# LTI systems

Many physical processes can be modeled as LTI systems.

An LTI system is completely characterized by its response to the unit impulse.

## DT LTI systems: the convolution sum

## CT LTI systems: the convolution integral

## Properties of LTI systems

## CT causal LTI systems described by differential equations

LCC differential equations model an extremely important class of CT systems.

Example.

DE provide an implicit specification of the system – they describe a relationship between input and output, rather than an explicit expression for the system output as a function of the input.

To obtain an explicit expression, we need to solve the DE.

To solve the DE, we need auxiliary conditions.

For example, if we apply a constant voltage of 1V to an RC circuit for 10 seconds, we cannot determine the final capacitor voltage without knowing the initial capacitor voltage.

To completely characterize a system, we need both the DE and the auxiliary conditions.

Going back to the example, let , where is a real number.

The complete solution to the DE is the sum of a particular solution and a homogeneous solution.

satisfies , while is a solution of the homogeneous DE

A common method for finding is to look for a “forced response” – i.e. a signal of the same form as the input. Since for , we guess a solution for of the form

is a number we must determine.

Substitute into the DE:

To find , we guess a solution of the form

Substitute into homogeneous DE:

Therefore is a solution to the homogeneous DE for any value of .

The complete solution of the DE for is

As noted earlier, the DE by itself does not uniquely specify in response to . There are infinitely many solutions for infinitely many values of . To determine , we need to specify an auxiliary condition – different choices for this auxiliary condition lead to different and different relationships between the input and output.

For the most part, we want our systems to be LTI and causal, and in this case, the auxiliary condition is the condition of initial rest.

Initial rest means that if for , then must also equal 0 for .

For our example, for . Initial rest implies that for .

Then for , . For , . Then the full solution is

**Important points concerning LCC differential equations and the systems they represent:**

* The response to input will generally consist of the sum of a particular solution to the DE and a homogeneous solution, which is the solution to the DE with the input set to 0. The homogeneous solution is often called the natural response of the system.
* To completely determine the input-output relationship of a system described by a DE, we must specify auxiliary conditions. Different auxiliary conditions lead to different input-output relationships. For the most part, we use the condition of initial rest, which means that the output of the system is zero until the input becomes nonzero. Under this condition, the system is LTI and causal.
* Mathematically, initial rest means that if for , then for . To solve the DE, we evaluate at to solve for the output for . For example, when charging an RC circuit, initial rest means that the capacitor is initially discharged. This also explains time invariance: when charging a capacitor, if you apply the same input voltage at different times, the capacitor voltage should have the same response except time-shifted.
* If the initial condition is nonzero, the resulting system is incrementally linear. The overall response is the superposition of the response to the initial conditions alone (input set to 0) and the response to the input with an initial condition of 0 (i.e., the response of the causal LTI system).

General th-order LCC differential equation:

When , reduces to an explicit function of .

When , the input-output relationship is implicitly specified. The solution to the DE consists of the particular solution and the homogeneous solution. The homogeneous solution is the solution to the homogeneous DE:

The solutions to this equation are also called the natural responses of the system.

Auxiliary conditions are needed to determine the explicit input-output relationship. We mostly use the at rest condition, which means if for , we assume for , and therefore, the response for can be calculated from the DE with the initial conditions

Under the condition of initial rest, the system is causal and LTI.

## DT causal LTI systems described by difference equations

LCC difference equations are the discrete-time counterpart of LCC differential equations.

The th-order LCC difference equation is

The solution in response to an input consists of a particular solution and a solution to the homogeneous equation

The solutions to the homogeneous equation are also known as the natural responses of the system.

To fully specify the system response, we need auxiliary conditions. We typically use the condition of initial rest, which means if for , then for . Under this condition, the system is LTI and causal.

In the CT case, we guessed the analytic forms of the particular and homogeneous solutions.

In DT, we can directly evaluate for by rearranging the LCC difference equation.

Starting with and knowing the auxiliary conditions , and knowing the input, we can recursively solve for .

In the special case of ,

The output is an explicit function of the input, so we don’t need auxiliary conditions. This is an LTI system with impulse response

That is, the expression for is exactly equal to the convolution sum. The impulse response has finite duration, so this type of system is often called a finite impulse response (FIR) system.

Let’s consider a case where .

Let the system be initially at rest, and let . Since for , then for , and we have the initial condition .

Since this system is LTI (initial rest), its behavior is completely characterized by its impulse response. For ,

The impulse response has infinite duration. In general, for , the LTI system corresponding to the LCC difference equation together with the condition of initial rest will have an infinite duration impulse response. These are called infinite impulse response (IIR) systems.

## Block diagrams of first-order LCCDE systems

Block diagram representations are very important.

1. The block diagram representation for CT systems can be directly translated into a program for simulating the system on a computer.
2. The block diagram representation for DT systems show how these systems can be implemented in digital hardware.

First-order LCC difference equation:

A diagram of a circuit

AI-generated content may be incorrect.

This is an example of a feedback system. The feedback is due to the recursive nature of the equation.

D is a single delay or memory element; the initial value of this memory element is the necessary initial condition for calculating the output. When the system is initially at rest, the initial value is 0.

First-order LCC differential equation:

A diagram of a circuit

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D is a differentiator.

In practice, we generally don’t use this representation since differentiators are difficult to implement and extremely sensitive to errors and noise.

Alternative: rewrite the equation and integrate from to . If , then .

A diagram of a mathematical equation

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Integrators are readily implemented using op amps. This is the basis of early analog computers and modern analog computation systems.

In CT, the integrator represents the memory storage element of the system. If we integrate from time instead of , we get

The specification of requires an initial condition, the value of , which is the value that the integrator stores at time .

## Singularity functions

# Time and frequency characterization of signals and systems

## First-order CT systems

Any system with a frequency response that can be written as a ratio of polynomials in (CT) or (DT) can be written as a product or sum of first- and second-order systems.

Therefore, high-order systems are frequently implemented by combining first- and second-order systems in cascade or parallel.

First-order system (causal, LTI, stable):

is the time constant of the system. It controls the rate at which the first-order system responds. At ,

* The impulse response has reached its value at
* The step response is within of its final value

As , impulse response decays more sharply and the rise time of the step response is shorter.

The step response does not exhibit any ringing.

We often plot the log-magnitude response on a log frequency scale. In decibels,

The low- and high-frequency asymptotes of the log magnitude are straight lines. Below , , and above , drops 20dB per decade (of ). This is the idea behind Bode plots.

We can make a straight-line approximation for phase as well:

Once again, we see the inverse relationship b/w time and frequency. As , the time response becomes more compressed toward the origin, while the 3dB cutoff of the magnitude response becomes broader.

is a function of , while is a function of 🡪 inverse scaling in time and frequency.

A graph of a function

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A graph of a line

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## Second-order CT systems

Second-order system (LTI, causal, stable):

RLC circuits are characterized by this type of equation.

Frequency response:

## First-order DT systems

First-order system (LTI, causal, stable):

plays a similar role to in first-order CT systems: determines the rate at which the system responds. As , the impulse and step responses converge more quickly.

Unlike the CT system, the step response of the first-order DT system can exhibit oscillatory behavior. When is negative, the step response both overshoots and exhibits ringing.

## Second-order DT systems

# The Laplace transform

## Analysis and characterization of LTI systems using the Laplace transform

The Laplace transform is used to analyze and characterize LTI systems. The role of the LT stems directly from the convolution property:

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Furthermore, if the input to an LTI system is , with in the ROC of , then the output is .

In other words, is an eigenfunction of the system with eigenvalue equal to .

If the ROC of includes , then is the frequency response of the LTI system.

is called the system function or transfer function. Many properties of LTI systems are closely associated with the characteristics of the system function in the -plane.

### Causality