}) = m\int u^\mu u^\nu \delta^{(4)}(\vec{x} - \vec{z}(\tau))\,d\tau$$
$$= m\gamma v^\mu(t) v^\nu(t)\delta^{(3)}(\boldsymbol{r} - \boldsymbol{q}(t)), \tag{4.18}$$

where $u^0 = \gamma = 1/\sqrt{1-v^2}$, $\vec{v} = (1, d\boldsymbol{q}/dt)$ is the velocity vector and $\delta^{(n)}$ is the n-dimensional Dirac delta function.

ii. For a "swarm" of particles: a region of spacetime filled with particles all of the same rest mass m and 4-velocity $\vec{u}$. Let n be the proper number density measured in a *comoving frame* where $\vec{u} = (1, \vec{\mathbf{0}})$, then

$$T^{\mu\nu} = mnu^\mu u^\nu. \tag{4.19}$$

iii. For a perfect fluid

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{4.20}$$

where ρ is the mass density in the *isotropic frame*, p is the isotropic pressure, [12] and u^μ is the fluid 4-velocity which satisfies $u^\mu u^\nu g_{\mu\nu} = -1$.

iv. For a source-free electromagnetic field

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha} g_{\alpha\beta} F^{\nu\beta} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right), \tag{4.21}$$

which is clearly traceless and we can easily see that given an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$ [10]

$$\begin{cases} T^{00} = (E^2 + B^2)/8\pi \\ T^{0j} = (\boldsymbol{E} \times \boldsymbol{B})_j/4\pi = S_j \\ T^{ij} = T^{00} \end{cases} \tag{4.22}$$

where $\boldsymbol{S}$ is the Poynting vector. In the presence of sources $\widetilde{\boldsymbol{J}}$

$$\begin{aligned} 4\pi T_\alpha{}^\beta{}_{,\beta} &= \left(F_{\alpha\mu}F^{\beta\mu} - \frac{1}{4}\delta_\alpha{}^\beta F_{\mu\nu}F^{\mu\nu}\right)_{,\beta} \\ &= F_{\alpha\mu,\beta}F^{\beta\mu} + F_{\alpha\mu}F^{\beta\mu}{}_{,\beta} - \frac{1}{2}\delta_\alpha{}^\beta F_{\mu\nu,\beta}F^{\mu\nu} \\ &= F_{\alpha\mu,\nu}F^{\nu\mu} - 4\pi F_{\alpha\mu}J^\mu - \frac{1}{2}F_{\mu\nu,\alpha}F^{\mu\nu} \\ &= -4\pi F_{\alpha\mu}J^\mu - \frac{1}{2}F^{\mu\nu}(F_{\alpha\mu,\nu} + F_{\nu\alpha,\mu} + F_{\mu\nu,\alpha}) \\ &= -4\pi F_{\alpha\mu}J^\mu, \end{aligned} \tag{4.23}$$

where in the third equality we used Maxwell equation (4.9) and in the last equality we used Maxwell equation (4.8). So that

$$T^{\alpha\beta}{}_{,\beta} = -F^{\alpha\mu}J_\mu. \tag{4.24}$$

[12] We will have $\rho = mn\gamma$ and $p = \rho v^2/3$, so that for example for photons $p \to (1/3)\rho$ (see the "black-body radiation" section §63 of Ref. [8]) and for non relativistic fluids $\rho \to mn(1 + v^2/2 + \ldots)$ so that $p \to mnv^2/3 \to (2/3)(\rho - mn) \ll \rho$ (remember that for the equipartition theorem [9], for an ideal gas, $\mathscr{E}_k = (3/2)k_BT$, where $\mathscr{E}_k$ is the average kinetic energy, and the equation of state is $p = nk_BT$) where $\rho - mn \to \mathscr{E}_k$.

The "comma goes to semicolon" rule

To implement the SEP one uses the "comma goes to semicolon" rule

$$, \qquad \longrightarrow \qquad ; \tag{4.25}$$

For example the stress-energy tensor $\mathbb{T}$ is divergenceless in SR

$$\nabla\mathbb{T} = 0 \quad \xrightarrow{SR} \quad T^{\hat{\mu}\hat{\nu}}{}_{,\hat{\nu}} = 0 \quad \xrightarrow{GR} \quad T^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}} = 0, \tag{4.26}$$

then it is divergenceless also in GR.

Another example are Maxwell equations of ElectroMagnetism described in the previous Section IV A which are written in the same form (4.8), (4.9) in a LLF and in $\mathscr{M}$.

Some caution is needed when applying the ", $\to$;" rule because in a LLF partial derivatives commute but globally covariant derivatives do not, as is shown in subection "Commutation of covariant derivatives" of Section III G. For example Faraday $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ with $\boldsymbol{A}$ the 4-potential. The 4 Maxwell equations in components

$$\begin{cases} F_{[\mu\nu,\rho]} = 0 \\ F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu \end{cases} \tag{4.27}$$

So the last 2 equations become in a LLF $A^{\nu,\mu}{}_{,\nu} - A^{\mu,\nu}{}_{,\nu} = 4\pi J^\mu$ or $A^\nu{}_{,\nu}{}^\mu - A^\mu{}_{,\nu}{}^\nu = 4\pi J^\mu$, but globally $A^{\nu;\mu}{}_{;\nu} = A^\nu{}_{;\nu}{}^\mu + R^\mu{}_\nu A^\nu$ according to Eq. (3.100) and the definition of Ricci (3.105). The last term is a curvature coupling which is experimentally negligible. In any case the former form is regarded as the correct one conventionally. Other examples are the ones where you cannot treat the system as localized in a LLF (Ex. 11.8 and 11.9 in Ref. [2]).

Einstein field equations

(Chapter 17 of Re. [1]) We want to determine the response of a gravitational field to matter. The mass-energy density, $\rho = u_\alpha u_\beta T^{\alpha\beta}$ for some observer $\vec{\boldsymbol{u}}$, is the source of gravity. We then generally think at the stress-energy tensor $\mathbb{T}$ as the machine encompassing all sources of gravity which will linearly determine the response of the gravitational field like so

$$\mathbb{H} = \chi\mathbb{T}, \tag{4.28}$$

where $\mathbb{H}$ will be a second rank symmetric tensor characterizing the spacetime geometry

$$\mathbb{H} = H(g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\delta}, \ldots). \tag{4.29}$$

Equation (4.28) cannot determine all 10 components of $g_{\alpha\beta}$ uniquely because exist 4 differentiable functions ($x^{\bar{\alpha}} = x^{\bar{\alpha}}(x^\alpha)$) to make coordinate transformation leaving $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ invariant. But $\nabla\mathbb{T} = 0$ are 4 equations. So we will have only 6 independent field equations. This solves the dilemma: 6 constraints on 10 components $g_{\alpha\beta}$ leaving only 4 components of $g_{\alpha\beta}$ to be determined by coordinate transformation.

Since in a LLF $g_{\alpha\beta,\gamma} = 0$ (see Section III D) in $\mathbb{H}$ we need at least second derivatives of $g_{\alpha\beta}$ otherwise it would reduce to $\boldsymbol{g}$ itself and multiples thereof and one would not be able to recover the Newtonian (weak field) limit where $\boldsymbol{\nabla}^2\phi = 4\pi G\rho$ with ϕ the gravitational field, ρ the mass density, and G Newton gravitation universal constant, as shown in the next subsection. The only tensor that can be constructed from $g_{\alpha\beta}, g_{\alpha\beta,\gamma}$, and $g_{\alpha\beta,\gamma\delta}$ is Riemann $\mathbb{R}$. But the only tensor that:

i. is second rank and symmetric;

ii. is constructed linearly from $\mathbb{R}$ and $\boldsymbol{g}$;

iii. has vanishing derivatives;

is Einstein tensor

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta}, \tag{4.30}$$

where Λ is a *cosmological constant* [13] and $\mathbb{G}$ is defined in subsection "The Einstein tensor" of Section III G. For $\mathbb{T} = 0$ spacetime must be flat and $\mathbb{G} = 0$ (also set $\Lambda = 0$). So it must be

$$G_{\alpha\beta} = \chi T_{\alpha\beta}, \tag{4.31}$$

which are *Einstein field equations*.

---

[13] That Einstein considered "its biggest mistake".

Newtonian limit

> **Simplicio** Quanto alle stelle nuove, l'Antiticone se ne sbriga benissimo in quattro parole, dicendo che tali moderne stelle nuove non son parti certe de i corpi celesti, e che bisogna che gli avversari, se voglion provare lassù esser alterazione e generazione, dimostrino mutazioni fatte nelle stelle descritte già tanto tempo, delle quali nissuno dubita che sieno cose celesti, il che non possono far mai in veruna maniera. Circa poi alle materie che alcuni dicono generarsi e dissolversi in faccia del Sole, ei non ne fa menzione alcuna; ond'io argomento ch'e' l'abbia per una favola, o per illusioni del *cannocchiale*, o al più per affezioncelle fatte per aria, ed in somma per ogni altra cosa che per materie celesti.
>
> **Salviati** Ma voi, signor Simplicio, che cosa vi sete immaginato di rispondere all'opposizione di queste macchie importune, venute a intorbidare il cielo, e più la peripatetica filosofia? egli è forza che, come intrepido difensor di quella, vi abbiate trovato ripiego e soluzione, della quale non dovete defraudarci.
>
> *Galileo Galilei*
> *Dialogo sopra i due massimi sistemi del mondo*

It remains to determine the constant χ in the Einstein field equations (4.31). We will accomplish this by proving how in the weak field and slow velocities limit the Einstein field equation must reduce to Newton equation of gravitation.

In the *weak gravity* regime [14]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1. \tag{4.32}$$

In the slow velocities regime ($c = 1$)

$$|v^i| \ll 1 \qquad \gamma = \frac{1}{\sqrt{1-\boldsymbol{v}^2}} \approx 1 \qquad \frac{dt}{d\tau} \approx 1, \tag{4.33}$$

where τ is the particle proper time.

Newton equation

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i}, \tag{4.34}$$

with ϕ Newton gravitational potential energy. Then

$$\begin{aligned}
& \frac{d^2x^i}{dt^2} \approx \frac{d^2x^i}{d\tau^2} = \\
\text{WEP} \quad & \frac{D^2x^i}{d\tau^2} = \frac{Du^i}{d\tau} = \\
\text{Eq. (3.19)} \quad & u^\beta \nabla_\beta u^i = \\
\text{Eq. (3.70) or “free fall”} \quad & -\Gamma^i{}_{\alpha\beta} u^\alpha u^\beta \approx \\
\text{slow velocity regime} \quad & -\Gamma^i{}_{00} \approx \\
g_{\mu\nu} \approx \eta_{\mu\nu} \quad & -\Gamma_{i00} \approx \\
\text{weak gravity regime and Eq. (3.35)} \quad & \frac{1}{2}(h_{00,i} - 2h_{0i,0}) \approx \\
\nabla \boldsymbol{h} = 0 \to h_{0i,0} \approx -h_{0i,j}v^j \ 2^{\text{nd}} \text{ order small} \quad & \frac{1}{2} h_{00,i}
\end{aligned} \tag{4.35}$$

So $h_{00,i} \approx -2\phi_{,i}$ or $h_{00} \approx -2\phi +$ ‘constant’ where the ‘constant’ must vanish since $h, \phi \to 0$ as $r \to \infty$. Then

$$g_{00} \approx -(1 + 2\phi). \tag{4.36}$$

[14] For the solar system for example from Eq. (4.36) follows $|h_{\mu\nu}| \sim |\phi| \sim GM_\odot/c^2R_\odot \sim 10^{-6}$ so linearized gravity should be adequate.

Now we can calculate Riemann components $R^{\alpha}{}_{\beta\gamma\delta}$ from $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $h_{00} \approx -2\phi$. Keeping in mind that on h $,0 \ll ,i$ one finds (Ex. 12.1 of Ref. [1])

$$R^{i}{}_{0j0} \approx \frac{\partial^2 \phi}{\partial x^i \partial x^j}. \tag{4.37}$$

Now from the definition (3.108) of Einstein tensor $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \chi T_{\mu\nu}$, contracting μ and ν and recalling the definition (3.107) of the scalar curvature, follows $R = -\chi T$ with $T = T^{\mu}{}_{\mu}$ the trace of the stress-energy tensor. So $R_{\mu\nu} = \chi(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$ and

$$\begin{aligned} R_{00} &= \chi \left(T_{00} - \frac{1}{2} g_{00} T \right) \\ &\approx \chi \left(T_{00} + \frac{1}{2} T^{0}{}_{0} \right) \\ &= \chi \left(\rho - \frac{1}{2} \rho \right) \\ &= \frac{1}{2} \chi \rho, \end{aligned} \tag{4.38}$$

where in the approximation we used the fact that $T^{i}{}_{i}$ is small in the small velocity regime.

But from the results (4.37) and (4.38) follows

$$R_{00} = R^{\mu}{}_{0\mu0} \approx R^{i}{}_{0i0} \approx \boldsymbol{\nabla}^2 \phi \approx \frac{1}{2} \chi \rho, \tag{4.39}$$

Since we know from Newton equation that

$$\boldsymbol{\nabla}^2 \phi = 4\pi G \rho, \tag{4.40}$$

we find

$$\chi = \frac{8\pi G}{c^4}, \tag{4.41}$$

where we restored the speed of light constant c.

Extension to the case of a charged particle

In a LLF a charged particle is subject to the Lorentz force (see Section IV A)

$$\frac{du^{\hat{\mu}}}{d\tau} = \frac{q}{m} F^{\hat{\mu}\hat{\beta}} u^{\hat{\alpha}} \eta_{\hat{\alpha}\hat{\beta}}, \tag{4.42}$$

where m is the particle rest mass, q its charge, $u^{\hat{\mu}} = dx^{\hat{\mu}}/d\tau$, and $\eta_{\hat{\alpha}\hat{\beta}} u^{\hat{\alpha}} u^{\hat{\beta}} = -1$. Then, according to SEP

$$\frac{Du^{\mu}}{d\tau} = \frac{q}{m} F^{\mu\beta} u^{\alpha} g_{\alpha\beta}, \tag{4.43}$$

and recalling the definition of the covariant dervative of Eq. (3.22)

$$\frac{du^{\mu}}{d\tau} = -\Gamma^{\mu}{}_{\alpha\beta} u^{\alpha} u^{\beta} + \frac{q}{m} F^{\mu\beta} u^{\alpha} g_{\alpha\beta}, \tag{4.44}$$

with $g_{\alpha\beta} u^{\alpha} u^{\beta} = -1$. This shows how the electromagnetic field that determines the curvature of spacetime through Einstein field equations (4.31) also produces an acceleration of the charged particle which will not be in "free fall" anymore. In particular, noticing that $\nabla(\mathbb{T}^{\text{matter}} + \mathbb{T}^{\text{em}}) = 0$ where $\mathbb{T}^{\text{matter}}$ is the stress-energy tensor of a charged particle of rest mass m of Eq. (4.18) and $\mathbb{T}^{\text{em}}$ is the stress-energy tensor of the electromagnetic field satisfying Eq. (4.24) with the 4-current density $\widetilde{\boldsymbol{J}} = q \int \vec{\boldsymbol{u}} \delta^{(4)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{z}}(\tau)) d\tau$ due to the charged particle at $\vec{\boldsymbol{z}}$ with velocity $\vec{\boldsymbol{u}}$, and using the rule (3.29) for the covariant derivative of a tensor, we recover Eq. (4.43).

We then conclude that among all the fields in which particles move the gravitational one has a privileged role in GR in the sense that it is the only one that only curves spacetime, according to the Einstein field equations (4.31), but it does not accelerate particles. Particles are "freely falling" *only* in a gravitational field.

In the weak gravitational field regime of Eq. (4.32) we find

$$G_{\mu\nu}=\frac{1}{2}[h_{\mu\alpha,\nu}{}^{\alpha}+h_{\nu\alpha,\mu}{}^{\alpha}-\Box h_{\mu\nu}-h_{,\mu\nu}-\eta_{\mu\nu}(h_{\alpha\beta,}{}^{\alpha\beta}-\Box h)], \tag{4.45}$$

where $h=h_{\mu}{}^{\mu}$ is the trace of the perturbation.

Now we observe that the perturbation $h_{\mu\nu}$ in Eq. (4.32) is not unique since it depends on the system of coordinates used on $\mathscr{M}$. We can always perform an infinitesimal coordinate transformation $x'_{\mu}(\mathscr{P})=x_{\mu}(\mathscr{P})+\xi_{\mu}(\mathscr{P})$ that must leave invariant the various tensorial equations. For example for the perturbation tensor we have $h'_{\mu\nu}=h_{\mu\nu}-(\xi_{\nu,\mu}+\xi_{\mu,\nu})$ and $h'=h-2\xi_{\mu,}{}^{\mu}$, to first order in ξ. This freedom of choice is the gauge freedom for gravity. If we now define $\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}h\eta_{\mu\nu}$ we will have $\bar{\bar{h}}_{\mu\nu}=h_{\mu\nu}$ or $h_{\mu\nu}=\bar{h}_{\mu\nu}-\frac{1}{2}\bar{h}\eta_{\mu\nu}$ so that Eq. (4.45) becomes

$$G_{\mu\nu}=\frac{1}{2}[-\Box\bar{h}_{\mu\nu}-\eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta}+\bar{h}_{\mu\alpha,\nu}{}^{\alpha}+\bar{h}_{\nu\alpha,\mu}{}^{\alpha}]. \tag{4.46}$$

We may then introduce $\bar{h}'_{\mu\nu}=h'_{\mu\nu}-\frac{1}{2}h'\eta_{\mu\nu}$ as well, so that for $\Box\xi_{\mu}=-\bar{h}_{\mu\nu,}{}^{\nu}$ [15] we find the condition $\bar{h}'_{\mu\nu,}{}^{\nu}=0$. This is the *harmonic gauge* (Box 18.2 in Ref, [1]) also known as *Lorenz gauge* [16] for gravity (to be compared with the Lorentz gauge for electromagnetism, $A_{\nu,}{}^{\nu}=0$, of subsection "Maxwell equations" of Section IV A). The Einstein tensor reduces then to $G_{\mu\nu}=-\frac{1}{2}\Box\bar{h}'_{\mu\nu}$ and Einstein field equations (4.31) reduce to the following *gravitational wave equation*

$$\Box\bar{h}'_{\mu\nu}=-2\chi T_{\mu\nu}. \tag{4.47}$$

On 14 September 2015, exactly 110 years from the *Annus Mirabilis* of Einstein setting up his SR theory [11] and 100 years from his publication of the 4 articles on GR [12–15] the first gravitational wave was detected by the twin LIGO (Laser Interferometer Gravitational-wave Observatory) observatories in the United Sates. The signal, named GW150914, came from two black holes merging about 1.3 billion light-years [17] away and confirmed a key prediction of Einstein theory of general relativity. This detection marked the beginning of gravitational wave astronomy and was notable for its unusually strong, "loud" signal.

The causal solution of Eq. (4.47) is then

$$\bar{h}'_{\mu\nu}(t,\boldsymbol{r})=4G\int\frac{T_{\mu\nu}(t_{\text{retarded}},\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d\boldsymbol{r}', \tag{4.48}$$

$$t_{\text{retarded}}=t-|\boldsymbol{r}-\boldsymbol{r}'|, \tag{4.49}$$

to be compared with the result (4.14), (4.15) in electromagnetism.

Schwarzschild metric in weak gravity

We will now use the weak gravity architecture to determine the perturbation tensor $h_{\mu\nu}$ for a static and spherical body of mass M at the origin $\boldsymbol{r}=0$. Since the body is static Eq. (4.47) becomes $\Box\bar{h}'_{\mu\nu}=\boldsymbol{\nabla}^2\bar{h}'_{\mu\nu}=-16\pi GT_{\mu\nu}$. The Poisson equation has solution

$$\bar{h}'_{\mu\nu}(\boldsymbol{r})=4G\int\frac{T_{\mu\nu}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d\boldsymbol{r}'. \tag{4.50}$$

For a spherical body of mass M at the origin the stress energy tensor is given by $T_{\mu\nu}(\boldsymbol{r})=\rho u_{\mu}u_{\nu}$ with $u^0\approx 1$ and $u^i=0$ and $\rho=M\delta^{(3)}(\boldsymbol{r})$ (see subsection "The stress-energy tensor" in Section IV B). Then Eq. (4.50) becomes $\bar{h}'_{00}=4GM/r$ and all other components vanish. Then $\bar{h}'=-4GM/r$. Recalling that $h'_{\mu\nu}=\bar{h}'_{\mu\nu}-\frac{1}{2}\bar{h}'\eta_{\mu\nu}$ we find $h'_{\mu\mu}=2GM/r$ and all other components vanish. Then we reach to the following metric (gauge invariant)

$$ds^2=-\left(1-\frac{r_s}{r}\right)c^2dt^2+\left(1+\frac{r_s}{r}\right)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2), \tag{4.51}$$

where $r_s=2GM/c^2$ is Schwarzchild radius, where we restored the speed of light constant, and we used spherical coordinates (see subsection "Spherical coordinates in flat space" in Section III J). This result agrees with the exact Schwarzchild solution [16–18] for $r_s\ll 1$.

---

[15] We will have an infinite number of solutions provided $\xi_{\mu}\to\xi_{\mu}+\xi^{add}_{\mu}$ with $\Box\xi^{add}_{\alpha}=0$.
[16] Without the "t".
[17] For $c=1$, 1 year $\approx\pi\times 10^7$s $\approx 10^{16}$m.

Eddington 1919 observation of bending of light by our Sun

In the equatorial plane (x, y) centered on the Sun, consider a light ray traveling, when unperturbed, along a straight line $y = b$ with equation of motion $x(t) = t$ (c=1), i.e. the ray 4-momentum $\vec{\boldsymbol{p}} = (p^0, p^0, 0, 0)$. The gravitational field of the Sun will curve the spacetime so that the light ray in the equatorial plane ($z = 0$) will follow the geodesic equation (3.70) whose y component becomes

$$\frac{dp^y}{d\tau} + \Gamma^y{}_{00}(p^0)^2 + \Gamma^y{}_{xx}(p^x)^2 + 2\Gamma^y{}_{x0}p^x p^0 = 0. \tag{4.52}$$

Foe a diagonal metric the Christoffel symbols are given in (Ex. 7.6 of Ref. [2]). Applied to our case of Eq. (4.51) one finds

$$\begin{cases} \Gamma^0{}_{0j} = \Gamma^j{}_{00} = \frac{GM}{(x^2+y^2)^{3/2}} x^j \\ \Gamma^j{}_{kl} = \frac{GM}{(x^2+y^2)^{3/2}} (x^j \delta_l{}^k - x^k \delta_l{}^j - x^l \delta_k{}^j) \\ \Gamma^j{}_{0k} = 0 \qquad\qquad\qquad\qquad\qquad\qquad j \neq k \end{cases} \tag{4.53}$$

For the massless photon

$$0 = g(\vec{\boldsymbol{p}}, \vec{\boldsymbol{p}}) = -(p^0)^2(1 - r_s/r) + (p^x)^2(1 + r_s/r) + (p^y)^2(1 + r_s/r). \tag{4.54}$$

For $|p^y| \ll |p^x|$ and to least orders in r_s/r we have $p^x \approx p^0$. Then, using result (4.53), the geodesic equation (4.52) becomes

$$\frac{dp^y}{d\tau} + \frac{2GMy}{(x^2 + y^2)^{3/2}}(p^x)^2 = 0, \tag{4.55}$$

A first perturbation effect due to the gravitational field of the Sun can be then found by

$$\frac{dp^y}{d\tau} + \frac{2GMb}{(x^2 + b^2)^{3/2}} p^x \frac{dx}{d\tau} = 0, \tag{4.56}$$

which can easily be integrated to find $\Delta p^y = p^y(x \to \infty) - p^y(x \to -\infty) = -4GMp^x/b$. Then we predict that the effect of the Sun gravitational field on the trajectory of the light ray will be to perturb its straight path bending it by an angle

$$\Delta\phi = \arctan(|\Delta p^y|/p^x) \approx 4GM/c^2 b = 2r_s/b, \tag{4.57}$$

where we restored the speed of light constant c and used the result of footnote 14. Observations of the total solar eclipse of 29 May 1919 carried out by two expeditions, one to the West African island of Príncipe and the other to the Brazilian town of Sobral, lead Sir Arthur Stanley Eddington (Kendal, 28 December 1882 – Cambridge, 22 November 1944) to estimate a *bending angle* $\Delta\phi \approx 1.75''$ [19].

V. FICTIONS

If I let my fingers wander idly over the keys of a typewriter it might happen that my screed made an intelligible sentence. If an army of monkeys were strumming on typewriters they might write all the books in the British Museum. The chance of their doing so is decidedly more favourable than the chance of the molecules returning to one half of the vessel.

Sir Arthur Stanley Eddington
The Nature of the Physical World

This thinker observed that all the books, no matter how diverse they might be, are made up of the same elements: the space, the period, the comma, the twenty-two letters of the alphabet. He also alleged a fact which travelers have confirmed: In the vast Library there are no two identical books. From these two incontrovertible premises he deduced that the Library is total and that its shelves register all the possible combinations of the twenty-odd orthographical symbols (a number which, though extremely vast, is not infinite). [...] The certitude that everything has been written negates us or turns us into phantoms. [...] I suspect that the human species - the unique species - is about to be extinguished, but the Library will endure: illuminated, solitary, infinite, perfectly motionless, equipped with precious volumes, useless, incorruptible, secret. [...] I venture to suggest this solution to the ancient problem: The Library is unlimited and cyclical.

Jorge Francisco Isidoro Luis Borges Acevedo
Ficciones (The Library of Babel)

But when I sit and gaze,
I imagine, in my thoughts, Endless spaces beyond the hedge,
An all encompassing silence and a deeply profound quiet,

Giacomo Leopardi
L' Infinito

In this Section we will describe some possible extension of GR which could prove useful routes to explain yet *unsolved mysteries* of Nature or to make the current theory *more accessible* (at least numerically) or to write an even *more elegant* theory.

A. Embedding of spacetime (eGR)

Would it be possible to imagine spacetime $\mathscr{M}$ *embedded* (eGR) in a 5-dimensional manifold $\mathscr{I}$? If some *entity* would live in such higher dimensional manifold $\mathscr{I}$ then *it* would be able to see the whole time line (of the peoples living in $\mathscr{M}$) [7], it would have complete knowledge of the past, the present, and the future. Even if this we all would really like it stayed a mystery, there are certain sort of *people* that would be ready to *pay you* enormous amount of money for the mere realization of this sort of projects. So it could politically come handy to use these financial support to carry on some other mathematical research, that is a good thing any case!

B. Imaginary time in General Relativity (wGR)

A *Wick rotation* changes time t into $-i\beta\hbar$ and the metric in a LLF changes from Minkowski to Euclidean $\eta_{\alpha\beta} = \mathrm{diag}(1,1,1,1)$. This allows to define a *statistical mechanics* of spacetime (wGR), where $\hbar$ is Planck constant and the *imaginary time* β is an inverse *temperature*. This is illustrated in the trilogy of Refs. [20–22] where wGR was called FEBB. Although we have no experimental evidence of the soundness of this recipe it could eventually offer a new *vision* into yet unresolved questions that nature poses us like the mystery of dark energy in cosmology.

We are interested in studying the density matrix for a particle of unit mass in thermal equilibrium at an inverse temperature β on a sphere of radius r [23–25]

$$\widehat{\rho} = e^{-\beta \widehat{H}}, \tag{5.1}$$

which satisfies the Bloch equation

$$\frac{\partial \widehat{\rho}}{\partial \beta} = -\widehat{H}\widehat{\rho}. \tag{5.2}$$

It is convenient to use position $x = (x^1, x^2)$ eigenstates normalized as

$$\langle x|x'\rangle = \frac{\delta^{(2)}(x - x')}{\sqrt{g}}, \tag{5.3}$$

so that the resolution of the identity is written as

$$\mathbb{1} = \int d^2x \sqrt{g}\, |x\rangle\langle x|. \tag{5.4}$$

The wave function $\psi(x) = \langle x|\psi\rangle$ corresponding to the vector $|\psi\rangle$ of the Hilbert space are scalars under arbitrary change of coordinates. And the position representation of the density matrix

$$\rho(x, x'; \beta) = \langle x|e^{-\beta \widehat{H}}|x'\rangle, \tag{5.5}$$

behaves as a biscalar under arbitrary change of coordinates, i.e. a scalar at both points x and x'. It satisfies the Bloch equation (for $\hbar = 1$)

$$\frac{\partial}{\partial \beta}\rho(x, x'; \beta) = \frac{1}{2}\Box_x \rho(x, x'; \beta), \tag{5.6}$$

$$\rho(x, x'; 0) = \frac{\delta^{(2)}(x - x')}{\sqrt{g}}. \tag{5.7}$$

where $\Box_x$ is the Laplacian of Eq. (3.57) acting on the x coordinates. The solution of this equation has a path integral representation [26].

In order to simplify the Bloch equation and the corresponding path integral we first transform the density matrix into a bidensity

$$\bar{\rho}(x, x'; \beta) = g^{1/4}(x)\rho(x, x'; \beta)g^{1/4}(x'), \tag{5.8}$$

so that Eq. (5.6) takes the form

$$\frac{\partial}{\partial \beta}\bar{\rho}(x, x'; \beta) = \frac{1}{2}g^{1/4}(x)\Box_x \left[g^{-1/4}(x)\bar{\rho}(x, x'; \beta)\right], \tag{5.9}$$

$$\bar{\rho}(x, x'; 0) = \delta^{(2)}(x - x'). \tag{5.10}$$

One may evaluate the differential operator appearing on the right hand side of Eq. (5.9) to obtain the identity

$$g^{1/4}\Box g^{-1/4} = \partial_i g^{ij}\partial_j - v_{\text{eff}}, \tag{5.11}$$

where in the first term derivatives act through, while the *effective scalar potential* is given by

$$v_{\text{eff}} = -g^{-1/4}\partial_i \left[g^{1/2}g^{ij}\left(\partial_j g^{-1/4}\right)\right]. \tag{5.12}$$

So Bloch equation can be rewritten as

$$\frac{\partial}{\partial \beta}\bar{\rho}(x, x'; \beta) = \left[\frac{1}{2}\partial_i g^{ij}(x)\partial_j - v_{\text{eff}}(x)\right]\bar{\rho}(x, x'; \beta). \tag{5.13}$$

Using a coordinate system with $x^1 = \theta$ and $x^2 = \varphi$ we find

$$v_{\text{eff}}(x) = -r^2\frac{2 + \cot^2(x^1)}{4}. \tag{5.14}$$

Note that at the point $x^1 = 0$, the north pole $\mathscr{N}$, the effective potential becomes singular. And this will affect the path of the quantum particle on the sphere [24, 25].

The sphere being a Riemannian manifold of dimension $n = 2$ is conformally flat, so it exists a coordinate frame where $g_{ij}(x') = \eta_{ij}e^{2\phi(x')}$, with $\eta_{ij} = \delta_{ij}$ is the Kronecker delta as in Eq. (3.115). On the other hand one can expand the metric tensor in *Riemann normal coordinates* [18] $\tilde{x}$ around the north pole $\mathscr{N}$, with $\tilde{x}^1 = \tilde{\theta} = 0$, on a local coordinate orthonormal frame, the tangent plane of the LLF, as explained in Section III D, recursively to second order. Then we discover that [19]

$$g_{ij}(\tilde{x}) = \eta_{ij} + f(|\tilde{x}|)P_{ij}, \tag{5.15}$$

$$P_{ij} = \eta_{ij} - \hat{x}_i \cdot \hat{x}_j, \quad \hat{x}_i = \tilde{x}_i/|\tilde{x}|, \quad |\tilde{x}| = \sqrt{\eta_{ij}\tilde{x}^i\tilde{x}^j}, \tag{5.16}$$

where P_{ij} is the projector, with $P_{ij}\hat{x}_i = 0$, from the *tangent bundle*, onto the *fiber bundle* of the sphere .

In Ref. [23, 27, 28] a form of f specific to the so called *normal neighborhood* of a point on the sphere, for example its north pole $\mathscr{N}$, is given. In the tangent LLF $\partial_i g_{jk} = 0$ as shown in Section III D. Nonetheless, it can be easily shown that with the metric tensor expression (5.15) introduced above $\tilde{\partial}_i g^{ij}(\tilde{x})\tilde{\partial}_j = \delta^{ij}\tilde{\partial}_i\tilde{\partial}_j$, for arbitrary f, and the Bloch Eq. (5.13) becomes

$$\frac{\partial}{\partial\beta}\bar{\rho}(\tilde{x},\tilde{x}';\beta) = \left[\frac{1}{2}\delta^{ij}\tilde{\partial}_i\tilde{\partial}_j - v_{\text{eff}}(|\tilde{x}|)\right]\bar{\rho}(\tilde{x},\tilde{x}';\beta), \tag{5.17}$$

which can be interpreted as the Bloch equation of a particle on a flat space (in cartesian coordinates) interacting with an effective scalar potential v_{eff} of quantum origin (it would be proportional to $\hbar^2$ in arbitrary units). In Ref. [23] is shown that the resulting effective potential v_{eff} becomes singular at the south pole $\mathscr{S}$, $\tilde{\theta} = \pi$. This is a consequence of the *hairy ball theorem*, where the Euler class is the obstruction to the tangent plane of the sphere having a nowhere vanishing *section*, i.e. fiber or hair. The theorem was first proven by Henri Poincaré for the sphere in 1885 [29], and extended to higher even dimensions in 1912 by Luitzen Egbertus Jan Brouwer [30]. The theorem has been expressed colloquially as “you can’t comb a hairy ball flat without creating a cowlick” or “you can’t comb the hair on a coconut” as shown in Fig. 3. If g is a continuous function that assigns a vector in the three dimensional space to every point $\mathscr{P}$ on a sphere such that $g(\mathscr{P})$ is always tangent to the sphere at $\mathscr{P}$, then there is at least one pole, a point where the field vanishes, i.e. a $\mathscr{P}$ such that $g(\mathscr{P}) = 0$. Every zero of a vector field has a (non-zero) *index*, and it can be shown that the sum of all of the indexes at all of the zeros must be two, because the Euler characteristic of the sphere is two. Therefore, there must be at least one zero. This is a consequence of the *Poincaré-Hopf theorem*. The theorem was proven for two dimensions by Henri Poincaré and later generalized to higher dimensions by Heinz Hopf [31].

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[18] Use the geodesics through a given point to define the coordinates for nearby points. Let the given point be $\mathscr{O}$ and consider some nearby point $\mathscr{P}$. If $\mathscr{P}$ is close enough to $\mathscr{O}$ then there exists a unique geodesic joining $\mathscr{O}$ to $\mathscr{P}$. Let a^μ be the components of the unit tangent vector to this geodesic at $\mathscr{O}$ and let λ be the geodesic arc length measured from $\mathscr{O}$ to $\mathscr{P}$. Then the Riemann normal coordinates of $\mathscr{P}$ are defined to be $x^\mu = \lambda a^\mu$. These coordinates are well defined provided the geodesics do not cross (which we can always ensure by choosing the neighborhood of $\mathscr{O}$ to be sufficiently small). From the geodesic equation (3.70) and its derivative respect to λ follows ${\Gamma^\mu}_{\alpha\beta} = 0$ and ${\Gamma^\mu}_{(\alpha\beta,\gamma)} = 0$. It is easy to see, by continuing in this way, that all symmetric derivatives of the connection vanish at the origin in Riemann normal coordinates.

[19] Is easy to see that the inverse $g^{ij}(\tilde{x}) = \eta^{ij} + h(|\tilde{x}|)P^{ij}$ with $h = -f/(1+f)$ and on the right hand side we are allowed to raise or lower indexes freely. We also have $g = \det||g_{ij}|| = 1 + f$.

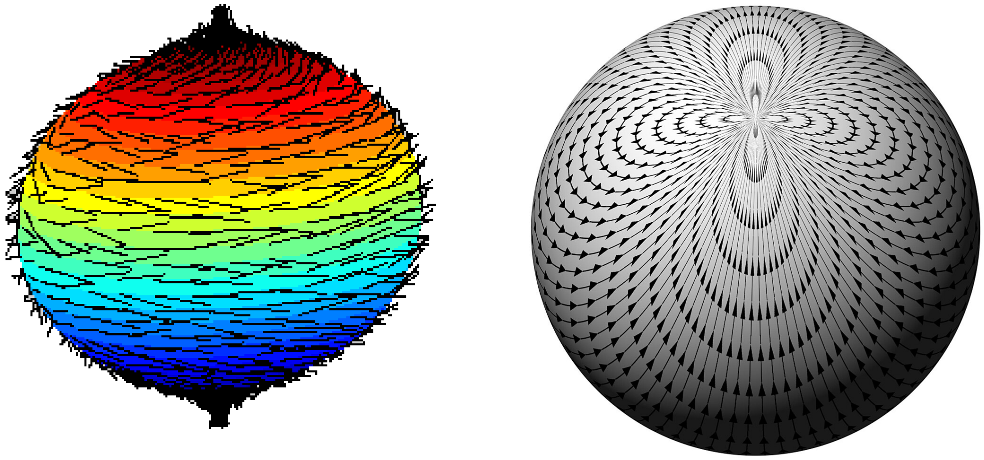

FIG. 3. On the left, pictorial view of a hairy ball; on the right, a continuous tangent vector field on a sphere with only one pole, in this case a dipole field with index 2.

Eq. (5.13) can now be solved by a standard path integral

$$\bar{\rho}(\tilde{x},\tilde{x}';\beta)=\int_{\tilde{x}(0)=\tilde{x}'}^{\tilde{x}(\beta)=\tilde{x}} e^{-S[\tilde{x}]}\,\mathscr{D}\tilde{x}, \tag{5.18}$$

where the action functional

$$S[\tilde{x}]=\int_0^\beta dt\left(\frac{1}{2}\delta_{ij}\dot{\tilde{x}}^i\dot{\tilde{x}}^j+v_{\rm eff}(|\tilde{x}|)\right). \tag{5.19}$$

In particular we see how, even a single free particle have a path which will be subject to some anisotropy due to the effective potential induced by the curvature of the sphere. This effect was studied in Refs. [24, 25] where instead of the path integral of Eq. (5.18) a covariant form $\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$ is used for the kinetic energy of the particle(s) as suggested in Ref. [26]. Of course one must tell how to compute it, i.e. how to discretize it. It is known that different prescriptions for discretizing the path integral correspond to different orderings of the momenta and coordinates in the quantum Hamiltonian. Eventually *affine quantization* could become essential here [32]. Given the fact that the metric $g_{ij}(x)$ is strictly positive definite.

C. Stochastic General Relativity (sGR)

We could consider the Einstein field deterministic equations (4.31) as the "Fokker-Planck-Einstein" equations for a yet to be written corresponding *stochastic differential field equations* of "Langevin-Einstein" according to Ito or Stratonovich calculus [33]. This route is expected to be rather promising in the field of *numerical relativity* [34, 35] since it would call to the rescue the powerful machinery of Markov processes and Monte Carlo methods [36].

D. A theory of "more General" Relativity (mGR)

As we saw in subsection "Extension to the case of a charged particle" of Section IV B it would be desirable to have a theory of *more General Relativity* that puts all fields (gravitational, electromagnetic, . . .) on the same ground. For example we could think at a higher dimensional space containing our spacetime and some other hidden dimensions allowing for "free fall" of a test particle in any field of nature. Projecting then its motion into our spacetime.

[1] Charles W. Misner and Kip S. Thorne and John A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).

[2] Alan P. Lightman and William H. Press and Richard H. Price and Saul A. Teukolsky, *Problem book in relativity and gravitation* (Princeton University Press, Princeton, 1975).
[3] John D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, 1962).
[4] Galileo Galilei, *Dialogo sui due massimi sistemi del mondo* (Stefano della Bella, 1632).
[5] Albert Einstein, *The Meaning of Relativity* (Routledge, 2003).
[6] A. Einstein and N. Rosen, The Particle Problem in the General Theory of Relativity, Phys. Rev. **48**, 76 (1935).
[7] E. A. Abbott, *Flatland: A Romance of Many Dimensions* (Seeley & Co., London, 1884).
[8] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Course of Theoretical Physics, Vol. 5 (Butterworth Heinemann, 1951) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.
[9] J. P. Hansen and I. R. McDonald, *Theory of simple liquids*, 2nd ed. (Academic Press, Amsterdam, 1986) section 3.3.
[10] W. Pauli, *Electrodynamics*, 1st ed., Pauli Lectures on Physics, Vol. 1 (MIT Press, Cambridge, Massachusetts, 1973).
[11] Albert Einstein, Zur Elektrodynamik bewegter Körper, Annalen der Physik , 891 (1905).
[12] Albert Einstein, On the General Theory of Relativity, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) , 778 (1915).
[13] Albert Einstein, On the General Theory of Relativity (addendum), Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) , 799 (1915).
[14] Albert Einstein, Explanation of the Perihelion Motion of Mercury from General Relativity Theory , Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) , 831 (1915).
[15] Albert Einstein, Die Feldgleichungen der Gravitation, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) , 844 (1915).
[16] K. Schwarzschild, Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **7**, 189 (1916).
[17] K. Schwarzschild, Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1**, 424 (1916).
[18] L. Flamm, Beiträge zur Einstein'schen Gravitationstheorie, Physikalische Zeitschrift **17**, 448 (1916).
[19] F. W. Dyson, A. S. Eddington, and C. Davidson, A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations Made at the Total Eclipse of May 29, 1919, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **220**, 291 (1920).
[20] R. Fantoni, Statistical Gravity through Affine Quantization, Quantum Rep. **6**, 706 (2024).
[21] R. Fantoni, Statistical Gravity and entropy of spacetime, Stats **8**, 23 (2025).
[22] R. Fantoni, Statistical Gravity, ADM splitting, and Affine Quantization, Gravitation and Cosmology **31**, 568 (2025).
[23] F. Bastianelli and O. Corradini, On the simplified path integral on spheres, Eur. Phys. J. C **77**, 731 (2017).
[24] R. Fantoni, One-component fermion plasma on a sphere at finite temperature, Int. J. Mod. Phys. C **29**, 1850064 (2018).
[25] R. Fantoni, One-component fermion plasma on a sphere at finite temperature. The anisotropy in the paths conformations, J. Stat. Mech. , 083103 (2023).
[26] L. S. Schulman, *Techniques and Applications of Path Integration* (John Wiley & Sons, Technion, Haifa, Israel, 1981).
[27] F. Bastianelli, The path integral for a particle in curved spaces and Weyl anomalies, Nucl. Phys. B **376**, 113 (1992).
[28] F. Bastianelli and N. D. H. Dass, Simplified method for trace anomaly calculations in $d \leq 6$, Phys. Rev. D **64**, 047701 (2001).
[29] H. Poincaré, Sur les courbes d́finies par les q́uations diffŕentielles, Journal de Mathématiques Pures et Appliqués **4**, 167 (1885).
[30] L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, Mathematische Annalen **71**, 97 (1912).
[31] H. Hopf, Vektorfelder in n-dimensionalen Mannigfaltigkeiten, Math. Ann. **96**, 209 (1926).
[32] J. R. Klauder and R. Fantoni, The Magnificent Realm of Affine Quantization: valid results for particles, fields, and gravity, Axioms **12**, 911 (2023).
[33] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, New York, 1983).
[34] R. Arnowitt, S. Deser, and C. Misner, The Dynamics of General Relativity, In: Witten, L., Ed., Gravitation: An Introduction to Current Research, Wiley & Sons, New York, 227 (1962), arXiv: gr-qc/0405109.
[35] Zachariah B. Etienne and Yuk T. Liu and Stuart L. Shapiro and Thomas W. Baumgarte, Relativistic Simulations of Black Hole-Neutron Star Mergers: Effects of black-hole spin, Phys. Rev. D. **76**, 104021 (2009).
[36] M. H. Kalos and P. A. Whitlock, *Monte Carlo Methods* (John Wiley & Sons Inc., New York, 1986).