

Cylindrical Tearing Mode Analysis

Richard Fitzpatrick

August 28, 2021

1 Normalization

All lengths are normalized to R_0 . All magnetic field-strengths are normalized to B_0 . All current densities are normalized to $B_0/(R_0 \mu_0)$.

2 Coordinates

Adopt cylindrical coordinates r, θ, z .

3 Plasma Equilibrium

The equilibrium magnetic field is written

$$\mathbf{B} = \nabla \Psi(r) \times \mathbf{e}_z + \mathbf{e}_z. \quad (1)$$

So,

$$B_r = 0, \quad (2)$$

$$B_\theta = -\frac{d\Psi}{dr}, \quad (3)$$

$$B_z = 1. \quad (4)$$

The safety-factor profile is

$$q(r) = \frac{r B_z}{B_\theta} = \frac{r}{B_\theta}. \quad (5)$$

The equilibrium current density is written

$$\mathbf{J} = J_z(r) \mathbf{e}_z, \quad (6)$$

where

$$J_z = \frac{1}{r} \frac{d(r B_\theta)}{dr} = \frac{1}{r} \frac{d}{dr} \left(\frac{r^2}{q} \right) = \frac{2-s}{q}, \quad (7)$$

where

$$s(r) = \frac{d \ln q}{dr}. \quad (8)$$

4 Cylindrical Tearing Mode Equation

Let

$$\mathbf{b} = \nabla [\psi(r) e^{i(m\theta - nz)}] \times \mathbf{e}_z \quad (9)$$

$$\mathbf{j} \simeq - \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) - \frac{m^2}{r^2} \psi \right] e^{i(m\theta - nz)} \mathbf{e}_z, \quad (10)$$

be the perturbed magnetic field and current density, respectively. Here, we have assumed that $r \ll 1$. Linearized force balance yields

$$(\mathbf{J} \cdot \nabla) \mathbf{b} + (\mathbf{j} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{j} - (\mathbf{b} \cdot \nabla) \mathbf{J} = 0. \quad (11)$$

The z component of the previous equation gives the cylindrical tearing mode equation:

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{m^2}{r^2} \psi - \frac{J'_z \psi}{r(1/q - n/m)} = 0, \quad (12)$$

where $' \equiv d/dr$.

5 Behavior in Vicinity of Rational Surface

Suppose that $q(r_s) = m/n$. Let $\rho = r - r_s$. Expanding the previous equation about $r = r_s$, we obtain

$$\frac{d^2\psi}{d\rho^2} + \frac{1}{r_s} \left(1 - \frac{\rho}{r_s} \right) \frac{d\psi}{d\rho} - \frac{m^2}{r_s^2} \left(1 - \frac{2\rho}{r_s} \right) \psi - \left(\frac{\alpha}{\rho} + \beta \right) \psi = 0, \quad (13)$$

where

$$\alpha = - \left(\frac{q J'_z}{s} \right)_{r=r_s}, \quad (14)$$

$$\beta = \alpha \left(\frac{J''_z}{J'_z} + \frac{s-1}{r_s} - \frac{q''}{2q'} \right)_{r=r_s}. \quad (15)$$

We obtain

$$\begin{aligned} \psi(\rho) = & \Psi \left[1 + \left(\frac{\Sigma' \pm \Delta'}{2} \right) \rho \right. \\ & + \left\{ \alpha \left(\frac{\Sigma' \pm \Delta'}{4} \right) \left(1 - \frac{1}{\alpha r_s} \right) + \frac{1}{2} \left[\frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left(3 - \frac{1}{\alpha r_s} \right) \right] \right\} \rho^2 \\ & \left. + \alpha \left\{ \rho + \frac{\alpha}{2} \left(1 - \frac{1}{\alpha r_s} \right) \rho^2 \right\} \ln |\rho| + \mathcal{O}(\rho^3) \right]. \end{aligned} \quad (16)$$

Here, the plus sign corresponds to $\rho > 0$, whereas the minus sign corresponds to $\rho < 0$. It follows that

$$\begin{aligned} \psi'(\rho) = \Psi & \left[\frac{\Sigma' \pm \Delta'}{2} + \alpha \right. \\ & + \left\{ \alpha \left(\frac{\Sigma' \pm \Delta'}{2} \right) \left(1 - \frac{1}{\alpha r_s} \right) + \frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left(3 - \frac{1}{\alpha r_s} \right) + \frac{\alpha^2}{2} \left(1 - \frac{1}{\alpha r_s} \right) \right\} \rho \\ & \left. + \alpha \ln |\rho| + \alpha^2 \left(1 - \frac{1}{\alpha r_s} \right) \rho \ln |\rho| + \mathcal{O}(\rho^2) \right]. \end{aligned} \quad (17)$$

Now,

$$\lambda_+ \equiv \left. \frac{\psi'}{\psi} \right|_{\rho=\delta} = \frac{a_{1+} + b_{1+} (\Sigma' + \Delta')}{a_{2+} + b_{2+} (\Sigma' + \Delta')}, \quad (18)$$

where

$$a_{1+} = \alpha (1 + \ln \delta) + \left\{ \frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left(3 - \frac{1}{\alpha r_s} \right) + \alpha^2 \left(1 - \frac{1}{\alpha r_s} \right) \left(\frac{1}{2} + \ln \delta \right) \right\} \delta, \quad (19)$$

$$b_{1+} = \frac{1}{2} + \frac{\alpha}{2} \left(2 - \frac{1}{\alpha r_s} \right) \delta, \quad (20)$$

$$a_{2+} = 1 + \alpha \delta \ln \delta, \quad (21)$$

$$b_{2+} = \frac{\delta}{2}. \quad (22)$$

It follows that

$$\Sigma' + \Delta' = \frac{a_{2+} \lambda_+ - a_{1+}}{b_{1+} - b_{2+} \lambda_+}. \quad (23)$$

Likewise,

$$\lambda_- \equiv \left. \frac{\psi'}{\psi} \right|_{\rho=-\delta} = \frac{a_{1-} + b_{1-} (\Sigma' - \Delta')}{a_{2-} + b_{2-} (\Sigma' - \Delta')}, \quad (24)$$

where

$$a_{1-} = \alpha (1 + \ln \delta) - \left\{ \frac{m^2}{r_s^2} + \beta - \frac{\alpha^2}{2} \left(3 - \frac{1}{\alpha r_s} \right) + \alpha^2 \left(1 - \frac{1}{\alpha r_s} \right) \left(\frac{1}{2} + \ln \delta \right) \right\} \delta, \quad (25)$$

$$b_{1-} = \frac{1}{2} - \frac{\alpha}{2} \left(2 - \frac{1}{\alpha r_s} \right) \delta, \quad (26)$$

$$a_{2-} = 1 - \alpha \delta \ln \delta, \quad (27)$$

$$b_{2-} = -\frac{\delta}{2}. \quad (28)$$

It follows that

$$\Sigma' - \Delta' = \frac{a_{2-} \lambda_- - a_{1-}}{b_{1-} - b_{2-} \lambda_-}. \quad (29)$$

6 Solution Launched from Magnetic Axis

Let

$$\psi(r) = r^m f(r). \quad (30)$$

It follows that

$$\psi'(r) = m r^{m-1} f + r^m f'. \quad (31)$$

The cylindrical tearing mode equation transforms into

$$f'' + (1 + 2m) \frac{f'}{r} - \frac{J'_z f}{r(1/q - n/m)} = 0. \quad (32)$$

The well-behaved solution launched from the magnetic axis is such that

$$f(r) = 1 + \left[\frac{c}{4(1+m)} \right] r^2, \quad (33)$$

where

$$c = \left(\frac{J''_z}{1/q - n/m} \right)_{r=0}. \quad (34)$$

7 Solution Launched from Plasma Boundary

Suppose that the plasma boundary lies at $r = a$. Let

$$\psi(r) = r^{-m} g(r). \quad (35)$$

It follows that

$$\psi'(r) = -m r^{-m-1} g + r^{-m} g'. \quad (36)$$

The cylindrical tearing mode equation transforms into

$$g'' + (1 - 2m) \frac{g'}{r} - \frac{J'_z g}{r(1/q - n/m)} = 0. \quad (37)$$

The well-behaved solution launched from the plasma boundary is such that

$$g(a) = 1, \quad (38)$$

$$g'(a) = 0. \quad (39)$$