Alg-Manip

Problem Set

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§1 Reading

There is not too much theory regarding "alg-manip" problems. The main thing to focus on while doing these problems is the **motivation** behind certain manipulations, as some of them may seem to be pulled out of thin air. Many of these problems involve global summations, telescoping, and manipulations which lead to massive cancellation; these are common techniques that can be used to simplify the problem.

Here's an example of this kind of problem from this year's olympiad:

Example 1.1 (USAMO 2021/5)

Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7 \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

§2 Problems

Here are a bunch of exercises, not really ordered by difficulty.

Problem 2.1 (IMO 2018/2). Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.

Problem 2.2 (ISL 2013/A4)**.** Let n be a positive integer, and consider a sequence $a_1, a_2, \cdots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \cdots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

Problem 2.3 (ISL 2001/A3)**.** Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

Problem 2.4 (ISL 2015/A1)**.** Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

Problem 2.5 (ELMOSL 2013/A2)**.** For positive real numbers a, b, c, prove that

$$\frac{1}{a+\frac{1}{b}+1} + \frac{1}{b+\frac{1}{c}+1} + \frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

Problem 2.6 (Mexico 2011/3)**.** Let n be a positive integer. Find all real solutions $(a_1, a_2, \ldots, a_n)$ to the system:

$$\begin{gathered} a_1^2 + a_1 - 1 = a_2 \\ a_2^2 + a_2 - 1 = a_3 \\ \vdots \\ a_n^2 + a_n - 1 = a_1. \end{gathered}$$

Problem 2.7 (TST 2009/7)**.** Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

Problem 2.8 (USMCA 2019/4)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(f(x) + y)^2 = (x - y)(f(x) - f(y)) + 4f(x)f(y).$$