Olympiad Intuition

Raymond Feng

February 19, 2021

Contents

§1 Introduction/Definition

§1.1 A Metaphor

I once read that solving an olympiad problem is like navigating the wilderness. Imagine you are in a forest, trying to locate a treasure chest. If you don't have any tools or aids, you can only blindly search/follow a random walk. This is so inefficient that by the time you come near the treasure chest (i.e. come close to a working solution), you may not even be alert enough to detect it.

Now, the solution to this problem is NOT to just learn more and more theory; think of that as enhancing your gear: if you had more durable boots or were more physically fit, of course you could trek faster and longer and this would indirectly improve your chances to reach the goal. But this is still very inefficient.

On the other hand, your job to find the treasure chest is so much easier if say, you had a map to aid your random walk: you are *much less likely to go in circles*. If you had a map and a metal detector, you are even more likely to succeed, as entire areas without metal *would not even have to be explored*. With this in mind, the goal of this handout is to help you learn to develop these tools and be aware that they exist.

§1.2 Defining Intuition

When solving a problem, there must be a balance between random flailing/exploring and using tools to guide your attacks and reduce the "search space" (if you want to think about problem solving as searching for a solution in a very, very large search space, we are "pruning" the search space). Intuition is then defined by me as the intangible tools that help guide us on our search.

§1.3 A Warning

I want to emphasize that one of the most dangerous things in olympiad math is to get sucked down a solution path that is **doomed to fail from the start**. If you don't realize that a solution path will fail, the problem is very unlikely to be solved. There are many reasons why something is just doomed to fail or nigh impossible to pull off; some reasons are subtle, but others are more obvious, and it is important to notice the signs of a doomed solution. What follows is a list of intuition strategies that can help guide/coax you toward a working solution path or away from a solution path that is doomed.

§2 "Intuition" strategies

- Often, thinking about what the official solution *should* look like can give you an idea of how to approach the problem.

- If you aren't using all of the problem conditions, you need to explain to yourself why the problem should still be true (for example, using degrees of freedom); if you can't, your solution path is most likely doomed. This tip applies mostly to olympiad problems; sometimes, computational problems add in conditions just to make it less guessable/engineerable.

- Make sure that all your claims are symmetric wrt things that are inherently symmetric in the problem. If they aren't, they must be false by "the definition of symmetry."

- In the context of a specific problem, if you can find any "handwavy" explanation for why the problem is true, this is often a good step towards hashing out structures that rigorize the argument. (This usually works since "handwavy" arguments are based on intuition.)
- Rephrase the problem in ways that make it easiest for you to think about it! This is different for everyone, but if you get stuck in a bunch of nasty notation or in what feels like an unnatural problem statement, just make up your own equivalent version; do what makes you most comfortable.
- When reflecting on problems that you solved and thought were nice, try to explain *why* you thought it was nice. I realize very hidden or deep connections to other problems based on why they were nice.

If this seems way too abstract right now, I will showcase some examples that I think are very illustrative.

§3 The Power of Intuition

Here's an example of a problem where intuition shifted my solution path to something that worked.

Example 3.1 (TSTST 2020/8)

For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \geq n \geq 2$ satisfying

$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}.$$

A natural starting point is to see if $\gcd(m, n) = 1$ gives any solutions, since that would allow us to use $\sigma(mn) = \sigma(m) \cdot \sigma(n)$. Spoiler: there aren't any solutions to this case. But why not?

A common explanation would be inequalities. Why inequalities? If we substitute $\sigma(mn) = \sigma(m) \cdot \sigma(n)$ into the equation, we end up getting something like

$$(m-1)\sigma(m)(\sigma(n)-n) - (\sigma(m)-m)(n-1) = 0,$$

and noting that the conditions also imply $(m-1)(\sigma(n)-n) = (n-1)(\sigma(m)-m)$, this becomes

$$(\sigma(m)-1)(m-1)(\sigma(n)-n) = 0.$$

But the expression is clearly greater than 0. OK, so the official solution probably involves inequalities/bounding.

After some testing of values of k into the fraction $\frac{\sigma(k)-1}{k-1}$, it became apparent to me that the solution set was m, n both powers of the same prime.

Combining, the next thing that came up in my thought process was expanding

$$\frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}$$

and trying to decompose by primes to get an inequality. I sunk almost 40 minutes in this thought process, and eventually took a pause to think: what am I missing?

This is when I realized: this appraoch is *doomed to fail*. Why? If I try to expand ONLY the second equals sign to get an inequality, the inequality will not always hold because that would be ingnoring half of the conditions. Additionally, I was only able to conclude in the $\gcd(m, n) = 1$ case because I ended up plugging in the other condition near the end. With this in mind, I know that the solution most likely follows this rough battle plan:

1. Expand the second equation out.
2. Do some algebra stuff to rearrange the equation similar to the $\gcd(m, n) = 1$ case to somehow be able to plug in the first equation.
3. Be alert for simple looking things that could represent NT inequalities, and check for whether equality holds at every step for the conjectured equality cases.

Following these steps, I solved the problem with no other novel ideas. Try it yourself! The hardest part of this solve was *coming up with the battle plan*. Psychologically, the battle plan helps a lot when having to "grind" things out; for example, expansion here. Another example would be the tangent line trick: it's a lot of computation, but when I know it will work, I enthusiastically carry out the computations. The execution is much easier when you know that the solution will most likely work out beforehand.

Here is another example where intuition helped motivate my solution path.

Example 3.2 (USEMO 2020/4)
A function f from the set of positive real numbers to itself satisfies

$$f(x + f(y) + xy) = xf(y) + f(x + y)$$

for all positive real numbers x and y. Prove that $f(x) = x$ for all positive real numbers x.

The first thing I tried after some preliminary plugging in of constants was applying the "cancellation trick." What happens if we try to reverse engineer $x + f(y) + xy = x + y$? This should never happen, because $xf(y)$ is strictly positive. Since we aren't allowed to have $f(y) + xy = y$, this means that $x = \frac{y - f(y)}{y}$ is disallowed, i.e. $y \le f(y)$ for all y. From here, it was clear to me that this would be more of an analytical/bounding flavored FE than an algebraic one, since plugging in constants didn't do much and the only progress I had so far was a bound.

OK, so let's suppose FSoC that some a exists with $f(a) \neq a \implies f(a) > a$. What goes wrong? Intuitively, a small error should build up and blow up for much larger inputs, which should result in a contradiction. I tried plugging in $P(x, a)$; this gave

$$f(x + f(a) + xa) = xf(a) + f(x + a) \ge xf(a) + x + a.$$

Thus, for large enough $t = x + f(a) + xa$, we have

$$f(t) \ge \frac{f(a) + 1}{a + 1} t + a - \frac{f(a)(f(a) + 1)}{a + 1}.$$

We can reformulate this as $f(t) \geq \lambda \cdot t + c$ for all large enough t, for some constants c, λ with $\lambda > 1$ and c in terms of a (but c is negative). The key realization is that for sufficiently large t, we can decrease the slope by a tiny bit while keeping it strictly greater than 1, and increase c to a big positive number! (Just imagine graphing the lines). Thus, I was actually able to say that for large enough t, we have

$$f(t) \geq \frac{1 + \frac{f(a)+1}{a+1}}{2} t + \text{BIG}.$$

Define

$$\frac{1 + \frac{f(a)+1}{a+1}}{2} = 1 + \varepsilon$$

so the above becomes

$$f(t) \geq (1 + \varepsilon)t + \text{BIG}.$$

Now, how can this be used to obtain a contradiction? After applying $P(x, t)$ for sufficiently large t and tiny x, we are close to finishing. (Hint: no real number is greater than all real numbers; this is a useful contradiction technique for analytical/bounding FE's).

Indeed, this results in

$$xf(t) + f(x + t) = f(x + f(t) + xt) \geq (1 + \varepsilon)(x + f(t) + xt) + \text{BIG}.$$

Then for all large enough t and $x < \varepsilon$:

$$\begin{aligned} f(x + t) &\geq (1 + \varepsilon - x)f(t) + (1 + \varepsilon)(x + xt) + \text{BIG} \\ &\geq f(t) + \text{BIG}. \end{aligned}$$

Now conclude by asking, what could the value of $f(t + 1)$ be?

This solution is a bit different from most of the other solutions, but at least for me, it was the most intuitive way to reformulate the problem; this shows that you should try to restate the problem in a way that makes it easiest for you to think about.

§4 Problems

These problems were selected because of how much I was guided by intuition rather than random flailing while solving them.

Problem 4.1 (IMO 2016/1)**.** Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

Problem 4.2 (Iran TST 2020/2/5)**.** For every positive integer $k > 1$ prove that there exist a real number x so that for every positive integer $n < 1398$:

$$\{x^n\} < \{x^{n-1}\} \iff k \mid n.$$

Problem 4.3 (Iran TST 2011/12)**.** Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function for which the expression $af(a) + bf(b) + 2ab$ for all $a, b \in \mathbb{N}$ is always a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.

Problem 4.4 (ISL 2018/C3)**.** Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n + 1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Problem 4.5 (Iran TST 2018/1/4)**.** Let ABC be a triangle ($\angle A \neq 90^\circ$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Problem 4.6 (TST 2015/5)**.** A tournament is a directed graph for which every (unordered) pair of vertices has a single directed edge from one vertex to the other. Let us define a proper directed-edge-coloring to be an assignment of a color to every (directed) edge, so that for every pair of directed edges $\overrightarrow{uv}$ and $\overrightarrow{vw}$, those two edges are in different colors. Note that it is permissible for $\overrightarrow{uv}$ and $\overrightarrow{uw}$ to be the same color. The directed-edge-chromatic-number of a tournament is defined to be the minimum total number of colors that can be used in order to create a proper directed-edge-coloring. For each n, determine the minimum directed-edge-chromatic-number over all tournaments on n vertices.

Problem 4.7 (IMO 2013/5)**.** Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x + y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 4.8 (USAMO 2020/3)**.** Let p be an odd prime. An integer x is called a *quadratic non-residue* if p does not divide $x - t^2$ for any integer t.

Denote by A the set of all integers a such that $1 \le a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

Problem 4.9 (ISL 2019/N6)**.** Let $H = \{\lfloor i\sqrt{2} \rfloor : i \in \mathbb{Z}_{>0}\} = \{1, 2, 4, 5, 7, \dots\}$ and let n be a positive integer. Prove that there exists a constant C such that, if $A \subseteq \{1, 2, \dots, n\}$ satisfies $|A| \ge C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)