Homework # 7 Solutions

set of units used: MKSA

-Problem 1- For a glass-to-air transition ($n_1 = 1.5, n_2 = 1$) at normal incidence, $E_T = 1.2E_I$. Show that energy is conserved at the interface, despite this apparent contradiction.

SOLUTION

In a glass-to-air transition the incident and refracted wave are in a media with an index of refraction ($n_1 \sim 1.5$) higher then the one ($n_2 \sim 1 < n_1$) of the media containing the refracted wave. For a plane wave $\mathbf{E}(x,t) = \mathbf{n}E_o e^{i(kx-wt)}$ at normal incidence on the surface of separation of the two media the average (in time) value of the electromagnetic energy stored in the wave is

$$\langle u(x,t)\rangle = \varepsilon\langle E(x,t)\rangle = \frac{1}{2}\varepsilon E_o \quad . \tag{1}$$

From the boundary condition on the component of the electric and magnetic field of the incident (I), reflected (R) and transmitted (T) wave parallel to the refrangence surface one gets

$$\begin{cases} \langle u_I(x,t)\rangle = \frac{1}{2}\varepsilon_1 E_{oI}^{\,2} \quad , \\ \langle u_R(x,t)\rangle = \frac{1}{2}\varepsilon_1 E_{oR}^{\,2} = \frac{1}{2}\varepsilon_1 \frac{n_1 - n_2}{n_1 + n_2} E_{oI}^{\,2} = \frac{1}{2}\varepsilon_1 (0.2)^2 E_{oI}^{\,2} \quad , \\ \langle u_T(x,t)\rangle = \frac{1}{2}\varepsilon_2 E_{oT}^{\,2} = \frac{1}{2}\varepsilon_2 \frac{2n_2}{n_1 + n_2} E_{oI}^{\,2} = \frac{1}{2}\varepsilon_2 (1.2)^2 E_{oI}^{\,2} \quad . \end{cases} \tag{2}$$

We have then the apparent contraddiction due to the fact that ($\varepsilon_1 \sim (1.5)^2\varepsilon_2$)

$$\langle u_I\rangle < \langle u_R\rangle + \langle u_T\rangle \quad . \tag{3}$$

Nevertheless the energy is conserved at the boundary surface. Consider the volume across the boundary region bounded by the surfaces S_1 and S_2 on opposite side respect to the boundary between the two media as shown in figure 1 (shaded region). We will show that the amount of energy entering the described volume through surface S_1 per unit time and per unit area

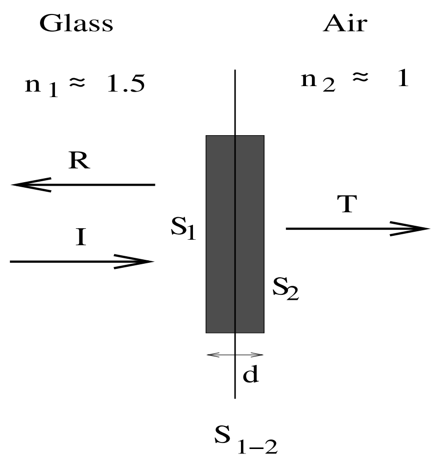

Figure 1: Reflection and transmission at normal incidence: case $n_1 > n_2$.

(namely the Poynting vector $\mathbf{S}_I$ of the incident wave) equals the amount of energy leaving thevolume through surface S_2 per unit time and per unit area (namely the sum of the Poynting vector $\mathbf{S}_R$ of the reflected wave and of the one $\mathbf{S}_T$ of the refracted). Since for a plane wave traveling in the $\hat{\mathbf{x}}$ direction with velocity v holds $\mathbf{S}(x,t) = u(x,t)v\hat{\mathbf{x}}$ then shrinking the volume to a narrow parallelepiped on the boundary ($d \to 0$ see figure 1) we get (remember $n_1/n_2 = \sqrt{\varepsilon_1/\varepsilon_2}$)

$$\begin{cases} S_I(0,t) = \dfrac{c}{n_1}\varepsilon_1 E_I^2(0,t) \quad , \\ S_R(0,t) = \dfrac{c}{n_1}\varepsilon_1 \left[\dfrac{n_1 - n_2}{n_1 + n_2}\right]^2 E_I^2(0,t) \quad , \\ S_T(0,t) = \dfrac{c}{n_2}\varepsilon_2 \left[\dfrac{2n_2}{n_1 + n_2}\right]^2 E_I^2(0,t) = \dfrac{c}{n_1}\varepsilon_1 \left[\dfrac{n_2}{n_1}\dfrac{2n_2}{n_1 + n_2}\right]^2 \quad , \end{cases} \tag{4}$$

from which easily follows the derired relation

$$S_I(0,t) = S_R(0,t) + S_T(0,t) \quad . \tag{5}$$

-Problem 2- Prove that in the problem of normal incidence of an electromagnetic plane wave on the boundary between two linear media, the reflected and transmitted wave must have the same polarization of the incident wave. (Let the polarizations of the transmitted and reflected wave be $\hat{\mathbf{n}}_T = \cos\theta_T \hat{\mathbf{y}} + \sin\theta_T \hat{\mathbf{z}}$, and $\hat{\mathbf{n}}_R = \cos\theta_R \hat{\mathbf{y}} + \sin\theta_R \hat{\mathbf{z}}$ respectively. Then prove from the boundary conditions that $\theta_T = \theta_R = 0$.)

SOLUTION

Suppose the yz plane forms the boundary between two different linear media. A plane wave of frequency ω, traveling in the $\hat{\mathbf{x}}-$direction and polarized in the $\hat{\mathbf{y}}-$direction, approaches the interface from the left (see figure 2).

The electric and magnetic fields of the incident (I) wave can be written as

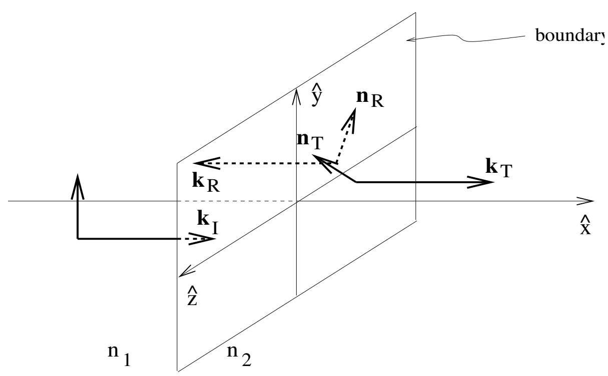

Figure 2: Reflection and transmission at normal incidence, without any apriori assumption on the polarization of the reflected and of the transmitted wave.

follows

$$\begin{cases} \mathbf{E}_I(x,t) = E_{oI} e^{i[k_I x - \omega t]} \hat{\mathbf{y}} \quad , \\ \mathbf{B}_I(x,t) = \hat{\mathbf{x}} \times \dfrac{\mathbf{E}_I}{v_1} \quad . \end{cases} \tag{1}$$

It gives rise to a transmitted (T) and a reflected (R) wave

$$\begin{cases} \mathbf{E}_{R,T}(x,t) = E_{oR,T} e^{i[(-)^{\alpha_{R,T}} k_{R,T} x - \omega t]} \hat{\mathbf{n}}_{R,T} \quad , \\ \mathbf{B}_{R,T}(x,t) = \hat{\mathbf{k}}_{R,T} \times \dfrac{\mathbf{E}_{R,T}}{v_{1,2}} \quad , \end{cases} \tag{2}$$

where $\alpha_T = -\alpha_R = 1$ and $\hat{\mathbf{k}}_T = -\hat{\mathbf{k}}_R = \hat{\mathbf{x}}$ since the transmitted and the reflected wave are travelling in opposite directions, $v_1 = c/n_1$ and $v_2 = c/n_2$ are the phase velocities of the waves in media 1 and media 2 and finally

$$\begin{cases} \hat{\mathbf{n}}_T = \cos\theta_T \hat{\mathbf{y}} + \sin\theta_T \hat{\mathbf{z}} \quad , \\ \hat{\mathbf{n}}_R = \cos\theta_R \hat{\mathbf{y}} + \sin\theta_R \hat{\mathbf{z}} \quad , \end{cases} \tag{3}$$

are the polarization vectors of the transmitted and the reflected wave.
The boundary conditions at the surface of separation of the two media for the parallel components of the electric fields and the parallel components of the magnetic fields, are the following

$$\begin{cases} E_I \hat{\mathbf{n}}_I + E_R \hat{\mathbf{n}}_R = E_T \hat{\mathbf{n}}_T \quad , \\ \dfrac{E_I}{\mu_1 v_1}(\hat{\mathbf{k}}_I \times \hat{\mathbf{n}}_I) + \dfrac{E_R}{\mu_1 v_1}(\hat{\mathbf{k}}_R \times \hat{\mathbf{n}}_R) = \dfrac{E_T}{\mu_2 v_2}(\hat{\mathbf{k}}_T \times \hat{\mathbf{n}}_T) \quad , \end{cases} \tag{4}$$

where $\hat{\mathbf{k}}_I = \hat{\mathbf{x}}$, $\hat{\mathbf{n}}_I = \hat{\mathbf{y}}$,

$$\begin{cases} \hat{\mathbf{k}}_R \times \hat{\mathbf{n}}_R = -\cos\theta_R \hat{\mathbf{z}} + \sin\theta_R \hat{\mathbf{y}} \quad , \\ \hat{\mathbf{k}}_T \times \hat{\mathbf{n}}_T = \cos\theta_T \hat{\mathbf{z}} - \sin\theta_T \hat{\mathbf{y}} \quad . \end{cases} \tag{5}$$

and $\mu v = \sqrt{\mu/\varepsilon} = Z$ is the impedence of the media.

Projecting the first boundary conditions of (4) along the $\hat{\mathbf{z}}$ direction and the second one along the $\hat{\mathbf{y}}$ direction, we get

$$\begin{pmatrix} E_R & -E_T \\ Z_2 E_R & Z_1 E_T \end{pmatrix} \begin{pmatrix} \sin\theta_R \\ \sin\theta_T \end{pmatrix} = 0 \quad . \tag{6}$$

Since the determinant of the 2×2 matrix is different from zero then $\sin\theta_R = \sin\theta_T = 0$. This shows that the polarizations of the reflected and transmitted wave (3) must be parallel or antiparallel to the polarization of the incident wave ($\hat{\mathbf{y}}$).

Choosing $\theta_R = \theta_T = 0$, ($\theta_R, \theta_T \in [-\pi/2, \pi/2]$) we get

$$\begin{cases} E_I + E_R \cos\theta_R = E_T \cos\theta_T \quad , \\ Z_2(E_I - E_R \cos\theta_R) = Z_1 E_T \cos\theta_T \quad , \end{cases} \tag{7}$$

from which follows

$$\begin{cases} E_R = \left(\dfrac{1-\beta}{1+\beta}\right) E_I \quad , \\ E_R = \left(\dfrac{2}{1+\beta}\right) E_I \quad , \end{cases} \tag{8}$$

where $\beta = Z_1/Z_2$. So choosing $\theta_R = \theta_T = 0$, if $\beta < 1$ and $E_I > 0$ then $E_R, E_T > 0$ (convention usually used).

$\boxed{\text{-Problem 3-}}$ At Brewster' s angle ($\theta_I = \theta_B$), there is no reflected wave of one polarization. Above the critical angle ($\theta_I > \theta_C$), there is no transmitted wave at all. Is it possible for both conditions to be satisfied simultaneously, implying nonconservation of energy ? (This is Heald, 6-10, p.223.)

SOLUTION

Consider the problem of the incidence at a certain angle θ_I, of an electromagnetic wave, on the on the boundary between two linear media (with $\mu_1 \sim \mu_2$) of different index of refraction n_1 and n_2. There will be a reflected wave at an angle θ_R with the normal to the surface of incidence, and a transmitted one at angle θ_T.

For polarizations *parallel* to the incident plane exist an angle of incidence called *Brewster' s angle* at which the reflected wave is zero. One can show that the amplitude of the reflected wave become zero for

$$\theta_I = \theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) \quad . \tag{1}$$

For example when $n_2/n_1 \sim 1.5$ then $\theta_B \sim 56'$.

In accord with Snell's law when $n_1 < n_2$ there is a critical angle

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \quad , \tag{2}$$

such that for $\theta_I > \theta_c$ there is no transmitted wave and we have total reflection (the rifracted wave propagates parallel to the rifrangent surface with an amplitude exponentially decaying as the distance in the second media from the refranget surface increases, and the energy transparent through the surface is 0).

For both conditions (2) and (1)to be satisfied simultaneously one should have a θ_I such that

$$\theta_I = \sin^{-1}\left(\sqrt{\frac{\beta}{1+\beta}}\right) > \sin^{-1}(\beta) \quad , \tag{3}$$

where $\beta = (n_2/n_1)$. Since the angles that we are considering always lay into the interval $[-\pi/2, \pi/2]$ then follows

$$\frac{\beta}{1+\beta} < \beta \Rightarrow 1 < \sqrt{1+\beta^2} \quad , \tag{4}$$

which can never be satisfied.

-Problem 4- Use the Poynting vector to calculate the transmission coefficient at the second surface of a conductor. Confirm that $T + R = 1$ (do *not* assume the conductor is a "good" one). [Griffiths, problem 8.22, p.377.]

SOLUTION

Reflection and transmission at a conducting surface (normal incidence). We

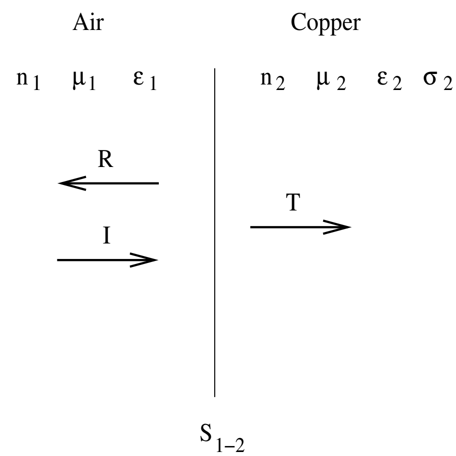

Figure 3: Schematic representation of reflection and transmission at a conducting surface (normal incidence). The wave vector of the rifracted plane wave traveling in copper is a complex number.

can write the Incident (I) and reflected (R) wave as

$$\begin{cases} \mathbf{E}_{R,I}(x,t) = E_{oR,I} e^{i[(-)^{\alpha_{R,I}} k_{R,I} x - \omega t]} \hat{\mathbf{y}} \quad , \\ \mathbf{B}_{R,I}(x,t) = \hat{\mathbf{k}}_{R,I} \times \dfrac{\mathbf{E}_{R,I}}{v_{1,2}} \quad , \end{cases} \tag{1}$$

where $\alpha_I = -\alpha_R = 1$ and $\hat{\mathbf{k}}_I = -\hat{\mathbf{k}}_R = \hat{\mathbf{x}}$ since the transmitted and the reflected wave are travelling in opposite directions, $v_1 = c/n_1$ is the phase velocity of the waves in the non conducting media The transmitted (T) wave can be written as

$$\begin{cases} \mathbf{E}_T(x,t) = E_{oT} e^{-k_{T-} x} e^{i[k_{T+} x - \omega t]} \hat{\mathbf{y}} \quad , \\ \mathbf{B}_T(x,t) = \dfrac{k_T}{\omega} e^{i\phi} e^{-k_{T-} x} e^{i[k_{T+} x - \omega t]} (\hat{\mathbf{x}} \times \hat{\mathbf{y}}) \quad . \end{cases} \tag{2}$$

where k_T unlike k_R and k_I is a complex number

$$k_T = k_{T+} + ik_{T-} \quad , \tag{3}$$

$$k_{T\pm} = \omega\sqrt{\frac{\varepsilon_2\mu_2}{2}}[\sqrt{1+\frac{\sigma_2}{\epsilon_2\omega}} \pm 1]^{1/2} \quad , \tag{4}$$

such that

$$|k_T| = \sqrt{k_{T+}^2 + k_{T-}^2} \quad , \phi = \tan_{-1}(k_{T-}/k_{T+}) \quad . \tag{5}$$

where ϕ represents the phase shift between the electric and the magnetic fields of the wave traveling in the conducting media.

Using the boundary conditions for the component of the magnetic and electric field parallel to the conducting surface one get

$$E_{oR} = \left(\frac{1-\beta}{1+\beta}\right) E_{oI} \quad , \tag{6}$$

$$E_{oR} = \left(\frac{2}{1+\beta}\right) E_{oI} \quad , \tag{7}$$

with β complex

$$\beta = \left(\frac{\mu_1 v_1 k_2}{\mu_2\omega}\right) \quad . \tag{8}$$

The transmission coefficient is defined as the ratio of the transmitted intensity to the incident one

$$T = \frac{I_T}{I_I} \quad . \tag{9}$$

The intensity of an electromagnetic plane wave is defined as the average (in time) power per unit area carried by the wave, namely

$$I = \langle S\rangle = c\langle U\rangle \quad , \tag{10}$$

where S is the modulus of the Poynting vector and U the electromagnetic energy carried by the wave. Thus one get for the incident and reflected waves

$$I_{R,I} = \frac{1}{2\mu_1}\frac{k_{I,R}}{\omega}|E_{oR,I}|^2 \quad , \tag{11}$$

and for the transmitted wave

$$T_T(x) = \frac{1}{2\mu_2}\frac{k_{T+}}{\omega}|E_{oT}|^2 e^{-2k_{T-}x} \quad , \tag{12}$$

where the moduli introduced in equations (11) and (12) are the moduli of complex numbers.

Finally from equations (12) and (9) we get

$$T = \frac{I_T(0)}{I_I} = \frac{\mu_1 k_{T+}}{\mu_2 k_I}\left|\frac{E_{oT}}{E_{oT}}\right|^2 = \mathcal{R}e(\beta)\left|\frac{2}{1+\beta}\right|^2 \quad , \tag{13}$$

and analogously for the reflection coefficient

$$R = \frac{I_R}{I_I} = \left|\frac{1-\beta}{1+\beta}\right|^2 \quad . \tag{14}$$

One can then easily show that $R + T = 1$

$$\left|\frac{1-\beta}{1+\beta}\right|^2 + \mathcal{R}e(\beta)\left|\frac{2}{1+\beta}\right|^2 \stackrel{?}{=} 1 \quad \Rightarrow \tag{15}$$

$$4\mathcal{R}e(\beta) + 1 + |\beta|^2 - 2\mathcal{R}e(\beta) \stackrel{?}{=} |1+\beta|^2 \Rightarrow \tag{16}$$

$$1 + |\beta|^2 + 2\mathcal{R}e(\beta) = |1+\beta|^2 \quad . \tag{17}$$