# Continued Fractions

## An Underappreciated Subject

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September 11, 2020, last updated January 16, 2021

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# §1 Definitions and Notation

**Definition 1.1** (Continued Fraction)**.** A **continued fraction** is denoted by $[a_1; a_2, a_3, \dots]$ (where $a_1 \in \mathbb{Z}$, $a_i \in \mathbb{N}$ for $i > 1$, and $a_n > 1$ if the continued fraction is of finite length $n$), which represents

$$a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots}}}.$$

A continued fraction may or may not terminate.

**Example 1.2**

$$\frac{151}{115} = [1; 3, 5, 7], \varphi = [1; 1, 1, \dots]$$

**Remark 1.3.** Every real number has a unique continued fraction (why?).

**Definition 1.4** (The $k$-th convergent)**.** For an infinite continued fraction or a continued fraction of the form $[a_1; a_2, a_3, \dots, a_n]$ for $n \geq k$, the **$k$-th convergent** of the continued fraction is the continued fraction $[a_1; a_2, a_3, \dots, a_k]$.

**Example 1.5**

The convergents of $\frac{151}{115}$ are

$$1, \frac{4}{3}, \frac{21}{16}, \frac{151}{115}.$$

The first few convergents of $\varphi$ are

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}.$$

# §2 Basics

**Proposition 2.1** (Rational iff terminating)

A continued fraction is rational iff it is terminating.

*Proof.* Let $\alpha \in \mathbb{R}$. If $\alpha \notin \mathbb{Q}$, then the continued fraction cannot terminate (if it did, just expand it out to a rational number).

Conversely, suppose FSoC that $\alpha \in \mathbb{Q}$ has an infinite continued fraction. By WOP take an $\alpha$ with the least denominator. Suppose $\alpha = [a_1; a_2, a_3, \dots]$. Then

$$\frac{1}{\alpha - a_1} = [a_2; a_3, a_4, \dots],$$

but $\frac{1}{\alpha - a_1}$ has a lesser denominator than $\alpha$, contradiction. $\square$

**Proposition 2.2** (Alternating direction of convergents)

If $\frac{P_k}{Q_k}$ is the $k$-th convergent to a real number $\alpha$, then these convergents of $\alpha$ alternate between overapproximations and underapproximations of $\alpha$ (under for $k$ odd, over for $k$ even).

*Proof.* Induction. □

**Proposition 2.3** (Eventually repeating continued fractions)

If a continued fraction $[a_1; a_2, a_3, \dots]$ is eventually periodic then it is a root of some quadratic equation.

*Proof.* Set the continued fraction equal to $x$. Solving for $x$ in the resulting equation gives $f(x) = \frac{g(x)}{h(x)}$ for some linear functions $f, g, h$ in $x$, which rearranges to a quadratic. □

**Example 2.4**

$$\sqrt{2} = [1; \overline{2}], \sqrt{3} = [1; \overline{1, 2}], \sqrt{5} = ?$$

# §3 Relating Successive Convergents

## §3.1 Calculating Convergent by Hand

**Example 3.1**

The "normal" way (some motivation) Suppose you wanted to calculate the simplified form of $[1; 2, 3, 4, \dots, 10]$. Normally, you start simplifying from the "bottom" up, i.e. start at 10 and work your way up.

What if you already knew the simplified form of $[1; 2, 3, 4, \dots, 9]$? With the current method of simplifying from the bottom up, *the $n$-th convergent doesn't help at all for calculating the $n+1$-th convergent.*

**Wish (Relating Successive Convergents):** Some way to calculate successive convergents based on knowledge of previous convergents.

## §3.2 The Granted Wish

**Claim 3.2** (Basis of Algorithm) — For a continued fraction $\alpha = [a_1; a_2, a_3, \dots, a_n]$, let $P_k, Q_k$ defined for $1 \le k \le n$ be 2 sequences of naturals such that $\frac{P_k}{Q_k}$ is equal to the $k$-th convergent of $\alpha$. Then $P_1 = a_1, Q_1 = 1$, and for $2 \le k \le n$, we have **(!)**

$$P_k = a_k P_{k-1} + P_{k-2},$$
$$Q_k = a_k Q_{k-1} + Q_{k-2}.$$

**Remark 3.3.** Up until now, we have assumed $a_i \in \mathbb{N}$. However, everything we have defined can be easily extended to arbitrary real numbers for $a_i$ (at the cost of losing uniqueness of continued fractions, but this won't matter in the following proof); we will use this extension in the proof.

*Proof.* We use strong induction on $n$, the length of the continued fraction. As mentioned earlier, extend our definitions to allow $a_i$ be real numbers. The base cases $n = 1, 2, 3$ are trivially true. Now assume the hypothesis holds for all $k < n$. We'll prove the hypothesis for $n$.

Consider the new continued fraction $\alpha' = \left[a_1; a_2, a_3, \ldots, a_{n-1} + \frac{1}{a_n}\right]$. Note that by definition, $\alpha' = \alpha$ (extension to reals has broken the uniqueness of a continued fraction). With $P_k, Q_k$ as defined earlier, define

$$P = \left(a_n + \frac{1}{a_{n-1}}\right) P_{n-2} + P_{n-3},$$
$$Q = \left(a_n + \frac{1}{a_{n-1}}\right) Q_{n-2} + Q_{n-3}.$$

Note by our inductive hypothesis (since $\alpha'$ is of length $n-1$) that $\frac{P}{Q} = \alpha'$. But we also have

$$\begin{aligned}\frac{P_n}{Q_n} &= \frac{a_n P_{n-1} + P_{n-2}}{a_n Q_{n-1} + Q_{n-2}} = \frac{(a_n a_{n-1} + 1)P_{n-2} + a_n P_{n-3}}{(a_n a_{n-1} + 1)Q_{n-2} + a_n Q_{n-3}} \\ &= \frac{\left(a_n + \frac{1}{a_{n-1}}\right) P_{n-2} + P_{n-3}}{\left(a_n + \frac{1}{a_{n-1}}\right) Q_{n-2} + Q_{n-3}} = \frac{P}{Q} = \alpha' = \alpha,\end{aligned}$$

completing the proof (since all other convergents follow immediately by the inductive hypothesis on $[a_1; a_2, a_3, \ldots, a_{n-1}]$). □

## §3.3 The "Magic Box"

**Example 3.4** (The "Magic Box")

Try to figure out how the convergents to the continued fraction $[1; 3, 5, 7]$ are calculated from the following table:

| | | 1 | 3 | 5 | 7 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 4 | 21 | 151 |
| 1 | 0 | 1 | 3 | 16 | 115 |

If you have convinced yourself of the algorithm behind the above and that it is justified by Claim 3.2, we can state the following corollary:

**Corollary 3.5** (Determinant $\pm 1$)

Determinant of successive $2 \times 2$ matrices is $\pm 1$, since it starts at $-1$ and each successive matrix is the result of adding a multiple of the right column to the left column, then switching the columns. In other words,

$$|P_k Q_{k-1} - P_{k-1} Q_k| = 1.$$

**Example 3.6** (Reciprocals of convergents are convergents of reciprocal)

Let $\alpha > 1$ be a real number. Then the reciprocal of the $k$-th convergent of $\alpha$ is the $(k+1)$-th convergent of $\frac{1}{\alpha}$.

# §4 How "good" are convergent approximations?

**Proposition 4.1** (Within $\frac{1}{\text{denom}^2}$)

For a real number $\alpha$, we have

$$\left|\alpha - \frac{P_k}{Q_k}\right| \le \frac{1}{Q_k Q_{k+1}} \le \frac{1}{Q_k^2}.$$

*Proof.* Note by the alternating nature of successive convergents, we have

$$\left|\alpha - \frac{P_k}{Q_k}\right| \le \left|\frac{P_k}{Q_k} - \frac{P_{k+1}}{Q_{k+1}}\right| = \frac{1}{Q_k Q_{k+1}} \le \frac{1}{Q_k^2}.$$

□

**Remark 4.2.** Note that we can only guarantee an error of at most $\frac{1}{2q}$ for an arbitrary denominator $q$, which makes convergents special!

# §5 Exercises

## §5.1 Theory Problems

**Exercise 5.1** (Convergents are best for their denominator)**.** Let $\frac{P_k}{Q_k}$ be a convergent to the real number $\alpha$. Suppose further that $P, Q \in \mathbb{N}$ satisfies

$$\left|\alpha - \frac{P}{Q}\right| \le \left|\alpha - \frac{P_k}{Q_k}\right|.$$

Prove that $Q \ge Q_k$.

**Exercise 5.2** (Rational vs. Irrational)**.** Prove that $\alpha \in \mathbb{R}$ is irrational iff there exist infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^2}$.

**Exercise 5.3** (Halving the error)**.** Prove that for any irrational number $\alpha$ there exist infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \le \frac{1}{2q^2}$.

**Exercise 5.4** ("Good" enough approximations are necessarily convergents (!!!))**.** Prove that for any real number $\alpha$ and fraction $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \le \frac{1}{2q^2}$ we necessarily have that $\frac{p}{q}$ is a convergent of the continued fraction of $\alpha$.

**Exercise 5.5** (How far can we go?)**.** Find, with proof, an integer $k$ not a perfect square such that no fraction $\frac{p}{q}$ exists with

$$\left|\sqrt{k} - \frac{p}{q}\right| \le \frac{1}{3q^2}.$$

**Exercise 5.6** (The best constant (!!!))**.** Prove that for any irrational number $\alpha$ there exist infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| \le \frac{1}{\sqrt{5}q^2}$. Furthermore, prove that if $\sqrt{5}$ is replaced by any greater constant, the assertion fails.

## §5.2 Related Contest Problems

**Exercise 5.7** (BMT Team 2015/13)**.** There exist right triangles with integer side lengths such that the legs differ by 1. For example, 3-4-5 and 20-21-29 are two such right triangles. What is the perimeter of the next smallest Pythagorean right triangle with legs differing by 1?

**Exercise 5.8** (TST 2016/4)**.** Let $\sqrt{3} = 1.b_1b_2b_3\dots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer $n$, at least one of the digits $b_n$, $b_{n+1}$, $\dots$, $b_{2n}$ equals 1.

**Exercise 5.9** (ISL 2003/C5)**.** Every point with integer coordinates in the plane is the center of a disk with radius $\frac{1}{1000}$.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

**Exercise 5.10** (ISL 2019/N6)**.** Let $H = \{\lfloor i\sqrt{2}\rfloor : i \in \mathbb{Z}_{>0}\} = \{1, 2, 4, 5, 7, \dots\}$ and let $n$ be a positive integer. Prove that there exists a constant $C$ such that, if $A \subseteq \{1, 2, \dots, n\}$ satisfies $|A| \ge C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to $z$.)

# §6 Acknowledgements

A special thank you to PROMYS for their extensive instruction on the topic of continued fractions. This handout was inspired by material covered during the camp.