Many Body in General Relativity

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We review the physics of many bodies in the context of General Relativity. Starting from the stress energy tensor for one body, for a swarm of bodies, for a perfect fluid, we review relativistic hydrodynamics, kinetic theory, and statistical physics.

Keywords: Particle; Swarm; Perfect Fluid; Hydrodynamicsm; Kinetic Theory; Statistical Mechanics; General Relativity

CONTENTS

I. INTRODUCTION

Consider *spacetime* as a smooth manifold $\mathscr{M}$ of dimension d and metric tensor $\boldsymbol{g}$ with covariant components $g_{\alpha\beta}$. We will denote with an arrow over a bold face letter the corresponding 4-vector and with just the bold face symbol the 3-dimensional vector. Greek indexes run over the d spacetime dimensions. Roman indexes run only over the $d-1$ space dimensions. We use Einstein summation convention of tacitly assuming a sum over repeated indexes.

One particle

For *one body* of mass m we have a self gravitating system with a stress-energy tensor given by

$$T^{\alpha\beta}(\vec{\boldsymbol{x}}) = m \int u^\alpha u^\beta \delta^{(4)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{z}}(\tau))\, d\tau, \tag{1.1}$$

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where τ is the body proper time, $d\vec{\boldsymbol{z}}/d\tau = \vec{\boldsymbol{u}} = (\gamma, \gamma \boldsymbol{v})$ with $u^0 = dt/d\tau = \gamma = (1 - v^2)^{-1/2}$ and

$$T^{\alpha\beta}(\vec{\boldsymbol{x}}) = m \frac{u^\alpha u^\beta}{u^0} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{z}(t)), \tag{1.2}$$

where the body is at $\boldsymbol{z}(t)$ with velocity $\boldsymbol{v}(t)$ at time t.

Swarm of particles

For a *swarm* of N bodies all of the same mass m and $\boldsymbol{v}$

$$\begin{aligned} T^{\alpha\beta}(\vec{\boldsymbol{x}}) &= m u^\alpha u^\beta \sum_{i=1}^{N} \int \delta^{(4)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{z}}_i(\tau_i)) \, d\tau_i \\ &= m \frac{u^\alpha u^\beta}{u^0} \sum_{i=1}^{N} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{z}_i(t)) \\ &= m u^\alpha u^\beta n, \end{aligned} \tag{1.3}$$

where

$$n = \frac{1}{u^0} \sum_{i=1}^{N} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{z}_i(t)), \tag{1.4}$$

is the proper number density of bodies measured in a *comoving frame* where $\vec{\boldsymbol{u}} = (1, \boldsymbol{0})$.

Perfect fluid

For a *perfect fluid* of proper number density n of non interacting bodies all of the same mass m and $v = |\boldsymbol{v}|$ but isotropic velocity profile $\boldsymbol{v} = v\boldsymbol{n}$

$$T^{\alpha\beta} = \chi \langle u^\alpha u^\beta \rangle_{\boldsymbol{n}}, \tag{1.5}$$

so that $T^{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $T^{00} = \chi\gamma^2$. Since $T^{00} = \rho = n(\gamma m)$ is the energy density of the fluid we require $\chi = mn/\gamma$. Then

$$\begin{aligned} T^{ij} &= \chi \gamma^2 v^2 \langle n^i n^j \rangle_{\boldsymbol{n}} \\ &= \chi \gamma^2 v^2 \frac{1}{3} \delta^{ij} \\ &= n(\gamma m) v^2 \frac{1}{3} \delta^{ij} \\ &= p \delta^{ij}, \end{aligned} \tag{1.6}$$

where δ is a Kronecker delta and in the second equality we used isotropy of $\boldsymbol{n}$ and

$$\begin{cases} \rho = n(m\gamma) \\ p = \frac{1}{3}\rho v^2 \end{cases} \tag{1.7}$$

are respectively the mass density and pressure in the *isotropic frame* of the fluid. Summarizing

$$T^{\alpha\beta} = (\rho + p) u^\alpha u^\beta + p \eta^{\alpha\beta}, \tag{1.8}$$

where $||\eta^{\alpha\beta}|| = \mathrm{diag}\{-1, 1, 1, 1\}$ is the metric in Minkowski spacetime. For photons $v = 1$ and $p = \rho/3$. For $v \ll 1$, $\rho = nm(1 + v^2/2 + \ldots)$, and $p \approx nmv^2/3 = (2/3)(\rho - nm) = (2/3)\epsilon$, where $\epsilon = (3/2)k_B T$ is the internal energy of a monatomic ideal gas in thermal equilibrium at a temperature T, k_B is Boltzmann constant, and $p = nk_B T$ is the ideal gas equation of state.

II. HYDRODYNAMICS

Hydrodynamics concerns itself with the study of the motion of fluids (liquids and gases). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. Therefore when we speak of the "point" of a fluid (or of an infinitesimal volume of it) we mean not a single molecule of the fluid but a volume element still containing very many molecules but yet small compared with the volume of the whole fluid.

A. Newtonian

A mathematical description of the state of a moving fluid consists in specifying the fluid velocity $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$ and any two thermodynamic functions pertaining to the fluid, for instance the pressure $p = p(t, \boldsymbol{x})$ and the density $\rho = \rho(t, \boldsymbol{x})$, from which one can determine all other thermodynamic quantities. These 5 quantities are functions of the coordinates $\boldsymbol{x} = (x, y, z)$ and of time t. Once again we stress that a point $\boldsymbol{r}$ in space at a given time t refers to a fixed point and not to specific particles of the fluid. From Chapter 1 of Ref. [1] we find

$$\frac{\partial \rho}{dt} + \boldsymbol{\nabla}(\rho \boldsymbol{v}) = 0, \tag{2.1}$$

$$\frac{\partial \boldsymbol{v}}{dt} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \frac{1}{\rho}\boldsymbol{\nabla} p, \tag{2.2}$$

$$\frac{\partial s}{dt} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) s = 0, \tag{2.3}$$

where the first equation is the *continuity equation*, the second is the *Euler equation*, and the third one is the *equation for the adiabatic flow* in which $s = s(t, \boldsymbol{x})$ is the entropy per particle.

From the first law of thermodynamics follows

$$d\epsilon = T\, ds - p\, d(m/\rho), \tag{2.4}$$

$$\epsilon = \epsilon(\rho, s), \tag{2.5}$$

$$p = \rho^2 \left. \frac{\partial \epsilon}{\partial \rho} \right|_s, \tag{2.6}$$

$$T = \left. \frac{\partial \epsilon}{\partial s} \right|_\rho, \tag{2.7}$$

where ϵ is the internal energy per particle. Eqs. (2.6) and (2.7) can be considered as algebraic relations for the right hand side of Eqs. (2.2) and (2.3) respectively.

For an ideal gas $\epsilon = \epsilon(T)$ and for a monatomic gas

$$s = k_B \ln(T^{3/2}/\rho) + \text{constant}. \tag{2.8}$$

B. Relativistic

We will work in a Local Lorentz Frame (LLF). Recalling that the stress energy tensor is divergenceless, from the stress energy tensor of a perfect fluid (1.8) we find

$$\begin{aligned} 0 = T^{\alpha\beta}{}_{,\beta} &= (\rho + p)_{,\beta} u^\alpha u^\beta + (\rho + p) u^\alpha{}_{,\beta} u^\beta + (\rho + p) u^\alpha u^\beta{}_{,\beta} + p_{,\beta} \eta^{\alpha\beta} \\ &= \frac{d(\rho + p)}{d\tau} u^\alpha + (\rho + p) a^\alpha + (\rho + p) u^\alpha u^\beta{}_{,\beta} + p_,{}^\alpha, \end{aligned} \tag{2.9}$$

where the comma stands for a partial derivative. Multiplying by $\vec{\boldsymbol{u}}$ and recalling that $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{a}} = 0$ we find

$$\frac{d\rho}{d\tau} = -(\rho + p) \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}}, \tag{2.10}$$

which is the relativistic continuity expression which extends Eq. (2.1).

To find the extension of the Euler equation we introduce the projector tensor

$$P^{\alpha\beta} = \eta^{\alpha\beta} + u^\alpha u^\beta \quad \text{for } \vec{\boldsymbol{u}} \text{ timelike} \quad \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1$$
$$P^{\alpha\beta} = \eta^{\alpha\beta} - n^\alpha n^\beta \quad \text{for } \vec{\boldsymbol{n}} \text{ spacelike} \quad \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} = +1$$

Then

$$0 = P_{\alpha\gamma} T^{\alpha\beta}{}_{,\beta} = (\rho + p) a_\gamma + P_{\alpha\gamma} p,^{\alpha}, \tag{2.11}$$

or

$$(\rho + p)\vec{\boldsymbol{a}} = -\vec{\boldsymbol{\nabla}} p - \vec{\boldsymbol{u}} \frac{dp}{d\tau}, \tag{2.12}$$

which is the relativistic Euler equation which extends Eq. (2.2).

It is easy to see that in the Newtonian limit $\vec{\boldsymbol{u}} = (\gamma, \gamma \boldsymbol{v}) \approx (1, \boldsymbol{v})$ with $v \ll 1$ and $p \ll \rho$, Eq. (2.10) reduces to Eq. (2.1) and Eq. (2.12) reduces to Eq. (2.2).

Let us now discuss the continuity Eq. (2.10). First of all we observe that the mass density is not conserved $d\rho/d\tau \neq 0$. But the baryon, lepton, charge,… numbers are conserved. For example if we call $n = N/V$ the baryon number density in the rest frame of the fluid with N baryons in a volume V, N is certainly constant but V will change, so that

$$0 = \frac{dN}{d\tau} = \frac{d(nV)}{d\tau}, \tag{2.13}$$

but $(dV/d\tau)/V = \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}}$ (see Ex. 22.1 in Ref. [2]). So

$$\begin{aligned} 0 &= \frac{1}{V} \frac{d(nV)}{d\tau} \\ &= \frac{dn}{d\tau} + n \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} \\ &= \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\nabla}} n + n \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} \\ &= \vec{\boldsymbol{\nabla}}(n\vec{\boldsymbol{u}}), \end{aligned} \tag{2.14}$$

where we may define the divergenceless current density

$$\vec{\boldsymbol{J}} = n\vec{\boldsymbol{u}}. \tag{2.15}$$

Let us now discuss the thermodynamics. The second law tells that $ds/d\tau \geq 0$ where s is the entropy per baryon. The first law becomes

$$d(\rho/n) = -p\, d(1/n) + T\, ds, \tag{2.16}$$

or

$$d\rho = \frac{\rho + p}{n}\, dn + nT\, ds, \tag{2.17}$$

which is the relativistic extension of Eq. (2.4). In this equation the differential d can be substituted either with an exterior derivative $\widetilde{\boldsymbol{d}}$, with a gradient $\vec{\boldsymbol{\nabla}}$, or with a directional derivative $\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{u}}} = u^\alpha \partial/\partial x^\alpha = d/d\tau$. Given an equation of state $\rho = \rho(n, s)$ we will have

$$p = n \left.\frac{\partial \rho}{\partial n}\right|_s - \rho, \tag{2.18}$$

$$T = \frac{1}{n} \left.\frac{\partial \rho}{\partial s}\right|_n, \tag{2.19}$$

which are the relativistic extensions of Eqs. (2.6) and (2.7).

It is easy to show that a perfect fluid flow is adiabatic. From the relativistic continuity Eqs. (2.10) and (2.14) follows

$$\frac{d\rho}{d\tau} = \frac{\rho + p}{n} \frac{dn}{d\tau}. \tag{2.20}$$

Then from the relativistic first thermodynamic Eq. (2.17) follows

$$\frac{ds}{d\tau} = 0. \tag{2.21}$$

Shock wave

Consider a homogeneous, static, perfect fluid. A sound wave in the fluid is an adiabatic perturbation. The speed of sound is

$$v_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_s \tag{2.22}$$

Expand

$$\begin{aligned} \rho &= \rho_0 + \rho_1, \\ p &= p_0 + p_1, \\ n &= n_0 + n_1, \end{aligned}$$

where ρ_0, p_0, n_0 are constant in space (uniform fluid) and in time (static fluid) and ρ_1, p_1, n_1 are small perturbations. Taking $\vec{\boldsymbol{u}} = (1, \boldsymbol{v}_1)$ with $v_1 \ll 1$ we find from the continuity Eq. (2.10)

$$\frac{\partial \rho_1}{\partial t} = -(\rho_0 + p_0)\boldsymbol{\nabla}\boldsymbol{v}_1, \tag{2.23}$$

and from the spatial part of Euler Eq. (2.12)

$$(\rho_0 + p_0)\frac{\partial \boldsymbol{v}_1}{\partial t} = -\boldsymbol{\nabla} p_1, \tag{2.24}$$

where we neglect the last term $\vec{\boldsymbol{u}}\, dp/d\tau = \vec{\boldsymbol{u}}\, u^\alpha \partial p/\partial x^\alpha$ because an infinitesimal of second order and $\partial p_0/\partial t = 0$. Therefore putting together Eqs. (2.23) and (2.24) we find

$$\frac{\partial^2 \rho_1}{dt^2} = -(\rho_0 + p_0)\boldsymbol{\nabla}\frac{\partial \boldsymbol{v}_1}{\partial t} = \boldsymbol{\nabla}^2 p_1. \tag{2.25}$$

In a perfect fluid $p = p(\rho, T)$ so that $p(\rho_0 + \rho_1, T) = p(\rho_0, T) + \partial p(\rho_0, T)/\partial\rho|_s \rho_1 = p_0 + p_1$ with $p_1 = v_c^2 \rho_1$ and we finally find

$$\frac{\partial^2 \rho_1}{dt^2} = v_s^2 \boldsymbol{\nabla}^2 \rho_1, \tag{2.26}$$

which is the shock wave equation.

Bernoulli equation

Consider a steady, adiabatic flow of a perfect fluid. Since in a steady state $\partial p/\partial t = 0$ from the relativistic Euler Eq. (2.12) follows

$$(\rho + p)\frac{du^0}{d\tau} = -u^0 \frac{dp}{d\tau}, \tag{2.27}$$

So

$$\begin{aligned} \frac{d}{d\tau}\left(u^0 \frac{\rho + p}{n}\right) &= \frac{du^0}{d\tau}\frac{\rho + p}{n} + u^0 \frac{d}{d\tau}\left(\frac{\rho + p}{n}\right) \\ &= -\frac{u^0}{n}\frac{dp}{d\tau} + u^0 \frac{d}{d\tau}\left(\frac{\rho + p}{n}\right) \\ &= \frac{u^0}{n}\left[\frac{d\rho}{d\tau} - \frac{\rho + p}{n}\frac{dn}{d\tau}\right] = 0, \end{aligned} \tag{2.28}$$

where in the second equality we used Eq. (2.27) and in the last equality we used the relativistic first law of thermodynamics Eq. (2.17). We then conclude that

$$u^0 \frac{\rho + p}{n} \quad \text{is constant along the fluid flow lines.} \tag{2.29}$$

In the Newtonian limit $\vec{\boldsymbol{u}} = (\gamma, \gamma\boldsymbol{v}) \approx (1, \boldsymbol{v})$ with $v \ll 1$, $u^0 = \gamma = (1 - v^2)^{-1/2} \approx 1 + v^2/2$, and $P \ll \rho$. We can take $\rho = \rho_0(1 + \pi)$ and $(\rho + p)/n \approx \rho_0(1 + \pi + p/\rho_0)/n$, so that

$$\frac{1}{2}v^2 + \pi + p/\rho_0 \quad \text{is constant along the fluid flow lines,} \tag{2.30}$$

where the sum of the last two terms is the *enthalpy*.

III. KINETIC THEORY APPROACH

The kinetic theory approach is based on a *one body* distribution function. We will construct a Lorentz invariant phase space distribution function as a number density of particles in phase space

$$f = \frac{d\mathscr{N}}{d\boldsymbol{x}\, d\boldsymbol{p}}, \qquad \int f\, d\boldsymbol{x}\, d\boldsymbol{p} = N, \tag{3.1}$$

for a fluid of N bodies, where $d\boldsymbol{x} = dx^1dx^2dx^3$ and $d\boldsymbol{p} = dp^1dp^2dp^3$. So that f/N can be considered as a probability distribution function. We will now prove that f as defined above is a Lorentz invariant distribution. We start defining a proper 3-volume. The 4-volume $d^4\Omega = dx^0\, dx^1\, dx^2\, dx^3$ is invariant under a Lorentz transformation. Dividing by $d\tau$ we find another Lorentz invariant

$$dV = u^0\, dx^1\, dx^2\, dx^3. \tag{3.2}$$

Then we want to define a 3-volume element in momentum space. The 4-volume $d^4p = dp^0\, dp^1\, dp^2\, dp^3$ is invariant. Since $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ we will define

$$\begin{aligned} d\Pi &= \int d^4p\, \delta\left(\sqrt{-\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}} - m\right) \\ &= \frac{\sqrt{(p^0)^2 - \boldsymbol{p}^2}}{p^0} dp^1\, dp^2\, dp^3 \\ &= \frac{m}{p^0}\, dp^1\, dp^2\, dp^3. \end{aligned} \tag{3.3}$$

And

$$dV\, d\Pi = dx^1\, dx^2\, dx^3\ dp^1\, dp^2\, dp^3, \tag{3.4}$$

is Lorentz invariant.

We will now prove conservation of volume in phase space in curved spacetime (see Liouville theorem in BOX 22.6 of Ref. [2]). Consider a very small bundle of identical particles that move through curved spacetime on a neghboring geodesics. We want to prove that $d(dV\, d\Pi)/d\lambda = 0$ where λ is an affine parameter along the central geodesic of the bundle. Given any function of phase space $g(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})$, if $m \neq 0$ [1] take $\tau = a\lambda + b$ for arbitrary a and b. Then

$$\frac{dg(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})}{d\lambda} = \frac{\partial g}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} + \frac{\partial g}{\partial p^\alpha}\frac{dp^\alpha}{d\lambda}, \tag{3.5}$$

on a geodesic $dp^\alpha/d\tau = 0$ so

$$\frac{dg(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})}{d\lambda} = \frac{\partial g}{\partial x^\alpha} p^\alpha \frac{a}{m}, \tag{3.6}$$

and for $g = dV\, d\Pi$ and $p^\alpha = m dx^\alpha/d\tau$

$$\frac{dg}{d\lambda} = a\frac{dg}{d\lambda}, \tag{3.7}$$

for any a. so $dg/d\lambda = 0$. Since $d\mathscr{N}$ and $dV\, d\Pi$ are unchanged then also $f = d\mathscr{N}/dV\, d\Pi$ is unchanged

$$\frac{df}{d\lambda} = 0. \tag{3.8}$$

This equation is at the heart of the collisionless Boltzmann equation and the Vlasov equation. All these approximate theories are valid at sufficiently low density. Whereas for the full Boltzmann equation [3, 4] one has

$$\frac{df}{d\lambda} = \left(\frac{\partial f}{\partial \lambda}\right)_{\text{collisions}}. \tag{3.9}$$

This is the most famous of all Kinetic equations and was obtained by Boltzmann more than a century ago.

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[1] If $m = 0$ see BOX 22.6 of Ref. [2].

The phase space probability density of a system in thermodynamic equilibrium at an inverse temperature $\beta = 1/k_B T$ with k_B Boltzmann constant, is not an explicit function of proper time. We shall use the symbol f_0 to denote the equilibrium probability density.

We can write [5], on a comoving frame with $\vec{\boldsymbol{u}} = (1, \mathbf{0})$,

$$f_0(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = \frac{d\mathscr{N}}{dV\, d\Pi} \propto \frac{g}{h^3} \frac{1}{e^{-\beta(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{u}}+\mu)} - \varepsilon}, \tag{3.10}$$

where h is Planck constant, μ is the chemical potential, $g = 2J + 1$ is the spin J degeneracy (2 polarizations for photons), and

$$\varepsilon = \begin{cases} +1 & \text{Bose-Einstein statistics} \\ 0 & \text{Maxwell-Boltzmann statistics} \\ -1 & \text{Fermi-Dirac statistics} \end{cases} \tag{3.11}$$

Next we can take moments of f_0 with respect to $\vec{\boldsymbol{p}}$

$$\int f_0 p^\mu \, d\Pi = J^\mu, \tag{3.12}$$

$$\int f_0 p^\mu p^\nu \, d\Pi = T^{\mu\nu}, \tag{3.13}$$

where $d\Pi = d\boldsymbol{p}/p^0$ with $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$. For $\vec{\boldsymbol{u}} = (1, \mathbf{0})$, from Eqs. (2.15) and (1.8), we must have

$$J^\mu = n u^\mu, \tag{3.14}$$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p \eta^{\mu\nu}. \tag{3.15}$$

So

$$\begin{aligned} n &= -J^\mu u_\mu = -\int f_0 p^\mu u_\mu \, d\Pi = \int f_0 \, d\boldsymbol{p} \\ &= \frac{g}{h^3} \int_0^\infty \frac{4\pi p^2 \, dp}{e^{\beta[\sqrt{\boldsymbol{p}^2+m^2}-\mu]} - \varepsilon}. \end{aligned} \tag{3.16}$$

Introduce the following change of variables

$$\begin{cases} p = m \sinh\chi \\ \bar{\beta} = m\beta \end{cases} \tag{3.17}$$

so that from Eq. (3.16) we find

$$n = \frac{4\pi g m^3}{h^3} \int_0^\infty \frac{\sinh^2\chi \, \cosh\chi \, d\chi}{e^{[\bar{\beta}\cosh\chi - \beta\mu]} - \varepsilon}. \tag{3.18}$$

For the pressure

$$\begin{aligned} p &= \frac{1}{3}(u_\mu u_\nu + \eta_{\mu\nu}) T^{\mu\nu} \\ &= \frac{1}{3} \int f_0 \boldsymbol{p}^2 \, d\Pi \\ &= \frac{1}{3} \int f_0 \boldsymbol{p}^4 \, \frac{d\boldsymbol{p}}{p^0} \\ &= \frac{4\pi g m^4}{3h^3} \int_0^\infty \frac{\sinh^4\chi \, d\chi}{e^{[\bar{\beta}\cosh\chi - \beta\mu]} - \varepsilon}. \end{aligned} \tag{3.19}$$

Also

$$\begin{aligned} \rho - 3p &= -T^\alpha{}_\alpha = m^2 \int f_0 \, \frac{d\boldsymbol{p}}{p^0} \\ &= \frac{4\pi g m^4}{h^3} \int_0^\infty \frac{\sinh^2\chi \, d\chi}{e^{[\bar{\beta}\cosh\chi - \beta\mu]} - \varepsilon}. \end{aligned} \tag{3.20}$$

Maxwell-Boltzmann statistics ($\varepsilon = 0$)

From Ref. [6] we learn that

$$K_n(\bar\beta) = \frac{\bar\beta^n}{(2n-1)!!}\int_0^\infty d\chi\, \sinh^{2n}\chi\, e^{-\bar\beta\cosh\chi} \tag{3.21}$$
$$= \frac{\bar\beta^{n-1}}{(2n-3)!!}\int_0^\infty d\chi\, \sinh^{2n-2}\chi\cosh\chi\, e^{-\bar\beta\cosh\chi}, \tag{3.22}$$

where K_n is a modified Bessel function of the second kind and in the second equality we performed an integration by parts. The asymptotic behaviors of the modified Bessel function are as follows

$$K_n(\bar\beta) = \sqrt{\frac{\pi}{2\bar\beta}}e^{-\bar\beta}\left[1+\frac{4n^2-1}{8\bar\beta}+O(\bar\beta^{-2})\right] \quad \bar\beta \gg 1, \tag{3.23}$$
$$K_n(\bar\beta) = \frac{1}{\bar\beta^n}\left[2^{n-1}(n-1)!+O(\bar\beta^2)\right] \quad \bar\beta \ll 1. \tag{3.24}$$

We then find

$$n = aK_2(\bar\beta)/\bar\beta, \tag{3.25}$$
$$p = amK_2(\bar\beta)/\bar\beta^2 = n\, k_BT, \tag{3.26}$$
$$\rho - 3p = amK_1(\bar\beta)/\bar\beta, \tag{3.27}$$

where $a = 4\pi g m^3 e^{\beta\mu}/h^3$. Note that the ideal gas equation of state (3.26) is a relativistic invariant.

For the internal energy per particle we then find

$$u(T) = \frac{\rho}{n} = m\frac{K_1(\bar\beta)}{K_2(\bar\beta)} + 3k_BT = \begin{cases} m\left[1+\frac{3}{2}\frac{k_BT}{m}+\ldots\right] & \bar\beta \gg 1, \\ 3k_BT & \bar\beta \ll 1, \end{cases} \tag{3.28}$$

where we used the asymptotic expansions (3.23) and (3.24).

For the ratio of the specific heats $\gamma(T) = c_p/c_v$ we then find

$$\gamma(T) = \frac{\left.\frac{du}{dT}\right|_p}{\left.\frac{du}{dT}\right|_v} = 1 + \frac{k_B}{\left.\frac{du}{dT}\right|_v} = \begin{cases} 5/3 & \bar\beta \gg 1, \\ 4/3 & \bar\beta \ll 1. \end{cases} \tag{3.29}$$

IV. STATISTICAL MECHANICS APPROACH

The statistical mechanics approach is based on a *many body* distribution function.

The aim of equilibrium statistical mechanics is to calculate observable properties of a system of interest either as averages over a phase trajectory (the method of Boltzmann), or as averages over an ensemble of systems, each of which is a replica of the system of interest (the method of Gibbs). In Gibbs's formulation of statistical mechanics the equilibrium distribution of phase points of systems of N bodies of the ensemble is described by a phase space probability density $f_0^{(N)}$. The quantity $f_0^{(N)}\prod_{i=1}^N d\boldsymbol{x}_i\, d\boldsymbol{p}_i$ is proportional to the probability that the physical system is in a microscopic state represented by a phase point lying in the infinitesimal, $6N$-dimensional phase space element $\prod_{i=1}^N d\boldsymbol{x}_i\, d\boldsymbol{p}_i = d^NV\, d^N\Pi$. So that now

$$f_0^{(N)} = \frac{d\mathscr{N}}{d^NV\, d^N\Pi}, \qquad \int f_0^{(N)}\, d^NV\, d^N\Pi = N!. \tag{4.1}$$

In the canonical Gibbs ensemble,

$$f^{(N)} \propto e^{-\beta\mathscr{H}}, \tag{4.2}$$

with $\mathscr{H}$ the Hamiltonian of the fluid. Integrating on the phase space of $N-1$ particles we find

$$\int f_0^{(N)}\, d^{N-1}V\, d^{N-1}\Pi = (N-1)! f_0^{(1)}, \tag{4.3}$$

where $f_0^{(1)} = f_0$ is the one body equilibrium distribution function (3.10) of the previous Section III.

FIG. 1. ...

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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