Continued Fractions Solutions

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Contents

§1 Solutions to Theory Problems

§1.1 Exercise 5.1 (Convergents are best for their denominator)

Let $\frac{P_k}{Q_k}$ be a convergent to the real number α. Suppose further that $P, Q \in \mathbb{N}$ satisfies

$$\left|\alpha - \frac{P}{Q}\right| \leq \left|\alpha - \frac{P_k}{Q_k}\right|.$$

Prove that $Q \geq Q_k$.

If $\frac{P_{k+1}}{Q_{k+1}} = \frac{P}{Q}$, then $Q \geq Q_{k+1} \geq Q_k$, so we are done. Otherwise assume $\frac{P_{k+1}}{Q_{k+1}} \neq \frac{P}{Q}$, so that

$$\left|\frac{P_{k+1}}{Q_{k+1}} - \frac{P}{Q}\right| \geq \frac{1}{QQ_{k+1}}.$$

From the given, the triangle inequality, the fact that convergents alternate between underestimations and overestimations, and $|P_kQ_{k+1} - Q_kP_{k+1}| = 1$, we have

$$\begin{aligned} \frac{1}{QQ_{k+1}} \leq \left|\frac{P_{k+1}}{Q_{k+1}} - \frac{P}{Q}\right| &\leq \left|\alpha - \frac{P_{k+1}}{Q_{k+1}}\right| + \left|\alpha - \frac{P}{Q}\right| \\ &\leq \left|\alpha - \frac{P_{k+1}}{Q_{k+1}}\right| + \left|\alpha - \frac{P_k}{Q_k}\right| = \left|\frac{P_{k+1}}{Q_{k+1}} - \frac{P_k}{Q_k}\right| = \frac{1}{Q_kQ_{k+1}} \implies Q \geq Q_k, \end{aligned}$$

as desired.

§1.2 Exercise 5.2 (Rational vs. Irrational)

Prove that $\alpha \in \mathbb{R}$ is irrational iff there exist infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}$.

If α is irrational, then it has infinitely many convergents $\frac{P_k}{Q_k}$, all of which satisfy $\left|\alpha - \frac{P_k}{Q_k}\right| \leq \frac{1}{Q_k^2}$.

If $\alpha = \frac{P}{Q}$ is rational, then the following claim will finish the problem.

Claim — No fractions $\frac{p}{q}$ with $\gcd(p, q) = 1$ and $q > Q$ satisfy $\left|\frac{P}{Q} - \frac{p}{q}\right| \leq \frac{1}{q^2}$.

Proof. Note that

$$\left|\frac{P}{Q} - \frac{p}{q}\right| = \frac{|Pq - Qp|}{Qq} \geq \frac{1}{Qq} > \frac{1}{q^2}.$$

□

§1.3 Exercise 5.3 (Halving the error)

Prove that for any irrational number α there exist infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2q^2}$.

Claim — For any two consecutive convergents $\frac{P_n}{Q_n}$ and $\frac{P_{n+1}}{Q_{n+1}}$ of α, one of them satisfies the condition.

Proof. FSoC suppose not. Then we have

$$\frac{1}{2Q_n^2}+\frac{1}{2Q_{n+1}^2} < \left|\alpha - \frac{P_n}{Q_n}\right| + \left|\alpha - \frac{P_{n+1}}{Q_{n+1}}\right| = \left|\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}\right| = \frac{1}{Q_k Q_{k+1}} \overset{\text{AM-GM}}{\leq} \frac{1}{2Q_n^2}+\frac{1}{2Q_{n+1}^2},$$

contradiction. □

§1.4 Exercise 5.4 ("Good" enough approximations are necessarily convergents (!!!))

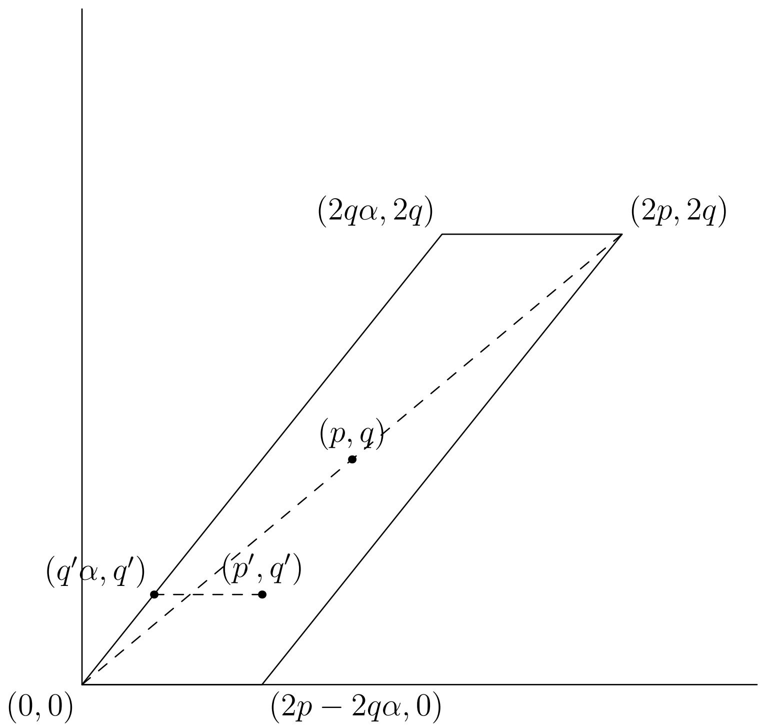
Prove that for any real number α and fraction $\frac{p}{q}$ satisfying

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2q^2}$$

we necessarily have that $\frac{p}{q}$ is a convergent of the continued fraction of α.

Claim — There are no lattice points other than (p, q) in the parallelogram with vertices $(0,0), (2p-2q\alpha, 0), (2p, 2q), (2q\alpha, 2q)$.

Proof. First assume $\alpha < \frac{p}{q}$, so that $\frac{1}{\alpha} > \frac{q}{p}$ and we can assume the configuration below. The case $\alpha > \frac{p}{q}$ follows similarly.



Suppose FSoC that there existed another lattice point (p', q') in the parallelogram. By the symmetry of the top triangle and bottom triangle with respect to lattice points, we may assume that (p', q') lies in the lower triangle; that is, $\alpha < \frac{p}{q} < \frac{p'}{q'}$. Furthermore, note that

$$\left|q'\alpha - p'\right| < |2q\alpha - 2p|$$

since the horizontal dotted length is shorter than the base of the parallelogram. Therefore, we have

$$\left|q'\alpha - p'\right| < |2q\alpha - 2p| < \frac{1}{q}.$$

On the other hand, note that $\alpha < \frac{p}{q} < \frac{p'}{q'}$ implies

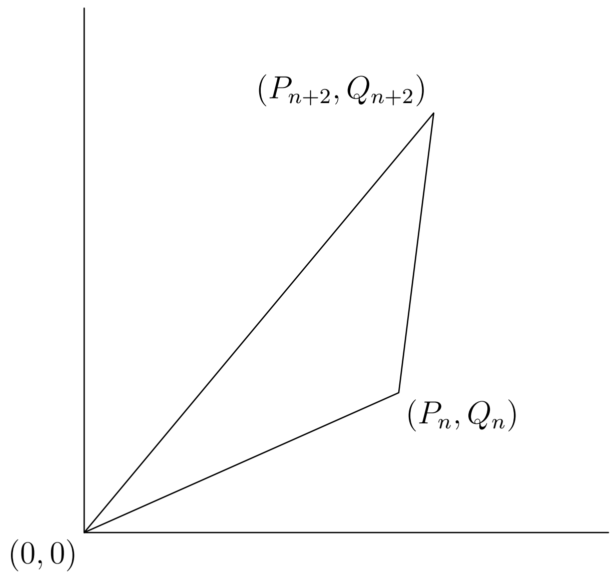
$$\left|q'\alpha - p'\right| > q'\left|\frac{p}{q} - \frac{p'}{q'}\right| = q' \cdot \frac{|pq' - p'q|}{qq'} \geq \frac{1}{q},$$

which gives the desired contradiction. Thus, the claim is proven. □

Claim — If $\frac{P_n}{Q_n}, \frac{P_{n+1}}{Q_{n+1}}, \frac{P_{n+2}}{Q_{n+2}}$ are consecutive convergents to α then there is no lattice point in the triangle with vertices $(0,0), (P_n, Q_n), (P_{n+2}, Q_{n+2})$.

Proof. First, it's well known that if $\alpha = [a_1; a_2, a_3, \dots]$ then $P_{n+2} = a_{n+2}P_{n+1} + P_n$ and $Q_{n+2} = a_{n+2}Q_{n+1} + Q_n$ and

$$|P_{n+2}Q_n - P_nQ_{n+2}| = a_{n+2}.$$



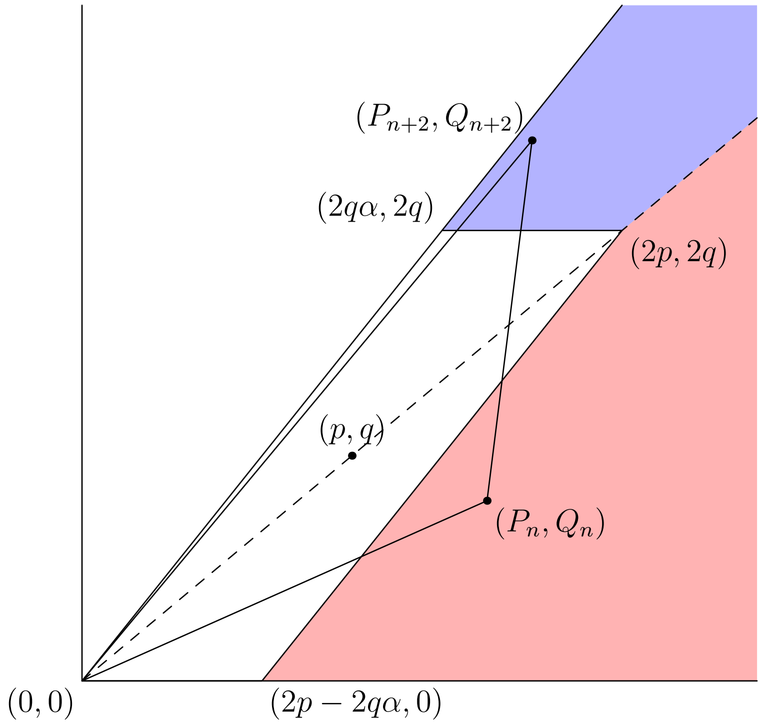
Therefore, by shoelace, the area of the triangle with vertices $(0,0), (P_n, Q_n), (P_{n+2}, Q_{n+2})$ is $\frac{a_{n+2}}{2}$.

On the other hand, from $P_{n+2} = a_{n+2}P_{n+1} + P_n$ and $Q_{n+2} = a_{n+2}Q_{n+1} + Q_n$ we have

$$\gcd(P_{n+2} - P_n, Q_{n+2} - Q_n) = \gcd(a_{n+2}P_{n+1}, a_{n+2}Q_{n+1}) = a_{n+2},$$

so by Pick's area formula with I representing the number of lattice points in the interior of the triangle, the area of the triangle is also $I + \frac{3+(a_{n+2}-1)}{2} - 1$. Equating the two expressions for the area of the triangle gives $I = 0$, so there are no interior points, as desired. □

With the above two claims, we are ready to prove the problem. Again impose $\alpha < \frac{p}{q}$ so we can assume the following configuration (the other case yields a similar proof).



Now suppose FSoC that $\frac{p}{q}$ is not a convergent of α. Then we may let n be the unique index such that

$$\frac{P_n}{Q_n} > \frac{p}{q} > \frac{P_{n+2}}{Q_{n+2}} > \alpha.$$

This is allowed by recalling that all convergents of the same parity are successively better approximations from the same direction (under for odd, over for even). But the previous inequality combined with the first claim implies that (P_n, Q_n) lies in the red region of the diagram and (P_{n+2}, Q_{n+2}) lies in the blue region of the diagram. However, this contradicts the second claim, because (p, q) is a lattice point lying inside the triangle with vertices $(0, 0), (P_n, Q_n), (P_{n+2}, Q_{n+2})$.

Thus, we finally conclude that $\frac{p}{q}$ must be a convergent of α, as desired.

§1.5 Exercise 5.5 (How far can we go?)

Find, with proof, an integer k not a perfect square such that no fraction $\frac{p}{q}$ exists with

$$\left|\sqrt{k} - \frac{p}{q}\right| \leq \frac{1}{3q^2}.$$

Claim — $k = 2$ satisfies the problem condition.

Proof. Suppose FSoC that $\frac{p}{q}$ satisfied $\left|\sqrt{2}-\frac{p}{q}\right| \le \frac{1}{3q^2}$. It's clear that $q \neq 1$, so assume $q > 1$. We then have $\left|\sqrt{2}-\frac{p}{q}\right| \le \frac{1}{12}$, so $\sqrt{2}+\frac{p}{q} \le 2\sqrt{2}+\frac{1}{12} < 3$. Then we have

$$0 < \left|2 - \frac{p^2}{q^2}\right| < 3 \cdot \frac{1}{3q^2} = \frac{1}{q^2} \implies 0 < \left|2q^2 - p^2\right| < 1,$$

contradiction. □

§1.6 Exercise 5.6 (The best constant (!!!))

Prove that for any irrational number α there exist infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \le \frac{1}{\sqrt{5}q^2}$. Furthermore, prove that if $\sqrt{5}$ is replaced by any greater constant, the assertion fails.

To show that $\sqrt{5}$ is optimal, take the number $\varphi = \frac{\sqrt{5}+1}{2} = [1; 1, 1, 1, \dots]$.

Let $\alpha = [a_1; a_2, a_3, \dots]$. Now, suppose FSoC that finitely many such $\frac{p}{q}$ existed. Then there exist three consecutive convergents $\frac{P_n}{Q_n}, \frac{P_{n+1}}{Q_{n+1}}, \frac{P_{n+2}}{Q_{n+2}}$ which all satisfy $\left|\alpha - \frac{p}{q}\right| > \frac{1}{\sqrt{5}q^2}$.

Claim — $Q_{n+1} < \varphi \cdot Q_n$. Similarly $Q_{n+2} < \varphi \cdot Q_{n+1}$.

Proof. It's well known that convergents alternate between overestimates and underestimates, and that $|P_k Q_{k+1} - Q_k P_{k+1}| = 1$ for all k. Therefore, we have

$$\frac{1}{\sqrt{5}Q_{n+1}^2} + \frac{1}{\sqrt{5}Q_n^2} < \left|\alpha - \frac{P_{n+1}}{Q_{n+1}}\right| + \left|\alpha - \frac{P_n}{Q_n}\right| = \left|\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}\right| = \frac{1}{Q_{n+1}Q_n}.$$

This rearranges to

$$0 > (Q_{n+1} - \varphi Q_n)\left(Q_{n+1} - \frac{Q_n}{\varphi}\right) \implies \frac{Q_n}{\varphi} < Q_{n+1} < \varphi Q_n.$$

□

Now, note that

$$\varphi \cdot Q_{n+1} > Q_{n+2} = a_{n+2}Q_{n+1} + Q_n > Q_{n+1} + \frac{Q_{n+1}}{\varphi} = \varphi \cdot Q_{n+1},$$

contradiction.

§2 Solutions to Contest Problems

§2.1 BMT Team 2015/13

There exist right triangles with integer side lengths such that the legs differ by 1. For example, 3-4-5 and 20-21-29 are two such right triangles. What is the perimeter of the next smallest Pythagorean right triangle with legs differing by 1?

We wish to find the next smallest solution to $n^2 + (n+1)^2 = m^2$, or

$$(2n+1)^2 - 2m^2 = -1.$$

Let $x = 2n+1$ and $y = m$. Note that solutions to $|x^2 - 2y^2| = 1$ satisfy

$$\left|2 - \frac{x^2}{y^2}\right| = \frac{1}{y^2} \implies \left|\sqrt{2} - \frac{x}{y}\right| = \frac{1}{\left(\sqrt{2} + \frac{x}{y}\right) y^2} \le \frac{1}{2y^2}.$$

Therefore, $\frac{x}{y}$ is a convergent to the continued fraction of $\sqrt{2}$. We can then compute (using "Magic Box") that the next convergent satisfying $x^2 - 2y^2 = -1$ is $(x, y) = (239, 169)$ which means the answer is $239 + 169 = \boxed{408}$.

§2.2 TST 2016/4

Let $\sqrt{3} = 1.b_1b_2b_3\ldots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits b_n, b_{n+1}, $\ldots$, b_{2n} equals 1.

Suppose for sake of contradiction that for some positive integer n, all the digits $b_n, b_{n+1}, \ldots, b_{2n}$ are zero.

Consider the fraction

$$\frac{p}{q} = 1.b_1b_2\ldots b_{n-1} = 1.b_1b_2\ldots b_{2n}.$$

Obviously, $q = 2^k$ for some $k \le n-1$. This implies that

$$\sqrt{3} \in \left(\frac{p}{q}, \frac{p}{q} + \frac{1}{2^{2n}}\right).$$

Then we get the 2 bounds

$$0 < q\sqrt{3} - p < \frac{q}{4^n} \le \frac{1}{4q}$$

and

$$0 < q\sqrt{3} + p < 2\sqrt{3}q.$$

Multiplying these 2 inequalities together, we get

$$0 < 3q^2 - p^2 < \frac{\sqrt{3}}{2} < 1,$$

contradiction.

§2.3 ISL 2003/C5

Every point with integer coordinates in the plane is the center of a disk with radius $\frac{1}{1000}$.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

For the first part, we may take $(0,0),(2n,0),(n,n\sqrt{3})$ for some convergent $\frac{m}{n}$ to the continued fraction of $\sqrt{3}$ with $n > 1000$. This would make

$$\left|n\sqrt{3}-m\right| < \frac{1}{n},$$

which suffices.

For the second part, note that the centers of the 3 disks cannot form an equilateral triangle, so shift one of the centers to the origin so that the distance from it to the other 2 centers is not equal. Let the other 2 centers be (a,b) and (c,d). Note that $a^2+b^2-c^2-d^2$ must be a nonzero integer since it is clearly an integer and cannot be 0 since we chose to shift the center which has different distances to the other 2 centers. WLOG we can also assume that it is positive. Thus, we have $a^2+b^2-c^2-d^2 \geq 1$. However, if the sidelength of the equilateral triangle is s, we have that $a^2+b^2 \leq s+0.002$ and $c^2+d^2 \geq s-0.002$ so

$$1 \leq a^2+b^2-c^2-d^2 \leq (s+0.002)^2-(s-0.002)^2 = 4\cdot 0.002\cdot s \implies s > 124 > 96,$$

as desired.

§2.4 ISL 2019/N6

Let $H = \{\lfloor i\sqrt{2}\rfloor : i \in \mathbb{Z}_{>0}\} = \{1,2,4,5,7,\dots\}$ and let n be a positive integer. Prove that there exists a constant C such that, if $A \subseteq \{1,2,\dots,n\}$ satisfies $|A| \geq C\sqrt{n}$, then there exist $a,b \in A$ such that $a-b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z\rfloor$ denotes the greatest integer less than or equal to z.)

Recall the following lemma:

Lemma (Beatty)
If $\frac{1}{a}+\frac{1}{b}=1$ and $a,b \notin \mathbb{Q}$ then $\{\lfloor na\rfloor : n \in \mathbb{N}\}$ and $\{\lfloor nb\rfloor : n \in \mathbb{N}\}$ partition $\mathbb{N}$.

With $a=\sqrt{2}$, we know that $\mathbb{N}\setminus H = \left\{\left\lfloor(2+\sqrt{2})n\right\rfloor : n \in \mathbb{N}\right\}$. Now, assume that for all $a,b \in A$ that $a-b \in \left\{\left\lfloor(2+\sqrt{2})n\right\rfloor : n \in \mathbb{N}\right\}$. We will show that $|A| \leq C\sqrt{n}$ for some constant C.

Let $A = \{a_1 < a_2 < \cdots < a_k\}$ and define $a_p - a_q = \left\lfloor i_{pq}(2+\sqrt{2})\right\rfloor$ where $i_{pq} \in \mathbb{Z}$. Since $a_3-a_1 = (a_2-a_1)+(a_3-a_2)$ then

$$\begin{aligned}(2+\sqrt{2})(i_{21}+i_{32}-i_{31}) &= \left\{i_{21}\sqrt{2}\right\}+\left\{i_{32}\sqrt{2}\right\}-\left\{i_{31}\sqrt{2}\right\} \in (-1,2)\\ &\implies i_{21}+i_{32}-i_{31}=0\\ \implies \left\{i_{21}\sqrt{2}\right\}+\left\{i_{32}\sqrt{2}\right\} &= \left\{i_{31}\sqrt{2}\right\}.\end{aligned}$$

By inductively applying this argument, we get that

$$\left\{i_{21}\sqrt{2}\right\} + \left\{i_{32}\sqrt{2}\right\} + \cdots + \left\{i_{k,k-1}\sqrt{2}\right\} = \left\{i_{k1}\sqrt{2}\right\} < 1.$$

Claim — For positive integers x,

$$\left\{x\sqrt{2}\right\} \geq \frac{1}{2\sqrt{2}x}.$$

Proof. Let $y = \lfloor x\sqrt{2} \rfloor < x\sqrt{2}$. Suppose FSoC that

$$0 < x\sqrt{2} - y < \frac{1}{2\sqrt{2}x}.$$

This implies

$$0 < 2x^2 - y^2 < \frac{x\sqrt{2}+y}{2\sqrt{2}x} < 1,$$

contradiction. □

By the above claim and the earlier inequality, we have

$$2\sqrt{2} > \frac{1}{i_{21}} + \frac{1}{i_{32}} + \cdots + \frac{1}{i_{k,k-1}}.$$

However, noting that $i_{pq} < \frac{a_p - a_q + 1}{2+\sqrt{2}}$, we now have by Jensen that

$$\begin{aligned} \frac{2\sqrt{2}}{2+\sqrt{2}} &> \frac{1}{a_2 - a_1 + 1} + \cdots + \frac{1}{a_k - a_{k-1} + 1} \\ &\geq \frac{k-1}{\frac{k+n-2}{k-1}} = \frac{(k-1)^2}{k+n-2}. \end{aligned}$$

This rearranges to $k \leq C\sqrt{n}$ for some constant C, so we are done.