Differential Geometry

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CONTENTS

I. PREAMBLE ON DIFFERENTIAL TOPOLOGY

Consider a Riemannian manifold $\mathscr{M}$ of dimension n.

A *curve* on $\mathscr{M}$

$$C(\lambda) \quad \lambda \text{ is an affine parameter.} \tag{1.1}$$

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The tangent *vector* to C

$$\vec{\boldsymbol{u}} = \frac{dC(\lambda)}{d\lambda} = \partial_{\vec{\boldsymbol{u}}}. \tag{1.2}$$

A *basis of vectors* $\{\vec{\boldsymbol{e}}_\alpha\}$ with

$$\vec{\boldsymbol{e}}_\alpha = \partial_{\vec{\boldsymbol{e}}_\alpha} = \partial_\alpha. \tag{1.3}$$

A change of basis vectors is realized as follows

$$\vec{\boldsymbol{e}}_{\alpha'} = L^\alpha{}_{\alpha'}\vec{\boldsymbol{e}}_\alpha, \tag{1.4}$$

where the primed indexes are for the vectors in the new basis and a summation over the repeated index is tacitly assumed here and everywhere else in these manuscript.

In a *coordinate basis*

$$\vec{\boldsymbol{e}}_\alpha = \partial_{\vec{\boldsymbol{e}}_\alpha} = \partial_\alpha = \frac{\partial}{\partial x^\alpha}. \tag{1.5}$$

For a transformation of coordinates $x^{\alpha'} = x^{\alpha'}(x^\beta)$

$$\vec{\boldsymbol{e}}_\alpha = \frac{\partial}{\partial x^\alpha} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\vec{\boldsymbol{e}}_{\alpha'} = L^{\alpha'}{}_\alpha\vec{\boldsymbol{e}}_{\alpha'}. \tag{1.6}$$

with

$$L^{\alpha'}{}_\alpha = \frac{\partial x^{\alpha'}}{\partial x^\alpha}, \quad L^\alpha{}_\beta L^\beta{}_\gamma = \delta^\alpha{}_\gamma, \tag{1.7}$$

where $||\delta^\alpha{}_\gamma|| = \mathrm{diag}(1,1,\dots,1)$ is the identity matrix and $||L^\alpha{}_\beta|| = ||L^\beta{}_\alpha||^{-1}$.

A transformation of coordinates of a vector

$$\vec{\boldsymbol{u}} = u^{\alpha'}\vec{\boldsymbol{e}}_{\alpha'} = u^\alpha\vec{\boldsymbol{e}}_\alpha = u^\alpha L^{\alpha'}{}_\alpha\vec{\boldsymbol{e}}_{\alpha'}, \tag{1.8}$$

with

$$u^{\alpha'} = L^{\alpha'}{}_\alpha u^\alpha. \tag{1.9}$$

The *1-form* $\widetilde{\boldsymbol{\sigma}}$ in the dual space of the tangent vector space

$$\langle\widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{u}}\rangle = \text{a real number}, \tag{1.10}$$

where $\langle\cdot,\cdot\rangle$ is a bilinear two slots machine such that

$$\langle\widetilde{\boldsymbol{\omega}}^\beta, \vec{\boldsymbol{e}}_\alpha\rangle = \delta^\beta{}_\alpha \tag{1.11}$$

with $\{\widetilde{\boldsymbol{\omega}}^\beta\}$ a *basis of 1-forms.*

So if

$$\vec{\boldsymbol{u}} = u^\alpha\vec{\boldsymbol{e}}_\alpha, \tag{1.12}$$

$$\widetilde{\boldsymbol{\sigma}} = \sigma_\beta\widetilde{\boldsymbol{\omega}}^\beta, \tag{1.13}$$

we will have

$$u^\alpha = \langle\widetilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{u}}\rangle, \tag{1.14}$$

$$\sigma_\beta = \langle\widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta\rangle, \tag{1.15}$$

$$\sigma_\alpha u^\alpha = \langle\widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{u}}\rangle. \tag{1.16}$$

A change of basis 1-forms is realized as follows

$$\widetilde{\boldsymbol{\omega}}^{\alpha'} = L^{\alpha'}_\alpha\widetilde{\boldsymbol{\omega}}^\alpha, \tag{1.17}$$

and for the 1-form components

$$\sigma_{\alpha'} = L^\alpha{}_{\alpha'}\sigma_\alpha. \tag{1.18}$$

A particularly important 1-form is the *gradient*, $\widetilde{\boldsymbol{d}}f$, with f a scalar (a function), defined like so

$$\langle\widetilde{\boldsymbol{d}}f,\vec{\boldsymbol{u}}\rangle=\partial_{\vec{\boldsymbol{u}}}f=u^\alpha\partial_\alpha f=u^\alpha f_{,\alpha}, \tag{1.19}$$

where we use the comma to denote a partial derivative

$$f_{,\alpha}=\langle\widetilde{\boldsymbol{d}}f,\vec{\boldsymbol{e}}_\alpha\rangle=\partial_{\vec{\boldsymbol{e}}_\alpha}f=\partial_\alpha f. \tag{1.20}$$

So

$$\widetilde{\boldsymbol{d}}f=f_{,\alpha}\widetilde{\boldsymbol{\omega}}^\alpha. \tag{1.21}$$

In a coordinate basis

$$f_{,\alpha}=\frac{\partial f}{\partial x^\alpha}, \tag{1.22}$$

and $\{\widetilde{\boldsymbol{d}}x^\alpha\}$ is dual to $\{\partial/\partial x^\alpha\}$

$$\langle\widetilde{\boldsymbol{d}}x^\alpha,\partial/\partial x^\beta\rangle=\partial_\beta x^\alpha=\frac{\partial x^\alpha}{\partial x^\beta}=\delta^\alpha{}_\beta. \tag{1.23}$$

A tensor $\mathbb{H}$ of rank $\binom{n}{m}$ is a linear machine with n input slots for 1-forms, $\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\lambda}},\ldots,\widetilde{\boldsymbol{\beta}}$, and m input slots for vectors, $\vec{\boldsymbol{u}},\vec{\boldsymbol{v}},\ldots,\vec{\boldsymbol{w}}$, which returns a real number

$$H(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\lambda}},\ldots,\widetilde{\boldsymbol{\beta}},\vec{\boldsymbol{u}},\vec{\boldsymbol{v}},\ldots,\vec{\boldsymbol{w}})=\text{real number}, \tag{1.24}$$

Up to here we did not use a metric at all so we woked in *differential topology*. We will introduce a metric only later. For the time being let us take a detour on *exterior calculus*.

II. EXTERIOR CALCULUS IN BRIEF

We may define a p-form pf our n-dimensional manifold $\mathscr{M}$ like so

$$\begin{aligned}\widetilde{\boldsymbol{\alpha}}&=\frac{1}{p!}\alpha_{\mu_1\mu_2\cdots\mu_p}\widetilde{\boldsymbol{\omega}}^{\mu_1}\wedge\widetilde{\boldsymbol{\omega}}^{\mu_2}\wedge\ldots\wedge\widetilde{\boldsymbol{\omega}}^{\mu_p}\\&=\alpha_{|\mu_1\mu_2\cdots\mu_p|}\widetilde{\boldsymbol{\omega}}^{\mu_1}\wedge\widetilde{\boldsymbol{\omega}}^{\mu_2}\wedge\ldots\wedge\widetilde{\boldsymbol{\omega}}^{\mu_p},\end{aligned} \tag{2.1}$$

where the vertical bars around the indexes means that the summation extends only over $\mu_1<\mu_2<\ldots<\mu_p$ and $\wedge$ is the *wedge product* which is defined by its action on any two 1-forms, $\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}}$ (or on any two vectors), as

$$\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}}=\widetilde{\boldsymbol{\alpha}}\otimes\widetilde{\boldsymbol{\beta}}-\widetilde{\boldsymbol{\beta}}\otimes\widetilde{\boldsymbol{\alpha}}, \tag{2.2}$$

where $\otimes$ denotes a *direct product*. So that $\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}}=-\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\alpha}}$ and $\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\alpha}}=0$. Given any three 1-forms, $\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\gamma}}$ (or any three vectors), the wedge product has the following properties

$$(a\widetilde{\boldsymbol{\alpha}}+b\widetilde{\boldsymbol{\beta}})\wedge\widetilde{\boldsymbol{\gamma}}=a\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\gamma}}+b\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\gamma}}, \tag{2.3a}$$

$$(\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}})\wedge\widetilde{\boldsymbol{\gamma}}=\widetilde{\boldsymbol{\alpha}}\wedge(\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\gamma}})=\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\gamma}}, \tag{2.3b}$$

$$\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}}=\alpha_\mu\beta_\nu\widetilde{\boldsymbol{\omega}}^\mu\wedge\widetilde{\boldsymbol{\omega}}^\nu=\frac{1}{2}(\alpha_\mu\beta_\nu-\alpha_\nu\beta_\mu)\widetilde{\boldsymbol{\omega}}^\mu\wedge\widetilde{\boldsymbol{\omega}}^\nu. \tag{2.3c}$$

and if $\widetilde{\boldsymbol{\alpha}}$ is a p-form and $\widetilde{\boldsymbol{\beta}}$ is a q-form with p and q greater than 1, then $\widetilde{\boldsymbol{\alpha}}\wedge\widetilde{\boldsymbol{\beta}}=(-1)^{pq}\widetilde{\boldsymbol{\beta}}\wedge\widetilde{\boldsymbol{\alpha}}$.

Analogously for a p-vector we will have

$$\vec{\boldsymbol{a}}=\frac{1}{p!}a_{\mu_1\mu_2\cdots\mu_p}\vec{\boldsymbol{e}}^{\mu_1}\wedge\vec{\boldsymbol{e}}^{\mu_2}\wedge\ldots\wedge\vec{\boldsymbol{e}}^{\mu_p}. \tag{2.4}$$

A contraction of the p-form $\widetilde{\boldsymbol{\alpha}}$ of Eq. (2.1) and the p-vector $\vec{\boldsymbol{a}}$ of Eq. (2.4) is

$$\langle\widetilde{\boldsymbol{\alpha}},\vec{\boldsymbol{a}}\rangle=\alpha_{|\mu_1\mu_2\cdots\mu_p|}a^{\mu_1\mu_2\cdots\mu_p}. \tag{2.5}$$

For example the jacobian determinant of a set of p functions $f^k(x^1,x^2,\ldots,x^n)$ with respect to p of their arguments is

$$\left\langle\widetilde{\boldsymbol{d}}f^1\wedge\widetilde{\boldsymbol{d}}f^2\wedge\ldots\wedge\widetilde{\boldsymbol{d}}f^p,\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\ldots\wedge\frac{\partial}{\partial x^p}\right\rangle=\det\left|\left|\left(\frac{\partial f^\mu}{\partial x^\nu}\right)\right|\right|=\frac{\partial(f^1,f^2,\ldots,f^p)}{\partial(x^1,x^2,\ldots,x^p)}. \tag{2.6}$$

A. Exterior derivative

The exterior derivative is defined by induction:

i. if $\tilde{\boldsymbol{\sigma}}$ is a p-form $\tilde{\boldsymbol{d}}\tilde{\boldsymbol{\sigma}}$ is a $(p+1)$-form;

ii. a function f is a 0-form and $\tilde{\boldsymbol{d}}f = f_{,\alpha}\tilde{\boldsymbol{\omega}}^{\alpha}$;

iii. if $\tilde{\boldsymbol{\alpha}}$ is a p-form and $\tilde{\boldsymbol{\beta}}$ is a q-form then $\tilde{\boldsymbol{d}}(\tilde{\boldsymbol{\alpha}} \wedge \tilde{\boldsymbol{\beta}}) = \tilde{\boldsymbol{d}}\tilde{\boldsymbol{\alpha}} \wedge \tilde{\boldsymbol{\beta}} + (-1)^p \tilde{\boldsymbol{\alpha}} \wedge \tilde{\boldsymbol{d}}\tilde{\boldsymbol{\beta}}$.

It can easily be verified that $\tilde{\boldsymbol{d}}\tilde{\boldsymbol{d}} = \tilde{\boldsymbol{d}}^2 = 0$.

B. Integration

In order to integrate a p-form in an n-dimensional manifold one should follow the following steps:

i. consider in a coordinate basis

$$\tilde{\boldsymbol{\sigma}} = \sigma_{|\mu_1\mu_2\cdots\mu_p|}(x^1, x^2, \ldots, x^n)\tilde{\boldsymbol{d}}x^{\mu_1} \wedge \tilde{\boldsymbol{d}}x^{\mu_2} \wedge \ldots \wedge \tilde{\boldsymbol{d}}x^{\mu_p}; \tag{2.7}$$

ii. substitute a parameterization of the p-dimensional surface of the form, $x^\mu(\lambda^1, \lambda^2, \ldots, \lambda^p)$, so that

$$\tilde{\boldsymbol{\sigma}} = \sigma(\lambda^1, \lambda^2, \ldots, \lambda^p)\tilde{\boldsymbol{d}}\lambda^1 \wedge \tilde{\boldsymbol{d}}\lambda^2 \wedge \ldots \wedge \tilde{\boldsymbol{d}}\lambda^p; \tag{2.8}$$

iii. integrate

$$\begin{aligned}\int \tilde{\boldsymbol{\sigma}} &= \int \left\langle \tilde{\boldsymbol{\sigma}}, \frac{\partial}{\partial\lambda^1} \wedge \frac{\partial}{\partial\lambda^2} \wedge \ldots \wedge \frac{\partial}{\partial\lambda^p} \right\rangle d\lambda^1 d\lambda^2 \ldots d\lambda^p, \\ &= \int \sigma(\lambda^1, \lambda^2, \ldots, \lambda^p) d\lambda^1 d\lambda^2 \ldots d\lambda^p,\end{aligned} \tag{2.9}$$

using the elementary definition of integration.

iv. Stokes theorem

$$\int_\Omega \tilde{\boldsymbol{d}}\tilde{\boldsymbol{\sigma}} = \int_{\partial\Omega} \tilde{\boldsymbol{\sigma}}, \tag{2.10}$$

and Gauss theorem

$$\int_\Omega \tilde{\boldsymbol{d}}\, {}^*\tilde{\boldsymbol{\sigma}} = \int_{\partial\Omega} {}^*\tilde{\boldsymbol{\sigma}}, \tag{2.11}$$

where $\partial\Omega$ is the closed p-dimensional boundary of the $(p+1)$-dimensional surface Ω.

C. Dual of a p-form

In an n-dimensional manifold $\mathscr{M}$, the dual of a p-form $\tilde{\boldsymbol{\sigma}}$ is an $(n-p)$-form ${}^*\tilde{\boldsymbol{\sigma}}$ with components

$${}^*\sigma_{\mu_1\mu_2\cdots\mu_{n-p}} = \sigma^{|\nu_1\nu 2\cdots\nu_p|}\varepsilon_{\nu_1\cdots\nu_p\mu_1\cdots\mu_{n-p}}, \tag{2.12}$$

where ε is the *Levi-Civita tensor*, a completely antisymmetric rank n tensor. On a positively oriented basis $\{\vec{\boldsymbol{e}}_\mu\}$, $\varepsilon_{12\cdots n} = \varepsilon(\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \ldots, \vec{\boldsymbol{e}}_n) = +1$ and

$$\varepsilon_{\mu_1\mu_2\cdots\mu_n} = [\mu_1, \mu_2, \ldots, \mu_n] = \begin{cases} 0 & \text{unless } \mu_1, \mu_2, \ldots, \mu_n \text{ are all different} \\ +1 & \text{for even permutations of } 1, 2, \ldots, n \\ -1 & \text{for odd permutations of } 1, 2, \ldots, n \end{cases}, \tag{2.13}$$

so that given any matrix Λ

$$\varepsilon_{\mu_1\mu_2\cdots\mu_n}\Lambda^{\mu_1}_{\bar{1}}\Lambda^{\mu_2}_{\bar{2}}\cdots\Lambda^{\mu_n}_{\bar{n}} = \det||\Lambda^{\mu}{}_{\bar{\nu}}||\,. \tag{2.14}$$

The dual has the following property

$$\tilde{\boldsymbol{\sigma}} \wedge^* \tilde{\boldsymbol{\sigma}} = ||\sigma||^2\varepsilon, \tag{2.15}$$

where

$$||\sigma||^2 = \sigma_{|\mu_1\mu_2\cdots\mu_p|}\sigma^{\mu_1\mu_2\cdots\mu_p}, \tag{2.16}$$

is the norm of the p-form.

III. THE METRIC TENSOR

Now we will introduce a metric and delve into differential geometry. The metric $\boldsymbol{g}$ is a rank 2 symmetric tensor. In its $\binom{0}{2}$ form

$$g(\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta) = \vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = g_{\alpha\beta}, \tag{3.1}$$

$$\boldsymbol{g} = \boldsymbol{ds}^2 = g_{\alpha\beta}\tilde{\boldsymbol{\omega}}^\alpha \otimes \tilde{\boldsymbol{\omega}}^\beta, \tag{3.2}$$

where in a coordinate basis $\tilde{\boldsymbol{\omega}}^\alpha = \tilde{\boldsymbol{d}}x^\alpha$. If $\vec{\boldsymbol{\xi}} = dx^\alpha\vec{\boldsymbol{e}}_\alpha$ is a displacement vector then

$$\begin{aligned} g(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}}) = \vec{\boldsymbol{\xi}} \cdot \vec{\boldsymbol{\xi}} &= g_{\alpha\beta}\tilde{\boldsymbol{\omega}}^\alpha \otimes \tilde{\boldsymbol{\omega}}^\beta(dx^\gamma\vec{\boldsymbol{e}}_\gamma, dx^\delta\vec{\boldsymbol{e}}_\delta) \\ &= g_{\alpha\beta}\langle\tilde{\boldsymbol{\omega}}^\alpha, dx^\gamma\vec{\boldsymbol{e}}_\gamma\rangle\langle\tilde{\boldsymbol{\omega}}^\beta, dx^\delta\vec{\boldsymbol{e}}_\delta\rangle \\ &= g_{\alpha\beta}dx^\gamma dx^\delta\langle\tilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{e}}_\gamma\rangle\langle\tilde{\boldsymbol{\omega}}^\beta, \vec{\boldsymbol{e}}_\delta\rangle \\ &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= ds^2. \end{aligned} \tag{3.3}$$

In other words

i. Interval between two unspecified displacements $\boldsymbol{ds}^2 = \boldsymbol{g}$;

ii. Interval between two unspecified displacements $ds^2 = g(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\xi}})$;

as for

i. Unspecified direction $\tilde{\boldsymbol{d}}f$;

ii. Specified direction $df = \langle\tilde{\boldsymbol{d}}f, \vec{\boldsymbol{v}}\rangle = \partial_{\vec{\boldsymbol{v}}}f = v^\alpha f_{,\alpha}$.

We use $\boldsymbol{g}$ to establish a correspondence between 1-forms and vectors

$$\tilde{\boldsymbol{u}} \leftrightarrow \vec{\boldsymbol{u}} \quad \text{if and only if} \quad \langle\tilde{\boldsymbol{u}}, \vec{\boldsymbol{a}}\rangle = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{a}} = g(\vec{\boldsymbol{u}}, \vec{\boldsymbol{a}}) \quad \forall\ \vec{\boldsymbol{a}}. \tag{3.4}$$

In components $\tilde{\boldsymbol{u}} = u_\beta\tilde{\boldsymbol{\omega}}^\beta$ and

$$u_\beta = \langle\tilde{\boldsymbol{u}}, \vec{\boldsymbol{e}}_\beta\rangle = g(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}_\beta) = g(u^\alpha\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta) = u^\alpha g_{\alpha\beta}, \tag{3.5}$$

so we use $g_{\alpha\beta}$ to *lower indexes*.

Also $\tilde{\boldsymbol{\omega}}^\alpha$ is dual to $\vec{\boldsymbol{e}}_\alpha$. Call $\tilde{\boldsymbol{e}}^\alpha$ the 1-form corresponding to $\vec{\boldsymbol{e}}_\alpha$, then

$$\langle\tilde{\boldsymbol{e}}^\alpha, \vec{\boldsymbol{e}}_\beta\rangle = \vec{\boldsymbol{e}}_\alpha \cdot \vec{\boldsymbol{e}}_\beta = g_{\alpha\beta} = \langle g_{\alpha\gamma}\tilde{\boldsymbol{\omega}}^\gamma, \vec{\boldsymbol{e}}_\beta\rangle, \tag{3.6}$$

so $\tilde{\boldsymbol{e}}^\alpha = g_{\alpha\gamma}\tilde{\boldsymbol{\omega}}^\gamma = \tilde{\boldsymbol{\omega}}^\alpha$.

$\boldsymbol{g}$ in its $\binom{1}{1}$ form

$$g^\alpha{}_\beta = g(\tilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{e}}_\beta) = \langle\tilde{\boldsymbol{\omega}}^\alpha, \vec{\boldsymbol{e}}_\beta\rangle = \delta^\alpha{}_\beta. \tag{3.7}$$

$\boldsymbol{g}$ in its $\binom{2}{0}$ form

$$g^{\alpha\beta} = g(\tilde{\boldsymbol{\omega}}^\alpha, \tilde{\boldsymbol{\omega}}^\beta), \tag{3.8}$$

and

$$g^{\alpha}{}_{\beta} = g^{\alpha\mu} g_{\mu\beta} = \delta^{\alpha}{}_{\beta} \tag{3.9}$$

or $||g^{\alpha\beta}|| = ||g_{\alpha\beta}||^{-1}$.

Consider for example a tensor $\mathbb{H}$ of rank $\binom{1}{2}$, then

$$H(\widetilde{\boldsymbol{\omega}}^{\alpha}, \vec{\boldsymbol{e}}_{\beta}, \vec{\boldsymbol{e}}_{\gamma}) = H^{\alpha}{}_{\beta\gamma}, \tag{3.10}$$

$$\mathbb{H} = H^{\alpha}{}_{\beta\gamma} \vec{\boldsymbol{e}}_{\alpha} \otimes \widetilde{\boldsymbol{\omega}}^{\beta} \otimes \widetilde{\boldsymbol{\omega}}^{\gamma}, \tag{3.11}$$

$$H_{\alpha\beta\gamma} = H^{\delta}{}_{\beta\gamma} g_{\delta\alpha} \tag{3.12}$$

$$H_{\alpha'\beta'\gamma'} = L^{\alpha}{}_{\alpha'} L^{\beta}{}_{\beta'} L^{\gamma}{}_{\gamma'} H_{\alpha\beta\gamma}, \tag{3.13}$$

where the last equation is the change of basis.

A. The global (non coordinate) orthonormal frame

Through a change of basis $L^{\hat{\mu}}{}_{\mu}$ we can always diagonalize the symmetric metric tensor globally, thanks to the spectral theorem, so to realize an orthogonal frame

$$L^{\mu}{}_{\hat{\mu}} L^{\nu}{}_{\hat{\nu}} g_{\mu\nu} = \eta_{\hat{\mu}\hat{\nu}}. \tag{3.14}$$

Furthermore, it is always possible to rescale each vector of the orthogonal basis to get $||\eta_{\hat{\mu}\hat{\nu}}|| = \mathrm{diag}\{1, 1, \ldots, 1\}$. This at the price of having a non coordinate basis. We will call this the *global Lorentz frame* (LF) or the *global (non coordinate) orthonormal frame* and we will denote it with a hat on the indexes.

Upon taking the determinant of Eq. (3.14) we find

$$\det ||L^{\mu}{}_{\hat{\mu}}||^2 \det ||g_{\mu\nu}|| = 1. \tag{3.15}$$

We will denote

$$g = \det||g_{\mu\nu}||. \tag{3.16}$$

In Section IV B we will see that in General Relativity (GR) $\mathscr{M}$ is a pseudo-Riemannian 4-dimensional manifold with $||\eta_{\hat{\mu}\hat{\nu}}|| = \mathrm{diag}\{-1, 1, 1, 1\}$ and $\det||L^{\mu}{}_{\hat{\mu}}|| = 1/\sqrt{-g}$.

The Levi-Civita tensor in a general basis becomes

$$\begin{aligned}
\varepsilon_{\mu_1\mu_2\cdots\mu_n} &= L^{\hat{\mu}_1}{}_{\mu_1} L^{\hat{\mu}_2}{}_{\mu_2} \cdots L^{\hat{\mu}_n}{}_{\mu_n} \varepsilon_{\hat{\mu}_1\hat{\mu}_2\cdots\hat{\mu}_n} \\
&= \det||L^{\hat{\mu}}{}_{\mu}|| \varepsilon_{\hat{\mu}_1\hat{\mu}_2\cdots\hat{\mu}_n} \\
&= \sqrt{|g|}\, \varepsilon_{\hat{\mu}_1\hat{\mu}_2\cdots\hat{\mu}_n},
\end{aligned} \tag{3.17}$$

where in the first equality we used the fact that the Levi-Civita tensor is defined as the completely antisymmetric tensor of Eq. (2.13) only in a LF, in the second equality we used property (2.14), and in the last equality we used properties (3.15) and (1.7).

B. Commutators

Consider two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. We want to prove that

$$[\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}] f = \partial_{\vec{\boldsymbol{u}}} \partial_{\vec{\boldsymbol{v}}} f - \partial_{\vec{\boldsymbol{v}}} \partial_{\vec{\boldsymbol{u}}} f = \text{vector}, \tag{3.18}$$

for example on a scalar f.

We will prove this in a coordinate basis and then extend the result in a general non coordinate basis:

i. In a coordinate basis $\vec{\boldsymbol{e}}_{\alpha} = \partial_{\alpha} = \partial/\partial x^{\alpha}$ and

$$\begin{aligned}
[\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}] &= u^{\alpha} \partial_{\alpha} (v^{\beta} \partial_{\beta}) - v^{\beta} \partial_{\beta} (u^{\alpha} \partial_{\alpha}) \\
&= u^{\alpha} v^{\beta} \frac{\partial^2}{\partial x^{\alpha} \partial x^{\beta}} + u^{\alpha} v^{\beta}{}_{,\alpha} \frac{\partial}{\partial x^{\beta}} - \\
&\quad v^{\beta} u^{\alpha} \frac{\partial^2}{\partial x^{\beta} \partial x^{\alpha}} - v^{\beta} u^{\alpha}{}_{,\beta} \frac{\partial}{\partial x^{\alpha}} \\
&= (u^{\beta} v^{\alpha}{}_{,\beta} - v^{\beta} u^{\alpha}{}_{,\beta}) \frac{\partial}{\partial x^{\alpha}},
\end{aligned} \tag{3.19}$$

where we used the commutation of the partial derivatives. For basis vectors $[\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta] = [\partial_\alpha, \partial_\beta] = 0$;

ii. In a non coordinate basis we will have instead

$$[\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta] = c_{\alpha\beta}{}^{\gamma}\vec{\boldsymbol{e}}_\gamma, \tag{3.20}$$

so that

$$\begin{aligned}[\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}] &= [u^\alpha \vec{\boldsymbol{e}}_\alpha, v^\beta \vec{\boldsymbol{e}}_\beta] \\ &= (u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta} + u^\gamma v^\beta c_{\gamma\beta}{}^\alpha)\vec{\boldsymbol{e}}_\alpha.\end{aligned} \tag{3.21}$$

C. Covariant derivative

When taking a derivative on $\mathscr{M}$ we need to take care also of how the basis vectors and 1-forms change. [1] Such a derivative is called a *covariant derivative* for which we will use interchangeably the following three symbols

$$\frac{D\ldots}{\partial\lambda}\ , \quad u^\alpha\nabla_\alpha \ldots\ , \quad u^\alpha(\ldots)_{;\alpha}. \tag{3.22}$$

Let us distinguish four cases:

i. On a scalar f

$$\nabla_\alpha f = \partial_\alpha f \quad \text{or} \quad f_{;\alpha} = f_{,\alpha}. \tag{3.23}$$

ii. On a vector $\vec{\boldsymbol{v}} v^\alpha \vec{\boldsymbol{e}}_\alpha$. We will prove later that

$$\nabla_\alpha \vec{\boldsymbol{e}}_\beta = \partial_\alpha \vec{\boldsymbol{e}}_\beta = \Gamma^\gamma{}_{\alpha\beta}\vec{\boldsymbol{e}}_\gamma, \tag{3.24}$$

where the Γ are some coefficients called *connection coefficients* for a non coordinate (anholonomic) basis and *Christoffel symbols* for a coordinate basis (holonomic). [2]

Then

$$\begin{aligned}\nabla_\alpha \vec{\boldsymbol{v}} &= \nabla_\alpha (v^\beta \vec{\boldsymbol{e}}_\beta) \\ &= (\partial_\alpha v^\beta)\vec{\boldsymbol{e}}_\beta + v^\beta \partial_\alpha \vec{\boldsymbol{e}}_\beta \\ &= v^\beta{}_{,\alpha}\vec{\boldsymbol{e}}_\beta + v^\beta \Gamma^\gamma{}_{\alpha\beta}\vec{\boldsymbol{e}}_\gamma \\ &= \left(v^\beta{}_{,\alpha} + v^\gamma \Gamma^\beta{}_{\alpha\gamma}\right)\vec{\boldsymbol{e}}_\beta,\end{aligned} \tag{3.25}$$

or

$$(\nabla_\alpha \vec{\boldsymbol{v}})^\beta = v^\beta{}_{;\alpha} = v^\beta{}_{,\alpha} + \Gamma^\beta{}_{\alpha\gamma}v^\gamma. \tag{3.26}$$

iii. On a 1-form $\widetilde{\boldsymbol{\sigma}} = \sigma_\alpha \widetilde{\boldsymbol{\omega}}^\alpha$

$$\langle \widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\alpha\rangle = \sigma_\beta \langle \widetilde{\boldsymbol{\omega}}^\beta, \vec{\boldsymbol{e}}_\alpha\rangle = \sigma_\beta \delta^\beta{}_\alpha = \sigma_\alpha, \tag{3.27}$$

taking the covariant derivative of this expression

$$\nabla_\alpha \langle \widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta\rangle = \sigma_{\beta,\alpha} \tag{3.28}$$

$$\langle \nabla_\alpha \widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta\rangle + \langle \widetilde{\boldsymbol{\sigma}}, \partial_\alpha \vec{\boldsymbol{e}}_\beta\rangle = \sigma_{\beta,\alpha} \tag{3.29}$$

$$\langle \nabla_\alpha \widetilde{\boldsymbol{\sigma}}, \vec{\boldsymbol{e}}_\beta\rangle = \sigma_{\beta,\alpha} - \langle \widetilde{\boldsymbol{\sigma}}, \Gamma^\gamma{}_{\alpha\beta}\vec{\boldsymbol{e}}_\gamma\rangle \tag{3.30}$$

$$(\nabla_\alpha \widetilde{\boldsymbol{\sigma}})_\beta = \sigma_{\beta;\alpha} = \sigma_{\beta,\alpha} - \Gamma^\gamma{}_{\alpha\beta}\sigma_\gamma, \tag{3.31}$$

where we see how the correction due to the change of the basis vector enters with a minus sign.

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[1] In other terms, when taking a derivative of a vector on $\mathscr{M}$ we come across the problem of comparing two vectors at two different points of $\mathscr{M}$. This is solved with the procedure of *parallel transport* where we simply compare the two vectors either at the initial or at the final point after having "copied" the components of the vector respect to the basis at one point on its components respect to the basis at the other point.

[2] Our definition for the connection coefficients is different from the one of Ref. [1] where $\nabla_\alpha \vec{\boldsymbol{e}}_\beta = \Gamma^\gamma{}_{\beta\alpha}\vec{\boldsymbol{e}}_\gamma$. This difference is only relevant for a non coordinate basis.

iv. On a tensor $\mathbb{H}$ of rank $\binom{r}{s}$

$$
\begin{aligned}
(\nabla_\alpha H)^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s} = {} & H^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s;\alpha} = H^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s,\alpha} \\
& +\Gamma^{\mu_1}{}_{\gamma\alpha}H^{\gamma\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\nu_s} + \ldots + \Gamma^{\mu_r}{}_{\gamma\alpha}H^{\mu_1\mu_2\cdots\gamma}{}_{\nu_1\nu_2\cdots\nu_s} \\
& -\Gamma^{\gamma}{}_{\nu_1\alpha}H^{\mu_1\mu_2\cdots\mu_r}{}_{\gamma\nu_2\cdots\nu_s} + \ldots - \Gamma^{\gamma}{}_{\nu_s\alpha}H^{\mu_1\mu_2\cdots\mu_r}{}_{\nu_1\nu_2\cdots\gamma}. \qquad (3.32)
\end{aligned}
$$

The connection coefficients

We now want to prove Eq. (3.24) and determine the expression of the connection coefficients in terms of the metric tensor. Start again from the definition (3.24)

$$
g(\nabla_\alpha\vec{e}_\beta,\vec{e}_\gamma) = g\left(\Gamma^\delta{}_{\alpha\beta}\vec{e}_\delta,\vec{e}_\gamma\right) = \Gamma^\delta{}_{\alpha\beta}\,g_{\delta\gamma}. \qquad (3.33)
$$

Then consider the partial derivative of the metric tensor

$$
g_{\beta\gamma,\alpha} = g(\nabla_\alpha\vec{e}_\beta,\vec{e}_\gamma) + g(\vec{e}_\beta,\nabla_\alpha\vec{e}_\gamma). \qquad (3.34)
$$

Rewrite Eq. (3.34) in the following 3 equivalent ways

$$
\begin{aligned}
g_{\beta\gamma,\alpha} &= g(\vec{e}_\beta,\nabla_\gamma\vec{e}_\alpha) + g(\vec{e}_\gamma,\nabla_\alpha\vec{e}_\beta) - g(\vec{e}_\beta,[\vec{e}_\gamma,\vec{e}_\alpha]), \qquad (3.35a)\\
g_{\gamma\alpha,\beta} &= g(\vec{e}_\alpha,\nabla_\beta\vec{e}_\gamma) + g(\vec{e}_\gamma,\nabla_\alpha\vec{e}_\beta) - g(\vec{e}_\gamma,[\vec{e}_\alpha,\vec{e}_\beta]), \qquad (3.35b)\\
g_{\alpha\beta,\gamma} &= g(\vec{e}_\alpha,\nabla_\beta\vec{e}_\gamma) + g(\vec{e}_\beta,\nabla_\gamma\vec{e}_\alpha) - g(\vec{e}_\alpha,[\vec{e}_\beta,\vec{e}_\gamma]), \qquad (3.35c)
\end{aligned}
$$

where we used the symmetry of the metric tensor $\boldsymbol{g}$ and the definition of the commutator $[\cdot,\cdot]$. Adding (3.35a) and (3.35b) and subtracting (3.35c), and using the definitions (3.24) and (3.20) for the Γ and c coefficients respectively, we find

$$
2g(\vec{e}_\gamma,\nabla_\alpha\vec{e}_\beta) = g_{\beta\gamma,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} - c_{\gamma\alpha\beta} - c_{\alpha\beta\gamma} + c_{\beta\gamma\alpha}. \qquad (3.36)
$$

Using Eqs. (3.33) and (3.20) we find for the connection coefficients

$$
\Gamma_{\gamma\alpha\beta} = \frac{1}{2}\{g_{\beta\gamma,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} + c_{\alpha\gamma\beta} + c_{\beta\alpha\gamma} - c_{\gamma\beta\alpha}\}. \qquad (3.37)
$$

In a coordinate basis all c are zero and we find the so called Christoffel symbols

$$
\Gamma_{\gamma\alpha\beta} = \frac{1}{2}\{g_{\beta\gamma,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}\}, \qquad (3.38)
$$

which is clearly symmetric in its last two indexes.

An important property of the metric tensor is to be covariantly constant, i.e. $\nabla\boldsymbol{g} = 0$. In fact in an orthonormal frame $g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$ and in the next Section III D we will see that it is also always possible to choose a local coordinate orthonormal frame on $\mathscr{M}$ such that $g_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = 0$ (and of course $c_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = 0$ on the coordinate frame), then $\Gamma_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = 0$ so that $g_{\hat{\alpha}\hat{\beta};\hat{\gamma}} = g_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = 0$.

We will now prove 3 properties of Γ:

i. Since the metric is covariantly constant

$$
\begin{aligned}
0 = g_{\alpha\beta;\gamma} &= g_{\alpha\beta,\gamma} - \Gamma^\mu{}_{\alpha\gamma}g_{\mu\beta} - \Gamma^\mu{}_{\beta\gamma}g_{\alpha\mu}\\
&= g_{\alpha\beta,\gamma} - \Gamma_{\beta\alpha\gamma} - \Gamma_{\alpha\beta\gamma}, \qquad (3.39)
\end{aligned}
$$

so that

$$
\frac{1}{2}g_{\alpha\beta,\gamma} = \Gamma_{(\alpha\beta)\gamma}, \qquad (3.40)
$$

where the round parenthesis contain indexes on which one symmetrizes. So Γ is antisymmetric on its first two indexes in the local coordinate orthonormal frame described in the next Section III D for which $g_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = 0$.

ii. From the definition of the commutator (3.20) and the connection coefficient (3.24) follows

$$
\begin{aligned}
[\vec{e}_\alpha, \vec{e}_\beta] &= \nabla_\alpha \vec{e}_\beta - \nabla_\beta \vec{e}_\alpha \\
&= (\Gamma^\gamma{}_{\alpha\beta} - \Gamma^\gamma{}_{\beta\alpha}) \vec{e}_\gamma \\
&= c_{\alpha\beta}{}^\gamma \vec{e}_\gamma,
\end{aligned} \tag{3.41}
$$

so that

$$
\frac{1}{2} c_{\alpha\beta\gamma} = \Gamma_{\gamma[\alpha\beta]}, \tag{3.42}
$$

where the square parenthesis contain indexes on which one antisymmetrizes. So Γ is symmetric on its last two indexes in a coordinate reference frame where $c_{\alpha\beta\gamma} = 0$.

iii. Γ is not a tensor. In fact let's see how Γ transforms

$$
\begin{aligned}
\nabla_{\alpha'} \vec{e}_{\beta'} = \Gamma^{\gamma'}{}_{\alpha'\beta'} \vec{e}_{\gamma'} = \nabla_{L^\alpha{}_{\alpha'} \vec{e}_\alpha} (L^\beta{}_{\beta'} \vec{e}_\beta) &= L^\alpha{}_{\alpha'} \nabla_\alpha (L^\beta{}_{\beta'} \vec{e}_\beta) \\
&= L^\alpha{}_{\alpha'} L^\beta{}_{\beta'} \nabla_\alpha \vec{e}_\beta + L^\alpha{}_{\alpha'} L^\beta{}_{\beta',\alpha} \vec{e}_\beta \\
&= L^\alpha{}_{\alpha'} L^\beta{}_{\beta'} \Gamma^\gamma{}_{\alpha\beta} \vec{e}_\gamma + L^\alpha{}_{\alpha'} L^\beta{}_{\beta',\alpha} L^{\gamma'}{}_\beta \vec{e}_{\gamma'},
\end{aligned} \tag{3.43}
$$

so that

$$
\Gamma^{\gamma'}{}_{\alpha'\beta'} = L^\alpha{}_{\alpha'} L^\beta{}_{\beta'} L^{\gamma'}{}_\gamma \Gamma^\gamma{}_{\alpha\beta} + L^\alpha{}_{\alpha'} L^{\gamma'}{}_\beta L^\beta{}_{\beta',\alpha}, \tag{3.44}
$$

where the last term is in general different from zero.

Useful identities

(Ex. 7.7 [1]) We will here enunciate and prove 7 useful identities:

i. From the definition (3.16) follows

$$
g_{,\alpha} = g g^{\mu\nu} g_{\mu\nu,\alpha} = -g g_{\mu\nu} g^{\mu\nu}{}_{,\alpha}. \tag{3.45}
$$

To prove this identity we first note that for any diagonalizable matrix A the following identity holds

$$
\det A = e^{\mathrm{tr}(\ln A)}, \tag{3.46}
$$

which clearly holds when A is in its diagonal form. So

$$
\begin{aligned}
g_{,\alpha} &= \left[e^{\mathrm{tr}(\ln ||g_{\mu\nu}||)} \right]_{,\alpha} \\
&= g[\mathrm{tr}(\ln ||g_{\mu\nu}||)]_{,\alpha} \\
&= g\,\mathrm{tr}[(\ln ||g_{\mu\nu}||)_{,\alpha}] \\
&= g\,\mathrm{tr}(||g_{\mu\nu}||^{-1} ||g_{\mu\nu,\alpha}||) \\
&= g g^{\mu\nu} g_{\mu\nu,\alpha} \\
&= -g g_{\mu\nu} g^{\mu\nu}{}_{,\alpha}
\end{aligned}
$$

where in the last equality we used Eq. (3.9).

All the remaining identities require a coordinate basis.

ii. Contraction of first two indexes of Christoffel symbol

$$
\Gamma^\alpha{}_{\beta\alpha} = \Gamma^\alpha{}_{\alpha\beta} = \left(\ln \sqrt{|g|} \right)_\beta. \tag{3.47}
$$

From Eq. (3.38) and identity [i.] follows

$$
\Gamma^\alpha{}_{\beta\alpha} = \frac{1}{2} g^{\alpha\nu} g_{\alpha\nu,\beta} = \frac{1}{2} g_{,\beta}/g = \left(\ln \sqrt{|g|} \right)_{,\beta}. \tag{3.48}
$$

iii. Contraction of last two indexes of Christoffel symbol

$$g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = -\frac{1}{\sqrt{|g|}}\left(g^{\alpha\nu}\sqrt{|g|}\right)_{\nu}. \tag{3.49}$$

From Eq. (3.38) follows

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\{g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\}, \tag{3.50}$$

using property [i.]

$$g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\{2g_{\beta\mu},{}^{\mu} - g_{,\beta}/g\}. \tag{3.51}$$

On the other hand sine the contracted index is mute

$$\frac{1}{\sqrt{|g|}}\left(g^{\alpha\nu}\sqrt{|g|}\right)_{\nu} = \frac{1}{\sqrt{|g|}}\left(g^{\alpha\nu}{}_{,\nu}\sqrt{|g|} + g^{\alpha\nu}g_{,\nu}/2\sqrt{|g|}\right) = g^{\alpha\nu}{}_{,\nu} + g^{\alpha\beta}g_{,\beta}/2g, \tag{3.52}$$

and using Eq. (3.9)

$$0 = (g^{\alpha\beta}g_{\beta\mu}),{}^{\mu} = g^{\alpha\beta}g_{\beta\mu},{}^{\mu} + g^{\alpha\nu}{}_{,\nu}. \tag{3.53}$$

Putting together (3.51), (3.52), and (3.53) gives identity [iii.].

iv. Divergence of a vector

$$A^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha}\right)_{,\alpha}. \tag{3.54}$$

From the definition of covariant derivative (3.25) and identity [ii.] follows

$$\begin{aligned} A^{\alpha}{}_{;\alpha} &= A^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}{}_{\beta\alpha}A^{\beta} \\ &= A^{\alpha}{}_{,\alpha} + \frac{\left(\sqrt{|g|}\right)_{,\alpha}}{\sqrt{|g|}}A^{\alpha} \\ &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha}\right)_{,\alpha}. \end{aligned} \tag{3.55}$$

v. Divergence of a rank $\binom{2}{0}$ antisymmetric tensor

$$A^{\alpha\beta}{}_{;\beta} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha\beta}\right)_{,\beta}. \tag{3.56}$$

From the definition of covariant derivative (3.32) and identity [ii.] follows

$$\begin{aligned} A^{\alpha\beta}{}_{;\beta} &= A^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\gamma\beta}A^{\gamma\beta}\Gamma^{\beta}{}_{\gamma\beta}A^{\alpha\gamma} \\ &= A^{\alpha\beta}{}_{,\beta} + \Gamma^{\beta}{}_{\gamma\beta}A^{\alpha\gamma} \\ &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^{\alpha\beta}\right)_{,\beta}, \end{aligned} \tag{3.57}$$

where in the first equality we used the symmetry of the Christoffel symbol respect to its last two indexes and in the last equality we used identity [ii.].

vi. Divergence of a rank $\binom{1}{1}$ tensor

$$A_\alpha{}^\beta{}_{;\beta} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A_\alpha{}^\beta\right)_{,\beta} - \Gamma^\lambda{}_{\alpha\mu}A_\lambda{}^\mu. \tag{3.58}$$

From the definition of covariant derivative (3.32) and identity [ii.] follows

$$\begin{aligned} A_\alpha{}^\beta{}_{;\beta} &= A_\alpha{}^\beta{}_{,\beta} + \Gamma^\beta{}_{\mu\beta}A_\alpha{}^\mu - \Gamma^\lambda{}_{\alpha\mu}A_\lambda{}^\mu \\ &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A_\alpha{}^\beta\right)_{,\beta} - \Gamma^\lambda{}_{\alpha\mu}A_\lambda{}^\mu. \end{aligned} \tag{3.59}$$

where again in the last equality we used identity [ii.].

vii. Laplacian

$$\Box S = S_{;\alpha}{}^\alpha = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}S_,{}^\beta\right)_{,\beta}, \tag{3.60}$$

where S is a scalar. Since the metric is covariantly constant

$$S_{;\alpha}{}^\alpha = (S_{,\alpha})_{;\beta}\, g^{\beta\alpha} = \left(S_{,\alpha}g^{\beta\alpha}\right)_{;\beta} = \left(S_,{}^\beta\right)_{;\beta} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}S_,{}^\beta\right)_{,\beta}. \tag{3.61}$$

where in the last equality we used identity [iv.].

D. The local (coordinate) orthonormal frame

(Ex. 13.3 [1]) A *local (coordinate) orthonormal frame* is "tangent" to the manifold $\mathscr{M}$ on its point $\mathscr{P}_0$. We will call this a *Local Lorentz Frame* (LLF). It is the closest thing to a global (non coordinate) orthonormal frame "near" $\mathscr{P}_0$. It satisfies the following recipes:

i. $g_{\alpha\beta}(\mathscr{P}_0) = \eta_{\alpha\beta}$;

ii. $g_{\alpha\beta,\gamma}(\mathscr{P}_0) = 0$;

iii. $g_{\alpha\beta,\gamma\delta}(\mathscr{P}_0) \neq 0$ in general;

We will now prove that it is always possible to choose a coordinate system such that [i.] and [ii.] hold at an arbitrary point.

Let's first count the number of independent components in a symmetric tensor of dimension n and rank r. For $r = 2$ we have $\binom{n}{2} + \binom{n}{1} = \frac{n(n+1)}{2}$ independent components. For $r = 3$ we have $\binom{n}{3} + 2\binom{n}{2} + \binom{n}{1} = \frac{n(n+1)(n+2)}{6}$. independent components. So for example in GR $n = 4$ and we find 10 for $r = 2$ and 20 for $r = 3$.

Consider now an arbitrary change of coordinates $x^{\alpha'} = f^{\alpha'}(x^\alpha)$. Taylor expand around $\mathscr{P}_0$ at the origin

$$x^{\alpha'} = f^{\alpha'}{}_{,\mu}x^\mu + \frac{1}{2}f^{\alpha'}{}_{,\mu\nu}x^\mu x^\nu + \frac{1}{6}f^{\alpha'}{}_{,\mu\nu\lambda}x^\mu x^\nu x^\lambda + \dots. \tag{3.62}$$

Then we can count the independent components of the various coefficients. For example in $n = 4$ the linear term $M^{\alpha'}{}_\mu = f^{\alpha'}{}_{,\mu}$ has $4 \times 4 = 16$ components, the quadratic term $N^{\alpha'}{}_{\mu\nu} = f^{\alpha'}{}_{,\mu\nu}$ has $4 \times 10 = 40$ components, and the cubic term $P^{\alpha'}{}_{\mu\nu\lambda} = f^{\alpha'}{}_{,\mu\nu\lambda}$ has $4 \times 20 = 80$ components. Recall that

$$L^{\alpha'}{}_\mu = \frac{\partial x^{\alpha'}}{\partial x^\mu} = M^{\alpha'}{}_\mu + N^{\alpha'}{}_{\mu\nu}x^\nu + \frac{1}{2}P^{\alpha'}{}_{\mu\nu\lambda}x^\nu x^\lambda + \dots. \tag{3.63}$$

At the origin we want $g_{\mu\nu}(\mathscr{P}_0) = \eta_{\mu\nu}$, but in general

$$\begin{aligned} g_{\mu\nu}(\mathscr{P}_0) = &\left[M^{\alpha'}{}_\mu + N^{\alpha'}{}_{\mu\nu}x^\nu + \frac{1}{2}P^{\alpha'}{}_{\mu\nu\lambda}x^\nu x^\lambda + \dots\right] \times \\ &\left[M^{\beta'}{}_\nu + N^{\beta'}{}_{\nu\lambda}x^\lambda + \frac{1}{2}P^{\beta'}{}_{\nu\mu\lambda}x^\mu x^\lambda + \dots\right] g_{\alpha'\beta'}. \end{aligned} \tag{3.64}$$

Then we conclude the following:

i. The condition on the metric requires

$$g_{\mu\nu}(\mathscr{P}_0) = \eta_{\mu\nu} = M^{\alpha'}{}_{\mu} M^{\beta'}{}_{\nu} g_{\alpha'\beta'}, \tag{3.65}$$

which can always be accommodated and for example in GR we have 10 independent components in $\eta_{\mu\nu}$ and $4 \times 4 = 16$ in $M^{\alpha}{}_{\beta}$. So we have 6 degrees of freedom left over for a Lorentz transformation (3 boosts and 3 rotations) to determine $M^{\nu}{}_{\mu}$.

ii. The condition on the first derivative of the metric requires

$$0 = g_{\mu\nu,\lambda}(\mathscr{P}_0) = M^{\alpha'}{}_{\mu} M^{\beta'}{}_{\nu} g_{\alpha'\beta',\lambda} + (N^{\alpha'}{}_{\mu\lambda} M^{\beta'}{}_{\nu} + N^{\beta'}{}_{\nu\lambda} M^{\alpha'}{}_{\mu}) g_{\alpha'\beta'}, \tag{3.66}$$

which can also be always accommodated with no degrees of freedom left. For example in GR $g_{\mu\nu,\lambda}$ has $4\times 10 = 40$ independent components then we will always be able to find the exactly 40 components of $N^{\alpha}{}_{\beta\gamma}$.

iii. The condition on the second derivative of the metric $g_{\mu\nu,\lambda\rho}(\mathscr{P}_0) = 0$ cannot in general be satisfied. For example in GR $g_{\mu\nu,\lambda\rho}$ has $10\times 10 = 100$ independent components but $P^{\alpha}{}_{\beta\gamma\delta}$ has only 80, so 20 degrees of freedom cannot be specified. We will see in Section III F that these are exactly the degrees of freedom of the Riemann curvature tensor.

So we can say that a LLF is the closest thing possible to a global orthonormal frame at a particular point $\mathscr{P}_0$ of the Riemannian manifold $\mathscr{M}$, being the tangent space to $\mathscr{M}$ at $\mathscr{P}_0$.

E. Geodesics

A *geodesic* is a curve on the manifold $\mathscr{M}$ that parallel transports its tangent vector along itself

$$\nabla_{\vec{u}} \vec{u} = 0, \tag{3.67}$$

i.e. the tangent vector $\vec{u}$ is covariantly constant along the curve

$$u^{\alpha} u^{\beta}{}_{;\beta} = 0, \tag{3.68}$$

$$u^{\alpha}\left(u^{\beta}{}_{,\beta} + \Gamma^{\beta}{}_{\alpha\gamma} u^{\gamma}\right) = 0, \tag{3.69}$$

$$\frac{d^2 x^{\beta}}{d\lambda^2} + \Gamma^{\beta}{}_{\alpha\gamma} \frac{dx^{\alpha}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} = 0, \tag{3.70}$$

where x^{α} is a coordinate system on $\mathscr{M}$ and $u^{\alpha} = dx^{\alpha}/d\lambda$ with $\lambda = a\bar{\lambda} + b$ is an affine parameter (the proper time in GR) with a and b two real numbers giving the units (of time) and the origin (of time) respectively.

The geodesic equation (3.70) is a second order differential equation. For a solution it is then necessary to give the initial conditions $x^{\alpha}(0)$ and $\dot{x}^{\alpha}(0)$, where the dot stands fro a derivative respect to λ . Through each point of $\mathscr{M}$ exists a unique geodesic in each direction.

All affine parameters are related by a linear transformation. In fact, let $\bar{\lambda} = \bar{\lambda}(\lambda)$, then $d/d\lambda = \dot{\bar{\lambda}} d/d\bar{\lambda}$ and $d^2/d\lambda^2 = \ddot{\bar{\lambda}} d/d\bar{\lambda} + (\dot{\bar{\lambda}})^2 d^2/d\bar{\lambda}^2$. So Eq. (3.70) becomes

$$\frac{d^2 x^{\beta}}{d\bar{\lambda}^2} + \frac{\ddot{\bar{\lambda}}}{(\dot{\bar{\lambda}})^2}\frac{dx^{\beta}}{d\bar{\lambda}} + \Gamma^{\beta}{}_{\alpha\gamma}\frac{dx^{\alpha}}{d\bar{\lambda}}\frac{dx^{\gamma}}{d\bar{\lambda}} = 0. \tag{3.71}$$

Since the change in the affine parameter must not change the geodetic equation then the second term in Eq. (3.71) must cancel. This occur if $\ddot{\bar{\lambda}} = 0$ or $\bar{\lambda} = a\lambda + b$.

From a variational principle

Alternatively we can define a geodesic as a curve of extremal length. The length of a curve $C(\lambda)$ is given by

$$\mathscr{S} = \int_C ds = \int_C \sqrt{g_{\alpha\beta}(x^{\gamma})\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda}}\, d\lambda = \int \mathscr{L}_C(x^{\gamma}, \dot{x}^{\gamma})\, d\lambda, \tag{3.72}$$

where λ is any parameter along the curve. The curve of extremal length is the one obtained through the stationary variational principle $\delta\mathscr{S}[x^\gamma, \dot{x}^\gamma] = 0$. We then find

$$\frac{d}{d\lambda}\left(\frac{\partial \mathscr{L}_\mathscr{C}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathscr{L}_\mathscr{C}}{\partial x^\alpha} = 0, \tag{3.73}$$

where $\mathscr{C}$ is the geodesic. We will from now on forget about this subscript. Since $\partial\mathscr{L}/\partial\dot{x}^\alpha = g_{\alpha\beta}\dot{x}^\beta/\mathscr{L}$ then Eq. (3.73) becomes

$$\frac{d}{d\lambda}\left(\frac{1}{\mathscr{L}} g_{\alpha\beta}\dot{x}^\beta\right) - \frac{1}{2\mathscr{L}} g_{\gamma\delta,\alpha}\dot{x}^\gamma\dot{x}^\delta = 0, \tag{3.74}$$

or

$$-\frac{1}{\mathscr{L}^2}\frac{d\mathscr{L}}{d\lambda} g_{\alpha\beta}\dot{x}^\beta + \frac{1}{\mathscr{L}} g_{\alpha\beta,\gamma}\dot{x}^\gamma\dot{x}^\beta + \frac{1}{\mathscr{L}} g_{\alpha\beta}\ddot{x}^\beta - \frac{1}{2\mathscr{L}} g_{\gamma\delta,\alpha}\dot{x}^\gamma\dot{x}^\delta = 0, \tag{3.75}$$

or, reordering terms,

$$g_{\alpha\beta}\ddot{x}^\beta + \frac{1}{2}\{g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\gamma\beta,\alpha}\}\dot{x}^\gamma\dot{x}^\beta = \frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{d\lambda} g_{\alpha\beta}\dot{x}^\beta, \tag{3.76}$$

recalling the definition (3.38) for the Christoffel symbol

$$g_{\alpha\beta}\ddot{x}^\beta + \Gamma_{\alpha\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = \frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{d\lambda}\dot{x}_\alpha, \tag{3.77}$$

On the extremal curve $\mathscr{L} = d\mathscr{S}/d\lambda$ =constant so $d\mathscr{L}/d\lambda = 0$ and we recover the geodesic Eq. (3.70).

F. Curvature

(Chapter 11 [1]) We will use a geometric introduction.

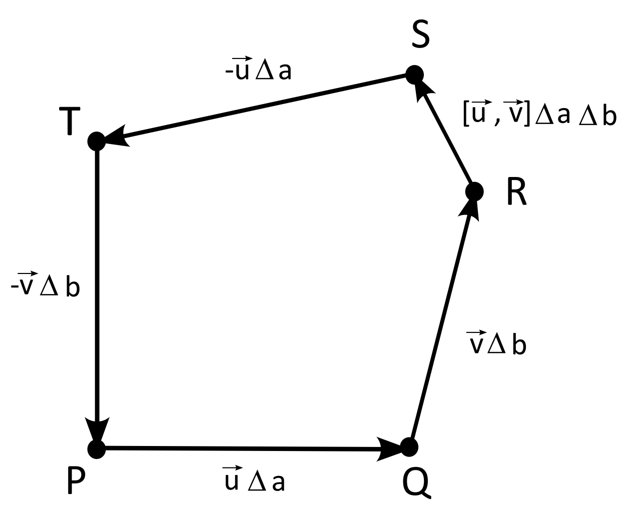

FIG. 1. Closed curve on $\mathscr{M}$ of infinitesimal area.

Consider a closed curve on $\mathscr{M}$ of infinitesimal area as in Fig. 1. We start at P then move to Q to end in R. We then move from P to T to S. The two paths are then closed by moving from R to S. We then consider the change $\delta\vec{\boldsymbol{A}}$ in a vector field $\vec{\boldsymbol{A}}^{\text{field}}$ from parallel transporting itself along the path P→T→S or parallel transporting itself along the path P→Q→R and closing the gap to reach S. At P we will have a vector $\vec{\boldsymbol{A}}_P = \vec{\boldsymbol{A}}_P^{\text{field}}$ where $\vec{\boldsymbol{A}}_P^{\text{field}}$ is our vector field at P. When we move from P to Q we can compare the vector field at Q with the parallel transported vector at Q, $\delta_{P\to Q}\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_Q^{\text{field}} - \vec{\boldsymbol{A}}_{P\overset{||}{\to}Q} = \Delta a\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{A}}$. We then move to R to find $\delta_{Q\to R}\delta_{P\to Q}\vec{\boldsymbol{A}} = \Delta a\Delta b\nabla_{\vec{\boldsymbol{v}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{A}}$. Going from P to T to S we find $\delta_{T\to S}\delta_{P\to T}\vec{\boldsymbol{A}} = \Delta a\Delta b\nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{v}}}\vec{\boldsymbol{A}}$. We then go from R to S to close the curve and we find $\delta_{R\to S}\vec{\boldsymbol{A}} = \Delta a\Delta b\nabla_{[\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}]}\vec{\boldsymbol{A}}$. So the change of the vector field around the curve is

$$\begin{aligned}\delta\vec{\boldsymbol{A}} &= \delta_{Q\to R}\delta_{P\to Q}\vec{\boldsymbol{A}} + \delta_{R\to S}\vec{\boldsymbol{A}} - \delta_{P\to T}\delta_{T\to S}\vec{\boldsymbol{A}} \\ &= \Delta a\Delta b\left(\nabla_{\vec{\boldsymbol{v}}}\nabla_{\vec{\boldsymbol{u}}} - \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{v}}} - \nabla_{[\vec{\boldsymbol{v}},\vec{\boldsymbol{u}}]}\right)\vec{\boldsymbol{A}} && (3.78)\\ &= \Delta a\Delta b\mathscr{R}(\vec{\boldsymbol{v}},\vec{\boldsymbol{u}})\vec{\boldsymbol{A}}, && (3.79)\end{aligned}$$

where

$$\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) = \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{v}}} - \nabla_{\vec{\boldsymbol{v}}}\nabla_{\vec{\boldsymbol{u}}} - \nabla_{[\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}]}, \tag{3.80}$$

is the *curvature (local) operator*.

We will now give 3 properties of this operator:

i. For any 3 vectors $\vec{\boldsymbol{u}},\vec{\boldsymbol{v}},\vec{\boldsymbol{w}}$, and a scalar f

$$\begin{aligned}\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})f\vec{\boldsymbol{w}} &= f\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}, && (3.81)\\ \mathscr{R}(f\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} &= f\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}, && (3.82)\\ \mathscr{R}(\vec{\boldsymbol{u}},f\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} &= f\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}. && (3.83)\end{aligned}$$

ii. $\mathscr{R}$ is linear

$$\begin{aligned}\mathscr{R}(\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} &= \mathscr{R}(\vec{\boldsymbol{a}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}} + \mathscr{R}(\vec{\boldsymbol{b}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}, && (3.84)\\ \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}})\vec{\boldsymbol{w}} &= \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{a}})\vec{\boldsymbol{w}} + \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{b}})\vec{\boldsymbol{w}}, && (3.85)\\ \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})(\vec{\boldsymbol{a}}+\vec{\boldsymbol{b}}) &= \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{a}} + \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{b}}. && (3.86)\end{aligned}$$

iii. $\mathscr{R}$ is local

$$\mathscr{R}(\vec{\boldsymbol{u}}+\vec{\boldsymbol{a}},\vec{\boldsymbol{v}}+\vec{\boldsymbol{b}})(\vec{\boldsymbol{w}}+\vec{\boldsymbol{c}}) \xrightarrow{\substack{\vec{\boldsymbol{a}}\to\vec{\boldsymbol{0}}\\ \vec{\boldsymbol{b}}\to\vec{\boldsymbol{0}}\\ \vec{\boldsymbol{c}}\to\vec{\boldsymbol{0}}}} \mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}. \tag{3.87}$$

These 3 properties imply that $\mathscr{R}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\vec{\boldsymbol{w}}$ is a tensor.

G. The Riemann tensor

The *Riemann tensor* $\mathbb{R}$ is defined in terms of the curvature tensor as follows

$$\mathbb{R}(\tilde{\boldsymbol{\sigma}},\vec{\boldsymbol{c}},\vec{\boldsymbol{a}},\vec{\boldsymbol{b}}) = \langle\tilde{\boldsymbol{\sigma}},\mathscr{R}(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}})\vec{\boldsymbol{c}}\rangle. \tag{3.88}$$

The components of Riemann are as follows

$$\begin{aligned}R^\alpha{}_{\beta\gamma\delta} &= \mathbb{R}(\tilde{\boldsymbol{\omega}}^\alpha,\vec{\boldsymbol{e}}_\beta,\vec{\boldsymbol{e}}_\gamma,\vec{\boldsymbol{e}}_\delta) && (3.89)\\ &= \langle\tilde{\boldsymbol{\omega}}^\alpha,\mathscr{R}(\vec{\boldsymbol{e}}_\gamma,\vec{\boldsymbol{e}}_\delta)\vec{\boldsymbol{e}}_\beta\rangle && (3.90)\\ &= \langle\tilde{\boldsymbol{\omega}}^\alpha,\nabla_\gamma\nabla_\delta\vec{\boldsymbol{e}}_\beta - \nabla_\delta\nabla_\gamma\vec{\boldsymbol{e}}_\beta - \nabla_{[\vec{\boldsymbol{e}}_\gamma,\vec{\boldsymbol{e}}_\delta]}\vec{\boldsymbol{e}}_\beta\rangle && (3.91)\\ &= \langle\tilde{\boldsymbol{\omega}}^\alpha,\nabla_\gamma\left(\Gamma^\mu{}_{\delta\beta}\vec{\boldsymbol{e}}_\mu\right) - \nabla_\delta\left(\Gamma^\mu{}_{\gamma\beta}\vec{\boldsymbol{e}}_\mu\right) - \nabla_{\left(c_{\gamma\delta}{}^\mu\vec{\boldsymbol{e}}_\mu\right)}\vec{\boldsymbol{e}}_\beta\rangle && (3.92)\\ &= \langle\tilde{\boldsymbol{\omega}}^\alpha,\Gamma^\mu{}_{\delta\beta,\gamma}\vec{\boldsymbol{e}}_\mu + \Gamma^\mu{}_{\delta\beta}\Gamma^\sigma{}_{\gamma\mu}\vec{\boldsymbol{e}}_\sigma - \Gamma^\mu{}_{\gamma\beta,\delta}\vec{\boldsymbol{e}}_\mu - \Gamma^\mu{}_{\gamma\beta}\Gamma^\sigma{}_{\delta\mu}\vec{\boldsymbol{e}}_\sigma - c_{\gamma\delta}{}^\mu\Gamma^\sigma{}_{\mu\beta}\vec{\boldsymbol{e}}_\sigma\rangle, && (3.93)\end{aligned}$$

so

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\delta\beta,\gamma} + \Gamma^\mu{}_{\delta\beta}\Gamma^\alpha{}_{\gamma\mu} - \Gamma^\alpha{}_{\gamma\beta,\delta} - \Gamma^\mu{}_{\gamma\beta}\Gamma^\alpha{}_{\delta\mu} - c_{\gamma\delta}{}^\mu\Gamma^\alpha{}_{\mu\beta}, \tag{3.94}$$

which for GR are $4\times4\times4\times4 = 256$ components and we expect a reduction to 20 (see Section III D).

Note that in a flat space $g_{\alpha\beta} = \eta_{\alpha\beta}$ globally, so all $R^\alpha{}_{\beta\gamma\delta} = 0$.

We give now 4 symmetry properties of Riemann. We will prove these working in a LLF (see Section III D). Since Riemann is local these properties will hold globally. In a LLF $g_{\alpha\beta,\gamma} = 0$ and the Christoffel symbols vanish, so we find

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\mu}R^\mu{}_{\beta\gamma\delta} = g_{\alpha\mu}(\Gamma^\mu{}_{\delta\beta,\gamma} - \Gamma^\mu{}_{\gamma\beta,\delta}) = \Gamma_{\alpha\delta\beta,\gamma} - \Gamma_{\alpha\gamma\beta,\delta}. \tag{3.95}$$

Using the symmetry of the Christoffel symbol (see Section III C) we easily prove the following:

i. Antisymmetry in the last two indexes

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}. \tag{3.96}$$

ii. Cyclic identity

$$R_{\alpha[\beta\gamma\delta]} = R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0. \tag{3.97}$$

iii. Antisymmetry in the first two indexes

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}. \tag{3.98}$$

iv. Pair symmetry

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}. \tag{3.99}$$

We can then count the number of independent components of Riemann in an n-dimensional manifold $\mathscr{M}$. Due to properties [i.] and [iii.] the number of independent components on these pair of indexes is $M = n(n-1)/2$; due to property [iv.] the number of independent components reduces to $M(M+1)/2$; and we yet have to subtract $\binom{n}{4}$ to the counting since due to properties [i.], [ii.], and [iii.] the 4 indexes cannot be all different. We are then left with $n^2(n^2-1)/12$ independent components. For example for $n = 2$ (sphere, see Section III J) we have only 1 component, for $n = 2$ we have 6, and for $n = 4$ (GR, see Section IV B) we have 20.

Commutation of covariant derivatives

(Ex. 16.3 [1]) Covariant derivatives do not generally commute. For any vector $\vec{\boldsymbol{B}}$ we will prove that

$$B^\mu{}_{;\alpha\beta} - B^\mu{}_{;\beta\alpha} = -R^\mu{}_{\gamma\alpha\beta}B^\gamma. \tag{3.100}$$

To prove this we work in a LLF

$$B^\mu{}_{;\alpha\beta} = (B^\mu{}_{;\alpha})_{,\beta} = (B^\mu{}_{,\alpha} + \Gamma^\mu{}_{\gamma\alpha}B^\gamma)_{,\beta} = B^\mu{}_{,\alpha\beta} + \Gamma^\mu{}_{\gamma\alpha,\beta}B^\gamma, \tag{3.101}$$

where in the last equality we used the fact that the Christoffel symbol vanish in a LLF. Then in a LLF

$$B^\mu{}_{;\alpha\beta} - B^\mu{}_{;\beta\alpha} = (\Gamma^\mu{}_{\gamma\alpha,\beta} - \Gamma^\mu{}_{\gamma\beta,\alpha})B^\gamma = R^\mu{}_{\gamma\beta\alpha}B^\gamma = -R^\mu{}_{\gamma\alpha\beta}B^\gamma. \tag{3.102}$$

Bianchi identities

The following *Bianchi identities* hold

$$R^\alpha{}_{\beta[\gamma\delta;\epsilon]} = 0. \tag{3.103}$$

These can be proven working in a LLF where

$$R^\alpha{}_{\beta\gamma\delta;\epsilon} = R^\alpha{}_{\beta\gamma\delta,\epsilon} = \Gamma^\alpha{}_{\delta\beta,\gamma\epsilon} - \Gamma^\alpha{}_{\gamma\beta,\delta\epsilon}, \tag{3.104}$$

and using the fact that partial derivatives commute.

The Ricci tensor

The *Ricci curvature tensor* is defined as

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}. \tag{3.105}$$

It is a symmetric tensor

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta} = g^{\gamma\epsilon}R_{\epsilon\alpha\gamma\beta} = g^{\epsilon\gamma}R_{\gamma\beta\epsilon\alpha} = R_{\beta\alpha}, \tag{3.106}$$

where we used the pair symmetry of Riemann (property [iv.] in Section III G).

The scalar curvature

The *scalar curvature* is the trace of Ricci

$$R = R^{\alpha}{}_{\alpha} \tag{3.107}$$

The Einstein tensor

The *Einstein tensor* $\mathbb{G}$ has the following components

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R. \tag{3.108}$$

The Einstein tensor is covariantly constant $\nabla\mathbb{G} = 0$ or

$$G^{\alpha\beta}{}_{;\beta} = 0, \tag{3.109}$$

which are also known as the *contracted Bianchi identities*. These can be proven using the Bianchi identities

$$R^{\alpha}{}_{\beta\gamma\delta;\epsilon} + R^{\alpha}{}_{\beta\epsilon\gamma;\delta} + R^{\alpha}{}_{\beta\delta\epsilon;\gamma} = 0. \tag{3.110}$$

Contract α and γ and use antisymmetry of Riemann in last two indexes (property [i.] of Section III G) in the second term

$$R_{\beta\delta;\epsilon} - R_{\beta\epsilon;\delta} + R^{\alpha}{}_{\beta\delta\epsilon;\alpha} = 0. \tag{3.111}$$

Contract β and δ and use antisymmetry of Riemann in first two indexes (property [iii.] of Section III G) in the third term

$$R_{;\epsilon} - R^{\beta}{}_{\gamma;\beta} - R_{\alpha\epsilon;}{}^{\alpha} = R_{;\epsilon} - 2R_{\alpha\epsilon;}{}^{\alpha} = 0, \tag{3.112}$$

so that

$$G_{\alpha\epsilon;}{}^{\alpha} = R_{\alpha\epsilon;}{}^{\alpha} - \frac{1}{2} g_{\alpha\epsilon} R_{;}{}^{\alpha} = 0. \tag{3.113}$$

The Weyl conformal tensor

(Ex. 13.131 [1]) The *Weyl conformal tensor* is defined as follows

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\delta_{[\gamma}{}^{[\alpha} R_{\delta]}{}^{\beta]} + \frac{1}{3}\delta_{[\gamma}{}^{[\alpha}\delta_{\delta]}{}^{\beta]} R. \tag{3.114}$$

The Weyl conformal tensor has the following properties:

i. Has same symmetries of Riemann;

ii. Is completely trace-free, i.e. contraction of $C_{\alpha\beta\gamma\delta}$ on any two indexes vanishes. It can be considered as the trace-free part of Riemann.

iii. In a manifold $\mathscr{M}$ of dimension n, its number of independent components can be inferred by the two properties above. Recalling the counting for Riemann of Section III G and noticing that property [ii.] above requires that contracting any two indexes we are left with only other two indexes with the proper symmetry constraints we conclude that the number of independent components of the Weyl tensor is given by $\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2}$ for $n \geq 3$ so it must be 0 for $n \leq 3$. Thus for $n \leq 3$ we may assume that the Weyl tensor is identically zero and the Riemann tensor is completely determined by its trace, the Ricci tensor,

iv. $C^{\alpha\beta}{}_{\gamma\delta} = 0$ if and only if $\mathscr{M}$ is *conformally flat*, i.e. if and only if it exists a coordinate frame where

$$ds^2 = e^{2\phi(x^{\alpha})}\eta_{\alpha\beta} dx^{\alpha} dx^{\beta}, \tag{3.115}$$

with ϕ a scalar.

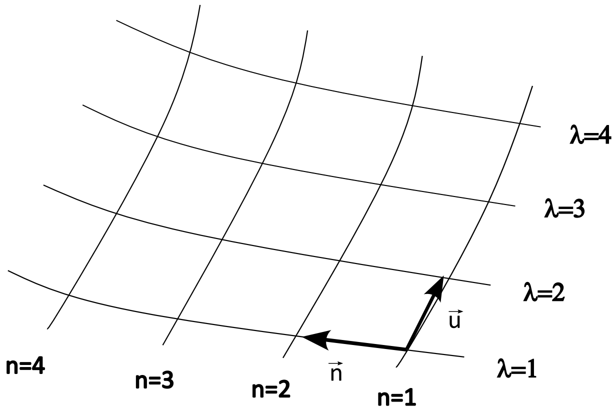

FIG. 2. Congruence of geodesics. λ =affine parameter (proper time in GR), $\vec{\boldsymbol{n}}$ =connecting vector which connects points of equal λ on different geodesics.

H. Geodesics deviation

Consider a congruence of geodesics $x^\alpha = x^\alpha(\lambda, m)$ with $\vec{\boldsymbol{u}} = \partial/\partial\lambda$, $\vec{\boldsymbol{m}} = \partial/\partial m$, and $\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{u}} = 0$. This is pictorially shown in Fig. 2.

By definition of a connecting vector $[\vec{\boldsymbol{n}}, \vec{\boldsymbol{u}}] = 0$, so $\vec{\boldsymbol{n}}$ and $\vec{\boldsymbol{u}}$ form a coordinate basis with coordinates n and λ respectively. Then $\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}} = \nabla_{\vec{\boldsymbol{n}}}\vec{\boldsymbol{u}}$ and $\mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}}) = \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{n}}} - \nabla_{\vec{\boldsymbol{n}}}\nabla_{\vec{\boldsymbol{u}}}$. So

$$\nabla_{\vec{\boldsymbol{u}}}(\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}}) = \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{n}}}\vec{\boldsymbol{u}} = \nabla_{\vec{\boldsymbol{n}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{u}} + \mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}})\vec{\boldsymbol{u}} = \mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}})\vec{\boldsymbol{u}}, \tag{3.116}$$

where in the last equality we used the geodesic equation $\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{u}} = 0$. We then reached the equation for the geodesics deviation

$$\nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}} = \mathscr{R}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{n}})\vec{\boldsymbol{u}}, \tag{3.117}$$

which in components $\langle\widetilde{\boldsymbol{\omega}}^\alpha, \nabla_{\vec{\boldsymbol{u}}}\nabla_{\vec{\boldsymbol{u}}}\vec{\boldsymbol{n}}\rangle + \langle\widetilde{\boldsymbol{\omega}}^\alpha, \mathscr{R}(\vec{\boldsymbol{n}}, \vec{\boldsymbol{u}})\vec{\boldsymbol{u}}\rangle = 0$ becomes

$$\frac{D^2 n^\alpha}{d\lambda^2} = u^\gamma u^\beta n^\alpha{}_{;\beta\gamma} = R^\alpha{}_{\beta\delta\gamma}u^\beta u^\delta n^\gamma. \tag{3.118}$$

I. Cartan structure equations

Cartan, taking profit from the forms language (see Section II), devised a very useful way to calculate the component of the Riemann tensor in a simple way. Cartan structure equations are 3, they need a metric and are true in any frame (coordinate or non coordinate). We will first enunciate them and then proceed to their proof.

i. Introduce the *connection 1-form*

$$\widetilde{\boldsymbol{\omega}}^\alpha{}_\beta = \langle\widetilde{\boldsymbol{\omega}}^\alpha, \nabla\vec{\boldsymbol{e}}_\beta\rangle = \Gamma^\alpha{}_{\beta\gamma}\widetilde{\boldsymbol{\omega}}^\gamma, \tag{3.119}$$

where the covariant derivative symbol ∇ has an empty index. Also the Γ we use here is the one of Ref. [1] with the last two indexes interchanged respect to our (this affects only a non coordinate basis).

Then the first Cartan structure equation is

$$\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^\alpha = -\widetilde{\boldsymbol{\omega}}^\alpha{}_\beta \wedge \widetilde{\boldsymbol{\omega}}^\beta. \tag{3.120}$$

We will now outline the proof. Let $\widetilde{\boldsymbol{\omega}}^\alpha = L^\alpha{}_{\bar\beta}\widetilde{\boldsymbol{\omega}}^{\bar\beta}$. Then, taking the exterior derivative,

$$\begin{aligned}
\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^\alpha &= L^\alpha{}_{\bar\beta,\bar\gamma}\widetilde{\boldsymbol{\omega}}^{\bar\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\bar\beta} \\
&= L^\alpha{}_{\bar\beta,\bar\gamma}L^{\bar\gamma}{}_\beta L^{\bar\beta}{}_\gamma\widetilde{\boldsymbol{\omega}}^\beta \wedge \widetilde{\boldsymbol{\omega}}^\gamma \\
&= L^\alpha{}_{\bar\beta,\bar\gamma}(L^{\bar\gamma}{}_\beta L^{\bar\beta}{}_\gamma - L^{\bar\gamma}{}_\gamma L^{\bar\beta}{}_\beta)\widetilde{\boldsymbol{\omega}}^\beta \otimes \widetilde{\boldsymbol{\omega}}^\gamma.
\end{aligned} \tag{3.121}$$

Now

$$
\begin{aligned}
-\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} \wedge \widetilde{\boldsymbol{\omega}}^{\beta} &= -\Gamma^{\alpha}{}_{\beta\gamma}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\beta} \\
&= (\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta})\widetilde{\boldsymbol{\omega}}^{\beta} \otimes \widetilde{\boldsymbol{\omega}}^{\gamma} \\
&= -c_{\beta\gamma}{}^{\alpha}\widetilde{\boldsymbol{\omega}}^{\beta} \otimes \widetilde{\boldsymbol{\omega}}^{\gamma}, \qquad (3.122)
\end{aligned}
$$

where in the last equality we used Eq. (3.42).

But

$$
[\vec{\boldsymbol{e}}_{\beta}, \vec{\boldsymbol{e}}_{\gamma}] = c_{\beta\gamma}{}^{\alpha}\vec{\boldsymbol{e}}_{\alpha} \qquad (3.123)
$$

$$
\begin{aligned}
&= [L^{\bar{\beta}}{}_{\beta}\partial_{\bar{\beta}}, L^{\bar{\gamma}}{}_{\gamma}\partial_{\bar{\gamma}}] \\
&= L^{\bar{\beta}}{}_{\beta}L^{\bar{\gamma}}{}_{\gamma,\bar{\beta}}\partial_{\bar{\gamma}} - L^{\bar{\gamma}}{}_{\gamma}L^{\bar{\beta}}{}_{\beta,\bar{\gamma}}\partial_{\bar{\beta}} \\
&= (L^{\bar{\gamma}}{}_{\beta}L^{\bar{\beta}}{}_{\gamma,\bar{\gamma}} - L^{\bar{\gamma}}{}_{\gamma}L^{\bar{\beta}}{}_{\beta,\bar{\gamma}})L^{\alpha}{}_{\bar{\beta}}\vec{\boldsymbol{e}}_{\alpha}, \qquad (3.124)
\end{aligned}
$$

and since $L^{\alpha}{}_{\bar{\beta}}L^{\bar{\beta}}{}_{\gamma} = \delta^{\alpha}{}_{\gamma}$ we have $L^{\alpha}{}_{\bar{\beta}}L^{\bar{\beta}}{}_{\gamma,\bar{\gamma}} = -L^{\alpha}{}_{\bar{\beta},\bar{\gamma}}L^{\bar{\beta}}{}_{\gamma}$ therefore

$$
c_{\beta\gamma}{}^{\alpha} = L^{\alpha}{}_{\bar{\beta},\bar{\gamma}}(L^{\bar{\gamma}}{}_{\gamma}L^{\bar{\beta}}{}_{\beta} - L^{\bar{\gamma}}{}_{\beta}L^{\bar{\beta}}{}_{\gamma}). \qquad (3.125)
$$

Substituting (3.125) in (3.121) and using (3.122) gives the desired equation (3.120).

ii. The second Cartan structure equation is

$$
\widetilde{\boldsymbol{d}}g_{\alpha\beta} = \widetilde{\boldsymbol{\omega}}_{\alpha\beta} + \widetilde{\boldsymbol{\omega}}_{\beta\alpha}. \qquad (3.126)
$$

This can easily be proven as follows

$$
\widetilde{\boldsymbol{\omega}}_{\alpha\beta} + \widetilde{\boldsymbol{\omega}}_{\beta\alpha} = (\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma})\widetilde{\boldsymbol{\omega}}^{\gamma} = g_{\alpha\beta,\gamma}\widetilde{\boldsymbol{\omega}}^{\gamma} = \widetilde{\boldsymbol{d}}g_{\alpha\beta}, \qquad (3.127)
$$

where in the second last equality we used Eq. (3.40). Note that even if we proved the Cartan Eq. (3.126) for a coordinate basis it holds generally.

iii. The *curvature 2-form* is defined as follows

$$
\mathscr{R}^{\alpha}{}_{\beta} = \widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} + \widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\sigma} \wedge \widetilde{\boldsymbol{\omega}}^{\sigma}{}_{\beta}. \qquad (3.128)
$$

The third Cartan structure equation is

$$
\mathscr{R}^{\alpha}{}_{\beta} = \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\delta}. \qquad (3.129)
$$

We will now outline the proof of this equation. Let us develop the two terms on the right hand side of Eq. (3.128). The first

$$
\begin{aligned}
\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} &= \widetilde{\boldsymbol{d}}\left(\Gamma^{\alpha}{}_{\beta\delta}\widetilde{\boldsymbol{\omega}}^{\delta}\right) \\
&= \Gamma^{\alpha}{}_{\beta\delta,\gamma}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\delta} \\
&= (\Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta})\widetilde{\boldsymbol{\omega}}^{\gamma} \otimes \widetilde{\boldsymbol{\omega}}^{\delta}. \qquad (3.130)
\end{aligned}
$$

The second

$$
\begin{aligned}
\widetilde{\boldsymbol{\omega}}^{\alpha}{}_{\sigma} \wedge \widetilde{\boldsymbol{\omega}}^{\sigma}{}_{\beta} &= \Gamma^{\alpha}{}_{\sigma\gamma}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \Gamma^{\sigma}{}_{\beta\delta}\widetilde{\boldsymbol{\omega}}^{\delta} \\
&= (\Gamma^{\alpha}{}_{\sigma\gamma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma})\widetilde{\boldsymbol{\omega}}^{\gamma} \otimes \widetilde{\boldsymbol{\omega}}^{\delta}. \qquad (3.131)
\end{aligned}
$$

The right hand side in the Cartan Eq. (3.129)

$$
\frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\widetilde{\boldsymbol{\omega}}^{\gamma} \wedge \widetilde{\boldsymbol{\omega}}^{\delta} = R^{\alpha}{}_{\beta\gamma\delta}\widetilde{\boldsymbol{\omega}}^{\gamma} \otimes \widetilde{\boldsymbol{\omega}}^{\delta}, \qquad (3.132)
$$

due to the antisymmetry of Riemann respect to its last two indexes.

Putting together Eqs. (3.130), (3.131), and (3.132) and recalling the expression (3.94) for the components of the Riemann tensor in a coordinate basis proves Cartan equation (3.129).

J. The 2 sphere

In spherical coordinates

$$\begin{cases} \vec{\boldsymbol{e}}_r = \frac{\partial}{\partial r} & r \quad \text{radius} \\ \vec{\boldsymbol{e}}_\theta = \frac{\partial}{\partial \theta} & \theta \quad \text{polar angle} \\ \vec{\boldsymbol{e}}_\varphi = \frac{\partial}{\partial \varphi} & \varphi \quad \text{azimuthal angle} \end{cases} \tag{3.133}$$

This is a coordinate basis and $[\vec{\boldsymbol{e}}_\alpha, \vec{\boldsymbol{e}}_\beta] = 0$ for any choice of α and β. The metric already diagonal and is given by

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \tag{3.134}$$

so that

$$||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{3.135}$$

with $g = r^4 \sin^2\theta$.

With a global (non coordinate) orthonormal basis

We find the orthonormal (non coordinate) frame by rescaling the basis vector (3.133) like so

$$\begin{cases} \vec{\boldsymbol{e}}_{\hat{r}} = \frac{\partial}{\partial r} \\ \vec{\boldsymbol{e}}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta} \\ \vec{\boldsymbol{e}}_{\hat{\varphi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{cases} \tag{3.136}$$

so that $g_{\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$ but $[\vec{\boldsymbol{e}}_{\hat{\alpha}}, \vec{\boldsymbol{e}}_{\hat{\beta}}] \neq 0$ whenever $\hat{\alpha} \neq \hat{\beta}$. It can easily be verified that

$$\begin{cases} [\vec{\boldsymbol{e}}_{\hat{r}}, \vec{\boldsymbol{e}}_{\hat{\theta}}] = -\frac{1}{r}\vec{\boldsymbol{e}}_{\hat{\theta}} = c_{\hat{r}\hat{\theta}}{}^{\hat{\theta}} \\ [\vec{\boldsymbol{e}}_{\hat{r}}, \vec{\boldsymbol{e}}_{\hat{\varphi}}] = -\frac{1}{r}\vec{\boldsymbol{e}}_{\hat{\varphi}} = c_{\hat{r}\hat{\varphi}}{}^{\hat{\varphi}} \\ [\vec{\boldsymbol{e}}_{\hat{\theta}}, \vec{\boldsymbol{e}}_{\hat{\varphi}}] = -\frac{1}{r\tan\theta}\vec{\boldsymbol{e}}_{\hat{\varphi}} = c_{\hat{\theta}\hat{\varphi}}{}^{\hat{\varphi}} \end{cases} \tag{3.137}$$

Note that J. D. Jackson book of "Classical Electrodynamics" [3] uses the orthonormal basis. For example the gradient of a scalar ψ

$$\begin{aligned} \boldsymbol{\nabla}\psi &= \vec{\boldsymbol{e}}_\alpha \psi_{,}{}^{\alpha} = \vec{\boldsymbol{e}}_\alpha g^{\alpha\beta}\psi_{,\beta} \\ &= \vec{\boldsymbol{e}}_{\hat{r}}\frac{\partial\psi}{\partial r} + \vec{\boldsymbol{e}}_{\hat{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \vec{\boldsymbol{e}}_{\hat{\varphi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}. \end{aligned} \tag{3.138}$$

where we used the fact that $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}$.

For the divergence of a vector $\vec{\boldsymbol{A}} = A^{\hat{\alpha}}\vec{\boldsymbol{e}}_{\hat{\alpha}} = A^\alpha \vec{\boldsymbol{e}}_\alpha$ with

$$\begin{cases} A^{\hat{r}} = A^r \\ A^{\hat{\theta}} = rA^\theta \\ A^{\hat{\varphi}} = r\sin\theta A^\varphi \end{cases} \tag{3.139}$$

we find

$$\begin{aligned} \boldsymbol{\nabla}\vec{\boldsymbol{A}} = A^\alpha{}_{;\alpha} &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}A^\alpha\right)_{,\alpha} \\ &= \frac{1}{r^2}\left(r^2 A^{\hat{r}}\right)_{,r} + \frac{1}{r\sin\theta}\left(\sin\theta A^{\hat{\theta}}\right)_{,\theta} + \frac{1}{r\sin\theta}A^{\hat{\varphi}}{}_{,\varphi}, \end{aligned} \tag{3.140}$$

where we used identity [iv.] in the subsection "Useful identities" of Section III C.

For the Laplacian of a scalar ψ

$$\begin{aligned} \boldsymbol{\nabla}^2\psi = \psi_{;\alpha}{}^{\alpha} &= \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\psi_{,}{}^{\alpha}\right)_{,\alpha} \\ &= \frac{1}{r^2}\left(r^2\psi_{,r}\right)_{,r} + \frac{1}{r\sin\theta}\left(\sin\theta\psi_{,\theta}\right)_{,\theta} + \frac{1}{r^2\sin^2\theta}\psi_{,\varphi\varphi}. \end{aligned} \tag{3.141}$$

where we used identity [vii.] in the subsection "Useful identities" of Section III C.

Embedded in 3 dimensional flat space

A sphere can be embedded in the 3 dimensional flat space. Here we can as usual choose a Cartesian coordinate system

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\sqrt{x^2+y^2}/z\right) \\ \varphi = \arctan\left(y/x\right) \end{cases} \tag{3.142}$$

From Eq. (1.6) we can determine the basis vectors

$$\begin{cases} \vec{\boldsymbol{e}}_r = \vec{\boldsymbol{e}}_x \frac{x}{r} + \vec{\boldsymbol{e}}_y \frac{y}{r} + \vec{\boldsymbol{e}}_z \frac{z}{r} \\ \vec{\boldsymbol{e}}_\theta = \vec{\boldsymbol{e}}_x \frac{xz}{r^2\sqrt{x^2+y^2}} + \vec{\boldsymbol{e}}_y \frac{yz}{r^2\sqrt{x^2+y^2}} + \vec{\boldsymbol{e}}_z \left(-\frac{\sqrt{x^2+y^2}}{r^2}\right) \\ \vec{\boldsymbol{e}}_\varphi = \vec{\boldsymbol{e}}_x \left(-\frac{y}{x^2+y^2}\right) + \vec{\boldsymbol{e}}_y \frac{x}{x^2+y^2} \end{cases} \tag{3.143}$$

and for the versors

$$\begin{cases} \hat{r} = \frac{\vec{\boldsymbol{e}}_r}{|\vec{\boldsymbol{e}}_r|} \quad |\vec{\boldsymbol{e}}_r| = \sqrt{\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}} = 1 \\ \hat{\theta} = \frac{\vec{\boldsymbol{e}}_\theta}{|\vec{\boldsymbol{e}}_\theta|} \quad |\vec{\boldsymbol{e}}_\theta| = \sqrt{\frac{x^2z^2}{r^4(x^2+y^2)} + \frac{y^2z^2}{r^4(x^2+y^2)} + \frac{x^2+y^2}{r^4}} = \frac{1}{r} \\ \hat{\varphi} = \frac{\vec{\boldsymbol{e}}_\varphi}{|\vec{\boldsymbol{e}}_\varphi|} \quad |\vec{\boldsymbol{e}}_\varphi| = \sqrt{\frac{y^2}{(x^2+y^2)^2} + \frac{x^2}{(x^2+y^2)^2}} = \frac{1}{r\sin\theta} \end{cases} \tag{3.144}$$

Curvature in a simple way

We will here use the results of Section III I to determine with the Cartan structure equations, in a rapid and simple way, the 1 independent component of the Riemann tensor for the sphere, the surface for which r is constant.

The 1-form orthonormal basis

$$\begin{cases} \widetilde{\boldsymbol{\omega}}^{\hat{\theta}} = r\widetilde{\boldsymbol{d}}\theta \\ \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}} = r\sin\theta\widetilde{\boldsymbol{d}}\varphi \end{cases} \tag{3.145}$$

so that $ds^2 = \eta_{\hat{\mu},\hat{\nu}}\widetilde{\boldsymbol{\omega}}^{\hat{\mu}} \otimes \widetilde{\boldsymbol{\omega}}^{\hat{\nu}}$. From Cartan structure equation (3.126) it must be $\widetilde{\boldsymbol{\omega}}_{\hat{\mu},\hat{\nu}} + \widetilde{\boldsymbol{\omega}}_{\hat{\nu},\hat{\mu}} = 0$ or $\widetilde{\boldsymbol{\omega}}^{\hat{\mu}}{}_{\hat{\nu}} + \widetilde{\boldsymbol{\omega}}_{\hat{\nu}}{}^{\hat{\mu}} = 0$ so that

$$\begin{cases} \widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\theta}} = \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\varphi}} = 0 \\ \widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} = -\widetilde{\boldsymbol{\omega}}_{\hat{\varphi}}{}^{\hat{\theta}} = -\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\theta}} = 0 \end{cases} \tag{3.146}$$

From the properties of the external derivative and from Cartan structure equation (3.120) it must be $\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\hat{\theta}} = \widetilde{\boldsymbol{d}}(r\widetilde{\boldsymbol{d}}\theta) = r\widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{d}}\theta = 0 = -\widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} \wedge \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}$. So it must be either $\widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} = 0$ or $\widetilde{\boldsymbol{\omega}}^{\hat{\theta}}{}_{\hat{\varphi}} \propto \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}$. The other basis 1-form gives

$$\begin{aligned} \widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}} &= \widetilde{\boldsymbol{d}}(r\sin\theta\widetilde{\boldsymbol{d}}\varphi) \\ &= r\cos\theta\widetilde{\boldsymbol{d}}\theta \wedge \widetilde{\boldsymbol{d}}\varphi \\ &= \frac{\cot\theta}{r}\widetilde{\boldsymbol{\omega}}^{\hat{\theta}} \wedge \widetilde{\boldsymbol{\omega}}^{\hat{\varphi}} \\ &= -\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\theta}} \propto \widetilde{\boldsymbol{\omega}}^{\hat{\theta}}. \end{aligned} \tag{3.147}$$

So we find that

$$\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}{}_{\hat{\theta}} = \frac{\cot\theta}{r}\widetilde{\boldsymbol{\omega}}^{\hat{\varphi}}. \tag{3.148}$$

From Cartan structure equation (3.129) and using the result of Eq. (3.146) then follows

$$\begin{aligned}
\mathscr{R}^{\hat\theta}{}_{\hat\varphi} &= \widetilde{\boldsymbol{d}}\widetilde{\boldsymbol{\omega}}^{\hat\varphi}{}_{\hat\theta} \\
&= \widetilde{\boldsymbol{d}}\left(-\frac{\cot\theta}{r}\widetilde{\boldsymbol{\omega}}^{\hat\varphi}\right) \\
&= \widetilde{\boldsymbol{d}}(-\cos\theta\,\widetilde{\boldsymbol{d}}\varphi) \\
&= \sin\theta\,\widetilde{\boldsymbol{d}}\theta\wedge\widetilde{\boldsymbol{d}}\varphi \\
&= \frac{1}{r^2}\widetilde{\boldsymbol{\omega}}^{\hat\theta}\wedge\widetilde{\boldsymbol{\omega}}^{\hat\varphi} \\
&= \frac{1}{2}R^{\hat\theta}{}_{\hat\varphi\hat\alpha\hat\beta}\widetilde{\boldsymbol{\omega}}^{\hat\alpha}\wedge\widetilde{\boldsymbol{\omega}}^{\hat\beta}.
\end{aligned} \tag{3.149}$$

So we reach the result that the only independent Riemann component is

$$R^{\hat\theta}{}_{\hat\varphi\hat\theta\hat\varphi} = 1/r^2. \tag{3.150}$$

The scalar curvature is then

$$R = R^{\hat\alpha\hat\beta}{}_{\hat\alpha\hat\beta} = 2R^{\hat\theta\hat\varphi}{}_{\hat\theta\hat\varphi} = 2/r^2. \tag{3.151}$$

IV. PHYSICS

A. Electromagnetism

B. Gravitation

[1] Charles W. Misner and Kip S. Thorne and John A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[2] Alan P. Lightman and William H. Press and Richard H. Price and Saul A. Teukolsky, *Problem book in relativity and gravitation* (Princeton University Press, Princeton, 1975).
[3] John D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, 1962).