# Highest-level summary

LTI systems:

* Described entirely by impulse response . Any output can be calculated as the convolution of the input with the impulse response .
* The step response is the integral or accumulation of the impulse response. The impulse response is the derivative or differencing of the step response.
* The steady-state response (system output) to a sinusoid at frequency is a sinusoid at multiplied by the frequency response of the system (for DT LTI, it’s and ).

LCCDE:

* Special class of LTI systems. Important because LCC differential equations represent real circuits, while LCC difference equations are necessary to implement IIR.
* Characterized entirely by their constant coefficients. By convention, are the coefficients for the input terms, and are the coefficients for the output terms. The highest order of is the order of the system.

# Topics

Reference: Signals and Systems for Dummies by Mark Wickert



# Math review

## Partial fraction expansion (PFE)

Important for inverse Laplace and z-transforms. PFE allows you to split up a ratio of polynomials into a sum of fractions, with each denominator containing a single or repeated root.

Consider Nth and Mth-order polynomials

For , is a proper rational function. If you assume has no repeated roots, then the PFE of is

If you don’t assume no repeated roots, and each of roots has multiplicity such that , then

When (improper rational function), you need to use long division to reduce the numerator order to be less than the denominator order. You need to first put your ratio of polynomials in the form

Where is a proper rational function.

For distinct (no repeats) roots, the coefficients are given by

For repeated roots, the coefficients are given by

## Trigonometry

Useful trig identities:

## Exponential and logarithmic functions

The logarithm of a number, , is the exponent by which another fixed value, the base , must be raised to produce that number.

Ex. .

Exponentiation and logarithm are inverses, that is,

Exponentiation turns sums into products, while logarithms turn products into sums.

Common bases:

* Base 10, aka decimal or common logarithm
* Base , aka natural logarithm, which corresponds to the exponential function
* Base 2, aka binary algorithm

Logarithmic scales reduce wide-ranging quantities to smaller scopes (like decibels).

To change from base to base ,

Identities:

Derivatives:

## Complex arithmetic

Rectangular form (x-y coordinates):

Polar form (magnitude/envelope and phase):

Rectangular to polar conversion:

Polar to rectangular conversion:

Let .

Euler’s identity:

Example:

### Phasor analysis

Phasor analysis: when sinusoidal signals of the same frequency are added together (superimposed), the result is a single sinusoidal signal having a composite amplitude and phase. Mathematically,

The composite amplitude and phase are given by the vector addition of the individual amplitudes and phases.

Proof:

## Calculus

Formally, has derivative at if this limit exists:

Differentiation formulas ( is constant and and are functions of ):

* (chain rule)

Indefinite integrals are also known as antiderivatives, i.e. if

Then

Useful indefinite integrals:

Useful definite integrals:

When a closed-form solution for integration is not available, use numerical integration (e.g. SciPy integrate module, quad function).

To find the minimum or maximum of a function , find

(As long as there is only one global minimum or maximum).

* If minimum,
* If maximum,

### Geometric series

Geometric series are often found in discrete-time signals and systems.

A finite geometric series, aka the sum of a finite initial segment of an infinite geometric series:

The convergence of the infinite sequence of partial sums of the infinite geometric series depends on , where may be complex.

When , the series converges:

When , the series diverges.

When , if , grows to infinity. When , the partial sums oscillate between . More generally, when for any integer , the series will circulate indefinitely with a period of .

Derivation of partial sum closed form:

Derivation of infinite series closed form for :

## Finding polynomial roots

For performing stability analysis of systems.

For 2nd-order polynomial, you can use the quadratic formula:

For higher-order, you will likely want to use numerical methods.

A special Nth-degree polynomial that occurs in discrete-time systems:

To find the roots, .

The roots equally spaced around the unit circle in the complex plane at separation angle .

If where is a real number, then the roots instead have magnitude .

# Intro to continuous-time signals and systems

## Complex exponential

General complex exponential (4 parameters):

The general complex exponential encompasses several important and special cases.

**Real exponential:**  and are real.

controls the amplitude and controls the decay rate.

In practice, we usually work with real exponentials that “turn on” after a certain point. We model this with the unit step function .

**Complex and real sinusoids:**  is complex and is imaginary.

is the signal amplitude, is frequency in rad/s, and is the phase.

, is frequency in Hz. is period in seconds.

**Damped complex and real sinusoids:** and are complex.

for damping.

## Singularity and other special signals

These signals are piecewise continuous – a formal derivative does not exist everywhere, and the signal may contain jumps.

**Rectangle pulse and triangle pulse:**

The width of the rectangle pulse is , but the width of the triangle pulse is . This is not an accident – the triangle pulse is the result of the rectangle pulse convolved with itself.

**Unit impulse (Dirac delta):**

Test waveform to find the impulse response of systems.

You can define this signal only in an operational sense, as in how it behaves. It is a spike with zero width and unity area, i.e.

It’s a function with unit area located at .

Sifting property of the Dirac delta:

Approximation of Dirac delta:

Dirac delta is drawn as a vertical line with an arrow at the top, with height equal to area.

**Unit step function:**

Model signals with on/off gates.

Derivative of rectangle pulse:

For derivative of triangle pulse, ignore the discontinuities at and define the derivative elsewhere.

A real-world step function doesn’t suddenly jump from 0 to 1 – it smoothly transitions over a period of time, which you can see when you zoom in. When you zoom out, it looks like a true mathematical step function. When you differentiate the real-world step, its derivative is defined everywhere, and the result is a pulse-like signal that looks like an impulse when viewed from a distance.

## Types of signals

**Deterministic vs. random:**

A deterministic signal is a completely specified function of time.

A random signal takes on values by chance according to a probabilistic model.

For example, is deterministic. However, if you make any of the parameters random, then it becomes a random signal.

**Periodic vs. aperiodic:**

A periodic signal satisfies , where is the period of the signal.

is an integer. For ,

**Power and energy signals:**

A signal has power and energy. Energy (joules) may be finite or infinite; power (watts) may be zero, finite, or infinite.

Circuit theory:

When or are functions of time (signals), then instantaneous power is a function of time.

In signals and systems, we typically normalize .

Average power and energy are defined as

is for complex signals.

**For a zero-mean signal, power is equivalent to variance.**

For a periodic signal, power becomes

For a **power signal**, and .

For an **energy signal**, and .

For signals that are neither (unbounded power and energy), .

Mathematically, a signal can have infinite power but this isn’t a practical reality. Infinite energy usually means signal duration is infinite, so it makes more sense to deal with power rather than energy.

**Single real sinusoid**: power signal with and .

because you’re integrating over two full periods.

**Two real sinusoids at different frequencies**: power signal with and . does not need to be periodic.

In the equation of power, as , the cosine terms are vanishingly small (the constant terms dominate). So

For to be periodic, and must be commensurate – you need to be able to find integers and s.t.

The ratio of the two periods/frequencies must be a rational number, and the fundamental period is . In algebraic terms, is the least common multiple (LCM) of the periods or is the greatest common divisor (GCD) of the frequencies.

Periodicity among multiple sinusoids is essential to Fourier series.

**In short, sinusoidal signals are power signals. For sinusoids at distinct frequencies, . If the sinusoids have the same frequency, you need to combine them using phasor addition before calculating power.**

**Real exponential:**

Energy signal if , power signal if , and neither if .

**Even and odd signals:**

Symmetry along the time axis relative to the origin, .

Even:

Odd:

(Or neither.)

is even, is odd.

## Signal transformations

Time shifting:

Time reversal:

Shift and reverse:

shifts by , and then reverses it around the point , or . (The origin is essentially shifted from 0 to .)

If is a finite-duration signal bounded by and , then is bounded by and . In the example below,

A graph of a function

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Superimposing signals:

## System properties

**Linear and nonlinear:**

A system is linear if superposition holds.

If the system is linear, then

In general,

If this isn’t true, then the system is nonlinear.

For example, you want a karaoke system to be linear so that the singer’s voice and the track can merge together in the audio amplifier without distortion. On the other hand, hard rock guitarists may intentionally send their signal through a nonlinear amplifier to get some distortion.

**Time-invariant and time-varying:**

A system is time-invariant if its properties or characteristics don’t change with time:

A system that doesn’t obey this condition is time-varying. For example, twisting the volume control on your car stereo means that the gain of your system is time varying.

Noise-removing filters are typically designed to be time-invariant – assuming the noise characteristics and desired signal characteristics are fixed, the filter design should be time-invariant.

A time-varying filter – aka an adaptive filter – is needed when the noise signal characteristics change over time, like in noise-canceling headphones.

**Causal and noncausal:**

A system is causal if all output values, , depend only on input values for . The current output depends only on past and current inputs.

A non-causal system can use future input values to generate the current output – essentially, it can predict the future because it’s anticipating future inputs. A causal system is also called non-anticipative.

A non-causal system is more of a mathematical concept than a practical reality (and is physically impossible for real-time processing). With discrete-time signals and systems, it’s possible to store a signal in memory and then process it later using a non-causal system, but in reality, you’re still using past input values because you’re working with a recorded signal.

Mathematical definition 1: If , then .

Mathematical definition 2: If is the impulse response of the system , then is causal if and only if .

**Memory and memoryless:**

In a memoryless system, each output depends only on the current input .

A system with memory (a filter) uses current and past values of the input to form the current output. For example, systems described by linear constant coefficient differential/difference equations, or an electronic circuit composed of resistors, capacitors, and inductors. The capacitors and inductors are the memory elements; a resistor-only network has no memory.

**Bounded-input bounded-output (BIBO):**

A signal that is bounded has magnitude less than infinity over all time. A signal is bounded if there exists a constant s.t. .

A system is BIBO-stable if every bounded input produces a bounded output.

**Choosing linear and time-invariant systems (LTI):**

Typically, we’re most interested in LTI systems because they are the easiest to analyze for system performance.

**Example systems:**

, so the system is nonlinear. The system is linear if there is no output offset.

, so the system is time-invariant. Also by inspection, none of the system parameters – 2 and 5 – vary with time.

The system is causal and memoryless.

If , then , so the system is BIBO stable.

**Triangle inequality:**

Next system.

Like the previous system, this is nonlinear. It is conditionally linear when the step function is off ().

This system is time-varying because the system coefficients – and – are functions of time.

This system is causal and memoryless.

This system is BIBO stable.

# Intro to discrete-time signals and systems

## Complex exponential

The equivalent continuous-time signal was written slightly differently: . But we could have written it as .

**Real exponential:**

**Complex exponential:**

**Complex sinusoid:**

**Real sinusoid:**

**Continuous-time vs. discrete-time frequency:**

Let be continuous-time frequency and be discrete-time frequency. Then

is in radians/second; is in radians/sample. They are not the same.

In practice, discrete-time signals are generated by uniformly sampling continuous-time signals at a sampling rate of or a sampling period of .

becomes :

Where for no aliasing.

## Special signals

**Unit impulse:**

Any sequence can be expressed as a linear combination of time-shifted impulses, which is important to convolution:

is nonzero only at .

**Unit step:**

Relationship with unit impulse:

A rectangle pulse of length can be expressed as .

**Window functions:**

is the window function.

**Pulse shapes:**

Pulse shapes create a digital communication waveform that’s bandwidth or spectrally efficient, e.g.

are data bits () and is the duration of each bit in samples. Popular pulse shapes: rectangle, half-sine, raised cosine, square-root raised cosine.

## Types of signals

Like CT, DT signals may be **deterministic or random** (common distributions include Gaussian and uniform).

**Periodic and aperiodic:**

If there exists s.t. , then is periodic with period .

In DT, a sinusoidal signal isn’t always periodic. To be periodic,

Since sinusoids are mod ,

Therefore, a sinusoid is periodic only if can be written as where and are integers. For multiple added sinusoids to be periodic, you need to find a common for all of them.

Again because sinusoids are mod ,

That is, a sinusoid of frequency is indistinguishable from a sinusoid of frequency . Therefore, the unique sinusoids that are periodic with period have frequencies given by

These are called distinguishable frequencies. They are distinct from all other frequencies that result in a sinusoidal sequence having period and lie on the fundamental frequency interval .

In CT, oscillation rate increases as increases. In DT, oscillation rate increases as increases for and decreases as increases for .

**Power and energy signals:**

Energy signal:

Power signal:

Neither:

An aperiodic signal of finite duration is an energy signal, and a sinusoidal signal is a power signal.

**For a zero-mean signal, power is equivalent to variance.**

## System properties

**Linearity condition:**

**Time-invariance condition:**

In CT, time-varying behavior can be due to uncontrollable like environmental conditions.

In DT, adaptive filters with time-varying attributes can optimize system performance. For example, the channel between your cell phone and the base station changes as you move, so your cell phone adapts to the environment by changing system attributes. This is part of adaptive signal processing.

**Causality condition:**

A system is causal if depends only on where . The present output depends only on past and present input values.

A system is causal if and .

A system is causal if .

**Memory:**

A system is memoryless if depends only on . Otherwise, it has memory.

**BIBO stability:**

A system is BIBO stable if and only if every bounded input produces a bounded output.

# CT LTI systems

## Impulse response and convolution integral

If you know the impulse response for an LTI system, you can calculate the output of that system for any input.

The impulse response of an LTI system is the output of that system when the input is , i.e. . The system must be **at rest** when it receives the impulse.

By the sifting property, any signal can be represented as

For an LTI system,

Convolution obeys standard commutative, associative, and distributive algebraic properties.

Commutative:

Associative (also known as series or cascading connection):

Distributive (aka parallel connection):

Useful identities:

**Convolution examples with finite extent signals:**

A graph with different colored lines

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A graph with a line graph

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If is bounded by and is bounded by , then is bounded by the “support interval” (the interval where ) and the duration of is , the durations of and .

**Exponential impulse response:**

A system with an exponential impulse response, or , models a first-order low-pass filter. This impulse response distorts the input pulse. When input pulses are sent at too high of a rate, this distortion can cause errors at the receiver (ISI). A rule of thumb is that the input pulse duration should be 10 times the time constant, which means that at the center of the input pulse, the tail from the previous pulse has decayed to .

Consider rectangular input pulses of duration and . The vertical dashed lines represent the centers of the pulses.

A graph of a function

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## Step response

Step response is the response of the system at-rest to the unit step.

That is, the step response is the integral of the impulse response.

On the flip side, differentiating the step response gives the impulse response:

## BIBO stability and causality

An LTI system is BIBO stable if .

An LTI system is causal if for .

# DT LTI systems

## Impulse response and convolution sum

DT LTI system is also known as digital filter.

The impulse response of a DT LTI system, , is the output produced by the at-rest system when given the input .

Any sequence can be expressed as

For an LTI system,

Convolution is commutative:

Associative (cascaded systems):

Distributive (parallel systems):

Useful convolutions:

Shift + reverse transformation for convolution (same as CT):

A graph with numbers and lines

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**Support intervals:**

If is bounded by and is bounded by , then is bounded by and its duration is , where and are the durations of and .

This easily extends to semi-infinite duration sequences.

E.g. has support and has support , then has support and duration .

E.g. has support and has support , then has support and duration .

**Example: unit step \* decaying exponential**

Consider two semi-infinite duration sequences: and where is real with magnitude less than 1.

By geometric series.

A graph of a function

AI-generated content may be incorrect.

**Example: rectangular pulse \* decaying exponential**

Let and . By LTI and using the previous example,

As in the CT case, a system with a decaying exponential impulse response is a low-pass filter – it slows down the edges of the rectangular pulse. The system charges up during and discharges during .

One case where this “slowing down” may be desirable: in the action of pressing a garage door opener, you need to hold down the button for some duration of time to make sure the charging up reaches the triggering threshold. This reduces the chance of false triggering.

A graph of a function

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## Step response

The step response of an LTI system at rest, , is its output in response to the input step, .

In other words, the step response is the accumulation of the impulse response.

For example, if ,

The inverse is given by

The impulse response is the “differencing” of the step response.

## BIBO stability

An LTI system is BIBO stable if

Proof:

Then if and .

Consider .

The system is BIBO stable only if . Note that by this definition, is not BIBO stable, but it’s still a useful building block.

## Causality

An LTI system is causal if for .

# LCCDE sinusoidal steady-state analysis

LCCDE are a special class of LTI systems:

* LCC differential equations for CT systems
* LCC difference equations for DT systems

LCC difference equations are relatively new to signals and systems; they are important for efficient DSP.

## CT LTI sinusoidal steady-state response

To solve LCC differential equations for the general solution, we need Laplace transforms. For now, we will focus on the sinusoidal steady-state solution.

The output of a BIBO-stable system, in response to a sinusoidal input as time goes to infinity, is a sinusoid of the same frequency.

The impulse and step responses of causal and stable LTI systems will usually include transient terms of the form . Transient terms die out as , so only the steady-state response is left.

Steady-state analysis is finding the output when the input is

For a causal and stable system,

**The output of the system in response to the complex sinusoid is the same sinusoid multiplied by , the frequency response of the system at .**

The frequency response of the system is

is the Fourier transform of the impulse response .

If is real, then

**Real sinusoid, real :**

## LCC differential equations

LCC differential equations have been around for a long time in electrical engineering, mechanical, chemical, biological, and other sciences. They are used to model time-varying signals in LTI systems, like a circuit composed of resistors, capacitors, and inductors.

Nth-order LCCDE:

and are the input and output, and N is the highest derivative in the output.

The coefficients and completely describe the system.

When the input and its derivatives are nonzero, this is a **nonhomogeneous differential equation**. When the input and its derivatives are zero, then it is a **homogeneous differential equation**, i.e.

**Frequency response:**

, corresponding to , are the numerator coefficients.

, corresponding to , are the denominator coefficients.

indicates the derivative order.

**Example:**

The coefficients are . You can use scipy.signal.freqs to calculate the frequency response. For a 1Hz real sinusoid, you can calculate the steady-state output using the frequency response, but you can also simulate the full response using scipy.signal.lsim, which includes the transient plus steady state response starting from .

A graph of a function

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A graph with blue and orange lines

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A graph of a full response

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## DT LTI sinusoidal steady-state response

The general output of DT LTI systems will contain transient terms of the form . For table systems, , so the transients die as , leaving only the steady-state response.

The frequency response of the system is

The frequency response is periodic with period :

**Real sinusoid, real :**

## LCC difference equations

Convolution sum may require an infinite number of calculations due to the doubly infinite sum, but LCC difference equations make DT LTI systems possible.

Nth-order LCCDE:

is the number of feedback terms (also, the system order), and is the number of feedforward terms. The system is completely defined by the coefficient sets and .

In continuous time, the LCC differential equation represents a circuit, but in discrete time, the LCC difference equation is exactly the system.

Generally, (or the coefficients are normalized by ). Then

The general form is an IIR filter.

When for , then it becomes an FIR filter,

Note that this is the same form as DT convolution, where the impulse response is

**Example: accumulator system**

An accumulator system is defined by .

This is an IIR, which means you need recursion (LCCDE) to represent it in the real world.

**Using recursion to find the impulse response:**

Recursion works for solving simple systems (z-transforms will be necessary for complex systems).

Let , a scaled step sequence. Let the initial condition be .

The output is found iteratively, calculating for each starting at 0.

For and ,

This is the step response of the at-rest system defined by .

You can also find the impulse response by setting the initial conditions to zero and letting .

**Frequency response:**

indicates the delay order.

**Example: FIR filter,**

A graph with lines and numbers

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A graph of a function

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# Fourier series of periodic CT signals

## Frequency domain and line spectra plots

Frequency domain representation is defined by the three parameters of a sinusoid: amplitude, phase, and frequency.

**Complex sinusoid:**

has a spectral amplitude of at and spectral phase of at . Characterized by , aka frequency-amplitude/phase pair.

**Real sinusoid:**

For real sinusoids,

For real signals,

The constant (DC) term is characterized by . If , then .

**Line spectra plot** = plot amplitude or phase against frequency as vertical lines. (Plot ).

* Two-sided plot shows both positive and negative frequencies
* Only for real signals: one-sided plot shows only nonnegative frequencies. Compared to two-sided plot, amplitudes are doubled (except for DC). is characterized by .

**Symmetry of real signals:**

When is real, the expansion into complex sinusoids always results in conjugate symmetry of coefficients, .

* Double-sided amplitude spectrum has even symmetry:
* Double-sided phase spectrum has odd symmetry:

## Fourier series

A signal that is a sum of sinusoids may be periodic or aperiodic. If the greatest common divisor (GCD) of the sinusoid frequencies exists, then is periodic with period equal to 1/GCD; otherwise, is aperiodic.

When a signal is periodic, every frequency in the signal is harmonically related to the fundamental frequency (which is equal to the GCD).

That is, , every frequency is an integer multiple of , and the signal is periodic with period .

**Fourier series:**

When a signal is periodic, you can convert between its time-domain and frequency-domain representation using Fourier series. Let be the fundamental frequency and be its period. The Fourier coefficients, , are the frequency-amplitude/phase pairs discussed earlier – if you can directly write as a sum of sinusoids, then you do not need to use the analysis equation to find .

Synthesis equation (frequency 🡪 time):

Analysis equation (time 🡪 frequency):

DC term is average waveform value:

### Deriving the analysis equation from the synthesis equation

The first step is to establish the orthogonality of when integrated over a interval.

Orthogonality means that the dot (inner) product of two functions is zero.

Dot product for complex functions is defined as

This evaluates to 0 when . When ,

Therefore, are orthogonal when .

With this, we can derive the analysis equation, starting from the synthesis equation:

### Gibbs phenomenon/ringing

For signals with jump discontinuities (like a square wave), the synthesis equation does NOT converge to the original signal in a point-by-point sense. At the discontinuities, it converges to the average of the left- and right-hand limits; for example, for a square wave that jumps between , the synthesis equation converges to 0.

Furthermore, the “reconstructed” signal oscillates around the discontinuity with a peak overshoot of around 9%. Increasing does not reduce the overshoot; it only increases oscillation frequency.

A graph of a pulse train

AI-generated content may be incorrect.

### Pulse train/square wave

Periodic pulse train over one period:

The Fourier coefficients of a periodic pulse train follow a sinc function. (sinc of 0 is 1.)

For a square wave,

For even, . For odd, .

**Example rectangular pulse train:**

A graph of a pulse train

AI-generated content may be incorrect.

Remember: the line spectra sit at multiples of the fundamental frequency, or .

are zero when , or , or . ( is an integer.) Thus, the nulls/sidelobes are defined by . For square waves, .

As decreases, sidelobe width increases.

Because of term, the slope of the phase is equal to w.r.t. frequency . At each null, the sinc term switches between , which causes a phase jump of .

**Example square wave signal:**

**A graph of a pulse train

AI-generated content may be incorrect.**

### Triangle wave