Homework # 2 Solutions

set of units used: MKSA

-Problem 1- A perfectly conducting spherical shell of radius R rotates about the $\hat{\mathbf{z}}$ axis with angular velocity ω, in a uniform magnetic field $\mathbf{B} = B_o\hat{\mathbf{z}}$.

(a) Calculate the emf developed between the "north pole" and the equator.

(b) Suppose that the device is used as the emf in a circuit which contains only a resistor as shown. Assume that the current flows from the pole to the equator along the dotted line as shown. Calculate the power

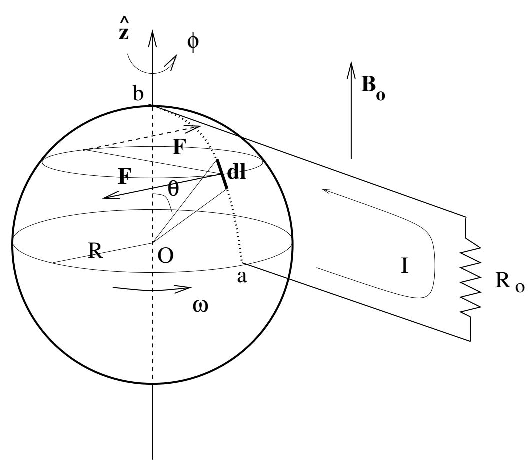

Figure 1: Rotatin, perfectly conducting spherical shell immersed in a uniform magnetic field and used as the emf in a circuit.

dissipated in the resistor (in terms of B_o, R, ω and R_o). Also calculate the torque that must be applied to the sphere to maintain a constant ω. Is energy conserved ?

SOLUTION

(a) In absence of a magnetic field each charge on the rotating, perfectly conducting spherical shell, has a uniform circular motion around axis $\hat{\mathbf{z}}$. If one switch on a magnetic field $\mathbf{B} = B_o\hat{\mathbf{z}}$ a charge at a distance $r_\perp(\theta)$ from the axis of rotation feels the Lorentz force

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B} = e\omega r_\perp(\theta) B_o[\hat{\theta}\cos\theta + \hat{\mathbf{r}}\sin\theta] \quad . \tag{1}$$

One can associate to $\mathbf{F}$ an apparent electric field $\mathbf{E}$ and electric potential V suches that

$$\mathbf{E} = -\nabla V = \mathbf{F}/e \quad . \tag{2}$$

Using the gradient theorem one can find the emf developed between the north pole (in fig. 1 point b) and the equator (in fig. 1 point a) as follows

$$\begin{aligned} V(b) - V(a) &= \int_a^b \nabla V \cdot \mathbf{dl} = -\int_a^b \mathbf{E} \cdot \mathbf{dl} \\ &\overset{(1)+(2)}{=} \int_0^{\pi/2} \omega B_o r_\perp(\theta) \cos\theta \; R d\theta \end{aligned} \tag{3}$$

where $r_\perp(\theta) = R\sin\theta$ (see fig. 1). Finally one obtains

$$\begin{aligned} \text{emf} &= V(b) - V(a) = \omega B_o R^2 \frac{1}{2} \int_0^{\pi/2} \sin 2\theta \; d\theta \\ &= \frac{1}{2}\omega R^2 B_o \end{aligned} \tag{4}$$

Notice that if the spherical shell inverts its rotation (i.e. $\omega < 0$) the emf changes sign.

The power dissipated in the resistor by Joule effect is then

$$P_R = \frac{V^2}{R} = \frac{\omega^2 R^4 B_o^2}{4R_o} \tag{5}$$

(b) As stated in the problem, one can assume that the current

$$I = \frac{V}{R_o} = \frac{\omega R^2 B_o}{2R_o} \quad . \tag{6}$$

flows from the pole to the equator along a meridian.

Consider an elementary length dl on the meridian determined by the contact b of the circuit with the shell and at an angle θ from the $\hat{\mathbf{z}}$ axis (see fig. 2). A current I flows through it, and the force

$$d\mathbf{F}(\theta) = I \ \mathbf{dl} \times \mathbf{B} = -IRd\theta B_o \cos\theta\hat{\phi} \tag{7}$$

will act on it [1].

To maintain ω constant one must then apply to the shell a torque that balances the one due to the Lorentz force, namely

$$\begin{aligned} \tau_z &= \hat{\mathbf{z}} \int_0^{\pi/2} \mathbf{r} \times d\mathbf{F}(\theta) \ d\theta = \int_0^{\pi/2} (\hat{\mathbf{z}} \times \mathbf{r}) \cdot d\mathbf{F}(\theta) \ d\theta \\ &= \int_0^{\pi/2} r_\perp \hat{\phi} \cdot \mathbf{F}(\theta) \ d\theta = -R^2 B_o I \frac{1}{2} \int_0^{\pi/2} \sin 2\theta \ d\theta \quad . \end{aligned} \tag{8}$$

Using eq. (6) one finally obtain

$$\tau_z = -\frac{B_o^2 R^4 \omega}{4R_o} \tag{9}$$

The energy is conserved. The energy spent in keeping ω constant by applying the torque $(-\tau_z\hat{\mathbf{z}})$ to the shell is dissipated by Joule effect in the resistor. This can be easily shown observing that the power necessary to maintain constant the angular velocity of the rotation

$$P_\omega = \frac{dE}{dt} \quad = \quad \frac{d(-\tau_z\phi)}{dt} = -\tau_z\omega = \frac{B_o^2 R^4 \omega^2}{4R_o}$$

coincides with the power dissipated in the resistor, P_R (5), thus the total energy is constant in time.

[1] The Lorentz force act on the singles charges within the dl considered. The component along $\hat{\phi}$ of the velocity of this charges, due to the rotation of the shell, doesn' t contribute to the Lorentz force.

$\boxed{\text{-Problem 2-}}$ A capacitor made made from parallel circular plates, of radius a and separation s, is inserted into a long straight wire carrying current I (see fig. 2). As the capacitor charges up, find the induced magnetic field midway between the plates, at a distance r ($r < a$) from the center. Assume $s \ll a$.

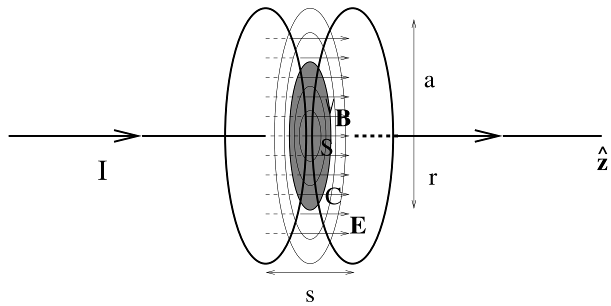

Figure 2: Parallel circular plates capacitor. The shaded surface S is the disk inside circle C midway between the plates with radius r and axes $\hat{\mathbf{z}}$.

SOLUTION

The corrent I charging the capacitor cause a variation on time of the charge surface density on the plates, and consequently of the electric field stored between the plates. Maxwell' s law tells that a magnetic field $\mathbf{B}(\mathbf{r})$ different from zero must exist between the plates. Let' s see how.

The whole physical system described in the problem, has a cylindrical symmetry around the $\hat{\mathbf{z}}$ axis (see fig. 2). This implies:

i. $\mathbf{B}$ cannot depend on ϕ, $\mathbf{B} = \mathbf{B}(\mathbf{r}, z)$

ii. The field lines of $\mathbf{B}$ (which has to be closed) must have axis $\hat{\mathbf{z}}$ as an axis of symmetry. Thus the field lines have to be circles (see fig. 2) with $\hat{\mathbf{z}}$ as their axes of symmetry.

To find the modulus of $\mathbf{B}$ midway between the plates at a distance r ($< a$) from the axis, one has to integrate the Ampere-Maxwell law on a suitable surface S bounded by the circle (C in the fig. 2) of radius r which lies midway between the plates and has the $\hat{\mathbf{z}}$ axis as its symmetry axis. The simplest choice is to take S as the surface which lies in the plane of the circle

C (surface S_2 in fig. 2). In this case one will have

$$\int_S \nabla \times \mathbf{B}\ \mathbf{da} = \oint_C \mathbf{B}\ \mathbf{dl} = B2\pi r = \varepsilon_o \mu_o \frac{d}{dt} \int_S \mathbf{E}(t)\mathbf{da} \tag{1}$$

If $\pm\sigma(t)$ is the surface charge density on the plates then the electric field inside the capacitor is found from Gauss' law (due to the condition $s \ll a$ one can assume $\mathbf{E}$ and σ to be uniform on the interior surfaces of the plates. They will be only functions of time) to be

$$\mathbf{E}(t) = \hat{\mathbf{z}} \frac{\sigma(t)}{\varepsilon_o} \quad . \tag{2}$$

Because of the continuity equation one has on each plate

$$\frac{d}{dt}\sigma(t) = -\nabla \mathbf{J}(t) \quad . \tag{3}$$

This eq. integrated over the volume of a cylinder with one basis coincident with the interior surface of a plate and the other outside of the capacitor and intersected by the wire carrying the current to such plate, gives

$$\frac{d}{dt} \int_{plate} \sigma(t)\ da = \pi a^2 \frac{d}{dt}\sigma(t) = I(t) \quad . \tag{4}$$

Using this result in eq (2) and eq. (2) in eq. (1) one gets

$$2\pi r B = \mu_o \frac{I(t)}{\pi a^2} \int_S da \quad , \tag{5}$$

which gives

$$B = \frac{\mu_o I r}{2\pi a^2} \quad . \tag{6}$$

$\boxed{\text{-Problem 3-}}$ Assuming that "Coulomb' s law" for magnetic charges (s_1 and s_2) reads

$$\mathbf{F} = \frac{\mu_o}{4\pi} \frac{s_q \; s_p}{|\mathbf{r}_q - \mathbf{r}_p|^2} \hat{\mathbf{r}} \quad ,$$

work out the "Lorentz force law" for a monopole s moving with velocity $\mathbf{v}$ through electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

SOLUTION

Assuming that "Coulomb laws" for magnetic charges (s_q and s_p) reads

$$\mathbf{F} = \frac{\mu_o}{4\pi} \frac{s_q \; s_p}{|\mathbf{r}_q - \mathbf{r}_p|^2} \hat{\mathbf{r}} \quad , \tag{1}$$

then the magnetic field generated by a magnetic charge s_q will be

$$\mathbf{B} = \mathbf{F}/s_p \quad . \tag{2}$$

The Maxwell eq. $\nabla \mathbf{B} = 0$ must be substituted with the new one

$$\nabla \mathbf{B} = \mu_o \delta(\mathbf{r} - \mathbf{r}_q) s_q \quad , \tag{3}$$

or for a generic magnetic distribution $\rho_m(\mathbf{r})$, by

$$\nabla \mathbf{B} = \mu_o \rho_m(\mathbf{r}) \quad . \tag{4}$$

This new equation has the same structure of the Gauss-Coulomb law thus one can immediately say that the new set of Maxwell equations will be

$$\partial_\mu F^{\mu,\nu} = \mu_o J_e^\nu \tag{5}$$

$$\partial_\mu G^{\mu,\nu} = \frac{1}{\varepsilon_o} J_m^\nu \tag{6}$$

where F, G and J_e are unchanged and the following magnetic charge-current four-vector has been introduced [1]

$$J_m^\nu(c\rho_m, \mathbf{J}_m) \quad \text{and} \quad \mathbf{J}_m = \mathbf{u}\rho_m \tag{7}$$

[1] Here one is tacitly ammitting that the magnetic densities satisfy a continuity equation of the same form of that satisfied by the electric densities.

with **uu** the velocity.

The tensor properties of F and G determine the connections between the field vector $\mathbf{E}$ and $\mathbf{B}$ as they appear in an unprimed system to the corresponding vectors in a primed system moving with velocity $\mathbf{v}$. When terms of relativistic magnitude (i.e. where $\beta^2 = (v/c)^2 \ll 1$ with c the speed of light) are neglected one has

$$\mathbf{B} = \mathbf{B}' - \mathbf{v} \times \mathbf{E}' \tag{8}$$

$$\mathbf{E} = \mathbf{E}' + \frac{\mathbf{v}}{c^2} \times \mathbf{B}' \tag{9}$$

If the primed system signifies the rest system of a magnetic charge s then in this system the force $sB = sB_o$ acts on the charge. From eq. (9) it is seen that this leads directly to the expression for the Lorentz force law for magnetic monopoles

$$\mathbf{F} = s(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}) \tag{10}$$

Comparing this eq. with the Lorentz force law for electric charges one concludes that interchanging $\mathbf{E}$ with $\mathbf{B}$ one goes in the otherone except for the minus sign between the contribution of $\mathbf{E}$ and the contribution of $\mathbf{B}$.

-Problem 4- In a **perfect conductor**, the conductivity is infinite, so $E = 0$, and any net charges resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*.

(a) Show that the magnetic field is constant (i.e., $\partial\mathbf{B}/\partial t = 0$), inside a perfect conductor.

A **superconductor** is a perfect conductor with the *additional* property that this constant $\mathbf{B}$ is always *zero*. (This "flux exclusion" is known as **Meissner effect**.)

(b) Show that the current in a superconductor is confined to the surface.

Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another.

(c) Suppose you had a sphere (radius R) above the critical temperature, and you held it in a uniform magnetic field $B_o\hat{\mathbf{z}}$ while cooling it below T_c. Find the induced surface current density $\mathbf{K}$, as a function of the polar angle θ.

(d) Suppose you made a loop of perfectly conducting wire, and a single magnetic monopole g passed through it. If the self inductance of the loop is L, what is the resulting current ?

SOLUTION

In a **perfect conductor**:

i. the conductivity $\sigma \to \infty$;

ii. the electric field inside is zero $\mathbf{E} \to 0$;

iii. net charges only on the surface.

(a) Inside a perfect conductor, Maxwell eq. $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ gives $\partial\mathbf{B}/\partial t = 0$.

A **superconductor**:

i. is a perfect conductor;

ii. the magnetic field inside is zero $\mathbf{B} \to 0$;

(b) Inside a superconductor, Maxwell eq. $\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \partial \mathbf{E}/\partial t$ gives $\mathbf{J} = 0$, since $\oint \mathbf{Bdl}$ is zero for every Amperian loop inside the superconductor. Current densities can exist but only confined on the surface.

(c) For $T > T_c$ the magnetic field lines which passed through the sphere at $T < T_c$ are bended out of the superconductor (see fig. 3). The

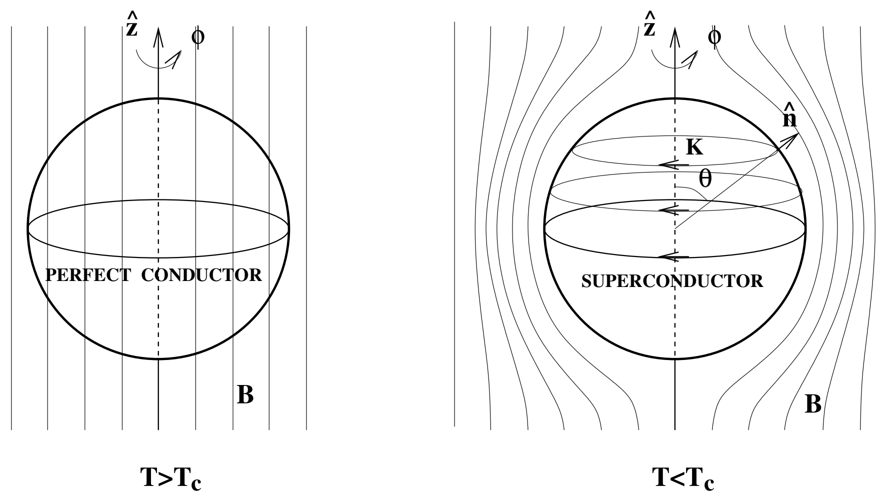

Figure 3: Schematic representation of the magnetic field lines in the neighborhood of a superconducting sphere above (on the left) and below (on the right) its critical temperature

process of cooling the superconducting sphere produces on the magnetic induction inside and outside the sphere, the same effect produced in a perfect conducting sphere immersed in a uniform magnetic field $B_o\hat{\mathbf{z}}$, by the switching on of a uniform magnetization $\mathbf{M} = -M_o\hat{\mathbf{z}}$ opposed to the field. It' s known that the magnetic field inside a uniformly magnetized sphere is

$$\mathbf{B} = \frac{2}{3}\mu_o \mathbf{M} \quad . \tag{1}$$

In order to have $B = 0$ inside the sphere the magnitude of the magnetization must be

$$M_o = \frac{3B_o}{2\mu_o} \quad . \tag{2}$$

The magnetization of the sphere will be produced by a volume current density

$$\mathbf{J} = \nabla \times \mathbf{M} = 0 \tag{3}$$

plus a surface current

$$\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}} = \hat{\phi} M_o \sin\theta = \hat{\phi}\frac{3B_o}{2\mu_o}\sin\theta \quad . \tag{4}$$

(d) A magnetic monopole of charge g generates around itself the following magnetic field

$$\mathbf{B} = \frac{\mu_o 4}{\pi}\frac{g}{r^2}\hat{\mathbf{r}} \quad . \tag{5}$$

To find the flux of the magnetic field generated by the monopole in $\mathbf{r}$

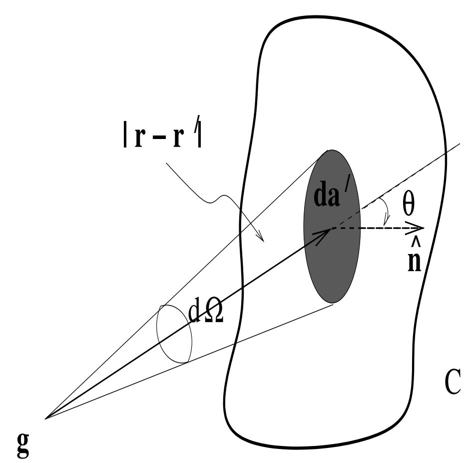

Figure 4: Magnetic monopole of charge g in $\mathbf{r}$ approaching the loop C from right. $\mathbf{da}'$ is an elementary surface around $\mathbf{r}'$.

through the loop C (see fig. 4), one has to fix a surface S bounded by the loop and to calculate

$$\Phi_g(\mathbf{r}) = \int_S \mathbf{B}\cdot\mathbf{da}' = \frac{\mu_o g}{4\pi}\int_S \frac{\cos\theta \; da'}{|\mathbf{r}-\mathbf{r}'|^2} = \frac{\mu_o g}{4\pi}\int_S d\Omega \tag{6}$$

where $d\Omega$ is the elementary solid angle under which g sees the elementary surface da' of surface S (see fig. 4).

Equation (6) shows that as the monopole passes through the loop the flux of its magnetic field through the loop has to have a jump of $\mu_o g$ (as shown in fig. 5). Infact for g approaching S from the right $d\Omega \to 2\pi$

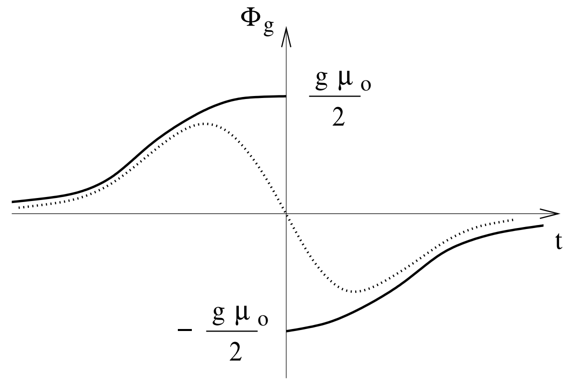

Figure 5: Schematic time dependence of the flux through loop C of a magnetic monopole approaching and leaving the loop. The solid line represents the case of the monopole passing thgough the loop. The dotted line represents the case of the monopole missing the loop.

and when g passes the elementary surface da', $d\Omega = -2\pi$.

This jump can be physically interpreted as due to the presence in the loop, of a current $I \neq 0$ which generate a magnetic field in the opposite direction of the one generated by the monopole and such that its flux through the loop (responsible for the jump in Φ_g) is [1]

$$\Phi_{loop} = LI = \mu_o g \tag{7}$$

Then I must be equal to $\mu_o g/L$.

COMMENT: An anologous problem dealing with electric field instead than magnetic field would be the parallel plate capacitor with surface charge $\pm\omega$ on the plates and *small* plate spacing d (the so called "double-layer"). In this case the role of Φ_g is played by the electric potential which must undergo a discontinuity of $4\pi D$ passing through the double-layer. $D = \omega d$ beeing the moment of the double-layer ($d \to 0$).

[1] Mathematically speaking the existence of the current I is necessary in order to make the total flux through the loop ($\Phi_g + \Phi_{loop}$) a single valued function.