<https://en.wikipedia.org/wiki/Spectral_density>

<https://en.wikipedia.org/wiki/Autocorrelation>

# Energy spectral density (ESD)

Fourier transform exists for energy signals.

has units of .

ESD, , has two mathematically equivalent definitions:

1. , i.e. the FT of the autocorrelation of

has units of . This is energy per Hz.

By Parseval’s theorem, total signal energy is

has units of . This is energy.

**Prove that .**

By the convolution theorem of FT,

Let . Then

Yes, energy spectral density is equal to the autocorrelation of . This autocorrelation definition is for energy signals.

# Power spectral density (PSD)

Typically, we work in units of power, not energy. ESD is only defined for pulse-like signals – i.e. energy signals, or signals that are square integrable/summable. ESD is not defined for signals that exist for all time – like noise, or in general, power signals – because these signals have infinite energy.

Therefore, we are typically interested in the power of signals and their power spectral density. To convert from energy to power, we simply normalize by , where is the duration of interest or the measurement duration. In the mathematical definitions, we take the limit as , but in practice, we choose to be long enough to capture several cycles of the lowest frequency of interest.

The average power of a signal is

has units of . This is power.

Alternatively, we may define , where is a brick wall in time with unity gain. Then

is an energy signal, which means we can define its FT

has units of .

Then PSD, , is given by two mathematically equivalent definitions:

1. – again, the autocorrelation of , but autocorrelation is normalized by

has units of . This is power per Hz.

Parseval’s theorem still holds:

**Prove that .**

From the proof for ESD,

This means

Then

Where autocorrelation for power signals is defined as

# One-sided vs. two-sided spectral density

Spectral density is always real, but it’s not always even (symmetric).

1. Real signal 🡪 autocorrelation is real and even 🡪 SD is real and even
2. Complex signal 🡪 autocorrelation is conjugate symmetric 🡪 SD is real but asymmetric

For real signals, since SD is real and even, you can define either a one-sided SD or a two-sided SD. One-sided SD has 2x power of two-sided SD (per Hz).

**Proof that SD is always real:**

Let’s take a look at autocorrelation for .

Change of variable:

Then

is conjugate symmetric.

What does this tell us about SD, which is the FT of ?

Since ,

SD is real.

If is real, then is real and symmetric, and SD is real and symmetric.

**Proof that one-sided SD is 2x the power of two-sided SD (per Hz):**

Let’s use the example of a real sinusoid.

Average power is given by

What is the average power in one tone?

The power in one tone is half of the total power.

# Periodic power signals

Periodic power signals can be represented as a sum of harmonics.

If is periodic with period , then the FS coefficients are

has units of .

The technique of FT in the limit tells us that

Then the FT of is

According to Parseval’s theorem, the power in is given by

Therefore, spectral density must have the form

(and ) has units of .

# Summing power of tones

When your signal consists of distinct tones, your signal’s average power is equal to the sum of the power of each tone.

Assuming and ,

If and , then