Statistical Gravity through Affine Quantization

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(Dated: November 29, 2024)

I propose a possible way to introduce the effect of temperature (defined through the virial theorem) into Einstein's theory of general relativity. This requires the computation of a path integral on a 10-dimensional *flat* space in a four dimensional spacetime *lattice*. Standard path integral Monte Carlo methods can be used to compute it.

Keywords: General Relativity; Einstein-Hilbert action; Statistical Physics; Path Integral; Monte Carlo; Virial Theorem

I. INTRODUCTION

A still unsolved problem in physics is the formulation of a well defined theory unifying gravity and quantum mechanics. In this paper we propose a possible way to estimate thermal statistical effects on the fabric of Einstein's spacetime. Once an action for the Einstein field equations of general relativity is found we use it to construct a path integral formulation for statistical gravity which will describe quantum effects at low temperatures. Recent progress on successful affine quantization [1] of euclidean relativistic scalar field theories can become important in making this path integral mathematically well defined and accessible numerically, for example through the usual path integral Monte Carlo method.

II. EINSTEIN'S FIELD EQUATIONS FROM A VARIATIONAL PRINCIPLE

Sempre caro mi fu quest'ermo colle,
e questa siepe, che da tanta parte
dell'ultimo orizzonte il guardo esclude.

Giacomo Leopardi
L' Infinito

The Einstein-Hilbert action in general relativity is the action that yields the Einstein field equations through the stationary-action principle. With the $(- + ++)$ metric signature, the gravitational part of the action is given as [2, 3]

$$S = \frac{1}{2\kappa}\int R\sqrt{-g}\,\mathrm{d}^4x, \tag{2.1}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor matrix, $\sqrt{-g}$ is the scalar density, $\sqrt{-g}\,\mathrm{d}^4x$ is the invariant volume element, R is the Ricci scalar, and $\kappa = 8\pi G c^{-4}$ is the Einstein gravitational constant (G is the gravitational constant and c is the speed of light in vacuum). If it converges, the integral is taken over the whole spacetime. If it does not converge, S is no longer well-defined, but a modified definition where one integrates over arbitrarily large, relatively compact domains, still yields the Einstein equation as the Euler-Lagrange equation of the Einstein-Hilbert action. The action was proposed [2] by David Hilbert in 1915 as part of his application of the variational principle to a combination of gravity and electromagnetism.

The stationary-action principle then tells us that to recover a physical law, we must demand that the variation of this action with respect to the inverse metric be zero, yielding

$$0 = \delta S = \int\left[\frac{1}{2\kappa}\frac{\delta\left(\sqrt{-g}R\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\,\mathrm{d}^4x \tag{2.2}$$

$$= \int\left[\frac{1}{2\kappa}\left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right)\right]\delta g^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \tag{2.3}$$

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Since this equation should hold for any variation $\delta g^{\mu\nu}$, it implies that

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = 0 \tag{2.4}$$

is the equation of motion for the metric field.

The variation of the Ricci scalar in Eq. (2.4) follows from varying the Riemann curvature tensor, and then the Ricci curvature tensor. The first step is captured by the Palatini identity

$$\delta R_{\sigma\nu} \equiv \delta R^{\rho}{}_{\sigma\rho\nu} = (\delta\Gamma^{\rho}_{\nu\sigma})_{;\rho} - \left(\delta\Gamma^{\rho}_{\rho\sigma}\right)_{;\nu}. \tag{2.5}$$

Using the product rule, the variation of the Ricci scalar $R = g^{\sigma\nu}R_{\sigma\nu}$ then becomes,

$$\begin{aligned}\delta R &= R_{\sigma\nu}\delta g^{\sigma\nu} + g^{\sigma\nu}\delta R_{\sigma\nu}\\ &= R_{\sigma\nu}\delta g^{\sigma\nu} + \left(g^{\sigma\nu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^{\mu}_{\mu\sigma}\right)_{;\rho},\end{aligned} \tag{2.6}$$

where we also used the metric compatibility $g^{\mu\nu}_{;\sigma} = 0$, and renamed the summation indices $(\rho,\nu) \to (\mu,\rho)$ in the last term. When multiplied by $\sqrt{-g}$, the term $\left(g^{\sigma\nu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^{\mu}_{\mu\sigma}\right)_{;\rho}$ becomes a total derivative, since for any vector A^{λ} and any tensor density $\sqrt{-g}\,A^{\lambda}$, we have

$$\sqrt{-g}\,A^{\lambda}_{;\lambda} = \left(\sqrt{-g}\,A^{\lambda}\right)_{;\lambda} = \left(\sqrt{-g}\,A^{\lambda}\right)_{,\lambda}. \tag{2.7}$$

By Stokes' theorem, this only yields a boundary term when integrated. The boundary term is in general non-zero, because the integrand depends not only on $\delta g^{\mu\nu}$, but also on its partial derivatives $\partial_{\lambda}\,\delta g^{\mu\nu} \equiv \delta\,\partial_{\lambda} g^{\mu\nu}$. However, when the variation of the metric $\delta g^{\mu\nu}$ vanishes in a neighbourhood of the boundary or when there is no boundary, this term does not contribute to the variation of the action. Thus, we can forget about this term and simply obtain

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}. \tag{2.8}$$

at events not in the closure of the boundary.

The variation of the determinant in Eq. (2.4) requires Jacobi's formula, the rule for differentiating a determinant:

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}. \tag{2.9}$$

Using this we get

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\delta g_{\mu\nu}\right) = -\frac{1}{2}\sqrt{-g}\left(g_{\mu\nu}\delta g^{\mu\nu}\right) \tag{2.10}$$

In the last equality we used the fact that from the symmetry of the metric tensor and $g_{\mu\nu}g^{\nu\mu} = \delta^{\mu}_{\mu} = 4$ follows

$$g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu} \tag{2.11}$$

Thus we conclude that

$$\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}. \tag{2.12}$$

Now that we have all the necessary variations at our disposal, we can insert Eq. (2.12) and Eq. (2.8) into the equation of motion (2.4) for the metric field to obtain

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \tag{2.13}$$

which is the Einstein field equations in vacuum.

Moreover, since Einstein's tensor $G_{\mu\nu}$ appears from a variational principle:

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2\kappa}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \frac{1}{2\kappa}G_{\mu\nu}, \tag{2.14}$$

its covariant divergence is necessarily zero [3].

Matter or electromagnetic fields will produce a curvature of spacetime. In order to take this into account it is necessary to add a term $\mathcal{L}_F$ as follows,

$$S = \int \left(\frac{1}{2\kappa}R + \mathcal{L}_{\rm F}\right)\sqrt{-g}\,{\rm d}^4x. \tag{2.15}$$

The equations of motion coming from the stationary-action principle now become

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \tag{2.16}$$

where

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm F})}{\delta g^{\mu\nu}} = -2\frac{\delta\mathcal{L}_{\rm F}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm F}, \tag{2.17}$$

is the stress-energy tensor and $\kappa = 8\pi G/c^4$ has been chosen such that the non-relativistic limit yields the usual form of Newton's gravity law.

III. PATH INTEGRAL FORMULATION OF STATISTICAL GRAVITY

> [...] e il suon di lei. Così tra questa
> immensità s'annega il pensier mio:
> e il naufragar m'è dolce in questo mare.
>
> *Giacomo Leopardi*
> *L' Infinito*

Then the action for Einstein's theory of general relativity is one for a particular field theory where the field is the metric tensor $g_{\mu\nu}(x)$ a symmetric tensor with 10 independent components, each of which is a smooth function of 4 variables. We will indicate all this components with the notation $\{g\}(x)$. We will also work in euclidean time $x^0 \equiv ct \to ict$ so that the metric signature becomes $(+ + + +)$.

The thermal average of an observable $\mathcal{O}[\{g\}(x)]$ will then be given by the following expression

$$\langle\mathcal{O}\rangle = \frac{\int \mathcal{O}[\{g\}(x)]\exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}{\int \exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}, \tag{3.1}$$

so that $\langle 1\rangle = 1$. Here $1/\upsilon$ is a constant of dimension of energy times length, $ct \in [0, \beta[$ where $\beta = 1/k_BT$, k_B is a Boltzmann constant of dimensions of one divided by length and by degree Kelvin, and T the absolute temperature in degree Kelvin. Since the thermal average involves taking a trace we must have $g_{\mu\nu}(t + \beta/c, \mathbf{x}) = g_{\mu\nu}(t, \mathbf{x})$, where we denote with $\mathbf{x} \equiv (x^1, x^2, x^3)$ a spatial point. S is the Einstein-Hilbert action of Eq. (2.15). We will also require periodic spatial boundary conditions on the finite volume $\Omega \subset \mathbb{R}^3$ which is the closest thing to a formal thermodynamic limit. As usual we will use $\mathcal{D}^{10}\{g\}(x) \equiv \prod_x d^{10}\{g\}(x)$ and the functional integrals will be calculated on a lattice using the path integral Monte Carlo method [4]. Moreover we will choose $d^{10}\{g\}(x) \equiv \prod_{\mu\leq\nu} dg^{\mu\nu}(x)$ where the 10-dimensional space of the 10 independent components of the symmetric metric tensor is assumed to be flat.

This makes sense since the Einstein theory does not predict a curvature of spacetime due to temperature so it does not take care of the virial theorem of statistical physics. According to the virial theorem the temperature of a portion of spacetime should be related to kinetic energy which in turn should enter the stress energy tensor and be responsible for the curvature of spacetime according to Eq. (2.16). Therefore we should take care of the effect of temperature in some alternative way. Here we propose to use the usual formalism of statistical physics to estimate thermal averages $\langle\ldots\rangle$ of observables that depend on the metric tensor like for example the Riemann curvature tensor or the Christoffel symbols. This requires to use a path integral over a 10-dimensional space like in Eq. (3.1). Then, both the curvature of spacetime and the geodesic equation of motion of a point particle will be influenced by temperature and one should replace them with their thermally averaged versions more correctly. In the classical high temperature limit applying a (real) time average to the contracted Einstein field equations (2.16), $-R = \kappa T_\mu^\mu$, assuming $\mathcal{L}_F$ independent of $g^{\mu\nu}$ so that $T_\mu^\mu = 4\mathcal{L}_F$, and replacing the time average with the ensemble average of Eq. (3.1) in the first member, we soon reach (see the Appendix) the following relation $2\kappa/\beta\upsilon' = \frac{\kappa}{2}\langle T_\mu^\mu\rangle_t$, with $\langle\ldots\rangle_t = \lim_{\tau\to\infty}\int_0^\tau \ldots dt/\tau$ the time average and υ' another constant of dimension length squared divided by energy. This gives a temperature definition

$k_B T = \frac{\upsilon'}{4}\langle T^\mu_\mu\rangle_t$ where κ has dimensions of length divided by energy and the stress energy tensor elements have dimensions of an energy density.

In Eq. (3.1) we assumed a constant temperature throughout the whole accessible spacetime. This can be a too restrictive condition and it could be necessary to think about a temperature scalar field $T(x)$ which gives the value of the temperature in a neighborhood of a given event x. Actually we are bound to choose $T(\mathbf{x})$ as the temperature in a neighborhood of a given spatial point $\mathbf{x}$ since we must require $x^0 \in [0, \beta(\mathbf{x})[$. And the temperature field $T(\mathbf{x}) = 1/k_B\beta(\mathbf{x})$ could be available experimentally.

In Eq. (3.1) we assumed that what gives rise to the thermal average is the product of the 10 independent components of the tensor path integral $\mathcal{I}^{\mu\nu} = \int \ldots \mathcal{D}g^{\mu\nu}(x)$. This may be regarded as a rather arbitrary receipt but it gives us something that is numerically amenable and directly accessible through path integral Monte Carlo. Moreover it gives the correct classical high temperature limit $\langle\mathcal{O}\rangle \to \langle\mathcal{O}_c\rangle$ where $\mathcal{O}_c$ is the value of the observable at the metric of the solution of the classical Einstein field equations (2.16).

In order to make precise these arguments and to treat correctly the integral in Eq. (3.1) we immediately recognize that it is necessary a splitting of the time component from the 3 spatial components in the spirit of the Arnowitt–Deser–Misner (ADM) foliation formalism. In this respect affine quantization as explained in Ref. [1] may become useful.

IV. CONCLUSIONS

We propose a way to include into Einstein's general relativity theory the effect of temperature. This involves the use of a path integral on a 10-dimensional flat space in a four dimensional spacetime lattice hypertorus. It may prove necessary introduce a temperature scalar field function of the position within the spatial volume under exam. From the definition of the thermal average we will have that at large temperature, i.e. small β, the quantum effects of statistics on the spacetime fabric will be irrelevant. On the other hand the quantum effects will become important at low temperatures, i.e. large β. This means that the quantum effects are negligible whenever the path in the metric components extends over a short imaginary time interval, i.e. the hypertorus 'radius' around the time curled dimension is small and it reduces to something like a round closed hyper'string' whose thickness varies along its length. We will call our theory the FEBB from the initials of the surnames of the protagonists of the 3 main advances in theoretical physics: Albert Einstein for physics of gravitation, Ludwig Boltzmann for statistical physics, and Niels Bohr for quantum physics.

The internal consistency of our FEBB theory requires a definition of temperature as

$$T = \frac{\upsilon'}{4k_B}\langle T^\mu_\mu\rangle_t, \tag{4.1}$$

where k_B is a Boltzmann constant with dimensions of one divided by length and by degree Kelvin.

Affine quantization as explained in Ref. [1] may become useful to make these arguments precise.

ACKNOWLEDGMENTS

I thank the two women of my life, the first for giving me the chance of learning path integral and general relativity and the second for giving me the chance of learning affine quantization.

Appendix A: Derivation of the virial theorem

Taking the thermal average of the trace of the Einstein field equations 2.16 we find

$$\langle -R\rangle = \kappa\langle T^\mu_\mu\rangle. \tag{A1}$$

Now we notice that we can evaluate the left hand side as follows

$$\langle -R\rangle \;=\; \frac{\int (-R)\exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}{\int \exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}. \tag{A2}$$

In the high temperature limit we will have, from Eq. (2.15),

$$\upsilon S \approx \beta\bar{\upsilon}\left(\frac{1}{2\kappa}R + \mathcal{L}_{\mathrm{F}}\right). \tag{A3}$$

Note that this is only an approximation and we would more correctly need a 3+1 splitting of space from imaginary time. If we do not worry about these details, for the time being, we reach the following result

$$\langle -R\rangle \;\approx\; \frac{2\kappa}{\bar{\upsilon}}\frac{(d/d\beta)\int \exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}{\int \exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}+2\kappa\langle\mathcal{L}_F\rangle. \tag{A4}$$

Now, assuming $\mathcal{L}_F$ independent from the metric tensor we find from Eq. (2.17) that $T_\mu^\mu = 4\mathcal{L}_F$ and also

$$\begin{aligned}\langle -R\rangle \;&\approx\; \frac{2\kappa}{\bar{\upsilon}}\frac{(d/d\beta)\int \exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}{\int \exp(-\upsilon S)\;\mathcal{D}^{10}\{g\}(x)}+\frac{\kappa}{2}\langle T_\mu^\mu\rangle &\text{(A5)}\\ &=\; -\frac{2\kappa}{\beta\bar{\upsilon}}+\frac{\kappa}{2}\langle T_\mu^\mu\rangle, &\text{(A6)}\end{aligned}$$

where in the last equality we took profit of the fact that $\mathcal{L}_F$ is independent from the metric tensor components $\{g\}$ so that its contribution simplifies from the two path integrals in the numerator and in the denominator [1].

Using this result into Eq. (A1) we reach the following result

$$-\frac{2\kappa}{\beta\bar{\upsilon}}+\frac{\kappa}{2}\langle T_\mu^\mu\rangle \approx \kappa\langle T_\mu^\mu\rangle. \tag{A7}$$

Calling $\upsilon' = -\bar{\upsilon}$ we then reach

$$\frac{2\kappa}{\beta\upsilon'} \approx \frac{\kappa}{2}\langle T_\mu^\mu\rangle. \tag{A8}$$

which gives the desired result upon replacing the thermal average with the (real) time average introduced in the main text $\langle\ldots\rangle \to \langle\ldots\rangle_t$.

AUTHOR DECLARATIONS

Conflict of interest

The author has no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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[1] Note that $\int_0^\infty e^{-\alpha y}\,dy = \frac{1}{\alpha}$ when $\alpha > 0$. So that $[(d/d\alpha)\int_0^\infty e^{-\alpha y}\,dy]/\int_0^\infty e^{-\alpha y}\,dy = -\frac{1}{\alpha}$.