The formula

$$e^{iy} = \cos(y) + i\sin(y) \tag{1}$$

is named for Leonard Euler (pronounced oiler) and is fundamental in mathematics, physics, signal processing, circuits, the mechanics of vibrations, and many other fields relevant to your education. Our goal is to understand Euler's formula both as a fact about complex numbers and as a tool for understanding linear differential equations. A good way to see that Euler's formula is important is to remember that the solution formula for

$$\dot{\mathbf{x}} = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + c_n e^{\lambda_n t}\mathbf{v}_n$$

where the λ are the eigenvalues, the $\mathbf{v}$ are the eigenvectors and the coefficients c are determined by the initial condition $\mathbf{x}(0)$. For a 2×2 matrix A, this solution looks like

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \tag{2}$$

Since we have seen that eigenvalues can be complex, the need to understand e^{it} becomes apparent.

1. Consider the matrix
$$A_\zeta = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}$$
with ζ unspecified. Let's look at the system of DEs $\dot{\mathbf{x}} = A_\zeta\mathbf{x}$ for values of ζ ranging between -2 and 2.
   (a) There is a tool called `pplane` for visualizing planar differential equations. It's not built in, but you can Google it to learn more and possibly download it for MATLAB. Use `pplane` (either the java applet or the MATLAB version) to visualize trajectories for the differential equation. Try a whole bunch of different values for ζ!
   (b) Compute the eigenvalues of this matrix (they will depend upon ζ). For some of the different values of ζ you plotted for part (a), what were the eigenvalues?
   (c) Describe the different kinds of behaviors you see for different values of ζ. Try to make an exhaustive list. Where do you see transitions between these behaviors? What connections do you see between the different behaviors and the corresponding eigenvalues of A_ζ?
   (d) Determine whether the following statements are correct or incorrect. If incorrect, salvage the statement by correcting it.:
      i. Whenever the all of the eigenvalues of A are greater than one in absolute value, the vector field points out toward infinity.
      ii. The eigenvalues of A are related by $\lambda_1 = i\lambda_2$.
      iii. The eigenvectors of A are related by $\mathbf{v}_1 = i\mathbf{v}_2$.

iv. Whevenver the real part of all of the eigenvalues of A is negative, the trajectories spiral inward.

v. Whenever the imaginary part of all of the eigenvalues of A is greater than one in absolute value the trajectories spiral counter-clockwise.

vi. The bigger the imaginary part of the eigenvalues, the tighter the spiral the trajectory makes.

vii. When the real part of the eigenvalues are large compared to the imaginary part, the trajectories are close to a circle.

viii. When the real and imaginary parts of the eigenvalues are about the same, the trajectories spend most of their close to the real part of the eigenvectors.

The next few problems get you working with polar coordinates and complex numbers. These ideas are essential to understanding Euler's formula.

2. Polar coordinates:

   Given a point in the plane, we call its representation in **cartesian coordinates** the pair (x, y) and its representation in **polar coordinates** the pair (r, θ). Here $x = r\cos\theta$ and $y = r\sin\theta$. Equivalently $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$ (with θ between $-\pi$ and π and with the sign of θ given by the sign of y).

   (a) For each of the following points given in cartesian coordinates, find its representation in polar coordinates: $(0, 1)$, $(-1, 1)$, $(2, 1)$.

   (b) For each of the following points given in polar coordinates, find its representation in cartesian coordinates: $(1, \pi)$, $(2, \pi/2)$, $(1/2, -\pi/4)$.

   (c) On an (x, y) plane, draw the lines (or curves) of constant $r = [1 : 1 : 4]$ (MATLAB notation for $r = 1, 2, 3, 4$) and also the lines (or curves) of constant $\theta = [-\pi : \pi/4 : \pi]$.

3. Polar coordinates and $\mathbb{C}$.

   (a) Write each of the following complex numbers in polar coordinates: i, $i - 1$ and $2 + i$.

   (b) Compute the products $(i)(i-1)$, $(i)(2+i)$ and $(i-1)(2+i)$ as well as their representations in polar coordinates.

   (c) Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be two complex numbers with polar representations (r_1, θ_1) and (r_2, θ_2). Find the representation in polar coordinates for the product $z_3 = z_1 z_2$.

4. Amplitude, phase and Frequency:

   For functions of the form $t \mapsto A\cos(\omega t + \phi)$ the constants A, ω and ϕ have specific names. A is called the **amplitude**, ω is called the **frequency**, and ϕ is called the **phase**.

   (a) Produce graphs (you need not record them in your portfolio unless you would like them for reference) of t against $A\cos(\omega t + \phi)$ for different values of the constants and describe why the constants have the names that they do.

   (b) The expression $a\cos(\omega t) + b\sin(\omega t)$ can always be rewritten as $A\cos(\omega t + \phi)$ for appropriate A and ϕ. Find A and ϕ in terms of a and b. The most straightforward way to do this symbolically is to write

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t + \phi),$$

square the left hand side, and make use of the formula $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ on the right hand side.

5. Solve

$$\dot{\mathbf{x}} = A_{1/2}\mathbf{x}$$

with initial condition $\mathbf{x}(0) = (1,0)^T$.

   (a) By using formula (2).

   (b) (optional) via diagonalization. (The eigenvalues, eigenvectors and coordinate transforms will all contain complex numbers here).

Optional Exploration

6. e^{it} via polar coordinate transformation

One way to define e^{it} is as the solution to the differential equation

$$\frac{dz}{dt} = iz; \quad z(0) = 1. \tag{3}$$

Define u and v to be the real and imaginary parts of z, i.e. $z(t) = u(t) + iv(t)$ and call its radial and angular coordinates r and θ, i.e. $u = r\cos\theta$ and $v = r\sin\theta$.

   (a) Write down $\dot{u}$ and $\dot{v}$ in terms of u and v.

   (b) Differentiate the identities $r^2 = u^2 + v^2$ and $\tan\theta = \frac{v}{u}$ with respect to t to get equations for $\dot{r}$ and $\dot{\theta}$ in terms of u, v, $\dot{u}$ and $\dot{v}$.

   (c) Use the equation in (a) above to get an equation for $\dot{r}$ and $\dot{\theta}$ in terms of u and v alone.

   (d) Use the definition of polar coordinates to substitute expressions involving r and θ for u and v in (c) above to get equations for $\dot{r}$ and $\dot{\theta}$ which involve only r and θ.

   (e) Solve the equations you found in (d) above to get functions $t \mapsto r(t)$ and $t \mapsto \theta(t)$.

   (f) Use the definition of polar coordinates together with the functions you found in (e) above to get functions $t \mapsto u(t)$ and $t \mapsto v(t)$.

   (g) Write $e^{it} = z(t) = u(t) + iv(t)$ to obtain (1).