# Representation of LTI systems by differential and difference equations

LTI systems are completely characterized by their impulse response.

Differential and difference equations are a different (and very important) way to model LTI systems.

* Differential equations link the rates of change of the input and output in CT systems.
* Difference equations link the present, future, and past values of the input and output in DT systems.

We focus on linear constant coefficient differential/difference equations (LCCDE).

## LCC differential equations

The constant parameters and quantify the contribution of each derivative to the behavior of the system.

In most practical systems, . is the order of the DE.

To solve the DE, we need to know auxiliary (initial) conditions at time .

An important set of initial conditions is when the LTI system is initially at rest at time , which means that and are zero for . Formally,

This implies that

In practical applications, , so

Because a DE relates rates of input and output change instead of an explicit equation relating input and output, there are infinitely many solutions to the DE, depending on the starting point of the system. Initial conditions allow us to solve for a unique solution.

## Representation of a CT LTI system by differential equations

A LCC differential equation, initially at rest, represents a CT causal LTI system.

For causality and LTI, the system must be initially at rest.

If the system is not initially at rest, the output is nonzero when the input is zero.

For the system to be linear, superposition must hold:

But it doesn’t.

When the system is initially at rest, we can write

Adding the two equations,

Similar reasoning for time invariance. If the system is not initially at rest, we have

If we apply , we have

The output hasn’t shifted.

If the system is initially at rest, we can write

## Solving LCC differential equations that represent LTI systems

The general solution is composed of

* Particular solution: the output for a given input
* Homogeneous solution: the solution of the equation when

Since the system is LTI, by superposition, the general solution is

### Finding the particular solution

Since the system is LTI, the analytical form of must be similar to the analytical form of .

Example:

Assume , then .

In general,

* If the input is an nth-order polynomial, the particular solution is another nth-order polynomial with parameters:
* If the input is an exponential function, the particular solution is another exponential function:
* If the input consists if trigonometric functions, the particular solution is also trigonometric functions:

### Finding the homogeneous solution

Homogeneous equation:

The homogeneous solution is an exponential function.

Then

cannot be 0, so we have

This is an algebraic equation with roots , which means actually has the form

With separate constants for each root.

Each term is a valid solution to the homogeneous equation, and the homogeneous solution is the superposition of these terms.

### Finding the general solution

Without any additional information, there are no constraints on the values of , which means we have infinitely many homogeneous solutions and consequently, general solutions.

That is why we need initial conditions.

For the system to be causal and LTI, it must be initially at rest.

With these initial conditions, we can solve the a unique set of in the general solution

Example:

Find the particular solution.