Program RESCALE

Richard Fitzpatrick

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1 Grad-Shafranov Equation

The Grad-Shafranov equation takes the form

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = -R^2\frac{dP}{d\psi} - \frac{1}{2}\frac{dT^2}{d\psi}, \tag{1}$$

where

$$j_\phi = -R\frac{dP}{d\psi} - \frac{1}{2\,R}\frac{dT^2}{d\psi}, \tag{2}$$

$$q = \frac{T}{2\pi}\oint_\psi \frac{dl}{R\,|\nabla\psi|}. \tag{3}$$

2 Type I Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\text{new}} = R_{\text{old}}, \tag{4}$$

$$Z_{\text{new}} = Z_{\text{old}}, \tag{5}$$

$$\psi_{\text{new}} = a_1\,\psi_{\text{old}}, \tag{6}$$

$$T_{\text{new}} = a_1\,T_{\text{old}}, \tag{7}$$

$$P_{\text{new}} = a_1^2\,P_{\text{old}}. \tag{8}$$

It follows that

$$I_{\phi\,\text{new}} = a_1\,I_{\phi\,\text{old}}, \tag{9}$$

$$q_{\text{new}} = q_{\text{old}}. \tag{10}$$

3 Type II Rescaling

The following rescaling of variable also leaves the Grad-Shafranov equation invariant:

$$R_{\text{new}} = R_{\text{old}}, \tag{11}$$

$$Z_{\text{new}} = Z_{\text{old}}, \tag{12}$$

$$\psi_{\text{new}} = \psi_{\text{old}}, \tag{13}$$

$$T_{\text{new}} = \text{sgn}(T_{\text{old}})\sqrt{T_{\text{old}}^2 + a_2}, \tag{14}$$

$$P_{\text{new}} = P_{\text{old}}. \tag{15}$$

It follows that

$$I_{\phi\,\text{new}} = I_{\phi\,\text{old}}, \tag{16}$$

$$q_{\text{new}} = \frac{T_{\text{new}}}{T_{\text{old}}} q_{\text{old}}. \tag{17}$$

4 Program RESCALE

Let

$$\Psi_N = \frac{\psi - \psi_{\text{axis}}}{\psi_{\text{separatrix}} - \psi_{\text{axis}}}. \tag{18}$$

Program RESCALE first performs a type II rescaling such that

$$a_2 = \left(\frac{q_{95\,\text{target}}^2}{q_{95\,\text{old}}^2} - 1 \right) T_{95\,\text{old}}^2, \tag{19}$$

where

$$q_{95} \equiv q(\Psi_N = 0.95), \tag{20}$$

$$T_{95} \equiv T(\Psi_N = 0.95). \tag{21}$$

It follows that

$$\psi_{\text{new}} = \psi_{\text{old}}, \tag{22}$$

$$T_{\text{new}} = T_{\text{old}}\sqrt{1 + \left(\frac{q_{95\,\text{target}}^2}{q_{95\,\text{old}}^2} - 1 \right) \frac{T_{95\,\text{old}}^2}{T_{\text{old}}^2}}, \tag{23}$$

$$P_{\text{new}} = P_{\text{old}}, \tag{24}$$

and

$$I_{\phi\,\text{new}} = I_{\phi\,\text{old}}, \tag{25}$$

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$$q_{\text{new}} = q_{\text{old}} \sqrt{1 + \left(\frac{q_{95\,\text{target}}^2}{q_{95\,\text{old}}^2} - 1 \right) \frac{T_{95\,\text{old}}^2}{T_{\text{old}}^2}}. \tag{26}$$

Program RESCALE then performs a type I rescaling such that

$$a_1 = \frac{T_{1\,\text{old}}}{T_{1\,\text{new}}}, \tag{27}$$

where

$$T_1 \equiv T(\Psi_N = 1). \tag{28}$$

It follows that

$$\psi_{\text{new new}} = a_1 \, \psi_{\text{old}}, \tag{29}$$

$$T_{\text{new new}} = a_1 \, T_{\text{new}} = T_{1\,\text{old}} \sqrt{\frac{T_{\text{old}}^2 + (q_{95\,\text{target}}^2/q_{95\,\text{old}}^2 - 1) \, T_{95\,\text{old}}^2}{T_{1\,\text{old}}^2 + (q_{95\,\text{target}}^2/q_{95\,\text{old}}^2 - 1) \, T_{95\,\text{old}}^2}}, \tag{30}$$

$$P_{\text{new new}} = a_1^2 \, P_{\text{old}}, \tag{31}$$

and

$$I_{\phi\,\text{new new}} = a_1 \, I_{\phi\,\text{old}}, \tag{32}$$

$$q_{\text{new new}} = q_{\text{new}} = q_{\text{old}} \sqrt{1 + \left(\frac{q_{95\,\text{target}}^2}{q_{95\,\text{old}}^2} - 1 \right) \frac{T_{95\,\text{old}}^2}{T_{\text{old}}^2}}. \tag{33}$$

Note that

$$q_{95\,\text{new new}} = q_{95\,\text{target}}, \tag{34}$$

$$T_{1\,\text{new new}} = T_{1\,\text{old}}. \tag{35}$$