Intuition
MA 5.0

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TSTST 2020/8

For every positive integer N, let $\sigma(N)$ denote the sum of the positive integer divisors of N. Find all integers $m \geq n \geq 2$ satisfying

$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} = \frac{\sigma(mn)-1}{mn-1}.$$

- Assuming $\gcd(m,n) = 1$:

$$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(m)\sigma(n)-1}{mn-1} \implies (m-1)\sigma(m)(\sigma(n)-n) - (\sigma(m)-m)(n-1) = 0.$$

- $$\frac{\sigma(m)-1}{m-1} = \frac{\sigma(n)-1}{n-1} \implies (\sigma(m)-m)(n-1) = (\sigma(n)-n)(m-1).$$

- This becomes

$$(\sigma(m)-1)(m-1)(\sigma(n)-n) = 0.$$

- Since *LHS* always positive, use inequalities somehow.
- Need to combine both equations into one to use inequalities successfully.
- (m,n) powers of the same prime.

- $$\begin{aligned}
\sigma(mn) - 1 &= (mn-1)\cdot\frac{\sigma(m)-1}{m-1} = \frac{(mn-1)\sigma(m) - mn + 1}{m-1} \\
\implies \sigma(mn) &= \frac{(mn-1)\sigma(m) - mn + m}{m-1} = \frac{(m-1)n\sigma(m) - mn + m + (n-1)\sigma(m)}{m-1} \\
&= n\sigma(m) + \frac{(n-1)(\sigma(m)-m)}{m-1} \\
&= n\sigma(m) + \frac{(n-1)(\sigma(n)-n)}{n-1} \\
&= n\sigma(m) + \sigma(n) - n.
\end{aligned}$$
- We get
$$\sigma(mn) = n\sigma(m) + \sigma(n) - n.$$
- **Claim:**
$$\sigma(mn) \geq n\sigma(m) + \sigma(n) - n.$$
- Proof: n times any factor of m is a factor of mn, and so is any factor of n. Intersection of the sets is exactly n, so done by PIE.

USEMO 2020/4

A function f from the set of positive real numbers to itself satisfies

$$f(x + f(y) + xy) = xf(y) + f(x + y)$$

for all positive real numbers x and y. Prove that $f(x) = x$ for all positive real numbers x.

- If $x + f(y) + xy = x + y$, then $x = \frac{y - f(y)}{y}$.
- Since this is impossible, $f(y) \geq y$.
- Intuition: "Small error will blow up at very large inputs and then the functional equation dies."
- Assume FSoC that some a with $f(a) > a$.
- $P(x, a)$ gives:
$$f(x + f(a) + xa) = xf(a) + f(x + a) \geq xf(a) + x + a.$$
- For large enough $t = x + f(a) + xa$, we have
$$f(t) \geq \frac{f(a) + 1}{a + 1} t + a - \frac{f(a)(f(a) + 1)}{a + 1}.$$

- For large enough t, we have

$$f(t) \geq \frac{1 + \frac{f(a)+1}{a+1}}{2} t + 32427394273904.$$

- Define

$$\frac{1 + \frac{f(a)+1}{a+1}}{2} = 1 + \varepsilon.$$

- Goal: How can we get contradiction from $f(x+t) \geq f(t) + \text{BIG}$ for arbitrarily small x?
  - What would $f(t+1)$ be?
- $P(x, t)$:

$$xf(t) + f(x+t) = f(x + f(t) + xt) \geq (1+\varepsilon)(x + f(t) + xt) + 192837402947.$$

- For all large enough t and $x < \varepsilon$:

$$\begin{aligned} f(x+t) &\geq (1 + \varepsilon - x) f(t) + (1+\varepsilon)(x + xt) + 2312312093 \\ &\geq f(t) + 1231290381230. \end{aligned}$$

Iran TST 2020/2/5

For every positive integer $k > 1$ prove that there exist a real number x so that for every positive integer $n < 1398$:

$$\{x^n\} < \{x^{n-1}\} \iff k \mid n.$$

- Regroup at 9:45pm to discuss ideas.
- Ideas:
  - Binomial theorem: $x = m + y$ where m is an integer and $0 < y < 1$. Then expand
  $$\{(m+y)^n\}.$$
  - Take $x = m + \frac{1}{N}$.
  - $$\begin{aligned}\{x\} &= \frac{1}{N}\\ \{x^2\} &= \left\{\frac{2m}{N} + \frac{1}{N^2}\right\}\\ \{x^3\} &= \left\{\frac{3m^2}{N} + \frac{3m}{N^2} + \frac{1}{N^3}\right\}\end{aligned}$$
  - How can we vary m while keeping $\{x^2\}$ fixed? Answer: We can add multiples of N to m.
  - How can we use above to assert control over the value of $\{x^3\}$?
  - We can change $\{x^3\}$ by $\frac{3N}{N^2}$ without affecting previous things!!!

- With N fixed from the beginning as some gigantic prime, we can actually change $\{x^3\}$ by multiples of $\frac{1}{N}$.
- How can we vary m while keeping $\{x^3\}$ AND $\{x^2\}$ fixed? Answer: We can add multiples of N^2 to m.

$$\{x^4\} = \left\{ \frac{4m^3}{N} + \frac{6m^2}{N^2} + \frac{4m}{N^3} + \frac{1}{N^4} \right\}.$$

- Extend until win! :D
- **Remark:** 1398 is arbitrary, and we can actually order the fractional parts however we want!

Very Useful NT Lemma

Suppose n is a quadratic residue modulo every prime number. Must n be a perfect square?

- Answer is yes.
- WLOG $1 < n = p_1 p_2 \cdots p_k$ is square-free. Want to find p such that n is not a quadratic residue mod p.

$$\prod \left(\frac{p_i}{p}\right) = -1,$$

  equivalent to

$$\prod \left(\frac{p}{p_i}\right) = (-1)^c.$$

- Motivation for flipping: When p is on the bottom, the modulus is varying. When p is on the top in the flipped version, the modulus is constant.
- Quadratic Reciprocity and Dirichlet!
- Since we can choose $p \pmod{p_i}$ for each i by Dirichlet, then we just win. (We can achieve the product equals 1 and -1 so doesn't matter what c is, we win in both cases).

Iran TST 2011/12

Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function for which the expression $af(a) + bf(b) + 2ab$ for all $a, b \in \mathbb{N}$ is always a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.

- $P(a, b)$ is the problem assertion.
- $P(a, p)$ for varying p gives? By above useful lemma this gives us $af(a)$ is a perfect square for all a.
- $g(a) := \sqrt{af(a)}$. Problem becomes $g(a)^2 + g(b)^2 + 2ab$ always a square.
- $p \mid g(p) \implies p \leq g(p)$.
- $P(1, p)$ gives $g(p)^2 + 2p + g(1)^2$ is always a square.
- $$4g(p) + 4 > 2p + g(1)^2 \geq 2g(p) + 1 \implies 2p + g(1)^2 = 2g(p) + 1$$

  for large enough p.
- By size considerations, we get $g(p) = p$ for all $p \gg g(1)^2$.
- This then forces $g(1) = 1$.
- Easy part of HW: Finish the problem from here.