Homework # 1 Solutions

all problems are from D.J.Griffiths: "introduction to electrodynamics" unless stated
set of units used: MKSA

-SOLUTION to problem 1.13-

Given the following definitions

$$\begin{aligned}
\mathbf{r} &\equiv (x-x_o)\hat{\mathbf{x}} + (y-y_o)\hat{\mathbf{y}} + (z-z_o)\hat{\mathbf{z}} \quad , \\
r &\equiv |\mathbf{r}| \equiv \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2} \quad , \\
\hat{\mathbf{r}} &\equiv \frac{\mathbf{r}}{r} \quad , \\
\nabla &\equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \quad .
\end{aligned}$$

it follows:

(a)

$$\begin{aligned}
\nabla r^2 &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2] \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2] \\
&\quad + \hat{\mathbf{y}}\frac{\partial}{\partial y}[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2] \\
&\quad + \hat{\mathbf{z}}\frac{\partial}{\partial z}[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2] \\
&= \hat{\mathbf{x}}2(x-x_o) + \hat{\mathbf{y}}2(y-y_o) + \hat{\mathbf{z}}2(z-z_o) \\
&= 2\mathbf{r} \quad ,
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla\frac{1}{r} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\frac{1}{\sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}} \\
&= \hat{\mathbf{x}}\left(-\frac{x-x_o}{[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2]^{3/2}}\right)
\end{aligned}$$

$$
\begin{aligned}
& + \hat{\mathbf{y}} \left(- \frac{y - y_o}{[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2]^{3/2}} \right) \\
& + \hat{\mathbf{z}} \left(- \frac{z - z_o}{[(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2]^{3/2}} \right) \\
= \quad & -\frac{\mathbf{r}}{r^3} = -\frac{\hat{\mathbf{r}}}{r^2} \quad ,
\end{aligned}
$$

(c)

$$
\begin{aligned}
\nabla r^n \quad = \quad & \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) [(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2]^{n/2} \\
= \quad & \hat{\mathbf{x}} n (x - x_o) [(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2]^{(n-2)/2} \\
& + \hat{\mathbf{y}} n (y - y_o) [(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2]^{(n-2)/2} \\
& + \hat{\mathbf{y}} n (z - z_o) [(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2]^{(n-2)/2} \\
= \quad & \mathbf{r} n r^{n-2} = \hat{\mathbf{r}} n r^{n-1} \quad .
\end{aligned}
$$

It' s worthwhile to observe at this point that the following simple rule generally hold for the gradient of a function of r

$$
\nabla f(r) = \hat{\mathbf{r}} \frac{df}{dr}(r) \quad .
$$

-SOLUTION to problems 1.15 (a),(b); 1.19 (a),(b)-

(a) Given $\mathbf{v} = (x^2, 3xz^2, -2xz)$ one gets for the divergence

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) \\ &= 2x - 2x = 0 \quad ,\end{aligned}$$

and for the curl

$$\begin{aligned}\nabla \times \mathbf{v} &\equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2)\right] \\ &\quad + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(-2xz)\right] \\ &\quad + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(x^2)\right] \\ &= \hat{\mathbf{x}}(-6xz) + \hat{\mathbf{y}}(2z) + \hat{\mathbf{z}}(3z^2) \quad ,\end{aligned}$$

(b) Given $\mathbf{v} = (xy, 2yz, 3zx)$ one gets for the divergence

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx) \\ &= y + 2z + 3x \quad ,\end{aligned}$$

and for the curl

$$\begin{aligned}\nabla \times \mathbf{v} &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(3zx) - \frac{\partial}{\partial z}(2yz)\right] \\ &\quad + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3zx)\right] \\ &\quad + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right] \\ &= \hat{\mathbf{x}}(-2y) + \hat{\mathbf{y}}(-3z) + \hat{\mathbf{z}}(-x) \quad .\end{aligned}$$

-SOLUTION to problem 1.31-

We have to test the divergence theorem for the function $\mathbf{v} = (xy, 2yz, 3zx)$.

$$\int_V \nabla \cdot \mathbf{v}\ d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a} \quad , \tag{1}$$

where V is the volume of the cube shown in fig. 1 and S is its surface. We

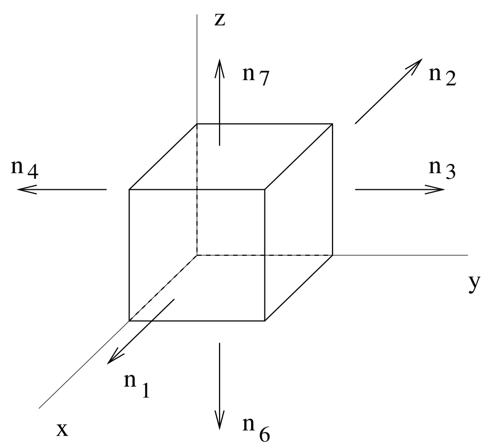

Figure 1: cube of side 2. $\mathbf{n}_i$ is the unitary vector orthogonal to the i^{th} face.

will calculate separately the right hand side (RHS) and the left hand side (LHS) of equation (1) and show that they are equal:

(LHS) As calculated in the solution to problem 1.15 (b) $\nabla \cdot \mathbf{v} = y + 2z + 3x$.

$$\begin{aligned}
\int_V \nabla \cdot \mathbf{v}\ d\tau &= \int_V (y + 2z + 3x)\ dxdydz \\
&= \int_0^2 dz \int_0^2 dy \int_0^2 dx(y + 2z + 3x) \\
&= \int_0^2 dz \int_0^2 dy(2y + 4z + 6) \\
&= \int_0^2 dz(4 + 8z + 12) = 8 + 16 + 24 = 48
\end{aligned}$$

(RHS)

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \sum_i \int_{i^{th} face} \mathbf{v} \cdot \mathbf{n}_i\ da$$

$$
\begin{aligned}
&= \left(\int_{3^{rd} face} 2yz\ dxdz - \int_{4^{th} face} 2yz\ dxdz\right) \\
&\quad + \left(\int_{5^{th} face} 3zx\ dxdy - \int_{6^{th} face} 3zx\ dxdy\right) \\
&\quad + \left(\int_{1^{st} face} xy\ dydz - \int_{2^{nd} face} xy\ dydz\right) \\
&= \int_0^2 \int_0^2 2y\ dydz + \int_0^2 \int_0^2 4z\ dxdz + \int_0^2 \int_0^2 6x\ dxdy \\
&= \int_0^2 4\ dz + \int_0^2 8\ dx + \int_0^2 12\ dy \\
&= 8 + 16 + 24 = 48
\end{aligned}
$$

-SOLUTION to problem 1.33-

We have to test Stokes' theorem for the function $\mathbf{v} = (xy, 2yz, 3zx)$.

$$\int_S \nabla \times \mathbf{v}\ d\mathbf{a} = \oint_L \mathbf{v} \cdot d\mathbf{l} \quad , \tag{1}$$

where S is the triangula shaded area shown in fig. 2 and L is its boundary.

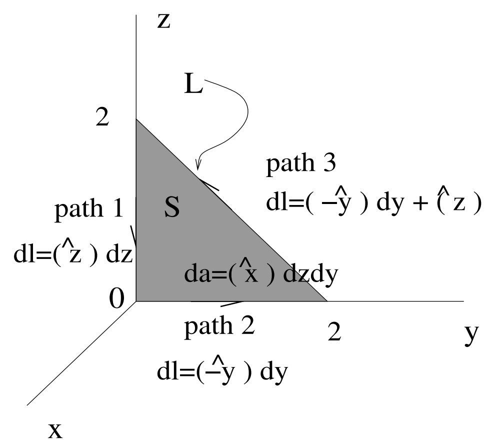

Figure 2: Triangular surface S.

We will calculate separately the right hand side (RHS) and the left hand side (LHS) of equation (1) and show that they are equal:

(LHS) As calculated in the solution to problem 1.15 (b) $\nabla\times\mathbf{v} = (-2y, -3z, -x)$.

$$\begin{aligned}
\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} &= \int_S (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{x}} dz dy = \int_S (-2y)\ \hat{\mathbf{x}} dz dy \\
&= \int_0^2 dy \int_0^{2-y} dz (-2y) = \int_0^2 2y(y-2)\ dy \\
&= \int_0^2 (4y - 2y^2)\ dy = \frac{2}{3} 8 - 8 = -\frac{8}{3} \quad ,
\end{aligned}$$

(RHS)

$$\oint_L \mathbf{v} \cdot d\mathbf{l} = \sum_{i=1}^{3} \int_{path-i} \mathbf{v} \cdot d\mathbf{l} \tag{2}$$

$$
\begin{aligned}
&= \left[\int_2^0 \mathbf{v}|_{y=0}\, \hat{\mathbf{z}}\ dz + \int_0^2 \mathbf{v}|_{z=0}\, (-\hat{\mathbf{y}})\ dy \right. && (3)\\
&\quad \left. + \int_{path-3} \mathbf{v}|_{y+z=2}\, (-\hat{\mathbf{y}} dy + \hat{\mathbf{z}} dz)\right]_{x=0} && (4)\\
&= \left[0 + 0 + \int_2^0 2y(2-y)\ dy\right] && (5)\\
&= \int_0^2 (4y - 2y^2)\ dy = \frac{2}{3}8 - 8 = -\frac{8}{3}\quad . && (6)
\end{aligned}
$$

-SOLUTION to problem 2.1 (a), (b)-

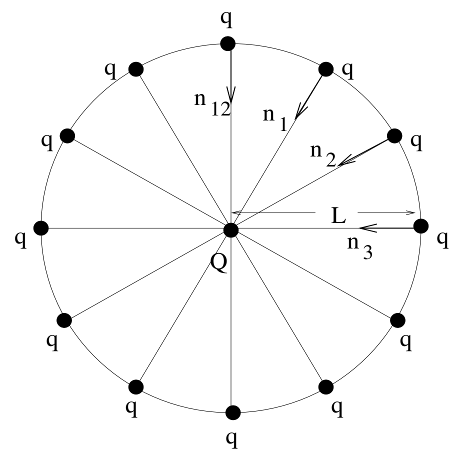

Figure 3: shows twelve equal charges, q, fixed at the corners of a regular 12-sided polygon (one on each numeral of a clock face). $\mathbf{n}_i$ is the versor poynting the center of the polygon from the i^{th} charge. A test charge Q is at the center of the polygon.

(a) Given the arrangement of charges depicted in fig. 3 and assuming fixed the charges q at the corners of the 12-sided regular polygon the force exercited by the i^{th}-charge (i.e. the charge at "i o'clock") on a test charge Q at the center of the polygon can be written as

$$\mathbf{F}_i = \frac{1}{4\pi\varepsilon_o}\frac{Qq}{L}\mathbf{n}_i \quad ,$$

where L is half dyagonal of the polygon and $\mathbf{n}_i$ is the unit vector pointing the test charge from the i^{th}-charge. The total force exercited on Q is then

$$\mathbf{F} = \sum_{i=1}^{12}\mathbf{F}_i = \frac{1}{4\pi\varepsilon_o}\frac{qQ}{L}\sum_{i=1}^{12}\mathbf{n}_i \tag{1}$$

$$= \frac{qQ}{L}\sum_{i=1}^{6}(\mathbf{n}_i + \mathbf{n}_{i+6}) = \frac{qQ}{L}\sum_{i=1}^{6}(\mathbf{n}_i - \mathbf{n}_i) = 0 \quad . \tag{2}$$

(b) when we remove the charge at 6 o'clock the force on Q will be

$$\mathbf{F} = \sum_{\substack{i=1 \\ i \neq 6}}^{12} \mathbf{F}_i = -\mathbf{F}_6 = \frac{1}{4\pi\varepsilon_o} \frac{Qq}{L} \mathbf{n}_6 \quad ,$$

as follows from eq. (1).

-SOLUTION to problem 2.2-

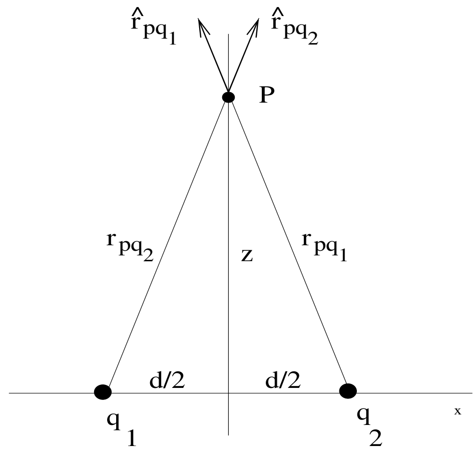

Figure 4: two charges q_1 and q_2 of equal charge q.

(a) When $q_1 = q_2 = q$

$$\begin{aligned}
\mathbf{E}_p &= \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2_{pq_1}}\hat{\mathbf{r}}_{pq_1} + \frac{q}{r^2_{pq_2}}\hat{\mathbf{r}}_{pq_2}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{z^2+d^2/4}(\hat{\mathbf{r}}_{pq_1}+\hat{\mathbf{r}}_{pq_2}) \\
&= \frac{1}{4\pi\varepsilon_o}\frac{q}{z^2+d^2/4}\left(\hat{\mathbf{z}}\frac{2z}{\sqrt{z^2+d^2/4}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{2zq}{(z^2+d^2/4)^{3/2}}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\varepsilon_o}\frac{2q}{z^2(1+(d/z)^2/4)^{3/2}}\hat{\mathbf{z}} \overset{d/z\ll 1}{\rightarrow} \hat{\mathbf{z}}\frac{1}{4\pi\varepsilon_o}\left(\frac{2q}{z^2}+O((d/z)^2)\right) \quad .
\end{aligned}$$

In the limit $z \gg d$ the dominant term in $\mathbf{E}_p$ resembles the electric field generated in P by a charge $2q$ at the origin.

(b) when $q_1 = -q_2 = q$

$$\begin{aligned}
\mathbf{E}_p &= \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2_{pq_1}}\hat{\mathbf{r}}_{pq_1} - \frac{q}{r^2_{pq_2}}\hat{\mathbf{r}}_{pq_2}\right) = \frac{1}{4\pi\varepsilon_o}\frac{q}{z^2+d^2/4}(\hat{\mathbf{r}}_{pq_1}-\hat{\mathbf{r}}_{pq_2}) \\
&= \frac{1}{4\pi\varepsilon_o}\frac{q}{z^2+d^2/4}\left(\hat{\mathbf{x}}\frac{d}{\sqrt{z^2+d^2/4}}\right) = \frac{1}{4\pi\varepsilon_o}\frac{qd}{(z^2+d^2/4)^{3/2}}\hat{\mathbf{x}}
\end{aligned}$$

$$= \frac{1}{4\pi\varepsilon_o} \frac{qd}{z^3(1+(d/z)^2/4)^{3/2}} \hat{\mathbf{x}} \stackrel{d/z \ll 1}{\rightarrow} \hat{\mathbf{x}} \frac{1}{4\pi\varepsilon_o} \left(\frac{qd}{z^3} + O((d/z)^2) \right) \quad .$$

In the limit $z \gg d$ the dominant term in $\mathbf{E}_p$ resembles the electric field generated in P by a "pure" dipole $-qd\hat{\mathbf{x}}$ at the origin [1].

[1]The assembly of charge used in this part (b) of the problem is called a "physical" dipole of magnitude $\mathbf{p} = -qd\hat{\mathbf{x}}$. The "pure" dipole is *defined* as a physical dipole with q infinitely big ($q \rightarrow e(\infty)$) and d infinitely small in such a way to keep the product $qd = p$ a finite quantity. p is the magnitude of the dipole.

-SOLUTION to problem 2.5-

The electric field at a point P on the axis of a circular loop of radius r carrying a uniform line charge λ (see fig. 5) can be calculated as follows.

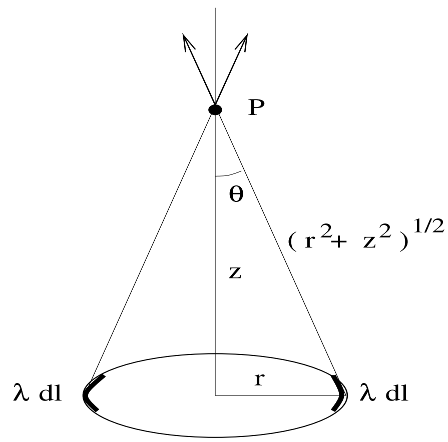

Figure 5: loop of radius r carrying a uniform line charge λ.

First consider the electric field due to two diametral opposite elementary pieces dl of the circular loop

$$\begin{aligned} d\mathbf{E}_p &= \frac{1}{4\pi\varepsilon_o}\frac{2\cos\theta}{r^2+z^2}(\lambda dl)\hat{\mathbf{z}} \quad , \\ \cos\theta &= \frac{z}{\sqrt{r^2+z^2}} \quad , \end{aligned}$$

then sum over half-loop to obtain the electric field due to all the uniformly charged loop

$$\begin{aligned} \mathbf{E}_p &= \int_{1/2-loop} d\mathbf{E}_p = \hat{\mathbf{z}}\int_{1/2-loop}\lambda dl\frac{1}{4\pi\varepsilon_o}\frac{2z}{(r^2+z^2)^{3/2}} \\ &= \hat{\mathbf{z}}\frac{1}{4\pi\varepsilon_o}\frac{2z\lambda}{(r^2+z^2)^{3/2}}\int_{1/2-loop} dl = \hat{\mathbf{z}}\frac{1}{4\pi\varepsilon_o}\frac{2\pi r\lambda z}{(r^2+z^2)^{3/2}} \quad . \end{aligned}$$

-SOLUTION to problem 2.10-

Using Gauss's law we know that the flux of the electric field due to the

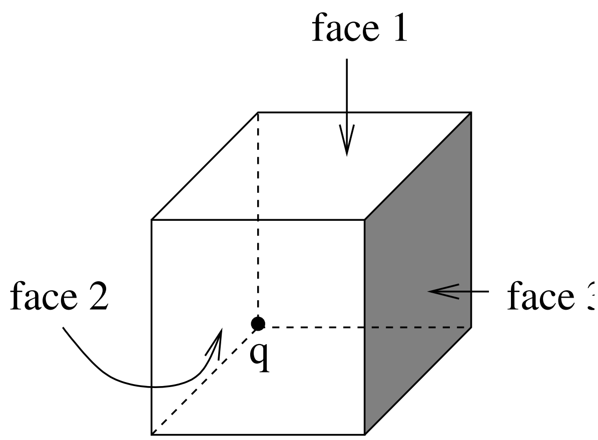

Figure 6: cube with a charge q at one of its corners.

charge q placed at one of the corner of the cube of fig. 6 through a surface enclosing the charge must be equal to q/ε_o . The flux of the electric field, for how it is defined ($\int \mathbf{E} \cdot d\mathbf{a}$), is different from zero only on the faces 1, 2 and 3 of the cube (see fig. 6; on the other 3 faces the electric field component orthogonal to the face vanishes) and for the symmetry of the problem has an *equal value* on each of the three faces. Since only 1/8 of the total flux q/ε_o goes through the cube one can then conclude that the flux of the electric field through the shaded face must be

$$\int_{face-3} \mathbf{E} \cdot d\mathbf{a} = \frac{q}{(8 * 3)\varepsilon_o} \quad .$$