System Issues: Constructing a Computer Algebra System

Lecture 16

Outline

1. Why do it?
   - CS /fun /it can be done.
   - Math/AI/Theology
   - Profit
2. General design goals
3. Strategies
4. The front end
5. Data
6. Algorithms

CS/fun

- Interest in building a system that can apply algorithmic mathematics. Enjoy algorithm development and generalization. With computers we can manipulate symbols not only numbers.
- but there are other goals
  - We will pursue this digression for a few slides.

Other Goals, different, maybe more ambitious: build a system that...

- "knows" mathematics
- is a repository for all mathematics
- can discover proofs (automated reasoning)
- can discover new mathematics
- invent interesting conjectures
- is an artificial intelligence (knows physics etc also!)

Sample version of the formalist / constructivist / mathematics theology

The goals of the QED project

- http://www.rbjones.com/rbjpub/logic/qedres00.htm

- to help mathematicians cope with the explosion in mathematical knowledge
- to help development of highly complex IT systems by facilitating the use of formal techniques
- to help in mathematical education
- to provide a cultural monument to "the fundamental reality of truth"
- to help preserve mathematics from corruption
- to help reduce the "noise level" of published mathematics
- to help make mathematics more coherent
- to add to the body of explicitly formulated mathematics
- to help improve the low level of self-consciousness in mathematics (?)

Reflection

- http://tunes.org/Review/Reflection.html
- discuss programs in the programming language.
- Easily done in some languages, impossible in others.
- Symbolic computing (especially in lisp, where data and programs can be made of the same "stuff") provides some opportunities not available in the usual CS curriculum.

- end of digression.

Can you get rich building a CAS?

- What is the market?
  - Fans of higher math?
  - Engineers, Scientists?
  - Wall St. Analysts?
  - Freshman Calculus students?
  - Dentists? (no.)
- How many people will pay how much for this facility?
- Find a "killer app". Probably education.
- Find people to gush about it (Steve Jobs? NYTimes?)

A few computer algebra systems

- Maple, Mathematica, Macsyma, Maxima, Derive, Reduce, Axiom, NTL, Cocoa, GiNaC, Cathode, GAP, Fermat, Form, Macaulay, Pari, Singular, Yacas, Jacal, Mupad ..
- In one sense it has become easier to build systems because many people have sufficient resources (one PC will do it) to attempt the task.
- In another sense it has become harder: there are systems with 1000 person-years' "head start".

- A list of most of them at http://symbolicnet.org

Shared Goals (well, not always)

- Generality (wide domain of discourse)
- Correctness, Robustness, orthogonality
- Ease of use (batch, interactive, web)
- Speed
- Portability (Linux, UNIXes, Windows, Mac)
- Conforming to standards
- Communication with other systems (OpenMath, MathML/XML, MP, RPC, OLE, Java beans)
- Ease of growth
- Parallel/distributed possibility

Strategies: Three traditional ways to focus attention on the task at hand

- Mathematics is neat
  - commutative algebra is neat
  - group theory is neat
  - number theory is neat
- Physics is neat
- Computers are neat

Mathematics is neat

- And computer programming is simple. Written by math types. Sometimes rather broad focus, e.g. Computational group theory; sometimes very specific: computations modulo 251. Sometimes very efficient, sometimes lacking in robustness or usability.

Physics is neat

- Written by physicists.
  - Everything is best done by physicists, who can cut through the nonsense of math and CS. Hence initially naive with respect to both math and CS. ''Let's do tensors''.
  - Sometimes pushes state of the art since there is (or has been) money for physics.

Computers are neat, math is simple.

- Written by CS types. Treat math as a programming language problem, or a data structure problem.
- Often mathematically unsophisticated. Sometimes grows in mathematical maturity as CS types perceive gaps and re-engineer.

CS types are not necessarily sophisticated

- Greenspun's Tenth Rule of Programming: "Any sufficiently complicated C or Fortran program contains an ad-hoc, informally-specified bug-ridden slow implementation of half of Common Lisp."
- i.e. complex systems implemented in low-level languages cannot avoid reinventing and/or reimplementing (maybe poorly on both counts) the facilities built into higher-level languages.

Strategies: the 2 language approach

- Build an interpreter/compiler for the user language which looks like math.
  - algebraic operations (+ * /)
  - Built-in functions (sin cos)
  - declarative style definitions
  - algebraic domains, geometric domains
  - Commands (solve, integrate, factor)
- but also should allow simple programming:
  - imperative style program definitions
  - Pattern-replacement rules

- The implementation language should be different and concentrate on data representation,
  - efficiency, portability, implementation of the user language as well as most algorithms.
- Simple algorithms in the user language are possible.
- (Macsyma: Macsyma+ Lisp
- Mathematica: Mathematica + C
- Most lisps: lisp + baby lisp + C + assembler)

Strategies: the library approach

- There is no user language. The implementation language, which is C++ or Java, is a library. GiNaC for example.
- Typically the first major "follow-on" effort is to provide a user-level front end, e.g. in Python or Java or Tcl or... thereby making the claim "we are only building a library" debatable. The library view therefore is really just deferring controversial decisions briefly.

Strategies: the one language. Design a new one

- Plan: The user language is sufficient, with an appropriate compiler, to write the whole system.
- The "any decent compiler" fallacy.
- What should the target for the compiler be?
- Often C is used as a "machine-independent assembly language."
- Some systems used Fortran that way (SAC-1,2)
- Axiom: The compiler Aldor for the Axiom language produced alternatively Lisp or C. In theory, no programs should be written in lisp or C directly.
- More discussion under "front end".

Implementation Language design issues

A person or team who invents a language must consider the costs of maintaining that implementation. In at least a few prominent cases, the inventor has little or no experience in languages and makes a muddle of it. Understanding SICP (Abelson/Sussman) would help, in my view.

Examples: Macsyma (but that was 1968!), Mathematica, Maple. Even Matlab.

A common approach

- An interpreter/byte-code + compiler or interface to C
- Tcl/TK, Reduce-CSL, Macsyma-CLISP, Mathematica

- Here's how it works:
- A small program is written in C or other "simple" language.
- (simple = simple statements in C are converted to
- code with relatively easily-predicted execution time. Often a substitute for assembler).
- This program is sometimes called a kernel.

What's in the kernel?

- The kernel includes…
- All the material that is necessary to raise the discourse to a reasonable level:
  - bignums,
  - storage allocation,
  - maybe polynomial arithmetic.
- an interpreter for a byte-code language.
- I/O and some OS dependent items
  - Web interface
  - Security
  - Efficient.. Numerics? Graphics? Database?

What is byte code?

- a design for a kind of simplified machine whose operations are just what the application needs, rather than the usual mix of assembly language.
- a framework (perhaps with stack management, storage management, security constraints) for this "virtual machine".
- its operation: a virtual machine evaluator or interpreter goes through these byte codes and calls subroutines in the kernel (or some other mechanism like threaded code) for execution.

Pros/Cons of byte code

- Advantages
  - usually much more compact than assembler. Important for cache memory.
  - usually portable across different architectures
  - could be intermediate code for further compilation steps (JIT).
  - could be used as a basis for distributing small patches.

- Disadvantages
  - Slower, usually.
  - VM design may restrict computation. e.g. Java VM can't do Lisp easily.

Mixing byte code and ordinary binary code

- CSL approach: (Codemist Standard Language)
  - While OptimizeMore do
    - {Load the whole system and run for a while as byte-code, profiling.
    - Compile and reload the parts that seem to be bottlenecks}
  - Used for implementing Reduce with a modest size memory.
  - Same data structures
- CLISP (implementation of Common Lisp)

Why is slow code sometimes OK?

It is usually acceptable to do infrequent parts of the computation at a slower speed. Even 100 times slower maybe acceptable. Trade off: size of code, fast turn-around (no need to run a compiler on it), ease of debugging. {cf. Matlab.. overhead to call invert(M)..}

- Lisp usually has interpreter and compiler both, with fast turn around for compiling:
- some Lisp systems always compile (and type check) before running (CMUCL), optional with Allegro CL.

Traditional strategy for numeric code

If you provide an opportunity to segment off numeric-data intensive operations into separate routines, you can also finesse efficiency.

- plotting z=f(x,y) on a grid of 100x100 points requires computing f 10000 times.
- integrating f(x) numerically from 0 to 1 may require computing f at many points.
- Compiling f may be plausible even if done at run time.
- Call to numeric library if appropriate: same speed as if called from C or Fortran. The time to check that an n £ n array consists entirely of double-floats is $O(n^2)$. If you are doing an $O(n^3)$ operation, the check is negligible. Or you may just have a package that assumes numerics. (Matlab, special packages e.g. for dense arrays, in Mathematica, Maple, Macsyma)

digression on Reduce

- Reduce is written in a dialect called RLISP, which is an infix kind of lisp. e.g. this makes syntactic sense:
- a := list(1, 2, 3); → (setq a (list 1 2 3))
- car a + car b ; → (+ (car a)(car b))

- There are two modes, algebraic and symbolic (not the clearest names...)
- RLISP is implementable in common lisp, CSL, PSL, Scheme ...

digression on SMP, pre-Mathematica

- (Brief biased history of SMP written by a group led by Stephen Wolfram when at Caltech:)
- SW decided that Macsyma was too slow and he could do much better by writing in C. He got together with some colleagues and wrote SMP. A legal hassle with Caltech made SW more cautious the next time he wrote a program.
- SMP was fatally flawed in several ways, but one was the unreliability of the underlying storage mechanism. (Neither GC nor reference count.)

digression on Mathematica

- Redid everything to produce Mathematica, which consists of some code written in a customized version of C (actually may be something like Objective C, with some kind of automatic reference counting.) and the user-language for Mathematica. For Version 4, the code for the kernel consists of about 650,000 lines of C and 30,000 lines of Mathematica.
- In the Mathematica 4 kernel the breakdown of different parts of the code is roughly as follows:
  - language and system: 30%;
  - numerical computation: 25%;
  - algebraic computation: 25%;
  - graphics and kernel output: 20%.
- Stats for version 5.2? Presumably much larger.

The front-end problem

- Input: how do we convey commands
  - factor
  - solve
- How do we convey mathematical data
  - w 2 [-1,1]
  - H is a Hilbert space
  - z is a complex number {clearly a lie. z is a letter!}
  - Introduce new notation?
- Output:
  - "scientific visualization"
  - Publication quality display

History of CAS Output

- First generation: line-display: integral(sin(x^2+1),x)
- Second generation: "glass teletype" with typeset display like this (Charybdis, 1966)

```
                                  /
                                  [      2
(D5)                              I SIN(x  + 1) dx
                                  ]
                                  /
```

- Third generation: typeset

(d5)

$$\int \sin\left(x^2 + 1\right)\, dx$$

How important is that fancy display?

- The answer to that problem is..

```
((sqrt(%pi) * (((sqrt(2) * %i - sqrt(2)) * sin(1)
+ ( - sqrt(2) * %i - sqrt(2)) * cos(1))
 * erf((((sqrt(2) * %i + sqrt(2)) * x)/2))
 + ((sqrt(2) * %i + sqrt(2)) * sin(1) + (sqrt(2) -
sqrt(2) * %i) * cos(1))
 * erf((((sqrt(2) * %i - sqrt(2)) * x)/2))))/8)
```

Or displayed in a fancy way (from Macsyma)

$$(d6)\quad -\frac{\sqrt{\pi}\left(\left(\left(\sqrt{2}\,i-\sqrt{2}\right)\sin(1)+\left(-\sqrt{2}\,i-\sqrt{2}\right)\cos(1)\right)*\operatorname{erf}\left(\frac{\left(\sqrt{2}\,i+\sqrt{2}\right)x}{2}\right)+\left(\left(\sqrt{2}\,i+\sqrt{2}\right)\sin(1)+\left(\sqrt{2}-\sqrt{2}\,i\right)\cos(1)\right)*\operatorname{erf}\left(\frac{\left(\sqrt{2}\,i-\sqrt{2}\right)x}{2}\right)\right)}{8}$$

Also Maple and TeX...

$$\frac{1}{2}\sqrt{2}\,\sqrt{\pi}\left(\cos(1)\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,x}{\sqrt{\pi}}\right)+\sin(1)\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,x}{\sqrt{\pi}}\right)\right)$$

$$1/2\,\sqrt{2}\sqrt{\pi}\left(\cos(1)\,FresnelS\left(\frac{\sqrt{2}x}{\sqrt{\pi}}\right)+\sin(1)\,FresnelC\left(\frac{\sqrt{2}x}{\sqrt{\pi}}\right)\right)$$

Typesetting is now easily solved at the demo-ware level. A few serious problems remain..

- Easy to hack together TeX and display
- BUT
- Serious solutions must address very large multi-line formulas, interactivity (selecting subexpressions)
- The spreadsheet idea
- Renaming
- Detailed control (macsyma demo $1/e^x$ or e^{-x} or …)
- MathML/XML.
  - output your formula to a browser and hope for the best

Our example in MathML (generated by Maple)

```
"<math xmlns='http://www.w3.org/1998/Math/MathML'> <semantics><mrow
xref='id33'><mrow><mrow><mfrac
xref='id1'><mn>1</mn><mn>2</mn></mfrac><mo>&InvisibleTimes;</mo><mrow
xref='id3'><msqrt><mn
xref='id2'>2</mn></msqrt></mrow></mrow><mo>&InvisibleTimes;</mo><mrow
xref='id5'><msqrt><mn
xref='id4'>&pi;</mn></msqrt></mrow></mrow><mo>&InvisibleTimes;</mo><mfence
d><mrow xref='id32'><mrow xref='id18'><mrow xref='id8'><mi
xref='id6'>cos</mi><mo>&ApplyFunction;</mo><mfenced><mn
xref='id7'>1</mn></mfenced></mrow><mo>&InvisibleTimes;</mo><mrow
xref='id17'><mi>S</mi><mo>&ApplyFunction;</mo><mfenced><mrow
xref='id16'><mfrac><mrow xref='id13'><mrow xref='id11'><msqrt><mn
xref='id10'>2</mn></msqrt></mrow><mo>&InvisibleTimes;</mo><mi
xref='id12'>x</mi></mrow><mrow xref='id15'><msqrt><mn
xref='id14'>&pi;</mn></msqrt></mrow></mfrac></mrow></mfenced></mrow></mrow
><mo>+</mo><mrow xref='id31'><mrow xref='id21'><mi
xref='id19'>sin</mi><mo>&ApplyFunction;</mo><mfenced><mn
xref='id20'>1</mn></mfenced></mrow><mo>&InvisibleTimes;</mo><mrow
xref='id30'><mi>C</mi><mo>&ApplyFunction;</mo><mfenced><mrow
xref='id29'><mfrac><mrow xref='id26'><mrow xref='id24'><msqrt><mn
xref='id23'>2</mn></msqrt></mrow><mo>&InvisibleTimes;</mo><mi
xref='id25'>x</mi></mrow><mrow xref='id28'><msqrt><mn
xref='id27'>&pi;</mn></msqrt></mrow></mfrac></mrow></mfenced></mrow></mrow
></mrow></mfenced></mrow><annotation-xml encoding='MathML-Content'><apply
id='id33'><times/><cn id='id1' type='rational'>1<sep/>2</cn><apply
id='id3'><root/><cn id='id2' type='integer'>2</cn></apply><apply
id='id5'><root/><pi id='id4'/></apply><apply id='id32'><plus/><apply
id='id18'><times/><apply id='id8'><cos id='id6'/><cn id='id7'
type='integer'>1</cn></apply><apply id='id17'><csymbol id='id9'
```

Our example in MathML (generated by Maple) continued

```
definitionURL='http://www.maplesoft.com/MathML/FresnelS'>FresnelS</csymbol
><apply id='id16'><divide/><apply id='id13'><times/><apply
id='id11'><root/><cn id='id10' type='integer'>2</cn></apply><ci
id='id12'>x</ci></apply><apply id='id15'><root/><pi
id='id14'/></apply></apply></apply></apply><apply id='id31'><times/><apply
id='id21'><sin id='id19'/><cn id='id20'
type='integer'>1</cn></apply><apply id='id30'><csymbol id='id22'
definitionURL='http://www.maplesoft.com/MathML/FresnelC'>FresnelC</csymbol
><apply id='id29'><divide/><apply id='id26'><times/><apply
id='id24'><root/><cn id='id23' type='integer'>2</cn></apply><ci
id='id25'>x</ci></apply><apply id='id28'><root/><pi
id='id27'/></apply></apply></apply></apply></apply></apply></annotation-
xml><annotation
encoding='Maple'>1/2*2^(1/2)*Pi^(1/2)*(cos(1)*FresnelS(2^(1/2)/Pi^(1/2)*x)
+sin(1)*FresnelC(2^(1/2)/Pi^(1/2)*x))</annotation></semantics></math>"
```

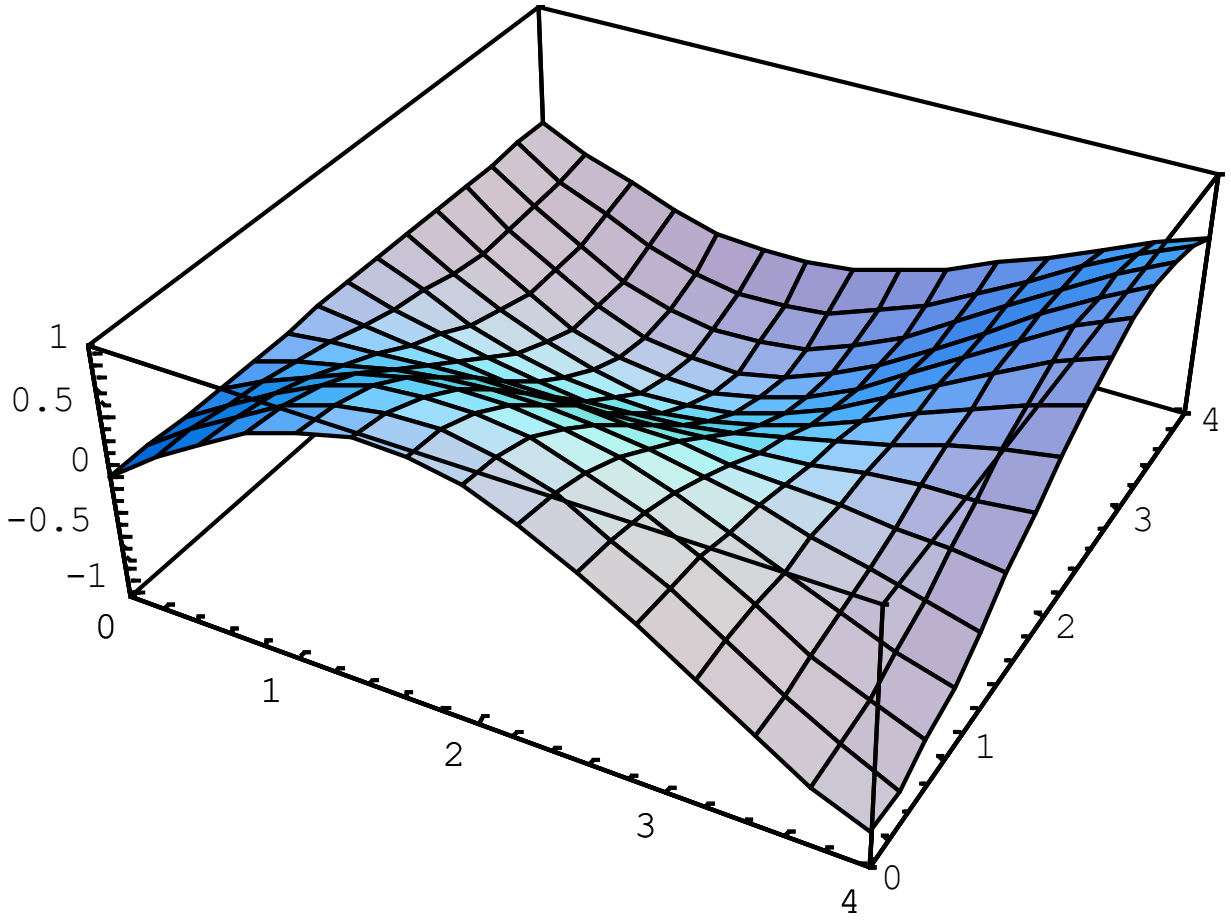
MathML as a standard could make everyone's output work with everyone's display

- That's the thought, anyway.
- Extensions for presentation and content
- OpenMath.org

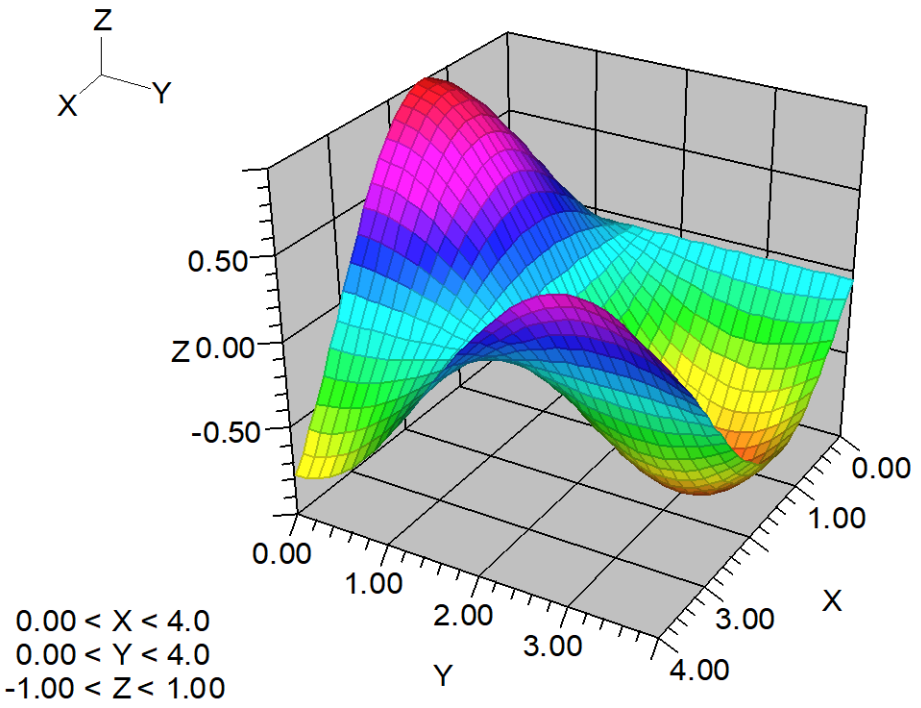
Front-end important for flash: graphics too

- Mathematica's introduction in the mid 1980s made the case clear that marketing was important for CAS. And a lot of the marketing required fancy displays on the computers then coming on the market: NeXt and the new Macintosh.
- For the front end on Windows, Mac, Unix (X), significant amount of specialized code is needed to support each different type of user interface environment.
- The **front end** for Mathematica (v4) contains ``about 600,000 lines of system-independent C source code, of which roughly 150,000 lines are concerned with expression formatting. Then there are between 50,000 and 100,000 lines of specific code customized for each user interface environment.''

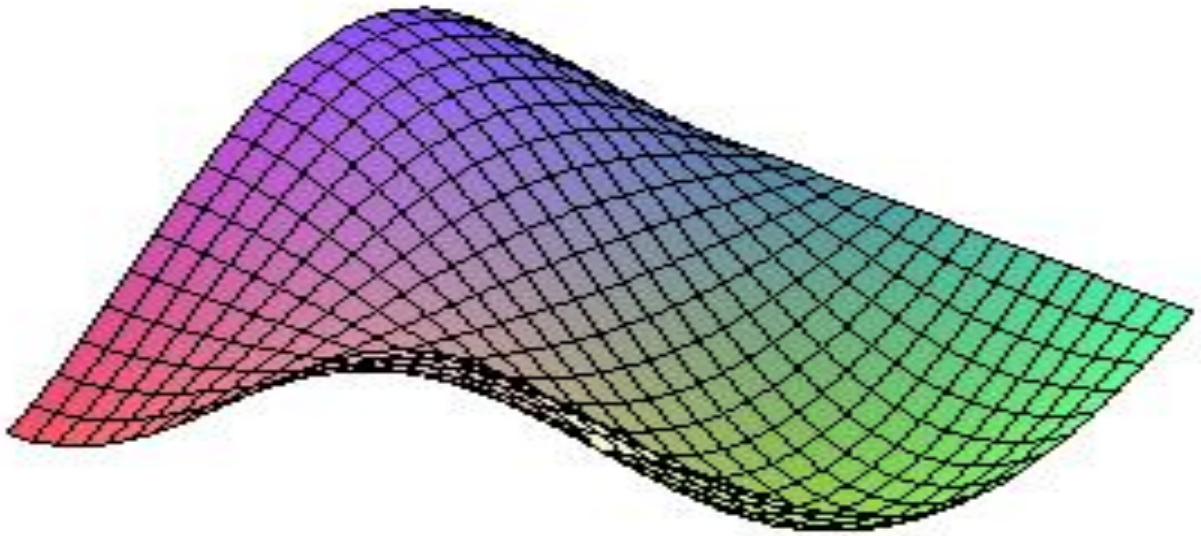
Mathematica Sin[x]*Cos[y]



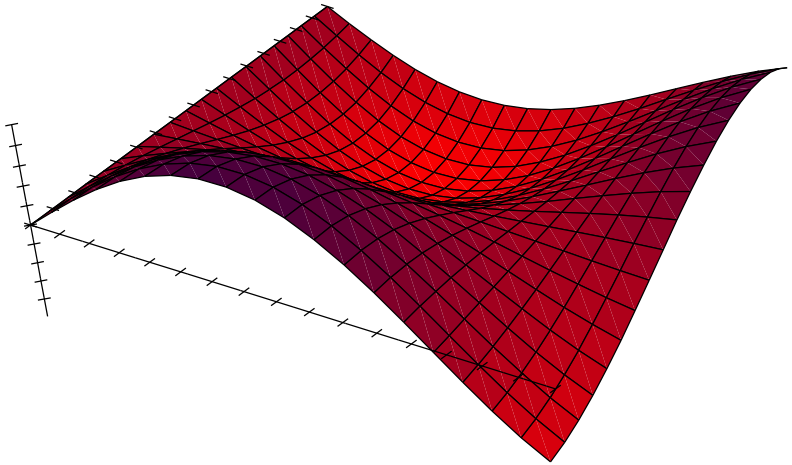
Macsyma sin(x)*cos(y)



Maple, ditto (no "options")



Mupad



Other systems (e.g. Reduce, open-source Maxima)

- Maxima uses utilities like Gnuplot in an attempt to be mostly portable. Hence:
  - Usually less "integrated" into the system.
  - Portable solutions don't take advantage of the special features of the interfaces... e.g. X window simulator on Microsoft Windows?

The Notebook "paradigm" input + output

- commands interspersed with displays
- text and pictures interspersed
- typeset displays
- command-line editing
- outlining (suppression of detail)
- save/restore/execute
- where do edited commands go? (not in-place)

Digression: Correctness (words from MMA book)

- ''The standards of correctness for Mathematica are certainly much higher than for typical mathematical proofs. But just as long proofs will inevitably contain errors that go undetected for many years, so also a complex software system such as Mathematica will contain errors that go undetected even after millions of people have used it. Nevertheless, particularly after all the testing that has been done on it, the probability that you will actually discover an error in Mathematica in the course of your work is extremely low. Doubtless there will be times when Mathematica does things you do not expect. But you should realize that the probabilities are such that it is vastly more likely that there is something wrong with your input to Mathematica or your understanding of what is happening than with the internal code of the Mathematica system itself. If you do believe that you have found a genuine error in Mathematica, then you should contact Wolfram Research at the addresses given in the front of this book so that the error can be corrected in future versions.''

Coming up:
Numeric Data and Algorithms in CAS

- Numeric, symbolic
- Some Sources:
- http://cs.berkeley.edu/~fateman/papers/
  - mac82.pdf mma.review.pdf (reviews by RJF of macsyma and mathematica)
- www.math.unm.edu/~wester (detailed comparisons of CAS and various other links)
- http://krum.rz.uni-mannheim.de/cabench/diractiv.html (another benchmark collection)