

Effect of temperature on the equilibrium of a white dwarf in general relativity

Riccardo Fantoni*

Università di Trieste, Dipartimento di Fisica, strada Costiera 11, 34151 Grignano (Trieste), Italy

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I. INTRODUCTION

A white dwarf has a central mass density $\rho_c < \rho_c^{\text{nd}}$ where $\rho_c^{\text{nd}} = 4 \times 10^{11} \text{ g/cm}^3$ is the mass density at which the leptons and baryons composing the star undergo the neutron drip process where by inverse β^+ decay one electron and one proton change into a neutron and a neutrino. A process which ultimately will change the white dwarf into a neutron star. Chandrasekhar [1–3] predicted an equilibrium mass for a white dwarf given by $M_{\text{Ch}} = 1.45639 M_\odot$ and independent from the central density. His argument is based on Newtonian hydrodynamics and the thermodynamics of an ideal (non interacting), perfect (adiabatic), and degenerate (at zero temperature) electron gas.

First Tolman [4], and later Oppenheimer and Volkoff [5] proposed a way to treat the hydrodynamic equations for the equilibrium and stability of a star within the framework of general relativity. Even if their hydrodynamic equations have been used mainly for the description of neutron stars for which general relativity effects cannot be neglected due to their extreme compactness, nonetheless they may be used also for the description of a white dwarf.

In this work we want to find a lower bound to the central mass density of a white dwarf necessary to observe a correction to the Chandrasekhar result for the equilibrium mass due to the general relativity description of Tolman-Oppenheimer-Volkoff larger than 2%.

In a previous work [6] we studied the effect of a non zero temperature on a white dwarf Newtonian equilibrium and structure. Here we will discuss how one could take care of a temperature gradient throughout the star due to heat transport and we will discover that the assumption of a degenerate electron gas throughout the whole star is fully legitimate.

II. DISCUSSION

In this work we consider spacetime as a smooth manifold \mathcal{M} of dimension $d = 4$ and metric tensor \mathbf{g} with covariant components $g_{\alpha\beta}$. We will denote with an arrow over a bold face letter the corresponding 4 vector and with just the bold face symbol the corresponding 3 dimensional vector. Greek indexes run over the d spacetime dimensions. Roman

* riccardo.fantoni@scuola.istruzione.it

indexes run only over the $d - 1$ space dimensions. We use Einstein summation convention of tacitly assuming a sum over repeated indexes. We will use geometrized units $c = G = 1$ throughout.

In Ref. [6] we determined how the Chandrasekhar argument for the limiting mass of a white dwarf at zero temperature could be modified to take into account the effects of a finite non zero temperature.

In Ref. [7] we showed that in general relativity hydrodynamics a *perfect fluid* with a stress energy tensor, given by $T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}$ in its isotropic frame, one finds in a local Lorentz frame

$$\frac{d\rho}{d\tau} + (\rho + p)\vec{\nabla}\vec{u} = 0, \quad (2.1)$$

$$(\rho + p)\vec{a} = -\vec{\nabla}p - \vec{u}\frac{dp}{d\tau}, \quad (2.2)$$

where ρ is the mass energy density, p is the pressure, τ is the fluid proper time, $\vec{u} = (\gamma, \gamma\mathbf{v})$ is the fluid 4 velocity, where $u^0 = dt/d\tau = \gamma = (1 - v^2)^{-1/2}$ is Lorentz factor, and $\vec{a} = d\vec{u}/d\tau$ is the 4 acceleration. The first Eq. (2.1) is the *continuity equation* and the second (2.2) is the *Euler equation*.

In §6.10 of the book [8] the effects of general relativity on the Newtonian Chandrasekhar argument are determined. They show that for white dwarfs it is often enough to approximate the general Euler equation (2.2) with its non relativistic and Newtonian limits. This does not appreciably alter the picture that includes general relativity effects on the star *equilibrium*. They find that general relativity affects only the *stability* of a white dwarf (see their (6.10.28) equation). Here we will show that albeit very small the general relativity effect on dense white dwarfs equilibrium is still appreciable. Moreover we will discuss the effect of taking care of a finite non zero temperature which, instead of being treated as uniform throughout the whole star, may in general be considered as a function of the distance r from the star center, to be integrated out together with the mass, the mass density, and the pressure.

In their book Shapiro and Teukolsky [8] treat the uniform zero temperature case. They start [9] from the non relativistic version of (2.1) and (2.2),

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0, \quad (2.3)$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p, \quad (2.4)$$

which are obtained from (2.1) and (2.2) in the non relativistic limit $v \ll 1$, $\gamma \approx 1$, and $p \ll \rho$ (since the thermal energy is much smaller than the rest mass of the fluid). And in the left hand side of Euler equation they assume the Newtonian result

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{m}{r^2}\hat{\mathbf{r}}, \quad (2.5)$$

where m is the mass of the star inside a radius r . They suggest that working within these approximation to the full general relativity picture is enough to extract the main general relativity effects on the stability of the star (see their (5.10.28) equation). The equilibrium properties of the white dwarf being unaltered from the Chandrasekhar analysis (see their (6.10.26) equation).

The equivalence principle allows to rewrite the spatial component of Euler Eq. (2.2) as follows

$$\rho\frac{D\mathbf{v}}{d\tau} = -\nabla p - \frac{D(\mathbf{v}p)}{d\tau}, \quad (2.6)$$

where $D \dots /d\tau$ is a covariant derivative with $Dv^i/d\tau = -\Gamma^i_{\alpha\beta}u^\alpha u^\beta$ and $D(v^i p)/d\tau = u^\beta\nabla_\beta(v^i p) = -p\Gamma^i_{\alpha\beta}u^\alpha u^\beta + v^i dp/d\tau$. Here Γ are the Christoffel symbols and the geodesic equation for the free fall of the particles of the fluid has been used. An important result of Newtonian gravitation is that at any point outside a spherical mass distribution, the gravitational field depends only on the mass interior to that point. Moreover, even if the mass interior is moving spherically symmetrically, the field outside is constant in time. This result is also true in general relativity, where it is known as *Birkhoff theorem*: the only vacuum, spherically symmetric gravitational field is static. It is the Schwarzschild metric solution of Einstein field equations for which the only non zero radial Christoffel symbols are $\Gamma^r_{00} = \Gamma^r_{rr} = f'(r)/2$, $\Gamma^r_{\theta\theta} = rf(r)$, and $\Gamma^r_{\phi\phi} = -rf(r)\sin^2\theta$, where $f(r) = 1 - 2m/r$. Using this solution and recalling that the star is static, so that $dp/d\tau = 0$, one reaches the *Tolman-Oppenheimer-Volkoff* (TOV) equation [4, 5]

$$\frac{dp}{dr} = -\frac{\rho m}{r^2} \left(1 + \frac{p}{\rho}\right) \left(1 + \frac{4\pi r^3 p}{m}\right) \left(1 - \frac{2m}{r}\right)^{-1}, \quad (2.7)$$

where the mass of the star must satisfy

$$\frac{dm}{dr} = 4\pi r^2 \rho \left(1 - \frac{2m}{r}\right)^{-1/2}, \quad (2.8)$$

with boundary condition $m(r=0) = 0$. The Newtonian limit is recovered by taking $p \ll \rho$ and $m \ll r$. A star made of a perfect fluid must also satisfy an adiabatic equation of state

$$p = K\rho^\Gamma, \quad (2.9)$$

known as a *polytrope* of polytropic index $1/(\Gamma - 1)$. As shown in Ref. [6] $\Gamma = d\ln p/d\ln \rho \rightarrow 4/3$ at high electron mass density, which corresponds to the *extremely relativistic regime* $v \gg 1$, i.e. when we can use a dispersion relation $\epsilon_{er}(k) = \hbar c k$ for the electrons. Assuming that the white dwarf is made up of a completely ionized plasma then we may write $\rho = \mu_e m_u n$ with n the electrons number density, m_u the atomic mass unit, and $\mu_e = A/Z$ with Z the atomic number and A the mass number, so that for example for a ${}^4\text{He}$ or a ${}^{12}\text{C}$ white dwarf $\mu_e = 2$ and for a ${}^{56}\text{Fe}$ white dwarf $\mu_e = 56/26 \approx 2.134$. In the extremely relativistic regime K in the polytrope Eq. (2.9) is only a function of the activity

$$K = \frac{\pi^{2/3} \hbar}{g^{1/3} (\mu_e m_u)^{4/3}} \frac{f_4(z)}{f_3^{4/3}(z)}, \quad (2.10)$$

where $g = 2$ is the electrons spin degeneracy and $f_\mu(z) = -\sum_{\nu=1}^{\infty} (-z)^\nu / \nu^\mu$ for an ideal Fermi-Dirac gas with an activity $z = \exp(\beta\mu)$, temperature $T = 1/k_B \beta$, with k_B Boltzmann constant, and μ the chemical potential of the electrons, as shown in §61 of Ref. [10]. Note that at $T \rightarrow 0$, $z \rightarrow \infty$ and in Eq. (2.10) we find $\lim_{z \rightarrow \infty} f_4(z)/f_3^{4/3}(z) = 3^{1/3}/2^{5/3}$. In Fig. 1 we show the polytrope exponent $\Gamma = d\ln p/d\ln n$ for the equation of state of the extremely relativistic electron gas as a function of density at zero temperature. This should be compared with Fig. 1 of Ref. [6] where the same plot is presented but along the now adiabatic equation of state at finite temperature of the *fully relativistic* case, i.e. assuming an electron dispersion relation $\epsilon_{fr}(k) = \sqrt{(\hbar k)^2 + m_e^2}$.

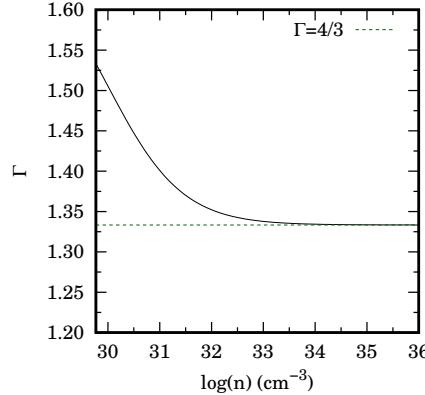


FIG. 1. We show $\Gamma = d\ln p/d\ln n$ for the equation of state of the extremely relativistic electron gas as a function of density at zero temperature $z \rightarrow \infty$. $g = 2$ and n is in cm^{-3} .

Let us denote with $p(r) = \bar{p}(\rho(r), T(r))$ the equation of state for each shell between r and $r + dr$ of the star. For the polytrope of Eq. (2.9) $T(r) = 0$ for any r and $\bar{p} = K\rho^\Gamma$. In order to integrate the Euler equation of TOV (2.7) one chooses $m(r=0) = 0$, a central mass density $\rho(r=0) = \rho_c$ and pressure $p(r=0) = \bar{p}(\rho_c, 0)$ and integrates it with Eqs. (2.8) and (2.9) till $r = R$ where $p(r=R) = 0$ and $m(r=R) = M$, with R, M respectively the radius and mass of the star.

We integrated numerically TOV equation and its Newtonian limit (N) in order to understand when the general relativity effects become important. In the N limit this reduces to the *Lane-Emden* problem as discussed in ??.

¹ Note that in the extremely relativistic case, unlike the fully relativistic case, the entropy per electron is just a function of the activity so on an adiabatic the activity is fixed.

At First we chose a high enough central electron number density $n_c = 10^{36}\text{cm}^{-3}$ with $\rho_c = \mu_e m_u n_c$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\Gamma = 4/3$. Moreover, following Chandrasekhar, we chose the case of degenerate electrons at $T \rightarrow 0$, i.e. $z \rightarrow \infty$ in Eq. 2.10. At this high central number density the electron gas adiabatic equation of state is, to a good approximation, a polytrope in the extreme relativistic $\Gamma \rightarrow 4/3$ limit. Our results are shown in Fig. 2. From the figure we see that the effects of general relativity are indeed non negligible. In the Newtonian case (N) we recover the Chandrasekhar result of a limiting mass independent from the central density, whereas the TOV equation, unlike the Lane-Emden equation, still has a limiting mass but it depends on the central density. For the existence of the limiting mass in both the TOV and its N limit it is crucial to choose a polytrope equation of state with index less than 5, i.e. $\Gamma > 6/5$. Nonetheless we see from our figure that in order to be able to observe some general relativity effects we had to have a central mass density $\rho_c \approx 3 \times 10^{12}\text{g/cm}^3$ which is one order of magnitude above the neutron drip density of $4 \times 10^{11}\text{g/cm}^3$.

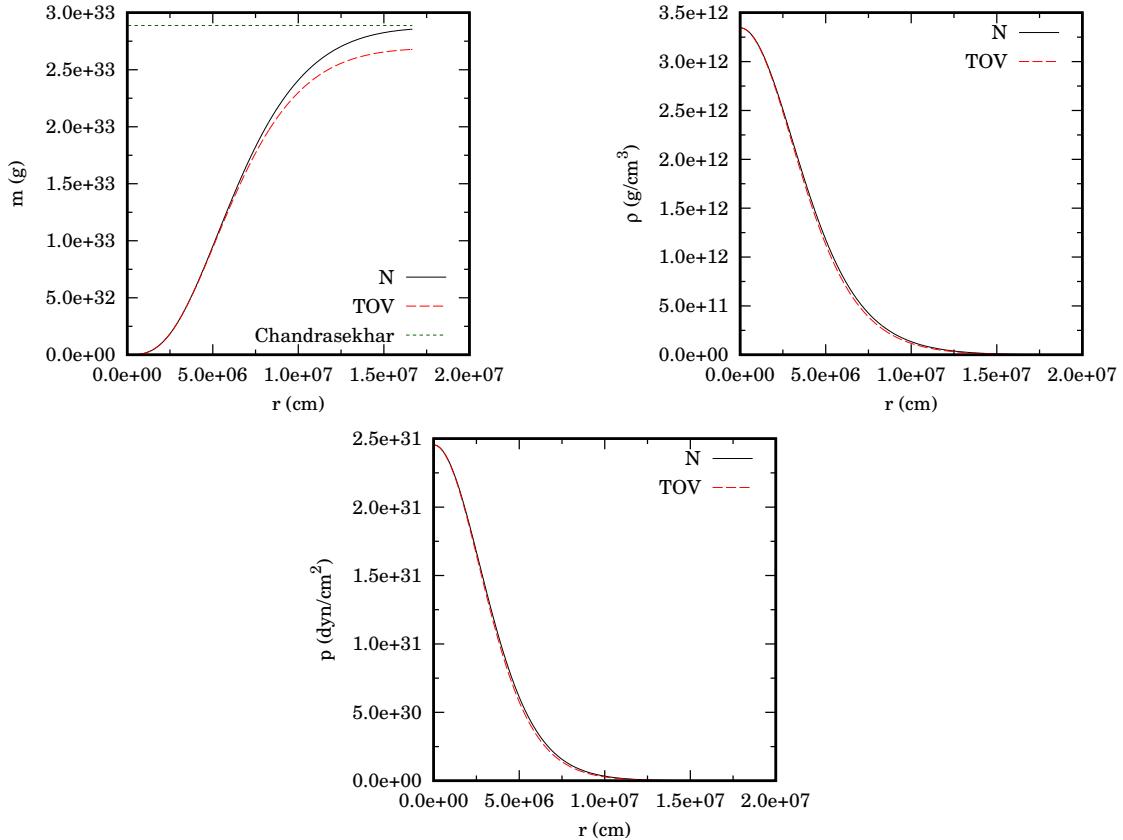


FIG. 2. We compare the numerical integration of the TOV equilibrium equation (TOV) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central electron number density $n_c = 10^{36}\text{cm}^{-3}$ and a central pressure $p_c = K\rho_c^\Gamma$ with $\rho_c = \mu_e m_u n_c$ at zero temperature $z \rightarrow \infty$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. In the first panel we also show the Chandrasekhar limiting value for the star mass. As one can see from the second panel the central mass density is an order of magnitude higher than the density for neutron drip. So the white dwarf would very likely turn into a neutron star starting from its core.

We therefore chose a central density $\rho_c = 10^{11}\text{g/cm}^3$, below neutron drip, with a central electron number density $n_c = \rho_c/(\mu_e m_u) \approx 10^{34}\text{cm}^{-3}$ for which we have a central Fermi temperature of $T_F = E_F/k_B = (\hbar^2/2m_e k_B)(6\pi^2 n_c/g)^{2/3} \approx 10^{12}\text{K}$, where m_e is the mass of the electron. From Fig. 1 of Ref. [6] we see that in this conditions we are well within the extremely relativistic $\Gamma \rightarrow 4/3$ regime. Choosing then again $z \rightarrow \infty$ in Eq. (2.10) and a central pressure $p_c = K\rho_c^{4/3}$ we observe the results of Fig. 3 which still shows a non negligible effect of general relativity on the white dwarf equilibrium.

As a lower bound $\rho_c^{\text{lb}} < \rho_c < \rho_c^{\text{nd}}$ to the central mass density of a white dwarf necessary to detect an equilibrium mass of the star such that $(M - M_{\text{Ch}})/M_{\text{Ch}} > 2\%$ due to general relativity effects we found $\rho_c^{\text{lb}} \approx 1.25 \times 10^{11}\text{g/cm}^3$ when $M \approx 2.75627 \times 10^{33}\text{g}$ and $R \approx 4.11333 \times 10^7\text{cm}$. at the neutron drip mass density the general relativity effects on the equilibrium mass are $\approx 3\%$.

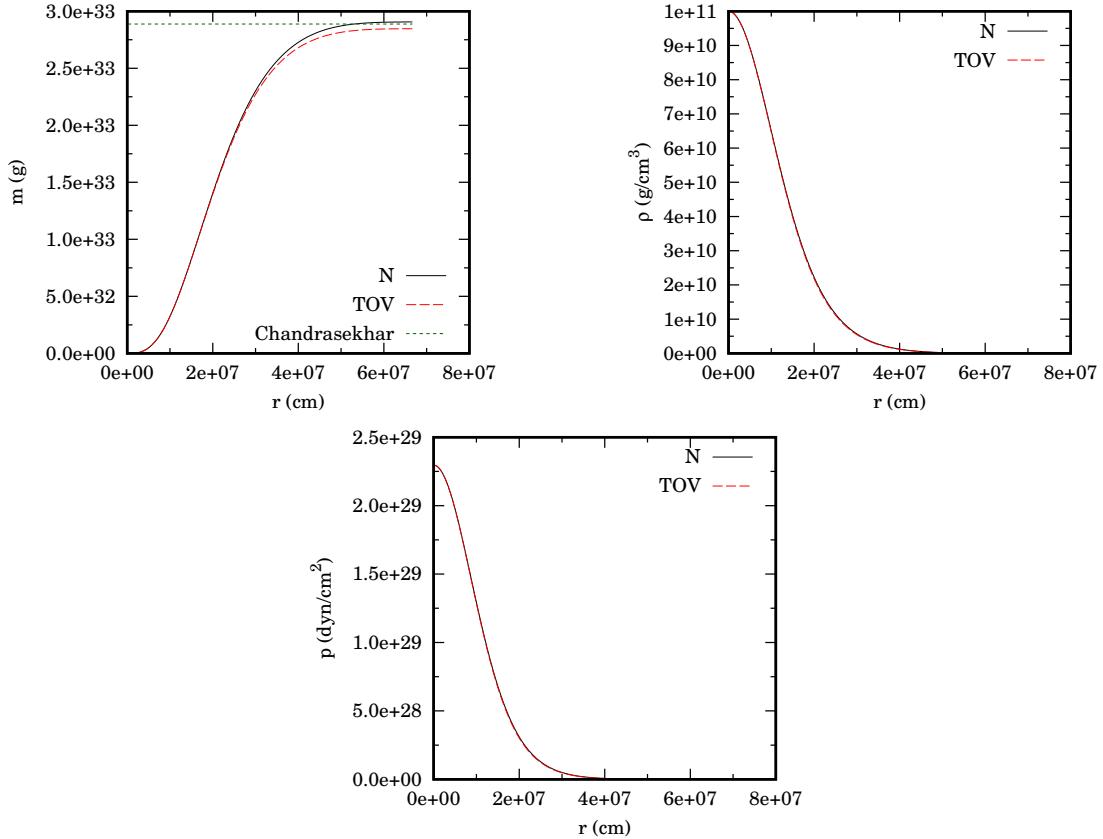


FIG. 3. We compare the numerical integration of the TOV equilibrium equation (TOV) with its Newtonian limit (N) for a white dwarf with $\mu_e = 2$ starting from a central mass density $\rho_c = 10^{11} \text{ g/cm}^3$, below neutron drip, and a central pressure $p_c = K \rho_c^{4/3}$ at $z \rightarrow \infty$. In the first panel we show the mass of the star, in the second the mass density, and in the third the pressure. In the first panel we also show the Chandrasekhar limiting value for the star mass. As we can see from this first panel the effect of general relativity is still not negligible.

As discussed in Ref. [6], at finite non zero temperature the adiabatic equation of state of the ideal Fermi gas of electrons will enter the extremely relativistic regime at high electron number density. In Fig. 1 of that reference it was shown for example that at a temperature of $T = 20000\text{K}$ the adiabatic equation of state becomes a polytrope with $\Gamma \approx 4/3$ already for $n \gtrsim 10^{25}\text{cm}^{-3}$. In order to take properly into account the effect of temperature on the star equilibrium we would need to introduce a *temperature profile* $T = T(r)$ dependent on the radial distance r within the star. The assumption of Chandrasekhar was to consider negligible the effect of temperature since the average temperatures of a white dwarf $100000\text{K} - 3000\text{K}$ are much smaller than the Fermi temperature so that the Fermi-Dirac distribution function can be considered everywhere as the one of a degenerate gas. The electron gas will be in its polytrope extremely relativistic regime in the inner star shells but it will gradually exit this regime in the outer shells where the number density approaches zero, and with it the Fermi temperature, at the star surface.

The process of integrating over temperature as a function of radial distance is part of solving the equation of energy transport, which is one of the four fundamental equations of stellar structure

$$\frac{dT}{dr} = -\frac{1}{\kappa} \frac{L}{4\pi r^2} \left(1 - \frac{2m}{r}\right)^{1/2}, \quad (2.11)$$

where for a ${}^4\text{He}$ white dwarf $\kappa \approx 10^4 - 10^7 \text{ erg(s cm K)}^{-1}$ is the thermal conductivity, and $L \approx 10^{-2} - 10^{-5} L_\odot$ is the average luminosity [11]². This equation determines the temperature gradient necessary to carry the star internal

² For example early estimates of the luminosity of Sirius B estimated from the observed flux and known distance set his luminosity to about 1/360 of that of the sun. In 1914 W. S. Adams [12] by assigning an effective temperature of 8000K to Sirius B from these spectral measurements and using the equation for black body emission, $L = 4\pi R^2 \sigma T_{\text{eff}}^4$, a radius R of $18.8 \times 10^8 \text{ cm}$ could be inferred for the star (this is just about four times bigger than the modern value).

luminosity outwards. It should be integrated together with the TOV equation (2.7), the mass equation (2.8), and the adiabatic equation of state. For an ideal fully relativistic electron gas (see the appendix of Ref. [6]), this is given in parametric form by

$$\beta p = g \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\ln z - \beta \epsilon(k)}{e^{\beta \epsilon(k)}/z + 1} + \frac{s\rho}{\mu_e m_u}, \quad (2.12)$$

$$\rho = g \mu_e m_u \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{e^{\beta \epsilon(k)}/z + 1}, \quad (2.13)$$

where the activity z is the parameter, $k_B s$ is the constant entropy per electron, and $\epsilon_{\text{fr}}(k) = \sqrt{(\hbar k)^2 + m_e^2}$ is the electron dispersion relation. In the appendix of Ref. [6] it was shown that in the extremely relativistic regime the entropy is only a function of the activity z so on an adiabatic one must have z constant, since the pressure is a homogeneous function of degree 4 in T and μ and the number density is homogeneous of degree 3 one sees that the adiabatic equation of state \bar{p} of Eqs. (2.12)-(2.13) it must reduce to the polytrope with index 3 of Eq. (2.9). Moreover at the densities of interest $z \rightarrow \infty$.

In order to understand if the effect of a finite non zero temperature is important or not we would need to integrate numerically TOV equation (2.7), the mass equation (2.8), the temperature equation (2.11), and the equation of state (2.12)-(2.13), choosing as boundary condition $m(r=0) = 0$, a central mass density $\rho(r=0) = \rho_c$, a central temperature $T(r=0) = T_c$, and a central pressure $p(r=0) = \bar{p}(\rho_c, T_c)$. But we soon realize that even if the core of the star is at a temperature T_c in the white dwarf range it will rapidly drop to zero at a radius of the order of $\Delta r \approx L/4\pi\kappa T_c \ll R$ according to the energy transport Eq. (2.11). Much more rapidly than the drop to zero of the mass density $d\rho/dr = (dp/dr)/K\Gamma\rho^{\Gamma-1}$ with dp/dr of the TOV equation (2.7). Note that at the center dm/dr , $d\rho/dr$, and dp/dr start from zero whereas dT/dr diverges to $-\infty$.

III. CONCLUSIONS

We reviewed the problem of the equilibrium of a white dwarf and determined the effect of general relativity on the limiting mass. In particular we saw that using the Tolman-Oppenheimer-Volkoff hydrodynamics instead of the Lane-Emden we have an equilibrium mass which depends on the central density. Moreover we determined that the star equilibrium mass deviations reach up to 3% for neutron drip central densities.

We also discussed the effect of considering a temperature gradient throughout the star interior due to heat transfer concluding that it is fully legitimate to assume a star uniformly at zero temperature with an everywhere degenerate electron gas.

We plan to drop the assumption of an ideal, non interacting, electron gas in order to be able to find a more accurate equation of state in the future. Some progress in this direction has already been made [7] which requires path integrals on curved spacetimes. We propose to use a path integral Monte Carlo method [13] with some rather subtle issues [14].

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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