Homework # 3 Solutions

set of units used: MKSA

-Problem 1- Consider this static vector potential given in spherical coordinates

$$\begin{cases} A_r = 0 \\ A_\theta = 0 \\ A_\phi = \dfrac{g}{r} \tan\left(\dfrac{\theta}{2}\right) \end{cases} \tag{1}$$

Note that $\mathbf{A}$ is not defined at $\theta = \pi$ (the $-\hat{z}$ axis). Calculate the $\mathbf{B}$ field that this $\mathbf{A}$ describes.

SOLUTION

We have to take the curl in spherical coordinates of the vector potential

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\ A_\phi) - \frac{\partial}{\partial\phi}A_\theta\right]\hat{r} + \tag{2}$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \tag{3}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\hat{\phi} \quad . \tag{4}$$

Inserting the vector potential (1) one gets

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{g}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\tan(\theta/2)\right)\hat{r} = \frac{g}{r^2}\hat{r} \quad . \tag{5}$$

One immediately recognizes in $\mathbf{B}$ the magnetic field of a magnetic monopole.

COMMENT: **the meaning of the singularity**. The vector potential given in the problem can be obtained from the following argument

$$\mathbf{A}(\mathbf{r}) = \int_L d\mathbf{A}(\mathbf{r}) \quad , \tag{6}$$

where

$$d\mathbf{A}(\mathbf{r}) = -g\mathbf{dl}' \times \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \quad , \tag{7}$$

is the vector potential in $\mathbf{r}$ due to an elementary magnetic dipole $\mathbf{dm} = g\mathbf{dl}'$ in $\mathbf{r}'$, and L is the negative $\hat{z}$ axis.

This construction suggests the following picture for a monopole: the magnetic charge g is the ending point of a "string" of elementary magnetic dipoles which extends to infinity as shown in figure 1. The vector potential gener-

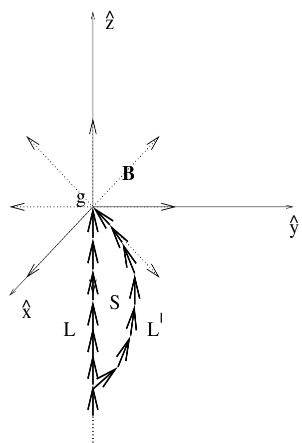

Figure 1: Description of a magnetic monopole g. S is a surface contained between the two strings L and L'.

ated by the magnetic charges g is singular on the string. as we know. This definition [1] allows to use for the electromagnetic interactions necessary to describe the dynamics and statics of a monopole, the same structure of the ones used in the decription of the electric monopole, $\mathbf{B} = \nabla \times \mathbf{A}$, etc.... In particular the strange fact that the total flux through a closed surface surrounding a magnetic charge g is $4\pi g$ (which wouldn' t allow to use $\nabla\mathbf{B} = 0$) is prevented assuming the existence of a magnetic field $\mathbf{B}'$ very intense on the string of dipoles and zero outside. $\mathbf{B}'$ is there just to compensate the flux exiting from the edge of of the string: the monopole. The magnetic field due

[1] Used by Dirac in his original argument to show how the electric charge quantization is a necessary consequenc of the assumption of the axistence of the magnetic monopole.

to the magnetic charge will be then

$$\mathbf{B}_{monopole} = \nabla \times \mathbf{A} - \mathbf{B}' \quad . \tag{8}$$

One can show that the freedom in choosing the string of dipoles from a point at infinite and the monopole is equivalent to the freedom due to the gauge tranformations. In particular changing the string L to the string L' (see figure 1) produces the following change in the vector potential describing the monopole

$$\mathbf{A}_{L'}(\mathbf{r}, t) = \mathbf{A}_L(\mathbf{r}, t) + g\nabla\Omega_S(\mathbf{r}, t) \tag{9}$$

where $\Omega_S(\mathbf{r}, t)$ is the solid angle with vertex in $\mathbf{r}$ which sees the surface S (see figure **??**). Try to derive formula (9) using equation (6) as an exercise !

-Problem 2- Suppose that a current $\mathbf{j}$ flows radially outward from a very small region at the origin

$$\mathbf{j} = \frac{J}{4\pi}\frac{\hat{r}}{r^2} \tag{1}$$

(a) Calculate the time derivative of the charge density, ρ, at some radius $r \neq 0$.

(b) Calculate the time derivative of the total charge contained in the small region at the origin.

(c) Currents produce magnetic fields, don' t they ? Use Ampere' s law to calculate the magnetic field at some $r \neq 0$. Which direction does it point ? (Hint: this is a trick question.)

SOLUTION

Let' s start with a **mathematical note**: We know from electrostatic that the more general solution of the equation [1]

$$\nabla^2\phi(\mathbf{r}) = -4\pi\delta(\mathbf{r}) \quad , \tag{2}$$

is, apart for a constant, the Coulomb potential

$$\phi(\mathbf{r}) = \frac{1}{|\mathbf{r}|} + constant \quad , \tag{3}$$

thus,

$$\mathbf{E} = -\nabla\phi(\mathbf{r}) = \frac{\hat{r}}{|\mathbf{r}|^2} \quad . \tag{4}$$

On the other way around we can say that (4) is the more general solution of the equation

$$\nabla\mathbf{E} = 4\pi\delta(\mathbf{r}) \quad , \tag{5}$$

[1] $\delta(\mathbf{r})$ is the Dirac delta function which physically represents the charge number density of a charge distribution composed by one charge on the origin. It is mathematically defined by the following properties:

i. $\delta(\mathbf{r}) = 0$ for $\mathbf{r} \neq 0$

ii. $\int_\Omega \delta(\mathbf{r})d\mathbf{r} = 1$ for every neighbourhoood Ω of the origin.

when we look for a solution which can be written as the gradient of some function.

Coming to the problem, for the current density in eq. (1) one has

(a) From the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \mathbf{j} = -J\delta(\mathbf{r}) \quad , \tag{6}$$

then $\partial\rho/\partial t = 0$ for $r \neq 0$.

(b) Integrating equation (6) over the small region at the origin containing the total charge Q one gets

$$\frac{\partial Q}{\partial t} = -J \quad . \tag{7}$$

(c) The Ampere Maxwell' s law tells

$$\nabla \times \mathbf{B} = \mu_o(\mathbf{j} + \mathbf{j}_D) \quad , \tag{8}$$

where we have indicated with $\mathbf{j}_D = \varepsilon_o[\partial E/\partial t]$ the displacement current. Taking the gradient of this eq. we get

$$\nabla \mathbf{j}_D = -\nabla \mathbf{j} = -J\delta(\mathbf{r}) \quad . \tag{9}$$

Since we know that exist ϕ such that $\mathbf{j}_D = \nabla\phi$ (in particular $\phi = \varepsilon_o[\partial E/\partial t]$ the mathematical note tells us that

$$\mathbf{j}_D = -J \quad , \tag{10}$$

from which follows $\nabla \times \mathbf{B} = 0$ and consequently $\mathbf{B} = 0$.

-Problem 3- A very long solenoid of radius a, with N turns per unit lenght, carries a current I_s. Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire, with resistance R. When the current in the solenoid is gradually decreased, a current I_r is induced in the ring.

(a) Calculate I_r, in terms of dI_s/dt.

(b) The power ($I_r^2 R$) delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid (the *electric field* is due to the changing flux in the solenoid; the *magnetic field* is due to the current in the ring). Integrate over the entire surface of the solenoid, and check that you recover the correct total power.

SOLUTION

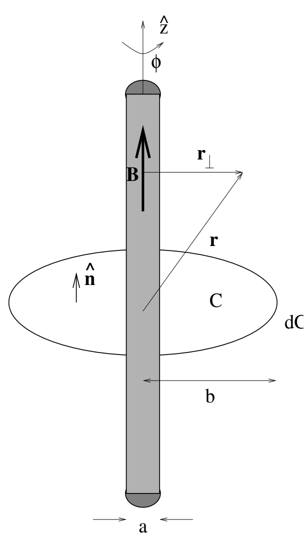

Figure 2: The apparatus described in the problem.

(a) The magnetic field inside the solenoid is

$$\mathbf{B}(t) = \mu_o N I_s(t) \hat{z} \quad . \tag{1}$$

Since $B(t)$ is varying with time an electric field with will be present with circles concentric with the solenoid as field lines. Then using Faraday' s law one can determine the electric potential felt by the electrons in the circular ring of wire ∂C (see figure 2)

$$V = \oint_C \mathbf{E} \cdot \mathbf{dl} = 2\pi b E = -\frac{d}{dt}\int_{\partial C} \mathbf{B} \cdot \mathbf{da} = -\pi a^2 \mu_o N \frac{dI_s}{dt} \quad . \tag{2}$$

Ohm law gives then for the current in the circular ring of resistance R

$$I_r = \frac{\mu_o N \pi a^2}{R} \frac{dI_s}{dt} \quad . \tag{3}$$

and for the power dissipated through Joule effect

$$P_r = R I_r^2 = I_r \mu_o N \pi a^2 \frac{dI_s}{dt} \quad . \tag{4}$$

(b) Let' s now calculate the Poynting vector just outside the solenoid. The electric field is due to the changing magnetic field inside the solenoid. Near the solenoids it will be

$$\mathbf{E} = -\mu_o \frac{\mu_o N \pi a}{2} \frac{dI_s}{dt} \hat{\phi} \quad . \tag{5}$$

The magnetic field is due to the current induced in the ring. Near the solenoids surface $r_\perp \approx a \ll b$, with $r_\perp$ the component of $\mathbf{r}$ orthogonal to $\hat{z}$, we have (see eq.(3) in problem 3 of Homework #1 Solutions)

$$\mathbf{B} \approx -\frac{\mu_o}{2} \frac{b^2}{(z^2+b^2)^{3/2}} I_r \hat{z} \quad . \tag{6}$$

Using equations (5) and (6) one can construct the Poynting vector (see figure 3)

$$\mathbf{S} = \frac{1}{\mu_o}(\mathbf{E} \times \mathbf{B}) = I_r \frac{\mu_o N a b^2}{4} \frac{1}{(z^2+b^2)^{3/2}} \hat{r_\perp} \quad . \tag{7}$$

The power P_r of eq. (4) dissipated in the resistor must come from the solenoid. Then integrating the Poynting vector over the surface of the solenoids we expect to get again P_r.

$$P = \int \mathbf{S} \cdot \mathbf{da} = I_r \frac{\mu_o N \pi a^2 b^2}{2} \int_{-\infty}^{\infty} \frac{1}{(z^2+b^2)^{3/2}} \, dz = P_r \quad . \tag{8}$$

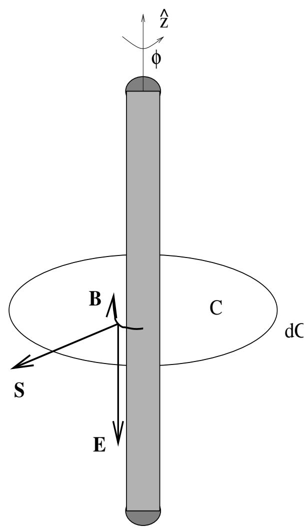

Figure 3: The Poynting vector near the surface of the solenoid.

-Problem 4- Show that it is always possible to choose

$$\nabla \mathbf{A} = -\mu_o \varepsilon_o \frac{\partial V}{\partial t} \quad , \tag{1}$$

as required for the Lorentz gauge, assuming you know how to solve the non homogeneous wave eqation. Is it always possible to pick $V = 0$? How about $A = 0$?

SOLUTION

It is alwaus possible, through a gauge transformation, choose the potentials $\mathbf{A}$ and V in such a way that they satisfy the Lorentz gauge (1). Suppose you have the potentials that don' t satisfy the Lorentz gauge. Through a gauge transformation

$$\left\{ \begin{array}{l} \mathbf{A}(\mathbf{r},t) \rightarrow \mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r},t) \\ V(\mathbf{r},t) \rightarrow V'(\mathbf{r},t) = V(\mathbf{r},t) - \dfrac{\partial}{\partial t}\chi(\mathbf{r},t) \end{array} \right. \tag{2}$$

we get the potentials in the new gauge $\mathbf{A}'$ and V' which describe the same physics as $\mathbf{A}$ and V; i.e. the electric and magnetic fields don' t change (are invariant under the gauge transformation). If we now impose that the potentials in the new gauge satisfy the Lorentz gauge (1) we get

$$\nabla \mathbf{A}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} = 0 = \nabla \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} + \left(\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}\right) \quad . \tag{3}$$

This shows that if we are able to find a gauge function $\chi(\mathbf{r}, t)$ which satisfies the equation

$$\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = -(\nabla \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}) \quad , \tag{4}$$

then the new potentials will be in the Lorentz gauge.

It is not always possible to have a gauge in which $V' = 0$. Immagine that you have found a gauge χ in which that condition holds. This means that in the preceding gauge one had $V = \partial\chi/\partial t$ then taking the gradient, $\mathbf{E} = -\nabla(\partial\chi/\partial t)$ and finally taking the curl, $\partial \mathbf{B}/\partial t$. This means that is possible to find a gauge in which $V' = 0$ only if originally we were in a situation in which the magnetic field wasn' t varying with time.

It is not always possible to have a gauge in which $\mathbf{A}' = 0$. Immagine that you have found a gauge χ in which that condition holds. This means that in the preceding gauge one had $\mathbf{A} = -\nabla\chi$ then $\mathbf{B} = 0$. This means that is possible to find a gauge in which $A' = 0$ only if originally were in a situation in which the magnetic field was zero.