Continued Fractions

An Underappreciated Subject

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September 11, 2020

Overview

Definitions and Notation

Definition (Continued Fraction)

A continued fraction is denoted by $[a_1; a_2, a_3, \ldots]$ (where $a_1 \in \mathbb{Z}$, $a_i \in \mathbb{N}$ for $i > 1$, and $a_n > 1$ if the continued fraction is of finite length n), which represents

$$a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}.$$

A continued fraction may or may not terminate.

Examples

$$\frac{151}{115} = [1; 3, 5, 7], \varphi = [1; 1, 1, \ldots]$$

Remark

Every real number has a unique continued fraction (why?).

Definitions and Notation (cont.)

Definition (The k-th convergent)

For an infinite continued fraction or a continued fraction of the form $[a_1; a_2, a_3, \ldots, a_n]$ for $n \geq k$, the k-th convergent of the continued fraction is the continued fraction $[a_1; a_2, a_3, \ldots, a_k]$.

Examples

The convergents of $\frac{151}{115}$ are

$$1, \frac{4}{3}, \frac{21}{16}, \frac{151}{115}.$$

The first few convergents of φ are

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}.$$

Some basic knowledge

Proposition (Rational iff terminating)

A continued fraction is rational iff it is terminating.

Proof.

Let $\alpha \in \mathbb{R}$. If $\alpha \notin \mathbb{Q}$, then the continued fraction cannot terminate (if it did, just expand it out to a rational number).
Conversely, suppose FSoC that $\alpha \in \mathbb{Q}$ has an infinite continued fraction. By WOP take an α with the least denominator. Suppose $\alpha = [a_1; a_2, a_3, \ldots]$. Then

$$\frac{1}{\alpha - a_1} = [a_2; a_3, a_4, \ldots],$$

but $\frac{1}{\alpha - a_1}$ has a lesser denominator than α, contradiction. □

Some basic knowledge (cont.)

Proposition (Alternating direction of convergents)

If $\frac{P_k}{Q_k}$ is the k-th convergent to a real number α, then these convergents of α alternate between overapproximations and underapproximations of α (under for k odd, over for k even).

Proof.

Induction. $\square$

Some basic knowledge (cont.)

Proposition (Eventually repeating continued fractions)

If a continued fraction $[a_1; a_2, a_3, \dots]$ is eventually periodic then it is a root of some quadratic equation.

Proof.

Set the continued fraction equal to x. Solving for x in the resulting equation gives $f(x) = \frac{g(x)}{h(x)}$ for some linear functions f, g, h in x, which rearranges to a quadratic. □

Examples

$$\sqrt{2} = [1; \overline{2}], \sqrt{3} = [1; \overline{1, 2}], \sqrt{5} = ?$$

Calculating Convergent by Hand

The “normal” way (some motivation)

Suppose you wanted to calculate the simplified form of $[1; 2, 3, 4, \ldots, 10]$. Normally, you start simplifying from the “bottom” up, i.e. start at 10 and work your way up.
What if you already knew the simplified form of $[1; 2, 3, 4, \ldots, 9]$? With the current method of simplifying from the bottom up, *the n-th convergent doesn't help at all for calculating the $n + 1$-th convergent*.

Wish (Relating Successive Convergents)

Some way to calculate successive convergents based on knowledge of previous convergents.

The Granted Wish

Basis of Algorithm

For a continued fraction $\alpha = [a_1; a_2, a_3, \ldots, a_n]$, let P_k, Q_k defined for $1 \leq k \leq n$ be 2 sequences of naturals such that $\frac{P_k}{Q_k}$ is equal to the k-th convergent of α. Then $P_1 = a_1, Q_1 = 1, P_2 = a_1 a_2 + 1, Q_2 = a_2$, and for $2 \leq k \leq n$, we have (!)

$$P_k = a_k P_{k-1} + P_{k-2},$$
$$Q_k = a_k Q_{k-1} + Q_{k-2}.$$

Remark

Up until now, we have assumed $a_i \in \mathbb{N}$. However, everything we have defined can be easily extended to arbitrary real numbers for a_i (at the cost of losing uniqueness of continued fractions, but this won't matter in the following proof); we will use this extension in the proof.

Why does it work?

Proof.

We use strong induction on n, the length of the continued fraction. As mentioned earlier, extend our definitions to allow a_i be real numbers. The base cases $n = 1, 2, 3$ are trivially true. Now assume the hypothesis holds for all $k < n$. We'll prove the hypothesis for n.

Consider the new continued fraction $\alpha' = \left[a_1; a_2, a_3, \ldots, a_{n-1} + \frac{1}{a_n}\right]$. Note that by definition, $\alpha' = \alpha$ (extension to reals has broken the uniqueness of a continued fraction). With P_k, Q_k as defined earlier, define

$$P = \left(a_{n-1} + \frac{1}{a_n}\right) P_{n-2} + P_{n-3},$$
$$Q = \left(a_{n-1} + \frac{1}{a_n}\right) Q_{n-2} + Q_{n-3}.$$

Why does it work? (cont.)

Proof.

Note by our inductive hypothesis (since α' is of length $n-1$) that $\frac{P}{Q} = \alpha'$. But we also have

$$\frac{P_n}{Q_n} = \frac{a_n P_{n-1} + P_{n-2}}{a_n Q_{n-1} + Q_{n-2}} = \frac{(a_n a_{n-1} + 1) P_{n-2} + a_n P_{n-3}}{(a_n a_{n-1} + 1) Q_{n-2} + a_n Q_{n-3}}$$
$$= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) P_{n-2} + P_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) Q_{n-2} + Q_{n-3}} = \frac{P}{Q} = \alpha' = \alpha,$$

completing the proof (since all other convergents follow immediately by the inductive hypothesis on $[a_1; a_2, a_3, \ldots, a_{n-1}]$). $\square$

The "Magic Box"

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		1	3	5	7
0	1	1	4	21	151
1	0	1	3	16	115

Corollary (Determinant ± 1)

Determinant of successive 2×2 matrices is ± 1, since it starts at -1 and each successive matrix is the result of adding a multiple of the right column to the left column, then switching the columns. In other words,

$$|P_k Q_{k-1} - P_{k-1} Q_k| = 1.$$

Exercise (Reciprocals of convergents are convergents of reciprocal)

Let $\alpha > 1$ be a real number. Then the reciprocal of the k-th convergent of α is the $(k+1)$-th convergent of $\frac{1}{\alpha}$.

How "good" are convergent approximations?

Proposition (Within $\frac{1}{\text{denom}^2}$)

For a real number α, we have

$$\left|\alpha - \frac{P_k}{Q_k}\right| \leq \frac{1}{Q_k Q_{k+1}} \leq \frac{1}{Q_k^2}.$$

Proof.

Note by the alternating nature of successive convergents, we have

$$\left|\alpha - \frac{P_k}{Q_k}\right| \leq \left|\frac{P_k}{Q_k} - \frac{P_{k+1}}{Q_{k+1}}\right| = \frac{1}{Q_k Q_{k+1}} \leq \frac{1}{Q_k^2}.$$

□

How "good" are convergent approximations? (cont.)

Remark

Note that we can only guarantee an error of at most $\frac{1}{2q}$ for an arbitrary denominator q, which makes convergents special!

Exercise (Convergents are best for their denominator)

Let $\frac{P_k}{Q_k}$ be a convergent to the real number α. Suppose further that $P, Q \in \mathbb{N}$ satisfies

$$\left|\alpha - \frac{P}{Q}\right| \leq \left|\alpha - \frac{P_k}{Q_k}\right|.$$

Prove that $Q \geq Q_k$.

More exercises

Exercise (Rational vs. Irrational)

Prove that $\alpha \in \mathbb{R}$ is irrational iff there exist infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}$.

Exercise (halving the error)

Prove that for any irrational number α there exist infinitely many convergents $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2q^2}$.

Exercise ("good" enough approximations are necessarily convergents)

Prove that for any real number α and fraction $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2q^2}$ we necessarily have that $\frac{p}{q}$ is a convergent of the continued fraction of α.

Acknowledgements

A special thank you to PROMYS for their extensive instruction on the topic of continued fractions. This presentation was inspired by material covered during the camp.