There are many questions still left to be answered:

* Simple question: why is it ? Why is it imaginary? Where does that come from? Is it because you decompose into complex sinusoids (that may add up to be real)?
* LTI systems not described by LCCDE
* Why do we use unilateral LT but bilateral ZT? I think in practical terms, we use one-sided. We’re talking about realizable systems, which are always causal. This is also why LT/ZT can handle initial conditions – there is an implicit understanding that the signal is right-sided, and it makes sense to conventionally make them causal. I guess technically you can use inverse LT/ZT to calculate system response, but at least for DT systems, it’s very simple to simulate in time domain – you did this in a notebook. Fourier transforms are for steady-state analysis, which is why we use infinity limits. Well, that’s not exactly true. FT are for energy signals, which are finite duration. FT are explicitly about translating between time and frequency representations, LT and ZT are also for translating between time and s/z domains, but with the added benefit of easily incorporating initial conditions for LCCDE. LT and ZT can also transform signals that aren’t L1 or L2, without having to go “in the limit”.
* As poles get closer to axis or unit circle, the system becomes less stable. For signals, this means the signal does not decay as quickly (same goes for impulse response).
* For ZT, is the number of poles = number of zeros thing only true for rational transfer function? For LT, is it also true that number of poles = number of zeros?
* All of these transforms and pairs are for developing intuition about real life signals and systems. In practice, we use computer tools to analyze real life signals and systems, and we use that intuition from these transforms to explain what we see.
* I think we typically talk about PSD, which is applicable to stationary processes. Even if a signal is time-limited, we assume it extends to infinity.
* DTFT in the limit
* I think all impulses – certainly for continuous time – need to be taken in the limit. E.g. to take the inverse CTFT of 1 to get dirac delta in time, or to take the CTFT of a constant or a sinusoid. It’s duality.

Random notes:

* PSD vs. ESD – notice that Fourier series is averaged over the period. Fourier transform is not because it is energy.
* Every integration/summation over one period is averaged
* Adding powers, orthogonality in freq domain
* Integrating powers/energy – Parseval’s or Plancherel’s theorem
* System must be initially at rest because . If there is no input, then output must be zero.

Summarize the most important points from memory.

Particular solution + homogeneous solution

Forced response + natural response

Exponential function = eigenfunction = forced response is an exponential of the same form just multiplied by the eigenvalue

Natural response coefficients depend on the input, but the natural response form is independent of input.

# Key points and examples

Time and frequency have inverse scaling – time-limited vs. band-limited (ISI)

System defined by LCCDE is LTI, causal, stable if and only if it has the condition of initial rest. If it is not initially at rest, it is incrementally linear (output is superposition of zero-input with initial conditions and input with initial rest).

Eigenfunctions of LCCDE

Transient response, steady-state response (do some derivations)

Particular solution + homogeneous solution (natural response)

Forced response

Particular solution with eigenfunction is a scaled eigenfunction.

Homogeneous solution is because of system poles. Always exists but the scaling depends on the specific input.

You can use LT/ZT to find closed-form solutions for certain types of signals (which signals?)

Transient response – decaying exponentials for order > 0. For order = 0? I think generally only exists in DT with FIR. Does FIR have transient response? Yes, but it’s limited to the length of the impulse response.

Actually, could multipath fading be an example an analog system with order=0?

Impulse response – transient response – homogeneous response. Applying an impulse is equivalent to nonzero initial conditions.

All of these transforms assume LTI systems?

Frequency response of rectangular pulse – there are nulls at and because when you multiply the pulse with a sinusoid at those frequencies and sum, the sum is 0. You are integrating over exactly an integer number of periods.

Important signals: complex exponential, sinusoids, unit step, impulse

Important systems: ideal LPF, impulse train, rectangular window

DT exponential:

CT exponential:

Important transforms: dirac delta in frequency, first-order, second-order

Any kind of discontinuity (either time or frequency) means that the representation in the other domain is neither absolutely nor square summable (I think)

If a signal can be written has a sum of exponentials, then its Z or Laplace transform will be a rational function

If coefficients are all real, then poles/zeros are real or come in complex conjugate pairs

# Signal properties

Energy/power

Periodic/aperiodic

Etc.

# System properties

**Linearity**

**Time invariance**

**Causality**

**Stability**

# Systems defined by LCCDE

th-order DE:

When , the input-output relationship is implicitly defined.

The response of the system to input consists of the particular solution and the homogeneous solution.

is the solution to the DE for the input . The most common method for solving for is looking for the “forced response” – i.e. assume that has the same form as .

is the solution to the homogeneous DE:

We assume has the form

Solving for yields a linear combination of exponentials with different time constants.

Where are the roots of the th-order characteristic equation

, which are also called the natural responses of the system, are all valid solutions to the homogeneous DE regardless of . This means there are infinitely many solutions.

To explicitly/uniquely define the input-output relationship, we need auxiliary conditions. Typically, we use the condition of initial rest, which means that if for , then for . This implies that

We plug these into the expression for and solve for the unknown parameters. This defines for .

When the system is initially at rest, it is also causal and LTI.

When the system is not initially at rest, it is incrementally linear. This means that the output of the system is the superposition of

1. The response of the system, with nonzero initial conditions, with zero input.
2. The response of the system, with the condition of initial rest, to the input. (In other words, the response of the causal LTI system to the input.)

These are also called the zero-input response and zero-state response.

Transfer function, response to exponential, frequency response + steady state

I believe that homogeneous response is the transient and particular response (for sinusoid) is the steady state.

BTW, difference equations model rate of change in DT systems just like differential equations model rate of change in CT systems.

If a stable, causal, LTI causal can be represented as a LCCDE, then its pole-zero geometry determines the frequency response. Assume the transfer function is defined by a rational fraction of polynomials in and a real gain. If all poles and zeros are either real or occur in complex conjugate pairs, then

* The polynomial coefficients are real
* The frequency response is conjugate symmetric
* The impulse response is real
* The complex conjugate pairs have complex conjugate residues when using PFE

LCC differential equations

With initial conditions

# Transforms

## Overview

Transforms are used for both signals and LTI systems.

Transforms exhibit duality because forward and inverse transforms are similar.

CT and DT Fourier transforms and Fourier series switch between time domain and frequency domain representations. Each frequency component in the frequency domain is defined by frequency, amplitude, and phase.

* FS map periodic power signals to harmonic coefficients. The harmonic frequencies are and for CT and DT, where and are the periods.
* FT map energy signals (L1 or L2) to a continuous spectrum. The limits of the forward transform are double infinity, but formally, FT only exists for signals that decay quickly enough. For L1 signals, the FT converges pointwise, and for L2 signals, the FT exhibits mean-square convergence.
* We can calculate the FT of periodic power signals (including a DC signal) using FT in the limit. The FT is composed of dirac deltas with area equal to the FS coefficients.
* FT also map the impulse response of stable LTI systems to their frequency response. FT is used for steady-state analysis.

CT Fourier series – take the CTFT of one period of a periodic signal. CTFS is a sampled version of that CTFT sampled @ .

DFT – essentially DFS but same trick as CTFS. Sampled version of DTFT sampled at .

Laplace and z-transforms switch between time domain and / domain.

* LT/ZT are a generalization of FT. They are typically used to solve for the complete solution – transient and steady state – of systems that are represented by LCCDE.
* Typically, we use the one-sided transforms (integration and summation limits from 0 to infinity) since realizable signals and systems are causal.
* When the transform takes the form of a rational function in /, we can calculate the poles and zeros. For ZT, the number of poles and zeros is always the same
* For causal signals and systems, the ROC has the form and , where is the real part of the most positive pole in the plane and is the magnitude of the largest pole in the plane.
* LT/ZT exist for signals that do not have a formal FT, like unit step and sinusoids, which makes / domain analysis easier because you can manipulate their transforms algebraically.
* LT/ZT allow us to solve for the complete solution of LCCDE using simple algebraic manipulation, and they allow us to easily include initial conditions of the system.
  + The poles and zeros of the transfer function allow us to easily determine whether the system is stable and if so, the shape of the frequency response.
  + the poles must be in the LHP of the plane and inside the unit circle of the plane for the FT to formally exist, which also means the system is stable. When this condition is satisfied, the FT is equal to the LT/ZT evaluated on the axis or unit circle.

## Continuous time transforms

**CT Fourier series:**

**CT Fourier transform:**

**Laplace transform:**

## Discrete time transforms

**DT Fourier transform:**

**z-transform:**

**Discrete Fourier series:**

**Discrete Fourier transform:**

Many of the properties observed for CT also hold for DT.

For causal signals and systems, the ROC is of the form , where is largest pole magnitude. For FT to exist, the ROC must contain the unit circle, which means that all poles must be inside the unit circle. This is the same condition as being absolutely summable (aka being in L1 space). For a system, this also means the system is BIBO stable. When this occurs, the FT will converge pointwise.

The FT also exists for signals that have mean square convergence (aka being in L2 space). This is a weaker condition than being absolutely integrable. In both cases, these signals are energy signals. For the L2 case, the FT converges in the mean square sense, meaning as the integration limits go to infinity, the mean-square error of the FT goes to zero. This is the reason for the Gibbs phenomenon.

Formally, the FT exists only for energy signals.

For energy signals, the square of the FT is the energy spectral density (ESD), which has units of or or . Integrating over the ESD yields the total energy of the signal, which is equal to integrating over the square of the signal over all time (Parseval’s theorem).

If there are poles on the unit circle, the FT may still exist – in the limit.

Periodic signals are periodic only if , where and are integers. is the fundamental period. Then can use DFS. DFS consists of frequencies at harmonics of .

For power signals, the square of the FT is the power spectral density (PSD), which has units of or . Integrating over the PSD yields the total power of the signal. The PSD is equal to the ESD taken to the limit (I think).

If a stable, causal, LTI causal can be represented as a LCCDE, then its pole-zero geometry determines the frequency response. Assume the transfer function is defined by a rational fraction of polynomials in and a real gain. If all poles and zeros are either real or occur in complex conjugate pairs, then

* The polynomial coefficients are real
* The frequency response is conjugate symmetric
* The impulse response is real
* The complex conjugate pairs have complex conjugate residues when using PFE

DFT:

Remember, the “underlying” signal is periodic with period , so that’s why it’s circular time shift, frequency shift, and convolution.

You can implement freq domain filtering. Make sure circular convolution is equal to linear convolution. FFT/IFFT.

LCCDE: number of poles equals number of zeros; poles and zeros can be at 0 or infinity to fill this out

## Transform properties

Important transform properties, continuous time and discrete time:

Linearity

Time shift

Symmetry

**Parseval’s theorem:**

For energy signals, the square of the FT is the energy spectral density (ESD), which has units of or or . Integrating over the ESD yields the total energy of the signal, which is equal to integrating over the square of the signal over all time (Parseval’s theorem).

For power signals, the square of the FT is the power spectral density (PSD), which has units of or . Integrating over the PSD yields the total power of the signal. The PSD is equal to the ESD taken to the limit.

# Transform pairs

Important transform pairs, continuous time and discrete time, poles, zeros, etc.

**Time domain impulse:**

**Delayed time domain impulse (assume :**

**Complex exponential:**

Usually represent the impulse response and frequency response of stable first-order systems. and represent the time constant of decay.

**Complex sinusoid:**

Sine:

Time domain impulse train:

Time domain sinc:

Rectangular pulse

Triangular pulse

Decaying exponential

Unit step

<https://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms>

<https://en.wikipedia.org/wiki/Z-transform#Table_of_common_Z-transform_pairs>

# Sampling

causes to be mod , aka periodic with period .

# Spectral density

<https://en.wikipedia.org/wiki/Spectral_density>

<https://en.wikipedia.org/wiki/Autocorrelation>

Energy signal

Parseval’s theorem

is the energy spectral density (ESD). It has units of .

By convolution theorem,

So yes, ESD is FT of autocorrelation of . This is known as Wiener-Kinchin theorem.

Typically, we work in units of power, not energy. That’s because some signals are not energy signals. For example, stationary processes like noise exist for all time. The FT/ESD is not defined for these kinds of signals.

For power signals, we need to work “in the limit” – FT in the limit. We can also extend this to energy signals by normalizing by time duration.

Let’s look at the case of a power signal.

Average power is given by

When is non-zero, the integral grows to infinity at least as fast as . That’s why we can’t use energy, which is a diverging integral.

Let’s define , a windowed version of that is equal to within a duration and is elsewhere. Then an alternative definition for average power is

Even if doesn’t formally exist, exists. Take it to the limit to get and .

Parseval’s theorem still applies here.

still has units of . Same with .

Let’s prove the autocorrelation thing again.

This means

Then

In this case,

Note that the definition of autocorrelation is different for energy and power signals.

If signal is real, autocorrelation is real and even, so PSD is real and even.

If signal is complex, autocorrelation is conjugate symmetric, so (I think) PSD is real but not necessarily even.

Autocorrelation is conjugate symmetric.

Yep.

One-sided vs. two-sided PSD: If you think in terms of dirac delta functions in frequency, then you are simply just adding up all the areas. So if PSD is even, then one-sided PSD has 2x value of two-sided.

PSD = FT of autocorrelation of a signal –

Autocorrelation of a signal is

What about for complex signal?

# DFT, spectrum analyzers, etc.

I think there are numerical integration ways to calculate CTFT, DTFT, etc. But the advantage of DFT is you don’t need to integrate.

How do spectrum analyzers measure power nowadays? Do they use DFT?

# Phasor analysis and lowpass equivalents

**Phasor analysis:**

The classical formulation of phasors is if you have sinusoids of the same frequency but varying amplitudes and phases, the summation of the sinusoids is equal to a sinusoid at the same frequency with amplitude and phase given by phasor addition.

A phasor is a complex number that represents the amplitude and phase of a sinusoid. If the sinusoid is , then the phasor is

Phasor analysis says that

Where and are given by

Proof:

**Lowpass equivalents (or lowpass-bandpass conversion):**

In the proof for phasor analysis, I used the technique of lowpass-bandpass conversion, which says that a real, narrowband, high frequency signal (a bandpass signal) can be represented in terms of a complex low frequency signal (the lowpass equivalent or the bandpass signal).

Phasors are a special case of lowpass equivalents where and , that is, the amplitude and phase do not vary with time, i.e. the bandpass signal is an unmodulated tone.

Lowpass equivalents simply extend this idea to modulated signals.

The same analysis applies: given bandpass signals, you may convert them to their lowpass equivalents and do your calculations/simulations at baseband. The only difference is that the lowpass equivalents are functions of time.