Many Body in General Relativity: A thermal equivalence principle

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We review the physics of many bodies in the context of general relativity. Starting from the stress energy tensor for one body, for a swarm of bodies, for a perfect fluid, we review relativistic hydrodynamics, kinetic theory, and statistical physics of N identical bodies. We conclude our excursion with a *thermal equivalence principle* in physics.

Keywords: Particle; Swarm; Perfect Fluid; Hydrodynamics; Kinetic Theory; Statistical Mechanics; General Relativity; Equivalence Principle

CONTENTS

I. INTRODUCTION

An interesting problem in physics is to study the properties of a (quantum) *many body* system at low (non-zero) temperature on a *curved surface*. For example colloidal particles may be adsorbed or confined on a substrate with nonzero curvature, be it the wall of a porous material, or a membrane, a vesicle, a micelle for example made of ampiphilic surfactant molecules such as lipids, or a biological membrane, or the surface of a large solid particle, or an interface in an oil-water emulsion [1]. For a fluid of ^{4}He atoms it would be interesting to study the superfluid fraction.

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For a fluid of electrons it would be interesting to study the superfluidity. Moreover it would be interesting to study the properties of the electron plasma on a sphere in presence of a magnetic field.

One important point to discuss is whether the space in which the particles live is exactly two dimensional, as it happens in the satirical novella of Edwin Abbott Abbott [2], or if it can be treated as *quasi* two dimensional. There is a profound difference between the two scenarios to the point that the form of the interaction between the particles also changes. For example for colloidal particles one may choose the polarizable hard sphere pair interaction or for the fluid of helium atoms one may use the Lennard Jones pair potential, but the distance between the two interacting particles may be chosen either as the geodesic distance between them or the Euclidean distance in the three dimensional space where the surface is embedded. For the electron gas the Coulomb pair potential as a solution to the Poisson equation has different forms in two or three dimensions and in general depends on the metric of the curved surface.

These properties can be studied exactly with the path integral (Monte Carlo)method and these studies certainly enrich the knowledge on many bodies in (quantum) general relativity [3–7]. Not even the two body problem can be treated analytically in general relativity [8]. The problem of gravitating many bodies should be separated by the problem of many bodies with non-gravitational interactions in general relativity. In fact mass curves spacetime through the Einstein field equations and gravitating bodies will behave as free particles on that curved spacetime, whereas non-gravitational interactions produce particles accelerations on the spacetime. So being able to treat many (quantum) bodies on a curved surface would be an important step forward for the much more complicated problem of gravitating many (quantum) bodies in general relativity.

We find it of fundamental importance issuing a bridge between the two scientific communities of the exact simulations of a many body (quantum) system and of general relativity. We foresee an important progress in the physics of (quantum) gravitating many body systems beyond the simple ideal gases or hydrodynamic systems that are usually treated [9, 10]. We here review the physics of many bodies in the context of general relativity. Starting from the stress energy tensor for one body, for a swarm of bodies, for a perfect fluid, we review relativistic hydrodynamics, kinetic theory, and statistical physics of N identical bodies. We conclude our excursion with a *thermal equivalence principle* in physics.

In this work we consider *spacetime* as a smooth manifold $\mathscr{M}$ of dimension d and metric tensor $\boldsymbol{g}$ with covariant components $g_{\alpha\beta}$. We will denote with an arrow over a bold face letter the corresponding 4-vector and with just the bold face symbol the 3-dimensional vector. Greek indexes run over the d spacetime dimensions. Roman indexes run only over the $d-1$ space dimensions. We use Einstein summation convention of tacitly assuming a sum over repeated indexes.

One particle

For *one body* of mass m we have a self gravitating system with a stress energy tensor given by

$$T^{\alpha\beta}(\vec{\boldsymbol{x}}) = m \int u^\alpha u^\beta \delta^{(4)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{z}}(\tau))\, d\tau, \tag{1.1}$$

where τ is the body proper time, $d\vec{\boldsymbol{z}}/d\tau = \vec{\boldsymbol{u}} = (\gamma, \gamma\boldsymbol{v})$ with $u^0 = dt/d\tau = \gamma = (1-v^2)^{-1/2}$ and

$$T^{\alpha\beta}(\vec{\boldsymbol{x}}) = m\frac{u^\alpha u^\beta}{u^0}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{z}(t)), \tag{1.2}$$

where the body is at $\boldsymbol{z}(t)$ with velocity $\boldsymbol{v}(t)$ at time t.

Swarm of particles

For a *swarm* of N bodies all of the same mass m and $\boldsymbol{v}$

$$\begin{aligned} T^{\alpha\beta}(\vec{\boldsymbol{x}}) &= m u^\alpha u^\beta \sum_{i=1}^N \int \delta^{(4)}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{z}}_i(\tau_i))\, d\tau_i \\ &= m\frac{u^\alpha u^\beta}{u^0}\sum_{i=1}^N \delta^{(3)}(\boldsymbol{x} - \boldsymbol{z}_i(t)) \\ &= m u^\alpha u^\beta n, \end{aligned} \tag{1.3}$$

where

$$n = \frac{1}{u^0}\sum_{i=1}^{N} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{z}_i(t)), \tag{1.4}$$

is the proper number density of bodies measured in a *comoving frame* where $\vec{\boldsymbol{u}} = (1, \boldsymbol{0})$.

Perfect fluid

For a *perfect fluid* of proper number density n of non interacting bodies all of the same mass m and $v = |\boldsymbol{v}|$ but isotropic velocity profile $\boldsymbol{v} = v\boldsymbol{n}$

$$T^{\alpha\beta} = \chi\langle u^\alpha u^\beta\rangle_{\boldsymbol{n}}, \tag{1.5}$$

so that $T^{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $T^{00} = \chi\gamma^2$. Since $T^{00} = \rho = n(\gamma m)$ is the energy density of the fluid we require $\chi = mn/\gamma$. Then

$$\begin{aligned} T^{ij} &= \chi\gamma^2 v^2 \langle n^i n^j\rangle_{\boldsymbol{n}} \\ &= \chi\gamma^2 v^2 \frac{1}{3}\delta^{ij} \\ &= n(\gamma m) v^2 \frac{1}{3}\delta^{ij} \\ &= p\delta^{ij}, \end{aligned} \tag{1.6}$$

where δ is a Kronecker delta and in the second equality we used isotropy of $\boldsymbol{n}$ and

$$\begin{cases} \rho = n(m\gamma) \\ p = \frac{1}{3}\rho v^2 \end{cases} \tag{1.7}$$

are respectively the mass density and pressure in the *isotropic frame* of the fluid. Summarizing

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p\eta^{\alpha\beta}, \tag{1.8}$$

where $||\eta^{\alpha\beta}|| = \mathrm{diag}\{-1, 1, 1, 1\}$ is the metric in Minkowski spacetime. For photons $v = 1$ and $p = \rho/3$. For $v \ll 1$, $\rho = nm(1 + v^2/2 + \ldots)$, and $p \approx nmv^2/3 = (2/3)(\rho - nm) = (2/3)\epsilon$, where $\epsilon = (3/2)k_BT$ is the internal energy of a monatomic ideal gas in thermal equilibrium at a temperature T, k_B is Boltzmann constant, and $p = nk_BT$ is the ideal gas equation of state.

II. HYDRODYNAMICS

Hydrodynamics concerns itself with the study of the motion of fluids (liquids and gases). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. Therefore when we speak of the "point" of a fluid (or of an infinitesimal volume of it) we mean not a single molecule of the fluid but a volume element still containing very many molecules but yet small compared with the volume of the whole fluid.

A. Newtonian

A mathematical description of the state of a moving fluid consists in specifying the fluid velocity $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$ and any two thermodynamic functions pertaining to the fluid, for instance the pressure $p = p(t, \boldsymbol{x})$ and the density $\rho = \rho(t, \boldsymbol{x})$, from which one can determine all other thermodynamic quantities. These 5 quantities are functions of the coordinates $\boldsymbol{x} = (x, y, z)$ and of time t. Once again we stress that a point $\boldsymbol{r}$ in space at a given time t refers to a fixed point and not to specific particles of the fluid. From Chapter I of Ref. [11] we find

$$\frac{\partial\rho}{dt} + \boldsymbol{\nabla}(\rho\boldsymbol{v}) = 0, \tag{2.1}$$

$$\frac{\partial\boldsymbol{v}}{dt} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \frac{1}{\rho}\boldsymbol{\nabla}p, \tag{2.2}$$

$$\frac{\partial s}{dt} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})s = 0, \tag{2.3}$$

where the first equation is the *continuity equation*, the second is the *Euler equation*, and the third one is the *equation for the adiabatic flow* in which $s = s(t, \boldsymbol{x})$ is the entropy per particle.

From the first law of thermodynamics follows

$$d\epsilon = T\,ds - p\,d(m/\rho), \tag{2.4}$$
$$\epsilon = \epsilon(\rho, s), \tag{2.5}$$
$$p = \rho^2 \left.\frac{\partial \epsilon}{\partial \rho}\right|_s, \tag{2.6}$$
$$T = \left.\frac{\partial \epsilon}{\partial s}\right|_\rho, \tag{2.7}$$

where ϵ is the internal energy per particle. Eqs. (2.6) and (2.7) can be considered as algebraic relations for the right hand side of Eqs. (2.2) and (2.3) respectively.

For an ideal gas $\epsilon = \epsilon(T)$ and for a monatomic gas

$$s = k_B \ln(T^{3/2}/\rho) + \text{constant}. \tag{2.8}$$

B. Relativistic

We will work in a Local Lorentz Frame (LLF). Recalling that the stress energy tensor is divergenceless, from the stress energy tensor of a perfect fluid (1.8) we find

$$\begin{aligned} 0 = T^{\alpha\beta}{}_{,\beta} &= (\rho+p)_{,\beta} u^\alpha u^\beta + (\rho+p) u^\alpha{}_{,\beta} u^\beta + (\rho+p) u^\alpha u^\beta{}_{,\beta} + p_{,\beta}\eta^{\alpha\beta} \\ &= \frac{d(\rho+p)}{d\tau} u^\alpha + (\rho+p) a^\alpha + (\rho+p) u^\alpha u^\beta{}_{,\beta} + p,^\alpha, \end{aligned} \tag{2.9}$$

where the comma stands for a partial derivative. Multiplying by $\vec{\boldsymbol{u}}$ and recalling that $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{a}} = 0$ we find

$$\frac{d\rho}{d\tau} = -(\rho+p)\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}}, \tag{2.10}$$

which is the relativistic continuity expression which extends Eq. (2.1).

To find the extension of the Euler equation we introduce the projector tensor

$$\begin{aligned} P^{\alpha\beta} &= \eta^{\alpha\beta} + u^\alpha u^\beta \quad \text{for } \vec{\boldsymbol{u}} \text{ timelike} \quad \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1 \\ P^{\alpha\beta} &= \eta^{\alpha\beta} - n^\alpha n^\beta \quad \text{for } \vec{\boldsymbol{n}} \text{ spacelike} \quad \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{n}} = +1 \end{aligned}$$

Then

$$0 = P_{\alpha\gamma} T^{\alpha\beta}{}_{,\beta} = (\rho+p) a_\gamma + P_{\alpha\gamma} p,^\alpha, \tag{2.11}$$

or

$$(\rho+p)\vec{\boldsymbol{a}} = -\vec{\boldsymbol{\nabla}} p - \vec{\boldsymbol{u}} \frac{dp}{d\tau}, \tag{2.12}$$

which is the relativistic Euler equation which extends Eq. (2.2).

It is easy to see that in the Newtonian limit $\vec{\boldsymbol{u}} = (\gamma, \gamma \boldsymbol{v}) \approx (1, \boldsymbol{v})$ with $v \ll 1$ and $p \ll \rho$, Eq. (2.10) reduces to Eq. (2.1) and Eq. (2.12) reduces to Eq. (2.2).

Let us now discuss the continuity Eq. (2.10). First of all we observe that the mass density is not conserved $d\rho/d\tau \neq 0$. But the baryon, lepton, charge,… numbers are conserved. For example if we call $n = N/V$ the baryon number density in the rest frame of the fluid with N baryons in a volume V, N is certainly constant but V will change, so that

$$0 = \frac{dN}{d\tau} = \frac{d(nV)}{d\tau}, \tag{2.13}$$

but $(dV/d\tau)/V = \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}}$ (see Ex. 22.1 in Ref. [10]). So

$$\begin{aligned} 0 &= \frac{1}{V}\frac{d(nV)}{d\tau} \\ &= \frac{dn}{d\tau} + n\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}} \\ &= \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\nabla}} n + n\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}} \\ &= \vec{\boldsymbol{\nabla}}(n\vec{\boldsymbol{u}}), \end{aligned} \tag{2.14}$$

where we may define the divergenceless current density

$$\vec{\boldsymbol{J}} = n\vec{\boldsymbol{u}}. \tag{2.15}$$

Let us now discuss the thermodynamics. The second law tells that $ds/d\tau \geq 0$ where s is the entropy per baryon. The first law becomes

$$d(\rho/n) = -p\,d(1/n) + T\,ds, \tag{2.16}$$

or

$$d\rho = \frac{\rho + p}{n}\,dn + nT\,ds, \tag{2.17}$$

which is the relativistic extension of Eq. (2.4). In this equation the differential d can be substituted either with an exterior derivative $\widetilde{\boldsymbol{d}}$, with a gradient $\vec{\boldsymbol{\nabla}}$, or with a directional derivative $\vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{u}}} = u^\alpha \partial/\partial x^\alpha = d/d\tau$. Given an equation of state $\rho = \rho(n, s)$ we will have

$$p = n\left.\frac{\partial \rho}{\partial n}\right|_s - \rho, \tag{2.18}$$

$$T = \frac{1}{n}\left.\frac{\partial \rho}{\partial s}\right|_n, \tag{2.19}$$

which are the relativistic extensions of Eqs. (2.6) and (2.7).

It is easy to show that a perfect fluid flow is adiabatic. From the relativistic continuity Eqs. (2.10) and (2.14) follows

$$\frac{d\rho}{d\tau} = \frac{\rho + p}{n}\frac{dn}{d\tau}. \tag{2.20}$$

Then from the relativistic first thermodynamic Eq. (2.17) follows

$$\frac{ds}{d\tau} = 0. \tag{2.21}$$

Shock wave

Consider a homogeneous, static, perfect fluid. A sound wave in the fluid is an adiabatic perturbation. The speed of sound is

$$v_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_s \tag{2.22}$$

Expand

$$\rho = \rho_0 + \rho_1,$$
$$p = p_0 + p_1,$$
$$n = n_0 + n_1,$$

where ρ_0, p_0, n_0 are constant in space (uniform fluid) and in time (static fluid) and ρ_1, p_1, n_1 are small perturbations. Taking $\vec{\boldsymbol{u}} = (1, \boldsymbol{v}_1)$ with $v_1 \ll 1$ we find from the continuity Eq. (2.10)

$$\frac{\partial \rho_1}{\partial t} = -(\rho_0 + p_0)\boldsymbol{\nabla}\boldsymbol{v}_1, \tag{2.23}$$

and from the spatial part of Euler Eq. (2.12)

$$(\rho_0 + p_0)\frac{\partial \boldsymbol{v}_1}{\partial t} = -\boldsymbol{\nabla} p_1, \tag{2.24}$$

where we neglect the last term $\vec{\boldsymbol{u}}\,dp/d\tau = \vec{\boldsymbol{u}}\,u^\alpha \partial p/\partial x^\alpha$ because an infinitesimal of second order and $\partial p_0/\partial t = 0$. Therefore putting together Eqs. (2.23) and (2.24) we find

$$\frac{\partial^2 \rho_1}{dt^2} = -(\rho_0 + p_0)\boldsymbol{\nabla}\frac{\partial \boldsymbol{v}_1}{\partial t} = \boldsymbol{\nabla}^2 p_1. \tag{2.25}$$

In a perfect fluid $p = p(\rho, T)$ so that $p(\rho_0 + \rho_1, T) = p(\rho_0, T) + \partial p(\rho_0, T)/\partial\rho|_s \rho_1 = p_0 + p_1$ with $p_1 = v_c^2 \rho_1$ and we finally find

$$\frac{\partial^2 \rho_1}{dt^2} = v_s^2 \boldsymbol{\nabla}^2 \rho_1, \tag{2.26}$$

which is the shock wave equation.

Bernoulli equation

Consider a steady, adiabatic flow of a perfect fluid. Since in a steady state $\partial p/\partial t = 0$ from the relativistic Euler Eq. (2.12) follows

$$(\rho + p)\frac{du^0}{d\tau} = -u^0 \frac{dp}{d\tau}, \tag{2.27}$$

So

$$\begin{aligned}
\frac{d}{d\tau}\left(u^0 \frac{\rho + p}{n}\right) &= \frac{du^0}{d\tau}\frac{\rho + p}{n} + u^0 \frac{d}{d\tau}\left(\frac{\rho + p}{n}\right) \\
&= -\frac{u^0}{n}\frac{dp}{d\tau} + u^0 \frac{d}{d\tau}\left(\frac{\rho + p}{n}\right) \\
&= \frac{u^0}{n}\left[\frac{d\rho}{d\tau} - \frac{\rho + p}{n}\frac{dn}{d\tau}\right] = 0,
\end{aligned} \tag{2.28}$$

where in the second equality we used Eq. (2.27) and in the last equality we used the relativistic first law of thermodynamics Eq. (2.17). We then conclude that

$$u^0 \frac{\rho + p}{n} \quad \text{is constant along the fluid flow lines.} \tag{2.29}$$

In the Newtonian limit $\vec{\boldsymbol{u}} = (\gamma, \gamma \boldsymbol{v}) \approx (1, \boldsymbol{v})$ with $v \ll 1$, $u^0 = \gamma = (1 - v^2)^{-1/2} \approx 1 + v^2/2$, and $P \ll \rho$. We can take $\rho = \rho_0(1 + \pi)$ and $(\rho + p)/n \approx \rho_0(1 + \pi + p/\rho_0)/n$, so that

$$\frac{1}{2}v^2 + \pi + p/\rho_0 \quad \text{is constant along the fluid flow lines,} \tag{2.30}$$

where the sum of the last two terms is the *enthalpy*.

III. KINETIC THEORY APPROACH

The kinetic theory approach is based on a *one body* distribution function.

Distribution function

We will construct a Lorentz invariant phase space distribution function as a number density of particles in phase space

$$f = \frac{d\mathcal{N}}{d\boldsymbol{x}\, d\boldsymbol{p}}, \qquad \int f\, d\boldsymbol{x}\, d\boldsymbol{p} = N, \tag{3.1}$$

for a fluid of N bodies, where $d\boldsymbol{x} = dx^1 dx^2 dx^3$ and $d\boldsymbol{p} = dp^1 dp^2 dp^3$. So that f/N can be considered as a probability distribution function. We will now prove that f as defined above is a Lorentz invariant distribution. We start defining a proper 3-volume. The 4-volume $d^4\Omega = dx^0\, dx^1\, dx^2\, dx^3$ is invariant under a Lorentz transformation. Dividing by $d\tau$ we find another Lorentz invariant

$$dV = u^0\, dx^1\, dx^2\, dx^3. \tag{3.2}$$

Then we want to define a 3-volume element in momentum space. The 4-volume $d^4p = dp^0\, dp^1\, dp^2\, dp^3$ is invariant. Since $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ we will define

$$\begin{aligned} d\Pi &= \int d^4p\, \delta\left(\sqrt{-\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{p}}} - m\right) \\ &= \frac{\sqrt{(p^0)^2 - \boldsymbol{p}^2}}{p^0} dp^1\, dp^2\, dp^3 \\ &= \frac{m}{p^0}\, dp^1\, dp^2\, dp^3. \end{aligned} \tag{3.3}$$

And

$$dV\, d\Pi = dx^1\, dx^2\, dx^3\;\, dp^1\, dp^2\, dp^3, \tag{3.4}$$

is Lorentz invariant.

We will now prove conservation of volume in phase space in curved spacetime (see Liouville theorem in BOX 22.6 of Ref. [10]). Consider a very small bundle of identical particles that move through curved spacetime on a neghboring geodesics. We want to prove that $d(dV\, d\Pi)/d\lambda = 0$ where λ is an affine parameter along the central geodesic of the bundle. Given any function of phase space $g(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})$, if $m \neq 0$ [1] take $\tau = a\lambda + b$ for arbitrary a and b. Then

$$\frac{dg(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})}{d\lambda} = \frac{\partial g}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} + \frac{\partial g}{\partial p^\alpha}\frac{dp^\alpha}{d\lambda}, \tag{3.5}$$

on a geodesic $dp^\alpha/d\tau = 0$ so

$$\frac{dg(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})}{d\lambda} = \frac{\partial g}{\partial x^\alpha} p^\alpha \frac{a}{m}, \tag{3.6}$$

and for $g = dV\, d\Pi$ and $p^\alpha = m dx^\alpha/d\tau$

$$\frac{dg}{d\lambda} = a\frac{dg}{d\lambda}, \tag{3.7}$$

for any a. so $dg/d\lambda = 0$. Since $d\mathscr{N}$ and $dV\, d\Pi$ are unchanged then also $f = d\mathscr{N}/dV\, d\Pi$ is unchanged

$$\frac{df}{d\lambda} = 0. \tag{3.8}$$

This equation is at the heart of the collisionless Boltzmann equation and the Vlasov equation. All these approximate theories are valid at sufficiently low density. Whereas for the full Boltzmann equation [12, 13] one has

$$\frac{df}{d\lambda} = \left(\frac{\partial f}{\partial \lambda}\right)_{\text{collisions}}. \tag{3.9}$$

This is the most famous of all Kinetic equations and was obtained by Boltzmann more than a century ago.

The phase space probability density of a system in thermodynamic equilibrium at an inverse temperature $\beta = 1/k_B T$ with k_B Boltzmann constant, is not an explicit function of proper time. We shall use the symbol f_0 to denote the equilibrium probability density.

From §55 of Ref. [14] we know that, on a comoving frame with $\vec{\boldsymbol{u}} = (1, \boldsymbol{0})$, we can write

$$f_0(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = \frac{d\mathscr{N}}{dV\, d\Pi} = \frac{g}{h^3}\frac{1}{e^{-\beta(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{u}}+\mu)} - \varepsilon}, \tag{3.10}$$

where h is Planck constant, μ is the chemical potential, $g = 2J + 1$ is the spin J degeneracy (2 polarizations for photons), and

$$\varepsilon = \begin{cases} +1 & \text{Bose-Einstein statistics} \\ 0 & \text{Maxwell-Boltzmann statistics} \\ -1 & \text{Fermi-Dirac statistics} \end{cases} \tag{3.11}$$

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[1] If $m = 0$ see BOX 22.6 of Ref. [10].

Moments of the distribution function

Next we can take moments of f_0 with respect to $\vec{\boldsymbol{p}}$

$$\int f_0 p^\mu \, d\Pi = J^\mu, \tag{3.12}$$

$$\int f_0 p^\mu p^\nu \, d\Pi = T^{\mu\nu}, \tag{3.13}$$

where here $d\Pi = d\boldsymbol{p}/p^0$ with $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$. For $\vec{\boldsymbol{u}} = (1, \boldsymbol{0})$, from Eqs. (2.15) and (1.8), we must have

$$J^\mu = n u^\mu, \tag{3.14}$$

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p \eta^{\mu\nu}. \tag{3.15}$$

So

$$\begin{aligned} n &= -J^\mu u_\mu = -\int f_0 p^\mu u_\mu \, d\Pi = \int f_0 \, d\boldsymbol{p} \\ &= \frac{g}{h^3} \int_0^\infty \frac{4\pi p^2 \, dp}{e^{\beta[\sqrt{\boldsymbol{p}^2+m^2}-\mu]} - \varepsilon}. \end{aligned} \tag{3.16}$$

Introduce the following change of variables

$$\begin{cases} p = m \sinh\chi \\ \bar{\beta} = m\beta \end{cases} \tag{3.17}$$

so that from Eq. (3.16) we find

$$n = \frac{4\pi g m^3}{h^3} \int_0^\infty \frac{\sinh^2\chi \, \cosh\chi \, d\chi}{e^{[\bar{\beta}\cosh\chi - \beta\mu]} - \varepsilon}. \tag{3.18}$$

For the pressure

$$\begin{aligned} p &= \frac{1}{3}(u_\mu u_\nu + \eta_{\mu\nu}) T^{\mu\nu} \\ &= \frac{1}{3} \int f_0 \boldsymbol{p}^2 \, d\Pi \\ &= \frac{1}{3} \int f_0 \boldsymbol{p}^4 \, \frac{d\boldsymbol{p}}{p^0} \\ &= \frac{4\pi g m^4}{3h^3} \int_0^\infty \frac{\sinh^4\chi \, d\chi}{e^{[\bar{\beta}\cosh\chi - \beta\mu]} - \varepsilon}. \end{aligned} \tag{3.19}$$

Also

$$\begin{aligned} \rho - 3p = -T^\alpha{}_\alpha &= m^2 \int f_0 \, \frac{d\boldsymbol{p}}{p^0} \\ &= \frac{4\pi g m^4}{h^3} \int_0^\infty \frac{\sinh^2\chi \, d\chi}{e^{[\bar{\beta}\cosh\chi - \beta\mu]} - \varepsilon}. \end{aligned} \tag{3.20}$$

Maxwell-Boltzmann statistics ($\varepsilon = 0$)

From Ref. [15] we learn that

$$K_n(\bar{\beta}) = \frac{\bar{\beta}^n}{(2n-1)!!} \int_0^\infty d\chi \, \sinh^{2n}\chi \, e^{-\bar{\beta}\cosh\chi} \tag{3.21}$$

$$= \frac{\bar{\beta}^{n-1}}{(2n-3)!!} \int_0^\infty d\chi \, \sinh^{2n-2}\chi \cosh\chi \, e^{-\bar{\beta}\cosh\chi}, \tag{3.22}$$

where K_n is a modified Bessel function of the second kind and in the second equality we performed an integration by parts. The asymptotic behaviors of the modified Bessel function are as follows

$$K_n(\bar\beta) = \sqrt{\frac{\pi}{2\bar\beta}}e^{-\bar\beta}\left[1+\frac{4n^2-1}{8\bar\beta}+O(\bar\beta^{-2})\right] \quad \bar\beta\gg 1, \tag{3.23}$$

$$K_n(\bar\beta) = \frac{(n-1)!}{\bar\beta^n}\left[2^{n-1}-\frac{2^{n-3}\bar\beta^2}{n-1}+O(\bar\beta^3)\right] \quad \bar\beta\ll 1. \tag{3.24}$$

We then find

$$n = aK_2(\bar\beta)/\bar\beta, \tag{3.25}$$

$$p = amK_2(\bar\beta)/\bar\beta^2 = n\,k_BT, \tag{3.26}$$

$$\rho-3p = amK_1(\bar\beta)/\bar\beta, \tag{3.27}$$

where $a=4\pi gm^3e^{\beta\mu}/h^3$. Note that the ideal gas equation of state (3.26) is a relativistic invariant.

For the internal energy per particle we then find

$$u(T)=\frac{\rho}{n}=m\frac{K_1(\bar\beta)}{K_2(\bar\beta)}+3k_BT=\begin{cases} m\left[1+\dfrac{3}{2}\dfrac{k_BT}{m}+\ldots\right] & \bar\beta\gg 1,\\ 3k_BT & \bar\beta\ll 1,\end{cases} \tag{3.28}$$

where we used the asymptotic expansions (3.23) and (3.24).

For the ratio of the specific heats $\gamma(T)=c_p/c_v$ we then find

$$\gamma(T)=\frac{\left.\frac{du}{dT}\right|_p}{\left.\frac{du}{dT}\right|_v}=1+\frac{k_B}{\left.\frac{du}{dT}\right|_v}=\begin{cases} 5/3 & \bar\beta\gg 1,\\ 4/3 & \bar\beta\ll 1.\end{cases} \tag{3.29}$$

IV. STATISTICAL MECHANICS APPROACH

The statistical mechanics approach is based on a *many body* distribution function.

Thermal equilibrium of the many bodies

The aim of equilibrium statistical mechanics is to calculate observable properties of a system of interest either as averages over a phase trajectory (the method of Boltzmann), or as averages over an ensemble of systems, each of which is a replica of the system of interest (the method of Gibbs). In Gibbs's formulation of statistical mechanics the equilibrium distribution of phase points of systems of N bodies of the ensemble is described by a phase space probability density $f_0^{(N)}$. The quantity $f_0^{(N)}\prod_{i=1}^N d\boldsymbol{x}_i\,d\boldsymbol{p}_i$ is proportional to the probability that the physical system is in a microscopic state represented by a phase point lying in the infinitesimal, $6N$-dimensional invariant phase space element $\prod_{i=1}^N d\boldsymbol{x}_i\,d\boldsymbol{p}_i=d^NV\,d^N\Pi$. So that now, for N identical bodies, in the canonical Gibbs ensemble [13],

$$f_0^{(N)}=\frac{d\mathscr{N}}{d^NV\,d^N\Pi}, \qquad Z_N=\mathrm{tr}\left(f_0^{(N)}\right)=\int f_0^{(N)}\,d^NV\,d^N\Pi=N!\mathscr{Z}_N\Lambda^{-3N}, \tag{4.1}$$

where $\mathrm{tr}(\ldots)$ denotes a trace, and $\Lambda=\sqrt{4\pi\lambda\beta}$ is the de Broglie thermal wavelength, with $\lambda=\hbar^2/2m$. Here [16]

$$f_0^{(N)}\propto\exp\left(-\int_{\partial\Omega}T^{\mu\nu}\beta_\nu\,dS_\mu\right), \tag{4.2}$$

where $\partial\Omega$ is a general, arbitrary, spacelike hypersurface bounding the 4-volume Ω and $\vec{\boldsymbol{\beta}}(\boldsymbol{x})$ is a 4-vector such that $\beta=\sqrt{\beta_\mu\beta^\mu}$ and as usual $1/k_B\beta(\boldsymbol{x})=T(\boldsymbol{x})$ the invariant absolute temperature, i.e. the temperature measured by a comoving thermometer. In particular $\int T^{00}\,dV=\mathscr{H}$ with $\mathscr{H}=\mathscr{K}+\mathscr{V}$ the Hamiltonian operator of the fluid where $\mathscr{K}$ is the kinetic energy operator of the N bodies. The covariant form of Eq. (4.2) of the equilibrium statistical operator was first used by Weldon [17] for the Belinfante symmetrized stress energy tensor.

And

$$\mathscr{Z}_N=\int e^{-\beta\mathscr{V}}\,d^NV. \tag{4.3}$$

Integrating on the phase space of $N-1$ particles we find

$$\int f_0^{(N)}\, d^{N-1}V\, d^{N-1}\Pi = (N-1)! f_0^{(1)}, \tag{4.4}$$

where $f_0^{(1)} = f_0$ is the one body equilibrium distribution function (3.10) of the previous Section III.

In a recent project [18] we studied an electron gas at low temperatures, the *Jellium*, on the surface of a sphere through the path integral Monte Carlo method. A unit sphere of radius is *the* surface [2] of constant positive scalar curvature 2. In Figure 1 we show some snapshots of the paths of the electrons during the simulation from which we can see how the conformation of the electron path changes shape moving from the equator to the poles. This is a

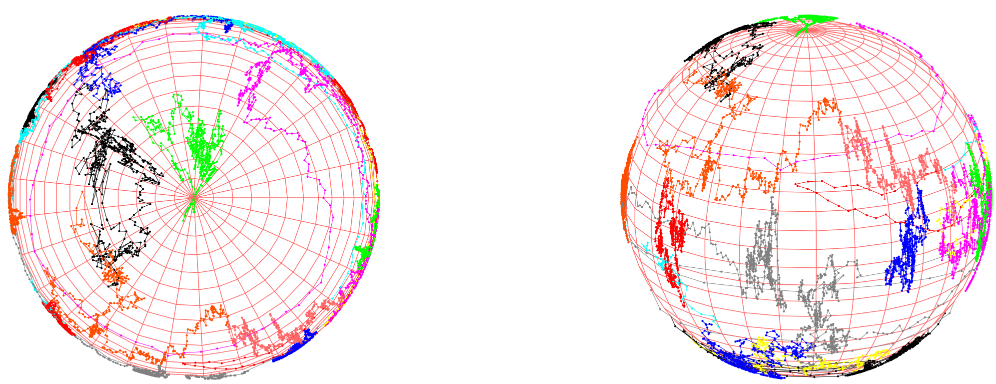

FIG. 1. Snapshot of the macroscopic path during a simulation of Ref. [18]. The different paths have different colors. As we can see the paths conformations differ from moving from the equator to the poles. We predict that this is a topological consequence of the hairy ball theorem.

consequence of the *hairy ball theorem*, according to which the Euler class is the obstruction to the tangent plane of the sphere having a nowhere vanishing *section*, i.e. fiber or hair. The theorem was first proven by Henri Poincaré for the sphere in 1885 [19], and extended to higher even dimensions in 1912 by Luitzen Egbertus Jan Brouwer [20]. The theorem has been expressed colloquially as "you can't comb a hairy ball flat without creating a cowlick" or "you can't comb the hair on a coconut" as shown in Fig. 1. If z is a continuous function that assigns a vector in the three dimensional space to every point $\mathscr{P}$ on a sphere such that $z(\mathscr{P})$ is always tangent to the sphere at $\mathscr{P}$, then there is at least one pole, a point where the field vanishes, i.e. a $\mathscr{P}$ such that $z(\mathscr{P}) = 0$. Every zero of a vector field has a (non-zero) *index* [3], and it can be shown that the sum of all of the indexes at all of the zeros must be two, because the Euler characteristic of the sphere is two. Therefore, there must be at least one zero. This is a consequence of the *Poincaré-Hopf theorem*. The theorem was proven for two dimensions by Henri Poincaré and later generalized to higher dimensions by Heinz Hopf [21]. In particular we see how, even a single free particle have a path which will be subject to some anisotropy due to the effective potential induced by the curvature of the sphere. This effect was studied in Refs. [18, 22].

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[2] Being a manifold of dimension $2 < 3$ it is conformally flat.

[3] The index of a bilinear function/al is the dimension of the space on which it is negative definite. According to Morse theorem, from the calculus of variations, there is a relation between the conjugate points (a point of the path where the path cease to be a minimum of the action) along a classical path to the negative eigenvalues of $\delta^2 S$. More precisely Morse index theorem states that, for an extremum $R(t), 0 < t < \beta$ of S, the index of $\delta^2 S$ is equal to the number of conjugate points to $R(0)$ along the path $R(t)$ (each such conjugate point is counted with its multiplicity). In the context of vector fields on a Riemannian manifold the index is equal to $+1$ around a source or a sink, and more generally equal to $(-1)^k$ around a saddle that has k contracting dimensions and $n-k$ expanding dimensions.

Thermal equilibrium of the metric tensor

A different story is to move the temperature from the stress energy tensor to the metric tensor as is done in Refs. [3, 7] and in the trilogy [4–6] also applied to study the vacuum in cosmic space in Ref. [23]. That is, to move the statistical physics description from the right hand side of Einstein field equations

$$G_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{4.5}$$

to the left hand side. Of course the two descriptions has to give the same picture. In Ref. [4] we took the statistical average of the trace of Einstein field equations

$$\langle -R\rangle_g=\kappa\langle {T_\mu}^\mu\rangle_t. \tag{4.6}$$

where $\kappa=8\pi G/c^4$ and $R=-{G_\mu}^\mu$ is the scalar curvature, $\langle\ldots\rangle_g$ is a statistical average on the metric tensor, and $\langle\ldots\rangle_t$ is a time average. On the right hand side, replacing the time average with an ensemble average, we find

$$\langle z_\mu T^{\mu\nu}\rangle_t=\frac{\mathrm{tr}\left(f_0^{(N)}z_\mu T^{\mu\nu}\right)}{Z_N}=-\frac{\delta}{\delta\beta_\nu(\boldsymbol{x})}\ln Z_N. \tag{4.7}$$

In the above formula, while the left hand side depends on a arbitrary vector $\vec{\boldsymbol{z}}$, the right hand side is not manifestly dependent on it. In fact, the functional derivative of Z_N of Eq. (4.1) includes a hidden dependence on the normal vector as the functional derivation implies the choice of a measure, hence of a hypersurface and a corresponding normal vector. We will also have

$$\langle {T_\mu}^\mu\rangle_t=-\frac{\delta}{\delta\beta}\ln Z_N, \tag{4.8}$$

where remember that $Z_N=N!\mathscr{Z}_N\Lambda^{-3N}$. Then the virial theorem of Eq. (4.6) can be rewritten as

$$\langle R\rangle_g=\kappa\frac{\delta}{\delta\beta}\ln Z_N, \tag{4.9}$$

For an ideal, non interacting, $\mathscr{V}=0$, gas we then find $\mathscr{Z}_N=V^N$ and from Eqs. (4.1) and (4.3)

$$\langle R\rangle_g=\kappa\frac{1}{V}\frac{\partial}{\partial\beta}\ln\left[N!\left(\frac{V}{\Lambda^3}\right)^N\right]=-3n\kappa\frac{\partial}{\partial\beta}\ln\Lambda=-\frac{3n}{2\beta}\kappa, \tag{4.10}$$

where the functional derivative has been replaced by a partial derivative and V/Λ^3 is the single particle translational partition function, familiar from elementary statistical mechanics.

In Ref. [4] we defined a *virial* inverse temperature $\tilde{\beta}$ stemming from the thermal fluctuations of the metric tensor, as

$$\tilde{\beta}^{-1}(\boldsymbol{x})=\frac{\tilde{v}}{4}\langle {T_\mu}^\mu\rangle_t, \tag{4.11}$$

where $\tilde{v}$ is a positive constant. Therefore we find the following equivalence

$$\tilde{\beta}^{-1}(\boldsymbol{x})=\frac{3n\tilde{v}}{8}\beta^{-1}(\boldsymbol{x}), \tag{4.12}$$

or $\tilde{T}(\boldsymbol{x})=T(\boldsymbol{x})$ with

$$\tilde{k}_B=\frac{3n\tilde{v}}{8}k_B, \tag{4.13}$$

where $n\tilde{v}$ is an intensive quantity, if fact $\tilde{v}$ is a local volume [4].

Thermal equivalence principle

This equivalence proves that Einstein field equations offer a symmetric way to study statistical physics where one can either work at the level of the many body system encoded in the stress energy tensor on the right hand side of (4.5) or at the level of the thermal fluctuations of the metric tensor on the left hand side of (4.5). The two descriptions are *equivalent*. This is a *thermal equivalence principle* in physics: "given a many body system in general relativity its thermal equilibrium properties derived from its statistical physics description are equivalent to the properties of a statistical physics description of the metric of the spacetime that it influences and viceversa." In other words: "A statistical ensemble of bodies goes into thermal equilibrium with the spacetime it occupies."

V. CONCLUSIONS

In this work we determined a thermal equivalence principle for a statistical theory of gravitation. First of all it is important to realize that at low temperatures a statistical theory of gravity will necessarily put together the quantum world with our Universe ruled by general relativity. Outer space, or simply space, is the expanse that exists beyond Earth's atmosphere and between celestial bodies. It contains very low particle densities, constituting a near perfect vacuum of predominantly hydrogen and helium plasma, permeated by electromagnetic radiation, cosmic rays, neutrinos, magnetic fields and dust. The baseline temperature of outer space, as set by the background radiation from the Big Bang, is ≈ 2.7 K. Intergalactic space takes up most of the volume of the universe, but even galaxies and star systems consist almost entirely of empty space. Most of the remaining mass-energy in the observable universe is made up of an unknown form, dubbed dark matter (60% of the Universe) and dark energy (27% of the Universe) [23].

The program of constructing a well defined statistical theory of our Universe is one of the greatest challenges of contemporary physics which had been foreseen by Einstein in his renown iconic phrase "God doesn't play dice". From the point of view of the challenge that it offers to mathematics one needs a way to create a bridge between the variational theory of functional integrals or more specifically path integrals and differential geometry or more specifically Riemannian geometry. From this point of view it seems natural to predict that differential topology will play a crucial role. Recently we carried out some path integral (Monte Carlo) simulations for Jellium (an electron plasma at low temperature) on the surface of a sphere, probably the simplest of all curved smooth manifolds. And already in that study we found important topological effects on the electrons paths. It is important to realize that this kind of calculations can be considered as toy simulations for a many body system on a more complex smooth manifold as needed by spacetime in general relativity.

In this work we show that the temperature as defined in this kind of statistical physics studies of many bodies plays the same role as the one that can be defined by a path integral on the spacetime metric that was introduced in Ref. [4]. This is perfectly natural from the point of view of the linear constraint given by the Einstein field equations. This symmetry between the statistical physics of many body matter in the Universe and a statistical physics theory of the metric tensor where the material lives, offers naturally a thermal equivalence principle that states that the material and the spacetime are in thermal equilibrium one another.

AUTHOR DECLARATIONS

Conflicts of interest

None declared.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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[1] R. Fantoni, J. W. O. Salari, and B. Klumperman, The structure of colloidosomes with tunable particle density: Simulation vs experiment, Phys. Rev. E **85**, 061404 (2012).
[2] E. A. Abbott, *Flatland: A Romance of Many Dimensions* (Seeley & Co., London, 1884).
[3] J. R. Klauder and R. Fantoni, The Magnificent Realm of Affine Quantization: valid results for particles, fields, and gravity, Axioms **12**, 911 (2023).
[4] R. Fantoni, Statistical Gravity through Affine Quantization, Quantum Rep. **6**, 706 (2024).
[5] R. Fantoni, Statistical Gravity and entropy of spacetime, Stats **8**, 23 (2025).
[6] R. Fantoni, Statistical Gravity, ADM splitting, and Affine Quantization, Gravitation and Cosmology **31** (2025).

[7] R. Fantoni, *Unifying Classical and Quantum Physics - How classical and quantum physics can pass smoothly back and forth* (Springer, New York, 2025).
[8] R. Fantoni, Relaxation in scalar gravitational field theory, Gravitation and Cosmology **32**, 2026 (2025).
[9] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars. The Physics of Compact Objects* (John Wiley & Sons Inc, New York, 1983).
[10] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[11] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, Course of Theoretical Physics, Vol. 6 (Butterworth Heinemann, 1959) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.
[12] P. Résibois and M. DeLeener, *Classical Kinetic Theory of Fluids* (John Wiley, New York, 1977).
[13] J. P. Hansen and I. R. McDonald, *Theory of simple liquids*, 2nd ed. (Academic Press, Amsterdam, 1986) section 3.3.
[14] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Course of Theoretical Physics, Vol. 5 (Butterworth Heinemann, 1951) translated from the Russian by J. B. Sykes and M. J. Kearsley, edited by E. M. Lifshitz and L. P. Pitaevskii.
[15] M. Abramowitz and A. Stegun, *"Handbook of mathematical functions"* (Dover, New York, 1965).
[16] F. Becattini, Covariant Statistical Mechanics and the Stress-Energy Tensor, Phys. Rev. Lett. **108**, 244502 (2012).
[17] H. A. Weldon, Covariant calculations at finite temperature: The relativistic plasma, Phys. Rev. D **26**, 1394 (1982).
[18] R. Fantoni, One-component fermion plasma on a sphere at finite temperature. The anisotropy in the paths conformations, J. Stat. Mech. , 083103 (2023).
[19] H. Poincaré, Sur les courbes dfinies par les qúations diffŕentielles, Journal de Mathématiques Pures et Appliqués **4**, 167 (1885).
[20] L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, Mathematische Annalen **71**, 97 (1912).
[21] H. Hopf, Vektorfelder in n-dimensionalen Mannigfaltigkeiten, Math. Ann. **96**, 209 (1926).
[22] R. Fantoni, One-component fermion plasma on a sphere at finite temperature, Int. J. Mod. Phys. C **29**, 1850064 (2018).
[23] R. Fantoni, Temperature of the Vacuum, Phys. Lett. B (2025), under submission; arXiv:2510.16383.