# High-level summary

LTI systems:

* Described entirely by impulse response . Any output can be calculated as the convolution of the input with the impulse response .
* The step response is the integral or accumulation of the impulse response. The impulse response is the derivative or differencing of the step response.
* The steady-state response (system output) to a sinusoid at frequency is a sinusoid at multiplied by the frequency response of the system (for DT LTI, it’s and ).

LCCDE:

* Special class of LTI systems. Important because LCC differential equations represent real circuits, while LCC difference equations are necessary to implement IIR.
* Characterized entirely by their constant coefficients. By convention, are the coefficients for the input terms, and are the coefficients for the output terms. The highest order of is the order of the system.

Frequency domain/CT Fourier series:

* Each frequency component is characterized by frequency, amplitude, and phase.
* A periodic signal with fundamental frequency and period can be decomposed into orthogonal frequency components. The frequencies are harmonics of , and the amplitude and phase of each component is the Fourier series coefficient, , where denotes the harmonic order.
* A delay, , causes a phase shift at each frequency component of .
* For a real signal, , . That is, are conjugate symmetric.
* For a real even signal, , are real.
* For a real odd signal, , are imaginary.
* For a real signal with odd half-wave symmetry, , even frequency components are null.
* The power of a periodic signal is equal to .
* For a pulse train, follow a sinc function, . Sidelobe width is given by or .
  + If you keep constant, the nulls are always at , etc. regardless of .
  + If you keep constant, decreasing pushes out the nulls.
  + For , we have a square wave, and even harmonics are 0. Odd harmonics roll off as .
* Due to discontinuous jump in pulse train, synthesizing pulse train from Fourier series does not converge point-by-point. Gibbs phenomenon/ringing.
* For a triangle wave, even harmonics are null, and odd harmonics roll off as .

CTFT:

* Formally, CTFT only exists for aperiodic energy signals, but with FT in the limit, it can be applied to periodic power signals.
* CTFS is a sampled version of CTFT when you analyze one period of a periodic signal.
* is the spectrum/energy spectral density of . If has units of V, then has units of . Integrating over or yields the total energy in the signal.
* Ideal filters are not realizable. Realizable filters include Butterworth, Chebyshev, Bessel, elliptic, and are defined by LCC differential equations.
* Same symmetry properties as CTFS

Sampling:

* CT-DT:
* Sampling = convolving with freq domain impulse train at . When converting to DT, DT spectrum is periodic, which means . Signals at appear at same DT frequency (aliasing), and signal at appears at .
  + CT freq domain: . are called the spectral translates.
* Nyquist-Shannon sampling theorem: , is the Nyquist rate and is the Nyquist frequency.
* Principle alias lies in . This is the alias that is reconstructed. .
* Quantization ( is number of bits)
  + , levels = to
  + Quantization noise power =
  + SQNR for full wave sinusoid =
* Reconstruction: return to sampled CT signal (signal + images) and then filter the images
  + ZOH (realizable DAC): . Nulls at provide partial image suppression.
  + Ideal reconstruction LPF is a time-domain sinc:
  + To relax the design of the analog interpolation filter, increase DAC sampling rate to push out the images.

Fourier transforms:

* Energy signals
  + If a signal is in L1 space, then its FT exists and converges pointwise (as the limits of integration or summation go to ).
    - L1 space means . is absolutely integrable or absolutely summable.
    - CT:
    - DT:
  + If a signal is in L2 space, then it is an energy signal, and its FT exists and converges in the mean-square sense. It’s not guaranteed to converge pointwise.
    - L2 space means . is square integrable or square summable.
    - CT:
    - DT:
  + is the energy spectral density and has units of or
* Power signals: you must use the technique of FT in the limit to calculate the FT
  + is the power spectral density and has units of or
  + PSD is ESD taken to the limit (I think)
* Convergence
* Autocorrelation: ESD/PSD is FT of autocorrelation function
* Symmetry properties for real signals
  + real 🡪 is conjugate symmetric, and are even, and are odd
  + real and even 🡪 is real and even
  + real and odd 🡪 is imaginary and odd

Program examples of each convergence type (L1, L2, power)

You can use scipy to plot the FT of a signal that is real and even because FFT treats the signal as infinite duration, so you can effectively get a “non-causal” signal.

Fourier transform theorems and pairs:

* Linearity
* Time shift
* Frequency shift
* Parseval’s/Plancherel’s – is this assuming energy signal?
* Downsample
* Upsample

CT-DT relationship:

* with the change of variable

Fourier transform convergence (convergence meaning as the time or frequency limits go to infinity):

* L1: the signal is absolutely integrable (CT) or absolutely summable (DT). The FT exists and converges pointwise. Can also say that the signal is part of L1 space.
* L2: the signal is an energy signal – square integrable (CT) or square summable (DT). The FT exists and converges in the mean-square sense (mean-square error goes to zero) but not pointwise. Can also say that the signal is part of L2 space.
* Power signals: use FT in the limit to get the FT, which will be impulses in the frequency domain. (Not necessarily – more generally it defines a PSD, which is continuous for stationary processes)
* Convergence goes the other way too. For example, if you start with a time-domain sinc, then you get the Gibbs phenomenon in frequency domain because of mean-square convergence. If you start with a freq-domain sinc, then you get the Gibbs phenomenon in time-domain. Another example of duality of FT.

Energy signals have ESD (energy spectral density) while power signals have PSD. These have different units. To compare ESD to PSD, you have to either limit the time window of the power signal, or you have to repeat the energy signal into infinity.

Autocorrelation

DFT: transform domain signal processing: in practice, at least for the digital front-end, we don’t do this. It’s still an approximation of convolution, which is easily realized with digital hardware. But I suppose in the baseband side, we have access to FFTs, which allows us to do freq-domain signal processing. Very useful for OFDM. FFT is basically like a filter. is the output signal, and each is created by multiplying the input signal with a bunch of coefficients (defined by ) and summing.

Convolution with impulse/sifting

Convergence requirements, FT in the limit

I guess I never really wrapped my head around the fact that energy signals are time-limited and that CTFT/DTFT represents the energy because the frequency domain doesn’t have any notion of time. Somehow all of these complex exponentials add up together to produce a signal that is time-limited.

Have a good picture of CT-DT relationship

* CTFS: continuous-time periodic signals, discrete spectrum, power signals
* CTFT: continuous-time signals, continuous spectrum, energy signals, energy spectral density
* DTFT: discrete-time signals, continuous spectrum, energy signals, energy spectral density
* DFT: discrete-time signals, discrete spectrum

# Topics

Reference: Signals and Systems for Dummies by Mark Wickert



# Math review

## Partial fraction expansion (PFE)

Important for inverse Laplace and z-transforms. PFE allows you to split up a ratio of polynomials into a sum of fractions, with each denominator containing a single or repeated root.

Consider Nth and Mth-order polynomials

For , is a proper rational function. If you assume has no repeated roots, then the PFE of is

If you don’t assume no repeated roots, and each of roots has multiplicity such that , then

When (improper rational function), you need to use long division to reduce the numerator order to be less than the denominator order. You need to first put your ratio of polynomials in the form

Where is a proper rational function.

For distinct (no repeats) roots, the coefficients are given by

For repeated roots, the coefficients are given by

## Trigonometry

Useful trig identities:

## Sinc function

Nulls at is non-zero integer.

TD sinc = ideal LPF

TD sinc used for ideal pulse shaping

FD sinc used for OFDM

## Exponential and logarithmic functions

The logarithm of a number, , is the exponent by which another fixed value, the base , must be raised to produce that number.

Ex. .

Exponentiation and logarithm are inverses, that is,

Exponentiation turns sums into products, while logarithms turn products into sums.

Common bases:

* Base 10, aka decimal or common logarithm
* Base , aka natural logarithm, which corresponds to the exponential function
* Base 2, aka binary algorithm

Logarithmic scales reduce wide-ranging quantities to smaller scopes (like decibels).

To change from base to base ,

Identities:

Derivatives:

## Complex arithmetic

Rectangular form (x-y coordinates):

Polar form (magnitude/envelope and phase):

Rectangular to polar conversion:

Polar to rectangular conversion:

Let .

Euler’s identity:

Example:

### Phasor analysis

Phasor analysis: when sinusoidal signals of the same frequency are added together (superimposed), the result is a single sinusoidal signal having a composite amplitude and phase. Mathematically,

The composite amplitude and phase are given by the vector addition of the individual amplitudes and phases.

Proof:

## Calculus

Formally, has derivative at if this limit exists:

Differentiation formulas ( is constant and and are functions of ):

* (chain rule)

Indefinite integrals are also known as antiderivatives, i.e. if

Then

Useful indefinite integrals:

Useful definite integrals:

When a closed-form solution for integration is not available, use numerical integration (e.g. SciPy integrate module, quad function).

To find the minimum or maximum of a function , find

(As long as there is only one global minimum or maximum).

* If minimum,
* If maximum,

### Geometric series

Geometric series are often found in discrete-time signals and systems.

A finite geometric series, aka the sum of a finite initial segment of an infinite geometric series:

The convergence of the infinite sequence of partial sums of the infinite geometric series depends on , where may be complex.

When , the series converges:

When , the series diverges.

When , if , grows to infinity. When , the partial sums oscillate between . More generally, when for any integer , the series will circulate indefinitely with a period of .

Derivation of partial sum closed form:

Derivation of infinite series closed form for :

### P series

<https://en.wikipedia.org/wiki/Convergence_tests#p-series_test>

This converges if .

Harmonic series () diverges.

Basel problem () converges to .

## Finding polynomial roots

For performing stability analysis of systems.

For 2nd-order polynomial, you can use the quadratic formula:

For higher-order, you will likely want to use numerical methods.

A special Nth-degree polynomial that occurs in discrete-time systems:

To find the roots, .

The roots equally spaced around the unit circle in the complex plane at separation angle .

If where is a real number, then the roots instead have magnitude .

# Intro to continuous-time signals and systems

## Complex exponential

General complex exponential (4 parameters):

The general complex exponential encompasses several important and special cases.

**Real exponential:**  and are real.

controls the amplitude and controls the decay rate.

In practice, we usually work with real exponentials that “turn on” after a certain point. We model this with the unit step function .

**Complex and real sinusoids:**  is complex and is imaginary.

is the signal amplitude, is frequency in rad/s, and is the phase.

, is frequency in Hz. is period in seconds.

**Damped complex and real sinusoids:** and are complex.

for damping.

## Singularity and other special signals

These signals are piecewise continuous – a formal derivative does not exist everywhere, and the signal may contain jumps.

**Rectangle pulse and triangle pulse:**

The width of the rectangle pulse is , but the width of the triangle pulse is . This is not an accident – the triangle pulse is the result of the rectangle pulse convolved with itself.

**Unit impulse (Dirac delta):**

Test waveform to find the impulse response of systems.

You can define this signal only in an operational sense, as in how it behaves. It is a spike with zero width and unity area, i.e.

It’s a function with unit area located at .

Sifting property of the Dirac delta:

Approximation of Dirac delta:

Dirac delta is drawn as a vertical line with an arrow at the top, with height equal to area.

**Unit step function:**

Model signals with on/off gates.

Derivative of rectangle pulse:

For derivative of triangle pulse, ignore the discontinuities at and define the derivative elsewhere.

A real-world step function doesn’t suddenly jump from 0 to 1 – it smoothly transitions over a period of time, which you can see when you zoom in. When you zoom out, it looks like a true mathematical step function. When you differentiate the real-world step, its derivative is defined everywhere, and the result is a pulse-like signal that looks like an impulse when viewed from a distance.

## Types of signals

**Deterministic vs. random:**

A deterministic signal is a completely specified function of time.

A random signal takes on values by chance according to a probabilistic model.

For example, is deterministic. However, if you make any of the parameters random, then it becomes a random signal.

**Periodic vs. aperiodic:**

A periodic signal satisfies , where is the period of the signal.

is an integer. For ,

**Power and energy signals:**

A signal has power and energy. Energy (joules) may be finite or infinite; power (watts) may be zero, finite, or infinite.

Circuit theory:

When or are functions of time (signals), then instantaneous power is a function of time.

In signals and systems, we typically normalize .

Average power and energy are defined as

is for complex signals.

**For a zero-mean signal, power is equivalent to variance.**

For a periodic signal, power becomes

For a **power signal**, and .

For an **energy signal**, and .

For signals that are neither (unbounded power and energy), .

Mathematically, a signal can have infinite power but this isn’t a practical reality. Infinite energy usually means signal duration is infinite, so it makes more sense to deal with power rather than energy.

**Single real sinusoid**: power signal with and .

because you’re integrating over two full periods.

**Two real sinusoids at different frequencies**: power signal with and . does not need to be periodic.

In the equation of power, as , the cosine terms are vanishingly small (the constant terms dominate). So

For to be periodic, and must be commensurate – you need to be able to find integers and s.t.

The ratio of the two periods/frequencies must be a rational number, and the fundamental period is . In algebraic terms, is the least common multiple (LCM) of the periods or is the greatest common divisor (GCD) of the frequencies.

Periodicity among multiple sinusoids is essential to Fourier series.

**In short, sinusoidal signals are power signals. For sinusoids at distinct frequencies, . If the sinusoids have the same frequency, you need to combine them using phasor addition before calculating power.**

**Real exponential:**

Energy signal if , power signal if , and neither if .

**Even and odd signals:**

Symmetry along the time axis relative to the origin, .

Even:

Odd:

(Or neither.)

is even, is odd.

## Signal transformations

Time shifting:

Time reversal:

Shift and reverse:

shifts by , and then reverses it around the point , or . (The origin is essentially shifted from 0 to .)

If is a finite-duration signal bounded by and , then is bounded by and . In the example below,

A graph of a function

AI-generated content may be incorrect.

Superimposing signals:

## System properties

**Linear and nonlinear:**

A system is linear if superposition holds.

If the system is linear, then

In general,

If this isn’t true, then the system is nonlinear.

For example, you want a karaoke system to be linear so that the singer’s voice and the track can merge together in the audio amplifier without distortion. On the other hand, hard rock guitarists may intentionally send their signal through a nonlinear amplifier to get some distortion.

**Time-invariant and time-varying:**

A system is time-invariant if its properties or characteristics don’t change with time:

A system that doesn’t obey this condition is time-varying. For example, twisting the volume control on your car stereo means that the gain of your system is time varying.

Noise-removing filters are typically designed to be time-invariant – assuming the noise characteristics and desired signal characteristics are fixed, the filter design should be time-invariant.

A time-varying filter – aka an adaptive filter – is needed when the noise signal characteristics change over time, like in noise-canceling headphones.

**Causal and noncausal:**

A system is causal if all output values, , depend only on input values for . The current output depends only on past and current inputs.

A non-causal system can use future input values to generate the current output – essentially, it can predict the future because it’s anticipating future inputs. A causal system is also called non-anticipative.

A non-causal system is more of a mathematical concept than a practical reality (and is physically impossible for real-time processing). With discrete-time signals and systems, it’s possible to store a signal in memory and then process it later using a non-causal system, but in reality, you’re still using past input values because you’re working with a recorded signal.

Mathematical definition 1: If , then .

Mathematical definition 2: If is the impulse response of the system , then is causal if and only if .

**Memory and memoryless:**

In a memoryless system, each output depends only on the current input .

A system with memory (a filter) uses current and past values of the input to form the current output. For example, systems described by linear constant coefficient differential/difference equations, or an electronic circuit composed of resistors, capacitors, and inductors. The capacitors and inductors are the memory elements; a resistor-only network has no memory.

**Bounded-input bounded-output (BIBO):**

A signal that is bounded has magnitude less than infinity over all time. A signal is bounded if there exists a constant s.t. .

A system is BIBO-stable if every bounded input produces a bounded output.

**Choosing linear and time-invariant systems (LTI):**

Typically, we’re most interested in LTI systems because they are the easiest to analyze for system performance.

**Example systems:**

, so the system is nonlinear. The system is linear if there is no output offset.

, so the system is time-invariant. Also by inspection, none of the system parameters – 2 and 5 – vary with time.

The system is causal and memoryless.

If , then , so the system is BIBO stable.

**Triangle inequality:**

Next system.

Like the previous system, this is nonlinear. It is conditionally linear when the step function is off ().

This system is time-varying because the system coefficients – and – are functions of time.

This system is causal and memoryless.

This system is BIBO stable.

# Intro to discrete-time signals and systems

## Complex exponential

The equivalent continuous-time signal was written slightly differently: . But we could have written it as .

**Real exponential:**

**Complex exponential:**

**Complex sinusoid:**

**Real sinusoid:**

**Continuous-time vs. discrete-time frequency:**

Let be continuous-time frequency and be discrete-time frequency. Then

is in radians/second; is in radians/sample. They are not the same.

In practice, discrete-time signals are generated by uniformly sampling continuous-time signals at a sampling rate of or a sampling period of .

becomes :

Where for no aliasing.

## Special signals

**Unit impulse:**

Any sequence can be expressed as a linear combination of time-shifted impulses, which is important to convolution:

is nonzero only at .

**Unit step:**

Relationship with unit impulse:

A rectangle pulse of length can be expressed as .

**Window functions:**

is the window function.

**Pulse shapes:**

Pulse shapes create a digital communication waveform that’s bandwidth or spectrally efficient, e.g.

are data bits () and is the duration of each bit in samples. Popular pulse shapes: rectangle, half-sine, raised cosine, square-root raised cosine.

## Types of signals

Like CT, DT signals may be **deterministic or random** (common distributions include Gaussian and uniform).

**Periodic and aperiodic:**

If there exists s.t. , then is periodic with period .

In DT, a sinusoidal signal isn’t always periodic. To be periodic,

Since sinusoids are mod ,

Therefore, a sinusoid is periodic only if can be written as where and are integers. For multiple added sinusoids to be periodic, you need to find a common for all of them.

Again because sinusoids are mod ,

That is, a sinusoid of frequency is indistinguishable from a sinusoid of frequency . Therefore, the unique sinusoids that are periodic with period have frequencies given by

These are called distinguishable frequencies. They are distinct from all other frequencies that result in a sinusoidal sequence having period and lie on the fundamental frequency interval .

In CT, oscillation rate increases as increases. In DT, oscillation rate increases as increases for and decreases as increases for .

**Power and energy signals:**

Energy signal:

Power signal:

Neither:

An aperiodic signal of finite duration is an energy signal, and a sinusoidal signal is a power signal.

**For a zero-mean signal, power is equivalent to variance.**

## System properties

**Linearity condition:**

**Time-invariance condition:**

In CT, time-varying behavior can be due to uncontrollable like environmental conditions.

In DT, adaptive filters with time-varying attributes can optimize system performance. For example, the channel between your cell phone and the base station changes as you move, so your cell phone adapts to the environment by changing system attributes. This is part of adaptive signal processing.

**Causality condition:**

A system is causal if depends only on where . The present output depends only on past and present input values.

A system is causal if and .

A system is causal if .

**Memory:**

A system is memoryless if depends only on . Otherwise, it has memory.

**BIBO stability:**

A system is BIBO stable if and only if every bounded input produces a bounded output.

# CT LTI systems

## Impulse response and convolution integral

If you know the impulse response for an LTI system, you can calculate the output of that system for any input.

The impulse response of an LTI system is the output of that system when the input is , i.e. . The system must be **at rest** when it receives the impulse.

By the sifting property, any signal can be represented as

For an LTI system,

Convolution obeys standard commutative, associative, and distributive algebraic properties.

Commutative:

Associative (also known as series or cascading connection):

Distributive (aka parallel connection):

Useful identities:

**Convolution examples with finite extent signals:**

A graph with different colored lines

AI-generated content may be incorrect.

A graph with a line graph

AI-generated content may be incorrect.

If is bounded by and is bounded by , then is bounded by the “support interval” (the interval where ) and the duration of is , the durations of and .

**Exponential impulse response:**

A system with an exponential impulse response, or , models a first-order lowpass filter. This impulse response distorts the input pulse. When input pulses are sent at too high of a rate, this distortion can cause errors at the receiver (ISI). A rule of thumb is that the input pulse duration should be 10 times the time constant, which means that at the center of the input pulse, the tail from the previous pulse has decayed to .

Consider rectangular input pulses of duration and . The vertical dashed lines represent the centers of the pulses.

A graph of a function

AI-generated content may be incorrect.

## Step response

Step response is the response of the system at-rest to the unit step.

That is, the step response is the integral of the impulse response.

On the flip side, differentiating the step response gives the impulse response:

## BIBO stability and causality

An LTI system is BIBO stable if .

An LTI system is causal if for .

# DT LTI systems

## Impulse response and convolution sum

DT LTI system is also known as digital filter.

The impulse response of a DT LTI system, , is the output produced by the at-rest system when given the input .

Any sequence can be expressed as

For an LTI system,

Convolution is commutative:

Associative (cascaded systems):

Distributive (parallel systems):

Useful convolutions:

Shift + reverse transformation for convolution (same as CT):

A graph with numbers and lines

AI-generated content may be incorrect.

**Support intervals:**

If is bounded by and is bounded by , then is bounded by and its duration is , where and are the durations of and .

This easily extends to semi-infinite duration sequences.

E.g. has support and has support , then has support and duration .

E.g. has support and has support , then has support and duration .

**Example: unit step \* decaying exponential**

Consider two semi-infinite duration sequences: and where is real with magnitude less than 1.

By geometric series.

A graph of a function

AI-generated content may be incorrect.

**Example: rectangular pulse \* decaying exponential**

Let and . By LTI and using the previous example,

As in the CT case, a system with a decaying exponential impulse response is a lowpass filter – it slows down the edges of the rectangular pulse. The system charges up during and discharges during .

One case where this “slowing down” may be desirable: in the action of pressing a garage door opener, you need to hold down the button for some duration of time to make sure the charging up reaches the triggering threshold. This reduces the chance of false triggering.

A graph of a function

AI-generated content may be incorrect.

## Step response

The step response of an LTI system at rest, , is its output in response to the input step, .

In other words, the step response is the accumulation of the impulse response.

For example, if ,

The inverse is given by

The impulse response is the “differencing” of the step response.

## BIBO stability

An LTI system is BIBO stable if

Proof:

Then if and .

Consider .

The system is BIBO stable only if . Note that by this definition, is not BIBO stable, but it’s still a useful building block.

## Causality

An LTI system is causal if for .

# LCCDE sinusoidal steady-state analysis

LCCDE are a special class of LTI systems:

* LCC differential equations for CT systems
* LCC difference equations for DT systems

LCC difference equations are relatively new to signals and systems; they are important for efficient DSP.

## CT LTI sinusoidal steady-state response

To solve LCC differential equations for the general solution, we need Laplace transforms. For now, we will focus on the sinusoidal steady-state solution.

The output of a BIBO-stable system, in response to a sinusoidal input as time goes to infinity, is a sinusoid of the same frequency.

The impulse and step responses of causal and stable LTI systems will usually include transient terms of the form . Transient terms die out as , so only the steady-state response is left.

Steady-state analysis is finding the output when the input is

For a causal and stable system,

**The output of the system in response to the complex sinusoid is the same sinusoid multiplied by , the frequency response of the system at .**

The frequency response of the system is

is the Fourier transform of the impulse response .

If is real, then

**Real sinusoid, real :**

## LCC differential equations

LCC differential equations have been around for a long time in electrical engineering, mechanical, chemical, biological, and other sciences. They are used to model time-varying signals in LTI systems, like a circuit composed of resistors, capacitors, and inductors.

Nth-order LCCDE:

and are the input and output, and N is the highest derivative in the output.

The coefficients and completely describe the system.

When the input and its derivatives are nonzero, this is a **nonhomogeneous differential equation**. When the input and its derivatives are zero, then it is a **homogeneous differential equation**, i.e.

**Frequency response:**

, corresponding to , are the numerator coefficients.

, corresponding to , are the denominator coefficients.

indicates the derivative order.

**Example:**

The coefficients are . You can use scipy.signal.freqs to calculate the frequency response. For a 1Hz real sinusoid, you can calculate the steady-state output using the frequency response, but you can also simulate the full response using scipy.signal.lsim, which includes the transient plus steady state response starting from .

A graph of a function

AI-generated content may be incorrect.

A graph with blue and orange lines

AI-generated content may be incorrect.

A graph of a full response

AI-generated content may be incorrect.

## DT LTI sinusoidal steady-state response

The general output of DT LTI systems will contain transient terms of the form . For table systems, , so the transients die as , leaving only the steady-state response.

The frequency response of the system is

The frequency response is periodic with period :

**Real sinusoid, real :**

## LCC difference equations

Convolution sum may require an infinite number of calculations due to the doubly infinite sum, but LCC difference equations make DT LTI systems possible.

Nth-order LCCDE:

is the number of feedback terms (also, the system order), and is the number of feedforward terms. The system is completely defined by the coefficient sets and .

In continuous time, the LCC differential equation represents a circuit, but in discrete time, the LCC difference equation is exactly the system.

Generally, (or the coefficients are normalized by ). Then

The general form is an IIR filter.

When for , then it becomes an FIR filter,

Note that this is the same form as DT convolution, where the impulse response is

**Example: accumulator system**

An accumulator system is defined by .

This is an IIR, which means you need recursion (LCCDE) to represent it in the real world.

**Using recursion to find the impulse response:**

Recursion works for solving simple systems (z-transforms will be necessary for complex systems).

Let , a scaled step sequence. Let the initial condition be .

The output is found iteratively, calculating for each starting at 0.

For and ,

This is the step response of the at-rest system defined by .

You can also find the impulse response by setting the initial conditions to zero and letting .

**Frequency response:**

indicates the delay order.

**Example: FIR filter,**

A graph with lines and numbers

AI-generated content may be incorrect.

A graph of a function

AI-generated content may be incorrect.

# Fourier series of periodic CT signals

## Frequency domain and line spectra plots

Frequency domain representation is defined by the three parameters of a sinusoid: amplitude, phase, and frequency.

**Complex sinusoid:**

has a spectral amplitude of at and spectral phase of at . Characterized by , aka frequency-amplitude/phase pair.

**Real sinusoid:**

For real sinusoids,

For real signals,

The constant (DC) term is characterized by . If , then .

**Line spectra plot** = plot amplitude or phase against frequency as vertical lines. (Plot ).

* Two-sided plot shows both positive and negative frequencies
* Only for real signals: one-sided plot shows only nonnegative frequencies. Compared to two-sided plot, amplitudes are doubled (except for DC). is characterized by .

**Symmetry of real signals:**

When is real, the expansion into complex sinusoids always results in conjugate symmetry of coefficients, .

* Double-sided amplitude spectrum has even symmetry:
* Double-sided phase spectrum has odd symmetry:

## Fourier series

A signal that is a sum of sinusoids may be periodic or aperiodic. If the greatest common divisor (GCD) of the sinusoid frequencies exists, then is periodic with period equal to 1/GCD; otherwise, is aperiodic.

When a signal is periodic, every frequency in the signal is harmonically related to the fundamental frequency (which is equal to the GCD).

That is, , every frequency is an integer multiple of , and the signal is periodic with period .

**Fourier series:**

When a signal is periodic, you can convert between its time-domain and frequency-domain representation using Fourier series. Let be the fundamental frequency and be its period. The Fourier coefficients, , are the frequency-amplitude/phase pairs discussed earlier – if you can directly write as a sum of sinusoids, then you do not need to use the analysis equation to find .

Synthesis equation (frequency 🡪 time):

Analysis equation (time 🡪 frequency):

DC term is average waveform value:

### Deriving the analysis equation from the synthesis equation

The first step is to establish the orthogonality of when integrated over a interval.

Orthogonality means that the dot (inner) product of two functions is zero.

Dot product for complex functions is defined as

This evaluates to 0 when . When ,

Therefore, are orthogonal when .

With this, we can derive the analysis equation, starting from the synthesis equation:

### Gibbs phenomenon/ringing

For signals with jump discontinuities (like a square wave), the synthesis equation does NOT converge to the original signal in a point-by-point sense. At the discontinuities, it converges to the average of the left- and right-hand limits; for example, for a square wave that jumps between , the synthesis equation converges to 0.

Furthermore, the “reconstructed” signal oscillates around the discontinuity with a peak overshoot of around 9%. Increasing does not reduce the overshoot; it only increases oscillation frequency.

A graph of a pulse train

AI-generated content may be incorrect.

### Pulse train/square wave

Periodic pulse train over one period:

The Fourier coefficients of a periodic pulse train follow a sinc function. (sinc of 0 is 1.)

For a square wave,

For even, . For odd, .

**Example rectangular pulse train:**

A graph of a pulse train

AI-generated content may be incorrect.

Remember: the line spectra sit at multiples of the fundamental frequency, or .

are zero when , or , or . ( is an integer.) Thus, the nulls/sidelobes are defined by or .

If you keep constant, the nulls are always at , etc. regardless of .

If you keep constant, decreasing pushes out the nulls.

For square waves, , so 🡪 even harmonics are null.

Because of term, the slope of the phase is equal to w.r.t. frequency . At each null, the sinc term switches between , which causes a phase jump of .

**Example square wave signal:**

**A graph of a pulse train

AI-generated content may be incorrect.**

### Triangle wave

As with the square wave, even-order harmonics are null. Since the triangle wave is smoother, the harmonics roll off more quickly as (compared to ). For the same reason, synthesizing the triangle wave using Fourier synthesis also works out much better than for the square wave.

In the plot below, harmonics are normalized to the fundamental (fundamental is at 0dB).

A graph of a square wave

AI-generated content may be incorrect.

A graph of a wave

AI-generated content may be incorrect.

### Rectified sine waves

Half-wave rectified sine-wave:

Full-wave rectified sine-wave:

### Properties

**DC level shifting and gain scaling:**

**Time shifting:**

**Symmetry properties for real signals:**

* is an even function of time, i.e. , then are real.
* is an odd function of time, i.e. , then are imaginary.
* has odd half-wave symmetry, i.e. , then for even. Ex: square waves and triangle waves with DC = 0.

**Parseval’s theorem:**

Power in periodic signal :

If is real,

The power of is equal to the sum of the power of each component because the components are orthogonal.

# CTFT

## CTFS to CTFT

Consider the periodic pulse train with fixed , centered around . The Fourier series coefficients are given by

Remember, the nulls are defined by 🡪 the nulls are always at regardless of .

If you let , becomes an aperiodic pulse. The spectral lines of the CTFS get closer together because the fundamental frequency , and in the limit, becomes a continuous spectrum.

A graph of a function

AI-generated content may be incorrect.

In the plot, is normalized by :

The line spectra is a sampling of the continuous spectrum at , where is the Fourier transform of the aperiodic signal .

## CTFT

For FT to exist, must be aperiodic and have finite energy.

is the spectrum of .

If has units of voltage, then has units of .

Alternative radian frequency form:

**Rectangular pulse:**

Nulls at

**Fourier transform is a linear operator:**

**Fourier transform is complex:**

**Symmetry properties for real signals:**

* is real 🡪 is conjugate symmetric. is even, is odd.
* is real and even, 🡪 is real
* is real and odd, 🡪 is imaginary

**Energy spectral density with Parseval’s theorem:**

Energy spectral density of energy signal is .

The unit of is , so the unit of is . is energy, so this is energy/Hz.

Total signal energy can be found by integrating over time or frequency.

### Properties

**Linearity:**

**Time delay:**

**Time scaling:**

: spreading in time = compressing in frequency

: compressing in time = spreading in frequency

**Frequency translation:**

**Modulation:**

**Convolution:**

**Multiplication:**

**Differentiation:**

**Integration:**

**Parseval’s theorem:**

**Duality of the Fourier transform:**

If you take the Fourier transform of , with replaced by , then you get with replaced by .

Proof:

This looks like the IFT (inverse Fourier transform), except is the variable of integration and is the independent variable. So

Duality examples:

* Time shift means multiplying by a complex exponential in freq domain, and freq shift means multiplying by a complex exponential in time domain
* Convolving in time equals multiplying in freq; multiplying in time equals convolving in freq
* Rectangular pulse = freq domain sinc; freq domain rectangle = time-domain sinc

Since sinc is even, .

### Transform pairs

Notes:

* Triangle pulse is convolution of rectangular pulse with itself
* repeated poles can easily be derived from convolution integral
* For DC, and representations are scaled differently b/c of the term in the synthesis equation
* Reciprocal spreading property: a signal with long duration will have narrower spectrum, and a signal with short duration will have wider spectrum

|  |  |  |  |
| --- | --- | --- | --- |
| Signal | Time Domain | Freq Domain | Freq Domain |
| Rectangular pulse |  |  |  |
| Sinc |  |  |  |
| Triangle pulse |  |  |  |
| Decaying exponential |  |  |  |
| repeated poles |  |  |  |
| Double-sided decaying exponential |  |  |  |
| Impulse |  |  |  |
| DC |  |  |  |
| Delayed impulse |  |  |  |
| Complex sinusoid |  |  |  |
| Cosine |  |  |  |
| Sine |  |  |  |
| Step |  |  |  |
| Impulse train (time-domain sampling) |  |  |  |
| Convolution with impulse train (frequency-domain sampling) |  |  |  |

### Fourier transforms in the limit for periodic power signals

Formally, FT requires to be an energy signal, but the technique called Fourier transforms in the limit allows you to take the FT of periodic power signals like sine/cosine/periodic pulse. This allows you to analyze energy signals and periodic power signals together.

**First, singularity functions:**

Consider the constant function . Rewrite in the limit form:

1. (this is true for any )

IFT:

**Periodic signals:**

A periodic signal with fundamental frequency can be decomposed into orthogonal frequencies:

The FT of a periodic signal is impulses scaled by the FS coefficients.

Alternatively, use the property of frequency domain sampling. Let be one period of , so

The FS coefficients for are given by

Then

The FT of a periodic signal is the FT of one period of the signal, sampled at multiples of the fundamental frequency.

These two interpretations are equivalent.

**Example: periodic pulse train**

### LTI systems in the frequency domain

is the frequency response of the LTI system. This is equivalent to the frequency response derived via sinusoidal steady-state response.

**Properties of the frequency response:**

If is real, .

Energy spectral density:

Example lowpass RC filter:

A diagram of a circuit

AI-generated content may be incorrect.

**Cascaded and parallel filters:**

**Ideal filters:**

Are not realizable. The impulse response for all ideal filters – lowpass, highpass, bandpass, bandstop – are nonzero for , which means they are non-causal.

**Realizable filters:**

Butterworth, Chebyshev, Bessel, elliptic filters – approximations of ideal filters. Defined by LCC differential equations.

# Sampling theory

## Sampling (ADC)

A diagram of a program

AI-generated content may be incorrect.

Converting analog signal to digital: sample and quantize.

**Example:**

has units of Hz (cycles/second), has units of radians/second, has units of radians/sample, has units of cycles/sample. in CT maps to in DT.

In DT, frequency is periodic. is equivalent to where is any integer.

This means that when converting from CT to DT, CT signals at all appear at the same DT frequency . This is called aliasing; this is why we need an antialiasing filter before the ADC.

This also means the DT spectrum is periodic. A signal at frequency appears at . In terms of CT frequency, a signal at appears at . In effect, the signal is repeated every .

What’s happening is you are convolving the CT spectrum with a frequency domain impulse train!

This leads to the Nyquist-Shannon sampling theorem, which says that in order to reconstruct from ,

The copy of the signal that lies within is the principle alias. When , the principle alias is not at the original frequency because and the signal aliases back onto itself at .

The principle alias is calculated by finding s.t.

is called the Nyquist rate, and is called the Nyquist frequency or folding frequency.

## Quantization error

Quantize to B bits (round or truncate):

If is the max voltage swing of the ADC, then

* Step size:
* Levels: to

Note: the output saturates. Voltages outside of saturate.

Quantization error is the difference between the quantizer input and output:

You can view as an additive noise source (which we call quantization noise). Technically, it is noise-like since it’s functionally related to the input.

For rounding, .

Average quantization noise power (assuming uniform distribution):

Full sine wave power:

SQNR:

## Frequency domain analysis

Frequency-domain convolution:

are called the spectral translates of , with being the principal translate.

If is bandlimited s.t. for , then we can recover from its samples by applying a LPF, as long as . is the Nyquist frequency; is the Nyquist rate.

If , then the signal and its translates overlap, and you cannot recover .

## Reconstruction (DAC)

Two steps:

1. Place signal samples, , back on the physical time axis,
2. Interpolate a continuum of signal values between the sample values

This is a mixed-signal design problem.

**Step 1:**

In frequency domain, this means going back to .

**Step 2:** pass through a zero-order hold (ZOH). ZOH models the output of a realizable DAC. As a filter, ZOH is modeled by a rectangular impulse response:

Remember, . We want while suppressing all other images at. The ZOH partially suppresses these images because it has nulls at .

The final step is to add a reconstruction filter to counteract the droop of the ZOH response (-3.9dB @ fs/2). The overall frequency response of the D-A conversion is or .

A graph of a frequency response

AI-generated content may be incorrect.

The ZOH only fully suppresses at , where is the DAC sampling rate. This means you will get significantly more distortion (from images) for frequencies farther from 0.

A graph of a line graph

AI-generated content may be incorrect.

A graph of a function

AI-generated content may be incorrect.

A graph of a function

AI-generated content may be incorrect.

### Ideal interpolation

Let’s ignore “realizable” and do D-A conversion using an ideal interpolation filter given by

What this says: If you add up a bunch of CT sinc functions, scaled and time-shifted according to the DT signal, you will recover the original CT function.

is the sampled version of that includes all aliases at where is any integer.

The ideal reconstruction filter removes all images except for the principle alias () – this is why we need to ensure no aliasing during sampling.

I simulated a digital reconstruction filter:

A graph of a reconstruction filter

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A graph with blue lines and a red line

AI-generated content may be incorrect.

A graph of a function

AI-generated content may be incorrect.

### System design

Real interpolation filters have finite image suppression. You can increase the filter order to improve suppression, but this increases complexity. Alternatively, you can use oversampling to first separate the spectral translates. This begins in the discrete-time domain, using efficient digital filters, and ends with a high DAC rate. A high DAC rate eases the design requirements on the analog interpolation filter.

# DTFT

Formally, DTFT exists if is absolutely summable (but there are exceptions for special signals):

**Example:** .

The DTFT only exists if .

## Relationship between CTFT and DTFT

In DTFT, the frequency variable always appears wrapped in a complex exponential, , which means the DTFT spectrum is periodic with period .

In practice, the DT sequence is commonly created by uniform sampling of the CT signal . This sampled waveform is

How do we relate the spectrum of to the spectrum of ?

First, take the FT of :

With a simple frequency conversion, is equal to !

As we established above, is a periodic version of . Mapping from to , we get

Therefore, if is bandlimited to , or equivalently, for , and we look only at the principle alias/translate, then and are equal except for frequency conversion and scaling by :

We can also do this for the radians/second form of the CTFT:

With these relationships, you can map CT spectrum to DT spectrum (sampling) and back (reconstruction).

## Symmetry properties for real signals

real 🡪 is conjugate symmetric, i.e. .

* is even
* is odd
* is even
* is odd

real and even 🡪 is real, i.e. .

real and odd 🡪 is imaginary, i.e. .

**Example:** . Real signal, which means in freq domain that magnitude and real part are even, phase and imaginary part are odd.

**A graph of a function

AI-generated content may be incorrect.**

**Example:**

Since is real and even, is real.

**Example:**

Since is real and odd, is imaginary.

## DTFT theorems and pairs

|  |  |  |
| --- | --- | --- |
| Property | Signal | Transform |
| Linearity |  |  |
| Time shift |  |  |
| Frequency shift |  |  |
| Convolution |  |  |
| Multiplication |  |  |
| Reversal |  |  |
| Differentiation |  |  |
| First difference |  |  |
| Accumulation |  |  |
| Downsample |  |  |
| Upsample |  |  |
| Parseval’s theorem |  |  |

|  |  |  |
| --- | --- | --- |
| Signal | Time | Frequency |
| Impulse |  |  |
| Delayed impulse |  |  |
| Decaying exponential/first-order LPF |  |  |
|  |  |  |
|  |  |  |
| Rectangular window |  |  |
| Sinc |  |  |
| Complex exponential |  |  |
| Cosine |  |  |
| DC |  |  |
| Step |  |  |
| Impulse train |  |  |

A graph of a function

AI-generated content may be incorrect.

A graph of a rectangular window

AI-generated content may be incorrect.

A graph of a windowed brickwall

AI-generated content may be incorrect.

A graph of a diagram

AI-generated content may be incorrect.

A graph of a function

AI-generated content may be incorrect.

## Mean-square convergence and time-domain sinc

Remember, DTFT only exists if absolutely converges:

A weaker form of convergence is mean-square convergence:

It is easier to satisfy this condition, but the DTFT may not converge pointwise in the frequency domain.

Example:

Since the terms of are of the form , the absolute sum is similar to the harmonic series, which diverges. The series does converge, so is square summable.

This is an infinite-duration signal, but you can truncate the coefficients to get an approximation to the ideal LPF. Increasing the number of taps sharpens the filter cutoff, but the ripple amplitude is unchanged (in both passband and stopband). This can be mitigated with windowing.

A graph of a function

AI-generated content may be incorrect.

A graph of a function

AI-generated content may be incorrect.

## Finding DTFT in the limit

Allows impulse functions to exist in the frequency domain.

Let’s say you have a DTFT of . With DTFT periodicity, the full representation is

(Remember that when is a function only of , the periodicity is built-in.)

When ,

When ,

## LTI systems in the frequency domain

LTI system defined by impulse response .

is the transfer function or frequency response.

**Energy spectral density:**

**Series and parallel connections:**

**Example:**

# DFT

DFT allows us to use a computer to easily compute the frequency domain representation of a DT signal (and invert it). The mathematical definition of the DFT involves only complex number multiplication and addition – no integrals.

FFT is a set of efficient algorithms for computing the DFT.

## DFT is a sampled DTFT

DTFT finds the continuous spectrum of a DT signal that is indexed into double infinity.

DFT operates on an -point signal and produces and -point frequency spectrum. As long as encompasses the entire finite-duration DT signal, the -point spectrum is a sampled version of the DTFT spectrum.

Start with the DTFT.

What are the values of at the discrete frequencies ?

is periodic with period , so this is simplified to

The complex weights are called the twiddle factors.

The DFT is the DTFT sampled at . In the z domain, the samples are on the unit circle. gives the DTFT, and gives the DFT.

## Forward/inverse transforms

Each sum is composed of N terms, so DFT/IDFT can be computed with a finite number of complex multiplications and additions. This is not true of CTFT/DTFT, which both require integrations along a continuous variable.

Proof of invertibility, starting from the synthesis equation:

Since is periodic with period ,

Then

## DFS mathematics underpin the DFT

DFS is the discrete-time version of Fourier series. is decomposed into periodic functions of the form , so is periodic with period . Just like in CTFS, the periodic functions are harmonically related.

Even though DFT is for finite-length signals, in the underlying mathematics, the signal is periodic. The sequence of length is repeated into double infinity, and the DFS contains unique coefficients that are also repeated into double infinity. But a periodic signal can be perfectly represented with just one period, so the DFT stores only one period of the signal and the DFS coefficients.

This is like convolving with impulse train , which is why DFT is sampled version of DTFT.

The repeated sequence of length – with its periodic extensions – can be represented as

The DFT is also periodic with period .

The two most important things about DFT:

* It is samples of the DTFT at
* The underlying signal is

It’s important to keep these two things in mind when using and plotting the DFT.

## Theorems and properties

|  |  |  |
| --- | --- | --- |
| Property | Sequence | -point DFT |
| Linearity |  |  |
| Duality |  |  |
| Circular shift |  |  |
| Circular shift frequency |  |  |
| Circular convolution |  |  |
| Multiplication |  |  |
| Symmetry | real |  |

Because of the periodic nature of the DFT, the time shift, frequency shift, and convolution theorems are slightly different from the DTFT. Circular convolution enables transform domain signal processing, which means we can implement a filter in the frequency domain by multiplying our signal spectrum by the filter frequency response and returning to time domain via IDFT.

It's circular because the “underlying” signal is periodic, and DFT is a window into one period of that signal.

Another difference is that all sequence lengths must be the same ().

**Circular shift:**

A graph of different sizes and colors

AI-generated content may be incorrect.

**Circular convolution:**

Shorthand notation:

A diagram of a function

AI-generated content may be incorrect.

## FFT

One of the pillars of modern signal processing – used in many different products and for many different applications.

* Easily calculates frequency response and spectra; able to provide real-time spectral analysis, e.g. electronic test equipment that displays spectra
* Transform domain filtering

From the DFT definition, the number of complex multiplications and additions required for an -point transform is and , respectively. FFT significantly reduces these numbers.

For a single , you have complex multiplications and complex additions. Since you need to compute for values of , this becomes and .

**Decimation-in-time (DIT) FFT algorithm:**

There are many FFT algorithms. The basic one is radix-2 DIT algorithm/radix-2 FFT, in which must be a power of 2.

If , then you can break down the DFT computation into stages of two-point DFTs. The two-point DFT is the fundamental computational unit.

example:

Multiplications: , down from (each two-point DFT only requires 1 complex multiplication)

A table with numbers and a few words

AI-generated content may be incorrect.

## Application example: transform domain filtering

You can use DFT/IDFT to perform filtering entirely in the frequency domain, but you need to answer two questions:

* How do you make circular convolution perform ordinary linear convolution (convolution sum)?
* How do you handle a very long sequence while keeping the DFT length manageable?

**Making circular convolution perform linear convolution:**

If is length and is length , then the linear convolution output is length .

In circular convolution, the input and output sequences are all length .

To implement linear convolution using circular convolution, you need to zero-pad and to length , where . (You still perform multiplication in the DFT domain to implement circular convolution.)

**Using overlap and add to continuously filter sequences:**

is a sequence of possibly infinite duration and is a length FIR filter. Divide into segments of finite length and then filter each segment using the DFT/IDFT as described above. The IDFT outputs are length , so afterward, when you add the filtered sections together, you need to add them with overlap.