Theoretical Part

1) EXAMple Question - Roll the dice

$$
\begin{aligned}
p(\theta|x) &= \frac{p(x|\theta)p(\theta)}{p(x)} \\
&= \frac{Mul(x;N,\boldsymbol{\theta}) \cdot Dir(\boldsymbol{\alpha},\boldsymbol{\theta})}{\int Mul(x;N,\boldsymbol{\theta}) \cdot Dir(\boldsymbol{\alpha},\boldsymbol{\theta})\ \ d\boldsymbol{\theta}} \\
&= \frac{\frac{N!}{\prod_{i=1}^{6} x_i!} \prod_{i=1}^{6} \theta_i^{x_i} \cdot \frac{1}{B(\boldsymbol{\alpha})} \prod_{i=1}^{6} \theta_i^{\alpha_i} - 1}{\int Mul(x;N,\boldsymbol{\theta}) \cdot Dir(\boldsymbol{\alpha},\boldsymbol{\theta})\ \ d\boldsymbol{\theta}} \\
&= \frac{N! \cdot \prod_{i=1}^{6} \theta_i^{x_i} (\theta_i^{\alpha_i} - 1)}{\prod_{i=1}^{6} x_i! \cdot B(\boldsymbol{\alpha}) \cdot p(x)}
\end{aligned}
$$

2) Theory Question - Gaussian posterior

(a)

We will show that $p(x^*|x) \propto p(x|x^*)p(x^*)$.

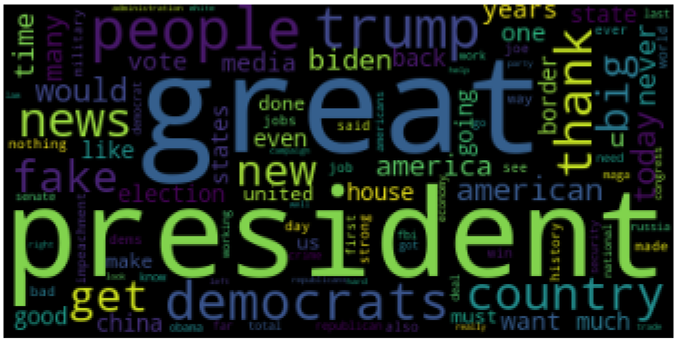
$$
\begin{aligned}
p(x^*|x) &= \frac{exp(\frac{-1}{2\frac{1}{\frac{1}{\sigma_0^2}+N}} \left(x^* - \frac{\sum_{i=1}^{N} x_i}{\frac{1}{\sigma_0^2}+N}) \right)}{\frac{1}{\frac{1}{\sigma_0^2}+N}\sqrt{2\pi}} \\
&\overset{\sigma_0^2=1}{\approx} \frac{(N+1)exp(-\frac{N+1}{2}(x^* - \frac{1}{N+1}\sum_{i=1}^{N} x_i)^2}{2\pi} \\
&\approx (N+1)\frac{exp(-\frac{N+1}{2}(x^* - \hat{\mu}))}{2\pi} \\
&\dots (?) \\
&\propto \frac{exp(-\frac{1}{2}(x^* - x)^2 - \frac{1}{2\sigma_0^2}{x^*}^2)}{\sigma_0 2\pi} \\
&= \frac{exp(-\frac{1}{2}(x - x^*)^2}{\sqrt{2\pi}} \cdot \frac{exp(-\frac{1}{2\sigma_0^2}{x^*}^2}{\sigma_0\sqrt{2\pi}} \\
&= p(x|x^*)p(x^*)
\end{aligned}
$$

(b) TODO

In [1]:
```
import pandas as pd
import re
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import dirichlet, entropy
import string
import gensim
from wordcloud import WordCloud, STOPWORDS, ImageColorGenerator
```

Modelling Trump tweets

In this exercise, we will create a simple model to generate a list of words representing a tweet. We will use Bayesian inference to infer the model parameters using tweets from ex-president Donald Trump. Samples from the posterior will represent "word frequencies" and can thus be visualized by a word cloud.



Finally, we will investigate how sequentially incorporating more tweets in our model reduces the entropy of the resulting posterior distribution.

In [2]:
```
df = pd.read_csv("trump_tweets.csv", index_col=0)
df.head()
```

Out[2]:

	date	tweet	tokens
0	2017-01-20 06:31	It all begins today! I will see you at 11:00 A...	['begins', 'today', 'see', '1100', 'swearingin...
1	2017-01-20 11:51	Today we are not merely transferring power fro...	['today', 'merely', 'transferring', 'power', '...
2	2017-01-20 11:51	power from Washington, D.C. and giving it back...	['power', 'washington', 'c', 'giving', 'back',...
3	2017-01-20 11:52	What truly matters is not which party controls...	['truly', 'matters', 'party', 'controls', 'gov...
4	2017-01-20 11:53	January 20th 2017, will be remembered as the d...	['january', '20th', '2017', 'remembered', 'day...

Tweets as a bag of words

In this exercise, we will assume that a tweet can be expressed as a **bag of words** i.e. a list only containing relevant words (excluding e.g. stop words, punctuations and more). We assume that these words come from a **dictionary** of a fixed length J.

To create a representative dictionary, we will use gensim to do this automatically.

In [3]:
```
J = 200 # Number of words in the dictionary
```

In [4]:
```
X_tokens = list(list(map(lambda x: eval(x), df.tokens)))
dictionary = gensim.corpora.Dictionary(X_tokens)
dictionary.filter_extremes(no_below=15, no_above=0.5, keep_n=J)
```

We can use the function "dictionary.doc2bow" to translate our tokens (i.e. a list of words) to a list of tuples

(i, j) where i is the index of the word in the dictionary and j are the counts of the word.

In [5]:
```
X_bow = [dictionary.doc2bow(x) for x in X_tokens]
df["index_counts"] = X_bow
```

In [6]:
```
# Remove tweets containing words that are not in the dictionary
df = df[df['index_counts'].map(lambda d: len(d)) > 0]
df.head()
```

Out[6]:

	date	tweet	tokens	index_counts
0	2017-01-20 06:31	It all begins today! I will see you at 11:00 A...	['begins', 'today', 'see', '1100', 'swearingin...	[(0, 1), (1, 1), (2, 1)]
1	2017-01-20 11:51	Today we are not merely transferring power fro...	['today', 'merely', 'transferring', 'power', '...	[(1, 1), (3, 1), (4, 2), (5, 2), (6, 1)]
2	2017-01-20 11:51	power from Washington, D.C. and giving it back...	['power', 'washington', 'c', 'giving', 'back',...	[(7, 1), (8, 1), (9, 1), (10, 1)]
3	2017-01-20 11:52	What truly matters is not which party controls...	['truly', 'matters', 'party', 'controls', 'gov...	[(6, 1), (9, 1), (11, 2)]
4	2017-01-20 11:53	January 20th 2017, will be remembered as the d...	['january', '20th', '2017', 'remembered', 'day...	[(9, 1), (12, 1), (13, 1)]

In [7]:
```
N = len(df)
print("number of valid tweets = " +str(N))
```

```
number of valid tweets = 20293
```

Bayesian inference via conjugate multinomial dirichlet

We will assume that all tweets are generated as follows:

- Each tweet is independently and identically distributed.
- A tweet is a list of K_i words $w_i = (w_{i1}, \dots, w_{iK_i})$ where each word is represented by the index in the dictionary $w_{ij} \in \{1, \dots, J\}$.
- Each word w_{ij} is drawn independently from the dictionary with a unknown probability $\theta = (\theta_1, \dots, \theta_J)$.

The likelihood under these assumptions is naturally given by the *Multinomial distribution*

$$p(w_i|\theta) = Mul(w_i; K_i, \theta) = \frac{K_i!}{\prod_{j=1}^{J} x_{ij}!} \prod_{j=1}^{J} \theta_j^{x_{ij}}$$

were x_{ij} does represent the number of times the word with index j appears in tweet i.

As you already saw in the first exercise of this sheet, a conjugate prior over θ is given by the Dirichlet distribution $p(\theta) = Dir(\alpha)$. Thus the posterior distribution is also a Dirichlet distribution

$$p(\theta|w_1, \dots, w_n) = Dir(\theta; \hat{\alpha}) \qquad \hat{\alpha} = \left(\alpha_1 + \sum_{i=1}^{n} x_{i1}, \cdots, \alpha_J + \sum_{i=1}^{n} x_{iJ}\right)$$

Exercise 1:

We will put all the word counts in a matrix to make the above calculation more convenient.

Compute a $X^{N \times J}$ matrix, which contains all the word counts for all tweets.

$$x_{ij} = \text{Number of times word with index j occurs in tweet i}$$

Note: In the dataframe, we already have a list of tuples containing the counts and index for each word.

In [8]:
```
# TODO Fill the matrix
X_matrix = np.zeros([N, J])
```

In [9]:
```
N,J
```

Out[9]:
```
(20293, 200)
```

In [10]:
```
for i,(dfindex, value) in enumerate(df['index_counts'].items()):
    vec = np.zeros(shape=(J,))
    for index,count in value:
        vec[index] = count
    X_matrix[i,:] = vec
```

Exercise 2

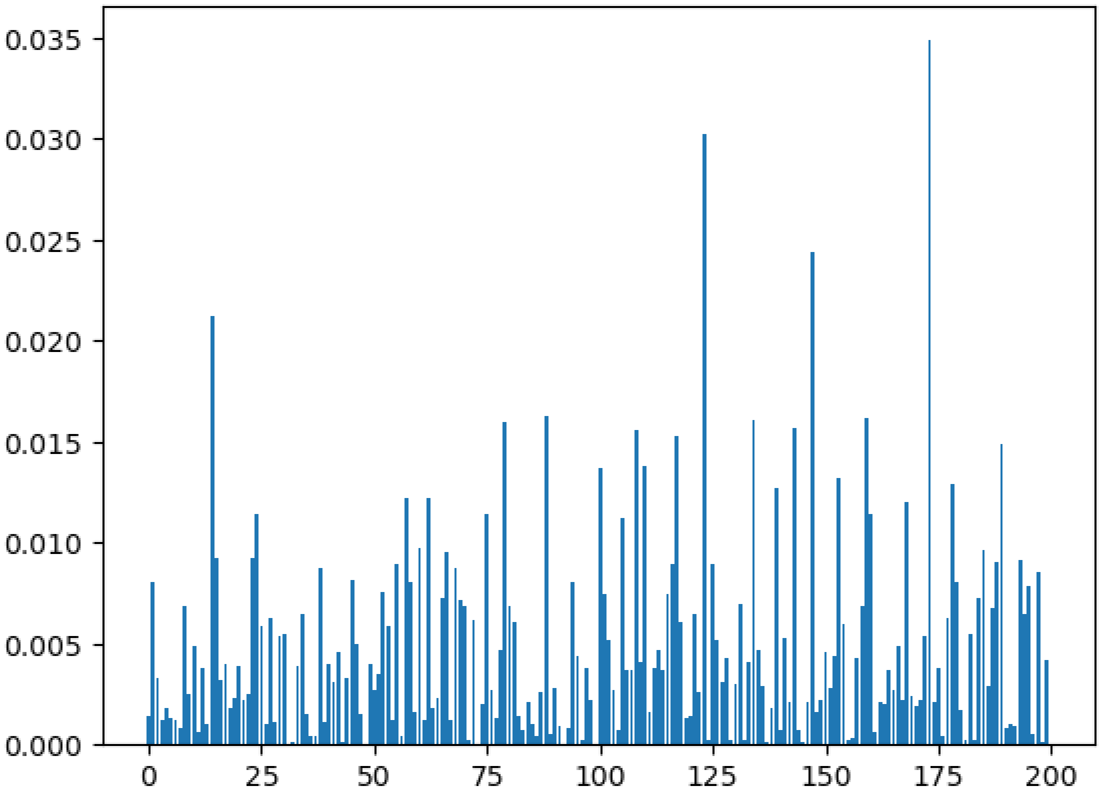
Below we have two priors with two different parameters α_1 and α_2. Draw three samples from the prior and visualize them using the 'plot_words' function.

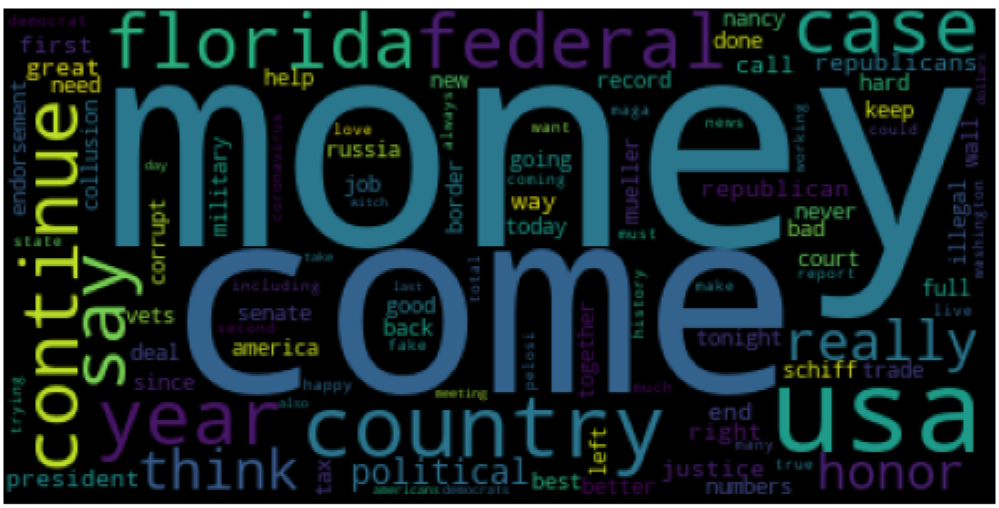
Hint: To sample from a Dirichlet distribution, you can use 'np.random.dirichlet'.

In [11]:
```
def plot_words(theta):
    """ Given a vector of word frequencies, this function produces a nice wordcloud."""
    fig = plt.figure()
    token_freq = dict(zip(list(dictionary.token2id.keys()), theta))
    wordcloud = WordCloud(max_words=100).fit_words(token_freq)
    plt.imshow(wordcloud)
    plt.axis("off")
    plt.show()
```

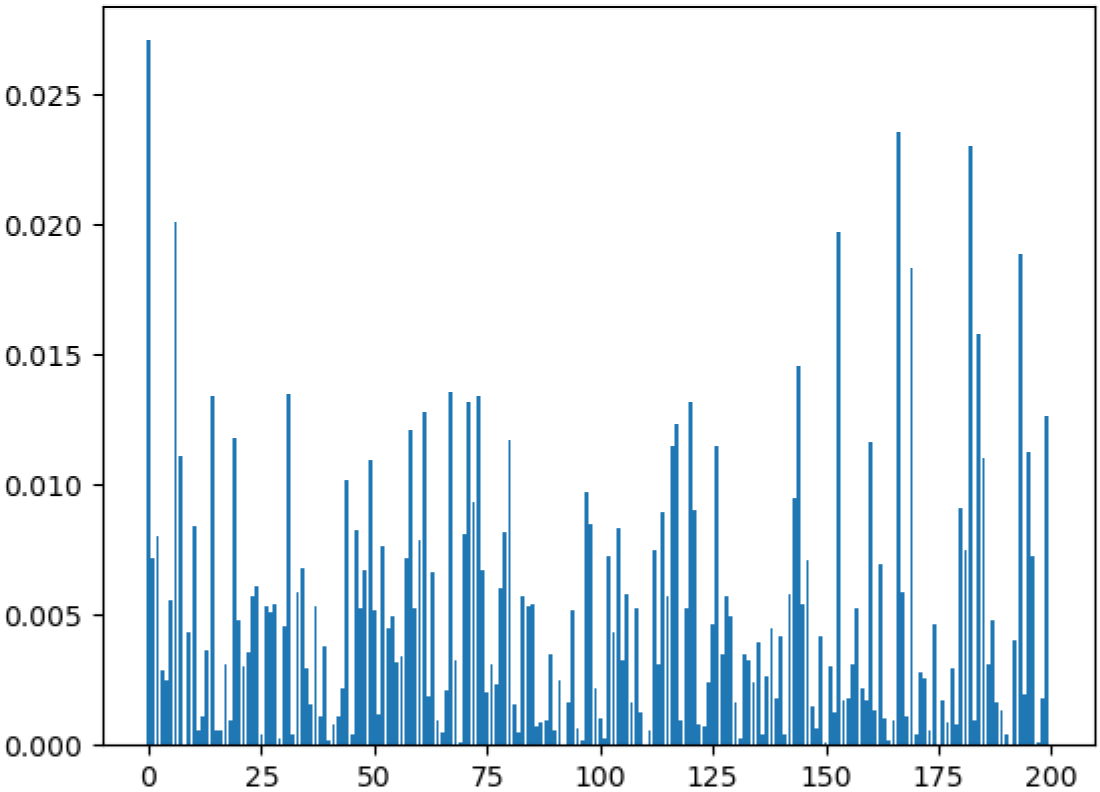
In [12]:
```
# Prior
alpha_prior1 = np.ones(J)
alpha_prior2 = np.arange(J) + 1

theta_alpha1 = np.random.dirichlet(alpha_prior1, size=3)
theta_alpha2 = np.random.dirichlet(alpha_prior2, size=3)
```

In [13]:
```
for th in theta_alpha1:
    plt.bar(range(J), th)
    plot_words(th)
    imax = np.argmax(th)
    print(f"max word: {dictionary.get(imax)}")
```






```
max word: money
```


0.025
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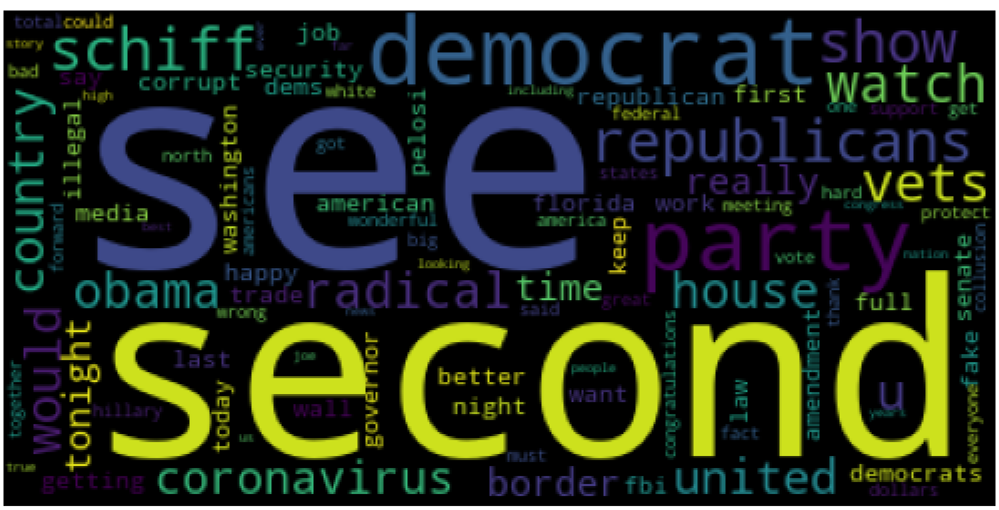
175
200


see
second
democrat
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really
republican
first
security
corrupt
job
dems
illegal
washington
pelosi
media
american
florida
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happy
trade
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senate
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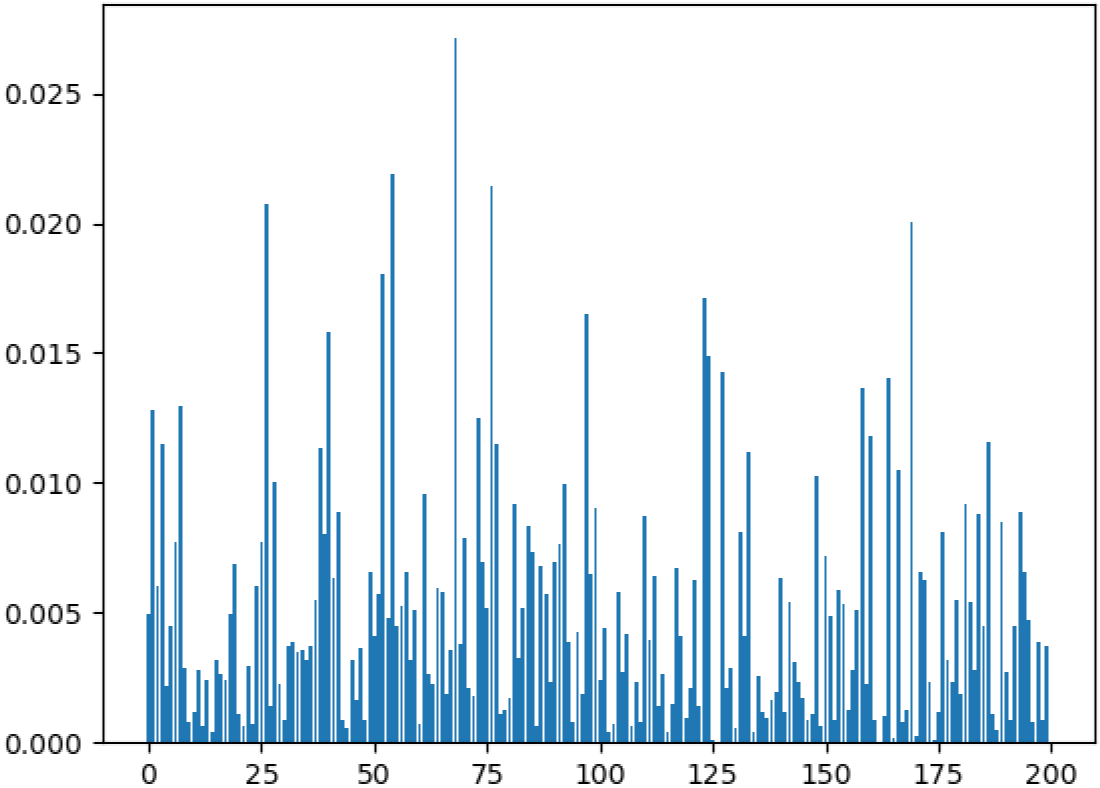
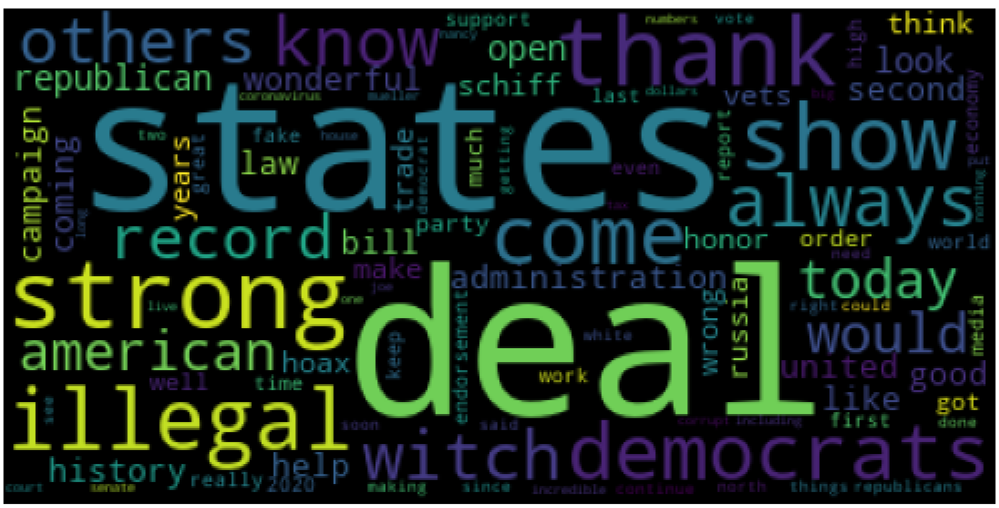
full
u

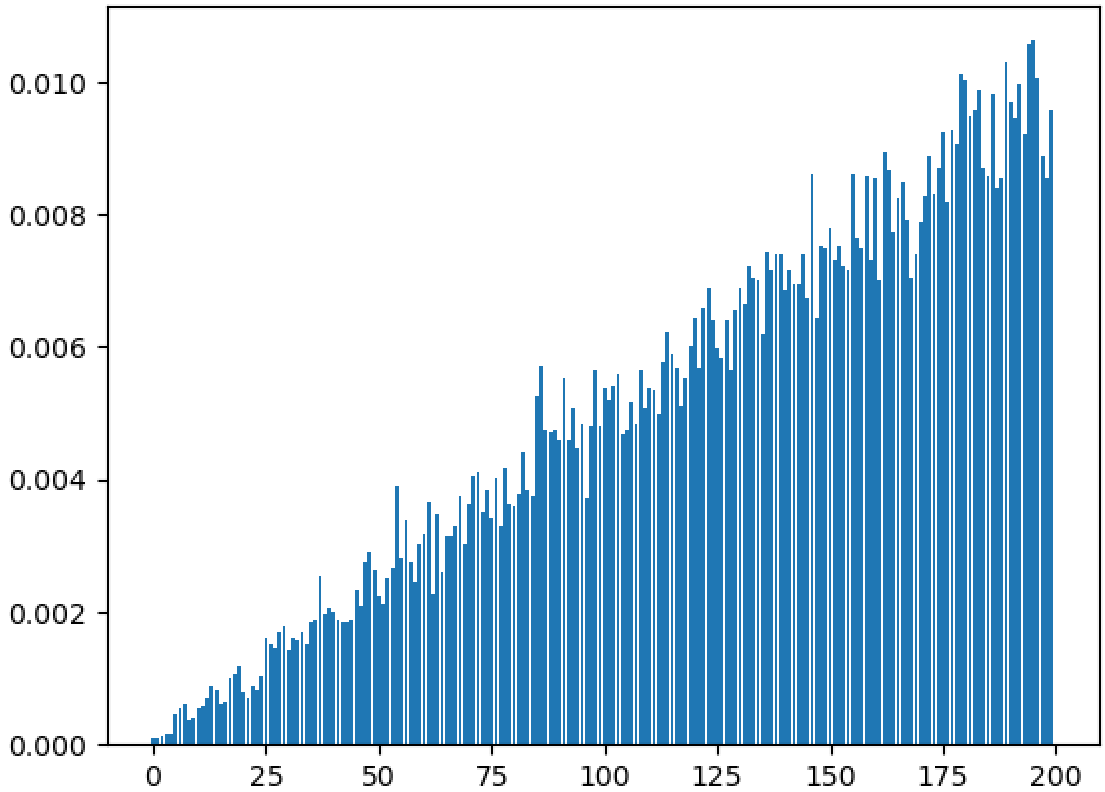
max word: see

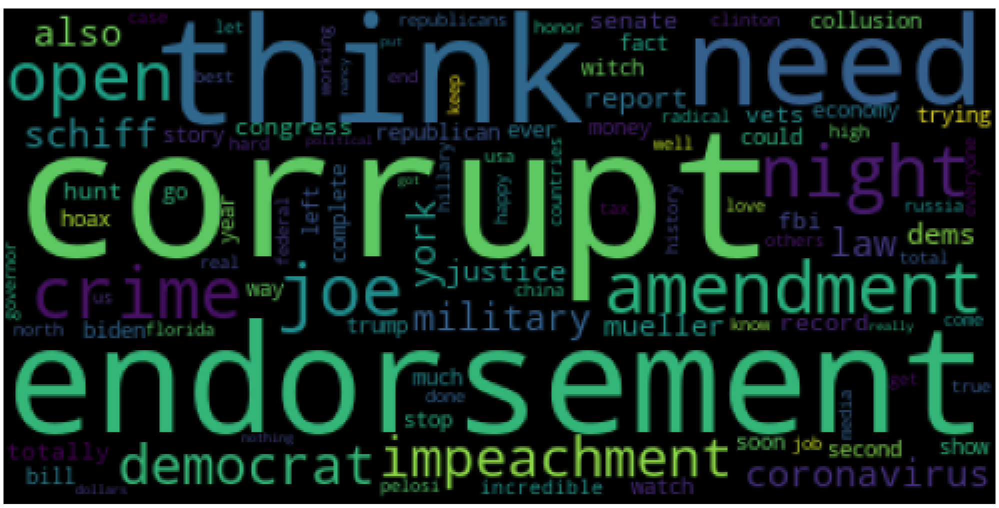




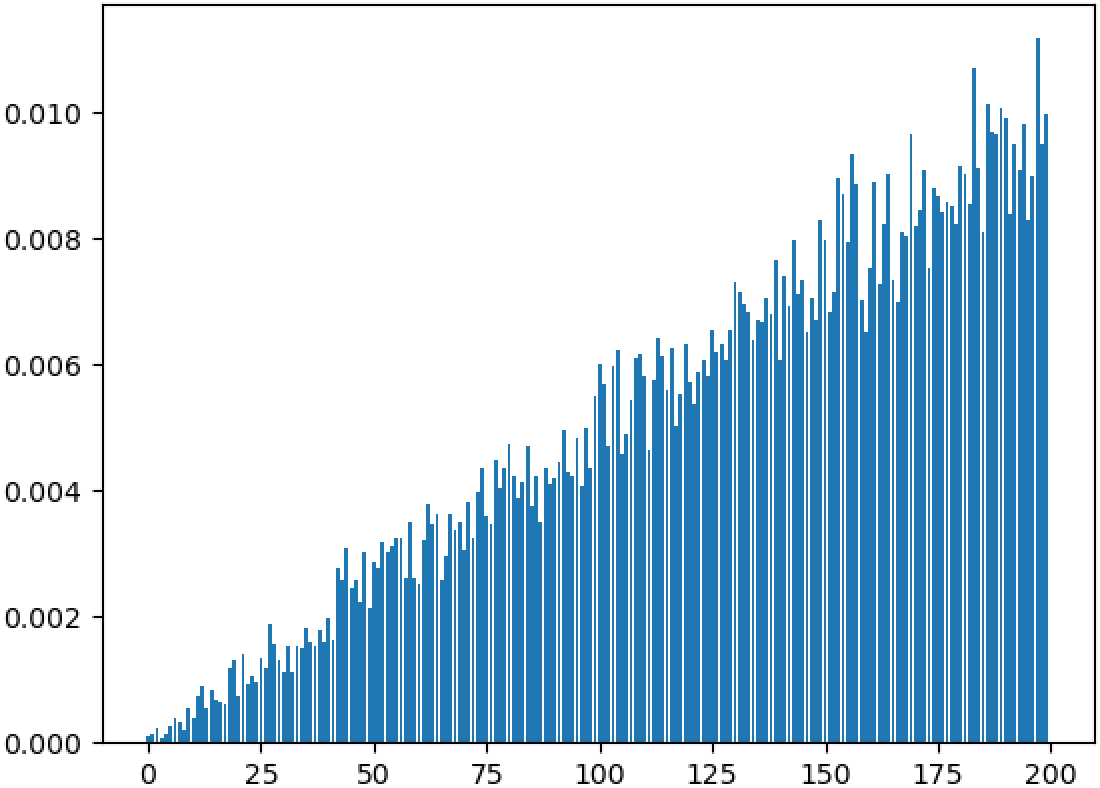
```
max word: deal
```

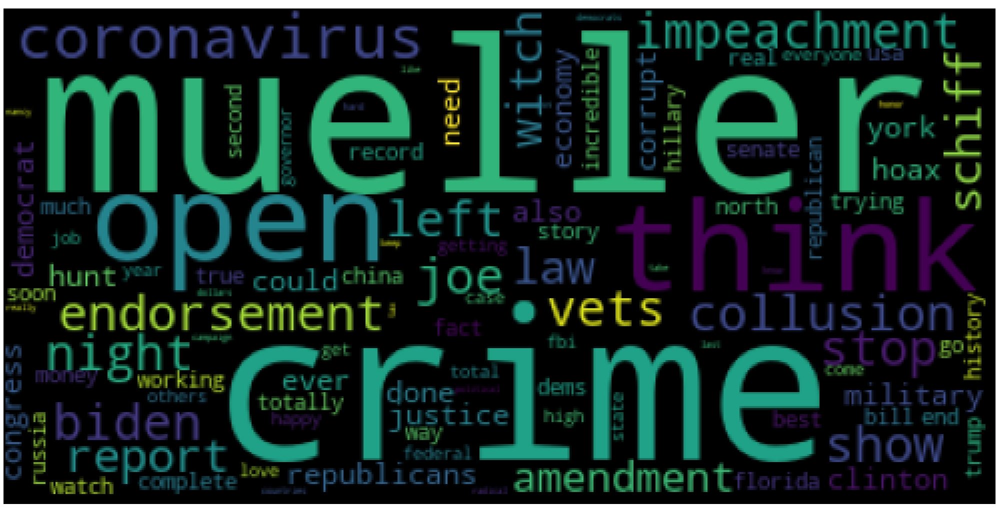
In [14]:
```
for th in theta_alpha2:
    plt.bar(range(J), th)
    plot_words(th)
    imax = np.argmax(th)
    print(f"max word: {dictionary.get(imax)}")
```



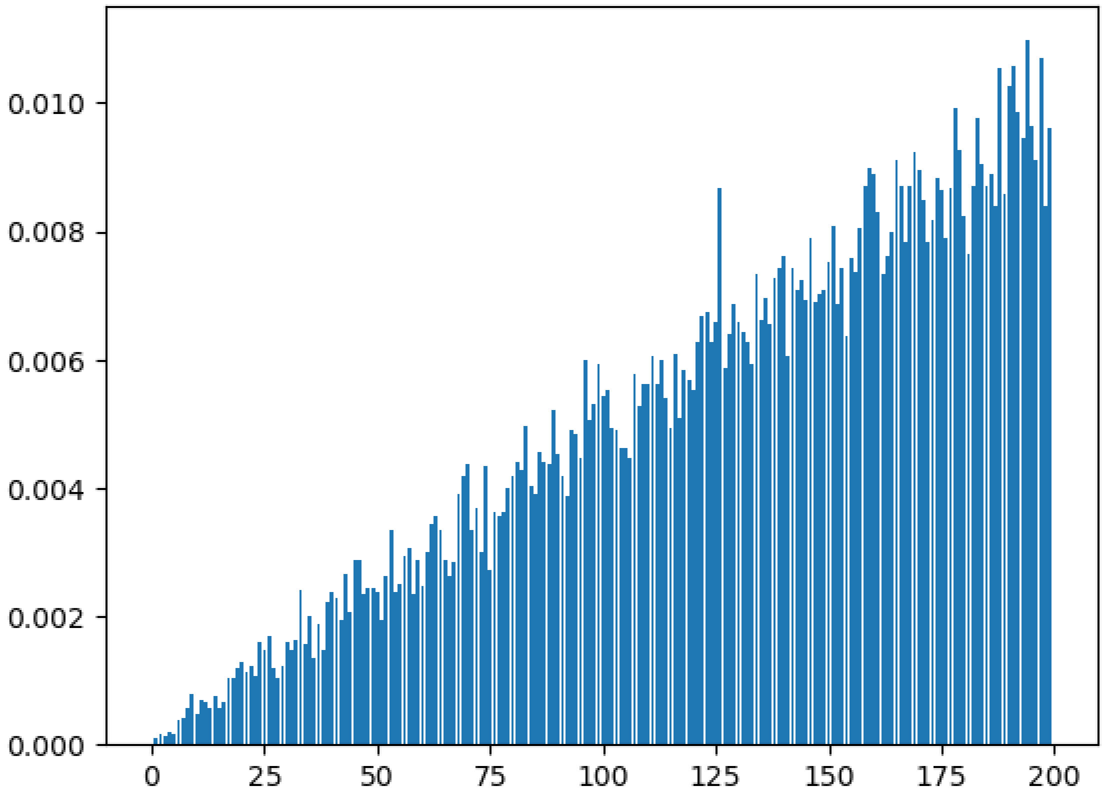


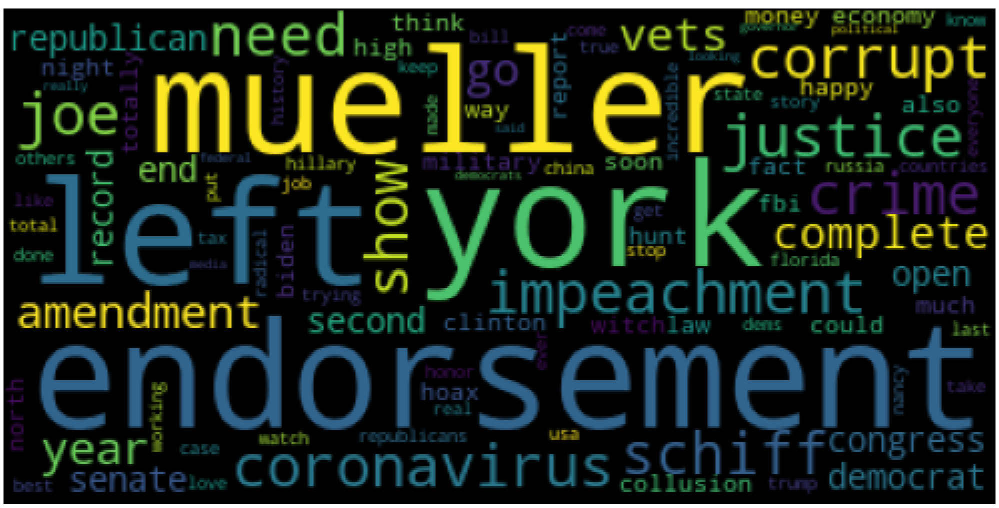
max word: corrupt





max word: mueller






```
max word: endorsement
```

Question: What are the different underling prior assumptions of α_1 and α_2 for the data?

Answer: with $\boldsymbol{\alpha} = \{1, 1, \ldots, 1\}$ the probability density function is a uniform distribution. For $\boldsymbol{\alpha} = \{1, 2, 3, \ldots, J\}$ the distribution we obtain is heavily skewed towards the latter parts of the dictionary.

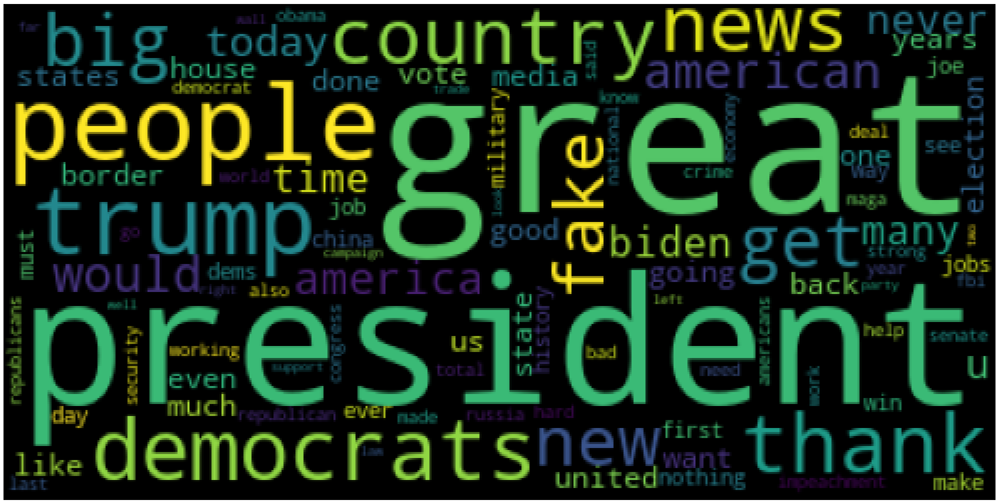
Exercise 3

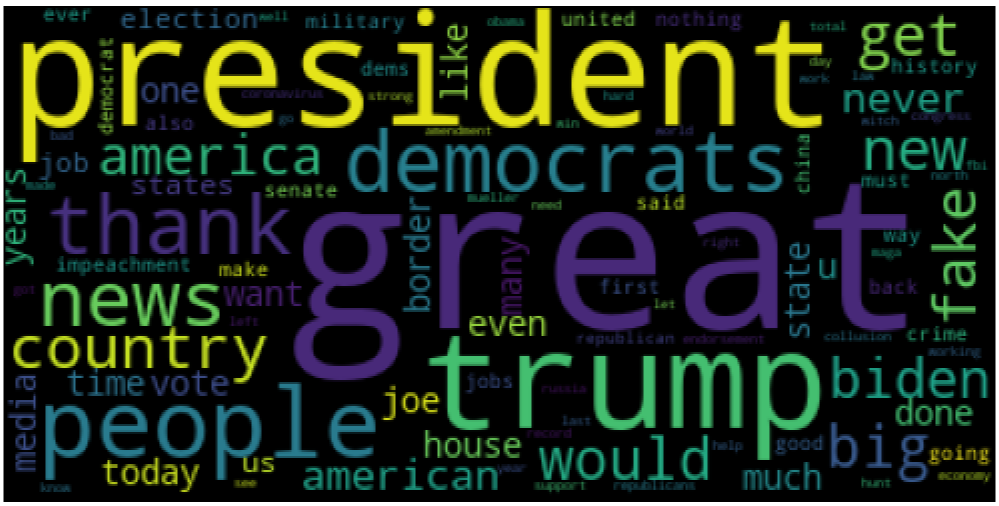
Compute the posterior distributions for both priors. This can be done efficiently using the matrix X you just computed.

Plot some posterior samples as you did for the prior.

$$p(x|\theta) = \frac{p(\theta|x)p(\theta)}{p(x)}$$

In [15]:
```
plot_words(np.random.dirichlet(np.cumsum(X_matrix, axis=0)[-1,:] + alpha_prior1))
```






```
In [16]: plot_words(np.random.dirichlet(np.cumsum(X_matrix, axis=0)[-1,:] + alpha_prior2))
```

Question: Do you see any difference? Explain why or why you don't see one.

Answer: As we update our belief about the frequencies of the words in our vocabulary with each tweet, the influence of our initial prior α diminishes. This becomes clear when we see, that even the two very different priors lead to the same posterior distribution of words.

Exercise 4:

You learned in the lecture that we can also *sequentially update* the Bayesian posterior. So let's compute all sequentially updated posteriors, i.e., compute

$$p(\theta|w_1),\ p(\theta|w_1, w_2),\ \ldots,\ p(\theta|w_1, \ldots, w_n)$$

You also learned about the **entropy** and **reduction in entropy**. So let's investigate how entropy changes.

- Compute the entropies i.e $H(\theta), H(\theta|w_1), H(\theta|w_1, w_2), \ldots, H(\theta|w_1, \ldots, w_n)$.
- Compute the reduction in entropies $H(\theta|w_1) - H(\theta), H(\theta|w_1, w_2) - H(\theta|w_1), \ldots$.

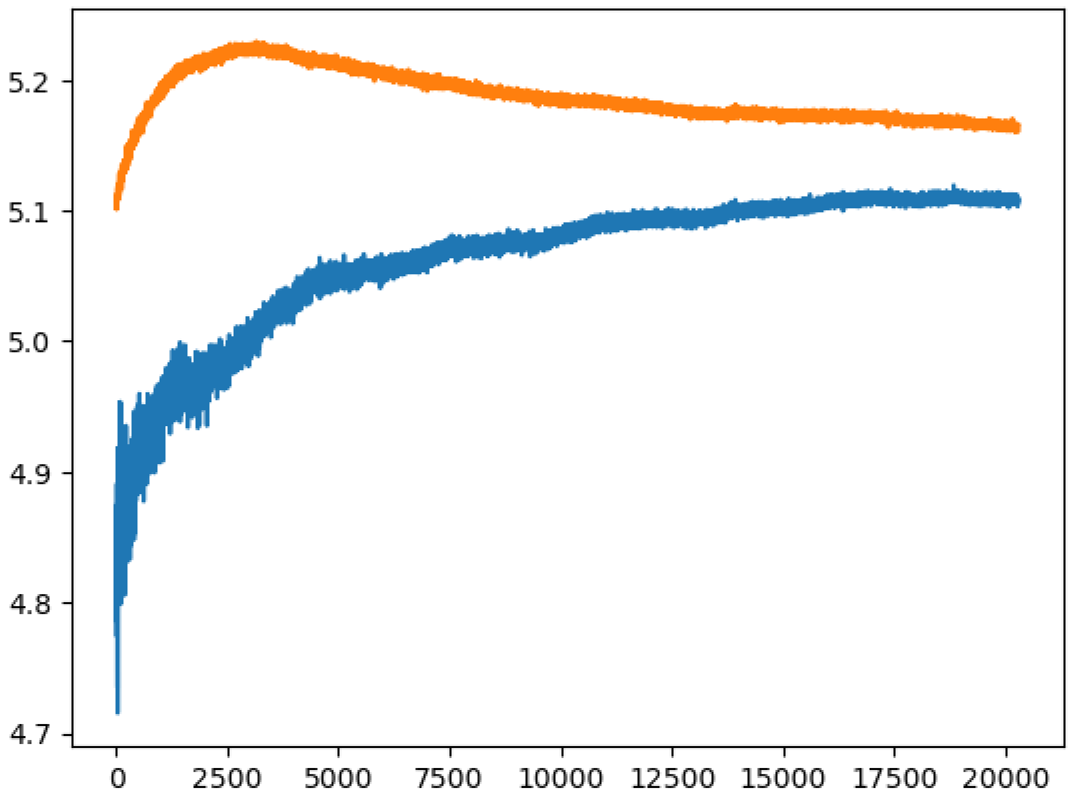
Note You can use *dirichlet.entropy* to compute the entropies.

```
In [17]: # Bayesian updating
alpha_post_sequential1 = list(map(np.random.dirichlet, np.cumsum(X_matrix, axis=0) + alp
alpha_post_sequential2 = list(map(np.random.dirichlet, np.cumsum(X_matrix, axis=0) + alp
```

In [18]:
```
entropies1 = entropy(alpha_post_sequential1, axis=1)
entropies2 = entropy(alpha_post_sequential2, axis=1)
```

In [19]:
```
plt.plot(entropies1)
plt.plot(entropies2)
```

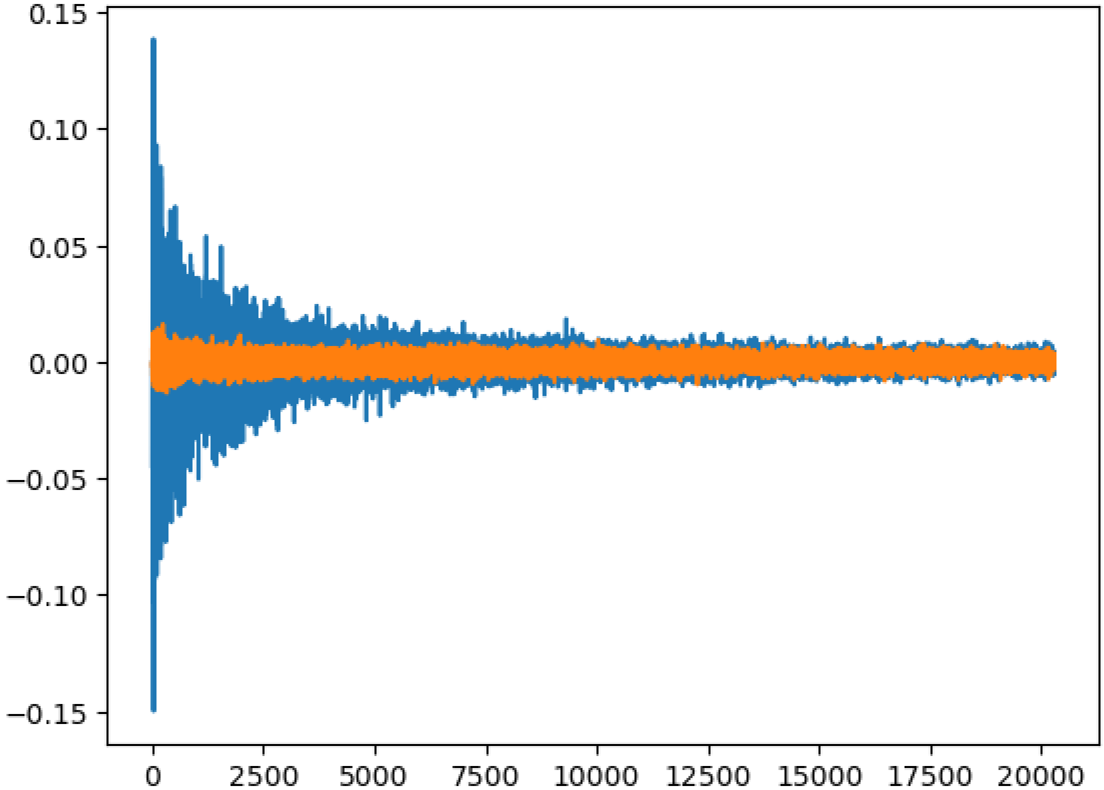
Out[19]:
```
[<matplotlib.lines.Line2D at 0x1bbeb51c220>]
```



In [20]:
```
reduction_in_entropy1 = entropies1[1:] - entropies1[:-1]
reduction_in_entropy2 = entropies2[1:] - entropies2[:-1]
```

In [21]:
```
plt.plot(reduction_in_entropy1)
plt.plot(reduction_in_entropy2)
```

Out[21]:
```
[<matplotlib.lines.Line2D at 0x1bbeb6c1b80>]
```



Question: Some tweets increase the entropy, why could this be?

Answer: When entropy is increased, it means that our distribution of word frequencies is converging towards a uniform distribution. A tweet containing low-frequency words for example might balance the overall frequencies out.

Question: Criticise the model. Point out at least two assumptions that the data likely does not satisfy

Answer: Probably the first flaw that comes to mind is the assumption, that the words found in a tweet are independent of each other, which is not true. The probability of a specific word appearing is highly dependant on the words preceding it. Additionally, the probability of observing a word w_i is dependant on the time the tweet was tweeted. The word "snow" is unlikely to appear in a summer-time tweet.

In [21]:
```

```