Linear Layer Physics

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1 Fundamental Quantities

Layer width - δ

Layer tearing stability index - $\varDelta$

Poloidal mode number - m

Toroidal mode number - n

Major radius - R_0

Minor radius - r

Toroidal magnetic field-strength - B_0

Magnetic shear - s

Electrostatic potential - $\varPhi$

Poloidal magnetic flux - ψ

Magnitude of electron charge - e

Electron mass - m_e

Electron number density - n_e

Electron temperature - T_e

Ion mass - m_i

Ion number density - n_i

Ion temperature - T_i

Parallel electrical resistivity - $\eta_\parallel$

Perpendicular particle diffusivity - $D_\perp$

Perpendicular energy diffusivity - χ_E

Perpendicular momentum diffusivity - χ_ϕ

2 Derived Quantities

Safety-factor - $q = m/n$

Scale length - $l = R_0\, q/s$

Electron pressure - $p_e = n_e\, T_e$

Ion pressure - $p_i = n_i\, T_i$

Alfvén velocity - $v_A = B_0/\sqrt{\mu_0\, n_e\, m_i}$

Alfvén time - $\tau_A = l/v_A = m\, \tau_H$

Hydromagnetic time - $\tau_H = R_0/(n\, s\, v_A)$

Resistive time - $\tau_R = \mu_0\, r^{\,2}/\eta_\parallel$

Particle/energy confinement time - $\tau_E = r^{\,2}/[D_\perp + (2/3)\,\chi_E]$

Momentum confinement time - $\tau_M = r^{\,2}/\chi_\phi$

Ion sound radius - $\rho_s = \sqrt{m_i\, T_e}/(e\, B_0)$

$\mathbf{E}\times\mathbf{B}$ frequency - $\omega_E = d\Phi/d\psi$

Electron diamagnetic frequency - $\omega_{\ast\, e} = (T_e/e)\, d\ln p_e/d\psi$

Ion diamagnetic frequency - $\omega_{\ast\, i} = -(T_i/e)\, d\ln p_i/d\psi$

$\mathbf{E}\times\mathbf{B}$ velocity - $v_E = (r/q)\, \omega_E$

Electron diamagnetic velocity - $v_{\ast\, e} = (r/q)\, \omega_{\ast\, e}$

Ion diamagnetic velocity - $v_{\ast\, i} = (r/q)\, \omega_{\ast\, i}$

3 Non-dimensional Quantities

Lundquvist number - $S = \tau_R/\tau_H$

Magnetic Prandtl number - $P_M = \tau_R/\tau_M$

Energy confinement number - $P_E = \tau_R/\tau_E$

4 Layer Theory Quantities

$$\left(\frac{k}{\eta}\right)^{1/3} = S^{1/3}\,\frac{l}{r}$$

$$\tau = -\frac{\omega_{\ast\,i}}{\omega_{\ast\,e}}$$

$$c_\beta^2 = P_E$$

$$P = P_M$$

$$D = \frac{5}{3}\,S^{1/3}\,\frac{\rho_s}{r}$$

$$Q_E = S^{1/3}\,n\,\omega_E\,\tau_H$$

$$Q_{e,i} = -S^{1/3}\,n\,\omega_{\ast\,e,i}\,\tau_H$$

$$\varDelta = \frac{\delta}{r}\,\tau_R\,\mathrm{i}\,n\,(\omega_E - \omega_{\ast\,i})$$

$$\hat{\varDelta} = \left(\frac{\eta}{k}\right)^{1/3}\,\frac{l}{r}\,\varDelta$$

5 Constant-ψ Layer Physics

Layer equation:

$$\frac{d^2\,Y}{dp^2} - \left[\frac{-Q_E\,(Q_E - Q_i) + \mathrm{i}\,(Q_E - Q_i)\,(P_M + P_E)\,p^2 + P_M\,P_E\,p^4}{\mathrm{i}\,(Q_E - Q_e) + \{P_E + \mathrm{i}\,(Q_E - Q_i)\,D^2\}\,p^2 + (1+\tau)\,P_M\,D^2\,p^4}\right]p^2\,Y = 0.$$

Suppose that small-p behavior of solution that is well-behaved as $p \to \infty$ is

$$Y(p) = Y_0\left[1 - c\,p + \mathcal{O}(p^2)\right].$$

Layer width is

$$\frac{\delta}{r} = \frac{\pi\,|c|}{S^{1/3}}.$$

Large-p limit of layer equation is

$$\frac{d^2 Y}{dp^2} - C\, p^2\, Y = 0,$$

where

$$C = \frac{P_E}{(1+\tau)\, D^2}.$$

Solutions are

$$p^{1/2}\, I_{1/4}\left(\frac{C^{1/2}\, p^2}{2}\right), \quad p^{1/2}\, K_{1/4}\left(\frac{C^{1/2}\, p^2}{2}\right).$$

Should need to integrate to $p \sim C^{-1/4}$.