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office hours 9-10 am

Algorithmic Analysis 1:

1) For each function below, describe how the function's value will change if the argument is increased fourfold.
   a. $\log_2 n$
   b. $\sqrt{n}$
   c. n
   d. n^2
   e. n^3
   f. 2^n

2) Use your understanding of O , θ , Ω to decide if the following are T or F:
   a. $\frac{n(n+1)}{2} \in O(n^3)$
   b. $\frac{n(n+1)}{2} \in O(n^2)$
   c. $\frac{n(n+1)}{2} \in O(n)$
   d. $\frac{n(n+1)}{2} \in \theta(n^3)$
   e. $\frac{n(n+1)}{2} \in \Omega(n)$

3) For each function below find the simplest $g(n)$ that indicates the class $\theta(g(n))$ that the function belongs to.
   a. $(n^2+5)^{20}$
   b. $4n\lg(n+4)^2 + 2(n+4)^2\lg\left(\frac{n}{2}\right)$
   c. $\sqrt{10n^2 + 4n + 7}$
   d. $2^{n+1} + 3^{n-1}$

4) Prove that the following functions are listed in increasing order of growth from left to right:

$$\log_2 n \ , \ n \ , \ n\log_2 n \ , \ n^2 \ , \ n^3 \ , \ 2^n \ , \ n!$$

5) Rearrange the following functions as needed so that they are listed from left to right in increasing order of growth:

$$(n-2)! \ , \ 5\lg(n+100)^{10} \ , \ 2^{2n} \ , \ 0.001n^4 + 3n^3 + 1 \ , \ ln^2(n) \ , \ \sqrt[3]{n} \ , \ 3^n$$

6) Prove that every polynomial degree k, $p(n) = a_k n^k + a_{k-1}n^{k-1} + \cdots + a_0$, with $a_k > 0$, belongs to $\theta(n^k)$

Solutions

1) For each function below, describe how the function's value will change if the argument is increased fourfold.
   a. $\log_2 n$ $\quad \log_2 4n = \log_2 4 + \log_2 n = 2 + \log n$ - The value increases by two.
   b. $\sqrt{n}$ $\quad \sqrt{4n} = \sqrt{4} \cdot \sqrt{n} = 2 \cdot \sqrt{n}$ - function grows by factor of 2
   c. n $\quad = 4n$ function grows by factor of 4
   d. n^2 $\quad = (4n)^2 = 4^2 \cdot n^2$ function grows by factor of 16
   e. n^3 $\quad = (4n)^3 = 4^3 \cdot n^3$ function grows by factor of 64
   f. 2^n $\quad = 2^{4n} = (2^4)^n = 16^n$ function grows by factor of 8

2) Use your understanding of O, θ, Ω to decide if the following are T or F:
   a. $\frac{n(n+1)}{2} \in O(n^3)$ TRUE $\quad \lim_{n\to\infty} \frac{n^2+n}{2} / n^3 = 0$
   b. $\frac{n(n+1)}{2} \in O(n^2)$ TRUE $\quad \lim_{n\to\infty} \frac{n^2+n}{2} / n^2 = 1$
   c. $\frac{n(n+1)}{2} \in O(n)$ False $\quad n^2 > n \to \lim_{n\to\infty} \frac{n^2+n}{2n} = \infty$
   d. $\frac{n(n+1)}{2} \in \theta(n^3)$ False $\quad n^2 < n^3$ θ must be equal
   e. $\frac{n(n+1)}{2} \in \Omega(n)$ TRUE $\quad n^2 > n$ true. n^2 is worse than best case.

3) For each function below find the simplest $g(n)$ that indicates the class $\theta(g(n))$ that the function belongs to.
   a. $(n^2+5)^{20}$
   b. $4n\lg(n+4)^2 + 2(n+4)^2 \lg\left(\frac{n}{2}\right)$
   c. $\sqrt{10n^2 + 4n + 7}$
   d. $2^{n+1} + 3^{n-1}$

3a. $(n^2+5)^{20}$

Prove: $g(n) \in n^{40}$

$$\lim_{n\to\infty} \frac{(n^2+5)^{20}}{n^{40}} = \frac{(n^2+5)^{20}}{(n^2)^{20}} = \left(\frac{n^2+5}{n^2}\right)^{20} = \left(\frac{n^2}{n^2} + \frac{5}{n^2}\right)^{20} = \left(1 + \frac{5}{n^2}\right)^{20} \to \lim_{n\to\infty} = 1 + 0 = \boxed{1}$$

Therefore $g(n) \in n^{40}$

3b is on the next page.

3b.

Premise: $4n \lg[(n+4)^2] + 2[(n+4)^2]\lg\left(\frac{n}{2}\right)$

Power Rule $\rightarrow$ $4n \cdot 2 \cdot \lg(n+4) + 2(n+4)^2 \lg\left(\frac{n}{2}\right)$

Distribution $2\left[4n \cdot \lg(n+4) + (n+4)^2 \lg\left(\frac{n}{2}\right)\right]$

Distribution & Quotient $2\left[4n \cdot \lg(n+4) + (n+4)(n+4)(\lg(n) - \lg(2))\right]$

Distribution $2[4n \cdot \lg(n+4) + (n+4)(n+4)(\lg(n)) - (n+4)(n+4)(\log(2))$

Hypothesis: Factor of growth is $n^2 \log n$

Proof: $$\frac{2[4n \cdot \lg(n+4) + (n+4)(n+4)(\lg(n)) - (n+4)(n+4)(\log 2)]}{n \cdot n \cdot \log n}$$

elimination $$2\left[\frac{4\cancel{n} \cdot \lg(n+4)}{\cancel{n} \cdot n \cdot \lg(n)} + \frac{(n+4)(n+4)(\cancel{\lg(n)})}{n \cdot n \cdot \cancel{\lg(n)}} - \frac{(n+4)(n+4)(\log 2)}{n \cdot n \cdot \lg(n)}\right]$$

distribution & elimination $$2\left[\frac{\lg(n+4)}{n \cdot \lg(n)} + \frac{(n+4)(n+4)}{n \cdot n} - \frac{(n^2+8n+16)(\log(2))}{n^2 \lg(n)}\right]$$

L'Hospitals Rule $$2\left[\frac{\frac{1}{\ln(2)(n+4)}}{n \cdot \frac{1}{\ln(2)(n)}} + \frac{2n+8}{2n} - \frac{(\log 2) + (2n+8)}{2n(\log(n)) + \left(\frac{1}{\ln(2)(n)}\right)(n^2)}\right]$$

Reduction $= \frac{\cancel{\ln(2)(n)}\,1}{\cancel{n \cdot \ln(2)}(n+4)}$

$= 0$

L'Hospitals Rule

$\frac{2}{2} = 1$

L'Hospitals Rule

$a(n) = \frac{2}{n(\log n) + \left(\frac{1}{\ln(2)(n)}\right)(2n) + \cancel{\left(\frac{0}{\ln(2)}\right)(2n)} + \left(\frac{1}{\ln(2)(n)}\right)(2)}$

$\lim_{n \to \infty} a(n) = 0$

$$\frac{\lim g(n)}{\lim f(n)} = 0 + 1 + 0 = 1$$

$g(n) \in n^2 \log n$

3c. $\sqrt{10n^2+4n+7}$

Hypothesis: $g(n) \in n$

$$\frac{\sqrt{10n^2+4n+7}}{n}$$

Split Sqr Root $$\frac{\sqrt{10n^2}+\sqrt{4n}+\sqrt{7}}{n}$$

factor out n & eliminate $$\frac{\cancel{n}\sqrt{10}}{\cancel{n}} + \frac{\sqrt{4n}}{n} + \frac{\sqrt{7}}{n}$$

top & Bottom $\times \sqrt{4n}$

top & Bottom $\times \sqrt{7n}$ $$\sqrt{10} + \frac{4\cancel{n}}{\sqrt{4n}\cdot \cancel{n}} + \frac{7}{\sqrt{7}\cdot n}$$

Eliminate $$\sqrt{10} + \frac{4}{\sqrt{4n}} + \frac{7}{\sqrt{7}\cdot n}$$

Take Limit $$\sqrt{10} + 0 + 0 = \sqrt{10}$$

$g(n) \in n$

3d. $2^{n+1}+3^{n-1}$

Hypothesis $g(n) \in 3^n$

$$\frac{2^{n+1}+3^{n-1}}{3^n}$$

$-n$

seperate & eliminate $$\frac{2^{n+1}}{3^n} + 3^{-1}$$

pull out a 2 $$= \frac{2\cdot 2^n}{3^n} + \frac{1}{3}$$

L'hospitals $$\frac{\cancel{n}\cdot 2^{n-1}}{\cancel{n}\cdot 3^{n-1}} = \boxed{\frac{2}{3}}$$

$\therefore g(n) \in 3^n$

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4) Prove that the following functions are listed in increasing order of growth from left to right:

$$\log_2 n \ , \ n \ , \ n\log_2 n \ , \ n^2 \ , \ n^3 \ , \ 2^n \ , \ n!$$

$\log_2 n < n$

$$\frac{\log_2 n}{n}$$

lhospitals $$\frac{\frac{1}{\ln 2(n)}}{1}$$

$$= \frac{1}{\ln^2(n)}$$

$$\lim_{n\to\infty} = 0$$

$$\lim_{n\to\infty} \log_2 n < \lim_{n\to\infty} n$$

$n < n\log_2 n$

$$\frac{\cancel{n}}{\cancel{n}\log_2 n}$$

$$\frac{1}{\log_2 n}$$

$$\lim_{n\to\infty} = 0$$

$$\lim_{n\to\infty} n < \lim_{n\to\infty} \log_2 n$$

$n\log_2 n < n^2$

$$\frac{\cancel{n}\log_2 n}{\cancel{n}\cdot n}$$

$$\frac{\log_2 n}{n}$$

l'hospitals Rule

$$\frac{1}{\ln(2)\cdot(n)\cdot(n)}$$

as $n\to\infty$

$f(n) \to 0$

$$\lim_{n\to\infty} \cancel{n}\log_2 n < \lim_{n\to\infty} n^2$$

$n^2 < n^3$

$$\frac{\cancel{n}\cdot\cancel{n}}{\cancel{n}\cdot\cancel{n}\cdot n}$$

$$= \frac{1}{n}$$

$$\lim_{n\to\infty} f(n) = 0$$

$$\lim_{n\to\infty} n^2 < \lim_{n\to\infty} n^3$$

$n^3 < 2^n$

$$\frac{n^3}{2^n}$$

Lhospitals Rule

$$\frac{3n^{\cancel{2}}}{\cancel{n}2^{n-1}} \quad 3\cdot n \cancel{\cdot n}$$

$$\frac{3n}{2^{n-1}}$$

Lhospitals Rule

$$\frac{3}{(n-1)(2^{n-2})}$$

$$\lim_{n\to\infty} f(n) = 0$$

$$\lim_{n\to\infty} n^3 < \lim_{n\to\infty} 2^n$$

$2^n < n!$

Prove $2^n < n!$

Let $K=2$

Let m = all #s ≥ 4 in $n!$ where $m \geq 2K$

Let R be the Remainder of $\left(\frac{n!}{m^{(n-3)}}\right)$.

Substitute 2 for K & #s >4 for m.

$$\frac{1 \cdot 2 \cdot 3 \cdot \left[m^{\wedge n-3}\right] \cdot R}{K^n}$$

$m \geq 2K$. Substitute $2K$. Since this new substitution is less than $n!$, should prove $n! > 2^n$.

$$\frac{1 \cdot 2 \cdot 3 \cdot (2K)^{(n-3)} \cdot R}{K^n}$$

$$\frac{1 \cdot 2 \cdot 3 \cdot 2^{n-3} \cdot \cancel{K^{n-3}} \cdot R}{K \cdot K \cdot K \cdot \cancel{K^{(n-3)}}}$$

R is dependent on n & Always Positive.

$$\lim_{n \to \infty} f(n) = \frac{1 \cdot 2 \cdot 3 \cdot 2^{n-3} \cdot R}{\cancel{K \cdot K \cdot K}\ 2^3} = \infty$$

$K=2$

$$\therefore \lim_{n \to \infty} 2^n < \lim_{n \to \infty} n!$$

5) Rearrange the following functions as needed so that they are listed from left to right in increasing order of growth:

$$(n-2)!\ ,\ 5\lg(n+100)^{10}\ ,\ 2^{2n}\ ,\ 0.001n^4+3n^3+1\ ,\ ln^2(n)\ ,\ \sqrt[3]{n}\ ,\ 3^n$$

1 2 3 4 5 6 7 8

$$\ln^2(n)\ ,\ 5\lg(n+100)^{10}\ ,\ \sqrt[3]{n}\ ,\ 0.001n^4+3n^3+1\ ,\ 3^n\ ,\ 2^{2n}\ ,\ (n-2)!$$

5.1

$$\frac{5\lg(n+10^2)^{10}}{\ln^2 n}$$

Change of Base to ln. Factor out a couple for elimination.

$$\frac{5\cdot \ln(n+10^2)^8\cdot \ln(n+10^2)\cdot \ln(n+10^2)}{(\ln 2)(\ln(n))(\ln(n))}$$

let $\ln(n+10^2)=K$ where

$$\lim_{n\to\infty}\frac{(\ln(n+10^2))'}{(\ln(n))'}$$

$$\frac{\frac{1}{n+100}}{\frac{1}{n}}$$

$$\frac{100}{0}=\infty$$

$$\lim_{n\to\infty}\ \ln(n+10^2)>\ln(n)$$

$$\frac{5\cdot K^8\cdot \cancel{K}\cdot \cancel{K}}{\ln(2)\,\cancel{(\ln(n))}\,\cancel{(\ln(n))}}$$

$$\lim_{n\to\infty}\frac{5\cdot K^8}{\ln 2}=\infty$$

$$\lim_{n\to\infty} 5\lg(n+100)^{10}>\lim_{n\to\infty}\ln^2(n)$$

5.2

$$\frac{5\log(n+100)^{10}}{n^{\frac{1}{3}}}$$

Base change

$$\left[\frac{5\ln(n+100)^{10}}{n^{\frac{1}{3}}\cdot\ln(2)}\right]'$$

L'Hospitals Rule

L'Hospitals Rule

$$\frac{\left[n^{\frac{1}{3}}\right]'}{\left[\ln(x)\right]'}$$

$\ln(n+100)^{10}\cdot 5$

L'Hospitals

$$\frac{\left(\frac{1}{3}n^{-\frac{2}{3}}\right)}{\left(\cancel{\frac{1}{n}}\right)'}\ 0$$

$\lim_{n\to\infty}$

$n^{\frac{1}{3}} > \ln(x)$

6) Prove that every polynomial degree k, $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0$, with $a_k > 0$, belongs to $\theta(n^k)$

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$$\frac{an^{k} + an^{k-1} + an^{k-2} + \dots an_{0}}{n^{k}}$$

Goal: 1. Prove $\lim_{n \to \infty} an^{k} = \lim_{n \to \infty} n^{k} > \lim_{n \to \infty} n^{k} \quad an^{k-1} \quad k-2 \ldots$

2. Prove the growth of n^k is the same as an^k, & that the limit of the Ratios of $\frac{n^k}{n^{k-1} \ldots}$ is 0 such that $an^k + a_n^{k-1} \ldots$ belongs to $\Theta(n^k)$.

Get Ratio of limits for first term.

$$\lim_{k \to \infty} \frac{(n^k)'}{n^{k\prime}} = \underline{1}$$

Ratio of limits for second term.

$$\lim_{k \to \infty} \frac{a(n^{k-1})}{n^k} = \frac{a(\cancel{n^{k-1}})}{k \cdot \cancel{n^{k-1}}} = \frac{a}{k} = \underline{0}$$

$\lim k \to \infty$

Prove all $n^{k-1} > n^{k-n}$

$\lim_{n \to \infty}$

$$\lim_{k \to \infty} \frac{a(n^{k-1})}{a(n^{k-2})} = \frac{k \cdot \cancel{n^{(k-2)}}}{\cancel{n^{k-2}}} = \frac{k}{0} = \infty$$

$$\therefore \lim_{k \to \infty} n^{k-1} > \lim_{k \to \infty} n^{k-2}$$

$$\lim_{k \to \infty} f(k)\; n^k = \underline{an^k > an^{k-1} > an^{k-2} \ldots 0}$$

$$\lim_{k \to \infty} \frac{an^k}{n^k} + \frac{an^{k-1}}{n^k} + \frac{an^{k-2}}{n^k} \ldots \frac{an}{n^k}$$

$$1 + 0 + 0 \ldots + 0 = \underline{1}$$

$$\frac{K \cdot 2^{R} \cdot 2^{\frac{n}{2}} \cdot \log_2 n \cdot Q}{2^2 \cdot \cancel{\log_2 n}}$$

$$\frac{K \cdot 2^{R} \cdot 2^{\left(\frac{n}{2}\right)} \cdot Q}{4}$$

$$\lim_{n \to \infty} = \frac{K \cdot 2^{R} \cdot 2^{\frac{n}{2}} \cdot Q}{4} = \infty$$

odd #s

$$\frac{(1, 3, 5, 7 \ldots)\,(n/2!)}{2^{n/2}}$$

Count # of 2s in $n!$ — $\frac{n!}{2^n}$

$$\lim_{n \to \infty} \frac{n!}{2^n}$$

$$\frac{(\text{odd nums})(n!/2)}{2 \cdot 2^{n-1}} \to (2^1\ 2^2\ 2^3 \ldots 2^{n-1})(2^{\frac{n}{2}})$$

All evens

A 2 from each even num

$$\frac{(\text{odd nums})\left(\cancel{2^{n}/2}\right)}{\cancel{2^n}}$$

$$4^{\log_6 n}$$

$$\log_2 16 = 4$$

$$2^4 = 16$$