Edwards Localization

Riccardo Fantoni[*]
Università di Trieste, Dipartimento di Fisica, strada Costiera 11, 34151 Grignano (Trieste), Italy

Félix Rose[†]
Laboratory of Theoretical Physics and Modelling, CY Cergy Paris Université., France
(Dated: November 9, 2025)

We study the localization problem in quantum stochastic mechanics. We start from the Edwards model for a particle in a bath of scattering centers and prove static localization of the ground state wavefunction of the particle in a one dimensional square well coupled to Dirac delta like scattering centers in arbitrary but fixed positions. We see how the localization increases for increasing coupling g. Then we choose the scattering centers positions as pseudo random numbers with a uniform probability distribution and observe an increase in the localization of the average of the ground state over the many positions realizations. We discuss how this averaging procedure is consistent with a picture of a particle in a Bose-Einstein condensate of of non interacting boson scattering centers interacting with the particle with Dirac delta functions pair potential. We then study the dynamics of the ground state wave function. We conclude with a discussion of the affine quantization version of the Lax model which reduces to a system of contiguous square wells with walls in arbitrary positions independently of the coupling constant g.

Keywords: Polaron localization; Anderson localization; Edwards localization

I. INTRODUCTION

...

II. EDWARDS MODEL

Consider [1] a d-dimensional system made of a particle of mass m in a periodic box of volume $\Omega = L^d$ interacting with N free spinless bosonic scattering centers of mass m_c at a density $\rho = N/\Omega$ and temperature $T < T_C$ with

$$T_C = \frac{2\pi\hbar^2}{m_c k_B}\sqrt{\frac{\rho}{\zeta(d/2)}}, \tag{2.1}$$

the critical temperature for Bose-Einstein condensation, where $\hbar$ is Planck constant, k_B is Boltzmann constant, and ζ is Riemann zeta function. Moreover let $v(\boldsymbol{r})$ be the pairwise interaction potential between the particle and the centers according to *Edwards model* [2].

Then the wave function of the whole system will be $\Psi(\boldsymbol{x}; \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)$ where $\boldsymbol{x}$ is the position of the particle and $\{\boldsymbol{r}_i\} = (\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)$ are the positions of the N scattering centers. If we neglect the interaction between the N bosons and the particle we may write the normalized wave function of the centers as follows

$$\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \frac{1}{\sqrt{N!}}\mathrm{perm}||\phi_j(\boldsymbol{r}_i)||, \tag{2.2}$$

in terms of the permanent of the N normalized wave functions of each center

$$\phi_j(\boldsymbol{r}_i) = \frac{1}{\sqrt{\Omega}} e^{i\boldsymbol{k}_j\cdot\boldsymbol{r}_i}, \tag{2.3}$$

with $\boldsymbol{k}_j = 2\pi\boldsymbol{n}_j/L$ and $\hbar\omega_j = (\hbar k_j)^2/2m_c$ his energy. Here $\boldsymbol{n}$ is a d-dimensional vector with integer components. Now below the critical temperature T_C the N centers will undergo condensation into the $\boldsymbol{n}_j = \boldsymbol{0}$ state. So that we will have

$$\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \propto \int_\Omega \prod_{j=1}^N \delta^{(d)}(\boldsymbol{r} - \boldsymbol{r}_j)\, d\boldsymbol{r}_1 d\boldsymbol{r}_2 \cdots d\boldsymbol{r}_N, \tag{2.4}$$

---

[*] riccardo.fantoni@scuola.istruzione.it
[†] felix.rose@cyu.fr

where $\delta^{(d)}$ is the Dirac d-dimensional delta function. In general we may expand the wave function Ψ into a basis of product states $\psi(\boldsymbol{x})\Phi(\{\boldsymbol{r}_i\})$. We may also define

$$\begin{aligned}\widetilde{\Psi}(\boldsymbol{x}) &= \langle\Psi(\boldsymbol{x};\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N)\rangle\\ &= \frac{1}{\Omega^N}\int_\Omega \Psi(\boldsymbol{x};\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N)\, d\boldsymbol{r}_1 d\boldsymbol{r}_2\ldots d\boldsymbol{r}_N.\end{aligned}\tag{2.5}$$

Clearly if we can neglect the interaction between the N bosons and the particle we will have $\widetilde{\Psi}\propto\psi$, but this is not true anymore in presence of a coupling between the particle and the scattering centers.

The Hamiltonian of the whole system

$$H=\frac{p^2}{2m}+\sum_j \hbar\omega_j+\sum_j v(\boldsymbol{x}-\boldsymbol{r}_j),\tag{2.6}$$

where $\boldsymbol{p}$ is the momentum of the particle, may also be rewritten, at $T<T_C$, as the following operator [1]

$$\begin{aligned}\hat{H} &= \frac{p^2}{2m}+\sum_j \hbar\omega_j \hat{b}_j^\dagger\hat{b}_j+\int_\Omega\sum_j v(\boldsymbol{x}-\boldsymbol{r}')\hat{b}_j^\dagger\hat{b}_j\, d\boldsymbol{r}'\\ &= \frac{p^2}{2m}+\int_\Omega\sum_j v(\boldsymbol{x}-\boldsymbol{r}')\hat{b}_j^\dagger\hat{b}_j\, d\boldsymbol{r}',\end{aligned}\tag{2.7}$$

where $\hat{b}_j^\dagger$ is the creation operator of scattering center j such that the number operator $\hat{n}_j=\hat{b}_j^\dagger\hat{b}_j$, for example, in his position $\boldsymbol{r}$ representation acts as follows $\hat{n}_j|0\rangle=\delta^{(d)}(\boldsymbol{r}-\boldsymbol{r}_j)$, with $|0\rangle$ the vacuum defined as the state that is annihilated by the destruction operator $\hat{b}_j$. In the second equality of Eq. (2.7) we explicitly used the fact that below the Bose-Einstein critical temperature the N boson scattering centers are all in their condensed phase at $\omega_j=0$ for all j. Here we are thinking at the condensed N scattering centers as being independent one from the other and non interacting among themselves so that Eq. (2.4) may be rewritten as

$$\Phi(\{\boldsymbol{r}_j\})=\left(\prod_j\int_\Omega \hat{n}_j\, d\boldsymbol{r}_j\right)|0\rangle.\tag{2.8}$$

Then the Hamiltonian of Eq. (2.6) is the result of the action of the operator $\hat{H}$ of Eq. (2.7) on the vacuum. Written in the form of Eq. (2.7) the Edwards Hamiltonian resembles the polaron Hamiltonian [3–5]. In particular the recipe of Eq. (2.5) of washing out the degrees of freedom of the scattering centers by averaging on their positions finds its justification in the need of a polaron description [5].

In the next Section III we will choose a particular form for v and will see how the averaging recipe of Eq. (2.5) favors localization.

III. THE LAX MODEL WITH CANONICAL QUANTIZATION

Lax and Phillips choose $d=1$ and $v(x)=g\delta(x)$ [6] with g the coupling constant between the particle and the boson scattering centers in their condensed phase so that the kinetic energy of the bosons can be neglected in the Hamiltonian of Eq. (2.6) as in Eq. (2.7)

$$H=-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+g\sum_{j=1}^N\delta(x-r_j),\tag{3.1}$$

where we set $\hbar=1$ and we may order $0<r_1<r_2<\ldots<r_N<L$ the positions of the N scattering centers in the periodic segment $[0,L[$ (a circle). We will then define $r_0=r_N$. In Appendix A of Ref. [1] it was shown that for an arbitrary choice of the N positions $\{r_i\}$, at large g, the eigenstates of the Hamiltonian (3.1), $\Psi_n(x;r_1,r_2,\ldots,r_N)$, become localized on the circle. Here $H\Psi_n(x;\{r_j\})=E_n(\{r_j\})\Psi_n(x;\{r_j\})$ where their eigenvalues, $E_n(\{r_j\})=k_n^2(\{r_j\})/2m$,

[1] Here we use the hat only for the operators acting on the scattering centers vacuum and not on the operators of the particle.

are determined from the periodic boundary condition $\Psi_n(0;\{r_i\}) = \Psi_n(L;\{r_i\})$, where $n = 0$ corresponds to the ground state and integers $n > 0$ to the excited states.

We here want to see whether the localization is *robust* against switching on of disorder, i.e. averaging over stochastic choices of the $\{r_i\}$. In a Monte Carlo spirit [7] we will then generate MN pseudo random numbers $\{r_i\}^k$, with $k = 1, 2, \dots, M$, ordered within $[0, L[$ and according to Eq. (2.5) we will measure

$$\begin{aligned}\widetilde{\Psi}_n(x) &= \langle \Psi_n(x; r_1, r_2, \dots, r_N)\rangle \\ &= \frac{1}{M}\sum_{k=1}^{M} \Psi_n(x; r_1^k, r_2^k, \dots, r_N^k),\end{aligned} \tag{3.2}$$

at fixed n. Together with

$$\begin{aligned}\widetilde{E}_n &= \langle E_n(r_1, r_2, \dots, r_N)\rangle \\ &= \frac{1}{M}\sum_{k=1}^{M} E_n(r_1^k, r_2^k, \dots, r_N^k).\end{aligned} \tag{3.3}$$

In Fig. 1 we show the ground state for three choices of increasing M at low $g = 1$ and high $g = 100$ for $N = 3$. As we can see the randomness in the positions of the scattering centers produces localization. And moreover increasing the coupling g with the scattering centers increases the localization of the averaged ground state wavefunction. From the figure we also see how at small g the amplitude of the localized averaged ground state wavefunction is much smaller than the amplitude of the unaveraged ground state, but the opposite behavior is observed at high coupling g.

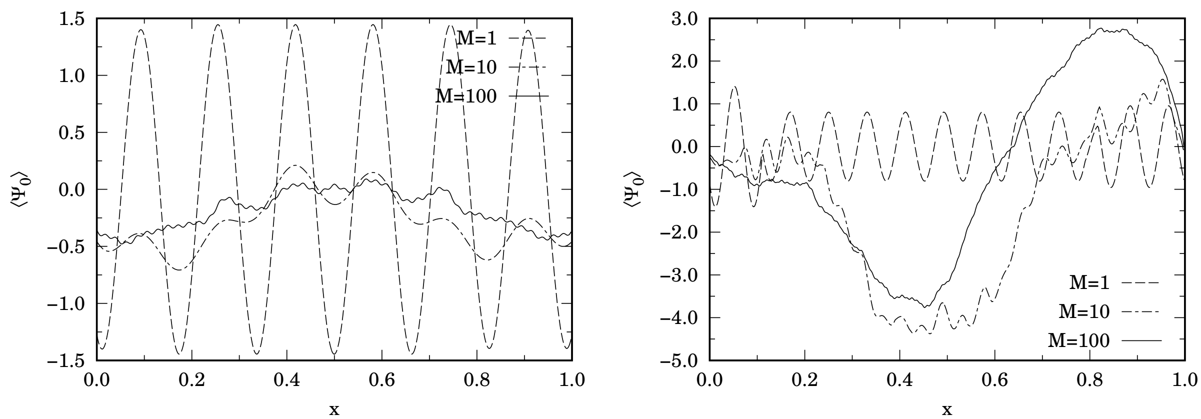

FIG. 1. We show the ground state from Eq. (3.2) for three choices of increasing M and $N = 3$. Low $g = 1$ in the left panel and high $g = 100$ in the right panel. From Eq. (3.3) the resulting ground state energy is as follows: for $g = 1 \longrightarrow (\widetilde{E}_0 = 741.956$ for $M = 1$, $\widetilde{E}_0 = 148.428$ for $M = 10$, and $\widetilde{E}_0 = 613.504$ for $M = 100)$ and for $g = 100 \longrightarrow (\widetilde{E}_0 = 3027.62$ for $M = 1$, $\widetilde{E}_0 = 2343.16$ for $M = 10$, and $\widetilde{E}_0 = 3698.51$ for $M = 100)$.

Dynamics

The dynamic evolution of the n-th eigenstate is as usual

$$\Psi_n(x;\{r_i\}|t) = e^{-iE_n(\{r_i\})t}\Psi_n(x;\{r_i\}), \tag{3.4}$$

according to the time, t, dependent Schrödinger equation $i\partial\Psi/\partial t = H\Psi$.

We may then initially think at the following severe short times Monte Carlo approximation

$$\begin{aligned}\widetilde{\Psi}_n(x|t) &= \langle \Psi_n(x;\{r_i\}|t)\rangle \\ &= \langle e^{-iE_n(\{r_i\})t}\Psi_n(x;\{r_i\})\rangle \\ &\approx \langle e^{-iE_n(\{r_i\})t}\rangle\langle\Psi_n(x;\{r_i\})\rangle \\ &\approx e^{-i\langle E_n(\{r_i\})\rangle t}\langle\Psi_n(x;\{r_i\})\rangle \\ &= e^{-i\widetilde{E}_n t}\widetilde{\Psi}_n(x),\end{aligned} \tag{3.5}$$

where the second approximation may be justified for a small times evolution. On the other hand, the first approximation is a rather severe one. Clearly it holds exactly only for $N = 0$. But from the approximation (3.5) follows that $|\widetilde{\Psi}_n(x|t)|^2 = |\widetilde{\Psi}_n(x)|^2$ independent of time.

We then see that in order to have a time dependent probability distribution it is essential to stick to the definition

$$|\widetilde{\Psi}_n(x|t)|^2 = |\langle \Psi_n(x; \{r_i\}|t)\rangle|^2. \tag{3.6}$$

In Fig. 2 we show the time evolution of the probability distribution for the ground state for $N = 3, g = 100$ with $M = 100$. As we can see from the figure the probability density initially localized around $x \sim 0.4$ and 0.8 gets localized only at $x \sim 0.8$ at intermediate times and then finally faints out at large times.

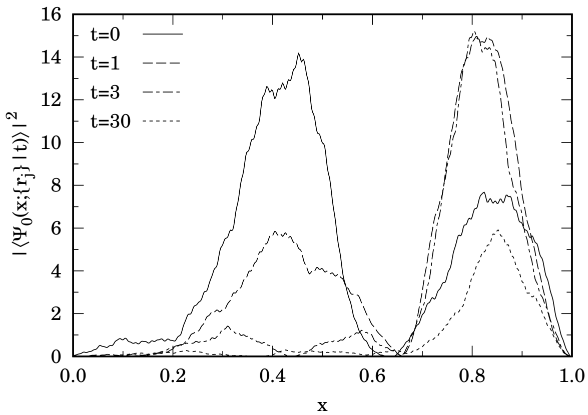

FIG. 2. We show the time evolution of the probability distribution of the ground state $|\widetilde{\Psi}_0(x|t)|^2$ from Eq. (3.6) for $N = 3, g = 100, M = 100$ at $t = 0, 1, 3, 30$.

IV. THE LAX MODEL WITH AFFINE QUANTIZATION

It would then be very interesting to repeat the calculation within *affine quantization* (see Appendix A in Ref. [8]) where

$$\mathscr{H} = \left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + v^{\text{aff}}(x)\right] + g\sum_{j=1}^{N}\delta(x - r_j), \tag{4.1}$$

$$v^{\text{aff}}(x) = \frac{3\left[\sum_j (x - a_j)\right]^2 + N\sum_j [b_j^2 - (x - a_j)^2]}{2m\left\{\sum_j [b_j^2 - (x - a_j)^2]\right\}^2}, \quad b_j = \frac{r_j - r_{j-1}}{2}, \quad a_j = \frac{r_j + r_{j-1}}{2}, \tag{4.2}$$

where the affine effective potential terms v_j^{aff} are the results of adopting the affine *dilation* operator $\mathscr{D}$, in place of the canonical momentum operator p,

$$\mathscr{D} = \sum_j \left\{p^\dagger[b_j^2 - (x - a_j)^2] + [b_j^2 - (x - a_j)^2]p\right\}/2, \tag{4.3}$$

in the affinely quantized Hamiltonian

$$\mathscr{H} = \mathscr{D}\left\{\sum_j [b_j^2 - (x - a_j)^2]\right\}^{-2}\mathscr{D} + g\sum_{j=1}^{N}\delta(x - r_j), \tag{4.4}$$

where the effective affine potential insures that the particle will live in any of the segments $(x - a_j)^2 < b_j^2$, i.e. $x \in]r_{j-1}, r_j[$, tunneling from one to the other through the delta function. From Eq. (4.1) follows the eigenstate

Schrödinger equation for an eigenvalue E

$$\left\{\sum_j [b_j^2 - (x-a_j)^2]\right\}^2 \left(\frac{\partial^2\Psi}{\partial x^2} + 2mE\Psi\right) = \left\{3\left[\sum_j (x-a_j)\right]^2 + N\sum_j [b_j^2 - (x-a_j)^2]\right\}\Psi, \tag{4.5}$$

where the delta function terms vanished. This decouples the particle from the bosonic scattering centers. This would tell that in affine quantization the eigenstates must have a continuous first derivative, i.e. they are smooth functions, and that the localization holds independently from g! Again we expect that making the scattering centers stochastic will increase the localization of the particle ground state wave function averaged over the disorder. But this is a complicated computational problem that we will leave open for the future.

V. CONCLUSIONS

...

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Funding

None declared.

[1] F. Rose and R. Schmidt, Disorder in order: Localization without randomness in a cold-atom system, Phys. Rev. A **105**, 013324 (2022).
[2] S. F. Edwards, A new method for the evaluation of electric conductivity in metals, Philosophical Magazine **3**, 33 (1958).
[3] R. Fantoni, Localization of acoustic polarons at low temperatures: A path integral Monte Carlo approach, Phys. Rev. B **86**, 144304 (2012).
[4] R. Fantoni, Low temperature acoustic polaron localization, Physica B **412**, 112 (2013).
[5] R. Fantoni, Polaron versus Anderson Localization, J. Chem. Phys. (2025), arXiV:2510.16434.
[6] M. Lax and J. C. Phillips, One-dimensional impurity bands, Phys. Rev. **110**, 41 (1958).
[7] M. H. Kalos and P. A. Whitlock, *Monte Carlo Methods* (John Wiley & Sons Inc., New York, 1986).
[8] J. R. Klauder and R. Fantoni, The Magnificent Realm of Affine Quantization: valid results for particles, fields, and gravity, Axioms **12**, 911 (2023).