# Representation of LTI systems by differential and difference equations

LTI systems are completely characterized by their impulse response.

Differential and difference equations are a different (and very important) way to model LTI systems.

* Differential equations link the rates of change of the input and output in CT systems.
* Difference equations link the present, future, and past values of the input and output in DT systems.

We focus on linear constant coefficient differential/difference equations (LCCDE).

## LCC differential equations

The constant parameters and quantify the contribution of each derivative to the behavior of the system.

In most practical systems, . is the order of the DE.

To solve the DE, we need to know auxiliary (initial) conditions at time .

An important set of initial conditions is when the LTI system is initially at rest at time , which means that and are zero for . Formally,

This implies that

In practical applications, , so

Because a DE relates rates of input and output change instead of an explicit equation relating input and output, there are infinitely many solutions to the DE, depending on the starting point of the system. Initial conditions allow us to solve for a unique solution.

## Representation of a CT LTI system by differential equations

A LCC differential equation, initially at rest, represents a CT causal LTI system.

Causality is baked into the definition of the equation.

The system is LTI if and only if the system is initially at rest.

If the system is not initially at rest, the output is nonzero when the input is zero.

For the system to be linear, superposition must hold:

But it doesn’t.

When the system is initially at rest, we can write

Adding the two equations,

Similar reasoning for time invariance. If the system is not initially at rest, we have

If we apply , we have

The output hasn’t shifted.

If the system is initially at rest, we can write

## Solving LCC differential equations that represent LTI systems

The general solution is composed of

* Particular solution: the output for a given input
* Homogeneous solution: the solution of the equation when

Since the system is LTI, by superposition, the general solution is

### Finding the particular solution

Since the system is LTI, the analytical form of must be similar to the analytical form of . This requires a guess.

Example:

Assume , then .

In general,

* If the input is an nth-order polynomial, the particular solution is another nth-order polynomial with parameters:
* If the input is an exponential function, the particular solution is another exponential function:
* If the input consists if trigonometric functions, the particular solution is also trigonometric functions:

### Finding the homogeneous solution

Homogeneous equation:

The homogeneous solution is an exponential function.

Then

cannot be 0, so we have

This is an algebraic equation (called the characteristic equation) with roots , which means actually has the form

With separate constants for each root.

Each term is a valid solution to the homogeneous equation, and the homogeneous solution is the superposition of these terms.

### Finding the general solution

Without any additional information, there are no constraints on the values of , which means we have infinitely many homogeneous solutions and consequently, general solutions.

That is why we need initial conditions.

For the system to be causal and LTI, it must be initially at rest.

With these initial conditions, we can solve the a unique set of in the general solution

Note that depend on both the initial conditions and the particular solution corresponding to the input.

#### Example of a first-order system initially at rest

Find the particular solution.

Note: is only defined within an integral, so we ignore the second term. This corresponds to considering the behavior of starting from .

Find the homogeneous solution.

Plugging in into , we get the characteristic equation

Therefore, the general solution has the form

Plug in the initial condition.

Since ,

The system is LTI and causal. The system is not memoryless.

#### Example of a first-order system not initially at rest

Same system and input, different initial conditions.

The general solution has the same form:

Plug in the initial conditions.

Then

The system is not LTI nor memoryless. Is the system causal?

#### Example of a second-order system initially at rest, non-degenerate case

Homogeneous solution:

Particular solution:

Note that depends on the decay rate, and when or , which are also the roots of the characteristic equation (that gave us ). When the roots of the characteristic equation are the same as the decay values , we get this degenerate case, and the particular solution needs a different analytic form.

Let .

General solution:

#### Example of a second-order system initially at rest, degenerate case

Same as previous example except . Homogeneous solution takes the same form, but we need to avoid the degenerate case with the particular solution.

In this case, we assume has the form

Plug into the LCCDE.

General solution.

Plug in initial conditions.

### Transfer function of a CT LTI system

Exponential inputs are especially important for LTI systems represented by LCCDE.

Plug into .

The coefficient of the particular solution, , is called the transfer function.

When , , i.e. the output is the input scaled by . The transfer function directly determines how much of the exponential is transferred to the output.

The general solution has the form

**Example:**

Homogeneous solution (previously solved):

Transfer function:

Particular solution:

General solution:

## LCC difference equations

is the order of the equation.

A system represented by a LCCDE is causal and LTI if and only if it is initially at rest, where being at rest means

Typically, .

In other words, a difference equation, with initial rest conditions, represents a discrete-time causal LTI system.

### Solving LCC difference equations

We can recursively find the solution. Rewrite the equation

Given the input and initial conditions, , we can iteratively find all values of .

**Example:**