Alg-manip

MA 5.0

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Mexico 2011/3

Let n be a positive integer. Find all real solutions $(a_1, a_2, \ldots, a_n)$ to the system:

$$a_1^2 + a_1 - 1 = a_2$$
$$a_2^2 + a_2 - 1 = a_3$$
$$\vdots$$
$$a_n^2 + a_n - 1 = a_1.$$

- Regroup at 8:50pm.
- Answer is all -1s or all 1s.
- One thing we can do is add everything together: then we get

$$\sum_{i=1}^{n} a_i^2 = n.$$

- $$\prod_{i=1}^{n} a_i(a_i+1) = \prod_{i=1}^{n}(a_i+1)$$
- Either all -1s or $\prod a_i = 1 \implies \prod a_i^2 = 1$. Since we have $\frac{1}{n} \cdot \sum a_i^2 = \sqrt[n]{\prod a_i^2} = 1$ then we are done.

ISL 2013/A4

Let n be a positive integer, and consider a sequence $a_1, a_2, \cdots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \cdots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

- Regroup at 9:05pm.
- Ideas?
    - Condition of $a_{a_i} \leq n + i - 1$ means a lot of the terms are "small."
    - Equality for $n = 2$ can hold at $\{1, 3\}$.
    - Equality for $n = 5$ can hold at $\{3, 4, 5, 6, 7\}$.
    - Equality for $n = 5$ can also hold at $\{4, 5, 5, 5, 6\}$.
- Let k be the greatest integer such that $k, a_k \leq n$.
    - Since the sequence is nondecreasing and we're given $a_{a_1} \leq n$, then $a_1 \leq k$.
- $a_{x+1} + \cdots + a_{x+y} \leq y \cdot a_{x+y}$, then when we plug in $\{3, 4, 5, 6, 7\}$ we should have $y = 1$.
- We have

$$a_{a_2+1} + a_{a_2+2} + \cdots + a_{a_3} \leq (a_3 - a_2)a_{a_3} \leq (a_3 - a_2)(n+2).$$

- Define ℓ to be the least integer such that $a_\ell \geq k$. (If we can't find such an ℓ, then we are done since sum of everything is at most kn which is at most n^2.)
- Note that $a_k < k$ then $a_1 + \cdots + a_k < k^2$ and $a_{k+1} + \cdots + a_n \leq (n-k)(n+k)$, so their sum is less than n^2.
- Otherwise, we have $a_k \geq k \implies \ell \leq k$.
- Bounds:

$$\begin{aligned} a_{k+1} + a_{k+2} + \cdots + a_{a_\ell} &\leq (a_\ell - k)a_{a_\ell} \leq (a_\ell - k)(n + \ell - 1) \\ a_{a_\ell+1} + a_{a_\ell+2} + \cdots + a_{a_{\ell+1}} &\leq (a_{\ell+1} - a_\ell)a_{a_{\ell+1}} \leq (a_{\ell+1} - a_\ell)(n+\ell) \\ a_{a_{\ell+1}+1} + a_{a_{\ell+1}+2} + \cdots + a_{a_{\ell+2}} &\leq (a_{\ell+2} - a_{\ell+1})a_{a_{\ell+2}} \leq (a_{\ell+2} - a_{\ell+1})(n+\ell+1) \\ &\vdots \\ a_{a_k+1} + a_{a_k+2} + \cdots + a_n &\leq (n - a_k)a_n \leq (n-a_k)(n+a_1) \leq (n-a_k)(n+k) \end{aligned}$$

- Adding this all together gives

$$(\ell - 1)k + a_\ell + \cdots + a_n \leq n^2.$$

USMCA 2019/4

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(f(x) + y)^2 = (x - y)(f(x) - f(y)) + 4f(x)f(y).$$

- Regroup at 10:05pm.
- Solution set is $f \equiv 0$ and $f(x) = x + c$ for any constant c.
- First, note that all xs are wrapped in an f other than a single instance of x. This means we should be able to get injectivity. If f is a constant function then $f \equiv 0$. Ignoring this case, we get injectivity.
- $y = x - f(x) \implies f(x)f(x - f(x)) = 0$.
- Above gives 0 in range. Suppose $f(c) = 0$. Then you get $x - f(x) = c$ for all $x \neq c$.