Rotation of Magnetic Islands in Tokamak Plasmas

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I. INTRODUCTION

Tearing modes are slowly growing instabilities of ideally-stable tokamak plasmas that reconnect magnetic field-lines at various resonant surfaces within the plasma, in the process forming magnetic island chains that degrade the plasma confinement.[1] *Mode locking* is a process by which the rotation of a magnetic island chain is braked due to electromagnetic interaction with the resistive vacuum vessel surrounding the plasma, causing the chain to eventually *lock* (i.e., become stationary in the laboratory frame) to an error-field.[2] Locked magnetic island chains are one of the principal causes of disruptions in tokamaks.[3]

It is well known that single-fluid, resistive magnetohydrodynamics (MHD) offers a very poor description of tearing mode dynamics in tokamak plasmas. For instance, the strong *diamagnetic* flows present in such plasmas decouple the electron and ion flows to some extent, necessitating a two-fluid treatment.[4] Previously, Fitzpatrick & Waelbroeck [5,6] used a generalized version of the four-field model of Hazeltine, et al.[7]) to determine the two-fluid response of a nonlinear magnetic island chain to a resistive vacuum vessel in a large aspect-ratio tokamak plasma. The aim of this paper is to revisit this analysis in the light of certain refinements to magnetic island theory introduced in Ref. 8.

II. PRELIMINARY ANALYSIS

A. Plasma Equilibrium

Consider a large aspect-ratio tokamak plasma equilibrium whose magnetic flux-surfaces map out (almost) concentric circles in the poloidal plane. Such an equilibrium can be approximated as a periodic cylinder.[9] Let r, θ, z be right-handed cylindrical coordinates. The magnetic axis corresponds to $r = 0$. The system is assumed to be periodic in the z direction with periodicity length $2\pi\, R_0$, where R_0 is the simulated major radius of the plasma. The safety-factor profile takes the form $q(r) = r\, B_z/[R_0\, B_\theta(r)]$, where B_z is the constant 'toroidal' magnetic field-strength, and $B_\theta(r)$ is the poloidal magnetic field-strength. The standard large aspect-ratio orderings, $r/R_0 \ll 1$ and $B_\theta/B_z \ll 1$, are adopted.

B. Perturbed Magnetic Field

Consider a tearing mode perturbation that has m periods in the poloidal direction, and n periods in the toroidal direction. The perturbed magnetic field associated with the tearing mode is written $\delta\mathbf{B} \simeq \nabla\delta\psi \times \mathbf{e}_z$, where $\delta\psi(r,\theta,\varphi,t) = \delta\psi(r,t)\,\exp[\,\mathrm{i}\,(m\,\theta - n\,\varphi)]$, and $\varphi = z/R_0$ is a simulated toroidal angle. Throughout most of the plasma, $\delta\psi(r,t)$ satisfies the *cylindrical tearing mode equation*:[1]

$$\frac{\partial^2\delta\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\delta\psi}{\partial r} - \frac{m^2}{r^2}\,\delta\psi - \frac{J_z'\,\delta\psi}{r\,(1/q - n/m)} = 0, \tag{1}$$

where $J_z(r) = R_0\,\mu_0\,j_z(r)/B_z$, and $j_z(r)$ is the equilibrium 'toroidal' current density. Here, $' \equiv d/dr$. Note that Eq. (1) is singular at the so-called *resonant* magnetic flux-surface, radius $r = r_s$, at which $q(r_s) = m/n$.

C. Outer Solution

Suppose that the plasma occupies the region $0 \leq r \leq a$, where a is the plasma minor radius. It follows that $J_z(r) = 0$ for $r > a$. Let the plasma be surrounded by a concentric, rigid, radially-thin, resistive wall of radius $r_w > a$. (Of course, the resistive wall represents the vacuum vessel.) An appropriate physical solution of the cylindrical tearing mode equation takes the separable form $\delta\psi(r,t) = \Psi_s(t)\,\hat{\psi}_s(r) + \Psi_w(t)\,\hat{\psi}_w(r)$, where the real function $\hat{\psi}_s(r)$ is a solution of

$$\frac{d^2\hat{\psi}_s}{dr^2} + \frac{1}{r}\frac{d\hat{\psi}_s}{dr} - \frac{m^2}{r^2}\,\hat{\psi}_s - \frac{J_z'\,\hat{\psi}_s}{r\,(1/q - n/m)} = 0 \tag{2}$$

that satisfies $\hat{\psi}_s(0) = 0$, $\hat{\psi}_s(r_s) = 1$, and $\hat{\psi}_s(r \geq r_w) = 0$, and the real function $\hat{\psi}_w(r)$ is a solution of

$$\frac{d^2\hat{\psi}_w}{dr^2} + \frac{1}{r}\frac{d\hat{\psi}_w}{dr} - \frac{m^2}{r^2}\,\hat{\psi}_w - \frac{J_z'\,\hat{\psi}_w}{r\,(1/q - n/m)} = 0 \tag{3}$$

that satisfies $\hat{\psi}_w(r \leq r_s) = 0$, $\hat{\psi}_w(r_w) = 1$, and $\hat{\psi}_w(\infty) = 0$.

As is well known, the general solution of Eq. (2) is such that $\delta\psi$ is continuous across the resonant surface, whereas $\partial\delta\psi/\partial r$ is discontinuous. The discontinuity in $\partial\delta\psi/\partial r$ implies the presence of a helical current sheet at the resonant surface. The complex quantity

$$\Psi_s(t) = \delta\psi(r_s,t) \tag{4}$$

parameterizes the amplitude and phase of the reconnected magnetic flux at the resonant surface.[9] The complex quantity

$$\Delta\Psi_s = \left[r\,\frac{\partial\delta\psi}{\partial r}\right]_{r_{s-}}^{r_{s+}} \tag{5}$$

parameterizes the amplitude and phase of the current sheet flowing at the resonant surface.

In general, $\delta\psi$ is continuous across the wall, whereas $\partial\delta\psi/\partial r$ is discontinuous. The discontinuity in $\partial\delta\psi/\partial r$ is caused by a helical current sheet induced in the wall. The complex quantity $\Psi_w(t)$ determines the amplitude and phase of the perturbed magnetic flux that penetrates the wall. The complex quantity

$$\Delta\Psi_w = \left[r\,\frac{\partial\delta\psi}{\partial r}\right]_{r_{w-}}^{r_{w+}} \tag{6}$$

parameterizes the amplitude and phase of the helical current sheet flowing in the wall.

Simultaneously matching our separable solution of Eq. (1) across the resonant surface and the wall yields[9]

$$\Delta\Psi_s = E_{ss}\,\Psi_s + E_{sw}\,\Psi_w, \tag{7}$$

$$\Delta\Psi_w = E_{ws}\,\Psi_s + E_{ww}\,\Psi_w. \tag{8}$$

Here, $E_{ss} = [r\,d\hat{\psi}_s/dr]_{r_{s-}}^{r_{s+}}$, $E_{ww} = [r\,d\hat{\psi}_w/dr]_{r_{w-}}^{r_{w+}}$, $E_{sw} = [r\,d\hat{\psi}_w/dr]_{r=r_{s+}}$, and $E_{ws} = -[r\,d\hat{\psi}_s/dr]_{r=r_{w-}}$ are real quantities determined by the solutions of Eqs. (2) and (3). It is easily demonstrated that $E_{sw} = E_{ws}$.[9]

Standard electromagnetic theory applied to the resistive wall reveals that[2,9]

$$\Delta\Psi_w = \tau_w\,\frac{d\Psi_w}{dt}, \tag{9}$$

where $\tau_w = \mu_0\,r_w\,\delta_w/\eta_w$. Here, η_w and δ_w are the electrical resistivity and radial thickness of the wall, respectively. In deriving Eq. (9), we have made use of the so-called *thin wall* approximation, according to which the radial variation of $\delta\psi$ across the wall is negligible. The thin wall approximation is valid as long as

$$\frac{r_w}{\delta_w} \gg \tau_w\left|\frac{d\ln\Psi_w}{dt}\right|. \tag{10}$$

Assuming an $\exp(-{\rm i}\,\omega\,t)$ time dependence of perturbed quantities, the previous four equations can be combined to give

$$\frac{\Delta\Psi_s}{\Psi_s} = \Delta_{pw} + \frac{\Delta_{nw}-\Delta_{pw}}{1-{\rm i}\,\omega\,\tau_{LR}}, \tag{11}$$

where $\Delta_{pw} = E_{ss}$ is the tearing stability index[10] when the wall is perfectly conducting, $\Delta_{nw} = E_{ss} + E_{sw}^2/(-E_{ww})$ is the tearing stability index when there is no wall, and $\tau_{LR} = \tau_w/(-E_{ww})$ is the effective L/R time of the wall. (Note that $E_{ww} < 0$.)

III. RESONANT PLASMA RESPONSE MODEL

A. Introduction

The helical current sheet at the resonant surface can only be resolved by solving a two-fluid, resistive-MHD, plasma response model in the inner region (i.e., the region of the plasma in the immediate vicinity of the resonant surface), and asymptotically matching the solution so obtained to the ideal-MHD solution in the outer region (i.e., everywhere else in the plasma). The particular plasma response model used in this paper is described in this section.

B. Useful Definitions

The plasma is assumed to consist of two species. First, electrons of mass m_e, electrical charge $-e$, number density n, and temperature T_e. Second, ions of mass m_i, electrical charge $+e$, number density n, and temperature T_i. Let $p = n\,(T_e+T_i)$ be the total plasma pressure.

It is helpful to define $n_0 = n(r_s)$, $p_0 = p(r_s)$,

$$\eta_e = \left.\frac{d\ln T_e}{d\ln n}\right|_{r=r_s}, \tag{12}$$

$$\eta_i = \left.\frac{d\ln T_i}{d\ln n}\right|_{r=r_s}, \tag{13}$$

$$\tau = \left(\frac{T_e}{T_i}\right)_{r=r_s}\left(\frac{1+\eta_e}{1+\eta_i}\right), \tag{14}$$

where $n(r)$, $p(r)$, $T_e(r)$, and $T_i(r)$ refer to density, pressure, and temperature profiles that are unperturbed by the tearing mode.

For the sake of simplicity, the perturbed electron and ion temperature profiles are assumed to be functions of the perturbed electron number density profile in the immediate vicinity of the resonant surface. In other words, $T_e = T_e(n)$ and $T_i = T_i(n)$. This implies that $p = p(n)$. The *MHD velocity*, which is the velocity of a fictional MHD fluid, is defined $\mathbf{V} = \mathbf{V}_E + V_{\parallel\,i}\,\mathbf{b}$, where $\mathbf{V}_E$ is the $\mathbf{E}\times\mathbf{B}$ drift velocity, $V_{\parallel\, i}$ is the parallel component of the ion fluid velocity, $\mathbf{b}=\mathbf{B}/|\mathbf{B}|$, and $\mathbf{B}$ is the magnetic field-strength.

C. Resonant Response Model

The fundamental fields in our resonant plasma reponse model—namely, ψ, ϕ, N, V, and J—have the following definitions:

$$\nabla\psi = \frac{\mathbf{n}\times\mathbf{B}}{r_s\,B_z}, \tag{15}$$

$$\nabla\phi = \frac{\mathbf{n}\times\mathbf{V}}{r_s\,V_A}, \tag{16}$$

$$N = -\hat{d}_i\left(\frac{p-p_0}{B_z^{\,2}/\mu_0}\right), \tag{17}$$

$$V = \hat{d}_i\left(\frac{\mathbf{n}\cdot\mathbf{V}}{V_A}\right), \tag{18}$$

$$J = -\frac{2\,\epsilon_s}{q_s}+\hat{\nabla}^2\psi. \tag{19}$$

Here, $\mathbf{n}=(0,\,\epsilon/q_s,\,1)$, $\epsilon=r/R_0$, $q_s=m/n$, $V_A=B_z/\sqrt{\mu_0\,n_0\,m_i}$, $d_i=\sqrt{m_i/(n_0\,e^{\,2}\,\mu_0)}$, $\hat{d}_i=d_i/r_s$, $\epsilon_s=r_s/R_0$, and $\hat{\nabla}=r_s\,\nabla$.

Our resonant plasma response model takes the form: [5]

$$\frac{\partial\psi}{\partial\hat{t}} = [\phi,\psi]-\left(\frac{\tau}{1+\tau}\right)(1+\lambda_e)\,[N,\psi]+\hat{\eta}_\parallel\,J+\hat{E}_\parallel, \tag{20}$$

$$\begin{aligned}\frac{\partial\hat{\nabla}^2\phi}{\partial\hat{t}} &= [\phi,\hat{\nabla}^2\phi]-\frac{1}{2\,(1+\tau)}\left(\hat{\nabla}^2[\phi,N]+[\hat{\nabla}^2\phi,N]+[\hat{\nabla}^2N,\phi]\right)+[J,\psi]\\ &\quad+\hat{\chi}_\varphi\,\hat{\nabla}^4\left(\phi+\frac{N}{1+\tau}\right),\end{aligned} \tag{21}$$

$$\frac{\partial N}{\partial\hat{t}} = [\phi,N]+c_\beta^{\,2}\,[V,\hat{\psi}]+\hat{d}_\beta^{\,2}\,[J,\psi]+\hat{D}_\parallel\,[[N,\hat{\psi}],\hat{\psi}]+\hat{D}_\perp\,\hat{\nabla}_\perp^{\,2}N, \tag{22}$$

$$\frac{\partial V}{\partial\hat{t}} = [\phi,V]+[N,\psi]+\hat{\chi}_\varphi\,\hat{\nabla}^2V. \tag{23}$$

Here, $[A,B]\equiv\hat{\nabla}A\times\hat{\nabla}B\cdot\mathbf{n}$, $\hat{t}=t/(r_s/V_A)$, $\hat{\eta}_{\parallel,\perp}=\eta_{\parallel,\perp}/(\mu_0\,r_s\,V_A)$, $\hat{E}_\parallel=E_\parallel/(B_z\,V_A)$, $\hat{\chi}_\varphi=\chi_\varphi/(r_s\,V_A)$, where $\eta_{\parallel,\perp}$ is the parallel/perpendicular plasma electrical resistivity at the resonant surface, $E_\parallel$ the parallel inductive electric field that maintains the equilibrium toroidal plasma current in the vicinity of the resonant surface, and χ_φ the (anomalous) perpendicular ion momentum diffusivity at the resonant surface. Moreover, $d_\beta=c_\beta\,d_i$, and

$\hat{d}_\beta = d_\beta/r_s$, where $c_\beta = \sqrt{\beta/(1+\beta)}$, and $\beta = (5/3)\,\mu_0\,p_0/B_z^2$. Here, d_β is usually referred to as the *ion sound radius*. Furthermore, $\lambda_e = 0.71\,\eta_e/(1+\eta_e)$. Finally,[5]

$$\hat{D}_\parallel \equiv \frac{D_\parallel}{r_s\,V_A} = \frac{2}{3}\,(1-c_\beta^2)\left(\frac{\eta_e}{1+\eta_e}\,\frac{\tau}{1+\tau}\,\hat{\chi}_{\parallel\,e} + \frac{\eta_i}{1+\eta_i}\,\frac{1}{1+\tau}\,\hat{\chi}_{\parallel\,i}\right), \tag{24}$$

$$\hat{D}_\perp \equiv \frac{D_\perp}{r_s\,V_A} = c_\beta^2\,\hat{\eta}_\perp + \frac{2}{3}\,(1-c_\beta^2)\left(\frac{\eta_e}{1+\eta_e}\,\frac{\tau}{1+\tau}\,\hat{\chi}_{\perp\,e} + \frac{\eta_i}{1+\eta_i}\,\frac{1}{1+\tau}\,\hat{\chi}_{\perp\,i}\right), \tag{25}$$

where $\hat{\chi}_{\parallel\,e,i} = \chi_{\parallel\,e,i}/(r_s\,V_A)$, $\hat{\chi}_{\perp\,e,i} = \chi_{\perp\,e,i}/(r_s\,V_A)$, $\chi_{\parallel\,e,i}$ is the parallel electron/ion energy diffusivity in the vicinity of the resonant surface, and $\chi_{\perp\,e,i}$ is the anomalous electron/ion perpendicular energy diffusivity in the vicinity of the resonant surface.

D. Boundary Conditions

The unperturbed plasma equilibrium is such that $\mathbf{B} = (0,\,B_\theta(r),\,B_z)$, $p = p(r)$, $\mathbf{V} = (0,\,V_E(r),\,V_z(r))$, where $V_E(r) \simeq E_r/B_z$ is the (dominant θ-component of the) $\mathbf{E}\times\mathbf{B}$ velocity.

It is convenient to work in a frame of reference that corotates with the tearing mode. In this reference frame, the reconnected flux, Ψ_s, is assumed to be a positive real quantity. If W is the full radial width of the magnetic island chain that develops at the resonant surface then it is helpful to define the reduced island width:

$$w = \frac{W}{4} = \left(\frac{L_s\,\Psi_s}{B_z}\right)^{1/2}. \tag{26}$$

It is assumed that $w \ll r_s$. Let $\hat{w} = w/r_s$. In the limit $|\hat{x}|/\hat{w} \gg 1$ (i.e., many island widths from the resonant surface), we have

$$\psi \to \frac{\hat{x}^{\,2}}{2\,\hat{L}_s} + \frac{\Psi_s}{r_s\,B_z}\,\cos\zeta, \tag{27}$$

$$\phi \to \left(\frac{\hat{\omega}}{m} - \hat{V}_E\right)\hat{x} - \frac{s}{4}\,\hat{V}_E'\,\hat{x}^{\,2}, \tag{28}$$

$$N \to -\hat{V}_*\,\hat{x}, \tag{29}$$

$$V \to \hat{V}_\parallel, \tag{30}$$

$$J \to -\frac{2}{q_s\,\hat{R}_0} + \frac{1}{\hat{L}_s}. \tag{31}$$

where $\hat{x} = (r-r_s)/r_s$, $\hat{L}_s = L_s/r_s$, $L_s = R_0\,q_s/s_s$, $\hat{R}_0 = R_0/r_s$, $\zeta = m\,\theta - n\,\varphi$, $\hat{V}_E = V_E(r_s)/V_A$, $\hat{V}_* = V_*(r_s)/V_A$, $V_*(r) = (dp/dr)/(e\,n_0\,B_z)$ is the (dominant θ-component of the) diamagnetic velocity, $\hat{V}_\parallel = \hat{d}_i\, V_z(r_s)/V_A$, $s_s = s(r_s)$, and $s(r) = d\ln q/d\ln r$. Here, $\omega = \hat{\omega}\, V_A/r_s$ is the rotation frequency of the tearing mode in the laboratory frame, $s = \mathrm{sgn}(\hat{x})$, and

$$\hat{V}_E' = \frac{1}{V_A}\left[r\,\frac{dV_E}{dr}\right]_{r_{s-}}^{r_{s+}}. \tag{32}$$

The parameter $\hat{V}_E'$ is introduced into the analysis in order to take into account the fact that the plasma rotation profile in the outer region develops a gradient discontinuity at the resonant surface in response to the nonlinear localized electromagnetic torque that emerges at the surface.[5] Finally, $\hat{E}_\parallel = (2/s_s - 1)\,(\hat{\eta}_\parallel/\hat{L}_s)$.

IV. NONLINEAR SOLUTION OF RESONANT PLASMA RESPONSE MODEL

A. Rescaled Response Model

Let $X = \hat{x}/\hat{w}$, and $T = \omega_\ast\, t$. It follows that $|X| \sim {\cal O}(1)$ in the immediate vicinity of the island chain. It is helpful to define the rescaled fields $\varPsi(X,\zeta,T)$, $\varPhi(X,\zeta,T)$, ${\cal N}(X,\zeta,T)$, ${\cal V}(X,\zeta,T)$, and ${\cal J}(X,\zeta,T)$, where

$$\psi = \left(\frac{\hat{w}^{\,2}}{\hat{L}_s}\right)\varPsi, \tag{33}$$

$$\phi = \left(\frac{\hat{\omega}_\ast\,\hat{w}}{m}\right)\varPhi, \tag{34}$$

$$N = \left(\frac{\hat{\omega}_\ast\,\hat{w}}{m}\right){\cal N}, \tag{35}$$

$$V = \hat{V}_\parallel + \left(\frac{\hat{L}_s\,\hat{\omega}_\ast^{\,2}}{m^2\,c_\beta^{\,2}}\right){\cal V}, \tag{36}$$

$$J = -\frac{2}{q_s\,\hat{R}_0} + \frac{1}{\hat{L}_s} + \left(\frac{\hat{L}_s\,\hat{\omega}_\ast^{\,2}}{m^2\,\hat{w}^{\,2}}\right){\cal J}, \tag{37}$$

where $\omega_\ast = -(m/r_s)\,V_\ast(r_s)$ and $\hat{\omega}_\ast = \omega_\ast/(V_A/r_s)$. The resonant response model specified in Sect. III C rescales to give

$$\frac{d(\ln\hat{w}^2)}{dT}\,\cos\zeta = \left\{\varPhi - \left(\frac{\tau}{1+\tau}\right)(1+\lambda_e)\,\mathcal{N},\varPsi\right\} + \epsilon_\beta\,\epsilon_d\,\epsilon_\eta\,\mathcal{J}, \tag{38}$$

$$\frac{\partial(\partial_X^2\,\varPhi)}{\partial T} = \partial_X\left\{\varPhi + \frac{\mathcal{N}}{1+\tau},\partial_X\varPhi\right\} + \{\mathcal{J},\varPsi\} + \epsilon_\chi\,\partial_X^4\left(\varPhi + \frac{\mathcal{N}}{1+\tau}\right), \tag{39}$$

$$\begin{aligned}\frac{\partial\mathcal{N}}{\partial T} &= \{\varPhi,\mathcal{N}\} + \{\mathcal{V},\varPsi\} + \epsilon_d\,\epsilon_n\,\{\mathcal{J},\varPsi\}\\ &\quad + \epsilon_D\left[\left(\frac{w}{w_c}\right)^4\{\{\mathcal{N},\varPsi\},\varPsi\} + \partial_X^2\mathcal{N}\right],\end{aligned} \tag{40}$$

$$\epsilon_d\,\frac{\partial\mathcal{V}}{\partial T} = \epsilon_d\,\{\varPhi,\mathcal{V}\} + \{\mathcal{N},\varPsi\} + \epsilon_d\,\epsilon_\chi\,\partial_X^2\mathcal{V}, \tag{41}$$

$$\partial_X^2\varPsi = 1 + \epsilon_\beta\,\epsilon_d\,\mathcal{J}. \tag{42}$$

Here, $\partial_X \equiv \partial/\partial X$,

$$\{A,B\} \equiv \frac{\partial A}{\partial X}\,\frac{\partial B}{\partial\zeta} - \frac{\partial A}{\partial\zeta}\,\frac{\partial B}{\partial X}, \tag{43}$$

and

$$w_c = \left(\frac{1}{\epsilon_s\,s_s\,n}\right)^{1/2}\left(\frac{D_\perp}{D_\parallel}\right)^{1/4} r_s. \tag{44}$$

Furthermore,

$$\epsilon_d = \left(\frac{L_s}{L_n}\right)^2\left(\frac{d_\beta}{w}\right)^2, \tag{45}$$

$$\epsilon_\beta = c_\beta^2, \tag{46}$$

$$\epsilon_n = \left(\frac{L_n}{L_s}\right)^2, \tag{47}$$

$$\epsilon_\eta = \frac{\eta_\parallel}{\mu_0\,\omega_\ast\,w^2}, \tag{48}$$

$$\epsilon_\chi = \frac{\chi_\phi}{\omega_\ast\,w^2}, \tag{49}$$

$$\epsilon_D = \frac{D_\perp}{\omega_\ast\,w^2}. \tag{50}$$

Finally,

$$L_n = \left[-\frac{1}{\varGamma\,(1-c_\beta^2)}\left(\frac{d\ln n}{dr}\right)_{r_s}\right]^{-1} \tag{51}$$

is the *density gradient scalelength* at the resonant surface.

Equations (38)–(42) must be solved subject to the boundary conditions [cf. Eqs. (27)–(31)]

$$\Psi \to \frac{X^2}{2} + \cos\zeta, \tag{52}$$

$$\Phi \to v\,X + \frac{s\,v'\,X^2}{2}, \tag{53}$$

$$\mathcal{N} \to X, \tag{54}$$

$$\mathcal{V} \to 0, \tag{55}$$

$$\mathcal{J} \to 0, \tag{56}$$

as $|X| \to \infty$. Here,

$$v = \frac{\omega - \omega_E}{\omega_\ast}, \tag{57}$$

$$v' = -\frac{\hat{w}}{2\,\omega_\ast}\left[r\,\frac{d\omega_E}{dr}\right]_{r_{s-}}^{r_{s+}}. \tag{58}$$

Note that Ψ, Φ, $\mathcal{N}$, $\mathcal{V}$, and $\mathcal{J}$ are all $\mathcal{O}(1)$ quantities. Note, further, that the boundary conditions (52)–(56), as well as the symmetry of Eqs. (38)–(42), ensure that Ψ, $\mathcal{V}$, and $\mathcal{J}$ are even functions of X, whereas Φ and $\mathcal{N}$ are odd functions.

B. Ordering Scheme

In the following, the dimensionless parameters ϵ_d, ϵ_β, ϵ_n, ϵ_η, ϵ_χ, and ϵ_D are all assumed to be of similar size, but much smaller than unity. The orderings $\epsilon_n \ll 1$ and $\epsilon_\beta \ll 1$ are consistent with the standard large-aspect ratio, low-β orderings used to describe conventional tokamak plasmas. As will become apparent, the ordering $\epsilon_d \ll 1$ ensures that the island chain is sufficiently wide that ion acoustic waves propagating parallel to the magnetic field are able to smooth out any variations in the normalized electron number density, $\mathcal{N}$, around magnetic flux-surfaces.[11] The fact that $\epsilon_d \ll 1$ implies that $w \gg \rho_\beta$, where

$$\rho_\beta = \left(\frac{L_s}{L_n}\right) d_\beta \tag{59}$$

is a characteristic lengthscale that is of order the ion poloidal gyroradius. The orderings $\epsilon_\eta \ll 1$, $\epsilon_\chi \ll 1$, and $\epsilon_D \ll 1$ ensure that the island chain is sufficiently wide that the perpendicular diffusion of magnetic flux, momentum, and particles are not dominant effects in the rescaled resonant response equations. We shall also assume that $w/w_c \sim \mathcal{O}(1)$. In other words, the island chain is sufficiently wide for the parallel diffusion of particles to compete with perpendicular diffusion.[12]

Suppose that

$$\frac{\partial}{\partial T} \sim \mathcal{O}(\epsilon_d^3). \tag{60}$$

Let us expand the various fields in our model as follows:

$$\Psi = \Psi_0 + \epsilon_d^2 \Psi_2 + \epsilon_d^3 \Psi_3 + \cdots, \tag{61}$$

$$\Phi = \Phi_0 + \epsilon_d^2 \Phi_2 + \epsilon_d^3 \Phi_3 + \cdots, \tag{62}$$

$$\mathcal{N} = \mathcal{N}_0 + \epsilon_d^2 \mathcal{N}_2 + \epsilon_d^3 \mathcal{N}_3 + \cdots, \tag{63}$$

$$\mathcal{V} = \mathcal{V}_0. + \epsilon_d \mathcal{V}_1 + \cdots, \tag{64}$$

$$\mathcal{J} = \mathcal{J}_0 + \epsilon_d \mathcal{J}_1 + \cdots. \tag{65}$$

Here, Ψ_0, Ψ_1, et cetera are assumed to be $\mathcal{O}(1)$.

C. Lowest-Order Solution

To lowest order in ϵ_d, Eq. (42) yields

$$\partial_X^2 \Psi_0 = 1. \tag{66}$$

Solving this equation subject to the boundary condition (52), we obtain

$$\Psi_0 = \Omega(X, \zeta) \equiv \frac{X^2}{2} + \cos\zeta. \tag{67}$$

Thus, we conclude that, to lowest order in our expansion, the magnetic flux-surfaces in the island region map out a helical magnetic island chain. The island O-points correspond to $\Omega = -1$ and $\zeta = (2k-1)\,\pi$ (where k is an integer), the X-points correspond to $\Omega = +1$ and $\zeta = 2k\,\pi$, and the magnetic separatrix corresponds to $\Omega = +1$.

To lowest order in ϵ_d, Eq. (41) yields

$$\{\mathcal{N}_0, \Omega\} = 0, \tag{68}$$

where use has been made of Eq. (67). Given that $\mathcal{N}$ is an odd function of X, it follows that

$$\mathcal{N}_0(X, \zeta, T) = s\,\mathcal{N}_{(0)}(\Omega, T). \tag{69}$$

We conclude that parallel ion acoustic waves, whose dynamics are described by Eq. (41), smooth out any variations in the lowest-order electron number density, $\mathcal{N}_0$, around magnetic flux-surfaces. By symmetry, $\mathcal{N}_0 = 0$ inside the magnetic separatrix of the island chain. In other words, the electron number density profile (and, by implication, the electron and ion temperature profile) is completely flattened inside the separatrix.

It is helpful to define

$$L(\Omega, T) = \frac{d\mathcal{N}_{(0)}}{d\Omega}. \tag{70}$$

It follows that $L(\Omega < 1, T) = 0$. Furthermore, Eqs. (54) and (67) imply that

$$L(\Omega \to \infty, T) = \frac{1}{\sqrt{2\,\Omega}}. \tag{71}$$

To lowest order in ϵ_d, Eq. (38) gives

$$\{\Phi_0, \Omega\} = 0, \tag{72}$$

where use has been made of Eqs. (67) and (69). Given that Φ is an odd function of X, it follows that

$$\Phi_0(X, \zeta, T) = s\,\Phi_{(0)}(\Omega, T). \tag{73}$$

We conclude that the lowest-order normalized electrostatic potential, Φ_0, is constant on magnetic flux-surfaces. By symmetry, $\Phi_0 = 0$ inside the magnetic separatrix of the island chain. In other words, the electrostatic potential profile is completely flattened inside the separatrix.

It is helpful to define

$$M(\Omega, T) = \frac{d\Phi_{(0)}}{d\Omega}. \tag{74}$$

It follows that $M(\Omega < 1, T) = 0$. Furthermore, Eqs. (53) and (67) imply that

$$M(\Omega \to \infty, T) = \frac{v}{\sqrt{2\,\Omega}} + v'. \tag{75}$$

To lowest order in ϵ_d, Eq. (40) yields

$$\{\mathcal{V}_0, \Omega\} = 0, \tag{76}$$

where use has been made of Eqs. (67), (69), and (73). Given that $\mathcal{V}$ is an even function of X, we can write

$$\mathcal{V}_0(X, \zeta, T) = \mathcal{V}_{(0)}(\Omega, T). \tag{77}$$

In other words, the lowest-order normalized parallel ion velocity, $\mathcal{V}_0$, is also constant on magnetic flux-surfaces.

Finally, to lowest order in ϵ_d, Eq. (39) yields

$$\{\mathcal{J}_0, \Omega\} = -\partial_X \left\{ \Phi_0 + \frac{\mathcal{N}_0}{1+\tau}, \partial_X \Phi_0 \right\} = \frac{1}{2} \left\{ d_\Omega \left[M \left(M + \frac{L}{1+\tau} \right) \right] X^2, \Omega \right\}, \tag{78}$$

where use has been made of Eqs. (67), (70), and (74). Moreover, $d_\Omega \equiv d/d\Omega$.

D. Flux-Surface Average Operator

The *flux-surface average operator*, $\langle \cdots \rangle$, is defined [13]

$$\langle A(s, \Omega, \zeta) \rangle \equiv \begin{cases} \int_{\zeta_0}^{2\pi-\zeta_0} \frac{A(s,\Omega\,\zeta)+A(-s,\Omega,\zeta)}{2\,[2\,(\Omega-\cos\zeta)]^{1/2}} \frac{d\zeta}{2\pi} & 0 \le \Omega \le 1 \\ \oint \frac{A(s,\Omega\,\zeta)}{[2\,(\Omega-\cos\zeta)]^{1/2}} \frac{d\zeta}{2\pi} & \Omega > 1 \end{cases}, \tag{79}$$

where $\zeta_0 = \cos^{-1}(\Omega)$ and $0 \le \zeta_0 \le \pi$. It follows that $\langle \{A, \Omega\} \rangle = 0$ for any $A(s, \Omega, \zeta, T)$. It is helpful to define $\tilde{A} \equiv A - \langle A \rangle / \langle 1 \rangle$. It follows that $\langle \tilde{A} \rangle = 0$ for any $A(s, \Omega, \zeta, T)$.

Equation (78) yields

$$\mathcal{J}_0(\Omega, \zeta, T) = \frac{1}{2} d_\Omega \left[M \left(M + \frac{L}{1+\tau} \right) \right] \widetilde{X^2} + \overline{\mathcal{J}}_0(\Omega, T), \tag{80}$$

where $\overline{\mathcal{J}}_0(\Omega, T)$ is an undetermined flux-surface function.

E. Higher-Order Solution

To lowest order in our expansion scheme, the rescaled resonant response model, (38)–(42), specifies the island solution in terms of four flux-surface functions: $\mathcal{N}_{(0)}(\Omega, T)$, $\Phi_{(0)}(\Omega, T)$, $\mathcal{V}_{(0)}(\Omega, T)$, and $\overline{\mathcal{J}}_0(\Omega, T)$. In order to determine the forms of these four functions, it is necessary to solve the rescaled model to higher order in our expansion. In particular, we need to include the terms that describe the perpendicular diffusion of magnetic flux, ion momentum, and particles in our analysis.

We need to evaluate Eq. (38) to third order in ϵ_d in order to include the perpendicular transport term. Doing so, we obtain

$$\frac{d(\ln \hat{w}^{\,2})}{dT} \cos\zeta = \{F_e, \Omega\} + \epsilon_\beta\, \epsilon_d\, \epsilon_\eta\, \mathcal{J}_0, \tag{81}$$

where

$$F_e(\Omega,\zeta,T) = \epsilon_d^2\,(\Phi_2+\epsilon_d\,\Phi_3) - \epsilon_d^2\left(\frac{\tau}{1+\tau}\right)(\mathcal{N}_2+\epsilon_d\,\mathcal{N}_3) - \epsilon_d^2\,s\,Y_e(\Psi_2+\epsilon_d\,\Psi_3), \tag{82}$$

and use has been made of Eqs. (61)–(63), (69), (70), (73), and (74). The flux-surface average of Eq. (81) yields

$$\overline{\mathcal{J}}_0(\Omega,T) = \frac{1}{\epsilon_\beta\,\epsilon_d\,\epsilon_\eta}\,\frac{d(\ln\hat{w}^2)}{dT}\,\frac{\langle\cos\zeta\rangle}{\langle 1\rangle}. \tag{83}$$

Hence, we can write

$$\mathcal{J}_0(\Omega,\zeta) = \frac{1}{2}\,d_\Omega(M\,Y_i)\,\widetilde{X^2} + \frac{1}{\epsilon_\beta\,\epsilon_d\,\epsilon_\eta}\,\frac{d(\ln\hat{w}^2)}{dT}\,\frac{\langle\cos\zeta\rangle}{\langle 1\rangle}, \tag{84}$$

where use has been made of Eq. (80). Here,

$$Y_i = M + \left(\frac{1}{1+\tau}\right)L. \tag{85}$$

The first term on the right-hand side of the previous equation represents the parallel return current driven by the perpendicular *polarization current* associated with the acceleration of the ion fluid around the magnetic separatrix of the island chain.[14,15] In fact, it is easily seen that if the ion fluid could pass freely through the separatrix (i.e., $\Phi_0 \propto X$ and $\mathcal{N}_0 \propto X$) then the term in question would be zero. The second term on the right-hand side of the previous equation represents the parallel current driven inductively when the reconnected flux at the resonant surfaces varies in time.

We need to evaluate Eq. (40) to first order in ϵ_d in order to include the perpendicular transport term. Doing so, we obtain

$$0 = \epsilon_d\,\{\mathcal{V}_1,\Omega\} + \epsilon_D\,(X^2\,d_\Omega L + L), \tag{86}$$

where use has been made of Eqs. (67), (69), and (70). The flux-surface average of the previous equation yields

$$\langle X^2\rangle\,d_\Omega L + \langle 1\rangle\,L = 0. \tag{87}$$

We can solve the previous equation, subject to the boundary condition (71), to give

$$L(\Omega) = \begin{cases} 0 & -1 \le \Omega \le 1 \\ 1/\langle X^2\rangle & \Omega > 1 \end{cases}. \tag{88}$$

Here, we have taken into account the previously mentioned fact that $L = 0$ within the island separatrix. Note that L is discontinuous across the island separatrix, which implies that the density gradient—and, hence, the diamagnetic velocity—are both also discontinuous across the separatrix. Of course, there is not a real discontinuity. Given that the discontinuity in L is mandated by the ordering $\epsilon_d = (\rho_\beta/w)^2 \ll 1$, which requires L to be a flux-surface function—and, hence, zero, by symmetry, inside the separatrix—it is plausible that the discontinuity is resolved by a thin layer of characteristic thickness ρ_β on the island separatrix.

We need to evaluate Eq. (39) to first order in ϵ_d in order to include the perpendicular transport term. Doing so, we obtain

$$0 = \epsilon_d \, \{\mathcal{J}_1, \Omega\} + \epsilon_\chi \, X \, d_\Omega \big[d_\Omega\big(X^3 \, d_\Omega Y_i\big)\big] \,. \tag{89}$$

where use has been made of Eqs. (67), (69), (70), (73), and (74). The flux-surface average of the previous equation yields

$$d_\Omega^2\big(\langle X^4\rangle \, d_\Omega Y_i\big) = 0. \tag{90}$$

We can solve the previous equation, subject to the boundary conditions (71) and (75), to give[5]

$$Y_i(\Omega, T) = \frac{v'(T)}{\int_1^\infty \frac{d\Omega}{\langle X^4\rangle}} \begin{cases} 0 & -1 \leq \Omega \leq 1 \\ \int_1^\Omega \frac{d\Omega'}{\langle X^4\rangle} & \Omega > 1 \end{cases} . \tag{91}$$

Here, we have rejected, as unphysical, the solution that blows up as $\Omega^{1/2}$ as $\Omega \to \infty$. We have also made use of the fact that $Y_i = 0$ within the magnetic separatrix of the island chain. Finally, we have demanded that the ion fluid velocity—and, hence, the function Y_i—be continuous across the separatrix, because the ion fluid possesses finite perpendicular viscosity. Note, however, that the discontinuity in the function $L(\Omega)$ across the separatrix [see Equation (88)] implies that the electron and MHD fluid velocities are discontinuous across the separatrix. As previously mentioned, these discontinuities are resolved in a layer of characteristic thickness ρ_β.

We need to evaluate Eq. (41) to second order in ϵ_d in order to include the perpendicular transport term. Doing so, we obtain

$$0 = \epsilon_d \, \{\mathcal{V}_1, \Omega\} + \epsilon_\chi \, (X^2 \, d_\Omega \mathcal{V}_{(0)} + \mathcal{V}_{(0)}), \tag{92}$$

where use has been made of Eqs. (67) and (77). The flux-surface average of the previous equation yields

$$\langle X^2\rangle \, d_\Omega \mathcal{V}_{(0)} + \langle 1\rangle \, \mathcal{V}_{(0)} = 0. \tag{93}$$

The previous equation can be solved, subject to the boundary condition (55), to give

$$\mathcal{V}_{(0)}(\Omega, T) = 0. \tag{94}$$

Hence, we conclude that the lowest-order ion parallel flow is unaffected by the presence of the island chain.

F. Asymptotic Matching

Now that we have solved the resonant response model in the immediate vicinity of the magnetic island chain, it is necessary to asymptotically match this solution to the solution in the outer region. It is easily seen from Eqs. (4) and (5) that

$$\mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = \frac{2}{\hat{w}}\int_{-\infty}^{\infty}\oint \partial_X^2\Psi\,\cos\zeta\,\frac{d\zeta}{2\pi}\,dX, \tag{95}$$

$$\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = -\frac{2}{\hat{w}}\int_{-\infty}^{\infty}\oint \partial_X^2\Psi\,\sin\zeta\,\frac{d\zeta}{2\pi}\,dX. \tag{96}$$

However, according to Eqs. (42) and (65),

$$\partial_X^2\Psi = 1 + \epsilon_\beta\,\epsilon_d\,\mathcal{J}_0 + \epsilon_\beta\,\epsilon_d^2\,\mathcal{J}_1. \tag{97}$$

Moreover, it is clear from Eqs. (84) and (89) that $\mathcal{J}_0$ has the symmetry of $\cos\zeta$, whereas $\mathcal{J}_1$ has the symmetry of $\sin\zeta$. Hence, we deduce that

$$\mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = \frac{2\,\epsilon_\beta\,\epsilon_d}{\hat{w}}\int_{-\infty}^{\infty}\oint \mathcal{J}_0\,\cos\zeta\,\frac{d\zeta}{2\pi}\,dX, \tag{98}$$

$$\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = -\frac{2\,\epsilon_\beta\,\epsilon_d^2}{\hat{w}}\int_{-\infty}^{\infty}\oint \mathcal{J}_1\,\sin\zeta\,\frac{d\zeta}{2\pi}\,dX. \tag{99}$$

The previous two equations can also be written

$$\mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = \frac{4\,\epsilon_\beta\,\epsilon_d}{\hat{w}}\int_{-1}^{\infty}\langle\mathcal{J}_0\,\cos\zeta\rangle\,d\Omega, \tag{100}$$

$$\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = -\frac{4\,\epsilon_\beta\,\epsilon_d^2}{\hat{w}}\int_{-1}^{\infty}\langle\mathcal{J}_1\,\sin\zeta\rangle\,d\Omega = -\frac{4\,\epsilon_\beta\,\epsilon_d^2}{\hat{w}}\int_{-1}^{\infty}\langle X\,\{\mathcal{J}_1,\Omega\}\rangle\,d\Omega. \tag{101}$$

Equations (89) and (101) can be combined to give [5,6]

$$\begin{aligned}\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) &= \frac{4\,\epsilon_\beta\,\epsilon_d\,\epsilon_\chi}{\hat{w}}\int_{1}^{\infty}\left\langle X^2\,d_\Omega\big[d_\Omega(X^3\,d_\Omega Y_i)\big]\right\rangle d\Omega\\ &= \frac{4\,\epsilon_\beta\,\epsilon_d\,\epsilon_\chi}{\hat{w}}\int_{1}^{\infty} d_\Omega\big(-\langle X\rangle\,Y_i + 2\,\langle X^3\rangle\,d_\Omega Y_i + \langle X^5\rangle\,d_\Omega^2 Y_i\big)\,d\Omega\\ &= \frac{4\,\epsilon_\beta\,\epsilon_d\,\epsilon_\chi}{\hat{w}}\lim_{\Omega\rightarrow\infty}\big(-\langle X\rangle\,Y_i + 2\,\langle X^3\rangle\,d_\Omega Y_i + \langle X^5\rangle\,d_\Omega^2 Y_i\big),\end{aligned} \tag{102}$$

where use has been made of the facts that Y_i is zero inside the island separatrix, and Y_i is continuous across the separatrix. Combining the previous equation with Eq. (91), we obtain [5]

$$\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = -\frac{4\,\epsilon_\beta\,\epsilon_d\,\epsilon_\chi\,v'}{\hat{w}}. \tag{103}$$

Equations (84) and (100) can be combined to give

$$\mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = \frac{2\,\epsilon_\beta\,\epsilon_d}{\hat{w}}\int_{1+}^{\infty} d_\Omega(M\,Y_i)\left\langle\widetilde{X^2}\,\cos\zeta\right\rangle d\Omega + \frac{4}{\epsilon_\eta\,\hat{w}}\,\frac{d(\ln\hat{w}^2)}{dT}\int_{-1}^{\infty}\frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\,d\Omega. \tag{104}$$

Making use of Eqs. (88) and (91), the previous equation yields

$$\begin{aligned}\mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = &-\frac{2\,\epsilon_\beta\,\epsilon_d\,v'}{(1+\tau)\,\hat{w}}\int_1^{\infty} d_\Omega\left(\frac{F_i}{\langle X^2\rangle}\right)\left\langle\widetilde{X^2}\,\cos\zeta\right\rangle d\Omega\\ &+\frac{2\,\epsilon_\beta\,\epsilon_d\,v'^{\,2}}{\hat{w}}\int_1^{\infty} d_\Omega\big(F_i^{\,2}\big)\left\langle\widetilde{X^2}\,\cos\zeta\right\rangle d\Omega\\ &+\frac{4}{\epsilon_\eta\,\hat{w}}\,\frac{d(\ln\hat{w}^2)}{dT}\int_{-1}^{\infty}\frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\,d\Omega,\end{aligned} \tag{105}$$

where

$$F_i(\Omega) = \int_1^{\Omega}\frac{d\Omega'}{\langle X^4\rangle}\Big/\int_1^{\infty}\frac{d\Omega}{\langle X^4\rangle} \tag{106}$$

Finally, Eqs. (103) and (105) can be combined to give [5,6]

$$\begin{aligned}\mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = &\,I_1\,\tau_R\,\frac{d}{dt}\left(\frac{4\,w}{r_s}\right) - \frac{I_2}{4}\left(\frac{c_\beta}{1+\tau}\right)\left(\frac{\rho_\beta}{r_s}\right)\left(\frac{\tau_\varphi}{\tau_H}\right)\left(\frac{w}{r_s}\right)^2\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right)\\ &-\frac{I_3}{16}\left(\frac{\tau_\varphi}{\tau_H}\right)^2\left(\frac{w}{r_s}\right)^7\left[\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right)\right]^2,\end{aligned} \tag{107}$$

where

$$I_1 = 2\int_{-1}^{\infty}\frac{\langle\cos\zeta\rangle^2}{\langle 1\rangle}\,d\Omega = 0.8227, \tag{108}$$

$$I_2 = \int_1^{\infty} d_\Omega\left(\frac{F_i}{\langle X^2\rangle}\right)\left(\langle X^4\rangle - \frac{\langle X^2\rangle^2}{\langle 1\rangle}\right)d\Omega = 0.1955, \tag{109}$$

$$I_3 = \int_1^{\infty} d_\Omega\big(F_i^{\,2}\big)\left(\langle X^4\rangle - \frac{\langle X^2\rangle^2}{\langle 1\rangle}\right)d\Omega = 0.4711. \tag{110}$$

V. ISOLATED MAGNETIC ISLAND CHAINS

A. Introduction

Suppose, for the sake of argument, that the wall surrounding the plasma is perfectly conducting (i.e. $\tau_{LR} \rightarrow \infty$). In this case, Eq. (11) yields

$$\frac{\Delta\Psi_s}{\Psi_s} = \Delta_{pw}, \tag{111}$$

where Δ_{pw} is the (real) perfect-wall tearing stability index. It is clear from the previous equation that

$$\mathrm{Im}\left(\frac{\Delta\Psi_s}{\Psi_s}\right) = 0, \tag{112}$$

which implies that zero electromagnetic torque is exerted at the resonant surface. In this respect, the magnetic island chain that develops at the resonant surface can be termed 'isolated'.

B. Rutherford Island Width Evolution Equation

Equation (111) implies that

$$\mathrm{Re}\left(\frac{\Delta\hat{\Psi}_s}{\hat{\Psi}_s}\right) = \Delta_{pw}. \tag{113}$$

Equations (26), (107), (112), and (113) yield the *Rutherford island width evolution equation*:[13]

$$I_1\,\tau_R\,\frac{d}{dt}\left(\frac{W}{r_s}\right) = \Delta_{pw}. \tag{114}$$

According to the Rutherford equation, if the perfect-wall tearing stability index is positive then the width of the magnetic island chain at the resonant surface grows algebraically on the resistive evolution timescale, τ_R.

C. Saturated Island Width

Equation (114) gives the impression that if $\Delta_{pw} > 0$ then the width of the magnetic island chain grows without limit. In fact, this is not the case. Instead, the width of the island chain eventually stops growing, and the tearing mode attains a saturated steady state. In order to model this effect, it is necessary to perform the asymptotic matching between the inner and outer regions to higher order, taking into account the finite width of the island chain.[17–20]

For the case of an island chain that is sufficiently wide to flatten the density and temperature profiles within its magnetic separatrix, the appropriate saturation theory is given in Ref. 21. According to this theory, $\varDelta_{pw}$ in Eq. (114) must be replaced by

$$\varDelta_{pw}(0) - \left(0.8\,\alpha_s^2 - 0.27\,\beta_s - 0.09\,\alpha_s\right)\frac{W}{r_s}, \tag{115}$$

where $\varDelta_{pw}(0)$ denotes the zero-island-width tearing stability index, α_s is defined in Eq. (4), and

$$\beta_s = -\left(\frac{r\,q\,J_z''}{s}\right)_{r=r_s}. \tag{116}$$

Equations (114) and (115) can be combined to give

$$I_1\,\tau_R\,\frac{d}{dt}\!\left(\frac{W}{r_s}\right) = \varDelta_{pw}(0) - \left(0.8\,\alpha_s^2 - 0.27\,\beta_s - 0.09\,\alpha_s\right)\frac{W}{r_s}. \tag{117}$$

If we define

$$W_{\rm sat} = \frac{\varDelta_{pw}(0)\,r_s}{0.8\,\alpha_s^2 - 0.27\,\beta_s - 0.09\,\alpha_s}, \tag{118}$$

$$\tau_{\rm sat} = \frac{I_1\,\tau_R}{\varDelta_{pw}(0)}\,\frac{W_{\rm sat}}{r_s}, \tag{119}$$

then Eq. (117) reduces to

$$\tau_{\rm sat}\,\frac{d}{dt}\!\left(\frac{W}{W_{\rm sat}}\right) = 1 - \frac{W}{W_{\rm sat}}. \tag{120}$$

The previous equation can be solved to give

$$W(t) = W_{\rm sat}\left(1 - {\rm e}^{-t/\tau_{\rm sat}}\right), \tag{121}$$

assuming that the island width is zero at $t = 0$. It follows that the width of the island chain does not grow without limit, but instead eventually attains the saturated value $W_{\rm sat}$. Moreover, the time required to attain saturation, $\tau_{\rm sat}$, is of order $\tau_R\,(W_{\rm sat}/r_s)$.

D. Natural Frequency

Equations (103) and (112) imply that

$$v' = 0 \tag{122}$$

for an isolated magnetic island chain. It follows from Equations (57), (75), (85), (88), (91), and (122) yield $v = -1/(1+\tau)$, or

$$\omega = \omega_{\perp\,i}, \tag{123}$$

where

$$\omega_{\perp\,i} = \omega_E + \omega_{\ast\,i}, \tag{124}$$

and

$$\omega_{\ast\,i} = -\left(\frac{1}{1+\tau}\right)\omega_\ast. \tag{125}$$

We conclude that, unlike a linear tearing mode, which is convected by the electron fluid at the resonant surface,[4] a nonlinear tearing mode is convected by the ion fluid.[5]

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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