

Conservation Laws in Physics

Riccardo Fantoni*

Università di Trieste, Dipartimento di Fisica, strada Costiera 11, 34151 Grignano (Trieste), Italy

(Dated: November 18, 2025)

We briefly review the conservation laws in physics.

Keywords: Time;Physics;Conservation

CONTENTS

I. Introduction	2
Intermezzo	2
Units	2
Fourier transform	3
Operators	3
II. In classical physics	3
A. Classical mechanics [1]	3
Homogeneity of time	3
Homogeneity of space	4
Isotropy of space	5
B. Fluid mechanics [2]	6
Conservation of matter	6
Euler equation	7
Bernoulli equation	7
Conservation of circulation	7
Incompressible fluids	7
C. Elasticity [3]	7
III. In quantum physics [4]	7
A. Invariance and time evolution	9
B. Galilean relativity	9
Spatial translations	9
Rotations	10
Galilean transformations	11
Galileo group	11
Parity invariance	12
Time reversal	13
C. Einstein Relativity	15
IV. In classical field theory [5]	19
A. Noether theorem	19
B. Conserved quantities	21
Translational invariance	21
Lorentz transformations invariance	21
U(1) transformations	22
Transformations under groups of internal symmetry	22
V. In general relativity [6]	22
A. Killing vectors	23
In flat spacetime	23
Conserved quantities in Minkowski spacetime	24

* riccardo.fantoni@scuola.istruzione.it

Globally conserved quantities	25
Internal symmetries	25
VI. Sum rules	26
A. Atom	26
Conservation of angular momentum ...	26
B. Liquids	26
Static structure	26
Dynamic structure	26
A. Commutators	26
Author declarations	28
Conflicts of interest	28
Data availability	28
Funding	28
References	28

I. INTRODUCTION

A reference frame is defined by a set of operative rules to measure physical quantities.

The same physical phenomenon can be observed from two different reference frames. In order for the two reference frames to be defined, the transformation between the quantities measured in the two frames must be known.

In a given reference frame a phenomenon obeys certain physical laws. A physical law is a relationship which poses conditions on the quantities measured at a given instant.

The frames are said to be equivalent respect to a class of phenomena if:

- a) Any physical situation realizable in one can also be realizable in the other.
- b) The time evolution laws are the same in the two frames.

The equivalence between frames produced by the invariance is an equivalence relationship in the mathematical sense: Given R, R', R'' three frames; R is equivalent to R , if R is equivalent to R' then R' is equivalent to R ; if R is equivalent to R' and R' is equivalent to R'' then R is equivalent to R'' .

The transformation laws between quantities in equivalent frames form a group:

- a) The identity transformation exists: The one between any frame and itself.
- b) Given any transformation, an inverse transformation exists which is itself an equivalence relationship respect to the class of phenomena in exam.
- c) The product of two equivalence relationships, defined as the application in succession and ordered of two transformations, is still an equivalence relationship.

The equivalence of a class of frames relative to a set of phenomena is called *invariance* of such phenomena relative to the group of transformations between the frames.

Intermezzo

We will skip Statistical Physics and work in *real time* except in the very last subsection VI B.

Units

We will always use relativistic units with $\hbar = 1, c = 1$. In these units, we have for the elementary charge $e^2/4\pi = 1/137$.

Fourier transform

The tridimensional Fourier transform is

$$f(\mathbf{p}) = \int f(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{p}} d\mathbf{q}, \quad (1.1)$$

$$f(\mathbf{q}) = \int f(\mathbf{p}) e^{i\mathbf{q}\cdot\mathbf{p}} \frac{d\mathbf{p}}{(2\pi)^3}, \quad (1.2)$$

and analogously for the four-dimensional case.

Operators

We will not introduce a different symbol for the operators on the Hilbert space and their eigenvalues. The reader should understand the difference from the context of the various equations introduced.

II. IN CLASSICAL PHYSICS

Consider a classical system of N particles of mass m_i in d dimensions with positions $\mathbf{q}_i = (q_i^1, q_i^2, \dots, q_i^d)$ with $i = 1, 2, \dots, N$. We will then have a total of $s = dN$ degrees of freedom. Call $\dot{\mathbf{q}}_i = d\mathbf{q}_i(t)/dt$ the velocities of the N particles where $\mathbf{q}_i = \mathbf{q}_i(t)$ are the N trajectories of the particles. The equation of motion of each particle can be obtained from a variational principle of least action from the following action functional

$$S[\{\mathbf{q}_i\}, \{\dot{\mathbf{q}}_i\}] = \int_0^T \mathcal{L}(\{\mathbf{q}_i\}, \{\dot{\mathbf{q}}_i\}, t) dt, \quad (2.1)$$

where \mathcal{L} is known as the Lagrangian function. In order to find agreement between the result of the least action principle $\delta S = 0$ and Newton equations of motion we need to choose

$$\mathcal{L} = \frac{1}{2} \sum_i m_i \dot{\mathbf{q}}_i^2 - U(\{\mathbf{q}_i\}, t) = T - U, \quad (2.2)$$

where U is the potential energy of the system of particles. Where we assumed that the variations $\mathbf{q}_i \rightarrow \mathbf{q}_i + \delta\mathbf{q}_i$ are such that $\delta\mathbf{q}_i(T) = \delta\mathbf{q}_i(0) = 0$. Euler-Lagrange equations of motion that results from the stationary action principle are then

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i}. \quad (2.3)$$

A. Classical mechanics [1]

Newton equations of motion are s coupled second order partial differential equations which admit a unique solution only after assigning $2s - 1$ constants that completely determine the initial conditions for $\{\mathbf{q}_i(t_0)\}$ and $\{\dot{\mathbf{q}}_i(t_0)\}$ for a fixed t_0 . These constants are known as *integrals of motion*.

The integrals of motion are usually not amenable of a physical interpretation but only of a mathematical one. In order to make some progress in the physical vision we will see that it is possible to find some physically relevant conserved quantities that reflect the homogeneity of time and space and the isotropy of space. We will then discuss these 3 important conserved quantities next.

Homogeneity of time

Let us consider the conservation law resulting from the homogeneity of time. This holds only for a closed¹ system for which the Lagrangian does not depend explicitly on time $\mathcal{L} = \mathcal{L}(\{\mathbf{q}_i\}, \{\dot{\mathbf{q}}_i\})$. The total derivative of the Lagrangian

¹ Note that we here just require that U does not depend explicitly on time, so we accept a system subject to a constant external field.

can therefore be written

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i} \cdot \dot{\mathbf{q}}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \cdot \ddot{\mathbf{q}}_i. \quad (2.4)$$

Using the equation of motion (2.3) we find

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \cdot \dot{\mathbf{q}}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \cdot \ddot{\mathbf{q}}_i = \sum_i \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \cdot \dot{\mathbf{q}}_i \right). \quad (2.5)$$

Hence we see that the quantity

$$\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \cdot \dot{\mathbf{q}}_i - \mathcal{L}, \quad (2.6)$$

remains constant during the motion of the closed system. This particular integral of motion is called the *energy* of the system. Mechanical systems whose energy is conserved are sometimes called *conservative*.

Since the energy depends linearly on the Lagrangian and the Lagrangian of a non interacting system, i.e. $U = 0$, is additive the same additivity property will be shared with the energy.

Since T in the Lagrangian (2.2) is a quadratic function of the velocities, using Euler theorem on homogeneous functions we have

$$\sum_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} \cdot \dot{\mathbf{q}}_i = \sum_i \frac{\partial T}{\partial \dot{\mathbf{q}}_i} \cdot \dot{\mathbf{q}}_i = 2T, \quad (2.7)$$

and

$$\mathcal{H} = T + U, \quad (2.8)$$

where T is the *kinetic energy*.

Homogeneity of space

A second conservation law follows from the *homogeneity of space*. By virtue of this homogeneity, the mechanical properties of a closed system are unchanged by any parallel displacement of the entire system in place.

A parallel displacement is a transformation in which every particle in the system is moved by the same amount ϵ . Since the mechanical properties of the system are unaltered by the parallel displacement, the change in \mathcal{L} resulting from an infinitesimal change in the coordinates, the velocities of the particles remaining fixed, must vanish

$$0 = \delta \mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i} \cdot \epsilon. \quad (2.9)$$

Since ϵ is arbitrary it must be

$$\sum_i \frac{\partial \mathcal{L}}{\partial \mathbf{q}_i} = 0. \quad (2.10)$$

From Euler-Lagrange equation (2.3) we therefore have

$$\frac{d}{dt} \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i} = 0. \quad (2.11)$$

Thus we conclude that, in a closed mechanical system, the vector

$$\mathcal{P} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}_i}, \quad (2.12)$$

remains constant during the motion. It is called the *momentum* of the system. The additivity of the momentum is evident. Moreover, unlike the energy, the momentum of the system is equal to the sum of its values $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$ for individual particles, whether or not the interaction between them can be neglected ².

² Note that some components of the momentum vector \mathcal{P} may be conserved even in the presence of an external field if U does not depend explicitly on all the Cartesian coordinates. The mechanical properties of the system are evidently unchanged by a displacement along the axis of a coordinate which does not appear in the potential, and so the corresponding component of the momentum is conserved.

Eq. (2.10) can be rewritten as

$$\sum_i \mathbf{F}_i = 0 \quad (2.13)$$

where $\mathbf{F}_i = \partial\mathcal{L}/\partial\dot{\mathbf{q}}_i = -\partial U/\partial\mathbf{q}_i$ is the force acting on particle i . In particular, for a system of only two particles $\mathbf{F}_1 + \mathbf{F}_2 = 0$, i.e. the force exerted by the first particle on the second is equal in magnitude and opposite in direction to that exerted by the second particle on the first. This equality of action and reaction is *Newton third law*.

We can always choose an *inertial* reference frame, i.e. a reference frame moving with constant velocity \mathbf{V} , were the momentum \mathbf{P} is zero. This is the *center of mass* reference frame such that

$$\mathbf{V} = \mathbf{P}/\sum_i m_i, \quad (2.14)$$

where

$$\mathbf{Q} = \sum_i m_i \mathbf{q}_i / \sum_i m_i, \quad (2.15)$$

is the center of mass point.

Isotropy of space

Let us now derive the conservation law which follows from the *isotropy of space*. This isotropy means that the mechanical properties of a closed system do not vary when it is rotated as a whole in any manner in space. Let us therefore consider an infinitesimal rotation of the system, and obtain the condition for the Lagrangian to remain unchanged.

We shall use the vector $\delta\phi$ of the infinitesimal rotation, whose magnitude is the angle of rotation $\delta\phi$, and whose direction is that of the axis of rotation (the direction of rotation being that of a right-handed screw driven along $\delta\phi$).

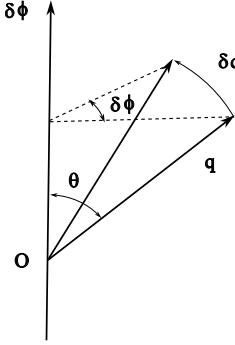


FIG. 1. ...

Let us find, first of all, the resulting increment in the radius vector from an origin on the axis to any particle in the system undergoing rotation. The linear displacement of the end of the radius vector is related to the angle by $|\delta\mathbf{q}| = r \sin \theta \delta\phi$ as shown in Fig. 1. The direction of $\delta\mathbf{q}$ is perpendicular to the plane of \mathbf{q} and $\delta\phi$. Hence $\delta\mathbf{q} = \delta\phi \times \mathbf{q}$. When the system is rotated, not only the radius vectors but also the velocities of the particles change direction, and all vectors are transformed in the same manner. The velocity increment relative to a fixed system of coordinates is $\delta\dot{\mathbf{q}} = \delta\phi \times \dot{\mathbf{q}}$. If these expressions are substituted in the condition that the Lagrangian is unchanged by the rotation

$$0 = \delta\mathcal{L} = \sum_i \left(\frac{\partial\mathcal{L}}{\partial\mathbf{q}_i} \cdot \delta\mathbf{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}_i} \cdot \delta\dot{\mathbf{q}}_i \right) \quad (2.16)$$

$$= \sum_i (\dot{\mathbf{p}}_i \cdot \delta\phi \times \mathbf{q}_i + \mathbf{p}_i \cdot \delta\phi \times \dot{\mathbf{q}}_i) \quad (2.17)$$

$$= \delta\phi \cdot \sum_i (\mathbf{q}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{q}}_i \times \mathbf{p}_i) \quad (2.18)$$

$$= \delta\phi \cdot \frac{d}{dt} \sum_i \mathbf{q}_i \times \mathbf{p}_i. \quad (2.19)$$

Since $\delta\phi$ is arbitrary, it follows that $(d/dt) \sum_i \mathbf{q}_i \times \mathbf{p}_i = 0$, and we conclude that the vector

$$\mathcal{M} = \sum_i \mathbf{q}_i \times \mathbf{p}_i, \quad (2.20)$$

called the *angular momentum* of the system, is conserved in the motion of a closed system³. Like the linear momentum \mathcal{P} it is additive, whether or not $U = 0$.

We immediately see that the angular momentum depends on the choice of origin except when the system is at rest as a whole, i.e. $\mathcal{P} = 0$. It is also easy to see that

$$\mathcal{M} = \mathcal{M}' + \mathcal{Q} \times \mathcal{P}, \quad (2.21)$$

where \mathcal{M}' is the angular momentum measured in the center of mass reference frame, i.e. the *intrinsic angular momentum*.

B. Fluid mechanics [2]

Fluid dynamics concerns itself with the study of the motion of fluids (liquids and gases). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. Therefore when we speak of the “point” of a fluid (or of an infinitesimal volume of it) we mean not a single molecule of the fluid but a volume element still containing very many molecules but yet small compared with the volume of the whole fluid.

A mathematical description of the state of a moving fluid consists in specifying the fluid velocity $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ and any two thermodynamic functions pertaining to the fluid, for instance the pressure $p = p(t, \mathbf{x}, t)$ and the density $\rho = \rho(t, \mathbf{x})$, from which one can determine all other thermodynamic quantities. These 5 quantities are functions of the coordinates $\mathbf{x} = (x, y, z)$ and of time t . Once again we stress that a point \mathbf{r} in space at a given time t refers to a fixed point and not to specific particles of the fluid.

The conservation laws will allow us to determine some fundamental equations of fluid dynamics

Conservation of matter

We consider some volume Ω of space. The mass of fluid in this volume is $\int_{\Omega} \rho d^3x$. The total mass of fluid flowing out of the volume in unit time is $\oint_{\partial\Omega} \rho \mathbf{v} \cdot d^2\mathbf{S}$ with $d^2\mathbf{S}$ a vector whose magnitude represents the area of an infinitesimal element of the surface $\partial\Omega$ bounding Ω (its frontier), whose direction points along the outward normal to such element. Clearly the decrease, per unit time, in the mass of the fluid in the volume Ω can be written as $-(\partial/\partial t) \int_{\Omega} \rho d^3x$. So we must have

$$\frac{\partial}{\partial t} \int_{\Omega} \rho d^3x = - \oint_{\partial\Omega} \rho \mathbf{v} \cdot d^2\mathbf{S} = - \int_{\Omega} \nabla(\rho \mathbf{v}) d^3x, \quad (2.22)$$

where in the last equality we used Gauss theorem. Since this equation must hold for any Ω , it must be

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0. \quad (2.23)$$

This is known as the *equation of continuity*. The vector $\mathbf{j} = \rho \mathbf{v}$ is called the *mass flux density*.

³ The law of conservation may hold in a more restricted form even for a system in an external field. It is evident from the above derivation that the component of angular momentum along an axis about which the field is symmetrical is always conserved, for the mechanical properties of the system are unaltered by any rotation about that axis. Here, of course, the angular momentum must be defined relative to an origin lying on the axis.

Euler equation

Bernoulli equation

Conservation of circulation

Incompressible fluids

C. Elasticity [3]

...

III. IN QUANTUM PHYSICS [4]

In quantum mechanics the invariance respect to a change of reference frame is defined as follows:

- a) The possible states in the two frames are the vectors of a same Hilbert space. The observables are the same. The transformation law is a mapping of the Hilbert space onto itself.
- b) Starting from the same initial state the time evolution is the same in the two frames.

The invariance transformations are a group. So an invariance transformation is a realization of the group on an Hilbert space.

Let $|a\rangle$ be a state, in a certain frame, defined by the simultaneous measure of a complete set of commuting observables. Any vector of the form $x_a|a\rangle$ where x_a is an arbitrary phase factor, is an eigenstate of the same observables with the same eigenvalues. So it represents the same physical state. The phase is not observable. A measurement on $|a\rangle$ means to observe the probability that $|a\rangle$ contains a state $|b\rangle$ defined by the measure instruments. What one measures is

$$P_{ab} = |\langle b|a \rangle|^2, \quad (3.1)$$

where the phases x_a and x_b cancel. A vector of the Hilbert space modulo a phase is called a “ray” of the Hilbert space and will be denoted $|\{a\}\rangle$.

Theorem III.1 (Wigner). *Given a bijective transformation between rays in a Hilbert space $|\{s\}\rangle \rightarrow |\{s'\}\rangle$ such that*

$$|\langle\{s'_2\}|\{s'_1\}\rangle|^2 = |\langle\{s_2\}|\{s_1\}\rangle|^2 \quad \forall |\{s_1\}\rangle, |\{s_2\}\rangle \quad (3.2)$$

it is always possible to choose the phases in such a way that the transformation is realized on the Hilbert space vectors as a unitary or antiunitary transformation.

Proof. We will prove Wigner theorem in 3 steps:

1. Let $|e_n\rangle$ be an orthonormal complete base of the Hilbert space and let $|\{e_n\}\rangle$ be the correspondent rays. The transformed rays are orthonormal

$$\langle e_i|e_j \rangle = \delta_{ij} \implies |\langle\{e'_i\}|\{e'_j\}\rangle|^2 = \delta_{ij} \quad (3.3)$$

Let us choose in an arbitrary way a set of phases on the rays $|\{e'_i\}\rangle$, i.e. a set of vectors $|e'_i\rangle$ that represent the states. Then

$$\langle e'_i|e'_j \rangle = \delta_{ij}, \quad (3.4)$$

The set of vectors so obtained is also a complete base of the Hilbert space. In fact, if there exists a vector $|v'\rangle$ such that $\langle v'|v'\rangle \neq 0$ and $\langle v'|e'_n\rangle = 0 \quad \forall n$, then, by hypothesis, there would exist a vector $|v\rangle$ such that $\langle v|v\rangle \neq 0$ and $\langle v|e_n\rangle = 0 \quad \forall n$, against the hypothesis of completeness of the base $|e_n\rangle$.

2. Let $|F_k\rangle = |e_1\rangle + |e_k\rangle$. The generic representative of the transformed ray $|\{F'_k\}\rangle$ will be

$$|F'_k\rangle = x_k(|e'_1\rangle + y_k|e'_k\rangle), \quad (3.5)$$

with x_k and y_k phases factors. In fact

$$|\langle F_k | e_n \rangle| = \delta_{n1} + \delta_{nk} \implies |\langle F'_k | e'_n \rangle| = \delta_{n1} + \delta_{nk}. \quad (3.6)$$

Next I can define the following S transformation

$$|Se_1\rangle = |e'_1\rangle \quad |Se_k\rangle = y_k|e'_k\rangle \quad (3.7)$$

$$|SF_k\rangle = \frac{1}{x_k}|F'_k\rangle = |e'_1\rangle + y_k|e'_k\rangle. \quad (3.8)$$

With this choice

$$|SF_k\rangle = |Se_1\rangle + |Se_k\rangle. \quad (3.9)$$

In other words we realized the transformation S as a linear transformation on vectors of kind $|F_k\rangle$. Let us next extend this construction to all vectors of the Hilbert space.

3. Consider a generic vector

$$|v\rangle = \sum_n a_n |e_n\rangle. \quad (3.10)$$

Let us assume, without loss of generality, a_1 real. The correspondent ray $|\{v\}\rangle$ will be transformed into a ray $|\{v'\}\rangle$ with the following generic representative

$$|v'\rangle = \sum_n a'_n |e'_n\rangle, \quad (3.11)$$

and since by hypothesis

$$|\langle v | e_n \rangle|^2 = |\langle v' | e'_n \rangle|^2, \quad (3.12)$$

we have

$$|a'_n| = |a_n|. \quad (3.13)$$

We define

$$|Se_1\rangle = |e'_1\rangle, \quad (3.14)$$

$$|Se_n\rangle = y_n|e'_n\rangle \quad \forall n \neq 1, \quad (3.15)$$

with y_n some phase factors, so that for any vector belonging to the transformed ray $|\{v'\}\rangle$

$$|v'\rangle = x \left\{ a_1 |Se_1\rangle + \sum_{n=2}^{\infty} \frac{a'_n}{y_n} |Se_n\rangle \right\}, \quad (3.16)$$

with x a phase factor. We then define

$$|Sv\rangle = \frac{1}{x} |v'\rangle. \quad (3.17)$$

By hypothesis it must be

$$|\langle F_k | v \rangle|^2 = |a_1 + a_k|^2 = |\langle SF_k | Sv \rangle|^2 = \left| a_1 + \frac{a'_k}{y_k} \right|^2. \quad (3.18)$$

Since we also have $|a_k| = |a'_k|$ we require

$$\mathbf{Re}(a_1 a_k) = \mathbf{Re} \left(a_1 \frac{a'_k}{y_k} \right). \quad (3.19)$$

Then there are only two possibilities:

- i. $a_k = a'_k/y_k$
- ii. $a_k = (a'_k/y_k)^*$

or

- i. $|Sv\rangle = S(\sum_n a_n |e_n\rangle) = \sum_n a_n |Se_n\rangle$
- ii. $|Sv\rangle = S(\sum_n a_n |e_n\rangle) = \sum_n a_n^* |Se_n\rangle$

In the first case the operator S is linear, in the second is antilinear. We also have

- i. $\langle Sv_1 | Sv_2 \rangle = \langle v_1 | v_2 \rangle \quad \forall |v_1\rangle, |v_2\rangle$
- ii. $\langle Sv_1 | Sv_2 \rangle = \langle v_2 | v_1 \rangle \quad \forall |v_1\rangle, |v_2\rangle$

In the first case S is unitary, in the second it is antiunitary.

□

A. Invariance and time evolution

The requirement b) for invariance tells us that the evolution of the transformed must coincide with the transformation of the evolved

$$U(t, t') S(t') |\psi\rangle = S(t) U(t, t') |\psi\rangle, \quad (3.20)$$

where $U(t, t')$ is the time evolution operator. Since $|\psi\rangle$ is arbitrary we must have

$$S^\dagger(t) U(t, t') S(t') = U(t, t'). \quad (3.21)$$

If the Hamiltonian H is independent of time

$$U(t, t') = e^{-iH(t-t')}, \quad (3.22)$$

and we require

$$S(t) = e^{-iH(t-t')} S(t') e^{iH(t-t')}. \quad (3.23)$$

B. Galilean relativity

We require invariance under translations, rotations, and velocity transformations for pointwise non relativistic particles.

Spatial translations

Let us consider a reference frame R' translated by \mathbf{a} relative to the frame R . If the spatial translations are a symmetry of the system it must exist a unitary transformation $U(\mathbf{a})$ which relates the dynamical variables \mathbf{q}' and \mathbf{p}' in R' to the variables \mathbf{q} and \mathbf{p} in R . The transformation law must be

$$\mathbf{q}' = \mathbf{q} - \mathbf{a}, \quad (3.24)$$

$$\mathbf{p}' = \mathbf{p}. \quad (3.25)$$

It is easy to see that the unitary operator exists and is

$$U(\mathbf{a}) = e^{i\mathbf{a} \cdot \mathbf{p}}. \quad (3.26)$$

Since the transformation is unitary the commutation relations do not change

$$[q'_i, p'_j] = [q_i, p_j] = i\delta_{ij}, \quad (3.27)$$

$$[q'_i, q'_j] = [q_i, q_j] = 0, \quad (3.28)$$

$$[p'_i, p'_j] = [p_i, p_j] = 0, \quad (3.29)$$

where $\mathbf{q} = U(\mathbf{a})^\dagger \mathbf{q} U(\mathbf{a})$ and $\mathbf{p} = U(\mathbf{a})^\dagger \mathbf{p} U(\mathbf{a})$. Moreover from Hadamard lemma (A11) follows immediately that Eqs. (3.24)-(3.25) are satisfied.

The invariance of the time evolution between two frames R and R' imposes

$$U^\dagger(\mathbf{a}, t)e^{-iH(t-t')}U(\mathbf{a}, t') = e^{-iH(t-t')}, \quad (3.30)$$

which means

$$[\mathbf{p}, H] = 0. \quad (3.31)$$

In other words, the momentum is a constant of motion. We can also write

$$\frac{\partial H}{\partial \mathbf{q}} = 0. \quad (3.32)$$

Rotations

A rotation is defined by a versor $\hat{\mathbf{n}}$ which indicates the axis of rotation and an angle θ . We define $\boldsymbol{\theta} = \theta \hat{\mathbf{n}}$. The angles are taken as positive for anti-clockwise rotations. Let us consider a frame R' rotated by $\boldsymbol{\theta}$ relative to frame R . The component of a vector \mathbf{v} will change according to

$$v'_i = R(\boldsymbol{\theta})_{ij} v_j, \quad (3.33)$$

where $R(\boldsymbol{\theta})$ is the rotation matrix. For infinitesimal transformations

$$\delta \mathbf{v} = \mathbf{v}' - \mathbf{v} \approx -\boldsymbol{\theta} \wedge \mathbf{v}. \quad (3.34)$$

If the quantum system is invariant under rotations it must be possible to construct a unitary transformation on the Hilbert space which realizes the transformation and commutes with the time evolution. Let us then consider the angular momentum

$$\mathbf{J} = \mathbf{q} \wedge \mathbf{p}. \quad (3.35)$$

It is easy to verify that for $\mathbf{v} = \mathbf{q}$ or $\mathbf{v} = \mathbf{p}$ we have

$$[\boldsymbol{\theta} \cdot \mathbf{J}, \mathbf{v}] = -i\boldsymbol{\theta} \wedge \mathbf{v}. \quad (3.36)$$

Then the transformation we are looking for is

$$U(\boldsymbol{\theta}) = e^{i\boldsymbol{\theta} \cdot \mathbf{J}}, \quad (3.37)$$

as can be readily verified for infinitesimal transformations

$$\mathbf{v}' = U^\dagger(\boldsymbol{\theta}) \mathbf{v} U(\boldsymbol{\theta}) \approx \mathbf{v} - i[\boldsymbol{\theta} \cdot \mathbf{J}, \mathbf{v}] = \mathbf{v} - \boldsymbol{\theta} \wedge \mathbf{v}. \quad (3.38)$$

The transformation commutes with the time evolution if

$$[\mathbf{J}, H] = 0 \quad (3.39)$$

which means that H must be a scalar and the angular momentum a constant of motion. Since the transformation is unitary it preserves the commutation relations.

If the particle has a spin the generator of the rotations is the total angular momentum

$$\mathbf{J} = \mathbf{q} \wedge \mathbf{p} + \mathbf{s}. \quad (3.40)$$

Galilean transformations

If we go from a frame R to a frame R' moving relative to R with a constant speed \mathbf{v} we must have

$$\mathbf{q}' = \mathbf{q} - t\mathbf{v}, \quad (3.41)$$

$$\mathbf{p}' = \mathbf{p} - m\mathbf{v}. \quad (3.42)$$

It is easy to verify that these laws of transformation are induced by the unitary operator

$$U(t, \mathbf{v}) = e^{i(\mathbf{p}t - \mathbf{q}m) \cdot \mathbf{v}}, \quad (3.43)$$

so that

$$U^\dagger(t, \mathbf{v})\mathbf{q}U(t, \mathbf{v}) = \mathbf{q} - t\mathbf{v}, \quad (3.44)$$

$$U^\dagger(t, \mathbf{v})\mathbf{p}U(t, \mathbf{v}) = \mathbf{p} - m\mathbf{v}. \quad (3.45)$$

If the Galilean transformation has to be an invariance we must also require

$$U(t, \mathbf{v}) = e^{-iH(t-t')}U(t', \mathbf{v})e^{iH(t-t')}, \quad (3.46)$$

or

$$t\mathbf{p} - m\mathbf{q} = e^{-iH(t-t')}(t'\mathbf{p} - m\mathbf{q})e^{iH(t-t')}. \quad (3.47)$$

If the system is invariant under translations $[\mathbf{p}, H] = 0$, so

$$(t - t')\mathbf{p} = m\mathbf{q} - me^{-iH(t-t')}\mathbf{q}e^{iH(t-t')}. \quad (3.48)$$

For infinitesimal time differences we get

$$\frac{\mathbf{p}}{m} = i[H, \mathbf{q}] = \frac{\partial H}{\partial \mathbf{p}}. \quad (3.49)$$

So

$$H = \frac{\mathbf{p}^2}{2m}. \quad (3.50)$$

Galileo group

We analyzed the symmetries under translations, rotations, and Galilean transformations for a non relativistic system. The corresponding unitary transformations are

$$U(\mathbf{a}) = e^{i\mathbf{a} \cdot \mathbf{p}}, \quad (3.51)$$

$$U(\boldsymbol{\theta}) = e^{i\boldsymbol{\theta} \cdot \mathbf{J}}, \quad (3.52)$$

$$U(\mathbf{v}) = e^{-i\mathbf{v} \cdot \mathbf{K}} \quad \mathbf{K} = m\mathbf{q} - t\mathbf{p} \quad (3.53)$$

The group corresponding to the set of these transformations is called “Galileo group” and the corresponding invariance “galilean invariance”.

From the canonical commutation relationships, the following algebra for the group generators, follows

$$[p_\mu, p_\nu] = 0 \quad P_0 = H \quad (3.54)$$

$$[\mathbf{J}, H] = 0 \quad [J_i, p_j] = i\epsilon_{ijk}p_k \quad (3.55)$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad [J_i, K_j] = i\epsilon_{ijk}K_k \quad (3.56)$$

$$[K_i, K_j] = 0 \quad [K_i, p_j] = im\delta_{ij} \quad [K_i, H] = ip_i \quad (3.57)$$

In the Hilbert space of the physical system is then defined an unitary representation of the group that transforms the spec into itself.

If this representation is reducible it is possible to write the Hilbert space as a direct sum of one or more orthogonal Hilbert spaces each one transforming in itself. The generators are written as sum of the generators acting in each subspace and generators acting on different irreducible subspaces commute. The states in each subspace evolve with their Hamiltonian each in states belonging to the same subspace.

A physical system can then be written as a sum of irreducible representations of the Galileo group.

The simplest case is a particle without internal structure. In this case the only internal variable is the spin which commutes with the orbital variables. A complete set of state is

$$|\mathbf{p}\rangle|s, s_z\rangle. \quad (3.58)$$

Assuming the usual metric

$$\langle \mathbf{p}' | \mathbf{p} \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'), \quad (3.59)$$

$$\langle s'_z | s_z \rangle = \delta_{s'_z s_z} \quad (3.60)$$

these states constitute an irreducible representation of the Galileo group if the states $|s_z\rangle$ are an irreducible representation of internal rotations. Let us show this explicitly:

$$\mathbf{p}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle, \quad (3.61)$$

$$\begin{aligned} \mathbf{p}e^{i\boldsymbol{\theta} \cdot \mathbf{J}}|\mathbf{p}\rangle &= e^{i\boldsymbol{\theta} \cdot \mathbf{J}}e^{-i\boldsymbol{\theta} \cdot \mathbf{J}}\mathbf{p}e^{i\boldsymbol{\theta} \cdot \mathbf{J}}|\mathbf{p}\rangle \\ &= R(\boldsymbol{\theta})\mathbf{p}e^{i\boldsymbol{\theta} \cdot \mathbf{J}}|\mathbf{p}\rangle, \end{aligned} \quad (3.62)$$

where \mathbf{p} on the right hand side denotes the momentum operator acting on the eigenstate $|\mathbf{p}\rangle$ and on the left denotes the eigenvalue. The eigenvalues of the rotated state is the rotated momentum. In the same way:

$$\begin{aligned} \mathbf{p}e^{-i\mathbf{v} \cdot \mathbf{K}}|\mathbf{p}\rangle &= e^{-i\mathbf{v} \cdot \mathbf{K}}e^{i\mathbf{v} \cdot \mathbf{K}}\mathbf{p}e^{-i\mathbf{v} \cdot \mathbf{K}}|\mathbf{p}\rangle \\ &= (\mathbf{p} - m\mathbf{v})e^{-i\mathbf{v} \cdot \mathbf{K}}|\mathbf{p}\rangle, \end{aligned} \quad (3.63)$$

so

$$e^{i\boldsymbol{\theta} \cdot \mathbf{J}}|\mathbf{p}\rangle = |R(\boldsymbol{\theta})\mathbf{p}\rangle, \quad (3.64)$$

$$e^{-i\mathbf{v} \cdot \mathbf{K}}|\mathbf{p}\rangle = |\mathbf{p} - m\mathbf{v}\rangle, \quad (3.65)$$

and we see that starting from any vector $|\mathbf{p}\rangle$ it is possible to reach any other vector $|\mathbf{p}'\rangle$ through successive applications of rotations or of Galileo transformations. The internal degrees of freedom only transform by rotations independently.

So a pointwise free particle is described by an irreducible unitary representation of the Galileo group.

Parity invariance

The parity transformation is defined by

$$\mathbf{p} \rightarrow -\mathbf{p} \quad \mathbf{q} \rightarrow -\mathbf{q} \quad \mathbf{s} \rightarrow \mathbf{s} \quad (3.66)$$

This is a canonical transformation since it does not change the commutation relations. The transformation operator is

$$U_P = e^{i\frac{\pi}{2}(\mathbf{p} + i\mathbf{q}) \cdot (\mathbf{p} - i\mathbf{q})}. \quad (3.67)$$

The parity transformation has square 1

$$U_P = U_P^{-1} = U_P^\dagger. \quad (3.68)$$

If the parity transformation is an invariance we must have

$$U_P^{-1} H U_P = H \quad (3.69)$$

or

$$[U_P, H] = 0. \quad (3.70)$$

Let us now prove Eq. (3.67) in the one-dimensional case

$$U_P = e^{i\frac{\pi}{2}(p^2 + q^2 - 1)}. \quad (3.71)$$

Apart from a phase this operator coincides with the time evolution operator of a harmonic oscillator of mass 1 and $\omega = 1$ from time $t = 0$ to time $t = \pi$. The Heisenberg equations for

$$q(t) = e^{iHt}q(0)e^{-iHt}, \quad (3.72)$$

$$p(t) = e^{iHt}p(0)e^{-iHt}, \quad (3.73)$$

are

$$\dot{q} = i[H, q], \quad (3.74)$$

$$\dot{p} = i[H, p], \quad (3.75)$$

with $H = (p^2 + q^2)/2$. They have solution

$$q(t) = q \cos t + p \sin t, \quad (3.76)$$

$$p(t) = p \cos t - q \sin t. \quad (3.77)$$

It follows for $t = \pi$

$$q(\pi) = U_P^\dagger q U_P = -q, \quad (3.78)$$

$$p(\pi) = U_P^\dagger p U_P = -p, \quad (3.79)$$

which is what we wanted.

Time reversal

The time reversal acts as follows

$$q \rightarrow q \quad p \rightarrow -p \quad s \rightarrow -s \quad t \rightarrow -t \quad (3.80)$$

This transformation cannot be realized by a unitary operator because in such case the commutation relations would be preserved. Instead we want, in one dimension,

$$[q, p] = i \rightarrow [q, -p] = -i \quad (3.81)$$

If the transformation is antiunitary this is possible:

$$[q', p'] = U_T^\dagger [q, p] U_T = U_T^\dagger i U_T = -i. \quad (3.82)$$

An antilinear operator is defined by

$$T|s_1\rangle = |Ts_1\rangle \quad T|s_2\rangle = |Ts_2\rangle \quad (3.83)$$

$$T(a|s_1\rangle + b|s_2\rangle) = a^*T|s_1\rangle + b^*T|s_2\rangle. \quad (3.84)$$

For a linear operator O

$$\langle a|Ob\rangle = \langle O^\dagger a|b\rangle = \langle b|O^\dagger a\rangle^*, \quad (3.85)$$

and the operator is Hermitian if

$$\langle a|Ob\rangle = \langle Oa|b\rangle. \quad (3.86)$$

For an antilinear operator T

$$\langle a|Tb\rangle = \langle b|T^\dagger a\rangle, \quad (3.87)$$

which is antilinear in $|a\rangle$ and in $|b\rangle$. An antilinear operator is antiunitary if

$$TT^\dagger = T^\dagger T = 1, \quad (3.88)$$

or

$$\langle a|T^\dagger Tb\rangle = \langle Tb|Ta\rangle = \langle a|b\rangle. \quad (3.89)$$

The transformed of O under T

$$O' = T^\dagger OT, \quad (3.90)$$

is still linear and

$$\langle b|T^\dagger OTa\rangle = \langle OTA|Tb\rangle = \langle Ta|O^\dagger Tb\rangle. \quad (3.91)$$

In particular for $O = i$ we find

$$T^\dagger iT = TiT^\dagger = -i. \quad (3.92)$$

The time reversal is realizable with an antiunitary operator:

$$T^\dagger \mathbf{q} T = \mathbf{q} \quad T^\dagger \mathbf{p} T = -\mathbf{p} \quad T^\dagger \mathbf{s} T = -\mathbf{s} \quad (3.93)$$

Moreover, in order to have invariance, we must require

$$T^\dagger H T = H. \quad (3.94)$$

If O is an observable

$$\langle b|OTa\rangle = \langle b|TT^\dagger OTa\rangle = \langle T^\dagger OTa|T^\dagger b\rangle. \quad (3.95)$$

So if $T^\dagger OT = \pm O$ we have

$$\langle b|OTa\rangle = \pm \langle Oa|T^\dagger b\rangle. \quad (3.96)$$

For eigenstates of O , $O|a\rangle = O_a|a\rangle$, we have

$$\langle b|OTa\rangle = \pm O_a \langle a|T^\dagger b\rangle = \pm O_a \langle b|Ta\rangle, \quad (3.97)$$

which means that $|Ta\rangle$ is an eigenstate of O with the transformed eigenvalue.

So for a state $|a\rangle = |\mathbf{p}, s_z\rangle$ we have

$$|Ta\rangle = |-\mathbf{p}, -s_z\rangle, \quad (3.98)$$

modulo a phase.

For a spinless particle with canonical variables \mathbf{q} and \mathbf{p} the time reversal is realized through

$$\langle \mathbf{q}|Ta\rangle = \psi_{Ta}(\mathbf{q}) = \psi_a^*(\mathbf{q}) = \langle \mathbf{q}|a\rangle^*, \quad (3.99)$$

on wave functions in coordinate representation. In fact we have

$$\langle a|T^\dagger \mathbf{p} Tb\rangle = \langle \mathbf{p} Tb|Ta\rangle = \int \psi_b(\mathbf{q})(-i\nabla)\psi_a^*(\mathbf{q}) d\mathbf{q} = - \int \psi_a^*(\mathbf{q})(-i\nabla)\psi_b(\mathbf{q}) d\mathbf{q} = -\langle a|\mathbf{p} b\rangle, \quad (3.100)$$

where we used an integration by parts. Analogously we verify

$$\langle a|T^\dagger \mathbf{q} Tb\rangle = \langle \mathbf{q} Tb|Ta\rangle = \langle a|\mathbf{q} b\rangle. \quad (3.101)$$

The Hamiltonian is an Hermitian function of \mathbf{q} and \mathbf{p} . In the coordinate representation, \mathbf{q} is a real variable and $\mathbf{p} = -i\nabla$. The transformation $\mathbf{p} \rightarrow -\mathbf{p}$ is equivalent to a complex conjugation. We will have invariance under T if $H(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}, -\mathbf{p})$ or if H is real. A Hamiltonian of the form

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}), \quad (3.102)$$

is invariant under T .

If the particle has spin, it is described by $2s+1$ functions of \mathbf{q}

$$\psi(\mathbf{q}) = \begin{pmatrix} \psi_1(\mathbf{q}) \\ \vdots \\ \psi_{2s+1}(\mathbf{q}) \end{pmatrix}. \quad (3.103)$$

The spin is represented by three matrices $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ independent from \mathbf{q} . We now take

$$\psi_{Ta}(\mathbf{q}) = U\psi_a^*(\mathbf{q}), \quad (3.104)$$

with U an unitary matrix independent from \mathbf{q} and acting on spin space. To have the correct spin transformations we must have

$$\langle a|T^\dagger \mathbf{s} Tb\rangle = \langle \mathbf{s} Tb|Ta\rangle = -\langle a|\mathbf{s} b\rangle, \quad (3.105)$$

or

$$-\int \psi_a^\dagger \Sigma \psi_b = \int \psi_b^{\text{Tr}} U^\dagger \Sigma U \psi_a^*, \quad (3.106)$$

which means

$$U^{\text{Tr}} \Sigma^{\text{Tr}} U^\dagger = -\Sigma, \quad (3.107)$$

and taking the complex conjugate, since $\Sigma^\dagger = \Sigma$, we find

$$U^\dagger \Sigma U = -\Sigma^*. \quad (3.108)$$

With the usual choice of phases in the angular momentum representation Σ_1 and Σ_3 are real matrices and Σ_2 is pure imaginary.

For example for spin 1/2 particles

$$\Sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad (3.109)$$

Then apart from an unessential phase we find

$$U = e^{i\pi\Sigma_2}, \quad (3.110)$$

a rotation of π around the 2 axis, which changes sign to Σ_1 and Σ_3 . In conclusions we have

$$\psi_{Ta} = e^{i\pi\Sigma_2} \psi_a^*. \quad (3.111)$$

C. Einstein Relativity

The invariance under the Galileo group is valid in the limit of small velocities. But, actually, physics is invariant under Lorentz transformations in addition to spatial translations. This invariance is known as Einstein relativity.

The Lorentz group is defined as the group of linear transformations which leaves invariant the quadratic form

$$ds^2 = dt^2 - d\mathbf{x}^2. \quad (3.112)$$

Let $d\mathbf{x} = (dx^0, dx^1, dx^2, dx^3) = (dt, d\mathbf{x})$ we can write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (3.113)$$

where Einstein summation convention is used with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad g^{\mu\nu} g_{\nu\alpha} = \delta_\alpha^\mu \quad (3.114)$$

The Lorentz transformations are defined as the linear transformations

$$dx'^\mu = \Lambda^\mu_\nu dx^\nu, \quad (3.115)$$

such that

$$g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta dx^\alpha dx^\beta. \quad (3.116)$$

Due to the arbitrariness of dx^μ we have

$$g_{\mu\nu} = g_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu, \quad (3.117)$$

or

$$\mathbf{g} = \mathbf{\Lambda}^{\text{Tr}} \mathbf{g} \mathbf{\Lambda}, \quad (3.118)$$

which defines the Lorentz group. Taking the 00 component in Eq. (3.117)

$$1 = g_{\alpha\beta} \Lambda^\alpha_0 \Lambda^\beta_0 = (\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2, \quad (3.119)$$

or

$$(\Lambda^0_0)^2 \geq 1, \quad (3.120)$$

or

$$\Lambda^0_0 \geq 1 \quad \text{or} \quad \Lambda^0_0 \leq -1. \quad (3.121)$$

Taking the determinant in Eq. (3.117) follows

$$(\det \mathbf{\Lambda})^2 = 1, \quad (3.122)$$

or

$$\det \mathbf{\Lambda} = \pm 1. \quad (3.123)$$

The transformations obtained continuously from the identity have $\Lambda^0_0 \geq 1$ and $\det \mathbf{\Lambda} = 1$ and constitute the *proper* Lorentz group. The transformations with $\Lambda^0_0 \geq 1$ and $\det \mathbf{\Lambda} = -1$ can be written as the product of the parity $P : \mathbf{x} \rightarrow -\mathbf{x}$ times a proper transformation. The ones with $\Lambda^0_0 \leq -1$ and $\det \mathbf{\Lambda} = 1$ as a product of the time reversal $T : x^0 \rightarrow -x^0$ times the proper transformations. The ones with $\Lambda^0_0 \leq 1$ and $\det \mathbf{\Lambda} = -1$ as PT times a proper transformation.

An infinitesimal proper transformation

$$\Lambda^\mu_{\mu'} = \delta^\mu_{\mu'} + \Omega^\mu_{\mu'}, \quad (3.124)$$

must satisfy Eq. (3.117). So

$$g_{\mu'\nu'} = g_{\mu'\nu'} + \Omega^\mu_{\mu'} g_{\mu\nu'} + \Omega^\nu_{\nu'} g_{\nu\mu'} + \mathcal{O}(\Omega^2). \quad (3.125)$$

Let

$$\Omega_{\mu\nu} = g_{\mu\alpha} \Omega^\alpha_\nu, \quad (3.126)$$

then we must have

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu}. \quad (3.127)$$

The group has 6 parameters as the number of components of an antisymmetric 4×4 matrix. The most general 4×4 antisymmetric matrix can be written as

$$\Omega_{\mu\nu} = \frac{1}{2} \sum_{\rho\sigma} \omega_{(\rho\sigma)} M_{\mu\nu}^{(\rho\sigma)}, \quad (3.128)$$

$$M_{\mu\nu}^{(\rho\sigma)} = \delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu = -M_{\mu\nu}^{(\sigma\rho)}. \quad (3.129)$$

We write

$$\left(M^{(\rho\sigma)} \right)_\nu^\mu = g^{\mu\alpha} M_{\alpha\nu}^{(\rho\sigma)}, \quad (3.130)$$

so

$$\Omega_\nu^\mu = g^{\mu\alpha} \Omega_{\alpha\nu} = g^{\mu\alpha} \frac{1}{2} \omega_{(\rho\sigma)} M_{\mu\nu}^{(\rho\sigma)} = \frac{1}{2} \omega_{(\rho\sigma)} \left(M^{(\rho\sigma)} \right)_\nu^\mu. \quad (3.131)$$

The matrices $M^{(\mu\nu)}$ satisfy the following algebra

$$[M^{(\alpha\beta)}, M^{(\mu\nu)}] = - \left(g^{\alpha\mu} M^{(\beta\nu)} + g^{\beta\nu} M^{(\alpha\mu)} - g^{\beta\mu} M^{(\alpha\nu)} - g^{\alpha\nu} M^{(\beta\mu)} \right). \quad (3.132)$$

We can then introduce

$$J^{(\mu\nu)} \equiv -i M^{(\mu\nu)}, \quad (3.133)$$

and

$$J^i = -\frac{1}{2} \epsilon_{0ijk} J^{(jk)}, \quad (3.134)$$

$$K^i = J^{(0i)}, \quad (3.135)$$

where $\epsilon_{\mu_0\mu_1\mu_2\mu_3}$ is the Levi-Civita symbol with $\epsilon_{0123} = 1$ ⁴. Then Eq. (3.132) is rewritten as

$$[J^i, J^j] = i\epsilon_{ijk}J^k, \quad (3.139)$$

$$[J^i, K^j] = i\epsilon_{ijk}K^k, \quad (3.140)$$

$$[K^i, K^j] = -i\epsilon_{ijk}J^k. \quad (3.141)$$

The generators J^i are the rotations generators, which constitute a subgroup of the Lorentz transformations. The K^i are the generators of the velocity (\mathbf{v}) transformations and are vectors, as follows from their commutation relations with the J^i . The infinitesimal transformations are then

$$\Lambda = 1 + i(\boldsymbol{\theta} \cdot \mathbf{J} - \boldsymbol{\alpha} \cdot \mathbf{K}). \quad (3.142)$$

The finite ones are

$$\Lambda = e^{\frac{i}{2}\sum_{\alpha\beta} J^{(\alpha\beta)}\omega_{(\alpha\beta)}} = e^{i(\boldsymbol{\theta} \cdot \mathbf{J} - \boldsymbol{\alpha} \cdot \mathbf{K})} \quad \mathbf{v} = (\tanh \alpha_1, \tanh \alpha_2, \tanh \alpha_3), \quad (3.143)$$

where $\boldsymbol{\theta}$ is the rotation angle vector and $\boldsymbol{\alpha}$ is the rapidity vector.

Under the Lorentz group the generators of the translations p_μ must transform as four-vectors

$$[J^{(\mu\nu)}, p^\alpha] = i(g^{\mu\alpha}p^\nu - g^{\alpha\nu}p^\mu), \quad (3.144)$$

or

$$[\mathbf{J}, p^0] = -\delta p^0 = -\mathbf{J}p^0 = 0, \quad (3.145)$$

which expresses the conservation of angular momentum, and

$$[J^i, p^j] = -\delta p^j = -J^i p^j = i\epsilon_{ijk}p^k, \quad (3.146)$$

which tells us that \mathbf{p} is a vector. On the momenta the generators of the velocity transformations act as follows

$$[K^i, p^0] = -\delta p^0 = -K^i p^0 = ig^{00}p^i, \quad (3.147)$$

$$[K^i, p^j] = -\delta p^j = -K^i p^j = -ig^{ij}p^0. \quad (3.148)$$

The invariance under translations is written as

$$[p^\mu, p^\nu] = 0. \quad (3.149)$$

The commutation relations between the generators are then

$$[p^\mu, p^\nu] = 0, \quad (3.150)$$

$$[J^{(\mu\nu)}, p^\alpha] = i(g^{\mu\alpha}p^\nu - g^{\alpha\nu}p^\mu), \quad (3.151)$$

$$[J^{(\alpha\beta)}, J^{(\mu\nu)}] = i \left(g^{\alpha\mu}J^{(\beta\nu)} + g^{\beta\nu}J^{(\alpha\mu)} - g^{\beta\mu}J^{(\alpha\nu)} - g^{\alpha\nu}J^{(\beta\mu)} \right). \quad (3.152)$$

They define the Lie algebra of a 10 parameters group known as the Poincaré group.

The Poincaré group is defined by the transformation laws

$$(\Lambda, a) : x \rightarrow x' = \Lambda x - a, \quad (3.153)$$

where a is a translation and Λ is a Lorentz transformation. We immediately find the multiplication properties of the group as

$$(\Lambda_1, a)(\Lambda_2, b) = (\Lambda_1\Lambda_2, -\Lambda_1 b - a), \quad (3.154)$$

from which immediately follows that the translations are an abelian invariant subgroup. In fact applying repetitively Eq. (3.154) we find that the transformed by similitude of a translation $(1, a)$,

$$(\Lambda, c)(1, a)(\Lambda^{-1}, -c) = (1, \Lambda(c - a) - c), \quad (3.155)$$

⁴ For any antisymmetric tensor $F^{\mu\nu}$ it is possible to use a decomposition of the following kind: $F^{\mu\nu} = (\mathbf{P}, \mathbf{A})$ with

$$A^1 = -F^{23} \quad A^2 = -F^{31} \quad A^3 = -F^{12} \quad (3.136)$$

$$P^1 = F^{01} \quad P^2 = F^{02} \quad P^3 = F^{03} \quad (3.137)$$

For the product of two tensors of this kind we have

$$\frac{1}{2}F_{\mu\nu}^{(1)}F^{(2)\mu\nu} = \mathbf{A}^{(1)} \cdot \mathbf{A}^{(2)} - \mathbf{P}^{(1)} \cdot \mathbf{P}^{(2)}. \quad (3.138)$$

is still a translation.

By Wigner theorem the states of a physical system are the basis of a unitary representation of the Poincaré group. An elementary system will be described by an irreducible representation of the Poincaré group.

We note that

$$\mathbf{J}_\pm = \frac{\mathbf{J} \pm i\mathbf{K}}{2}, \quad (3.156)$$

obey the following commutation relations

$$[J_+^i, J_+^j] = i\epsilon_{ijk}J_+^k, \quad (3.157)$$

$$[J_-^i, J_-^j] = i\epsilon_{ijk}J_-^k, \quad (3.158)$$

$$[J_+^i, J_-^j] = 0. \quad (3.159)$$

So the generators of \mathbf{J}_+ and \mathbf{J}_- obey to the algebra $SU(2) \otimes SU(2)$. Let us show now that an irreducible representation of the Poincaré group, i.e. an elementary particle, is determined by the mass and the spin.

An irreducible representation is characterized by the value of the *invariants*, i.e. of the operators built with the generators of the group that commute with all the group generators. We then define

$$\Gamma_\mu = \frac{1}{2}\epsilon_{\mu\alpha\beta\sigma}J^{(\alpha\beta)}p^\sigma, \quad (3.160)$$

$$\Gamma_\mu p^\mu = 0, \quad (3.161)$$

$$g^\mu = J^{(\mu\nu)}p_\nu, \quad (3.162)$$

$$g^\mu p_\mu = 0. \quad (3.163)$$

One can prove [7–9]⁵ that

$$p^2 J^{(\mu\nu)} = g^\mu p^\nu - g^\nu p^\mu - \epsilon^{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda. \quad (3.165)$$

This tells us that $J^{(\mu\nu)}$ can be expressed in terms of p_μ , Γ_μ , and g_μ if $p^2 = p_\mu p^\mu \neq 0$.

Moreover we have

$$[\Gamma_\mu, \Gamma_\nu] = i\epsilon_{\rho\mu\nu\lambda}\Gamma^\rho p^\lambda, \quad (3.166)$$

$$[g_\mu, \Gamma_\sigma] = -i\Gamma_\mu p_\sigma, \quad (3.167)$$

$$[g_\mu, p_\nu] = i(g_{\mu\nu}p^2 - p_\mu p_\nu), \quad (3.168)$$

$$[g_\mu, g_\nu] = -i(g^\mu p^\nu - g^\nu p^\mu - \epsilon^{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda) \quad (3.169)$$

$$[p_\mu, \Gamma_\sigma] = 0. \quad (3.170)$$

An invariant should be constructed with the vectors p_μ , Γ_μ , and g_μ . Recalling that $g_\mu p^\mu = 0$ and $\Gamma_\mu p^\mu = 0$ the only independent invariants under the Lorentz group are

$$p^2, \Gamma^2, g^2, \Gamma_\mu g^\mu. \quad (3.171)$$

But g^2 and $\Gamma_\mu g^\mu$ do not commute with translations. Then the representation is determined by p^2 , Γ^2 , and by the sign of p^0 , which is also invariant under the proper Lorentz group and commutes with translations, if $p^2 \geq 0$.

The physical interpretation of the two invariants is obvious:

i. For the invariant p^2 we have 4 cases

$$p^2 > 0, \quad (3.172)$$

$$p^2 = 0 \quad p \neq 0, \quad (3.173)$$

$$p^2 = 0 \quad p = 0, \quad (3.174)$$

$$p^2 < 0. \quad (3.175)$$

Since $p^2 = m^2$ we will be interested only in the first two cases. In these two cases, for the representations of the proper group ($\Lambda_0^0 \geq 0$ and $\det \Lambda = 1$) we will have another invariant, namely the sign of p^0 .

⁵ One can use the identity

$$\epsilon^{\sigma\mu\nu\lambda}\epsilon_{\sigma\alpha\beta\rho} = \det \begin{pmatrix} \delta_\alpha^\mu & \delta_\beta^\mu & \delta_\rho^\mu \\ \delta_\alpha^\nu & \delta_\beta^\nu & \delta_\rho^\nu \\ \delta_\alpha^\lambda & \delta_\beta^\lambda & \delta_\rho^\lambda \end{pmatrix}, \quad (3.164)$$

and the definition of Γ_σ to calculate the product $\epsilon^{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda$.

ii. The invariant Γ^2 can be calculated in the reference frame where $\mathbf{p} = 0$. In such a frame

$$\Gamma = (\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3) = (\Gamma^0, \mathbf{\Gamma}) = (0, m\mathbf{J}) \quad \Gamma^2 = -m^2 J(J+1). \quad (3.176)$$

The modulus of \mathbf{J} in the rest frame is by definition the particle spin, so $\Gamma^2 = -m^2 s(s+1)$

Then the representation is determined by the mass m and by the spin s , exactly as in the nonrelativistic happens for the Galileo group.

IV. IN CLASSICAL FIELD THEORY [5]

In this section we will work in Minkowski spacetime $x = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$ with the metric (3.114) $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. We will denote with $\partial_\mu = \partial/\partial x^\mu$ and adopt Einstein convention to subtend a sum over repeated indexes. We will use greek indexes for all 4 spacetime components and roman indexes for only the 3 spatial components.

Theorem IV.1. *If the action is invariant under a local transformation $x \rightarrow x'$, $\varphi(x) \rightarrow \varphi'(x') = T\varphi(x)$ in the sense that, for any domain D and the corresponding transformed domain D'*

$$S_D = \int_D \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)] d^4x = S_{D'} = \int_{D'} \mathcal{L}[\varphi'(x'), \partial'_\mu \varphi'(x')] d^4x', \quad (4.1)$$

where the Lagrangian \mathcal{L} is the same function of the initial fields and of the transformed fields, then the equations of motion are invariant in form under the transformation, i.e. on the two reference frames one experiences the same physics.

Proof. An arbitrary variation $\delta\varphi(x)$ vanishing on the frontier of domain D produces a variation $\delta\varphi'(x') = T\delta\varphi(x)$ vanishing on the frontier of D' (see Fig. 2). Since we identically have $S_D = S_{D'}$, the stationary principle $\delta S_D = 0$ implies $\delta S_{D'} = 0$ and the resulting equations of motion for $\varphi(x)$ and $\varphi'(x')$ have the same form. \square

If additionally to the Lagrangian form invariance

$$\mathcal{L}[\varphi(x), \partial_\mu \varphi(x)] = \mathcal{L}[\varphi'(x'), \partial'_\mu \varphi'(x')], \quad (4.2)$$

we also have that the Jacobian of the transformation $J(x, x') = 1$ then $S_D = S_{D'}$ and we will have invariance under the local transformation.

A Lagrangian that transforms as a scalar density under the Poincaré group (3.153), parity (3.66), and time reversal (3.80) will then be invariant under these transformations, since their Jacobian is 1.

A. Noether theorem

Amalie Emmy Noether (Erlangen, 23 March 1882 – Bryn Mawr, 14 April 1935) formulated the following theorem:

Theorem IV.2 (Noether). *If S_D is an invariant action under a group of continuous transformations and*

$$x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu \quad \text{with} \quad \delta x^\mu = A_a^\mu \delta\omega^a \quad (4.3)$$

$$\varphi_i \rightarrow \varphi'_i(x') = (\delta_{ij} + M_{ij}^a \delta\omega^a) \varphi_j(x), \quad (4.4)$$

are the infinitesimal transformations of parameters $\delta\omega^a$, then the currents

$$J_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} [A_a^\nu \partial_\nu \varphi_i(x) - M_{ij}^a \varphi_j(x)] - A_a^\mu \mathcal{L}, \quad (4.5)$$

are conserved, in the sense that

$$\partial_\mu J_a^\mu(x) = 0. \quad (4.6)$$

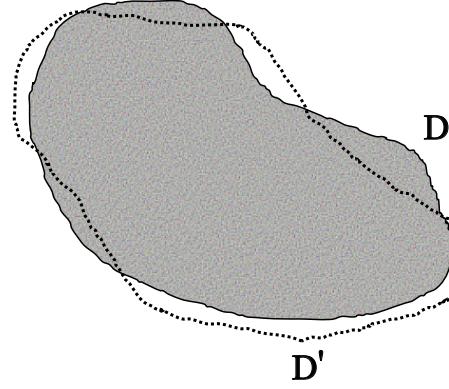


FIG. 2. Domains of integration in the initial action and its transformed.

Proof. The invariance under the infinitesimal transformation implies

$$\int_{D'} \mathcal{L}[\varphi'(x'), \partial'_\mu \varphi'(x')] d^4 x' = \int_D \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)] d^4 x, \quad (4.7)$$

where the domains of integration are shown in Fig. 2. We may also write

$$\int_{D'} \mathcal{L}[\varphi'(x'), \partial'_\mu \varphi'(x')] d^4 x' \approx \int_D \mathcal{L}[\varphi'(x), \partial_\mu \varphi'(x)] d^4 x + \int_{\partial D} \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)] \delta x^\alpha d^3 \Sigma_\alpha, \quad (4.8)$$

where in the second integral in the right hand side φ' has been approximated with φ since the deformations δx^α are already infinitesimal of the first order and we want to neglect infinitesimals of higher order than the first.

On the other hand $\varphi'(x') = (1 + M_a \delta \omega^a) \varphi(x)$ and neglecting terms of second order⁶

$$\begin{aligned} \varphi'(x) &\approx (1 + M_a \delta \omega^a) \varphi(x) - \delta x^\mu \partial_\mu \varphi(x) \\ &= \varphi(x) + [M_a \varphi(x) - A_a^\mu \partial_\mu \varphi(x)] \delta \omega^a = \varphi(x) + \bar{\delta} \varphi(x). \end{aligned} \quad (4.9)$$

Starting from (4.8) and using Gauss theorem

$$0 = S_D - S_{D'} = \int_D \left[-\frac{\partial \mathcal{L}}{\partial \varphi_i} \bar{\delta} \varphi_i - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_i)} \partial_\mu \bar{\delta} \varphi_i - \partial_\mu (\mathcal{L} \delta x^\mu) \right] d^4 x. \quad (4.10)$$

Using the equations of motion

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_i)}, \quad (4.11)$$

we find

$$\begin{aligned} 0 &= \int_D \partial_\mu \left[-\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_i)} \bar{\delta} \varphi_i - \mathcal{L} \delta x^\mu \right] d^4 x \\ &= \delta \omega^a \int_D \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_i)} (A_a^\nu \partial_\nu \varphi_i - M_{ij}^a \varphi_j) - A_a^\mu \mathcal{L} \right] d^4 x, \end{aligned} \quad (4.12)$$

from which follows the thesis (4.6) from the arbitrariness of D and $\delta \omega^a$. \square

Lemma IV.3. *An immediate consequence of Theorem IV.2 is that the charge*

$$Q_a(t) = \int J_a^0(t, \mathbf{x}) d^3 \mathbf{x}, \quad (4.13)$$

is independent of time

$$\frac{dQ_a(t)}{dt} = 0. \quad (4.14)$$

⁶ Note that $\varphi'(x') = \varphi'(x + \delta x) \approx \varphi'(x) + \delta x^\mu \partial_\mu \varphi'(x) = \varphi'(x) + \delta x^\mu \partial_\mu \varphi(x)$ where in the first equality we expanded in a Taylor series truncated to first order and in the second equality we replaced $\varphi'(x)$ with $\varphi(x)$ since we neglect terms of order higher than the first.

Proof. To prove Lemma IV.3 we note that from Eq. (4.6) follows

$$\frac{dQ_a(t)}{dt} = - \int_{\Omega} \partial_k J^k(x) d^3x = - \int_{\partial\Omega} J^k d^2S_k = 0 \quad (4.15)$$

where in the second equation we used Gauss theorem and in the last equality we note that since there are no sources of current within the domain Ω the current flux through the surface enclosing the domain $\partial\Omega$ must be zero. \square

B. Conserved quantities

Let us now consider some examples of invariance that give rise to conserved currents and to their corresponding conserved quantities. The first two examples refer to spacetime transformations, i.e. the translations and the Lorentz transformations that give rise, as conserved quantities, to the energy-momentum tensor and to the density of angular momentum. The other two examples refer to “internal” symmetries.

Translational invariance

For translations

$$\delta\omega^\nu = a^\nu, \quad M_{ij}^a = 0, \quad A_a^\nu = \delta_a^\nu. \quad (4.16)$$

If there is invariance the conserved quantity is

$$Q^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_i)} \partial^\nu \varphi_i - \mathcal{L} g^{\mu\nu}, \quad (4.17)$$

that is the energy-momentum tensor. The conserved 4-momentum is

$$P^\nu = \int Q^{0\nu}(t, \mathbf{x}) d^3x = \int \left[\frac{\partial \mathcal{L}}{\partial(\partial_0\varphi_i)} \partial^\nu \varphi_i - \mathcal{L} g^{0\nu} \right] d^3x. \quad (4.18)$$

For a free scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha\varphi\partial^\alpha\varphi - m^2\varphi^2), \quad (4.19)$$

we have

$$Q_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}(\partial_\alpha\varphi\partial^\alpha\varphi - m^2\varphi^2) \quad (4.20)$$

Lorentz transformations invariance

Under a Lorentz transformation

$$\delta x^\mu = \omega^{\mu\alpha} x_\alpha = \frac{1}{2}(\delta^\mu{}_\rho\delta^\nu{}_\sigma - \delta^\mu{}_\sigma\delta^\nu{}_\rho)x_\nu\omega^{\rho\sigma}. \quad (4.21)$$

The conserved quantity is the angular momentum tensor

$$S^{\mu,\nu\tau} = x^\tau Q^{\mu\nu} - x^\nu Q^{\mu\tau} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_i)} M_{ij}^{\nu\tau} \varphi_j, \quad (4.22)$$

where $Q^{\mu\nu}$ is the energy-momentum tensor of Eq. (4.17).

For a scalar field $M_{ij}^{\nu\tau} = 0$ and

$$S^{\mu,\nu\tau} = x^\tau Q^{\mu\nu} - x^\nu Q^{\mu\tau}. \quad (4.23)$$

The conservation of the energy-momentum tensor

$$\partial_\mu Q^{\mu\nu} = 0, \quad (4.24)$$

together with the Lorentz invariance implies

$$0 = \partial_\mu S^{\mu,\nu\tau} = Q^{\tau\nu} - Q^{\nu\tau}, \quad (4.25)$$

i.e. the energy-momentum tensor is symmetric. The constants of motion associated to the Lorentz invariance are

$$\int [x^\tau Q^{0\nu} - x^\nu Q^{0\tau}] d^3\mathbf{x}. \quad (4.26)$$

For τ, ν spatial, these represent the density of orbital angular momentum of the system. For $\tau = 0$ and $\nu = i$ they impose that

$$p^i = \frac{d}{dt} \int Q^{00} x^i d^3\mathbf{x}, \quad (4.27)$$

where

$$p^i = \int Q^{0i} d^3\mathbf{x}, \quad (4.28)$$

is the conserved total momentum. Eq. (4.27) tells that the center of mass of the system undergoes uniform rectilinear motion.

U(1) transformations

If the action is invariant under the transformation

$$\varphi_i(x) \rightarrow e^{-i\alpha q_i} \varphi_i(x), \quad \delta\varphi_i = -i\alpha q_i \varphi_i, \quad (4.29)$$

parametrized by the phase α , with q_i a number that depends on the kind of field i , $A_a^\mu = 0$ and $M_{ij}^a = -iq_i \delta_{ij}$, the current is

$$J_\mu = i \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_i)} q_i \varphi_i, \quad (4.30)$$

and the corresponding charge

$$Q = \int J^0(t, \mathbf{x}) d^3\mathbf{x}. \quad (4.31)$$

For example for a (complex) scalar field

$$\varphi \rightarrow \varphi e^{ie\alpha}, \quad \varphi^* \rightarrow \varphi^* e^{-ie\alpha}, \quad (4.32)$$

with charges $\pm e$.

Transformations under groups of internal symmetry

If the action is invariant under a group of internal symmetry

$$\delta x^\mu = 0, \quad \varphi \rightarrow e^{iT^a \omega^a} \varphi, \quad (4.33)$$

where T^a are the generators of the group, then the conserved currents are

$$J_\mu^a = -i \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} T^a \varphi. \quad (4.34)$$

V. IN GENERAL RELATIVITY [6]

Spacetime is described by a pseudo Riemannian manifold of dimension 4 with a metric tensor $g_{\mu\nu}$ on which we have a system of coordinates $\tilde{\mathbf{x}} = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$.

An *isometry* is a transformation that leaves the metric tensor unchanged. We will deal with continuous transformations.

A. Killing vectors

Consider a vector field $\vec{\xi}$ and choose spacetime coordinates such that $\vec{\xi} = \partial/\partial t$. $\vec{\xi}$ is a *killing vector* if $\partial g_{\mu\nu}/\partial t = g_{\mu\nu,0} = 0$, where as usual we indicate with a comma a partial derivative.

We will now prove that $\vec{\xi}$ is a killing vector if and only if

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \quad (5.1)$$

where as usual we indicate with a semicolon the covariant derivative. Eq. (5.1) is known as the *killing equation* and this property of ξ is the coordinate characterization of the killing vector.

Consider then

$$\xi_{\mu;\nu} = g_{\mu\alpha}\xi^\alpha_{;\nu} = g_{\mu\alpha}(\xi^\alpha_{,\nu} + \Gamma^\alpha_{\beta\nu}\xi^\beta) = \Gamma_{\mu 0\nu} = \frac{1}{2}(g_{\mu 0,\nu} + g_{\mu\nu,0} - g_{0\nu,\mu}), \quad (5.2)$$

where as usual we use Einstein convention of tacitly assuming a summation over repeated indexes, in the first equality we used the covariant invariance of the metric tensor, in the second equation we used the definition of the covariant derivative, where we denoted with Γ the Christoffel symbol, in the third equation we used the fact that $\xi^\alpha = \delta^\alpha_0$, and in the last equality we used the definition of the Christoffel symbol. Using then the fact that the metric tensor is symmetric we find $\xi_{\mu;\nu} + \xi_{\nu;\mu} = g_{\mu\nu,0} = 0$. This proves the killing equation (5.1).

The killing vectors have the following 3 properties. If $\vec{\xi}$ and \vec{p} are killing vectors then:

- i. $[\vec{\xi}, \vec{p}] = \vec{\xi} \cdot \vec{p} - \vec{p} \cdot \vec{\xi}$ is a killing vector;
- ii. $a\vec{\xi} + b\vec{p}$, with a, b constants, is a killing vector;
- iii. if \vec{p} is the 4-momentum of a free particle of rest mass m then $\vec{p} \cdot \vec{\xi}$ is a constant of motion.

The first two properties are proven in Exercise 10.3 of [10]. We are here interested in the third property

$$\nabla_{\vec{p}}(\vec{p} \cdot \vec{\xi}) = (\nabla_{\vec{p}}\vec{p}) \cdot \vec{\xi} + \vec{p} \cdot (\nabla_{\vec{p}}\vec{\xi}) = \vec{p} \cdot (\nabla_{\vec{p}}\vec{\xi}) = p^\mu p^\nu \xi_{\mu;\nu} = 0, \quad (5.3)$$

where in the second equality we used the fact that the particle parallel transports herself along her geodetic, $\nabla_{\vec{p}}\vec{p} = 0$, and in the last equality we used the killing equation (5.1). This third property then tells us that $\vec{p} \cdot \vec{\xi}$ is constant along the particle world line since $D/d\tau = \vec{p} \cdot \nabla_{\vec{p}}$, with τ the particle proper time, is a covariant derivative.

Moreover, at an event \mathcal{P} of spacetime, we may write $\vec{p} \cdot \vec{\xi} = m \lim_{\tau \rightarrow 0} [\vec{x}'(\mathcal{P}) - \vec{x}(\mathcal{P})] \cdot \vec{\xi}/\tau = q$ with q a constant. Then, for an infinitesimal proper time τ , we have

$$\vec{x}'(\mathcal{P}) \approx \vec{x}(\mathcal{P}) + \epsilon \vec{\xi}, \quad (5.4)$$

with $\epsilon = \tau q/(m\xi^\mu \xi_\mu)$.

In flat spacetime

Choose an inertial Local Lorentz Frame (LLF) at an event \mathcal{P} of spacetime so that $g_{\mu\nu} = \eta_{\mu\nu}$ with $\|\eta_{\mu\nu}\| = \text{diag}(1, -1, -1, -1)$ Minkowski metric. Then the Christoffel symbols Γ vanish and the killing equation simply becomes

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \quad (5.5)$$

from which follows that there are only 10 independent constraints whose solutions can be found as follows:

- i There will be 4

$$\xi^\mu(\alpha) = a^\mu(\alpha), \quad \alpha = 0, 1, 2, 3. \quad (5.6)$$

These represents 4-translations. For example, we may choose the 4 constant vectors as $a^\mu(\alpha) = \delta^\mu_\alpha$, pointing in each of the 4 directions of the frame. For example, for $a^\mu(1) = \delta^\mu_1$, $\vec{\xi}(1) = \partial/\partial x$ and (5.4) represents a translation along the x direction, $x' \approx x + \epsilon$.

ii There will be 6

$$\xi^\mu(\alpha) = \omega^{\mu\nu}(\alpha)x^\nu, \quad (5.7)$$

where $\omega^{\mu\nu}$ are antisymmetric constants. In fact

$$\xi^\mu_{,\alpha} + \xi^\alpha_{,\mu} = \omega^{\mu\nu}x^\nu_{,\alpha} + \omega^{\alpha\nu}x^\nu_{,\mu} = \omega^{\mu\alpha} + \omega^{\alpha\mu} = 0, \quad (5.8)$$

so the choice (5.7) satisfies the killing equation. We will now show how the infinitesimal coordinate transformations induced by the $\vec{\xi}$ of (5.7) constitute the *Lorentz group*: 3 rotations and 3 boosts:

i. The 3 rotations are generated by the choice

$$\xi^0(k) = 0, \quad \xi^i(k) = \epsilon^{ikm}x^m, \quad k = 1, 2, 3, \quad (5.9)$$

where as usual we use roman indexes to denote the spatial components. So, for example, $\vec{\xi}(1) = y\partial/\partial z - z\partial/\partial x$ represents a rotation (5.4) around the x axis of an infinitesimal angle ϵ .

ii. The 3 boosts are generated by choosing

$$\xi^\mu(k) = 2g^{\mu\nu}\delta_\nu^{[0]}x^{k]}, \quad k = 1, 2, 3, \quad (5.10)$$

where the square parenthesis around two indexes denotes their antisymmetrization. So, for example, $\vec{\xi}(1) = x\partial/\partial t + t\partial/\partial x$ represents a boost (5.4) along the x direction of infinitesimal velocity $\beta = \epsilon$ for which $\gamma = 1/\sqrt{1-\beta^2} \approx 1$.

So the killing vectors of flat spacetime induce the infinitesimal coordinate transformations of the *Poincaré group* (the inhomogeneous Lorentz group).

In the global spacetime there are in general no killing vectors.

In correspondence of the 10 killing vectors there are 10 currents and then 10 conserved scalar quantities.

Conserved quantities in Minkowski spacetime

Given the stress-energy tensor $T_{\mu\nu}$, we may define the following currents

$$J^\mu(\alpha) = T^{\mu\nu}\xi_\nu(\alpha), \quad \alpha = 1, 2, 3, \dots, 10, \quad (5.11)$$

such that

$$J^\mu_{,\mu} = T^{\mu\nu}\xi_{\nu,\mu} = 0, \quad (5.12)$$

where in the first equality we used that the stress-energy tensor is divergence-less due to the particles, fluids, fields ... equations of motion and in the last equality the killing equation and the symmetry of the stress-energy tensor. We may also define as many scalar quantities, for each α

$$Q(\alpha) = \int_{t=\text{constant}} J^\mu(\alpha) d^3\Sigma_\mu = \int J^0(\alpha) dx dy dz. \quad (5.13)$$

such that

$$\frac{dQ(\alpha)}{dt} = \int J^0_{,0} dx dy dz = - \int J^k_{,k} dx dy dz = - \int_{\text{closed surface}} J^k d^2S_k = 0, \quad (5.14)$$

where in the second equality we used Eq. (5.12), in the third equality we used Gauss theorem, and in the last equality we used the fact that there is no current source within the closed surface. Then we see that the 10 scalars $Q(\alpha)$ are *conserved quantities*.

So we will have explicitly:

i. for $\vec{\xi} = \partial/\partial t$

$$\int J^0 d^3x = \int T^{00} d^3x = \text{constant}, \quad (5.15)$$

which is the conservation of total mass-energy.

ii. for the 3 $\vec{\xi} = \partial/\partial x^k$

$$\int J^0 d^3\mathbf{x} = - \int T^{0k} d^3\mathbf{x} = \text{constant}, \quad (5.16)$$

which is the conservation of the k -component of the momentum density.

iii. for $\vec{\xi} = x\partial/\partial y - y\partial/\partial x$

$$\int J^0 d^3\mathbf{x} = - \int (xT^{02} - yT^{01}) d^3\mathbf{x} = \text{constant}, \quad (5.17)$$

which is the conservation of the z component of the angular momentum density. The same holds for the other 2 components.

iv. for $\vec{\xi} = x\partial/\partial t + t\partial/\partial x$

$$\int J^0 d^3\mathbf{x} = \int (xT^{00} - tT^{0x}) d^3\mathbf{x} = \text{constant}. \quad (5.18)$$

The same holds for the other 2 boosts along y and z . If we define

$$x_{\text{c.m.}} = \frac{\int xT^{00} d^3\mathbf{x}}{\int T^{00} d^3\mathbf{x}}, \quad v_{\text{c.m.}}^x = \frac{\int T^{0x} d^3\mathbf{x}}{\int T^{00} d^3\mathbf{x}}, \quad (5.19)$$

then the conservation laws (5.18) tells that the center of mass has uniform motion along the x direction

$$x_{\text{c.m.}} = v_{\text{c.m.}}^x t + \text{constant}. \quad (5.20)$$

Globally conserved quantities

Imagine we found a killing vector $\vec{\xi}$ then the current $J^\mu = T^{\mu\nu}\xi_\nu$ is certainly covariantly constant $J^\mu_{;\mu} = 0$, since the stress-energy tensor is covariantly constant and symmetric and the killing equation 5.1 holds. Integrating then on a region Ω of spacetime

$$0 = \int_{\Omega} J^\mu_{;\mu} \sqrt{|g|} d^4\vec{x} = \int_{\Omega} (\sqrt{|g|} J^\mu)_{,\mu} d^4\vec{x} = \int_{\text{time } t_2} \int_{\text{time } t_1} J^\mu d^3\Sigma_\mu - \int_{\text{time } t_1} J^\mu d^3\Sigma_\mu, \quad (5.21)$$

where in the third equality we used Gauss theorem. So the scalar quantity

$$Q = \int_{t = \text{constant}} J^\mu d^3\Sigma_\mu, \quad (5.22)$$

is conserved.

Internal symmetries

We saw in Eq. (5.18) that in a LLF

$$J^{\alpha\beta}(0) = \int_{t = \text{constant}} (x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}) d^3\Sigma_\gamma, \quad (5.23)$$

is the conserved total angular momentum density about the origin. So we may define a tensor

$$J^{\alpha\beta\gamma} = x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}, \quad (5.24)$$

and prove that $J^{\alpha\beta\gamma}_{;\gamma} = 0$ by going into an inertial LLF (so that \rightarrow) and using the symmetry of the stress-energy tensor. Analogously we may define the conserved total angular momentum density about an event \vec{c} as

$$\begin{aligned} J^{\alpha\beta}(c) &= J^{\alpha\beta}(0) - c^\alpha \int_{t = \text{constant}} T^{\beta\gamma} d^3\Sigma_\gamma + c^\beta \int_{t = \text{constant}} T^{\alpha\gamma} d^3\Sigma_\gamma \\ &= J^{\alpha\beta}(0) - c^\alpha p^\beta + c^\beta p^\alpha, \end{aligned} \quad (5.25)$$

where p^μ is the conserved 4-momentum density defined in (5.15) and (5.16). But we note that $J^{\alpha\beta}$ is not invariant under this coordinate transformation.

In order to define “an angular momentum” that is invariant under the coordinate transformation, i.e. the *internal angular momentum* or *spin* we may define $S^{\alpha\beta}$ as $J^{\alpha\beta}$ about points on the center of mass worldline.

While this program can always be carried out in an inertial LLF, i.e. in special relativity, it is not globally on the whole spacetime manifold, i.e. in general relativity. In fact in general relativity there may exist no killing vector and one cannot in general define $J^{\alpha\beta}$. This can be done only for *isolated* gravitating systems or for *small* objects.

VI. SUM RULES

A. Atom

Conservation of angular momentum ...

B. Liquids

Static structure

Dynamic structure

Appendix A: Commutators

The commutator of two operators A and B is defined as

$$[A, B] = AB - BA. \quad (\text{A1})$$

The commutator satisfies to the following Lie algebra relations

$$[A, A] = 0, \quad (\text{A2})$$

$$[A, B] = -[B, A], \quad (\text{A3})$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \quad (\text{A4})$$

where the third one is known as the Jacobi identity.

For three operators A , B , and C we also have

$$[A, B + C] = [A, B] + [A, C], \quad (\text{A5})$$

$$[A, BC] = B[A, C] + [A, B]C. \quad (\text{A6})$$

If $[A, B] = \alpha \in \mathbb{C}$ then

$$[A, B^2] = B[A, B] + [A, B]B = 2\alpha B, \quad (\text{A7})$$

$$[A, B^3] = B[A, B^2] + [A, B]B^2 = 3\alpha B^2, \quad (\text{A8})$$

...

$$[A, B^n] = n\alpha B^{n-1}. \quad (\text{A9})$$

Then, given a smooth function f , using its Taylor series expansion, we readily obtain

$$[A, f(B)] = \alpha \frac{df(B)}{dB}. \quad (\text{A10})$$

In general we can prove the following lemma:

Theorem A.1 (Hadamard lemma). *Given any two operators A and B we have*

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots \quad (\text{A11})$$

Proof. Consider the function $f(s) = e^{sA}Be^{-sA}$. We want $f(1)$. Taylor expand $f(s)$ around $s = 0$

$$f(s) = f(0) + sf'(0) + \frac{1}{2!}s^2f''(0) + \frac{1}{3!}s^3f'''(0) + \dots, \quad (\text{A12})$$

but it is easy to see that

$$f'(s) = e^{sA}ABe^{-sA} - e^{sA}BAe^{-sA} = e^{sA}[A, B]e^{-sA}, \quad (\text{A13})$$

$$f''(s) = e^{sA}[A, [A, B]]e^{-sA}, \quad (\text{A14})$$

$$f'''(s) = e^{sA}[A, [A, [A, B]]]e^{-sA}, \quad (\text{A15})$$

and so on. \square

Another important result is the Baker–Campbell–Hausdorff formula:

Theorem A.2 (Baker–Campbell–Hausdorff formula). *Given any two operators A and B we have*

$$\ln(e^A e^B) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) - \frac{1}{24}[B, [A, [A, B]]] + \dots \quad (\text{A16})$$

Proof. Consider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad (\text{A17})$$

or

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad (\text{A18})$$

integrate respect to x

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad (\text{A19})$$

or

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots \quad (\text{A20})$$

Now

$$\begin{aligned} \ln(e^A e^B) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{m,n=0}^{\infty} \frac{A^m B^n}{m!n!} - 1 \right)^k \\ &= \left(A + B + AB + \frac{A^2 + B^2}{2} \dots \right) - \frac{1}{2} (A^2 + B^2 + AB + BA \dots) + \dots \\ &= A + B + \frac{1}{2}[A, B] + \dots \end{aligned} \quad (\text{A21})$$

\square

Commutators are of fundamental importance in a *Lie algebra*: a vector space \mathfrak{g} together with an operation called the Lie bracket, an alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, that satisfies the Jacobi identity. In other words, a Lie algebra is an algebra over a field for which the multiplication operation (called the Lie bracket) is alternating and satisfies the Jacobi identity. The Lie bracket of two vectors x and y is denoted $[x, y]$. A Lie algebra is typically a non-associative algebra. However, every associative algebra gives rise to a Lie algebra, consisting of the same vector space with the commutator Lie bracket, $[x, y] = xy - yx$.

Lie groups are smooth differentiable manifolds and as such can be studied using differential calculus, in contrast with the case of more general topological groups. One of the key ideas in the theory of Lie groups is to replace the global object, the group, with its local or linearized version, which Lie himself called its “infinitesimal group” and which has since become known as its Lie algebra or the tangent space to the manifold at the identity.

The following theorem is also of great importance:

Theorem A.3. *Given two hermitian operators A and B which commutes, $[A, B] = 0$, they can be diagonalized simultaneously on the same orthonormal basis of vectors of the Hilbert space.*

Proof. Thanks to the spectral theorem one can always find an orthonormal basis $\{e_i\}$ where A is diagonal. Then $Ae_i = \lambda_i e_i$ for each e_i . But from the hypothesis we also have $ABe_i = BAe_i = \lambda_i Be_i$ so Be_i is an eigenstate of A relative to the same eigenvalue λ_i . Then if the eigenvalue is non degenerate Be_i must be proportional to e_i . If the eigenvalue is degenerate one can always choose a linear combination of the basis of the subspace relative to it, \bar{e}_i such that it is a common eigenvector of A and B . \square

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Funding

None declared.

- [1] L. D. Landau and E. M. Lifshitz, *Mechanics*, Course of Theoretical Physics, Vol. 1 (Butterworth Heinemann, 1960) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.
- [2] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, Course of Theoretical Physics, Vol. 6 (Butterworth Heinemann, 1959) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.
- [3] L. D. Landau and E. M. Lifshitz, *Theory of elasticity*, Course of Theoretical Physics, Vol. 7 (Butterworth Heinemann, 1959) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.
- [4] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics. Non-relativistic Theory*, Vol. 3 (Pergamon Press Ltd, Oxford, 1977) course of Theoretical Physics. Section §6.
- [5] L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, Course of Theoretical Physics, Vol. 2 (Butterworth Heinemann, 1971) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.
- [6] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
- [7] Y. M. Shirokov, JETP **6**, 664,919,929 (1958).
- [8] Y. M. Shirokov, JETP **7**, 493 (1958).
- [9] Y. M. Shirokov, JETP **9**, 620 (1959).
- [10] Alan P. Lightman and William H. Press and Richard H. Price and Saul A. Teukolsky, *Problem book in relativity and gravitation* (Princeton University Press, Princeton, 1975).