Homework # 4 Solutions

set of units used: MKSA

-Problem 1- Consider two equal point charges q, separated by a distance $2d$ as in figure 1. Construct the plane equidistant from the two charges. By

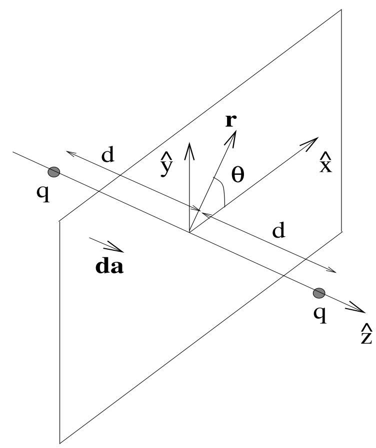

Figure 1: Charge configuration

integrating Maxwell' s stress tensor over this plane,

(a) determine the force of one chearge on the other.

(b) Do the same for charges that are opposite in sign.

(c) Explain how one abtains $\mathbf{F}_{1,2} = -\mathbf{F}_{2,1}$.

SOLUTION

The force experienced by charge q in $(0, 0, -d)$ from the charge q in $(0, 0, d)$ is equivalent to the total force exercited on the volume $V = \{z < 0\}$ by the electromagnetic fields generated by the charges configuration, namely

$$\mathbf{F} = \oint_S \mathcal{T}\mathbf{da} - \varepsilon_o\mu_o \frac{d}{dt}\int_V \mathbf{S}d\tau \tag{1}$$

where we have indicated with S the surface bounding volume V, with $\mathbf{S}$ the Poynting vector and $\mathcal{T}$ is the Maxwell' s stress tensor. Since we are dealing with two static charges then: (i) the last term in the right hand side of equation (1) is zero; (ii) the electrostatic fields decay as $1/r^2$ at $r \to \infty$. The Maxwell' s stress tensor decays as $1/r^4$. Then we conclude that in the first term of the right hand side of equation (1), the integral gives zero when calculated on the parts of S at infinity. We remain then with the following equation for the force $\mathbf{F}$

$$\mathbf{F} = \int_{\{z=0\}} \mathcal{T}\mathbf{da} \quad , \tag{2}$$

which in components can be rewritten as

$$F_i = \int_{\{z=0\}} \sum_j \mathcal{T}_{i,j} da_j \quad , \tag{3}$$

where

$$\mathcal{T}_{i,j} = \varepsilon_o (E_i E_j - \frac{1}{2}\delta_{i,j} E^2) \quad , \tag{4}$$

since there is no magnetic field entering in the problem. We observe also that, following the notations in figure 1, we have $\mathbf{da} = (rdr\ d\theta)\hat{z}$ then we can simplify equation (3) to

$$F_i = \int_{\{z=0\}} \mathcal{T}_{i,z}\ rdr\ d\theta \quad . \tag{5}$$

Since the force between two point charges can only be in the direction of the axes containing the two charges (axes $\hat{z}$ for us: $\mathbf{F} = F\hat{z}$) we get in the end

$$F = \int_{\{z=0\}} \mathcal{T}_{z,z}\ rdr\ d\theta \quad . \tag{6}$$

(a) In this case we have for the electric field generated by the two charges of equal sign

$$\begin{aligned} E_z &= 0 & (7)\\ E_x^2 + E_y^2 &= \frac{2q}{4\pi\varepsilon_o}\frac{1}{(r^2+d^2)}\frac{r}{\sqrt{r^2+d^2}} \quad . & (8)\end{aligned}$$

From equation (4) we get

$$\mathcal{T}_{z,z} = -\frac{\varepsilon_o}{2}(E_x^2 + E_y^2) \quad . \tag{9}$$

Substituting equation (8) into (9) and then (9) into equation (6) we get

$$F = -\frac{q^2}{4\pi\varepsilon_o}\int_0^\infty \frac{r^2}{(r^2+d^2)^3}\; rdr = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2} \quad . \tag{10}$$

(b) In this case we have for the electric field generated by the two charges of opposite sign

$$\begin{aligned} E_x &= E_y = 0 && (11)\\ E_z &= \frac{2q}{4\pi\varepsilon_o}\frac{1}{(r^2+d^2)}\frac{d}{\sqrt{r^2+d^2}} \quad . && (12) \end{aligned}$$

From equation (4) we get

$$\mathcal{T}_{z,z} = \frac{\varepsilon_o}{2}E_z^2 \tag{13}$$

Substituting equation (12) into (13) and then (13) into equation (6) we get

$$F = -\frac{q^2}{4\pi\varepsilon_o}\int_0^\infty \frac{d^2}{(r^2+d^2)^3}\; rdr = \frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2} \quad . \tag{14}$$

(c) $\mathbf{F}_{1,2} = -\mathbf{F}_{2,1}$ because when we calculate $\mathbf{F}_{1,2}$, the force exercited by charge 2 on charge 1, we use a closed surface containing particle 1, while when we calculate $\mathbf{F}_{2,1}$ we use a closed surface containing particle 2. In both way we can find the force integrating the Maxwell' s stress tensor (which will be the same in both cases, since only dependent on the charges configuration) over the plane midway between the charges, but if in one case the elementary surface of the plane is $\propto \hat{z}$ in the other is $\propto -\hat{z}$. Equation (5) shows then that we must have $\mathbf{F}_{1,2} = -\mathbf{F}_{2,1}$.

$\boxed{\text{-Problem 2-}}$ Calculate the power (energy per unit time) transported down a long coaxial cable of figure 2 assuming the two conductors are held at a potential difference V and carry a current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b).

SOLUTION

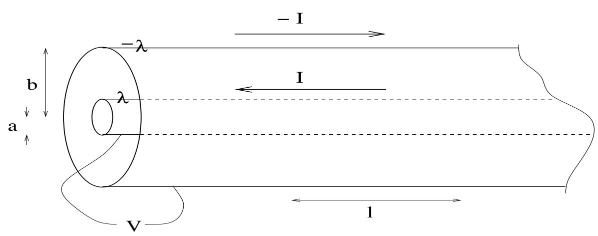

Figure 2: Coaxial cable.

The coaxial cable is built in such a way to confine all the electromagnetic field generated by itself between the two conducting cylinders. According to Ampere' s law the magnetic field between the cylinders, that carry a current I each, can be written as

$$\mathbf{B} = -\frac{\mu_o I}{2\pi r}\hat{\phi} \quad . \tag{1}$$

Since the two cylinders are held at a potential difference V, we will have for the electric field

$$\mathbf{E} = \frac{V}{(b-a)}\hat{r} \quad . \tag{2}$$

The Poynting vector between the cylinders will be

$$\mathbf{S} = -\frac{1}{\mu_o}(\mathbf{E}\times\mathbf{B}) = \frac{VI}{2\pi r(b-a)}\hat{z} \quad . \tag{3}$$

From the Poynting' s theorem applied to the surface S coming from $z = -\infty$ and cutting the cable at some point z, we get the power transported down the coaxial cable by the static fields, namely

$$P = \frac{dW_{E,B}}{dt} = -\oint_S \mathbf{S}\cdot\mathbf{da} = \int_a^b \int_0^{2\pi} \frac{VI\ dr\ d\theta}{2\pi(b-a)} = VI \quad . \tag{4}$$

-Problem 3- Picture the electron as a uniformly charged spherical shell, with charge e and radius R, spinning at angular velocity ω.

(a) Calculate the total energy contained in the electromagnetic fields.

(b) Calculate the total angular momentum contained in the fields.

(c) According to Einstein formula ($E = mc^2$), the energy in the field should contribute to the mass of the electron. Lorentz and other speculated that the *entire* mass of the electron might be accounted for in this way: $W_{E,B} = m_e c^2$. Suppose moreover that the electron' s spin angular momentum is entirely attributable to the electromagnetic fields: $L_{E,B} = \hbar/2$. On these two assumptions, determine the radius and the angular velocity of the electron

(d) What is their product ωR ?

(e) Does this classical model make sense ?

SOLUTION

In a uniformly charged spherical shell of radius R and total charge e, the

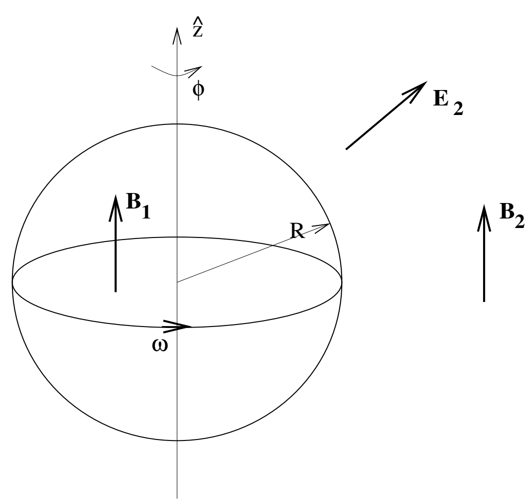

Figure 3: The “classical” (non-QM) electron.

surface charge density is

$$\sigma = \frac{e}{4\pi R^2} \quad . \tag{1}$$

For the Gauss theorem, inside and outside the sphere holds

$$r^2 E_r = constant \quad . \tag{2}$$

If we indicate with 1 the inner region and with 2 the outer region, we can write

$$E_r = \begin{cases} E_{r1} = \frac{c_1}{r^2} & r < R \\ E_{r2} = \frac{c_2}{r^2} & r > R \end{cases} \tag{3}$$

Since the electric field cannot be singular (explode) at the center of the sphere we have to impose $c_1 = 0$. Then from the boundary condition

$$E_{r2} - E_{r1} = \frac{\sigma}{\varepsilon_o} = \frac{e}{4\pi\varepsilon_o R^2} = (E_{r2})_{r=R} \quad , \tag{4}$$

follows that $c_2 = e/(4\pi\varepsilon_o)$ and then

$$E_r = \begin{cases} 0 & r < R \\ \frac{e}{4\pi\varepsilon_o r^2} & r > R \end{cases} \tag{5}$$

Then the total energy stored in the electric field of the electron will be [1]

$$W_E = \frac{1}{2}\int \varepsilon_o E^2 \; dV = 4\pi \int_R^\infty \varepsilon_o \frac{e^2}{(4\pi\varepsilon_o)^2 r^2} \; dr = \frac{e^2}{8\pi\varepsilon_o R} \tag{6}$$

For the magnetic field of a uniform spherical shell spinning with angular velocity ω around axis $\hat{z}$ we get from **example 11** in chapter 5 of D. J. Griffiths, the following result

$$\mathbf{B} = \begin{cases} \mathbf{B}_1 = \frac{2}{3}\mu_o \sigma R \omega \hat{z} & r < R \\ \mathbf{B}_2 = \left[\frac{3(\mathbf{m}\cdot\mathbf{r})\cdot\mathbf{r}}{r^2} - \mathbf{m}\right]\frac{\mu_o}{4\pi r^3} & r > R \end{cases} \tag{7}$$

[1] In the limit of a puntiform electron $W_E \to \infty$.

where we have indicated with $\mathbf{m} = \hat{z}(eR^2\omega)/3$ the magnetic dipole moment of the electron.

Then the total energy stored in the magnetic field of the electron (inside W_{B1}, and outside W_{B2}, the spherical shell) will be

$$W_B = W_{B1} + W_{B2} \tag{8}$$

$$W_{B1} = \frac{1}{2}\int \frac{B_1^2}{\mu_o}\, dV = \frac{1}{2\mu_o}\left(\frac{4}{3}\pi R^3\right)\left(\frac{2}{3}\mu_o \sigma R\omega\right)^2 = 2\frac{mu_o}{4\pi}\frac{e^2}{27} \quad , \tag{9}$$

$$\begin{aligned} W_{B2} &= \frac{1}{2}\int \frac{B_2^2}{\mu_o}\, dV \\ &= \frac{1}{2\mu_o}2\pi \int_R^\infty \int_{-1}^1 \left(\frac{\mu_o}{4\pi r^3}\right)^2 m^2[4\cos^2\theta + \sin^2\theta]^2\ d(\cos\theta)\ r^2 dr \\ &= \frac{\mu_o}{2(4\pi)^2}\frac{m^2}{3R^3}2\pi \int_{-1}^1 (3\cos^2\theta + 1) d(cos\theta) = \frac{\mu_o}{4\pi}\frac{e^2}{27}(R\omega^2) \quad . \end{aligned} \tag{10}$$

(a) The total energy contained in the electromagnetic fields is the sum of the energy contained in the magnetic field inside the shell W_{B1} and the energy contained in the electromagnetic field outside the shell $W_E + W_{B2}$. From equations (6), (9) and (10) we get

$$W_{E,B} = \frac{e^2}{8\pi\varepsilon_o R} + \frac{\mu_o}{4\pi}\frac{e^2}{9}(R\omega^2) \quad . \tag{11}$$

(b) The Poynting vector $\mathbf{S}$ is different from zero only outside of the spherical shell (inside $\mathbf{E} = 0$). Then from equations (5) and (7) we get for the volume density of the linear momentum $\mathbf{p}$ of the electromagnetic fields (momentum density = energy current density $/c^2$, $c^2 = (\varepsilon_o\mu_o)^{-1}$ = speed of light.) the following expression

$$\mathbf{p} = \frac{\mathbf{S}}{c^2} = \frac{e(\mathbf{r}\times\mathbf{m})}{\varepsilon_o(4\pi c)^2}\frac{1}{r^6} = \frac{r_o}{(e/m)4\pi r^6}(\mathbf{r}\times\mathbf{m}) \quad , \tag{12}$$

where we have introduced the classical radius of the electron [2]

$$r_o = \frac{e^2}{4\pi\varepsilon_o mc^2} \sim 2.82 fm \quad . \tag{13}$$

[2] Remember that $1fm = 10^{-13}cm$.

The volume density of angular momentum is given by

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = \frac{r_o}{(e/m)4\pi r^6}\mathbf{r} \times (\mathbf{r} \times \mathbf{m}) = \frac{r_o}{(e/m)4\pi r^6}[\mathbf{r}(\mathbf{r} \cdot \mathbf{m}) - \mathbf{m}r^2] . \quad (14)$$

Since we are taking $\mathbf{m}$ alligned with the $\hat{z}$ axis, the only component of $\mathbf{l}$ which will contribute to the total angular momentum will be the $\hat{z}$ component, namely [3]

$$l_z = -\frac{r_o}{(e/m)4\pi r^6} m r^2 \sin^2\theta \quad . \quad (15)$$

For the total angular momentum $\mathbf{S} = S\hat{z}$ we get then

$$S = \int l_z \, dV = -\frac{2}{3} r_o \frac{m}{(e/m)R} = -\frac{2}{9}\frac{e r_o}{(e/m)}(\omega R) \quad . \quad (16)$$

(d) From (16) and the definition of r_o (13) we get

$$R\omega = -S\frac{18\pi}{\mu_o e^2} \quad . \quad (17)$$

If we put $S = \hbar/2$ we find the following result

$$R\omega \sim 9.26 \times 10^{10} m/s \approx 300c \quad . \quad (18)$$

(c) If we imagine that the *entire* mass of the electron is contained in its field, i.e. $W_{E,B} = mc^2$, we get from equation (11), the following value for the radius of the elctron

$$R = r_o \left[\frac{1}{2} + \frac{(R\omega)^2}{9c^2}\right] \quad , \quad (19)$$

then from result (18) and from equation (13) we get

$$R \approx 3 \times 10^{-9} cm \gg r_o \quad . \quad (20)$$

From equation (18) we get for the angular velocity

$$\omega \approx 3 \times 10^{21} rad/s \quad . \quad (21)$$

(e) The result (18) shows that the classical model considered for the electron doesn' t make sense: a point on the equator of the rotating shell has a velocity of 300 times the speed of light !

[3] The component orthogonal to the $\hat{z}$ axis will be proportional to $\sin\theta\cos\theta$ which gives zero when integrated over all the space from $r = R$ to infinity.