Cylindrical Tearing Mode Analysis

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1 Normalization

All lengths are normalized to R_0. All magnetic field-strengths are normalized to B_0. All current densities are normalized to $B_0/(R_0\,\mu_0)$.

2 Coordinates

Adopt cylindrical coordinates r, θ, z.

3 Plasma Equilibrium

The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\Psi(r)\times\mathbf{e}_z + \mathbf{e}_z. \tag{1}$$

So,

$$B_r = 0, \tag{2}$$

$$B_\theta = -\frac{d\Psi}{dr}, \tag{3}$$

$$B_z = 1. \tag{4}$$

The safety-factor profile is

$$q(r) = \frac{r\,B_z}{B_\theta} = \frac{r}{B_\theta}. \tag{5}$$

The equilibrium current density is written

$$\mathbf{J} = J_z(r)\,\mathbf{e}_z, \tag{6}$$

where

$$J_z = \frac{1}{r}\,\frac{d(r\,B_\theta)}{dr} = \frac{1}{r}\,\frac{d}{dr}\left(\frac{r^{\,2}}{q}\right) = \frac{2-s}{q}, \tag{7}$$

and

$$s(r) = \frac{d\ln q}{dr}. \tag{8}$$

4 Cylindrical Tearing Mode Equation

Let

$$\mathbf{b} = \nabla \left[\psi(r)\, \mathrm{e}^{\,\mathrm{i}\,(m\,\theta - n\,z - \omega\,t)}\right] \times \mathbf{e}_z \tag{9}$$

$$\mathbf{j} \simeq -\left[\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{d\psi}{dr}\right) - \frac{m^{\,2}}{r^{\,2}}\,\psi\right] \mathrm{e}^{\,\mathrm{i}\,(m\,\theta - n\,z - \omega\,t)}\,\mathbf{e}_z, \tag{10}$$

be the perturbed magnetic field and current density, respectively. Here, we have assumed that $r \ll 1$. Linearized force balance yields

$$(\mathbf{J}\cdot\nabla)\,\mathbf{b} + (\mathbf{j}\cdot\nabla)\,\mathbf{B} - (\mathbf{B}\cdot\nabla)\,\mathbf{j} - (\mathbf{b}\cdot\nabla)\,\mathbf{J} = 0. \tag{11}$$

The z component of the previous equation gives the cylindrical tearing mode equation:

$$\frac{d^{\,2}\psi}{dr^{\,2}} + \frac{1}{r}\,\frac{d\psi}{dr} - \frac{m^{\,2}}{r^{\,2}}\,\psi - \frac{J_z'\,\psi}{r\,(1/q - n/m)} = 0, \tag{12}$$

where $' \equiv d/dr$. In the following, $q(r)$ and $J_z(r)$ are taken from the EQDSK equilibrium, rather than being related according to Eq. (7).

5 Behavior in Vicinity of Rational Surface

Suppose that $q(r_s) = m/n$. Let $\rho = r - r_s$. Expanding the previous equation about $r = r_s$, we obtain

$$\frac{d^{\,2}\psi}{d\rho^{2}} + \frac{1}{r_s}\left(1 - \frac{\rho}{r_s}\right)\frac{d\psi}{d\rho} - \frac{m^{\,2}}{r_s^{\,2}}\left(1 - \frac{2\,\rho}{r_s}\right)\psi - \left(\frac{\alpha}{\rho} + \beta\right)\psi = 0, \tag{13}$$

where

$$\alpha = -\left(\frac{q\,J_z'}{s}\right)_{r=r_s}, \tag{14}$$

$$\beta = \alpha\left(\frac{J_z''}{J_z'} + \frac{s-1}{r_s} - \frac{q''}{2\,q'}\right)_{r=r_s}. \tag{15}$$

We obtain

$$\begin{aligned}\psi(\rho) = \Psi\Bigg[1 &+ \left(\frac{\Sigma' \pm \Delta'}{2}\right)\rho \\ &+ \left\{\alpha\left(\frac{\Sigma' \pm \Delta'}{4}\right)\left(1 - \frac{1}{\alpha\,r_s}\right) + \frac{1}{2}\left[\frac{m^{\,2}}{r_s^{\,2}} + \beta - \frac{\alpha^{\,2}}{2}\left(3 - \frac{1}{\alpha\,r_s}\right)\right]\right\}\rho^{\,2} \\ &+ \alpha\left\{\rho + \frac{\alpha}{2}\left(1 - \frac{1}{\alpha\,r_s}\right)\rho^{\,2}\right\}\ln|\rho| + \mathcal{O}(\rho^{\,3})\Bigg].\end{aligned} \tag{16}$$

Here, the plus sign corresponds to $\rho>0$, whereas the minus sign corresponds to $\rho<0$. It follows that

$$\begin{aligned}\psi'(\rho) = \Psi\Bigg[&\frac{\Sigma'\pm\Delta'}{2}+\alpha\\ &+\left\{\alpha\left(\frac{\Sigma'\pm\Delta'}{2}\right)\left(1-\frac{1}{\alpha\,r_s}\right)+\frac{m^2}{r_s^2}+\beta-\frac{\alpha^2}{2}\left(3-\frac{1}{\alpha\,r_s}\right)+\frac{\alpha^2}{2}\left(1-\frac{1}{\alpha\,r_s}\right)\right\}\rho\\ &+\alpha\,\ln|\rho|+\alpha^2\left(1-\frac{1}{\alpha\,r_s}\right)\rho\,\ln|\rho|+{\cal O}(\rho^2)\Bigg]. \end{aligned}\tag{17}$$

Now,

$$\lambda_+\equiv\left.\frac{\psi'}{\psi}\right|_{\rho=\delta}=\frac{a_{1+}+b_{1+}\,(\Sigma'+\Delta')}{a_{2+}+b_{2+}\,(\Sigma'+\Delta')},\tag{18}$$

where

$$a_{1+}=\alpha\,(1+\ln\delta)+\left\{\frac{m^2}{r_s^2}+\beta-\frac{\alpha^2}{2}\left(3-\frac{1}{\alpha\,r_s}\right)+\alpha^2\left(1-\frac{1}{\alpha\,r_s}\right)\left(\frac{1}{2}+\ln\delta\right)\right\}\delta,\tag{19}$$

$$b_{1+}=\frac{1}{2}+\frac{\alpha}{2}\left(2-\frac{1}{\alpha\,r_s}\right)\delta,\tag{20}$$

$$a_{2+}=1+\alpha\,\delta\,\ln\delta,\tag{21}$$

$$b_{2+}=\frac{\delta}{2}.\tag{22}$$

It follows that

$$\Sigma'+\Delta'=\frac{a_{2+}\,\lambda_+-a_{1+}}{b_{1+}-b_{2+}\,\lambda_+}.\tag{23}$$

Likewise,

$$\lambda_-\equiv\left.\frac{\psi'}{\psi}\right|_{\rho=-\delta}=\frac{a_{1-}+b_{1-}\,(\Sigma'-\Delta')}{a_{2-}+b_{2-}\,(\Sigma'-\Delta')},\tag{24}$$

where

$$a_{1-}=\alpha\,(1+\ln\delta)-\left\{\frac{m^2}{r_s^2}+\beta-\frac{\alpha^2}{2}\left(3-\frac{1}{\alpha\,r_s}\right)+\alpha^2\left(1-\frac{1}{\alpha\,r_s}\right)\left(\frac{1}{2}+\ln\delta\right)\right\}\delta,\tag{25}$$

$$b_{1-}=\frac{1}{2}-\frac{\alpha}{2}\left(2-\frac{1}{\alpha\,r_s}\right)\delta,\tag{26}$$

$$a_{2-}=1-\alpha\,\delta\,\ln\delta,\tag{27}$$

$$b_{2-}=-\frac{\delta}{2}.\tag{28}$$

It follows that

$$\Sigma'-\Delta'=\frac{a_{2-}\,\lambda_--a_{1-}}{b_{1-}-b_{2-}\,\lambda_-}.\tag{29}$$

Hence,

$$\Sigma' = \frac{1}{2}\left(\frac{a_{2+}\,\lambda_+ - a_{1+}}{b_{1+} - b_{2+}\,\lambda_+} + \frac{a_{2-}\,\lambda_- - a_{1-}}{b_{1-} - b_{2-}\,\lambda_-}\right), \tag{30}$$

$$\Delta' = \frac{1}{2}\left(\frac{a_{2+}\,\lambda_+ - a_{1+}}{b_{1+} - b_{2+}\,\lambda_+} - \frac{a_{2-}\,\lambda_- - a_{1-}}{b_{1-} - b_{2-}\,\lambda_-}\right). \tag{31}$$

6 Solution Launched from Magnetic Axis

Let

$$\psi(r) = r^{\,m}\,f(r). \tag{32}$$

It follows that

$$\psi'(r) = m\,r^{\,m-1}\,f + r^{\,m}\,f'. \tag{33}$$

The cylindrical tearing mode equation transforms into

$$f'' + (1+2\,m)\,\frac{f'}{r} - \frac{J_z'\,f}{r\,(1/q - n/m)} = 0. \tag{34}$$

The well-behaved solution launched from the magnetic axis is such that

$$f(r) = 1 + \left[\frac{c}{4\,(1+m)}\right] r^{\,2}, \tag{35}$$

where

$$c = \left(\frac{J_z''}{1/q - n/m}\right)_{r=0}. \tag{36}$$

7 Solution Launched from Plasma Boundary

Suppose that the plasma boundary lies at $r = a$. Let

$$\psi(r) = r^{-m}\,g(r). \tag{37}$$

It follows that

$$\psi'(r) = -m\,r^{-m-1}\,g + r^{-m}\,g'. \tag{38}$$

The cylindrical tearing mode equation transforms into

$$g'' + (1-2\,m)\,\frac{g'}{r} - \frac{J_z'\,g}{r\,(1/q - n/m)} = 0. \tag{39}$$

In the absence of a wall, or external currents flowing in non-axisymmetric field-coils, the well-behaved solution launched from the plasma boundary is such that

$$g(a) = 1, \tag{40}$$

$$g'(a) = 0. \tag{41}$$

8 Resistive Wall

Suppose that a thin resistive wall of radius $r_w > a$, uniform thickness δ_w, and uniform resistivity η_w, is located outside the plasma. We can now launch two different solutions from the plasma boundary. The *no wall* solution is such that

$$g_{nw}(a) = 1, \tag{42}$$

$$g_{nw}'(a) = 0. \tag{43}$$

Let the associated Σ' and Δ' values be denoted Σ'_{nw} and Δ'_{nw}, respectively. The *perfect wall* solution is such that

$$g_{pw}(a) = 1, \tag{44}$$

$$g_{pw}'(a) = -\frac{2\,m/a}{(r_w/a)^{\,2m}-1}. \tag{45}$$

Let the associated Σ' and Δ' values be denoted Σ'_{pw} and Δ'_{pw}, respectively.

Asymptotic matching in the presence of the wall yields

$$\Delta\Psi_s = E_{ss}\,\Psi_s + E_{sw}\,\Psi_w, \tag{46}$$

$$\Delta\Psi_w = E_{ww}\,\Psi_w + E_{sw}\,\Psi_s. \tag{47}$$

Here, $\Psi_s \equiv \psi(r_s)$, $\Psi_w \equiv \psi(r_w)$, and

$$\Delta\Psi_s \equiv \left[r\,\frac{d\psi}{dr}\right]_{r_s-}^{r_s+}, \tag{48}$$

$$\Delta\Psi_w \equiv \left[r\,\frac{d\psi}{dr}\right]_{r_w-}^{r_w+}. \tag{49}$$

Moreover, E_{ss}, E_{sw}, and E_{ww} are real parameters. We can make the identifications

$$\hat{\Delta}'_{pw} = \left.\frac{\Delta\Psi_s}{\Psi_s}\right|_{\Psi_w=0} = E_{ss}, \tag{50}$$

$$\hat{\Delta}'_{nw} = \left.\frac{\Delta\Psi_s}{\Psi_s}\right|_{\Delta\Psi_w=0} = E_{ss} - \frac{E_{sw}^{\,2}}{E_{ww}}, \tag{51}$$

$$E_{sw} = \frac{2m\,(r_s/r_w)^m}{[1-(a/r_w)^{2m}]\,g_{pw}(r_s)}, \tag{52}$$

where $\hat{\Delta}'_{pw} \equiv r_s\,\Delta'_{pw}$, et cetera. Hence,

$$E_{ww} = -\frac{E_{sw}^{\,2}}{\hat{\Delta}'_{nw} - \hat{\Delta}'_{pw}}. \tag{53}$$

Standard analysis reveals that

$$\Delta\Psi_w = -\mathrm{i}\,\omega\,\tau_w\,\Psi_w, \tag{54}$$

$$\int_{r_w-}^{r_w+} j_z\,dr = \frac{\mathrm{i}\,\omega\,\tau_w\,\Psi_w\,\mathrm{e}^{\,\mathrm{i}\,(m\,\theta-n\,z-\omega\,t)}}{\mu_0\,r_w}, \tag{55}$$

where

$$\tau_w = \frac{\mu_0\,\delta_w\,r_w}{\eta_w}. \tag{56}$$

Let

$$\hat{\Delta}'(\omega) = \mathrm{Re}\left(\frac{\Delta\Psi_s}{\Psi_s}\right). \tag{57}$$

It follows that

$$\hat{\Delta}'(\omega) = \hat{\Delta}'_{nw} - (\hat{\Delta}'_{nw} - \hat{\Delta}'_{pw})\left(\frac{\omega^2}{\omega^{\,2}+\omega_w^{\,2}}\right), \tag{58}$$

where

$$\omega_w = -\frac{E_{ww}}{\tau_w}. \tag{59}$$

The net poloidal torque acting on the rational surface is minus that acting on the wall. Hence,

$$T_\theta = -\frac{1}{2}\,r_w\int_{r_w-}^{r_w+}\oint\int_0^{2\pi\,R_0} j_z\,b_r^\ast\,r\,dr\,d\theta\,dz. \tag{60}$$

It follows that

$$T_\theta = -\frac{2\pi^{\,2}\,R_0}{\mu_0}\,m\,(\Delta'_{nw}-\Delta'_{pw})\left(\frac{\omega\,\omega_w}{\omega^{\,2}+\omega_w^{\,2}}\right)|\Psi_s|^{\,2}. \tag{61}$$