# Closed Forms for Summations

Lecture 19

# Two categories (maybe 3) of Summations

- Indefinite summation:
  - $\Sigma_{1 \cdot i \cdot n}$ f(i); f not dependent on n
    f dependent on n
  - -- finite difference calculus has a long history: initial work done by Newton, Euler, Bernoulli, Boole.

  Definite summation (particular solutions): Zeilberger, Gosper, Ramanujan, etal
  - The Book ***A=B*** by Petkovsek, Wilf, Zeilberger**)**
  - **Online at http://www.cis.upenn.edu/~wilf/AeqB.html**

# Indefinite summation parallels to integration

- integration of polynomials
- integration of rational functions
- difference operator $\Delta$ parallels the derivative
- $\Sigma$ and s are similar

- But not similar enough for some purposes!

# Some simple examples (Macsyma, in this case)

$$\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2}$$

$$\sum_{i=1}^{n} \frac{i}{2^i} = -\frac{n}{2^n} - \frac{2}{2^n} + 2$$

# A more elaborate example

$$\sum_{n=0}^{n} \frac{n^4\,4^n}{\binom{2\,n}{n}} = \frac{2\,(n+1)\,\left(63\,n^4+112\,n^3+18\,n^2-22\,n+3\right)\,4^n}{693\,\binom{2\,n}{n}} - \frac{2}{231}$$

# Start simply

- if we need $g(n) = \sum_{i=a}^{n} f(i)$ we approach by finding *the indefinite summation*

$h(x) = \sum_{i=0}^{x-1} f(i)$

Then we can find the *definite summation* from a to n:

$g(n) = h(n+1) - h(a)$.

Sidestepping any issues of singularities.

Note that this means sum from a to b + sum from b+1 to c = sum from a to c, even if $a<b<c$ is false..

$\Delta$ h(x) *is defined* by $h(x+1)-h(x)$, is $f(x)$

# Also

- $\Delta^{-1} f(x) = h(x)$, and as a reminder, note that
  $$h(x) = \sum_{i=0}^{x-1} f(i)$$
- Note parallel: we can obtain an expression for the summation by anti-differencing; compare to integration by anti-differentiation.

# Simple Properties of $\Delta$

- Unique up to addition of functions whose first difference is zero
  - Constants, since $h(x):=C$ means $h(x+1)-h(x)= 0$
  - functions with period 1, e.g. $h(x):=\sin(\pi x)$

# It is also convenient to define shift operator, E

- $E\ f(x) := f(x+1)$
- hence
  - $\Delta f(x) = Ef(x)-f(x)$

# $\Delta(f¢g)=Eg¢\Delta f+f\Delta g$

Proof that $\Delta(f¢g)=Eg\Delta f+f\Delta g$
Expand the right side,
g(x+1)*(f(x+1)-f(x))+f(x)*(g(x+1)-g(x))

.. Multiply out..
g(x+1)*f(x+1)-<u>g(x+1)*f(x)+f(x)*g(x+1)</u>-f(x)*g(x)
.. The underlined terms cancel, leaving
f(x+1)*g(x+1)-f(x)*g(x),
Which is $\Delta$ (f(x)¢ g(x)).
QED

# $\Delta(1/g)=\Delta\ g\ /\ (g\ E\ g)$

Proof that $\Delta(1/g)=\Delta\ g\ /\ (g\ E\ g)$
Expand the right side,
$1/g(x+1)-1/(g(x)$, combine over common denominator:

$(g(x+1)-g(x))/(g(x)*g(x+1))$
Which corresponds to
$\Delta g\ /\ (g\ E\ g)$

QED

# Summary of properties of Δ

$$\Delta k\, f(x) = k\, \Delta f(x) \qquad k \in \;F$$

$$\Delta(g(x) + f(x)) = \Delta g(x) + \Delta f(x)$$

$$\Delta(f(x) \cdot g(x)) = Eg(x) \cdot \Delta f(x) + f(x) \cdot \Delta g(x)$$

$$\Delta\left(\frac{1}{g(x)}\right) = -\frac{\Delta g(x)}{g(x)\, Eg(x)}$$

$$\Delta\left(\frac{f(x)}{g(x)}\right) = -\frac{f(x)\,\Delta g(x) + g(x)\,\Delta f(x)}{g(x)\, Eg(x)}$$

# Occasionally useful property

- The chain rule
- $\underset{1}{\overset{x}{\Delta}} f(g(x)) = \underset{\Delta g(x)}{\overset{g(x)}{\Delta}} f(x)$
- where $\underset{h}{\overset{x}{\Delta}} f(x) = f(x+h)-f(x)$

# Proof..

- The chain rule

- $$\Delta_{1}^{x} f(g(x)) = \Delta_{\Delta g(x)}^{g(x)} f(x)$$

$f(g(x+1))-f(g(x))$ on left
$f(g(x)+\Delta g(x))-f(g(x))$ on right =
$f(g(x)+g(x+1)-g(x))-f(g(x))$… QED

## The simplest non-trivial form to sum is a polynomial

- $A(x) = \sum a_i x^i$
- The analogy to differential calculus is to integrate, term by term:
  - easy since $Dx^n = nx^{n-1}$.
- Differences of powers are not so concise:

$\Delta(x^n) = (x+1)^n - x^n = \sum(\text{binomial}(n,i)\ x^i, i=0..n-1)$... has n-1 terms.

INSTEAD consider *factorial functions*, defined by

$[x]_n = x(x-1)(x-2)....(x-n+1)$.

# What is the difference of a factorial function?

- $\Delta\,[x]_n = n[x]_{n-1}$
- Proof:

$\Delta\,[x]_n = E\,[x]_n - [x]_n$ .

$E[x]_n = (x+1)x(x-1)(x-2)\ldots.\ (x-n+2)$

$[x]_n = x(x-1)(x-2)\ldots.(x-n+2)(x-n+1).$

All the terms in red are the same, and one can factor them out. they are $[x]_{n-1}$. The remaining factor is simply $(x+1)-(x-n+1) = n$.

- The product is $n[x]_{n-1}$ . QED

# To sum a polynomial of degree d, convert it to factorial form:

- one way is to set up a table by which we expand d factorial functions $[x]_1$+x, $[x]_2= x^2-x$, etc, solve for powers of x, e.g. $x^2= [x]_2-[x]_1$, and we can substitute into any polynomial, collect [] terms and provide the result.
- Another is to use Newton's *divided difference interpolation formula* , which as a special case gives:
- $f(x)=sum([x]_i/i!\ \Delta^i f(0))$ where we mean by $\Delta f(0)$, is $\Delta f(x)$ where x=0. We define higher differences in the obvious way: $(\Delta^2 f(0) = \Delta (\Delta(f(0)) = \Delta (f(1)-f(0)) = \Delta(f(2)-f(1)-(f(1)-f(0))= f(2)-2f(1)+f(0).\ )$ This is almost always written in a table.

# Divided difference table for f= 3*x^3-2*x+1

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 18 | 18 |
| 1 | 2 | 19 | 36 | |
| 2 | 21 | 55 | | |
| 3 | 76 | | | |

# Divided difference table for f= 3*x^3-2*x+1

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|---|---|---|---|---|
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$\Delta^{-1} f = \mathrm{sum}([x]_i/i!\ \Delta^i f(0)) = 1*[x]_1 + 1/2*[x]_2 + 18/3!*[x]_3 + 18/4!*[x]_4$ .

# Converting BACK to conventional polynomial form can be done by expanding $[x]_i$ and combining terms

Total is 1*x

+1/2* x*(x-1)

+18/3!* x*(x-1)*(x-2)

+18/4!* x*(x-1)*(x-2)*(x-3) =

$$\frac{3\,x^4}{4} - \frac{3\,x^3}{2} - \frac{x^2}{4} + 2\,x$$

# Sums of rational functions

- Define factorial operators on functions…
- $[f(x)]_k = f(x)$ ¢ $f(x-1)$ ¢ … ¢ $f(x-k+1)$ for $k > 0$
- extend the operator by noticing
- $[f(x)]_k = [f(x)]_r$¢ $[f(x-r)]_{k-r}$
- Define $[f(x)]_0$ to be 1 and use the previous line as an identity. Then for k=0 we get
- $[f(x)]_{-r} = 1/[f(x+r)]_r$

# Differences of factorials

(11) $$[f(x)]_{-\ell} = \frac{1}{[f(x+\ell)]_{\ell}}$$

We will call the value of k or $\ell$ in equations 9 and 11, the factorial degree
function, because of its parallel to the "power" degree. We now proceed to
examine the differences of factorials.

(12) $$\Delta[f(x)]_{k} = [f(x)]_{k-1} \underset{k}{\triangle\!\!\!\!\!x}\ f(x-k+1)\ , \ k>0\ .$$

A special case of this is eq. 5 for factorial polynomials.

(13) $$\Delta[f(x)]_{-\ell} = -[f(x)]_{-(\ell+1)} \underset{\ell}{\triangle\!\!\!\!\!x}\ Ef(x)$$

$$= \frac{\underset{\ell}{\triangle\!\!\!\!\!x}\ Ef(x)}{[f(x+\ell+1)]_{\ell+1}}$$

Notice that the factorial degree is decreased (resp. increased) by 1 on diff
encing factorials (resp. reciprocal factorials).

R. Moenck, Macsyma Users' Conf. 1977

# What does this mean for summation ($\Delta^{-1}$)?

- If we can get rational expressions so they look like the RHS of that equation, we can find their summation, namely $[f(x)]_{-1}$

# We need to use Shift Free Decomposition to go further.

Given a product of functions, we can decompose it into a product of factorial functions.

Let S=a ¢ b ¢ c where a,b,c are mutually relatively prime and Ea=b. Then shift S:

ES = (Ea)¢(Eb)¢(Ec) = b ¢ Eb ¢ Ec

GCD(S,ES) = b

So we can divide out b and a from S and express

S=$[b]_2$¢ c .

# If we apply this observation repeatedly, we can get S to be shift free

- $S=[s_1]_1 ¢ [s_2]_2 ¢ \ldots ¢ [s_k]_k$ where the individual $s_k$ are shift-free.
- *Analogous* to partial fraction decomposition in the differential calculus Hermite integration process, we can form a shift-free partial fraction for some rational function we wish to sum. That is,
- $A(x)/S(x) = \sum (A_i/[s_i]_i)$, i=1..k
- and a "complete" decomposition
- $A(x)/S(x) = \sum \sum(A_{ij}/[s_i]_j)$, i=1..k,j=1..i

# Shift-1 independence is not enough. We need to show S(x) is k-shift-free

- Compute resultant of S(x) and S(x+k) with respect to k. If there is an integer k>1 shift, then fill in the terms for numerator and denominator. e.g. if S = x*(x+3), change it to

$[x+3]_4$ and multiply numerator by (x+1)(x+2).

# Summation by parts

- Similar to Hermite integration using

$\Delta^{-1}(u¢\ \Delta\ v) = u\ ¢\ v - \Delta^{-1}(Ev\ ¢\ \Delta\ u)$

can be used to reduce denominators of the form $[x_i]_j$ to $[x_i]_1$

- EVENTUALLY... one gets a rational function plus an indefinite summation of terms with shift-free denominators of factorial degree 1.

# The transcendental part

- Define $\psi_m(x) = D^m(\log\Gamma(x+1))$, $m>0$ where $n! = \Gamma(n+1)$ is the well-known gamma function

$\Delta\psi_m(x) =$

$D^m(\Delta \log\Gamma(x+1)) =$

$D^m(\log(\Gamma(x+2)/\Gamma(x+1))) =$

$D^m \log(x+1) =$

$D^{m-1}(1/(x+1)) =$

$((-1)^{m-1}$¢ $(m-1)!$¢$(x+1)^{-m}$.

## The sum of a negative power of x+1 finishes the task

- $\Delta^{-1}(x+1)^{-m} = (-1)^{m-1}/(m-1)!\ \psi_m(x)$

- The $\psi_m$ functions are known as polygamma functions and serve a role similar to logs in Hermite integration.
- Rational summation is pretty much solved, though people still look for fast ways of doing some of the steps (shift-free decomposition).

# This is not the end of the story: what about more elaborate summands?

- Gosper's algorithm looks at $\sum a_i = \Delta^{-1} a_i$ by seeking a "telescoping function" f(n).
- Let $a_n = \Delta\, g(n) = g(n+1)-g(n)$
- then suppose $g(n)=f(n)*a_n$.
- We have to solve the functional equation
- $C(n)=a_{n+1}/a_n = (f(n)+1)/f(n+1)$

- Only the ratio of 2 terms is used (easily computed). If C(n) is rational in n, then this is called hypergeometric summation.

## Restrictions/ Extensions

- Note that the terms $a_n$ can be far more general than rational; the requirement is on the RATIO: $a_{n+1}/a_n$ is rational.
- Gosper's work is the basis for a decision procedure, widely used in computer algebra systems.
- Excellent reference: *A=B*, by Petkovsec, Wilf, Zeilberger