Conversion of the von Mises-Fisher concentration parameter to an equivalent angle

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July 2021

The von Mises-Fisher probability distribution function is written for $X \in \mathcal{S}^2$ (unit 3-dimensional sphere):

$$vmf_{\mu,\kappa}(X) = C(\kappa)\, exp(\kappa \mu^T X)$$
$$\text{With } C(\kappa) = \frac{\kappa}{4\pi sinh(\kappa)}$$

where $\mu \in \mathcal{S}^2$ represents the mean direction of the distribution and $\kappa \in \mathbb{R}^{+*}$ is the concentration parameter of the distribution (analogue to the invert variance of a Gaussian).

Taking advantage of the isotropy of the von Mises-Fisher distribution (cylindrical symmetry around μ) we derive here a formula for converting the concentration parameter κ to an equivalent more interpretable angle θ_α such that a fraction α of the probability distribution function centered on μ is contained within an angle θ_α of μ.

By rewritting X in polar coordinates i.e.

$$X = r \begin{bmatrix} cos(\phi)sin(\theta) & sin(\phi)sin(\theta) & cos(\theta) \end{bmatrix}^T \tag{1}$$

and setting μ to $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ (the same result can be derived for different values of μ using a rotation) we obtain for a given value of α:

$$\begin{aligned}
\alpha &= \int_{\mathcal{S}^2,\, \theta < \theta_\alpha} vmf_{\mu,\kappa}(X)\, dS \\
&= \int_{\theta=0}^{\theta_\alpha} \int_{\phi=0}^{2\pi} vmf_{\mu,\kappa}(X)\, sin(\theta)\, d\theta\, d\phi \\
&= C(\kappa)\, 2\pi \int_{\theta=0}^{\theta_\alpha} exp(\kappa\, cos(\theta))\, sin(\theta)\, d\theta \\
&= \frac{exp(\kappa) - exp(\kappa\, cos(\theta_\alpha))}{exp(\kappa) - exp(-\kappa)}
\end{aligned} \tag{2}$$

After a few simplifications, we obtain:

$$\theta_\alpha = arccos \left[1 + \frac{log[1 - \alpha + \alpha\, exp(-2\,\kappa)]}{\kappa} \right] \tag{3}$$