# $\nu_p$ aka Unique Prime Factorization

## Problem Set

Raymond Feng

March 19, 2021

## §1 Reading

Refer to the reading at https://rfeng2004.github.io/files/NRU-Prime.pdf. One of the main take aways is the following claim (which I'll reproduce here):

> **Claim 1.1** ($\nu_p$ defines integers (and rationals!)) — If $a, b \in \mathbb{Q}^+$ with $\nu_p(a) = \nu_p(b)$ for all primes $p$, then $a = b$.

If this is not immediately obvious to you, take a moment to consider why the claim holds.

## §2 Problems

These problems are ordered in *roughly* increasing order of difficulty, but that is only based on my personal judgement.

**Problem 2.1** (https://rfeng2004.github.io/files/NRU-Prime.pdf)**.** The problems here are good exercises. Recommended: CMC 12B 2021/23, RMM TST 2010/1/5, USAMO 1985/1, China 2015/4, Kosovo 2020/12.4.

**Problem 2.2** (ISL 2011/N1)**.** For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly $d$ positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

**Problem 2.3** (ISL 2015/N1)**.** Determine all positive integers $M$ such that the sequence $a_0, a_1, a_2, \ldots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \ldots$$

contains at least one integer term.

**Problem 2.4** (IMO 2019/4)**.** Find all pairs $(k, n)$ of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

**Problem 2.5** (Iran TST 2013/1/5)**.** Do there exist natural numbers $a, b$ and $c$ such that $a^2 + b^2 + c^2$ is divisible by $2013(ab + bc + ca)$?

**Problem 2.6** (IMO 2010/3)**.** Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$\left(g(m) + n\right)\left(g(n) + m\right)$$

is a perfect square for all $m, n \in \mathbb{N}$.