Homework # 1 Solutions

set of units used: MKSA

-Problem 1- By what ratio does the inductance change when an inductor is immersed in linear material of permeability μ ?

SOLUTION

In a linear (isotropic) [1] material of permeability μ, the following relation holds between the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$

$$\mathbf{B} = \mu\mathbf{H} \quad . \tag{1}$$

If ϕ_B is the flux of the magnetic induction through a surface S bounded by the wire of an inductor carrying a steady current I, the self-inductance (inductance) L of the inductor is given by

$$L \equiv \frac{\phi_B}{I} \equiv \frac{\int_S \mathbf{B} \cdot d\mathbf{a}}{I} \quad . \tag{2}$$

The Maxwell eq. used to find the flux of the magnetic induction through the inductor is the one containing the conduction current (free current) density, namely

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad . \tag{3}$$

In the inductor then one must have $H \propto I$ and using eq. (1), $B \propto \mu I$. Finally from the definition of the magnetic flux (see eq. (2)) one easily infers that also $\phi_B \propto \mu I$ and then L has to be

$$L = L_o \frac{\mu}{\mu_o} \quad . \tag{4}$$

where L_o is the inductance of the inductor in vacuum. One then conculde that when an inductor is immersed in a linear material of permeability μ the inductance change by the ratio μ/μ_o.

[1] In a non linear (high magnetic field) and non isotropic material eq. (1) would have the more general form $B_\alpha = \mu^{(1)}_{\alpha,\beta} H_\beta + \mu^{(2)}_{\alpha,\beta,\gamma} H_\beta H_\gamma + \ldots$.
In a linear and non isotropic material one would have instead $B_\alpha = \mu_{\alpha,\beta} H_\beta$

-Problem 2- A small loop of wire (radius b) lies a distance h above the center of a large loop (radius a), as in figure. The planes of the two loops are parallel, and perpendicular to a common axis.

(a) Suppose current I flows in the big loop. Find the flux through the little loop (the little loop is so small that you may consider the field of the big loop to be essentially constant).

(b) Suppose current I flows in the little loop. Find the flux through the big loop (the little loop is so small that you may treat it as a magnetic dipole).

(c) Find the mutual inductance, and confirm that $M_{1,2} = M_{2,1}$.

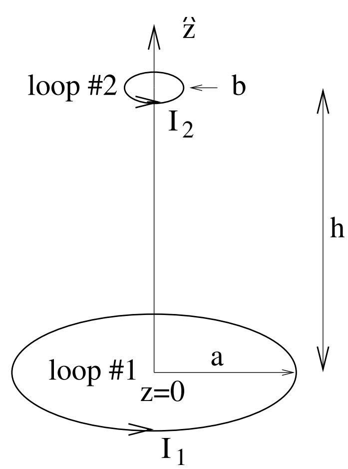

Figure 1: Two current loops

SOLUTION

Suppose that current I_1 flows in loop 1 generating a magnetic field $\mathbf{B}_1$ and I_2 in loop 2 generating a magnetic field $\mathbf{B}_2$, then one can write

$$\phi_{2,1} = \text{flux of } \mathbf{B}_1 \text{ through loop } 2 \equiv M_{2,1} I_1 \tag{1}$$

$$\phi_{1,2} = \text{flux of } \mathbf{B}_2 \text{ through loop } 1 \equiv M_{1,2} I_2 \tag{2}$$

(a) The magnetic field generated by the current in loop 1 has the following exact expression on the axis of the loops

$$\mathbf{B}_1(0,0,z) = \left[\frac{\mu_o}{2}\frac{a^2}{(z^2+a^2)^{3/2}}I\right]\hat{\mathbf{z}} \quad . \tag{3}$$

If one assumes (as suggested by the problem) that the loop2 is sufficiently small that the field $\mathbf{B}_1$ can be considered constant on the surface enclosed by such loop, then the following equation holds

$$\phi_{2,1} = \pi b^2 B_1(0,0,h) = \frac{\mu_o \pi}{2}\frac{a^2 b^2}{(h^2+a^2)^{3/2}}I_1 \quad . \tag{4}$$

(b) The magnetic dipole moment of the loop 2 is given by

$$\mathbf{m}_2 = I_2(\pi b^2)\hat{\mathbf{z}} \quad . \tag{5}$$

The magnetic field generated by the dipole $\mathbf{m}_2$ is

$$\mathbf{B}_2 = \frac{\mu_o}{4\pi}\frac{1}{r^3}[3(\mathbf{m}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_2] \quad . \tag{6}$$

The flux of $\mathbf{B}_2$ through the surface bounded by the big loop 1 can be calculated as follows

$$\begin{aligned}
\phi_{1,2} &= \frac{\mu_o}{4\pi}\int_0^{2\pi} d\theta \int_0^a \rho\, d\rho \frac{m}{(h^2+\rho^2)^{3/2}}\left[3\frac{h^2}{h^2+\rho^2} - 1\right] && (7)\\
&= \frac{\mu_o \pi}{4} b^2 I_2 \int_0^{a^2} dx \left(3\frac{h^2}{(h^2+x)^{5/2}} - \frac{1}{(h^2+x)^{3/2}}\right) && (8)\\
&= \frac{\mu_o \pi}{4} b^2 I_2 \left[-2\frac{h^2}{(x+h^2)^{3/2}} + 2\frac{1}{(x+h^2)^{1/2}}\right]_{x=0}^{x=a^2} && (9)\\
&= \frac{\mu_o \pi}{2}\frac{a^2 b^2}{(h^2+a^2)^{3/2}} I_2 \quad . && (10)
\end{aligned}$$

(c) Using eq. (4) and taking into account definition (1) one finds the following expression for the mutual inductance $M_{2,1}$

$$M_{2,1} = \frac{\mu_o \pi}{2}\frac{a^2 b^2}{(a^2+h^2)^{3/2}} \quad . \tag{11}$$

The same expression is found for for the mutual inductance $M_{1,2}$ from the result (10) and taking into account definition (2)

$$M_{1,2} = M_{2,1} = M \quad . \tag{12}$$

$\boxed{\text{-Problem 3-}}$ Calculate the energy stored in the toriodal coil described in the figure, knowing that the energy is "stored in the magnetic field" in the amount $(1/2\mu_o)B^2$ per unit volume. Use your answer to calculate the self-inductance of the coil.

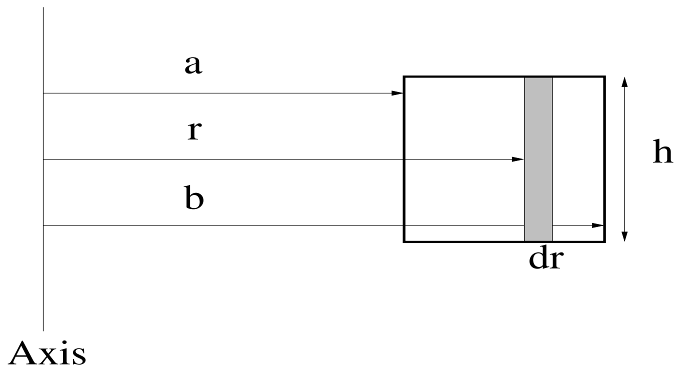

Figure 2: Toroidal coil with rectangular cross section (inner radius a, outer radius b and height h), which carries a total of N turns.

SOLUTION

We have to find the total energy "stored in the magnetic field" inside the toroidal coil, namely

$$W = \frac{1}{2\mu_o}\int_{\text{all space}} \mathbf{B}^2 \, d\tau \quad . \tag{1}$$

The cross sectional shape of the coil is immaterial as long as it remains the same all the way around the ring ($\phi \in [0, 2\pi]$). In this case follows that the magnetic field of the toroid is **circumferential** at all points and is given by the following expression

$$\mathbf{B}(r) = \begin{cases} \dfrac{\mu_o NI}{2\pi r} & \text{for points inside the coil} \\ 0 & \text{for point outside the coil} \end{cases} \quad . \tag{2}$$

Calling V_{coil} the space region occupied by the coil one can write

$$\int_{\text{all space}} \mathbf{B}^2 \, d\tau \overset{(2)}{=} \int_{V_{coil}} \mathbf{B}^2(r) \; rdr \; d\phi \; dz = \frac{\mu_o^2}{2\pi}(NI)^2[h\ln(\frac{b}{a})] \quad . \tag{3}$$

From the definition of energy associated to a magnetic field (eq. (1)) one finds the following expression for the total energy "stored in the magnetic field" inside the toroidal coil

$$W = \frac{\mu_o}{4\pi}(NI)^2[h \ln(\frac{b}{a})] \quad . \tag{4}$$

From this information one can compute the self-inductance of the coil. Looking at W as the energy "stored in the current distribution" of the toroidal coil (i.e. the work spent to raise the current in the coil from 0 to I) one comes up with the relation

$$W = \frac{1}{2}LI^2 \quad . \tag{5}$$

Finally from the result (4) and the expression (5) one obtains the self-inductance of the toroidal coil as

$$L = \frac{\mu_o N^2 h}{2\pi} \ln(\frac{b}{a}) \quad . \tag{6}$$

$\boxed{\text{-Problem 4-}}$ An electromagnetic crane is constructed af a U-shaped steel yoke with 1000 turns of wire carrying current I, as shown in the figure below. The permeability of the steel is 1000 μ_o. We want to lift a steel bar of dimensions 30 $cm \times 30$ cm and having the same permeability. Note that the dimensions of the bar are exactly such that the yoke and the bar form a square donut.
Estimate the magnitude of I in order that we are just able to lift the bar. The density of the steel is 8 g/cm^3.
[HINT $\mathbf{F} = -\nabla$ Energy]

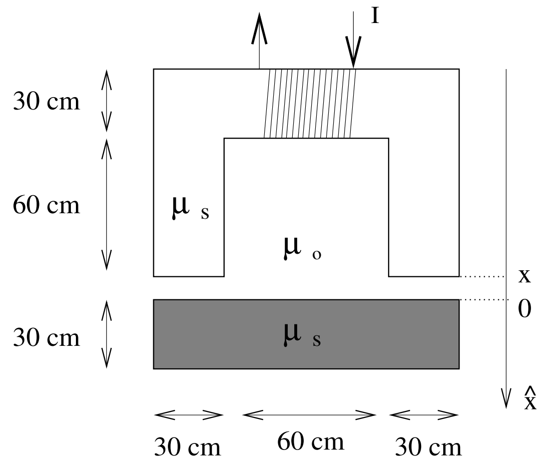

Figure 3: Both the yoke and the bar have a depth of 30 cm.

SOLUTION

Usefull quantities to solve the problem in MKSA units:

$\rho_s = 8 \times 10^3 \ kg/m$	density of the steel
$\mu_s = 1000 \ \mu_o$	magnetic permeability of the steel
$\mu_o = 4\pi \times 10^{-7} m \ kg/(A^2 sec^2)$	magnetic permeability of the vacuum
$V_y = 2v_b$	volume of the yoke
$V_b = 0.108 \ m^3$	volume of the steel bar to lift
$M_b = \rho V_b = 8.64 \times 10^2 \ kg$	mass of the steel bar to lift
$g \sim 10 \ m/sec$	acceleration of gravity
$N = 1000$	turns of wire
$n = N/(0.6 \ m) = 1.67 \times 10^3$ turns$/m$	turns per unit length

To lift the bar one has to overcome the gravitational force acting on the steel bar, namely $\mathbf{F}_g = M_s \mathbf{g} = (8.64 \times 10^3 N)\hat{\mathbf{x}}$.

The current flowing in the N turns of wire induce a constant magnetic induction $B = \mu_s n I$ inside the U-shaped steel yoke. Due to the high permeability of the steel the force lines of the magnetic induction will be trapped inside the donut composed by the yoke and the bar (see fig. 3). As long as the distance of the yoke from the bar ($|x|$ in the fig. 3) is small enough one can assume that **all the force lines** of the magnetic induction will remain inside the donut. Then the energy associated to the magnetic field of all the system can be evaluated as follows

$$W(x) \equiv \int_{\text{all space}} \frac{1}{2\mu} \mathbf{B}^2 \, d\tau = \frac{1}{2\mu_s}(V_b + V_y)B^2 + \frac{1}{2\mu_o} 2(Sx)B^2 \tag{1}$$

where $S = 9 \times 10^{-2} m$ is the area of the constant cross section of the donut. The last term of the right hand side of eq. (1) is obtained from the following considerations: one can assume that the magnetic induction in the region

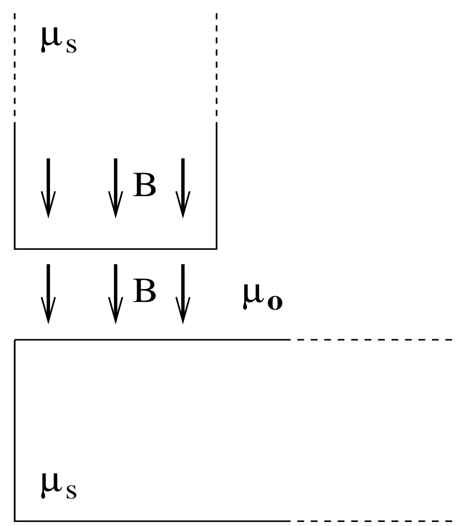

Figure 4: Behaviour of the magnetic induction through the contact region between the yoke and the bar.

of the yoke near to the contact with the bar, doesn' t have any components parallel to the contact surface between the yoke and the bar (infact since in the problem $\mu_s \gg \mu_o$, one can assume that the force lines of $\mathbf{B}$ are constrained to follow the circuit formed by the yoke), as is sketched in fig. 4 . Then using the boundary conditions deriving from the maxwell equation $\nabla \cdot \mathbf{B} = 0$ one can conclude that the amplitude of the magnetic induction in the vacuum regions has to be the same as the one of the magnetic induction in the yoke. Returning to eq. (1) one observes that W is a function of the position x of the steel bar (the yoke is fixed) and can finally evaluate the force that the yoke exercise on the bar, following the HINT of the problem

$$\mathbf{F}_B = -\hat{\mathbf{x}}\frac{dW(x)}{dx} = -\hat{\mathbf{x}}\frac{SB^2}{\mu_o} = -\hat{\mathbf{x}}\frac{\mu_s^2 S}{\mu_o}(nI)^2 \quad . \tag{2}$$

Then F_B result to be opposed to F_g and in order to lift the bar one must have

$$F_B > F_g \qquad \Rightarrow \qquad I > \sqrt{F_g/(\frac{\mu_s^2 S}{\mu_o} n^2)} \tag{3}$$

Inserting the numerical values one gets

$$I > 1.66 \times 10^-1 \ A \quad . \tag{4}$$