Neoclassical Physics

Richard Fitzpatrick

November 16, 2020

1 Lowest-Order Force Balance

Let

$$\vec{B} = I(\psi)\,\vec{\nabla}\zeta + \vec{\nabla}\zeta\times\vec{\nabla}\psi, \tag{1}$$

$$\vec{E} = -\vec{\nabla}\Phi(\psi) + E_\parallel\,\vec{b}, \tag{2}$$

where $\vec{b}=\vec{B}/B$. Expect $n_a = n_a(\psi)$, $p_a = p_a(\psi)$, and $T_a = T_a(\psi)$. We can write

$$\vec{u}_j^{\,a} = u_{\parallel\,j}^{\,a}\,\vec{b} + \vec{u}_{\perp\,j}^{\,a}, \tag{3}$$

and

$$\vec{u}_0^{\,a} = \vec{V}_a, \tag{4}$$

$$\vec{u}_1^{\,a} = -\frac{2}{5}\,\frac{\vec{q}_a}{p_a}, \tag{5}$$

with

$$\vec{u}_{\perp\,0}^{\,a} = \frac{\vec{E}\times\vec{B}}{B^{\,2}} + \frac{\vec{B}\times\vec{\nabla}p_a}{n_a\,e_a\,B^{\,2}}, \tag{6}$$

$$\vec{u}_{\perp\,1}^{\,a} = -\frac{\vec{B}\times\vec{\nabla}T_a}{e_a\,B^{\,2}}. \tag{7}$$

It follows that

$$u_{\theta\,j}^{\,a} = \frac{u_{\parallel\,j}^{\,a}}{B} - \frac{V_{\psi\,j}^{\,a}}{B^{\,2}}, \tag{8}$$

where

$$u_{\theta\,j}^{\,a}(\psi) = \frac{\vec{u}_j^{\,a}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}, \tag{9}$$

$$V_{\psi\,0}^{\,a}(\psi) = -\left(\frac{\vec{u}_{\perp\,0}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}\right)B^{\,2} = -\frac{I\,T_a}{e_a}\left(\frac{d\ln p_a}{d\psi} + \frac{e_a}{T_a}\,\frac{d\Phi}{d\psi}\right), \tag{10}$$

$$V_{\psi\,1}^{\,a}(\psi) = -\left(\frac{\vec{u}_{\perp\,1}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}\right)B^{\,2} = \frac{I\,T_a}{e_a}\,\frac{d\ln T_a}{d\psi}. \tag{11}$$

2 Friction Forces

$$\begin{pmatrix} \vec{F}_{a\,0} \\ \vec{F}_{a\,1} \end{pmatrix} = \sum_b \begin{pmatrix} l_{00}^{ab}, & l_{01}^{ab} \\ l_{10}^{ab}, & l_{11}^{ab} \end{pmatrix} \begin{pmatrix} \vec{u}_0^{\,b} \\ \vec{u}_1^{\,b} \end{pmatrix}, \tag{12}$$

$$l_{jk}^{ab} = \left(\sum_c \frac{n_a\, m_a}{\tau_{ac}}\, M_{ac}^{jk} \right) \delta_{ab} + \frac{n_a\, m_a}{\tau_{ab}}\, N_{ab}^{jk}, \tag{13}$$

$$\frac{1}{\tau_{ab}} = \frac{4}{3\sqrt{\pi}}\, \frac{4\pi\, n_b\, e_a^2\, e_b^2\, \ln \Lambda}{(4\pi\, \epsilon_0)^2\, m_a^2\, v_{T\,a}^3}, \tag{14}$$

$$v_{T\,a} = \sqrt{\frac{2\, T_a}{m_a}}, \tag{15}$$

$$M_{ab}^{00} = -\frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{\,3/2}}, \tag{16}$$

$$M_{ab}^{01} = -\frac{3}{2}\, \frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{\,5/2}} = M_{ab}^{10}, \tag{17}$$

$$M_{ab}^{11} = -\frac{13/4 + 4\, x_{ab}^2 + (15/2)\, x_{ab}^4}{(1 + x_{ab}^2)^{\,5/2}}, \tag{18}$$

$$N_{ab}^{00} = \frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{\,3/2}}, \tag{19}$$

$$N_{ab}^{01} = \frac{3}{2}\, \frac{T_a}{T_b}\, \frac{1 + m_b/m_a}{x_{ab}\, (1 + x_{ba}^2)^{\,5/2}}, \tag{20}$$

$$N_{ab}^{10} = \frac{3}{2}\, \frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{\,5/2}}, \tag{21}$$

$$N_{ab}^{11} = \frac{27}{4}\, \frac{T_a}{T_b}\, \frac{x_{ab}^2}{(1 + x_{ab}^2)^{\,5/2}}, \tag{22}$$

where

$$x_{ab} = \frac{v_{T\,b}}{v_{T\,a}}. \tag{23}$$

Species are e, i, and I. Let

$$Z_{\rm eff} = \frac{n_i\, e_i^2 + n_I\, e_I^2}{n_e\, e_e^2}. \tag{24}$$

Follows that

$$\frac{n_e}{n_i} = \frac{Z_I - 1}{Z_I - Z_{\rm eff}}, \tag{25}$$

$$\frac{n_I}{n_i} = \frac{Z_{\rm eff} - 1}{Z_I\, (Z_I - Z_{\rm eff})}, \tag{26}$$

$$\frac{n_I}{n_e} = \frac{Z_{\rm eff}-1}{Z_I\,(Z_I-1)}. \tag{27}$$

Thus,

$$M_{aa}^{00} = -\frac{1}{\sqrt{2}}, \tag{28}$$

$$M_{aa}^{01} = -\frac{3}{4\sqrt{2}} = M_{aa}^{10}, \tag{29}$$

$$M_{aa}^{11} = -\frac{59}{16\sqrt{2}}, \tag{30}$$

$$N_{aa}^{00} = \frac{1}{\sqrt{2}}, \tag{31}$$

$$N_{aa}^{01} = \frac{3}{4\sqrt{2}}, \tag{32}$$

$$N_{aa}^{10} = \frac{3}{4\sqrt{2}}, \tag{33}$$

$$N_{aa}^{11} = \frac{27}{16\sqrt{2}}, \tag{34}$$

for $a = e$, i, I. Now,

$$M_{iI}^{00} = -\frac{1+m_i/m_I}{(1+x_{iI}^{\,2})^{\,3/2}}, \tag{35}$$

$$M_{iI}^{01} = -\frac{3}{2}\,\frac{1+m_i/m_I}{(1+x_{iI}^{\,2})^{5/2}} = M_{iI}^{10}, \tag{36}$$

$$M_{iI}^{11} = -\frac{13/4+4\,x_{iI}^{\,2}+(15/2)\,x_{iI}^{\,4}}{(1+x_{iI}^{\,2})^{\,5/2}}, \tag{37}$$

$$N_{iI}^{00} = \frac{1+m_i/m_I}{(1+x_{iI}^{\,2})^{\,3/2}}, \tag{38}$$

$$N_{iI}^{01} = \frac{3}{2}\,\frac{T_i}{T_I}\,\frac{1+m_I/m_i}{x_{iI}\,(1+x_{Ii}^{\,2})^{\,5/2}}, \tag{39}$$

$$N_{iI}^{10} = \frac{3}{2}\,\frac{1+m_i/m_I}{(1+x_{iI}^{\,2})^{\,5/2}}, \tag{40}$$

$$N_{iI}^{11} = \frac{27}{4}\,\frac{T_i}{T_I}\,\frac{x_{iI}^{\,2}}{(1+x_{iI}^{\,2})^{\,5/2}}. \tag{41}$$

Furthermore,

$$M_{Ii}^{00} = -\frac{1+m_I/m_i}{(1+x_{Ii}^{\,2})^{\,3/2}}, \tag{42}$$

$$M_{Ii}^{01} = -\frac{3}{2}\,\frac{1+m_I/m_i}{(1+x_{Ii}^2)^{5/2}} = M_{Ii}^{10}, \tag{43}$$

$$M_{Ii}^{11} = -\frac{13/4+4\,x_{Ii}^2+(15/2)\,x_{Ii}^4}{(1+x_{Ii}^2)^{\,5/2}}, \tag{44}$$

$$N_{Ii}^{00} = \frac{1+m_I/m_i}{(1+x_{Ii}^2)^{\,3/2}}, \tag{45}$$

$$N_{Ii}^{01} = \frac{3}{2}\,\frac{T_I}{T_i}\,\frac{1+m_i/m_I}{x_{Ii}\,(1+x_{iI}^2)^{\,5/2}}, \tag{46}$$

$$N_{Ii}^{10} = \frac{3}{2}\,\frac{1+m_I/m_i}{(1+x_{Ii}^2)^{\,5/2}}, \tag{47}$$

$$N_{Ii}^{11} = \frac{27}{4}\,\frac{T_I}{T_i}\,\frac{x_{Ii}^2}{(1+x_{Ii}^2)^{\,5/2}}. \tag{48}$$

Also,

$$M_{ea}^{00} = -1, \tag{49}$$

$$M_{ea}^{01} = -\frac{3}{2} = M_{ea}^{10}, \tag{50}$$

$$M_{ea}^{11} = -\frac{13}{4}, \tag{51}$$

$$N_{ea}^{00} = 1, \tag{52}$$

$$N_{ea}^{01} = 0, \tag{53}$$

$$N_{ea}^{10} = \frac{3}{2}, \tag{54}$$

$$N_{ea}^{11} = 0, \tag{55}$$

for $a = i,\, I$, where we have neglected terms of order $(m_e/m_i)^{\,1/2}$. Finally,

$$M_{ae}^{00} = 0, \tag{56}$$

$$M_{ae}^{01} = 0 = M_{ae}^{10}, \tag{57}$$

$$M_{ae}^{11} = 0, \tag{58}$$

$$N_{ae}^{00} = 0, \tag{59}$$

$$N_{ae}^{01} = 0, \tag{60}$$

$$N_{ae}^{10} = 0 \tag{61}$$

$$N_{ae}^{11} = 0, \tag{62}$$

for $a = i,\, I$.

Now,

$$l^{ii}_{jk} = \frac{n_i\,m_i}{\tau_{ii}}\left[\left(\sum_a \frac{n_a\,e_a^{\,2}}{n_i\,e_i^{\,2}}\,M^{jk}_{ia}\right)+N^{jk}_{ii}\right] \tag{63}$$

$$= \frac{n_i\,m_i}{\tau_{ii}}\left(M^{jk}_{ii}+N^{jk}_{ii}+\alpha\,M^{jk}_{iI}\right), \tag{64}$$

where

$$\alpha = \frac{n_I\,e_I^{\,2}}{n_i\,e_i^{\,2}} = (Z_{\rm eff}-1)\left(\frac{Z_I}{Z_I-Z_{\rm eff}}\right). \tag{65}$$

Thus,

$$l^{ii}_{00} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^{\,2})^{\,3/2}}, \tag{66}$$

$$l^{ii}_{01} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^{\,2})^{5/2}}, \tag{67}$$

$$l^{ii}_{10} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^{\,2})^{5/2}}, \tag{68}$$

$$l^{ii}_{11} = -\frac{n_i\,m_i}{\tau_{ii}}\,\left\{\sqrt{2}+\frac{\alpha\,[13/4+4\,x_{iI}^{\,2}+(15/2)\,x_{iI}^{\,4}]}{(1+x_{iI}^{\,2})^{\,5/2}}\right\}. \tag{69}$$

Now,

$$l^{iI}_{jk} = \frac{n_i\,m_i}{\tau_{ii}}\,\alpha\,N^{jk}_{iI}, \tag{70}$$

so

$$l^{iI}_{00} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^{\,2})^{\,3/2}}, \tag{71}$$

$$l^{iI}_{01} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha\,(1+m_I/m_i)}{x_{iI}\,(1+x_{Ii}^{\,2})^{\,5/2}}, \tag{72}$$

$$l^{iI}_{10} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^{\,2})^{\,5/2}}, \tag{73}$$

$$l^{iI}_{11} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{27}{4}\,\frac{T_i}{T_I}\,\frac{\alpha\,x_{iI}^{\,2}}{(1+x_{iI}^{\,2})^{\,5/2}}. \tag{74}$$

Now,

$$l^{II}_{jk} = \frac{n_I\,m_I}{\tau_{II}}\left[\left(\sum_a \frac{n_a\,e_a^{\,2}}{n_I\,e_I^{\,2}}\,M^{jk}_{Ia}\right)+N^{jk}_{II}\right] \tag{75}$$

$$= \frac{n_I\,m_I}{\tau_{II}}\left(M^{jk}_{II}+N^{jk}_{II}+\alpha^{-1}\,M^{jk}_{Ii}\right)$$

$$= \frac{n_i\, m_i}{\tau_{ii}}\, \alpha^{\,2}\, \frac{m_i}{m_I}\, x_{Ii}^3 \left(M_{II}^{jk} + N_{II}^{jk} + \alpha^{-1}\, M_{Ii}^{jk}\right), \tag{76}$$

so

$$l_{00}^{II} = -\frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3 \frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{\,3/2}}, \tag{77}$$

$$l_{01}^{II} = -\frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3\, \frac{3}{2}\, \frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}}, \tag{78}$$

$$l_{10}^{II} = -\frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3\, \frac{3}{2}\, \frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}}, \tag{79}$$

$$l_{11}^{II} = -\frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3\, \left\{\sqrt{2}\,\alpha^{\,2} + \frac{\alpha\,[13/4 + 4\,x_{Ii}^2 + (15/2)\,x_{Ii}^4]}{(1+x_{Ii}^2)^{\,5/2}}\right\}. \tag{80}$$

Finally,

$$l_{jk}^{Ii} = \frac{n_i\, m_i}{\tau_{ii}}\, \alpha\, \frac{m_i}{m_I}\, x_{Ii}^3\, N_{Ii}^{jk}, \tag{81}$$

so

$$l_{00}^{Ii} = \frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3\, \frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{\,3/2}}, \tag{82}$$

$$l_{01}^{Ii} = \frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3\, \frac{3}{2}\, \frac{T_I}{T_i}\, \frac{\alpha\,(1+m_i/m_I)}{x_{Ii}\,(1+x_{iI}^2)^{\,5/2}}, \tag{83}$$

$$l_{10}^{Ii} = \frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3\, \frac{3}{2}\, \frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{\,5/2}}, \tag{84}$$

$$l_{11}^{Ii} = \frac{n_i\, m_i}{\tau_{ii}}\, \frac{m_i}{m_I}\, x_{Ii}^3\, \frac{27}{4}\, \frac{T_I}{T_i}\, \frac{\alpha\, x_{Ii}^2}{(1+x_{Ii}^2)^{\,5/2}}. \tag{85}$$

Thus,

$$(F^{\,i}) = \frac{n_i\, m_i}{\tau_{ii}}\, \left\{-\left[F^{\,ii}\right]\,(u^{\,i}) + \left[F^{\,iI}\right]\,(u^{\,I})\right\}, \tag{86}$$

$$(F^{\,I}) = \frac{n_i\, m_i}{\tau_{ii}}\, \left\{\left[F^{\,Ii}\right]\,(u^{\,i}) - \left[F^{\,II}\right]\,(u^{\,I})\right\}, \tag{87}$$

where

$$(F^{\,a}) = \begin{pmatrix} \vec{F}_{a\,0} \\ \vec{F}_{a\,1} \end{pmatrix}, \tag{88}$$

$$(u^{\,a}) = \begin{pmatrix} \vec{u}_0^{\,a} \\ \vec{u}_1^{\,a} \end{pmatrix}. \tag{89}$$

Here,

$$F_{00}^{\,ii} = \frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI})^{\,3/2}}, \tag{90}$$

$$F_{01}^{ii} = \frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{5/2}}, \tag{91}$$

$$F_{10}^{ii} = \frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{5/2}} = F_{01}^{ii}, \tag{92}$$

$$F_{11}^{ii} = \sqrt{2} + \frac{\alpha\,[13/4 + 4\,x_{iI}^2 + (15/2)\,x_{iI}^4]}{(1+x_{iI}^2)^{\,5/2}}, \tag{93}$$

$$F_{00}^{iI} = \frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{\,3/2}} = F_{00}^{ii}, \tag{94}$$

$$F_{01}^{iI} = \frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha\,(1+m_I/m_i)}{x_{iI}\,(1+x_{Ii}^2)^{\,5/2}}, \tag{95}$$

$$F_{10}^{iI} = \frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{\,5/2}} = F_{01}^{ii}, \tag{96}$$

$$F_{11}^{iI} = \frac{27}{4}\,\frac{T_i}{T_I}\,\frac{\alpha\,x_{iI}^2}{(1+x_{iI}^2)^{\,5/2}}, \tag{97}$$

$$F_{00}^{Ii} = \frac{m_i}{m_I}\,x_{Ii}^3\,\frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{\,3/2}} = \frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{\,3/2}} = F_{00}^{ii}, \tag{98}$$

$$F_{01}^{Ii} = \frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{T_I}{T_i}\,\frac{\alpha\,(1+m_i/m_I)}{x_{Ii}\,(1+x_{iI}^2)^{\,5/2}} = \frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{\,5/2}} = F_{01}^{ii}, \tag{99}$$

$$F_{10}^{Ii} = \frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{\,5/2}} = \frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha\,(1+m_I/m_i)}{x_{iI}\,(1+x_{Ii}^2)^{\,5/2}} = F_{01}^{iI}, \tag{100}$$

$$F_{11}^{Ii} = \frac{m_i}{m_I}\,x_{Ii}^3\,\frac{27}{4}\,\frac{T_I}{T_i}\,\frac{\alpha\,x_{Ii}^2}{(1+x_{Ii}^2)^{\,5/2}} = \frac{27}{4}\,\frac{\alpha\,x_{iI}^2}{(1+x_{iI}^2)^{\,5/2}}, \tag{101}$$

$$F_{00}^{II} = \frac{m_i}{m_I}\,x_{Ii}^3\,\frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{\,3/2}} = \frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{\,3/2}} = F_{00}^{ii}, \tag{102}$$

$$F_{01}^{II} = \frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}} = \frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha\,(1+m_I/m_i)}{x_{iI}\,(1+x_{Ii}^2)^{\,5/2}} = F_{01}^{iI}, \tag{103}$$

$$F_{10}^{II} = \frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}} = \frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha\,(1+m_I/m_i)}{x_{iI}\,(1+x_{Ii}^2)^{\,5/2}} = F_{01}^{iI}, \tag{104}$$

$$\begin{aligned} F_{11}^{II} &= \frac{m_i}{m_I}\,x_{Ii}^3\,\left\{\sqrt{2}\,\alpha^{\,2} + \frac{\alpha\,[13/4 + 4\,x_{Ii}^2 + (15/2)\,x_{Ii}^4]}{(1+x_{Ii}^2)^{\,5/2}}\right\} \\ &= \frac{T_i}{T_I}\left\{\sqrt{2}\,\alpha^{\,2}\,x_{Ii} + \frac{\alpha\,[15/2 + 4\,x_{iI}^2 + (13/4)\,x_{iI}^4]}{(1+x_{iI}^2)^{\,5/2}}\right\}. \end{aligned} \tag{105}$$

Note that $F_{0j}^{ii} = F_{0j}^{Ii}$ and $F_{0j}^{iI} = F_{0j}^{II}$, for $j = 0,\,1$, which implies that $(F^{\,I})_0 + (F^{\,i})_0 = 0$. This is a statement of collisional momentum conservation. Furthermore, $F_{j0}^{ii} = F_{j0}^{iI}$ and $F_{j0}^{Ii} = F_{j0}^{II}$.

Now,

$$l_{jk}^{ee} = \frac{n_e\,m_e}{\tau_{ee}}\left[\left(\sum_a \frac{n_a\,e_a^2}{n_e\,e_e^2}\,M_{ea}^{jk}\right) + N_{ee}^{jk}\right] \tag{106}$$

$$= \frac{n_e\,m_e}{\tau_{ee}}\left(M_{ee}^{jk} + N_{ee}^{jk} + Z_{\rm eff}\,M_{ei}^{jk}\right), \tag{107}$$

where $a = e,\, i,\, I$. It follows that

$$l_{00}^{ee} = -\frac{n_e\,m_e}{\tau_{ee}}\,Z_{\rm eff}, \tag{108}$$

$$l_{01}^{ee} = -\frac{n_e\,m_e}{\tau_{ee}}\,\frac{3}{2}\,Z_{\rm eff}, \tag{109}$$

$$l_{10}^{ee} = -\frac{n_e\,m_e}{\tau_{ee}}\,\frac{3}{2}\,Z_{\rm eff}, \tag{110}$$

$$l_{11}^{ee} = -\frac{n_e\,m_e}{\tau_{ee}}\left(\sqrt{2} + \frac{13}{4}\,Z_{\rm eff}\right). \tag{111}$$

Likewise,

$$l_{jk}^{ea} = \frac{n_e\,m_e}{\tau_{ee}}\,Z_{{\rm eff}\,a}\,N_{ea}^{jk}, \tag{112}$$

where

$$Z_{{\rm eff}\,a} = \frac{n_a\,e_a^2}{n_e\,e_e^2}, \tag{113}$$

for $a = i,\, I$. In particular,

$$Z_{{\rm eff}\,i} = \frac{Z_I - Z_{\rm eff}}{Z_I - 1}, \tag{114}$$

$$Z_{{\rm eff}\,I} = \frac{Z_I\,(Z_{\rm eff} - 1)}{Z_I - 1}. \tag{115}$$

Hence,

$$l_{00}^{ei} = \frac{n_e\,m_e}{\tau_{ee}}\,Z_{{\rm eff}\,i}, \tag{116}$$

$$l_{01}^{ei} = 0, \tag{117}$$

$$l_{10}^{ei} = \frac{n_e\,m_e}{\tau_{ee}}\,\frac{3}{2}\,Z_{{\rm eff}\,i}, \tag{118}$$

$$l_{11}^{ei} = 0, \tag{119}$$

and

$$l_{00}^{eI} = \frac{n_e\,m_e}{\tau_{ee}}\,Z_{{\rm eff}\,I}, \tag{120}$$

$$l^{eI}_{01} = 0, \tag{121}$$

$$l^{eI}_{10} = \frac{n_e\, m_e}{\tau_{ee}}\,\frac{3}{2}\,Z_{\mathrm{eff}\,I}, \tag{122}$$

$$l^{eI}_{11} = 0. \tag{123}$$

Thus,

$$(F^{\,e}) = \frac{n_e\, m_e}{\tau_{ee}}\left\{-\left[F^{\,ee}\right](u^{\,e}) + \left[F^{\,ei}\right](u^{\,i}) + \left[F^{\,eI}\right](u^{\,I})\right\}, \tag{124}$$

where

$$F^{ee}_{00} = Z_{\mathrm{eff}}, \tag{125}$$

$$F^{ee}_{01} = \frac{3}{2}\,Z_{\mathrm{eff}}, \tag{126}$$

$$F^{ee}_{10} = \frac{3}{2}\,Z_{\mathrm{eff}}, \tag{127}$$

$$F^{ee}_{11} = \sqrt{2} + \frac{13}{4}\,Z_{\mathrm{eff}}, \tag{128}$$

$$F^{ei}_{00} = Z_{\mathrm{eff}\,i}, \tag{129}$$

$$F^{ei}_{01} = 0, \tag{130}$$

$$F^{ei}_{10} = \frac{3}{2}\,Z_{\mathrm{eff}\,i}, \tag{131}$$

$$F^{ei}_{11} = 0, \tag{132}$$

$$F^{eI}_{00} = Z_{\mathrm{eff}\,I}, \tag{133}$$

$$F^{eI}_{01} = 0, \tag{134}$$

$$F^{eI}_{10} = \frac{3}{2}\,Z_{\mathrm{eff}\,I}, \tag{135}$$

$$F^{eI}_{11} = 0. \tag{136}$$

Note that $F^{ee}_{j0} = F^{ei}_{j0} + F^{eI}_{j0}$.

3 Neoclassical Viscosities

$$\left\langle \vec{B}\cdot\vec{\nabla}\cdot(\overleftrightarrow{\Pi}^{\,a})\right\rangle = \frac{n_a\, m_a}{\tau_{aa}}\,[\mu^{\,a}]\,(u^a_\theta), \tag{137}$$

where

$$\langle A\rangle = \oint \frac{A(\theta)\,d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\Bigg/\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}, \tag{138}$$

$$(\overleftrightarrow{\Pi}^{\,a}) = \begin{pmatrix} \overleftrightarrow{\Pi}_{\parallel\,a} \\ \overleftrightarrow{\Theta}_{\parallel\,a}\end{pmatrix}, \tag{139}$$

$$(u_\theta^a) = \begin{pmatrix} u_{\theta\,0}^a\,\langle B^2\rangle \\ u_{\theta\,1}^a\,\langle B^2\rangle \end{pmatrix}, \tag{140}$$

$$[\mu^a] = \begin{pmatrix} \mu_{00}^a, & \mu_{01}^a \\ \mu_{10}^a, & \mu_{11}^a \end{pmatrix}. \tag{141}$$

Let

$$\mu_{00}^a = K_{00}^a, \tag{142}$$

$$\mu_{01}^a = \frac{5}{2}\,K_{00}^a - K_{01}^a = \mu_{10}^a, \tag{143}$$

$$\mu_{11}^a = K_{11}^a - 5\,K_{01}^a + \frac{25}{4}\,K_{00}^a, \tag{144}$$

and

$$\omega_{t\,a} = \frac{v_{T\,a}}{L_c}, \tag{145}$$

$$\nu_{*\,a} = \frac{8}{3\pi}\,\frac{\langle B^2\rangle}{\langle(\vec{b}\cdot\vec{\nabla}B)^2\rangle}\,\frac{g\,\omega_{t\,a}}{v_{T\,a}^2\,\tau_{aa}}, \tag{146}$$

$$g = \frac{1-f_c}{f_c}, \tag{147}$$

$$f_c = \frac{3}{4}\,\langle B^2\rangle \int_0^{1/B_{\rm max}} \frac{\lambda\,d\lambda}{\langle\sqrt{1-\lambda\,B}\rangle}. \tag{148}$$

We have

$$K_{ij}^a = g\,\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{e^{-x}\,x^{4+i+j}\,\hat{\nu}_D^a(x)\,dx}{[x^2+\nu_{*\,a}\,\hat{\nu}_D^a(x)]\,[x^2+(5\pi/8)\,(\omega_{t\,a}\,\tau_{aa})^{-1}\,\hat{\nu}_T^a(x)]}, \tag{149}$$

and

$$\hat{\nu}_D^a(x) = \frac{3\sqrt{\pi}}{4}\sum_b \frac{n_b\,e_b^2}{n_a\,e_a^2}\left[\left(1-\frac{x_{ab}}{2\,x}\right)\psi\!\left(\frac{x}{x_{ab}}\right)+\psi'\!\left(\frac{x}{x_{ab}}\right)\right]\frac{1}{x}, \tag{150}$$

$$\hat{\nu}_\epsilon^{\,a}(x) = \frac{3\sqrt{\pi}}{2}\sum_b \frac{n_b\,e_b^2}{n_a\,e_a^2}\left[\frac{m_a}{m_b}\,\psi\!\left(\frac{x}{x_{ab}}\right)-\psi'\!\left(\frac{x}{x_{ab}}\right)\right]\frac{1}{x}, \tag{151}$$

$$\hat{\nu}_T^a(x) = 3\,\hat{\nu}_D^a(x)+\hat{\nu}_\epsilon^{\,a}(x), \tag{152}$$

$$\psi(x) = \frac{2}{\sqrt{\pi}}\int_0^{x^2}\sqrt{t}\,{\rm e}^{-t}\,dt = {\rm erf}(x) - \frac{2}{\sqrt{\pi}}\,x\,{\rm e}^{-x^2}, \tag{153}$$

$$\psi'(x) = \frac{2}{\sqrt{\pi}}\,x\,{\rm e}^{-x^2}. \tag{154}$$

In particular,

$$K_{ij}^i = g\,\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{e^{-x}\,x^{2+i+j}\,\hat{\nu}_D^i(x)\,dx}{[x+\nu_{*\,i}\,\hat{\nu}_D^i(x)]\,[x+(5\pi/8)\,(\omega_{t\,i}\,\tau_{ii})^{-1}\,\hat{\nu}_T^i(x)]}, \tag{155}$$

$$\hat{\nu}_D^{\,i} = \frac{3\sqrt{\pi}}{4}\left[\left(1-\frac{1}{2\,x}\right)\psi(x)+\psi'(x)\right]\frac{1}{x}$$
$$+\frac{3\sqrt{\pi}}{4}\,\alpha\left[\left(1-\frac{x_{iI}}{2\,x}\right)\psi\!\left(\frac{x}{x_{iI}}\right)+\psi'\!\left(\frac{x}{x_{iI}}\right)\right]\frac{1}{x}, \tag{156}$$
$$\hat{\nu}_\epsilon^{\,i} = \frac{3\sqrt{\pi}}{2}\left[\psi(x)-\psi'(x)\right]\frac{1}{x}$$
$$+\frac{3\sqrt{\pi}}{2}\,\alpha\left[\frac{m_i}{m_I}\,\psi\!\left(\frac{x}{x_{iI}}\right)-\psi'\!\left(\frac{x}{x_{iI}}\right)\right]\frac{1}{x}, \tag{157}$$

and

$$K_{ij}^{I} = g\,\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{e^{-x}\,x^{\,2+i+j}\,\hat{\nu}_D^{I}(x)\,dx}{[x+\nu_{*\,I}\,\hat{\nu}_D^{I}(x)]\,[x+(5\pi/8)\,(\omega_{t\,I}\,\tau_{II})^{-1}\,\hat{\nu}_T^{I}(x)]}, \tag{158}$$
$$\hat{\nu}_D^{\,I} = \frac{3\sqrt{\pi}}{4}\left[\left(1-\frac{1}{2\,x}\right)\psi(x)+\psi'(x)\right]\frac{1}{x}$$
$$+\frac{3\sqrt{\pi}}{4}\,\alpha^{-1}\left[\left(1-\frac{x_{Ii}}{2\,x}\right)\psi\!\left(\frac{x}{x_{Ii}}\right)+\psi'\!\left(\frac{x}{x_{Ii}}\right)\right]\frac{1}{x}, \tag{159}$$
$$\hat{\nu}_\epsilon^{\,I} = \frac{3\sqrt{\pi}}{2}\left[\psi(x)-\psi'(x)\right]\frac{1}{x}$$
$$+\frac{3\sqrt{\pi}}{2}\,\alpha^{-1}\left[\frac{m_I}{m_i}\,\psi\!\left(\frac{x}{x_{Ii}}\right)-\psi'\!\left(\frac{x}{x_{Ii}}\right)\right]\frac{1}{x}, \tag{160}$$

and

$$K_{ij}^{e} = g\,\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{e^{-x}\,x^{\,4+i+j}\,\hat{\nu}_D^{e}(x)\,dx}{[x^{\,2}+\nu_{*\,e}\,\hat{\nu}_D^{e}(x)]\,[x^{\,2}+(5\pi/8)\,(\omega_{t\,e}\,\tau_{ee})^{-1}\,\hat{\nu}_T^{e}(x)]}, \tag{161}$$
$$\hat{\nu}_D^{\,e} = \frac{3\sqrt{\pi}}{4}\left[\left(1-\frac{1}{2\,x}\right)\psi(x)+\psi'(x)\right]+\frac{3\sqrt{\pi}}{4}\,Z_{\rm eff}, \tag{162}$$
$$\hat{\nu}_\epsilon^{\,e} = \frac{3\sqrt{\pi}}{2}\left[\psi(x)-\psi'(x)\right]. \tag{163}$$

4 Parallel Force and Heat Balance Equations

$$\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Pi}_a\rangle = \langle\vec{B}\cdot\vec{F}_{a\,0}\rangle+\langle\vec{B}\cdot\vec{F}_{a\,0}^{\,cx}\rangle+n_a\,e_a\,\langle\vec{E}\cdot\vec{B}\rangle, \tag{164}$$
$$\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle = \langle\vec{B}\cdot\vec{F}_{a\,1}\rangle+\langle\vec{B}\cdot\vec{F}_{a\,1}^{\,cx}\rangle. \tag{165}$$

Now,

$$\langle\vec{B}\cdot\vec{F}^{\,i}\rangle = \frac{n_i\,m_i}{\tau_{ii}}\left\{-\left[F^{\,ii}\right](u_\parallel^{i})+\left[F^{\,iI}\right](u_\parallel^{I})\right\}, \tag{166}$$

$$\langle \vec{B}\cdot\vec{F}^{\,I}\rangle = \frac{n_i\,m_i}{\tau_{ii}}\left\{\left[F^{\,Ii}\right](u_\parallel^{\,i}) - \left[F^{\,II}\right](u_\parallel^{\,I})\right\}, \tag{167}$$

$$\langle \vec{B}\cdot\vec{F}^{\,e}\rangle = \frac{n_e\,m_e}{\tau_{ee}}\left\{-\left[F^{\,ee}\right](u_\parallel^{\,e}) + \left[F^{\,ei}\right](u_\parallel^{\,i}) + \left[F^{\,eI}\right](u_\parallel^{\,I})\right\}, \tag{168}$$

$$\langle \vec{B}\cdot\vec{F}^{\,i\,cx}\rangle = -n_i\,m_i\,\langle\sigma\,v\rangle_i^{cx}\,[X^{\,i}](\tilde{u}_\parallel^{\,i}), \tag{169}$$

$$\langle \vec{B}\cdot\vec{F}^{\,I\,cx}\rangle = 0, \tag{170}$$

$$\langle \vec{B}\cdot\vec{F}^{\,e\,cx}\rangle = 0, \tag{171}$$

where

$$(u_\parallel^{\,a}) = \left(\begin{array}{c}\langle u_{\parallel\,0}^{\,a}\,B\rangle\\ \langle u_{\parallel\,1}^{\,a}\,B\rangle\end{array}\right), \tag{172}$$

$$(\tilde{u}_\parallel^{\,a}) = \left(\begin{array}{c}\langle n_n\,u_{\parallel\,0}^{\,a}\,B\rangle\\ \langle n_n\,u_{\parallel\,1}^{\,a}\,B\rangle\end{array}\right), \tag{173}$$

$$[X^{\,i}] = \left(\begin{array}{cc}1, & 0\\ 0, & E_n/T_i\end{array}\right). \tag{174}$$

Here, E_n is the neutral kinetic energy, n_n the neutral number density, and $\langle\sigma\,v\rangle_i^{cx}$ the majority ion charge exchange rate coefficient.

Let

$$(E_\parallel) = \left(\begin{array}{c}\langle E_\parallel\,B\rangle/\langle B\rangle\\ 0\end{array}\right), \tag{175}$$

$$\Omega_a = \frac{e_a\,\langle B\rangle}{m_a}, \tag{176}$$

$$[\tilde{\mu}^{\,I}] = \alpha^{\,2}\,\frac{m_i}{m_I}\,x_{Ii}^3\,[\mu^{\,I}] = \alpha^{\,2}\,\frac{T_i}{T_I}\,x_{Ii}\,[\mu^{\,I}]. \tag{177}$$

Follows that

$$\left[\begin{array}{rr}[F^{\,ii}], & -[F^{\,iI}]\\ -[F^{\,Ii}], & [F^{\,II}]\end{array}\right]\left[\begin{array}{c}(u_\parallel^{\,i})\\ (u_\parallel^{\,I})\end{array}\right] = -\left[\begin{array}{c}[\mu^{\,i}]\,(u_\theta^{\,i})\\ [\tilde{\mu}^{\,I}]\,(u_\theta^{\,I})\end{array}\right] + \left[\begin{array}{c}\tau_{ii}\,\langle\sigma\,v\rangle_i^{cx}\,[X^{\,i}]\,(\tilde{u})_{\parallel\,i},\\ 0\end{array}\right], \tag{178}$$

and

$$[\mu^{\,e}]\,(u_\theta^{\,e}) = -[F^{\,ee}]\,(u_\parallel^{\,e}) + [F^{\,ei}]\,(u_\parallel^{\,i}) + [F^{\,eI}]\,(u_\parallel^{\,I}) + \Omega_e\,\tau_{ee}\,(E_\parallel). \tag{179}$$

Follows from Eq. (8) that

$$(u_\parallel^{\,a}) = (u_\theta^{\,a}) + (V_\psi^{\,a}). \tag{180}$$

where

$$V_\psi^{\,a} = \left(\begin{array}{c}V_{\psi\,0}^{\,a}\\ V_{\psi\,1}^{\,a}\end{array}\right). \tag{181}$$

Thus,

$$\left[\begin{array}{rr}[F^{\,ii}+\mu^{\,i}+Y^{\,i}/y], & -[F^{\,iI}]\\ -[F^{\,Ii}], & [F^{\,II}+\tilde{\mu}^{\,I}]\end{array}\right]\left[\begin{array}{c}(u_\theta^{\,i})\\ (u_\theta^{\,I})\end{array}\right]=-\left[\begin{array}{rr}[F^{\,ii}+Y^{\,i}], & -[F^{\,iI}]\\ -[F^{\,Ii}], & [F^{\,II}]\end{array}\right]\left[\begin{array}{c}(V_\psi^{\,i})\\ (V_\psi^{\,I})\end{array}\right], \tag{182}$$

where

$$[Y^{\,i}]=\tau_{ii}\,\langle\sigma v\rangle_i^{cx}\,\langle n_n\rangle\,[X^{\,i}], \tag{183}$$

$$y=\frac{\langle n_n\rangle\,\langle B^{\,2}\rangle}{\langle n_n\,B^{\,2}\rangle}. \tag{184}$$

Can write

$$(V_\psi^{\,a})=(V_*^{\,a})+(V_E), \tag{185}$$

where

$$V_{*\,0}^{\,a}(\psi)=-\frac{I\,T_a}{e_a}\,\frac{d\ln p_a}{d\psi}, \tag{186}$$

$$V_{*\,1}^{\,a}(\psi)=\frac{I\,T_a}{e_a}\,\frac{d\ln T_a}{d\psi}, \tag{187}$$

$$V_{E\,0}(\psi)=-I\,\frac{d\varPhi}{d\psi}, \tag{188}$$

$$V_{E\,1}(\psi)=0. \tag{189}$$

Fact that $F_{j0}^{\,ii}=F_{j0}^{\,iI}$ and $F_{j0}^{\,Ii}=F_{j0}^{\,II}$ means that we can rewrite Eq. (182) as

$$\left[\begin{array}{rr}[F^{\,ii}+\mu^{\,i}+Y^{\,i}/y], & -[F^{\,iI}]\\ -[F^{\,Ii}], & [F^{\,II}+\tilde{\mu}^{\,I}]\end{array}\right]\left[\begin{array}{c}(u_\theta^{\,i})\\ (u_\theta^{\,I})\end{array}\right]=-\left[\begin{array}{rr}[F^{\,ii}+Y^{\,i}], & -[F^{\,iI}]\\ -[F^{\,Ii}], & [F^{\,II}]\end{array}\right]\left[\begin{array}{c}(V_*^{\,i})\\ (V_*^{\,I})\end{array}\right]. \tag{190}$$

Let

$$\left[\begin{array}{cc}[L^{\,ii}], & [L^{\,iI}]\\ [L^{\,Ii}], & [L^{\,II}]\end{array}\right]=\left[\begin{array}{rr}[F^{\,ii}+\mu^{\,i}+Y^{\,i}/y], & -[F^{\,iI}]\\ -[F^{\,Ii}], & [F^{\,II}+\tilde{\mu}^{\,I}]\end{array}\right]^{-1}\left[\begin{array}{rr}[F^{\,ii}+Y^{\,i}], & -[F^{\,iI}]\\ -[F^{\,Ii}], & [F^{\,II}]\end{array}\right]. \tag{191}$$

Follows that

$$(u_\theta^{\,i})=-[L^{\,ii}]\,(V_*^{\,i})-[L^{\,iI}]\,(V_*^{\,I}), \tag{192}$$

$$(u_\theta^{\,I})=-[L^{\,II}]\,(V_*^{\,I})-[L^{\,Ii}]\,(V_*^{\,i}), \tag{193}$$

$$(u_\parallel^{\,i})=(V_E)+(1-[L^{\,ii}])\,(V_*^{\,i})-[L^{\,iI}]\,(V_*^{\,I}), \tag{194}$$

$$(u_\parallel^{\,I})=(V_E)+(1-[L^{\,II}])\,(V_*^{\,I})-[L^{\,Ii}]\,(V_*^{\,i}). \tag{195}$$

Equation (179) becomes

$$[F^{\,ee}+\mu^{\,e}]\,(u_\theta^{\,e})=-\left([F^{\,ee}]-[F^{\,ei}]-[F^{eI}]\right)(V_E)-[F^{\,ee}]\,(V_*^{\,e})$$

$$+\left\{[F^{ei}]\,(1-[L^{ii}])-[F^{eI}]\,[L^{Ii}]\right\}(V_*^{\,i})$$
$$+\left\{[F^{\,eI}]\,(1-[L^{\,II}])-[F^{\,ei}]\,[L^{\,iI}]\right\}(V_*^{\,I})+\Omega_{ee}\,\tau_{ee}\,(E_\parallel). \qquad (196)$$

However, because $F_{j0}^{ee}=F_{j0}^{ei}+F_{j0}^{eI}$, the previous equation becomes

$$[F^{\,ee}+\mu^{\,e}]\,(u_\theta^{e})=-[F^{\,ee}]\,(V_*^{\,e})+\left\{[F^{\,eI}]\,(1-[L^{\,ii}])-[F^{\,II}]\,[L^{\,Ii}]\right\}(V_*^{\,i})$$
$$+\left\{[F^{eI}]\,(1-[L^{\,II}])-[F^{\,ei}]\,[L^{\,iI}]\right\}(V_*^{\,I})+\Omega_{ee}\,\tau_{ee}\,(E_\parallel). \qquad (197)$$

Let

$$[Q^{\,ee}]=[F^{\,ee}+\mu^{\,e}]^{\,-1}, \qquad (198)$$
$$[L^{\,ee}]=[Q^{\,ee}]\,[F^{\,ee}], \qquad (199)$$
$$[L^{\,ei}]=-[Q^{\,ee}]\left\{[F^{\,ei}]\,(1-[L^{\,ii}])-[F^{\,eI}]\,[L^{\,Ii}]\right\}, \qquad (200)$$
$$[L^{\,eI}]=-[Q^{\,ee}]\left\{[F^{\,eI}]\,(1-[L^{\,II}])-[F^{\,ei}]\,[L^{\,iI}]\right\}. \qquad (201)$$

Follows that

$$(u_\theta^{e})=-[L^{\,ee}]\,(V_*^{\,e})-[L^{\,ei}]\,(V_*^{\,i})-[L^{\,eI}]\,(V_*^{\,I})+\Omega_e\,\tau_{ee}\,[Q^{\,ee}]\,(E_\parallel), \qquad (202)$$
$$(u_\parallel^{e})=(V_E)+(1-[L^{\,ee}])\,(V_*^{\,e})-[L^{\,ei}]\,(V_*^{\,i})-[L^{\,eI}]\,(V_*^{\,I})+\Omega_e\,\tau_{ee}\,[Q^{\,ee}]\,(E_\parallel), \qquad (203)$$

Finally,

$$(j_\parallel)=\sum_a n_a\,e_a\,(u_\parallel^{a})$$
$$=\left\{n_i\,e_i\,(1-[L^{\,ii}])-n_I\,e_I\,[L^{\,Ii}]-n_e\,e_e\,[L^{\,ei}]\right\}(V_*^{\,i})$$
$$+\left\{n_I\,e_I\,(1-[L^{\,II}])-n_i\,e_i\,[L^{\,iI}]-n_e\,e_e\,[L^{\,eI}]\right\}(V_*^{\,I})$$
$$+n_e\,e_e\,(1-[L^{\,ee}])\,(V_*^{\,e})+n_e\,e_e\,\Omega_e\,\tau_{ee}\,[Q^{\,ee}]\,(E_\parallel), \qquad (204)$$

where we have made use of the fact that $n_i\,e_i+n_I\,e_I+n_e\,e_e=0$.

5 Fluid Velocities

Let

$$\omega_E(\psi)=-\frac{d\Phi}{d\psi}, \qquad (205)$$
$$\omega_{*\,a}(\psi)=-\frac{T_a}{e_a}\,\frac{d\ln p_a}{d\psi}, \qquad (206)$$
$$\eta_a(\psi)=\frac{d\ln T_a}{d\ln n_a}. \qquad (207)$$

It follows that

$$V_{*\,0}^{\,a}(\psi) = I\,\omega_{*\,a}, \tag{208}$$

$$V_{*\,1}^{\,a}(\psi) = -I\left(\frac{\eta_a}{1+\eta_a}\right)\omega_{*\,a}, \tag{209}$$

$$V_{E\,0}(\psi) = I\,\omega_E, \tag{210}$$

$$V_{E\,1}(\psi) = 0. \tag{211}$$

We have $e_e = -e$, $e_i = e$, $e_I = Z_I\,e$. Thus,

$$\frac{V_\theta^{\,i}}{B_\theta}\,\frac{\langle B^2\rangle}{I} = -L_{00}^{ii}\,\omega_{*\,i} - L_{00}^{iI}\,\omega_{*\,I} + L_{01}^{ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i} + L_{01}^{iI}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}, \tag{212}$$

$$\frac{V_\theta^{\,I}}{B_\theta}\,\frac{\langle B^2\rangle}{I} = -L_{00}^{II}\,\omega_{*\,I} - L_{00}^{Ii}\,\omega_{*\,i} + L_{01}^{II}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I} + L_{01}^{Ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}, \tag{213}$$

$$\frac{\langle V_\parallel^{\,i}\,B\rangle}{I} = \omega_E + \omega_{*\,i} - L_{00}^{ii}\,\omega_{*\,i} - L_{00}^{iI}\,\omega_{*\,I} + L_{01}^{ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i} + L_{01}^{iI}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}, \tag{214}$$

$$\frac{\langle V_\parallel^{\,I}B\rangle}{I} = \omega_E + \omega_{*\,I} - L_{00}^{II}\,\omega_{*\,I} - L_{00}^{Ii}\,\omega_{*\,i} + L_{01}^{II}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I} + L_{01}^{Ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}. \tag{215}$$

Also,

$$\frac{V_\phi^{\,i}}{R} = \omega_E + \omega_{*\,i} + \frac{I^{\,2}}{R^2\,\langle B^2\rangle}\left[-L_{00}^{ii}\,\omega_{*\,i} - L_{00}^{iI}\,\omega_{*\,I} + L_{01}^{ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i} + L_{01}^{iI}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}\right], \tag{216}$$

$$\frac{V_\phi^{\,I}}{R} = \omega_E + \omega_{*\,I} + \frac{I^{\,2}}{R^2\,\langle B^2\rangle}\left[-L_{00}^{II}\,\omega_{*\,I} - L_{00}^{Ii}\,\omega_{*\,i} + L_{01}^{II}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I} + L_{01}^{Ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}\right]. \tag{217}$$

6 Fluid Angular Rotation Velocities

Let

$$\omega_{\theta\,i}(\psi) = \frac{V_\theta^{\,i}}{B_\theta}\,\frac{I}{R^{\,2}}, \tag{218}$$

$$\omega_{\theta\,I}(\psi) = \frac{V_\theta^{\,I}}{B_\theta}\,\frac{I}{R^{\,2}}, \tag{219}$$

$$\omega_{\phi\,i}(\psi) = \frac{V_\phi^{\,i}}{R}, \tag{220}$$

$$\omega_{\phi\,I}(\psi) = \frac{V_\phi^{\,I}}{R}, \tag{221}$$

where the right-hand sides are evaluated on the outboard mid-plane. It follows that

$$\omega_{\theta\,i} = K_\theta\,\omega_{\mathrm{nc}\,i}, \tag{222}$$

$$\omega_{\theta\,I} = K_\theta\,\omega_{\mathrm{nc}\,I}, \tag{223}$$

$$\omega_{\phi\,i} = \omega_E + \omega_{*\,i} + K_\theta\,\omega_{\mathrm{nc}\,i}, \tag{224}$$

$$\omega_{\phi\,I} = \omega_E + \omega_{*\,I} + K_\theta\,\omega_{\mathrm{nc}\,I}, \tag{225}$$

where

$$K_\theta = \frac{I^{\,2}}{R^{\,2}\,\langle B^{\,2}\rangle}, \tag{226}$$

and

$$\omega_{\mathrm{nc}\,i} = -L_{00}^{ii}\,\omega_{*\,i} - L_{00}^{iI}\,\omega_{*\,I} + L_{01}^{ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i} + L_{01}^{iI}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}, \tag{227}$$

$$\omega_{\mathrm{nc}\,I} = -L_{00}^{II}\,\omega_{*\,I} - L_{00}^{Ii}\,\omega_{*\,i} + L_{01}^{II}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I} + L_{01}^{Ii}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}. \tag{228}$$

7 Natural Frequencies

Let

$$\omega_{0\,\mathrm{ln}} = \omega_E + \omega_{*\,e}, \tag{229}$$

$$\omega_{0\,\mathrm{eb}} = \omega_E, \tag{230}$$

$$\omega_{0\,\mathrm{nl}} = \omega_E + \omega_{*\,i} + \omega_{\mathrm{nc}\,i}. \tag{231}$$

The corresponding natural frequencies are

$$\varpi_{0\,\mathrm{ln}} = -n\,\omega_{0\,\mathrm{ln}}, \tag{232}$$

$$\varpi_{0\,\mathrm{eb}} = -n\,\omega_{0\,\mathrm{eb}}, \tag{233}$$

$$\varpi_{0\,\mathrm{nl}} = -n\,\omega_{0\,\mathrm{nl}}. \tag{234}$$

8 Current Densities

Can write

$$j_\parallel = j_{\rm bootstrap} + j_{\rm ohmic}, \tag{235}$$

where

$$\begin{aligned}
\frac{\langle j_{\rm bootstrap}\,B\rangle}{I} = &-\left\{1 - L_{00}^{ii} - \left(\frac{Z_{\rm eff}-1}{Z_I - Z_{\rm eff}}\right) L_{00}^{Ii} + \left(\frac{Z_I-1}{Z_I-Z_{\rm eff}}\right) L_{00}^{ei}\right\}\frac{dp_i}{d\psi}\\
&-\left\{1 - L_{00}^{II} - \left(\frac{Z_I-Z_{\rm eff}}{Z_{\rm eff}-1}\right) L_{00}^{iI} + \left(\frac{Z_I-1}{Z_{\rm eff}-1}\right) L_{00}^{eI}\right\}\frac{dp_I}{d\psi}\\
&-(1-L_{00}^{ee})\,\frac{dp_e}{d\psi}\\
&-\left\{L_{01}^{ii} + \left(\frac{Z_{\rm eff}-1}{Z_I-Z_{\rm eff}}\right) L_{01}^{Ii} - \left(\frac{Z_I-1}{Z_I-Z_{\rm eff}}\right) L_{01}^{ei}\right\} n_i\,\frac{dT_i}{d\psi}\\
&-\left\{L_{01}^{II} + \left(\frac{Z_I-Z_{\rm eff}}{Z_{\rm eff}-1}\right) L_{01}^{iI} - \left(\frac{Z_I-1}{Z_{\rm eff}-1}\right) L_{01}^{eI}\right\} n_I\,\frac{dT_I}{d\psi}\\
&-L_{01}^{ee}\,n_e\,\frac{dT_e}{d\psi}
\end{aligned} \tag{236}$$

and

$$\langle j_{\rm ohmic}\,B\rangle = Q_{00}^{ee}\,\sigma_{ee}\,\langle E_\parallel\,B\rangle, \tag{237}$$

where

$$\sigma_{ee} = \frac{n_e\,e^2\,\tau_{ee}}{m_e}. \tag{238}$$