Homework # 8 Solutions

set of units used: MKSA

-Problem 1- Assuming negligible damping ($\gamma_j = 0$) calculate the group velocity ($v_g = d\omega/dk$) of an electromagnetic waves traveling into a dispersive medium. Show that $v_g < c$ even when $v > c$.

SOLUTION

Using the model of the "elastically bound electron" one can give a first rough description of *dispersion* in matter, i.e. the dependence of the permittivity, ε, on the frequency of the electromagnetic excitation perturbing the material. Thid model gives the following dynamic value for the permittivity and the wave vector of an electromagnetic wave traveling into a *dispersive* medium

$$\varepsilon(\omega) = \varepsilon_o \left[1 + \frac{Nq^2}{m\varepsilon_o} \sum_j \frac{f_j}{\omega_o^2 - \omega^2 - i\gamma_j \omega}\right] \quad , \tag{1}$$

$$k = k_+ + ik_- = \omega\sqrt{\varepsilon(\omega)\mu_o} \quad , \tag{2}$$

where k_+ and k_- are real numbers, N is the number of molecule per unit volume, f_j is the number of electrons of charge q and mass m oscillating into the $j-th$ molecule with a characteristic frequency ω_j and a damping factor γ_j, ε_o is the permittivity of the vacuum and c the speed of light.

Assuming $\gamma_j = 0$ for all molecule one gets

$$k = k_+ = \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\varepsilon_o} \sum_j \frac{f_j}{\omega_o^2 - \omega^2}\right] \quad , \tag{3}$$

$$\tag{4}$$

and for the group velocity

$$v_g = \left(\frac{dk_+}{d\omega}\right)^{-1} \tag{5}$$

$$= \left[\frac{1}{c}\left(1 + \frac{Nq^2}{2m\varepsilon_o}\sum_j \frac{f_j}{\omega_o^2 - \omega^2}\right) + \frac{\omega}{c}\frac{Nq^2}{2m\varepsilon_o}\sum_j \frac{2\omega f_j}{(\omega_o^2 - \omega^2)^2}\right]^{-1} \tag{6}$$

$$= \quad c\left(1+\frac{Nq^2}{2m\varepsilon_o}\sum_j\frac{(\omega_j^2+\omega^2)f_j}{(\omega_j^2-\omega^2)^2}\right)^{-1} \quad . \tag{7}$$

Since all terms $(\omega_j^2+\omega^2)f_j/(\omega_j^2-\omega^2)^2$ of the sum in (7) are positive, one has $v_g < c$ also when for sufficiently high ω ($\omega > \omega_j$ for a sufficient number of j), the phase velocity is greater then c

$$v = \frac{\omega}{k} = c\left[1+\frac{Nq^2}{2m\varepsilon_o}\sum_j\frac{f_j}{\omega_j^2-\omega^2}\right]^{-1} > c \quad . \tag{8}$$

$\boxed{\text{-Problem 2-}}$ Calculate the attenuation length (in terms of N, f, q, γ, m, etc.) in a plasma at low frequencies: $\omega \ll \gamma$.

SOLUTION

Electromagnetic waves in a conductor or in plasmas must be described using a complex wave vector $k = k_+ + ik_-$, namely

$$k^2 = \mu\varepsilon\omega^2 + i\sigma\mu\omega \quad , \tag{1}$$

where μ is the permeability, ε the permittivity, σ the conductivity of the material and ω the frequency of the electromagnetic perturbation. Modelling the conductor as N molecule per unit volume containing f *free* electrons of cherge q and mass m subject to a damping attributable to the cumulative effect of many collisions and modeled by the damping factor γ, one can write the conductivity of the medium as

$$\sigma = \frac{Nfq^2/m}{\gamma - i\omega} \quad . \tag{2}$$

As long as $\omega \ll \gamma$ the conductivity, σ, remains nearly real and independent from the frequency of the perturbation, namely

$$\sigma_o = \frac{Nfq^2/m}{\gamma} \quad . \tag{3}$$

Moreover in a plasma $\varepsilon \sim \varepsilon_o$ and $\mu \sim \mu_o$ and one can rewrite k^2 as ($c^2 = 1/\mu_o\varepsilon_o$)

$$k^2 = \frac{\omega^2}{c^2}\left(1 + i\frac{\sigma_o}{\omega\varepsilon_o}\right) \tag{4}$$

The *attenuation length* d of the plasma is defined by $d \equiv 1/k_-$. From equation (1) one gets

$$\begin{cases} k_+^2 - k_-^2 = (\omega/c)^2 \\ 2k_+k_- = \mu_o\sigma_o\omega \end{cases} \tag{5}$$

and solving in $k_\pm$ (σ_o, c, μ_o, ω are all real quantities)

$$k_\pm = \frac{1}{\sqrt{2}}\frac{\omega}{c}\left(\sqrt{1 + \left(\frac{\sigma_o}{\omega\varepsilon_o}\right)^2} \pm 1\right)^{1/2} \quad . \tag{6}$$

The attenuation length can then be written as

$$d = \frac{1}{k_-} = \frac{c\sqrt{2}}{\sqrt{[\omega^4 + \omega_p^4(\omega^2/\gamma^2)]^{1/2} - \omega^2}} \quad , \tag{7}$$

where we have introduced the plasma frequency defined as $\omega_p \equiv q\sqrt{Nf/m\varepsilon_o} = \sqrt{\gamma\sigma_o/\varepsilon_o}$.

Since a plasma of electrons is a good conductor then one can assume that

$$\frac{\sigma_o}{\omega\varepsilon_o} \gg 1 \quad . \tag{8}$$

To the lowest order in $(\omega\varepsilon_o/\sigma_o)$ equation (6) gives

$$d = \frac{1}{k_-} \sim \frac{c}{\sqrt{\sigma_o\omega/(2\varepsilon_o)}} \tag{9}$$

$$= \frac{c\sqrt{2\gamma}}{\omega_p\sqrt{\omega}} \quad . \tag{10}$$

For example in Copper $\sigma_o = 5.4 \times 10^{17} S^{-1}$ and these $d-$values also provide

λ_o	$1cm$	$1m$	$100m$	$10km$
d	0.37μ	3.7μ	37μ	$0.37mm$

Table 1: Skin depth d in copper for electromagnatic radiation with vacuum wavelengths λ_o

something of an estimate of the metal thickness necessary for screening or shielding against the respective wavelengths.

-Problem 3- Interstellar space is a very low density ($N \sim 1cm^{-3}$) plasma of free protons and electrons. What is the lowest frequency electromagnetic wave can propagate there.

SOLUTION

In a dilute plasma (ionized gas) the damping of the *free* charge motion is negligible, and the conductivity σ is purely imaginary,

$$\sigma = i\left(\frac{Nq^2}{m\omega}\right) \quad , \tag{1}$$

where N is the number density of the free charges q and mass m, ω is the frequency of the perturbation. Since, moreover, $\mu \simeq \mu_o$ and $\varepsilon \simeq \varepsilon_o$ in a plasma, the wave vector of an electromagnetic wave propagating into the low density plasma is

$$k^2 = \frac{1}{c^2}(\omega^2 - \omega_p^2) \quad , \tag{2}$$

where

$$\omega_p = q\sqrt{\frac{N}{m\varepsilon_o}} \quad , \tag{3}$$

is the **plasma frequency**. For frequencies in excess of ω_p, the wave number is real, and the waves propagate without attenuation. At frequencies *below* ω_p, on the other hand, k is purely imaginary and the waves are attenuated away. Accordingly, the plasma is *opaque* to waves of frequency less then ω_p and *transparent* to those above ω_p.

Interstellar space is a very dilute plasma of *free* protons (of density $N_p = N/2 =\sim 0.5cm^{-3}$ and mass m_p) and of *free* electrons (of density $N_e = N/2 =\sim 0.5cm^{-3}$ and mass m_e). There will be then two plasma frequencies with which one has to compare the wave frequency of the electromagnetic perturbation, in order to understand if it can propagate there, one for the electrons $\omega_{p,el}$, and one for the protons $\omega_{p,pr}$. Since a proton is almost 2000 havier then an electron one gets

$$\omega_{p,el} = q\sqrt{\frac{N_e}{m_e\varepsilon_o}} \sim 3.98 \times 10^4 S^{-1} \tag{4}$$

$$\omega_{p,pr} = q\sqrt{\frac{N_p}{m_p\varepsilon_o}} = \frac{\omega_{p,el}}{\sqrt{2000}} \quad . \tag{5}$$

Then interstellar space will be transparent to electromagnetic waves of frequencies

$$\omega > max\{\omega_{p,el}, \omega_{p,pr}\} = \omega_{p,el} \sim 3.98 \times 10^4 S^{-1} \quad , \tag{6}$$

and of wavelengths

$$\lambda < 4.6 \times 10^7 m \quad . \tag{7}$$

$\boxed{\text{-Problem 4-}}$ We have ignored magnetic forces (see, e.g., Griffith eq. 8.175). Consider a material with $\omega_o = 10^{15}S^{-1}$ and $\gamma = 0$. For $\omega = 10^{14}S^{-1}$, estimate the value that E_o must have for magnetic forces (qvB) to be significant. What is the intensity ($Watts/m^2$) of this wave ?

SOLUTION

When studying the behaviour of a *dispersive* medium perturbed by an electromagnetic wave one can use the model of the *elastically bound electron*, namely

$$m\frac{d^2y(t)}{dt^2} = -m\gamma\frac{dy(t)}{dt} - m\omega_o^2 y(t) = qE_o\cos(\omega t) \quad , \tag{1}$$

where $y(t)$ is the displacement of the electron of mass m and charge q inside a particular molecule of the medium, $\gamma = 0$ the damping factor, $\omega_o = 10^{15}S^{-1}$ the natural oscillation frequency and $\omega = 10^{14}S^{-1}$ the frequency of the electromagnetic wave. We have to estimate which value the electric field, E_o must have for the Lorentz force ($q\mathbf{r} \times \mathbf{B}$) to be significant into Newton' s second law for the motion of the elctron (1).

Solving equation (1) one gets

$$\begin{aligned} y(t) &= \frac{q/m}{\omega_o^2 - \omega^2}E_o\cos(\omega t) \quad , & (2)\\ \frac{dy(t)}{dt} &= -\frac{\omega q/m}{\omega_o^2 - \omega^2}E_o\sin(\omega t) \quad . & (3)\end{aligned}$$

An estimate of the magnitude of the Lorentz force would then be

$$F_l = \frac{\omega q^2/m}{\omega_o^2 - \omega^2}B_o \quad , \tag{4}$$

where we used the maximum value for the velocity (3) and

$$\begin{aligned} B_o &= \frac{|k|}{c}E_o \quad , & (5)\\ k &= 1 + \frac{1}{2}X \quad , & (6)\\ X &= \frac{Nq^2}{m\varepsilon_o}\frac{1}{\omega_o^2 - \omega^2} \sim N(m^{-3}) \cdot 1.6 \times 10^{-27}m^3 \quad . & (7)\end{aligned}$$

where N is the number density of electrons in the medium. Since generally [1] one has $N \ll 10^{27} m^{-3}$ which imply $X \ll 1$ and then $B_o \sim E_o/c$.

The Lorentz force will be relevant in the equation of motion (1) when it is comparable with the driving force due to the electric component of the electromagnetic perturbation, namely when

$$F_l \sim qE_o \Rightarrow E_o \sim \frac{cm}{q} \frac{(\omega_o^2 - \omega^2)}{\omega} \sim 1.7 \times 10^{13} V/m \quad . \tag{8}$$

The intensity (average power per unit area) of the corrisponding wave is $I = \varepsilon(\omega) v(\omega) E_o^2/2$ and can be evaluated observing that

$$\begin{aligned} \varepsilon(\omega) &= \varepsilon_o(1 + X) \sim \varepsilon_o \simeq 8.85 \times 10^{-12} coulomb^2/(Nm^2) \quad , && (9) \\ v(\omega) &\sim c \simeq 3 \times 10^8 m/S \quad . && (10) \end{aligned}$$

Finally using the value for the electric amplitude found in (8) we get

$$I \sim \frac{1}{2} \varepsilon_o c E_o^2 \sim 3.8 \times 10^{23} \quad . \tag{11}$$

[1] In the ionosphere $N \sim 10^{15} - 10^{22} m^{-3}$. In low density plasmas $N \sim 10^6 m^{-3}$