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Wigner Crystallization Versus Goldstone Theorem in a 2D Surface One Component Plasma

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We present a self contained derivation of the Wigner crystallization in a bidimensional one-component classical plasma.

Keywords: Wigner crystallization; Goldstone theorem; one component plasma; jellium.

1. Introduction

1.1. *The model*

A bidimensional one-component classical plasma (2D-OCP) is a system of N point classical particles of charge e and mass m living in a bidimensional space in an area Λ, immersed in a homogeneous neutralizing background and interacting with the usual tridimensional Coulomb potential $v(r) = e^2/r$.

This system can be parametrized by four constants: the mass m, the charge e, the area per particle $\Lambda/N \equiv \rho^{-1}$, and the temperature $T = (\beta k_B)^{-1}$ (where β^{-1} is the mean kinetic energy per particle). These parameters can be cast into two characteristic constants: (i) the mean interparticle distance r_o, such that $\pi r_o^2 \rho = 1$,[a] (ii) the coupling constant, defined as the ratio of the mean Coulomb energy to the mean kinetic energy per particle, namely $\Gamma = \beta e^2/r_o$.[b]

The 2D-OCP turns out to be a good model for example, for the description of surface electrons on liquid He.[1] The range of densities experimentally accessible in these systems is particularly wide, i.e. at $T = 0.5\,K$ $\Gamma \in [2, 200]$.

Some recent studies on the electron gas or *the jellium* are about two-dimensional systems[2–6] or three-dimensional ones.[7–10]

1.2. *The Wigner crystallization*

Wigner[11,12] emphasized that at low densities, i.e. large r_s, the Hartree–Fock value of the ground state energy of a $3D$-jellium must become a poor approximation, as the Slater determinant wave function used in calculating it minimizes the kinetic energy scaling as an inverse square lenght r_s^{-2}, while at low density the energy is dominated by the potential energy scaling as an inverse length r_s^{-1}. He noticed that to minimize the potential energy as $r_s \to \infty$, electrons would avoid one another optimally which is achieved by their going onto the sites of a crystal lattice.

The argument of Wigner, while given explicitly for the quantum electron fluid, obviously is basically

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[a]In the quantum system (the Jellium) r_o is usually expressed in units of the Bohr radius ($a_B = \hbar^2/me^2$) and called $r_s = r_o/a_B$.
[b]In the quantum system the coupling constant is $r_s = (e^2/r_o)(\hbar^2/mr_o^2)^{-1}$ itself.

a comparison of electrostatic potential energy per electron (Madelung energy) with kinetic energy. In the classical system, the Wigner crystal, i.e. the high coupling state ($\Gamma \gg 1$) would occur either for sufficiently low temperatures or sufficiently high densities.

Such phase transition was first experimentally observed by Grimes and Adams[13] in the system of electrons trapped on the surface of liquid Helium (in accord with the predictions of Crandall and Williams[14]). They found that the Coulomb liquid crystallized into a triangular lattice in correspondence of a value for the coupling constant of $\Gamma = 137 \pm 15$ in agreement with the previous Monte Carlo calculation by Gann, Chakravarty, and Chester.[15] The crystallization of electrons trapped on the liquid Helium surface corresponds to a classical regime.

1.3. *The statistical physics problem*

The theoretical possibility of crystalline order is notoriously related to poor clustering of particle correlations and has drawn considerable attention for Coulomb systems in low dimensionalities.

Given a physical system with short-range forces and a continuous symmetry, if the ground state is not invariant under the symmetry, the Goldstone theorem states that the system possesses excitations of arbitrarily low energy[16] (the original Goldstone, Salam, Weinberg paper[16] treated the relativistic case only in detail, although the non-relativistic case has been extensively discussed after that[17]).

In non-zero temperature classical system, what is important in connection with symmetry breakdown are the clustering properties (the law of the decay) of the correlations between the particles. In this case, one may then reformulate the Goldestone theorem as follows: given a system with short-range forces and a continuous symmetry group, if the thermodynamic equilibrium state is not invariant under the symmetry then the system does not possess exponential clustering (but a non-integrable clustering[18]).

In contrast to what happens for systems of classical particles interacting via short-range potentials, where crystallinity is ruled out by the original Mermin argument in one or two dimensions,[19,20]

matters are not so transparent in the case of long-range interactions (e.g. Coulomb).

The purpose of this project is to show how in a 2D-OCP, Goldstone's theory manifests itself through the statement that a symmetry breakdown of translational invariance (appearance of a crystalline phase with positional long-range order) is destroyed by the transverse sound like phonons if the two point correlation functions decay faster than $|\mathbf{r}_1 - \mathbf{r}_2|^{-1/2}$ as $|\mathbf{r}_1 - \mathbf{r}_2| \to \infty$.

Requardt and Wagner[21] proved this property using the Bogoliubov inequality (a modified version of the original Mermin inequality) improving and presumably optimizing various known bounds. Their strategy was to introduce an external localizing one body potential $\alpha V_{\text{ext}}(\mathbf{r})$ with peaks at the sites of a given lattice in order to start with the system with broken translational invariance and to take *first* the thermodynamic limit[c] and let *afterwards* $\alpha \to 0$. In Sec. 5, I will briefly retrace their argument.

It's worth mentioning that the situation is here analogous to the Ising model on a lattice where in order to prove or disprove a symmetry breakdown of the state at low T (existence of spontaneous magnetization), the Hamiltonian symmetry should be broken, introducing a small external magnetic field $h \geq 0$ at every lattice point, computing the magnetization, and removing the field afterwards.

A simple argument that can be used to realize that the transverse phonons are the ones which matter is due to Peierls.[22,23] Start from the assumption that a crystal of electrons exists. Calculate the mean square particle displacement around the equilibrium position within the harmonic approximation, namely

$$\langle u^2 \rangle = \frac{\hbar}{\pi N m} \sum_{i=1}^{2} \sum_{\mathbf{k}} \left(n_i(\mathbf{k}) + \frac{1}{2} \right) \omega_i(\mathbf{k}), \quad (1)$$

where $n_i(\mathbf{k})$ is the Bose occupation number of the ith phonon polarization of frequency $\omega_i(\mathbf{k})$ and in-plane wave vector $\mathbf{k}$. So that (1) can be rewritten as

$$\langle u^2 \rangle = \frac{\hbar}{2\pi N m} \sum_{i=1}^{2} \sum_{\mathbf{k}} \frac{\coth\left[\frac{\beta \hbar \omega_i(\mathbf{k})}{2} \right]}{\omega_i(\mathbf{k})}. \quad (2)$$

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[c]This is an uncontrollable point of their treatment. Notice that for the 2D-OCP the correlation functions have been calculated exactly in the thermodynamic limit only for the system with logarithmic interparticle potential and at $\Gamma = 2$ by Jancovici.[28]

In the long wavelength limit ($kr_o \ll 1$), the system behaves as if it were weakly coupled. In such a limit, the longitudinal excitation mode frequency $\omega_1(k)$ (the plasmon) turns out to be equal to $\sqrt{\rho v(k)k^2/m} \propto k^{1/2}$ to leading order in k while the transverse one behaves as $\omega_2(k) \propto k$. Using this asymptotic behavior in Eq. (2), one can immediately check that if the contribution to the mean square displacement due to the plasmon remains finite, the one due to the transverse phonon diverges logarithmically at any finite temperature. So within the harmonic approximation, one cannot assume the existence of an electron crystal. It is helpful to stress that the divergence appears only at the thermodynamic limit when the confining external potential goes to zero (for finite confining potential, $\omega(k = 0)$ is finite).

1.4. *What do we know about the homogeneous phase?*

For a homogeneous fluid confined to a single layer, it is known that while the charge sum rule suffices to ensure that the structure factor $S(k)$ vanishes as $k \to 0$, a dipole moment arising from the three-body correlation function ensures that $S(k)/k$ is a finite constant in the same long wavelength limit. Bearing in mind the possibility of other singularities[24] arising at finite k, the conclusion is that the pair correlation functions, independently from the thermodynamic parameters (ρ and T), cannot decay asymptotically faster than r^{-3} at large r.[25] This property shows in particular, the importance of dimensionality, keeping in mind how in the 3D-OCP the decay must be faster than any inverse power (due to the re-established harmonicity of the potential).

Moreover, in the weak coupling regime $\Gamma \ll 1$ (high T or low ρ), either the mean field approximation (derived in an early work by Fetter[26] within a hydrodynamic approach which reduces in the static case to the Debye-Hückel approximation) or the Gaussian approximation (or Random Phase Approximation[27]) predicts a clustering $1/r^3$. This result and the one of Requardt and Wagner that I am going illustrate in the following section are in accord with the Wigner ersatz: if there has to be a phase transition, this has to occur in the strong coupling regime $\Gamma \gg 1$.

2. Clustering and Crystallization

2.1. *The Requardt and Wagner argument*

I will decompose all vectors into their in-plane and z components, with the notations $\mathbf{R} \equiv (\mathbf{r}, z)$ and $\mathbf{K} \equiv (\mathbf{k}, k_z)$. If the N electrons are supposed to form a crystal with Bravais lattice $\mathcal{L}$ generated by ($\mathbf{a}_1$, $\mathbf{a}_2$) ($\mathcal{L}^\star$ will be the reciprocal lattice generated by ($\mathbf{b}_1$, $\mathbf{b}_2$) with $\mathbf{b}_i \cdot \mathbf{a}_j = 2\pi\delta_{i,j}$), I can take a plane Λ of sides $L\mathbf{a}_1$ and $L\mathbf{a}_2$, and wrap it around a torus to ensure periodic boundary conditions. As anticipated, the electron localization is achieved by introducing a periodic external potential with periods $\mathbf{a}_1$ and $\mathbf{a}_2$. Taking the thermodynamic limit of the system ($N \to \infty$ and $\Lambda \to \infty$), this potential will ensure an equilibrium state nontrivially periodic.

The condition for crystallinity (long range order) is that the one-body density, namely

$$\rho(\mathbf{r}) = \left\langle \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) \right\rangle, \qquad (3)$$

where $\langle \cdots \rangle$ denotes the canonical equilibrium average and $\mathbf{r}_i$ is the position of the ith particle, be a (nonconstant) periodic function with periods $\mathbf{a}_1$ and $\mathbf{a}_2$. In the finite system, one can define the Fourier transform of $\rho(\mathbf{r})$ as follows[d]:

$$\rho(\mathbf{k}) = \frac{1}{N} \int_\Lambda \rho(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^2\mathbf{r} \qquad (4)$$

$$= \frac{1}{N} \left\langle \sum_i e^{-i\mathbf{k}\cdot\mathbf{r}_i} \right\rangle, \qquad (5)$$

where $\mathbf{k}$ is a vector consistent with the periodic boundary conditions, i.e. $\mathbf{k} \in \mathcal{L}^\star/L$ ($\mathbf{k} = (n_1/L)\mathbf{b}_1 + (n_2/L)\mathbf{b}_2$ with n_1 and n_2 — two integers). The existence of a nontrivial periodic equilibrium state then, means that $\rho(\mathcal{K}) \neq 0$ for at least one non-zero vector $\mathcal{K}$ of the reciprocal lattice. Note that since I'm assuming that in the thermodynamic limit, the equilibrium state and its correlation functions have the periodicity of the Bravais lattice, one can also write in that limit

$$\rho(\mathbf{r}) = \rho \sum_{\mathcal{K}} \rho(\mathcal{K})e^{i\mathcal{K}\cdot\mathbf{r}}, \qquad (6)$$

where the sum is now over all the reciprocal lattice vectors $\mathcal{K}$ and the Fourier coefficients are

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[d]Implying $\rho(\mathbf{r})/\rho = \sum_k \rho(k) \exp(i\mathbf{kr})$.

given by

$$\rho(\mathcal{K}) = \frac{1}{\rho C_o} \int_{C_o} d^2\mathbf{r} \rho(\mathbf{r}) e^{-i\mathcal{K}\cdot\mathbf{r}}, \qquad (7)$$

where $C_o = \Lambda/L^2$ is the area of the primitive cell.

2.1.1. *Potential energy*

Take the dielectric constant of the medium to be one so that image forces are absent. The particles on the plane $z = 0$ interact via the Coulomb potential $e^2\phi(r,z) = e^2(r^2 + z^2)^{-1/2}$ with two-dimensional Fourier transform $\phi(k,z) = (2\pi/k)\exp(-k|z|)$.

The total potential energy of the 2D-OCP subject to the external potential $\alpha V_{\text{ext}}(\mathbf{r})$ can be written as

$$U = \frac{1}{2}\int_\Lambda d^2\mathbf{r}\left(\sum_{i\neq j}\delta(\mathbf{r} - \mathbf{r}_i + \mathbf{r}_j) - \frac{N}{\Lambda}\right)e^2\phi(r,0)$$

$$- \frac{1}{2}\int_\Lambda d^2\mathbf{r}\sum_i \delta(\mathbf{r} - \mathbf{r}_i)e^2\phi(|\mathbf{r} - \mathbf{r}_i|,0)$$

$$+ \alpha\int_\Lambda d^2\mathbf{r}\left(\sum_i \delta(\mathbf{r} - \mathbf{r}_i) - \frac{N}{\Lambda}\right)V_{\text{ext}}(\mathbf{r}), \qquad (8)$$

where the second (formally infinite) integral represents the self-energy counterterm occurring in the electrodynamics of ideal point particles and the last integral is the contribution from the external potential. Extracting from U, the terms with explicit dependence on the particle position, one gets

$$U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$$

$$= U_o + \frac{1}{2}\sum_{i\neq j}\frac{1}{\Lambda}\sum_{\mathbf{k}\neq 0}\frac{2\pi e^2}{k}e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)}$$

$$+ \alpha\sum_i V_{\text{ext}}(\mathbf{r}_i), \qquad (9)$$

where $\mathbf{k} \in \mathcal{L}^\star/L$.

2.1.2. *Mermin inequality*

Define one-body functions of the in-plane position $\mathbf{r}$ by

$$\psi(\mathbf{r}) = e^{i(\mathbf{k}+\mathcal{K})\cdot\mathbf{r}}, \qquad (10)$$

and

$$\varphi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad (11)$$

where $\mathcal{K} \in \mathcal{L}^\star$, $\mathcal{K} \neq 0$ and as usual $\mathbf{k} \in \mathcal{L}^\star/L$ so that $\psi(\mathbf{r})$ and $\varphi(\mathbf{r})$ obey periodic boundary conditions on the perimeter of the 2D-OCP. Choose next

$$A = \sum_i \psi(\mathbf{r}_i), \qquad (12)$$

$$B = e^{\beta U(\mathbf{r}_1,\ldots,\mathbf{r}_N)}\sum_j(\mathbf{e}_t \cdot \nabla_j)[\varphi(\mathbf{r}_j)e^{-\beta U(\mathbf{r}_1,\ldots,\mathbf{r}_N)}], \qquad (13)$$

where $(\mathbf{e}_t \cdot \nabla_j)$ denotes the derivative with respect to $\mathbf{r}_j$ along the versor $\mathbf{e}_t$. By plugging A and B into the Schwartz inequality

$$\langle|A|^2\rangle \geq \frac{|\langle A^*B\rangle|^2}{\langle|B|^2\rangle}, \qquad (14)$$

and performing an integration by parts in the numerator on the right-hand side and two in the denominator (in all of which the line integral terms vanish due to the periodic boundary conditions), one reaches Mermin inequality (see the Appendix)

$$S(\mathbf{k} + \mathcal{K}) \geq \frac{[(\mathbf{k} + \mathcal{K}) \cdot \mathbf{e}_t]^2|\rho(\mathcal{K})|}{N(\mathbf{k} \cdot \mathbf{e}_t)^2 + D_{tt}(\mathbf{k})}, \qquad (15)$$

where

$$S(\mathbf{q}) - 1 = \frac{1}{N}\left\langle\sum_{i\neq j}e^{i\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\right\rangle - N\delta(\mathbf{q}), \qquad (16)$$

and

$$D_{tt}(\mathbf{q}) = \frac{\beta}{N}\left\langle\sum_{i\neq j}e^{i\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_j)}(\mathbf{e}_t \cdot \nabla_j)\right.$$

$$\left. \times (\mathbf{e}_t \cdot \nabla_i)U(\mathbf{r}_1,\ldots,\mathbf{r}_N)\right\rangle. \qquad (17)$$

Using the expression for the total potential energy (9) into (17), one gets

$$D_{tt}(\mathbf{k}) = \frac{\beta}{\Lambda}\sum_{\mathbf{p}\neq 0}\phi(p,0)(\mathbf{p}\cdot\mathbf{e}_t)^2[S(\mathbf{p} - \mathbf{k}) - S(\mathbf{p})]$$

$$+ \beta\rho\sum_{\mathbf{p}\neq 0}\phi(p,0)(\mathbf{p}\cdot\mathbf{e}_t)^2[\delta(\mathbf{p} - \mathbf{k}) - \delta(\mathbf{p})]$$

$$+ \frac{\alpha}{N}\beta\sum_{i=1}^N\langle(\mathbf{e}_t \cdot \nabla_i)^2V_{\text{ext}}(\mathbf{r}_i)\rangle. \qquad (18)$$

This is apparently a very complicated term but can be greatly simplified by: (i) restricting ourselves to treat only the transverse soundlike

phonons (remember Sec. 3)), $(\mathbf{k} \cdot \mathbf{e}_t) = 0$, so that the second sum in (18) vanishes; (ii) taking the thermodynamic limit. In this limit, the particle–background interaction, which is of order $\rho e^2/L$, vanishes, making unrestricted the first sum in (18). Moreover, rewriting the last term of (18) as $[\beta \int_\Lambda \rho(\mathbf{r})(\mathbf{e}_t \nabla)^2 V_{\text{ext}}(\mathbf{r}) d^2\mathbf{r}]\alpha/N$ is apparent how this term vanishes when one takes first the thermodynamic limit and after letting $\alpha \to 0$.

The final expression for D_{tt} becomes

$$D_{tt}(\mathbf{k}) = \beta\rho \sum_{\mathbf{p}} \phi(p,0)(\mathbf{p} \cdot \mathbf{e}_t)^2 [S(\mathbf{p} - \mathbf{k}) - S(\mathbf{p})], \tag{19}$$

and the Mermin inequality becomes

$$S(\mathbf{k} + \mathcal{K}) \geq \frac{[\mathcal{K} \cdot \mathbf{e}_t]^2 |\rho(\mathcal{K})|}{D_{tt}(\mathbf{k})}. \tag{20}$$

2.1.3. *Clustering hypothesis*

To give some physical insight to the meaning of the structure factor $S(\mathbf{q})$ is useful to show its relationship to the radial distribution function

$$\rho g(\mathbf{r}) \equiv \frac{1}{N} \int_\Lambda d\mathbf{r}' \rho^{(2)}(\mathbf{r}', \mathbf{r}' - \mathbf{r}) \tag{21}$$

$$= \frac{1}{N} \left\langle \sum_{i \neq j} \delta(\mathbf{r}_i - \mathbf{r}_j - \mathbf{r}) \right\rangle. \tag{22}$$

Calling $S(\mathbf{r})$ the Fourier transform of the structure factor, one gets

$$S(\mathbf{r}) = \rho g(\mathbf{r}) - \rho + \delta(\mathbf{r}). \tag{23}$$

This identity offers a simple electrostatic interpretation: imagine to fix a charge at the origin, then $S(\mathbf{r})$ represents the total charge density on the plane which is made up by the fixed charge ($\delta(\mathbf{r})$), the neutralizing background ($-\rho$), and the "electronic cloud" screening the fixed charge ($\rho g(\mathbf{r})$). So that, for example, the property of the two-body correlation function $\rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2)$, to be equal to N times the particle density, when integrated over $\mathbf{r}_1$ or $\mathbf{r}_2$ on all Λ, is generally called charge or electroneutrality sum rule and expressed as

$$\int_\Lambda \rho[g(r) - 1]d^2\mathbf{r} \equiv \int_\Lambda \rho h(\mathbf{r})d^2\mathbf{r} = -1. \tag{24}$$

I will show now following Requardt and Wagner, that since our system exhibits crystalline order in the thermodynamic limit, $h(\mathbf{r})$ is asymptotically nontrivially periodic as $r \to \infty$ (in particular not decaying) in this limit. Write

$$\rho h(\mathbf{r}) = \frac{1}{N} \int_\Lambda d^2\mathbf{r}' \rho^{(2)}(\mathbf{r}', \mathbf{r}' - \mathbf{r}) - \rho \tag{25}$$

$$= \frac{1}{N} \int_\Lambda d^2\mathbf{r}' \rho_T^{(2)}(\mathbf{r}', \mathbf{r}' - \mathbf{r})$$

$$+ \frac{1}{N} \int_\Lambda d^2\mathbf{r}' \rho(\mathbf{r}')\rho(\mathbf{r}' - \mathbf{r}) - \rho, \tag{26}$$

and let $\Lambda \to \infty$. In the state, at thermodynamic equilibrium (in a pure phase), the truncated correlation function $\rho_T^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = \rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2) - \rho^{(1)}(\mathbf{r}_1)\rho^{(1)}(\mathbf{r}_2)$ has to decay to zero in the difference variable $(\mathbf{r}_1 - \mathbf{r}_2)$ on *a priori* grounds. The second term in (25) acquires the form (C_o is the unit cell)

$$\frac{1}{\rho C_o} \int_{C_o} d^2\mathbf{r}' \rho(\mathbf{r}')\rho(\mathbf{r}' - \mathbf{r}) - \rho \tag{27}$$

$$= \rho \sum_{\mathcal{K} \neq 0} |\rho(\mathcal{K})|^2 e^{-i\mathcal{K}\mathbf{r}}, \tag{28}$$

with $\mathcal{K} \in \mathcal{L}^\star$ [see Eqs. (6) and (7)]. If the equilibrium state is nontrivially periodic (i.e. $\rho(\mathcal{K}) \neq 0$ for at least one $\mathcal{K} \neq 0$), one can infer that $\rho h(\mathbf{r})$ is asymptotically nontrivially periodic as $r \to \infty$.

Following Requardt and Wagner, one can then split off in $\rho h(\mathbf{r})$ the term asymptotically oscillating in the thermodynamic limit,

$$\rho h^{(o)} = \frac{1}{N} \int_\Lambda d^2\mathbf{r}' \rho(\mathbf{r}')\rho(\mathbf{r}' - \mathbf{r}) - \rho, \tag{29}$$

and assume that the remaining part, $\rho h^{(d)}(\mathbf{r})$, can be bound uniformly in Λ both like

$$|\rho h^{(d)}(\mathbf{r})| \leq C_1 \quad \text{and}$$

$$X|\rho h^{(d)}(\mathbf{r})| \leq \frac{C_2}{r^{1/2+\varepsilon}} \quad \varepsilon > 0. \tag{30}$$

2.1.4. *Small wavelength limit in Mermin inequality*

Rewriting in real space $D_{tt}(\mathbf{k})$ defined in (19), one gets

$$D_{tt}(\mathbf{k}) = \beta\rho \int_\Lambda d^2\mathbf{r} h(\mathbf{r})[1 - \cos(\mathbf{k} \cdot \mathbf{r})](\mathbf{e}_t \cdot \nabla)^2 \phi_\Lambda(\mathbf{r}), \tag{31}$$

with the effective potential

$$\phi_\Lambda(\mathbf{r}) = \frac{1}{\Lambda} \sum_{p \in \mathcal{L}^\star/L} \phi(p) \exp(i\mathbf{p}\mathbf{r}).$$

R. Fantoni

Call $D_{tt}^{(o)}(\mathbf{k})$, $D_{tt}(\mathbf{k})$ calculated using $h^{(o)}$ instead of h, and $D_{tt}^{(d)}(\mathbf{k})$, $D_{tt}(\mathbf{k})$ calculated using $h^{(d)}$. It can be easily shown that in the thermodynamic limit, the large wavelength $(k \to 0)$ behaviour of such functions is, uniformly in Λ, as follows:

$$D_{tt}^{(o)}(\mathbf{k}) \propto k^2, \tag{32}$$

and under the clustering assumption (30)

$$D_{tt}^{(d)}(\mathbf{k}) \propto k^{3/2}. \tag{33}$$

The conclusion is that in the thermodynamic limit, the Mermin inequality (20) becomes

$$S(\mathbf{k} + \mathcal{K}) \geq \text{const}(\mathcal{K} \cdot \mathbf{e}_t)^2 |\rho(\mathcal{K})|^2 \frac{1}{k^{3/2}}. \tag{34}$$

I'm now able to show, using (34), that the poor decay of (30) is still too fast to allow the existence of a $\rho(\mathcal{K}) \neq 0$ for a $\mathcal{K} \neq 0$. To this end, one starts extracting from the structure factor $S(\mathbf{k} + \mathcal{K})$ (recall that $S(\mathbf{r}) = 1 + \rho h^{(o)}(\mathbf{r}) + \rho h^{(d)}(\mathbf{r})$) the delta functions coming from the oscillating part of $1 + \rho h(\mathbf{r})$, namely

$$S^{(o)}(\mathbf{k} + \mathcal{K}) = 2\pi\delta(\mathbf{k} + \mathcal{K})$$
$$+ 2\pi\rho \sum_{\mathcal{K}' \neq 0} |\rho(\mathcal{K}')|^2 \delta(k - \mathcal{K}' + \mathcal{K}). \tag{35}$$

This part evidently does not contribute in (34).

One then introduces a smooth nonnegative cut-off function $f_\varepsilon(k)$, localized in an ε-neighborhood of $k = 0$. Since the usual Mermin argument relies on nonintegrable singularities, one rewrites (34), with $S^{(d)} = S - S^{(o)}$, as

$$S^{(d)}(\mathbf{k} + \mathcal{K})\frac{f_\varepsilon(k)}{k^{1/2}} \geq \text{const}(\mathcal{K} \cdot \mathbf{e}_t)^2 |\rho(\mathcal{K})|^2 \frac{f_\varepsilon(k)}{k^2}. \tag{36}$$

Integrating now with respect to $\mathbf{k}$, one gets on the left-hand side

$$\int d^2\mathbf{k} S^{(d)}(\mathbf{k} + \mathcal{K})\frac{f_\varepsilon(k)}{k^{1/2}}$$

$$= \int d^2\mathbf{r} e^{i\mathcal{K}\mathbf{r}} h^{(d)}(\mathbf{r})F(\mathbf{r}) \tag{37}$$

$$\leq \int_{r \geq R_o} d^2\mathbf{r} \frac{1}{r^{1/2+\varepsilon}} \frac{1}{r^{3/2}} < \infty, \tag{38}$$

where F, the Fourier transform of $f_\varepsilon(k)/k^{1/2}$, behaves as $r^{-3/2}$ for large r. On the other hand, Eq. (36) was built in such a way that an integration

with respect to $\mathbf{k}$ would lead to an infinite right-hand side if $|\rho(\mathcal{K})| \neq 0$ for some $\mathcal{K} \neq 0$. In other words, crystallinity is incompatible with a decay $|h^{(d)}(\mathbf{r})| \propto r^{-(1/2+\varepsilon)}$, for $\varepsilon > 0$. Since the decay of $h^{(d)}(\mathbf{r})$ as $r \to \infty$ is the same as the asymptotic behaviour of $\rho_T^{(2)}(\mathbf{r}_1 + \mathbf{r}_2)$ in the difference variable $(\mathbf{r}_1 - \mathbf{r}_2)$, one can say that crystallinity is incompatible with a clustering faster then $1/r^{1/2}$.

3. Conclusions

We present a self-contained derivation of the Wigner crystallization in a bidimensional one-component classical plasma and its relation to the Goldstone theorem. This derivation can be used in statistical physics books for graduate students.

Appendix A

Mermin Inequality

Inserting Eq. (10) and Eqs. (12) and (13) into Schwartz inequality (14), we find

$$\langle |A|^2 \rangle = \left\langle \sum_{i,j} e^{i(\mathbf{k}+\mathcal{K})\cdot(\mathbf{r}_i - \mathbf{r}_j)} \right\rangle$$
$$= NS(\mathbf{k} + \mathcal{K}), \tag{A.1}$$

$$\langle |B|^2 \rangle = N^2(\mathbf{k} \cdot \mathbf{e}_t)^2 + ND_{tt}(\mathbf{k}), \tag{A.2}$$

$$\langle A^*B \rangle = N^2[(\mathbf{k} + \mathcal{K}) \cdot \mathbf{e}_t]^2 |\rho(\mathcal{K})|, \tag{A.3}$$

where in the last Eqs. (A.2) and (A.3) we used integration by parts and in Eq. (A.3) we used Eq. (7). When inserted into Schwartz inequality (14), these results give Mermin inequality (15).

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References

1. T. R. Brown and C .C. Grimes, Observation of cyclotron resonance in surface-bound electrons on liquid helium, *Phys. Rev. Lett.* **29**, 1233 (1972).
2. R. Fantoni and M. P. Tosi, Decay of correlations and related sum rules in a layered classical plasma, *Nuovo Cimento* **17D**, 155 (1995).
3. R. Fantoni, B. Jancovici and G. Téllez, Pressures for a one-component plasma on a pseudosphere, *J. Stat. Phys.* **112**, 27 (2003).

4. R. Fantoni and G. Téllez, Two dimensional one-component plasma on a flamm's paraboloid, *J. Stat. Phys.* **133**, 449 (2008).

5. R. Fantoni, Two component plasma in a flamm's paraboloid, *J. Stat. Mech.* 04015 (2012).

6. R. Fantoni, Jellium at finite temperature, *Mol. Phys.* **120**, 4 (2021).

7. R. Fantoni and M. P. Tosi, Coordinate space form of interacting reference response function of d-dimensional jellium, *Nuovo Cimento* **17D**, 1165 (1995).

8. R. Fantoni, Radial distribution function in a diffusion monte carlo simulation of a fermion fluid between the ideal gas and the jellium model, *Eur. Phys. J. B* **86**, 286 (2013).

9. R. Fantoni, Jellium at finite temperature using the restricted worm algorithm, *Eur. Phys. J. B* **94**, 63 (2021).

10. R. Fantoni, Form invariance of the moment sum-rules for jellium with the addition of short-range terms in the pair-potential, *Indian J. Phys.* **95**, 1027 (2021).

11. E. Wigner, On the interaction of electrons in metals, *Phys. Rev.* **46**, 1002 (1934).

12. E. Wigner, Effects of the electron interaction on the energy levels of electrons in metals, *Trans. Faraday Soc.* **34**, 678 (1938).

13. C. C. Grimes and G. Adams, Evidence for a liquid-to-crystal phase transition in a classical, two-dimensional sheet of electrons, *Phys. Rev. Lett.* **42**, 795 (1979).

14. R. S. Crandall and R. Williams, Crystallization of electrons on the surface of liquid helium, *Phys. Lett. A* **34**, 404 (1971).

15. R. C. Gann, S. Chakravarty and G. V. Chester, Monte Carlo simulation of the classical two-dimensional one-component plasma, *Phys. Rev.* **20**, 326 (1979).

16. J. Goldstone, A. Salam and S. Weinberg, Broken symmetries, *Phys. Rev.* **127**, 965 (1972).

17. D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Perseus Books Group, 1975).

18. L. Landau, J. Fernando Perez and W. F. Wreszinski, Energy gap, clustering, and the Goldstone theorem in statistical mechanics, *J. Stat. Phys.* **26**, 755 (1981).

19. N. D. Mermin, Crystalline order in two dimensions, *Phys. Rev.* **176**, 250 (1968).

20. N. D. Mermin, Erratum: Crystalline order in two dimensions, *Phys. Rev. B* **20**, 4762 (1979).

21. M. Requardt and H. J. Wagner, Wigner crystallization and its relation to the poor decay of pair correlations in one-component plasmas of arbitrary dimension, *J. Stat. Phys.* **58**, 1165 (1990).

22. R. E. Peierls, *Helv. Phys. Acta* **7**, 81 (1936).

23. A. V. Chaplik, Possible crystallization of charge carriers in low-density inversion layers, *Sov. Phys. JETP* **35**, 395 (1972).

24. A. Alastuey and P. A. Martin, Decay of correlations in classical fluids with long-range forces, *J. Stat. Phys.* **39**, 405 (1985).

25. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions* (Cambridge University Press, Cambridge, England, 1958).

26. A. L. Fetter, Electrodynamics of a layered electron gas. II. Periodic array, *Ann. Phys.* **88**, 1 (1974).

27. A. L. Fetter, Electrodynamics and thermodynamics of a classical electron surface layer, *Phys. Rev. B* **10**, 3739 (1974).

28. B. Jancovici, Exact results for the two-dimensional one-component plasma, *Phys. Rev. Lett.* **46**, 386 (1981).

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Aim of his research is to develop analytical and computational methods for condensed and soft matter starting from the fundamental many-body equations. Apart from the few analytically exactly solvable models our principal instruments, guided by the various

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