PSYCH-GA.2211/NEURL-GA.2201 – Fall 2017
Mathematical Tools for Neural and Cognitive Science

Homework 3

Due: 26 Oct 2017
(late homeworks penalized 10% per day)

See the course web site for submission details. Please: don't wait until the day before the due date... start *now*!

1. **Principal components.** Load the file `PCA.mat` into your MATLAB environment. You'll find a matrix M containing responses of a population of 12 neurons, under 150 different experimental conditions (each column contains the estimated firing rate of one neuron under each of the conditions). We cannot directly visualize data of this many dimensions, but we can use linear algebra tools to project them into lower dimensional space.

   (a) Compute the principal components of the 12-dimensional population responses. First, center the data by subtracting the mean response $mean(M)$ from every row of the matrix (hint: you might find the function `repmat` helpful). Call this re-centered data matrix $\tilde{M}$. Then compute the eigenvectors and eigenvectors of $\tilde{M}^T\tilde{M}$ (alternatively, you can compute the singular values of $\tilde{M}$). Plot the eigenvalues (or singular values). What do you think is the "true" dimensionality of the responses?

   (b) Project the data in $\tilde{M}$ onto the first principal component (i.e., the eigenvector corresponding to the maximal eigenvalue). Plot a histogram (using `hist`) of these values. Show that the sum of squares of these values is equal to the first eigenvalue λ_1. What proportion of the total variability of the data (sum of squared lengths of all data vectors, which is just the sum of squares of all entries of $\tilde{M}$) does this component account for?

   (c) Show a scatter plot of the data projected onto the first two principal components (that is, plot the inner product of the data with the first component versus the inner product with the second component). You can use `plot` (with circular plot symbols and no connecting lines), or use `scatter`. Use `axis('equal')` to set the two axes to use equal scales. Show that the sum of the squared lengths of these projected vectors is equal to $\lambda_1 + \lambda_2$. What proportion of the total variability of the data do these two components account for?

   (d) It appears that much of the response in this 12-neuron population can be explained by the projection into this 2-dimensional space. Now we'd like to interpret this result back in the space of the original responses. Plot the two eigenvectors that you computed in the previous answer, on a single graph. What combinations of neurons do they suggest? In particular, could you categorize the neurons based on the weights indicated by the elements of these two eigenvectors? What if another researcher discovered that these neurons were wired up in a way that each neuron drove one of four muscle groups, each responsible for rotating the eye in one of the four cardinal directions (right, left, up, down)?

2. **Linear shift-invariant (time-invariant) systems**. Written exercises: Oppenheim & Schafer, problems 2.35 and 2.36 [see attached pages]. Please *explain* your answers! Note: Index values

n in this problem represent time, and can be positive or negative. The $\delta[n]$ denotes the standard "impulse" vector (zero everywhere except at index $n = 0$, where it is one).

3. **LSI system characterization.** You are experimenting with three unknown systems, embodied in compiled matlab functions `unknownSystemX.p`, (with X=1,2,3), that each take an input column vector of length $N = 64$. The response of each is a column vector (of the same length). Your task is to examine them to see if they behave like they're linear and/or shift-invariant with circular (periodic) boundary-handling. For each system,

   (a) "Kick the tires" by measuring the response to an impulse in the first position of an input vector. Check that this impulse response is shift-invariant by comparing to the response to an impulse in a few later positions. Also check that the response to a sum of two of these impulses is equal to the sum of their individual responses.

   (b) If the previous tests were positive, examine the response of the system to sinusoids with frequencies $\{2\pi/N, 4\pi/N, 6\pi/N, 12\pi/N\}$, and random phases, and check whether the outputs are sinusoids of the same frequency (i.e., verify that the output vector lies completely in the subspace containing all the sinusoids of that frequency).

   (c) If the previous tests were positive, verify that the change in amplitude and phase from input to output is predicted by the amplitude (`abs`) and phase (`angle`) of the corresponding terms of the Fourier transform of the impulse response. If not, explain which property (linearity, or shift-invariance, or both) seems to be missing from the system.

4. **Gabor filter.**
   (a) Create a one-dimensional linear filter that is a product of a Gaussian and a sinusoid, $\exp\left(-\frac{n^2}{2\sigma^2}\right)\cos(\omega n)$, with parameters $\sigma = 3.5$ and $\omega = 2\pi * 10/64$ samples. The filter should contain 25 samples, and should be centered on the middle (13th) sample. Plot the filter to verify that it looks like what you'd expect. Plot the amplitude of the Fourier transform of this filter, sampled at 64 locations (MATLAB's `fft` function takes an optional additional argument). What kind of filter is this? If you were to convolve this filter with sinusoids of different frequencies, which of them would produce a response with the largest amplitude? Obtain this answer by reasoning about the equation defining the filter (above), and also by finding the maximum of the computed Fourier amplitudes (using the `max` function), and verify that the answers the same. Compute the *period* of this sinusoid, measured in units of sample spacing (hint: this is the inverse of its frequency, in cycles/sample), and verify by eye that this is roughly matched to the oscillations in the graph of the filter itself. Which sinusoids would produce responses with about 25% of this maximal amplitude?

   (c) Create three unit-amplitude 64-sample sinusoidal signals at the three frequencies (low, mid, high) that you found in part (b). Convolve the filter with each, and verify that the amplitude of the response is approximately consistent with the answers you gave in part (b). (hint: to estimate amplitude, you'll either need to project the response onto sine and cosine of the appropriate frequency, or compute the DFT of the response and measure ampitude at the appropriate frequency).

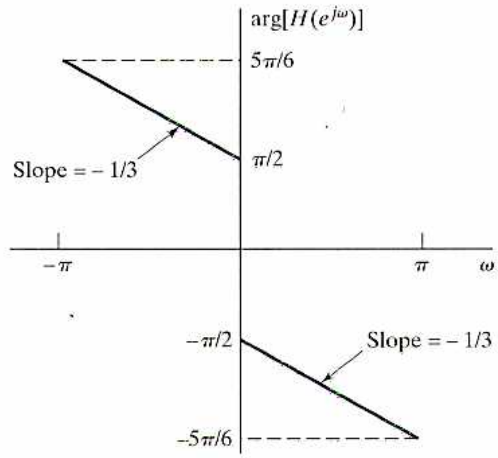

Figure P2.33-1

2.34. The input–output pair shown in Figure P2.34-1 is given for a stable LTI system.

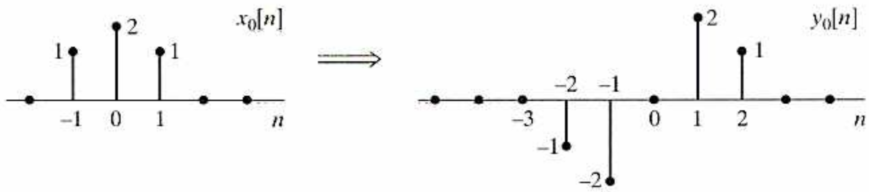

Figure P2.34-1

(a) Determine the response to the input $x_1[n]$ in Figure P2.34-2.

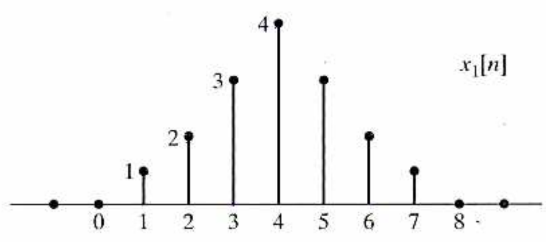

Figure P2.34-2

(b) Determine the impulse response of the system.

Advanced Problems

2.35. The system T in Figure P2.35-1 is known to be *time invariant*. When the inputs to the system are $x_1[n]$, $x_2[n]$, and $x_3[n]$, the responses of the system are $y_1[n]$, $y_2[n]$, and $y_3[n]$, as shown.

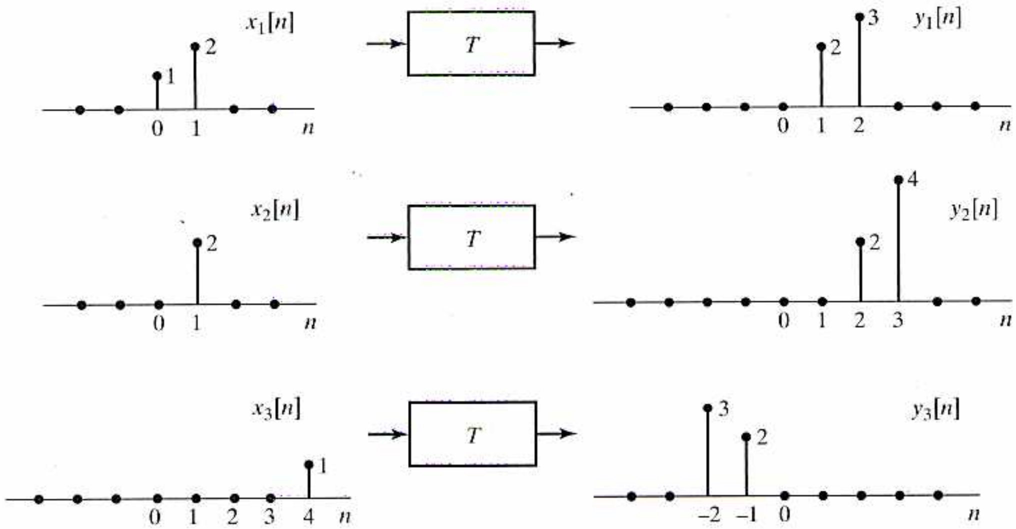

Figure P2.35-1

(a) Determine whether the system T could be linear.
(b) If the input $x[n]$ to the system T is $\delta[n]$, what is the system response $y[n]$?
(c) What are all possible inputs $x[n]$ for which the response of the system T can be determined from the given information alone?

2.36. The system L in Figure P2.36-1 is known to be *linear*. Shown are three output signals $y_1[n]$, $y_2[n]$, and $y_3[n]$ in response to the input signals $x_1[n]$, $x_2[n]$, and $x_3[n]$, respectively.

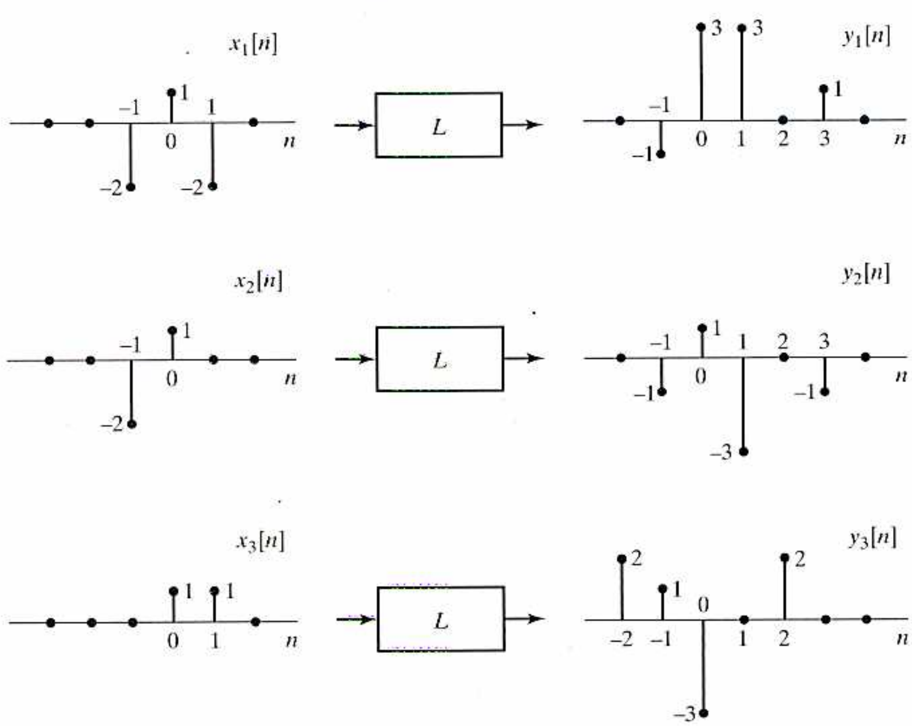

Figure P2.36-1

(a) Determine whether the system L could be time invariant.

(b) If the input $x[n]$ to the system L is $\delta[n]$, what is the system response $y[n]$?

2.37. Consider a discrete-time linear time-invariant system with impulse response $h[n]$. If the input $x[n]$ is a periodic sequence with period N (i.e., if $x[n] = x[n+N]$), show that the output $y[n]$ is also a periodic sequence with period N.

2.38. In Section 2.5, we stated that the solution to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0 \tag{P2.38-1}$$

is of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n, \tag{P2.38-2}$$

with the A_m's arbitrary and the z_m's the N roots of the polynomial

$$\sum_{k=0}^{N} a_k z^{-k} = 0; \tag{P2.38-3}$$

i.e.,

$$\sum_{k=0}^{N} a_k z^{-k} = \prod_{m=1}^{N} (1 - z_m z^{-1}). \tag{P2.38-4}$$

(a) Determine the general form of the homogeneous solution to the difference equation

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = 2x[n-1]. \tag{P2.38-5}$$

(b) Determine the coefficients A_m in the homogeneous solution if $y[-1] = 1$ and $y[0] = 0$.

(c) Now consider the difference equation

$$y[n] - y[n-1] + \tfrac{1}{4}y[n-2] = 2y[n-1]. \tag{P2.38-6}$$

If the homogeneous solution contains only terms of the form of Eq. (P2.38-2), show that the initial conditions $y[-1] = 1$ and $y[0] = 0$ cannot be satisfied.

(d) If Eq. (P2.38-3) has two roots that are identical, then, in place of Eq. (P2.38-2), $y_h[n]$ will take the form

$$y_h[n] = \sum_{m=1}^{N-1} A_m z_m^n + nB_1 z_1^n, \tag{P2.38-7}$$

where we have assumed that the double root is z_1. Using Eq. (P2.38-7), determine the general form of $y_h[n]$ for Eq. (P2.38-6). Verify explicitly that your answer satisfies Eq. (P2.38-6) with $x[n] = 0$.

(e) Determine the coefficients A_1 and B_1 in the homogeneous solution obtained in Part (d) if $y[-1] = 1$ and $y[0] = 0$.

2.39. Consider a system with input $x[n]$ and output $y[n]$. The input–output relation for the system is defined by the following two properties:

1. $y[n] - ay[n-1] = x[n]$,
2. $y[0] = 1$.

(a) Determine whether the system is time invariant.

(b) Determine whether the system is linear.