Program FLUX

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1 Coordinate Systems

- Cartesian coordinates: X, Y, Z.

- Cylindrical coordinates: $R \equiv (X^2 + Y^2)^{1/2}$, $\phi \equiv \tan^{-1}(Y/X)$, Z, so

$$|\nabla\phi| = \frac{1}{R}. \tag{1}$$

- Flux coordinates: $r(R, Z)$, $\theta(R, Z)$, ϕ, where

$$(\nabla r \cdot \nabla\theta \times \nabla\phi)^{-1} = \mathcal{J}, \tag{2}$$

$$\mathcal{J} = \frac{r\, R^2}{R_0}, \tag{3}$$

where R_0 is a convenient scale major radius.

2 Useful Identities

Easily demonstrated that:

$$\mathbf{A} = A^r\, \mathcal{J}\, \nabla\theta \times \nabla\phi + A^\theta\, \mathcal{J}\, \nabla\phi \times \nabla r + A^\phi\, \mathcal{J}\, \nabla r \times \nabla\theta, \tag{4}$$

$$\mathbf{A} = A_r\, \nabla r + A_\theta\, \nabla\theta + A_\phi\, \nabla\phi, \tag{5}$$

$$\mathbf{A} \cdot \mathbf{B} = A_r\, B^r + A_\theta\, B^\theta + A_\phi\, B^\phi = A^r\, B_r + A^\theta\, B_\theta + A^\phi\, B_\phi, \tag{6}$$

$$(\mathbf{A} \times \mathbf{B})_r = \mathcal{J}\, (A^\theta\, B^\phi - A^\phi\, B^\theta), \tag{7}$$

$$(\mathbf{A} \times \mathbf{B})_\theta = \mathcal{J}\, (A^\phi\, B^r - A^r\, B^\phi), \tag{8}$$

$$(\mathbf{A} \times \mathbf{B})_\phi = \mathcal{J}\, (A^r\, B^\theta - A^\theta\, B^r), \tag{9}$$

$$\mathcal{J}\, (\mathbf{A} \times \mathbf{B})^r = A_\theta\, B_\phi - A_\phi\, B_\theta, \tag{10}$$

$$\mathcal{J}\, (\mathbf{A} \times \mathbf{B})^\theta = A_\phi\, B_r - A_r\, B_\phi, \tag{11}$$

$$\mathcal{J}\,(\mathbf{A}\times\mathbf{B})^{\phi}=A_r\,B_\theta-A_\theta\,B_r, \tag{12}$$

$$\mathcal{J}\,\nabla\cdot\mathbf{A}=\frac{\partial\,(\mathcal{J}\,A^r)}{\partial r}+\frac{\partial\,(\mathcal{J}\,A^\theta)}{\partial\theta}+\frac{\partial\,(\mathcal{J}\,A^\phi)}{\partial\phi}, \tag{13}$$

$$\mathcal{J}\,(\nabla\times\mathbf{A})^r=\frac{\partial A_\phi}{\partial\theta}-\frac{\partial A_\theta}{\partial\phi}, \tag{14}$$

$$\mathcal{J}\,(\nabla\times\mathbf{A})^\theta=\frac{\partial A_r}{\partial\phi}-\frac{\partial A_\phi}{\partial r}, \tag{15}$$

$$\mathcal{J}\,(\nabla\times\mathbf{A})^\phi=\frac{\partial A_\theta}{\partial r}-\frac{\partial A_r}{\partial\theta}. \tag{16}$$

3   Equilibrium Magnetic Field

Equilibrium magnetic field:

$$\mathbf{B}=B_0\,R_0\left[f(r)\,\nabla\phi\times\nabla r+g(r)\,\nabla\phi\right], \tag{17}$$

$$q(r)=\frac{r\,g}{R_0\,f}, \tag{18}$$

where B_0 is a convenient scale magnetic field strength. So,

$$B^r=0, \tag{19}$$

$$B^\theta=B_0\,R_0^2\,\frac{f}{r\,R^2}, \tag{20}$$

$$B^\phi=B_0\,R_0^2\,\frac{q\,f}{r\,R^2}=B_0\,B_0\,\frac{g}{R^2}, \tag{21}$$

$$B_r=-B_0\,r\,f\,\nabla r\cdot\nabla\theta, \tag{22}$$

$$B_\theta=B_0\,r\,f\,|\nabla r|^2, \tag{23}$$

$$B_\phi=B_0\,R_0^2\,\frac{q\,f}{r}=B_0\,R_0\,g, \tag{24}$$

and

$$\mathbf{B}\cdot\nabla=B_0\,R_0^2\,\frac{f}{r\,R^2}\left(\frac{\partial}{\partial\theta}+q\,\frac{\partial}{\partial\phi}\right). \tag{25}$$

4   Equilibrium Current

The relation $\mu_0\,\mathbf{J}=\nabla\times\mathbf{B}$ yields

$$\mu_0\,\mathcal{J}\,J^r=0, \tag{26}$$

$$\mu_0\,\mathcal{J}\,J^\theta=-B_0\,R_0\,\frac{dg}{dr}, \tag{27}$$

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$$\mu_0 \, \mathcal{J} \, J^\phi = B_0 \, \frac{\partial}{\partial r} \left(r \, f \, |\nabla r|^2 \right) + B_0 \, \frac{\partial}{\partial \theta} \left(r \, f \, \nabla r \cdot \nabla \theta \right). \tag{28}$$

5   Grad-Shafranov Equation

The force balance relation

$$\mathbf{J} \times \mathbf{B} = \nabla P \tag{29}$$

yields

$$\mathcal{J} \left(J^\theta \, B^\phi - J^\phi \, B^\theta \right) = \frac{dP}{dr}, \tag{30}$$

which reduces to

$$\frac{f}{r} \, \frac{\partial}{\partial r} \left(r \, f \, |\nabla r|^2 \right) + \frac{f}{r} \, \frac{\partial}{\partial \theta} \left(r \, f \, \nabla r \cdot \nabla \theta \right) + g \, \frac{dg}{dr} + \frac{\mu_0}{B_0^2} \left(\frac{R}{R_0} \right)^2 \frac{dP}{dr} = 0. \tag{31}$$

6   High-q Limit

Now,

$$\nabla \cdot \delta \mathbf{B} = 0 \tag{32}$$

yields

$$\frac{\partial (\mathcal{J} \, \delta B^r)}{\partial r} + \frac{\partial (\mathcal{J} \, \delta B^\theta)}{\partial \theta} + \frac{\partial (\mathcal{J} \, \delta B^\phi)}{\partial \phi} = 0. \tag{33}$$

Furthermore,

$$\delta \mathbf{B} = \delta B_r \, \nabla r + \delta B_\theta \, \nabla \theta + \delta B_\phi \, \nabla \phi. \tag{34}$$

So,

$$\delta B^r = \delta \mathbf{B} \cdot \nabla r = |\nabla r|^2 \, \delta B_r + (\nabla r \cdot \nabla \theta) \, \delta B_\theta, \tag{35}$$

$$\delta B^\theta = \delta \mathbf{B} \cdot \nabla \theta = (\nabla r \cdot \nabla \theta) \, \delta B_r + |\nabla \theta|^2 \, \delta B_\theta, \tag{36}$$

$$\delta B^\phi = \delta \mathbf{B} \cdot \nabla \phi = \frac{\delta B_\phi}{R^2}. \tag{37}$$

Follows that

$$\delta B_r = \left(\frac{1}{|\nabla r|^2} \right) \delta B^r - \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B_\theta, \tag{38}$$

and

$$\delta B^\theta = \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B^r + \left[|\nabla \theta|^2 - \frac{(\nabla r \cdot \nabla \theta)^2}{|\nabla r|^2} \right] \delta B_\theta. \tag{39}$$

But, Eqs. (2) and (3) imply that

$$|\nabla r|^2 \, |\nabla \theta|^2 - (\nabla r \cdot \nabla \theta)^2 = \frac{R_0^2}{r^2 \, R^2}. \tag{40}$$

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Hence,

$$\delta B^{\theta} = \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2}\right) \delta B^r + \left(\frac{R_0^2}{r^2 \, R^2 \, |\nabla r|^2}\right) \delta B_\theta. \tag{41}$$

Assume that high-q limit is equivalent to $\delta \mathbf{B}$ being curl-free. Follows that

$$\frac{\partial \, \delta B_\phi}{\partial \theta} = \frac{\partial \, \delta B_\theta}{\partial \phi}, \tag{42}$$

$$\frac{\partial \, \delta B_r}{\partial \phi} = \frac{\partial \, \delta B_\phi}{\partial r}, \tag{43}$$

$$\frac{\partial \, \delta B_\theta}{\partial r} = \frac{\partial \, \delta B_r}{\partial \theta}. \tag{44}$$

Previous three equations imply that

$$\delta B_\phi \sim \frac{n}{m} \, \delta B_\theta, \; \frac{n}{m} \, r \, \delta B_r \tag{45}$$

and, hence, that

$$\delta B^\phi \sim \frac{n}{m} \left(\frac{r}{R}\right)^2 \delta B^\theta, \; \frac{n}{m} \left(\frac{r}{R}\right)^2 \frac{\delta B^r}{r}, \tag{46}$$

Consequently, final term on right-hand side of (33) is of order $(n/m)^2 \, (r/R)^2$ smaller than other two terms, and, therefore, negligible. Thus, we get

$$\frac{\partial(\mathcal{J} \, \delta B^r)}{\partial r} + \frac{\partial(\mathcal{J} \, \delta B^\theta)}{\partial \theta} = 0. \tag{47}$$

Follows from (41) and previous equation that

$$r \, \frac{\partial}{\partial r} \left(\frac{r \, R^2 \, \delta B^r}{R_0^2}\right) = -\frac{\partial}{\partial \theta} \left[\left(\frac{r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2}\right) \left(\frac{r \, R^2 \, \delta B^r}{R_0^2}\right) + \left(\frac{1}{|\nabla r|^2}\right) \delta B_\theta\right]. \tag{48}$$

Follows from (3), (38) and (44) that

$$r \, \frac{\partial \, \delta B_\theta}{\partial r} = \frac{\partial}{\partial \theta} \left[\left(\frac{R_0^2}{R^2 \, |\nabla r|^2}\right) \left(\frac{r \, R^2 \, \delta B^r}{R_0^2}\right) - \left(\frac{r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2}\right) \delta B_\theta\right]. \tag{49}$$

Let

$$\frac{r \, R^2 \, \delta B^r(r, \theta, \phi)}{R_0^2} = \sum_j \delta \hat{B}_j^r(r) \, e^{i \, (m_j \, \theta - n\phi)}, \tag{50}$$

$$\delta B_\theta(r, \theta, \phi) = \sum_j \delta \hat{B}_{\theta \, j}(r) \, e^{i \, (m_j \, \theta - n \, \phi)}. \tag{51}$$

Follows that

$$\sum_{j'} r \, \frac{d \, \delta \hat{B}_{j'}^r}{dr} \, e^{i \, m_{j'} \, \theta} = -\frac{\partial}{\partial \theta} \sum_{j'} \left[\left(\frac{r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2}\right) e^{i \, m_{j'} \, \theta} \, \delta \hat{B}_{j'}^r + \left(\frac{1}{|\nabla r|^2}\right) e^{i \, m_{j'} \, \theta} \, \delta \hat{B}_{\theta \, j'}\right], \tag{52}$$

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$$\sum_{j'} r \, \frac{d\,\delta\hat{B}_{\theta\,j'}}{dr} \, \mathrm{e}^{\mathrm{i}\,m_{j'}\,\theta} = \frac{\partial}{\partial\theta} \sum_{j'} \left[\left(\frac{R_0^2}{R^2\,|\nabla r|^2} \right) \mathrm{e}^{\mathrm{i}\,m_{j'}\,\theta}\,\delta\hat{B}_{j'}^r - \left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2} \right) \mathrm{e}^{\mathrm{i}\,m_{j'}\,\theta}\,\delta\hat{B}_{\theta\,j'} \right]. \tag{53}$$

Operating with $\oint(\cdots)\,\mathrm{e}^{-\mathrm{i}\,m_j\,\theta}\,d\theta/(2\pi)$, we get

$$r\,\frac{d\,\delta\hat{B}_j^r}{dr} = -\mathrm{i}\,m_j \sum_{j'} \left(-\mathrm{i}\,c_{jj'}\,\delta\hat{B}_{j'}^r + a_{jj'}\,\delta\hat{B}_{\theta\,j'} \right), \tag{54}$$

$$r\,\frac{d\,\delta\hat{B}_{\theta\,j}}{dr} = -\mathrm{i}\,m_j \sum_{j'} \left(-\mathrm{i}\,c_{jj'}\,\delta\hat{B}_{\theta\,j'} - b_{jj'}\,\delta\hat{B}_{j'}^r \right), \tag{55}$$

where

$$a_{jj'} = \oint \frac{1}{|\nabla r|^2}\,\mathrm{e}^{-\mathrm{i}\,(m_j-m_{j'})\,\theta}\,\frac{d\theta}{2\pi}, \tag{56}$$

$$b_{jj'} = \oint \frac{R_0^2}{R^2\,|\nabla r|^2}\,\mathrm{e}^{-\mathrm{i}\,(m_j-m_{j'})\,\theta}\,\frac{d\theta}{2\pi}, \tag{57}$$

$$c_{jj'} = \oint \frac{\mathrm{i}\,r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\mathrm{e}^{-\mathrm{i}\,(m_j-m_{j'})\,\theta}\,\frac{d\theta}{2\pi}. \tag{58}$$

Let

$$\delta\hat{B}_j^r(r) = \mathrm{i}\,\psi_j(r), \tag{59}$$

$$\delta\hat{B}_{\theta\,j}(r) = -\chi_j(r). \tag{60}$$

It follows that

$$r\,\frac{d\psi_j}{dr} = m_j \sum_{j'} \left(-c_{jj'}\,\psi_{j'} + a_{jj'}\,\chi_{j'} \right), \tag{61}$$

$$r\,\frac{d\chi_j}{dr} = m_j \sum_{j'} \left(-c_{jj'}\,\chi_{j'} + b_{jj'}\,\psi_{j'} \right). \tag{62}$$

Hence,

$$\frac{r\,R^2\,\delta B^r(r,\theta,\phi)}{R_0^2} = \mathrm{i} \sum_j \psi_j(r)\,\mathrm{e}^{\mathrm{i}\,(m_j\,\theta - n\phi)}, \tag{63}$$

$$\frac{r\,R^2\,\delta B^\theta(r,\theta,\phi)}{R_0^2} = -\sum_j \frac{1}{m_j}\,\frac{d\psi_j}{dr}\,\mathrm{e}^{\mathrm{i}\,(m_j\,\theta - n\,\phi)}, \tag{64}$$

$$R^2\,\delta B^\phi(r,\theta,\phi) = n \sum_j \frac{\chi_j(r)}{m_j}\,\mathrm{e}^{\mathrm{i}\,(m_j\,\theta - n\,\phi)}. \tag{65}$$

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where use has been made of (42). Also have

$$\delta B_r(r,\theta,\phi) = \mathrm{i}\sum_j \frac{1}{m_j}\frac{d\chi_j}{dr}\,\mathrm{e}^{\mathrm{i}(m_j\,\theta - n\,\phi)}, \tag{66}$$

$$\delta B_\theta(r,\theta,\phi) = -\sum_j \chi_j(r)\,\mathrm{e}^{\mathrm{i}(m_j\,\theta - n\,\phi)}, \tag{67}$$

$$\delta B_\phi(r,\theta,\phi) = n\sum_j \frac{\chi_j(r)}{m_j}\,\mathrm{e}^{\mathrm{i}(m_j\,\theta - n\,\phi)}, \tag{68}$$

We can write

$$\mathcal{J}\,\mu_0\,\delta J^r = \frac{\partial\,\delta B_\phi}{\partial\theta} - \frac{\partial\,\delta B_\theta}{\partial\phi}, \tag{69}$$

$$\mathcal{J}\,\mu_0\,\delta J^\theta = \frac{\partial\,\delta B_r}{\partial\phi} - \frac{\partial\,\delta B_\phi}{\partial r}, \tag{70}$$

$$\mathcal{J}\,\mu_0\,\delta J^\phi = \frac{\partial\,\delta B_\theta}{\partial r} - \frac{\partial\,\delta B_r}{\partial\theta}. \tag{71}$$

Normally, all three components are zero. However, at kth rational surface ψ_k, $\psi_{j\neq k}$, and $\chi_{j\neq k}$ are continuous, whereas χ_k is discontinuous. Hence,

$$\mathcal{J}\,\mu_0\,\delta J^r(r,\theta,\phi) = 0, \tag{72}$$

$$\mathcal{J}\,\mu_0\,\delta J^\theta(r,\theta,\phi) = -\sum_k \frac{n}{m_k}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\delta(r-r_k)\,\mathrm{e}^{\mathrm{i}(m_k\,\theta - n\,\phi)}, \tag{73}$$

$$\mathcal{J}\,\mu_0\,\delta J^\phi(r,\theta,\phi) = -\sum_k [\chi_k]_{r_{k-}}^{r_{k+}}\,\delta(r-r_k)\,\mathrm{e}^{\mathrm{i}(m_k\,\theta - n\,\phi)}. \tag{74}$$

Now,

$$(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\theta = \frac{1}{4}\left(\mathcal{J}\,\mu_0\,\delta J^\phi\,\delta B^{r\,*} + \text{c.c.}\right), \tag{75}$$

$$(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\phi = \frac{1}{4}\left(-\mathcal{J}\,\mu_0\,\delta J^\theta\,\delta B^{r\,*} + \text{c.c.}\right), \tag{76}$$

which implies that

$$\mathcal{J}\,(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\theta = \frac{R_0}{2}\sum_j\sum_k \text{Re}\left\{\mathrm{i}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\psi_j^*(r_k)\,\mathrm{e}^{\mathrm{i}(m_k - m_j)\,\theta}\right\}\delta(r-r_k), \tag{77}$$

$$\mathcal{J}\,(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\phi = -\frac{R_0}{2}\sum_j\sum_k \frac{n}{m_k}\,\text{Re}\left\{\mathrm{i}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\psi_j^*(r_k)\,\mathrm{e}^{\mathrm{i}(m_k - m_j)\,\theta}\right\}\delta(r-r_k). \tag{78}$$

Let

$$\Psi_k = \frac{\psi_k(r_k)}{m_k}, \tag{79}$$

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$$\Delta\Psi_k = [\chi_k]_{r_{k-}}^{r_{k+}}, \tag{80}$$

$$\delta T_{\theta\,k} \equiv \delta\mathbf{T}\cdot\mathbf{e}_\theta = \int_{r_{k-}}^{r_{k+}} \oint\oint (\delta\mathbf{J}\times\delta\mathbf{B})_\theta\,\mathcal{J}\,dr\,d\theta\,d\phi, \tag{81}$$

$$\delta T_{\phi\,k} \equiv \delta\mathbf{T}\cdot\mathbf{e}_\phi = \int_{r_{k-}}^{r_{k+}} \oint\oint (\delta\mathbf{J}\times\delta\mathbf{B})_\phi\,\mathcal{J}\,dr\,d\theta\,d\phi, \tag{82}$$

where $\delta\mathbf{T}(r)\,dr$ is the net electromagnetic torque acting on the plasma between r and $r+dr$. Here, $\mathbf{e}_\theta = (R^2/R_0)\,\nabla\phi\times\nabla r$ and $\mathbf{e}_\phi = R\,\nabla\phi$. Follows that

$$\delta T_{\theta\,k} = -\frac{2\pi^2\,R_0\,m_k}{\mu_0}\,\mathrm{Im}\,(\Psi_k^*\,\Delta\Psi_k), \tag{83}$$

$$\delta T_{\phi\,k} = \frac{2\pi^2\,R_0\,n}{\mu_0}\,\mathrm{Im}\,(\Psi_k^*\,\Delta\Psi_k). \tag{84}$$

7 Homogeneous Solution

We can write

$$\delta\mathbf{B} = \nabla\times\delta\mathbf{A}. \tag{85}$$

Suppose that $r\,\delta A_r$, δA_θ are negligible with respect to δA_ϕ. In fact, it is easily demonstrated from $\nabla\cdot\delta\mathbf{A} = 0$ that $r\,\delta A_r$, $\delta A_\theta \sim (n/m)\,(r/R)^2\,\delta A_\phi$. It follows that

$$\mathcal{J}\,\delta B^r \simeq \frac{\partial\,\delta A_\phi}{\partial\theta}, \tag{86}$$

$$\mathcal{J}\,\delta B^\theta \simeq -\frac{\partial\,\delta A_\phi}{\partial r}, \tag{87}$$

$$\mathcal{J}\,\delta B^\phi \simeq 0. \tag{88}$$

The neglected terms in the previous three equations are $(n/m)^2\,(r/R)^2$ smaller than the dominant terms. The previous three expressions are consistent with (63), (64), and (65) provided

$$\delta A_\phi(r,\theta,\phi) \simeq R_0 \sum_j \frac{\psi_j(r)}{m_j}\,\mathrm{e}^{\mathrm{i}\,(m_j\,\theta - n\,\phi)}. \tag{89}$$

According to the Biot-Savart law:

$$\delta A_\phi(\mathbf{x}) = \frac{1}{4\pi}\int \frac{R\,R'\,\mu_0\,\delta\mathbf{J}(\mathbf{x}')\cdot\nabla\phi}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'. \tag{90}$$

Assume that

$$\delta A_\phi(R,\phi,Z) = \delta A_\phi(R,0,Z)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi}. \tag{91}$$

Hence, can evaluate integral at $\phi = 0$ without loss of generality. Follows that

$$\mathbf{x} = (R,\,0,\,Z), \tag{92}$$

$$\mathbf{x}' = (R'\,\cos\phi',\ R'\,\sin\phi',\ Z'), \tag{93}$$

and

$$|\mathbf{x}-\mathbf{x}'| = \left[R^2 + R'^2 + (Z-Z')^2 - 2\,R\,R'\,\cos\phi'\right]^{1/2}. \tag{94}$$

Now,

$$\delta\mathbf{J}(R',\phi',Z')\cdot\nabla\phi = \delta J^\phi(R',0,Z')\,\mathrm{e}^{-\mathrm{i}\,n\,\phi'}\,\cos\phi', \tag{95}$$

so

$$\delta A_\phi(r,\theta,0) = \frac{1}{4\pi}\int_0^\infty \oint\oint \frac{R\,R'\,\mu_0\,\delta J^\phi(r',\theta',0)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi'}\,\cos\phi'\,\mathcal{J}'\,dr'\,d\theta'\,d\phi'}{[R^2 + R'^2 + (Z-Z')^2 - 2\,R\,R'\,\cos\phi']^{1/2}}, \tag{96}$$

which can be written

$$\delta A_\phi(r,\theta,0) = \frac{1}{4\pi}\int_0^\infty \oint R\,R'\,\mu_0\,\delta J^\phi(r',\theta',0)\,G(r,\theta;r',\theta')\,\mathcal{J}'\,dr'\,d\theta', \tag{97}$$

where

$$G(r,\theta;r',\theta') = \oint \frac{\cos\phi'\,\cos(n\,\phi')\,d\phi'}{[R^2 + R'^2 + (Z-Z')^2 - 2\,R\,R'\,\cos\phi']^{1/2}}, \tag{98}$$

or

$$G(r,\theta;r',\theta') = \frac{1}{2}\oint \frac{(\cos[(n-1)\,\phi'] + \cos[(n+1)\,\phi'])\,d\phi'}{[R^2 + R'^2 + (Z-Z')^2 - 2\,R\,R'\,\cos\phi']^{1/2}}. \tag{99}$$

Now (GR 8.85.1),

$$\begin{aligned}
P^n_{-1/2}(\cosh\eta) &= \frac{(-1)^n}{2\pi}\,\frac{\Gamma(1/2)}{\Gamma(1/2-n)}\oint \frac{\cos(n\,\varphi)\,d\varphi}{(\cosh\eta + \sinh\eta\,\cos\varphi)^{1/2}} \\
&= \frac{1}{2\pi}\,\frac{\Gamma(1/2)}{\Gamma(1/2-n)}\oint \frac{\cos(n\,\varphi)\,d\varphi}{(\cosh\eta - \sinh\eta\,\cos\varphi)^{1/2}} \\
&= \frac{(-1)^n\,\Gamma(1/2)\,\Gamma(1/2+n)}{2\pi^2}\oint \frac{\cos(n\,\varphi)\,d\varphi}{(\cosh\eta - \sinh\eta\,\cos\varphi)^{1/2}}.
\end{aligned} \tag{100}$$

It follows that

$$\alpha\,\cosh\eta = R^2 + R'^2 + (Z-Z')^2, \tag{101}$$

$$\alpha\,\sinh\eta = 2\,R\,R'. \tag{102}$$

Hence,

$$\eta = \tanh^{-1}\left[\frac{2\,R\,R'}{R^2 + R'^2 + (Z-Z')^2}\right], \tag{103}$$

and

$$\alpha = \frac{R^2 + R'^2 + (Z-Z')^2}{\cosh\eta}. \tag{104}$$

It follows that

$$G(r,\theta;r',\theta') = \pi\left[\frac{\cosh\eta}{R^2 + R'^2 + (Z-Z')^2}\right]^{1/2}$$

$$\times \left[\frac{\Gamma(3/2 - n)}{\Gamma(1/2)} \, P_{-1/2}^{n-1}(\cosh\eta) + \frac{\Gamma(-1/2 - n)}{\Gamma(1/2)} \, P_{-1/2}^{n+1}(\cosh\eta) \right]. \tag{105}$$

However,

$$\Gamma(3/2 - n) = \frac{(-1)^{n+1} \, \pi \, (n - 1/2)}{\Gamma(n + 1/2)}, \tag{106}$$

$$\Gamma(-1/2 - n) = \frac{(-1)^{n+1} \, \pi}{\Gamma(n + 1/2) \, (n + 1/2)}, \tag{107}$$

so

$$G(r, \theta; r', \theta') = \frac{(-1)^{n+1} \, \pi^2}{\Gamma(1/2) \, \Gamma(n + 1/2)} \left[\frac{\cosh\eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2}$$
$$\times \left[(n - 1/2) \, P_{-1/2}^{n-1}(\cosh\eta) + \frac{P_{-1/2}^{n+1}(\cosh\eta)}{n + 1/2} \right]. \tag{108}$$

Now, from (74) and (80),

$$\mathcal{J} \, \mu_0 \, \delta J^\phi(r, \theta, 0) = - \sum_k \Delta \Psi_k \, \delta(r - r_k) \, \mathrm{e}^{\mathrm{i} \, m_k \, \theta}. \tag{109}$$

Furthermore, from (79) and (89),

$$\Psi_k = \frac{1}{R_0} \oint \delta A_\phi(r_k, \theta, 0) \, \mathrm{e}^{-\mathrm{i} \, m_k \, \theta} \, \frac{d\theta}{2\pi}. \tag{110}$$

Hence,

$$\Psi_k = \sum_{k'} F_{kk'} \, \Delta \Psi_{k'}, \tag{111}$$

where

$$F_{kk'} = \oint \oint \mathcal{G}(r_k, \theta; r_{k'}, \theta') \, \mathrm{e}^{-\mathrm{i} \, (m_k \, \theta - m_{k'} \, \theta')} \, \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}, \tag{112}$$

and

$$\mathcal{G}(r, \theta; r', \theta') = \frac{(-1)^n \, \pi^2 \, R \, R'/R_0}{2 \, \Gamma(1/2) \, \Gamma(n + 1/2)} \left[\frac{\cosh\eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2}$$
$$\times \left[(n - 1/2) \, P_{-1/2}^{n-1}(\cosh\eta) + \frac{P_{-1/2}^{n+1}(\cosh\eta)}{n + 1/2} \right]. \tag{113}$$

Note that

$$\mathcal{G}(r', \theta'; r, \theta) = \mathcal{G}(r, \theta; r', \theta'), \tag{114}$$

which implies that

$$F_{k'k} = F_{kk'}^*. \tag{115}$$

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8 Inomogeneous Solution

Suppose that there are currents flowing in a number of external poloidal field coils. Let I_l, R_l, and Z_l be the peak current, and coordinates of the lth field coil. The currents are assumed to modulate like $e^{-i n \phi}$. It follows that

$$\delta J^{\phi}_{\text{ext}}(R, 0, Z) = \sum_j \frac{I_l}{R_l} \, \delta(R - R_l) \, \delta(Z - Z_l). \tag{116}$$

Hence,

$$\Psi_k = \sum_{k'} F_{kk'} \, \Delta\Psi_k - \sum_l g_{kl} \, \mu_0 \, I_l, \tag{117}$$

where

$$g_{kj} = \frac{1}{2\pi} \oint \mathcal{G}(r_k, \theta; R_j, Z_j) \, e^{-i m_k \theta} \, \frac{d\theta}{2\pi}. \tag{118}$$

It follows that

$$\Delta\Psi_k = \sum_{k'} E_{kk'} \, \Psi_{k'} + |E_{kk}| \, \chi_k \tag{119}$$

where $E_{kk'}$ is the inverse of the $F_{kk'}$ matrix, and

$$\chi_k = \frac{1}{|E_{kk}|} \sum_j h_{kl} \, \mu_0 \, I_l, \tag{120}$$

$$h_{kl} = \sum_{k'} E_{kk'} \, g_{k'l}. \tag{121}$$

9 Island Width

We have

$$\mathcal{J} \, \delta B^r \simeq \frac{\partial \, \delta A_{\phi}}{\partial \theta}, \tag{122}$$

$$\mathcal{J} \, \delta B^{\theta} \simeq -\frac{\partial \, \delta A_{\phi}}{\partial r}, \tag{123}$$

$$\mathcal{J} \, \delta B^{\phi} \simeq 0. \tag{124}$$

It follows that

$$\delta \mathbf{B} \cdot \nabla \delta A_{\phi} = \delta B^r \, \frac{\partial \, \delta A_{\phi}}{\partial r} + \delta B^{\theta} \, \frac{\partial \, \delta A_{\phi}}{\partial \theta} + \delta B^{\phi} \, \frac{\partial \, \delta A_{\phi}}{\partial \phi} \simeq 0. \tag{125}$$

Suppose that

$$\delta A_{\phi}(r, \theta, \phi) \simeq \delta \hat{A}_{\phi} \, e^{i \, (m_k \theta - n \phi)} \tag{126}$$

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in the vicinity of the kth rational surface. It follows that

$$\mathbf{B} \cdot \nabla \delta A_\phi = \mathrm{i}\, B_0\, R_0^2\, \frac{f}{r\, R^2}\, (m_k - n\, q)\, \delta \hat{A}_\phi\, \mathrm{e}^{\mathrm{i}\,(m_k\, \theta - n\, \phi)}. \tag{127}$$

Let us search for a function,

$$F(r, \theta, \phi) = F_0(r) + \delta A_\phi, \tag{128}$$

which is such that

$$(\mathbf{B} + \delta \mathbf{B}) \cdot \nabla F = 0. \tag{129}$$

It follows that

$$\delta B^r\, \frac{dF_0}{dr} + \mathrm{i}\, B_0\, R_0^2\, \frac{f}{r\, R^2}\, (m_k - n\, q)\, \delta \hat{A}_\phi\, \mathrm{e}^{\mathrm{i}\,(m_k\, \theta - n\, \phi)} = 0, \tag{130}$$

or

$$\mathrm{i}\, m_k\, \frac{R_0}{r\, R^2}\, \delta \hat{A}_\phi\, \frac{dF_0}{dr} + \mathrm{i}\, B_0\, R_0^2\, \frac{f}{r\, R^2}\, (m_k - n\, q)\, \delta \hat{A}_\phi = 0, \tag{131}$$

which implies that

$$\frac{dF_0}{dr} = -\frac{B_0\, R_0\, f}{m_k}\, (m_k - n\, q), \tag{132}$$

or

$$F_0(r) \simeq \frac{B_0}{2} \left(\frac{g\, s}{q} \right)_{r_k} (r - r_k)^2. \tag{133}$$

Hence,

$$F(r, \theta, \phi) = \frac{B_0}{2} \left(\frac{g\, s}{q} \right)_{r_k} (r - r_k)^2 + R_0\, \Psi_k\, \cos(m_k\, \theta - n\, \phi) \tag{134}$$

is a flux surface function in the island region. Thus,

$$\frac{F}{R_0\, |\Psi_k|} = 2\, X^2 + \cos(m_k\, \theta - n\, \phi), \tag{135}$$

where

$$X = \frac{2\, (r - r_k)}{W_k}, \tag{136}$$

and

$$\frac{W_k}{4\, R_0} = \left[\left(\frac{q}{g\, s} \right)_{r_k} \frac{|\Psi_k|}{B_0\, R_0} \right]^{1/2}. \tag{137}$$

It follows that W_k is the full radial island width (which is constant in r).

10 Flux Coordinate System

Let all lengths be normalized to R_0, and all magnetic field-strengths to B_0. We have

$$\mathbf{B} = \nabla\phi \times \nabla\psi_p + g(\psi_p)\,\nabla\phi, \tag{138}$$

and

$$\nabla\psi_p \times \nabla\theta \cdot \nabla\phi = \frac{g}{q\,R^2}, \tag{139}$$

where $q = q(\psi_p)$.

Let $\Psi = \psi_p/\psi_c = 1 - \Psi_N$, where ψ_c is the value of ψ_p on the magnetic axis. (It is assumed that $\psi_p = 0$ on the plasma boundary.) The previous equation implies that

$$\frac{d\theta}{dl} = \frac{g}{q} \frac{1}{|\psi_c|\,R\,\sqrt{\Psi_R^2 + \Psi_Z^2}}, \tag{140}$$

where dl is an element of poloidal path length on a magnetic flux-surface, and $\Psi_R \equiv \partial\Psi/\partial R$, etc. Furthermore,

$$dR = -\frac{\Psi_Z\,dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}, \tag{141}$$

$$dZ = \frac{\Psi_R\,dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}. \tag{142}$$

It follows that

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi\,|\psi_c|} \oint \frac{dl}{R\,\sqrt{\Psi_R^2 + \Psi_Z^2}}. \tag{143}$$

If we define

$$\tan\zeta = \frac{Z - Z_{\mathrm{axis}}}{R_{\mathrm{axis}} - R} \tag{144}$$

then

$$\frac{dR}{d\zeta} = -\Psi_Z\,F, \tag{145}$$

$$\frac{dZ}{d\zeta} = \Psi_R\,F, \tag{146}$$

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi\,|\psi_c|} \oint \frac{F}{R}\,d\zeta, \tag{147}$$

where

$$F = \frac{(R_{\mathrm{axis}} - R)^2 + (Z - Z_{\mathrm{axis}})^2}{-(Z - Z_{\mathrm{axis}})\,\Psi_Z + (R_{\mathrm{axis}} - R)\,\Psi_R}. \tag{148}$$

It is helpful to define the length-like flux-surface coordinate r, according to

$$\nabla r \times \nabla\theta \cdot \nabla\phi = \frac{1}{r\,R^2}. \tag{149}$$

It follows that

$$r(\Psi) = \left[2\,|\psi_c| \int_\Psi^1 \frac{q(\Psi')}{g(\Psi')}\,d\Psi' \right]^{1/2}. \tag{150}$$

We can calculate $R(r,\theta)$ and $Z(r,\theta)$ from

$$\frac{dR}{d\theta} = -|\psi_c|\,\frac{q}{g}\,R\,\Psi_Z, \tag{151}$$

$$\frac{dZ}{d\theta} = |\psi_c|\,\frac{q}{g}\,R\,\Psi_R. \tag{152}$$

Now,

$$r\,\frac{dr}{d\Psi} = -|\psi_c|\,\frac{q(r)}{g(r)}. \tag{153}$$

So

$$\nabla r = \frac{dr}{d\Psi}\,\nabla\Psi = -|\psi_c|\,\frac{q(r)}{r\,g(r)}\,\nabla\Psi. \tag{154}$$

Hence,

$$a_{jj} = \left(\oint \frac{1}{|\nabla r|^2}\,\frac{d\theta}{2\pi} \right)_{r_j} = \left(\frac{r\,g}{|\psi_c|\,q} \right)_{r_j}^2 \oint \frac{1}{\Psi_R^2 + \Psi_Z^2}\,\frac{d\theta}{2\pi}. \tag{155}$$

Note that

$$\frac{d\Psi_N}{dr} = \frac{r\,g(r)}{|\psi_c|\,q(r)}. \tag{156}$$

Hence, if $\overline{W}_k$ is the full magnetic island width in Ψ_N at the kth rational surface then

$$\overline{W}_k = \frac{W_k}{R_0}\,\frac{d\Psi_N}{dr}. \tag{157}$$

11   GPEC Coupling

We have

$$\frac{d\psi_p}{dr} = B_0\,R_0\,f(r), \tag{158}$$

$$\mathbf{B}\cdot\nabla\phi = \frac{B_0\,R_0}{R^2}\,g(r), \tag{159}$$

$$\delta B^r = \frac{i}{r}\left(\frac{R_0}{R}\right)^2 \sum_j \psi_j\,e^{i(m_j\,\theta - n\,\phi)}. \tag{160}$$

According to PRL **99**, 195003 (2007),

$$\Delta_j\,e^{i(m_j\,\theta - n\,\phi)} = \left[\frac{\partial}{\partial\psi_p}\,\frac{\delta\mathbf{B}\cdot\nabla\psi_p}{\mathbf{B}\cdot\nabla\phi} \right]_{r_j} = \left[\frac{\partial}{\partial r}\,\frac{\delta\mathbf{B}\cdot\nabla r}{\mathbf{B}\cdot\nabla\phi} \right]_{r_j} = \left[\frac{\partial}{\partial r}\,\frac{\delta B^r}{\mathbf{B}\cdot\nabla\phi} \right]_{r_j}. \tag{161}$$

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It follows that

$$\Delta_j = \frac{i}{r_j} \left(\frac{R_0}{R}\right)^2 \frac{R^2}{B_0 \, R_0 \, g_j} \left[\frac{d\psi_j}{dr}\right]_{r_j}, \tag{162}$$

But,

$$\left[r \frac{d\psi_j}{dr}\right]_{r_j} = m_j \, a_{jj} \, [\chi_j]_{r_j} = m_j \, a_{jj} \, \Delta\Psi_j, \tag{163}$$

and

$$\chi_j = \frac{\Delta\Psi_j}{|E_{jj}|}, \tag{164}$$

which implies that

$$\Delta_j = i \left(\frac{R_0}{r_j}\right)^2 \frac{m_j \, a_{jj}}{g_j} \frac{\Delta\Psi_j}{B_0 \, R_0}, \tag{165}$$

$$\frac{\chi_j}{R_0 \, B_0} = -i \left(\frac{r_j}{R_0}\right)^2 \frac{g_j}{m_j \, a_{jj}} \frac{\Delta_j}{|E_{jj}|}. \tag{166}$$

Now, $d\psi_p/dr = B_0 \, R_0 \, f$. It follows that

$$(\nabla\psi_p \times \nabla\theta \cdot \nabla\phi)^{-1} = \frac{R^2 \, q}{B_0 \, R_0 \, g}. \tag{167}$$

Hence,

$$\mathbf{B} = \nabla\phi \times \nabla\psi_p + B_0 \, R_0 \, g \, \nabla\phi = \nabla\phi \times \nabla\psi_p + q \, \nabla\psi_p \times \nabla\theta. \tag{168}$$

Let $\Psi_p = 2\pi \, \psi_p$ and $d\Psi_p/d\Psi_t = 1/q$. Hence,

$$\frac{d\Psi_p}{dr} = 2\pi \, B_0 \, R_0 \, f, \tag{169}$$

$$\frac{d\Psi_t}{dr} = 2\pi \, B_0 \, r \, g, \tag{170}$$

and [cf. PoP **13**, 102501 (2006), Eq. (41)]

$$2\pi \, \mathbf{B} = q^{-1} \, \nabla\phi \times \nabla\Psi_t + \nabla\Psi_t \times \nabla\theta. \tag{171}$$

We have

$$\delta J^r = 0, \tag{172}$$

$$\mathcal{J} \, \mu_0 \, \delta J^\theta = -\sum_j \frac{\Delta\Psi_j}{q_j} \, e^{i(m_j \theta - n \phi)} \, \delta(r - r_j), \tag{173}$$

$$\mathcal{J} \, \mu_0 \, \delta J^\phi = -\sum_j \Delta\Psi_k \, e^{i(m_j \theta - n \phi)} \, \delta(r - r_j), \tag{174}$$

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which implies that

$$\mu_0\,\delta\mathbf{J} = -2\pi\sum_j \varDelta\varPsi_j\,\mathrm{e}^{\mathrm{i}\,(m_j\,\theta - n\,\phi)}\,\delta(\psi_t - \psi_{t\,j})\,\mathbf{B}. \tag{175}$$

However, according to PoP **13**, 102501 (2006),

$$\varDelta\varPsi_j = -\mathrm{i}\,\frac{\mu_0\,J_c\,\varDelta_j}{2\pi\,m_j}, \tag{176}$$

where

$$\frac{1}{\mu_0\,J_c} = \left(\oint \frac{B^2}{|\nabla\psi_t|^2}\,\frac{d\theta\,d\phi}{2\pi\,\mathbf{B}\cdot\nabla\phi}\right)_{r_j}. \tag{177}$$

It is easily demonstrated that

$$\frac{1}{\mu_0\,J_c} = \left(\frac{R_0}{r_j}\right)^2 \frac{1}{2\pi\,B_0\,R_0\,g_j}\left[a_{jj} + \left(\frac{r_k}{R_0\,q_j}\right)^2\right]. \tag{178}$$

Hence,

$$\frac{\varDelta\varPsi_j}{B_0\,R_0} = -\mathrm{i}\,\varDelta_j\left(\frac{r_j}{R_0}\right)^2 \frac{g_j}{m_j\,[a_{jj} + (r_k/R_0\,q_j)^2]}. \tag{179}$$

12 Neoclassical Coordinate System

It is also helpful to define the geometric poloidal angle

$$\mathbf{b}\cdot\nabla\varTheta = \gamma(r). \tag{180}$$

It follows that

$$\frac{d\varTheta}{dl} = \frac{\gamma\,B\,R}{|\psi_c|\,\sqrt{\varPsi_R^2 + \varPsi_Z^2}}, \tag{181}$$

where

$$B\,R = \left[g^2 + |\psi_c|^2\,(\varPsi_R^2 + \varPsi_Z^2)\right]^{1/2}. \tag{182}$$

Hence,

$$\frac{1}{\gamma(r)} = \frac{1}{2\pi\,|\psi_c|}\oint \frac{B\,R\,dl}{\sqrt{\varPsi_R^2 + \varPsi_Z^2}} = \frac{1}{2\pi\,|\psi_c|}\oint B\,R\,F\,d\zeta. \tag{183}$$

We can calculate $R(r,\varTheta)$ and $Z(r,\varTheta)$ from

$$\frac{dR}{d\varTheta} = -|\psi_c|\,\frac{\varPsi_Z}{\gamma\,B\,R}, \tag{184}$$

$$\frac{dZ}{d\varTheta} = |\psi_c|\,\frac{\varPsi_R}{\gamma\,B\,R}. \tag{185}$$

Note that

$$\frac{d\Theta}{d\theta} = \left(\frac{\gamma\,q}{g}\right) B\,R^2.\tag{186}$$

Thus,

$$\frac{1}{\gamma} = \frac{q}{g}\oint B\,R^2\,\frac{d\theta}{2\pi}.\tag{187}$$

13   Neoclassical Parameters

The flux-surface average operator has the following properties:

$$\langle 1\rangle = 1,\tag{188}$$

$$\langle \mathbf{B}\cdot\nabla A\rangle = 0.\tag{189}$$

It follows that

$$\langle A\rangle = \oint R^2\,A\,\frac{d\theta}{2\pi}\bigg/\oint R^2\,\frac{d\theta}{2\pi} = \oint \frac{A}{B}\,\frac{d\Theta}{2\pi}\bigg/\oint \frac{1}{B}\,\frac{d\Theta}{2\pi}.\tag{190}$$

Let

$$I_0 = \oint \frac{1}{B\,R^2}\,\frac{d\Theta}{2\pi} = \frac{\gamma\,q}{g},\tag{191}$$

$$I_1 = \oint \frac{1}{B}\,\frac{d\Theta}{2\pi},\tag{192}$$

$$I_2 = \oint B\,\frac{d\Theta}{2\pi},\tag{193}$$

$$I_3 = \oint \left(\frac{\partial B}{\partial\Theta}\right)^2\frac{1}{B}\,\frac{d\Theta}{2\pi},\tag{194}$$

$$I_{4,k} = \sqrt{\frac{2}{k}}\oint \frac{\sin(k\,\Theta)}{B^2}\frac{\partial B}{\partial\Theta}\,\frac{d\Theta}{2\pi} = \oint \frac{\sqrt{2\,k}\,\cos(k\,\Theta)}{B}\,\frac{d\Theta}{2\pi},\tag{195}$$

$$I_{5,k} = \sqrt{\frac{2}{k}}\oint \frac{\sin(k\,\Theta)}{B^3}\frac{\partial B}{\partial\Theta}\,\frac{d\Theta}{2\pi} = \oint \frac{\sqrt{2\,k}\,\cos(k\,\Theta)}{2\,B^2}\,\frac{d\Theta}{2\pi},\tag{196}$$

$$I_6(\lambda) = \oint \frac{\sqrt{1 - \lambda\,B/B_{\max}}}{B}\,\frac{d\Theta}{2\pi}.\tag{197}$$

It follows that

$$\langle B^2\rangle = \frac{I_2}{I_1},\tag{198}$$

$$\langle(\mathbf{b}\cdot\nabla B)^2\rangle = \gamma^2\,\frac{I_3}{I_1},\tag{199}$$

$$|\langle \mathbf{B} \cdot \nabla \theta \rangle| = \frac{g}{|q|} \frac{I_0}{I_1} = \frac{|\gamma|}{I_1}, \tag{200}$$

$$\left\langle \sqrt{\frac{2}{k}} \, \sin(k\,\Theta)\,(\mathbf{b} \cdot \nabla \ln B) \right\rangle = \gamma \, \frac{I_{4,k}}{I_1}, \tag{201}$$

$$\left\langle \sqrt{\frac{2}{k}} \, \sin(k\,\Theta) \, \frac{(\mathbf{b} \cdot \nabla \ln B)}{B} \right\rangle = \gamma \, \frac{I_{5,k}}{I_1}. \tag{202}$$

Hence,

$$L_c = \frac{1}{|\gamma|} \frac{I_2^2}{I_1^2 \, I_3} \sum_{k>0} I_{5,k} \, I_{6,k}, \tag{203}$$

$$\omega_{t\,a} \equiv \frac{v_{T\,a}}{L_c} = K_t \, |\gamma| \, v_{T\,a}, \tag{204}$$

$$\nu_{*\,a} \equiv \frac{8}{3\pi} \frac{\langle B^2 \rangle}{\langle (\mathbf{b} \cdot \nabla B)^2 \rangle} \frac{g_t \, \omega_{t\,a}}{v_{T\,a}^2 \, \tau_{aa}} = K_* \, \frac{g_t}{\omega_{t\,a} \, \tau_{aa}}, \tag{205}$$

$$f_c = \frac{3}{4} \frac{I_2}{B_{\max}^2} \int_0^1 \frac{\lambda \, d\lambda}{I_6(\lambda)}, \tag{206}$$

where

$$K_t = \frac{I_1^2 \, I_3}{I_2^2 \, \sum_{k>0} I_{4,k} \, I_{5,k}}, \tag{207}$$

$$K_* = \frac{8}{3\pi} \frac{I_2}{I_3} K_t^2. \tag{208}$$