

# Program PHASE

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## 1 Plasma Angular Velocity Evolution

### 1.1 Plasma Angular Equations of Motion

We can write

$$\Omega_\theta(r, t) = \Omega_{\theta 0}(r) + \Delta\Omega_\theta(r, t), \quad (1)$$

$$\Omega_\phi(r, t) = \Omega_{\phi 0}(r) + \Delta\Omega_\phi(r, t), \quad (2)$$

where  $\Omega_\theta(r, t)$  and  $\Omega_\phi(r, t)$  are the poloidal and toroidal plasma angular velocity profiles, respectively, whereas  $\Omega_{\theta 0}(r)$  and  $\Omega_{\phi 0}(r)$  are the corresponding unperturbed profiles, and  $\Delta\Omega_\theta(r, t)$  and  $\Delta\Omega_\phi(r, t)$  are the respective modifications to the profiles induced by the electromagnetic torques exerted at the various resonant surfaces within the plasma. The modifications to the angular velocity profiles are governed by the poloidal and toroidal angular equations of motion of the plasma, which take the respective forms

$$4\pi^2 R_0 \left[ (1 + 2q^2) \rho r^3 \frac{\partial \Delta\Omega_\theta}{\partial t} - \frac{\partial}{\partial r} \left( \mu r^3 \frac{\partial \Delta\Omega_\theta}{\partial r} \right) + \frac{\rho}{\tau_\theta} r^3 \Delta\Omega_\theta \right] = \sum_{k=1, K} \delta T_{\theta k} \delta(r - r_k), \quad (3)$$

$$4\pi^2 R_0^3 \left[ \rho r \frac{\partial \Delta\Omega_\phi}{\partial t} - \frac{\partial}{\partial r} \left( \mu r \frac{\partial \Delta\Omega_\phi}{\partial r} \right) + \frac{\rho}{\tau_\phi} r \Delta\Omega_\phi \right] = \sum_{k=1, K} \delta T_{\phi k} \delta(r - r_k), \quad (4)$$

and are subject to the spatial boundary conditions

$$\frac{\partial \Delta\Omega_\theta(0, t)}{\partial r} = \frac{\partial \Delta\Omega_\phi(0, t)}{\partial r} = 0, \quad (5)$$

$$\Delta\Omega_\theta(a, t) = \Delta\Omega_\phi(a, t) = 0. \quad (6)$$

Here,  $\mu(r)$  is the anomalous plasma perpendicular ion viscosity (due to plasma turbulence), whereas  $\rho(r)$  is the plasma mass density profile. Furthermore,  $1/\tau_\theta(r)$  is the neoclassical

poloidal flow-damping rate. Finally,  $1/\tau_\phi(r)$  is the neoclassical toroidal flow-damping rate (which is, presently, set to zero in the code).

According to standard neoclassical theory,

$$\frac{1}{\tau_\theta(r)} = \left( \frac{q R_0}{r} \right)^2 \frac{\mu_{00}^i}{\tau_{ii}}. \quad (7)$$

## 1.2 Simplified Plasma Angular Equations of Motion

It is convenient to write

$$\Delta\Omega_\theta(r, t) = \sum_{k=1, K} \Delta\Omega_{\theta k}(r, t), \quad (8)$$

$$\Delta\Omega_\phi(r, t) = \sum_{k=1, K} \Delta\Omega_{\phi k}(r, t), \quad (9)$$

where

$$4\pi^2 R_0 \left[ (1 + 2q^2) \rho r^3 \frac{\partial \Delta\Omega_{\theta k}}{\partial t} - \frac{\partial}{\partial r} \left( \mu r^3 \frac{\partial \Delta\Omega_{\theta k}}{\partial r} \right) + \frac{\rho}{\tau_\theta} r^3 \Delta\Omega_{\theta k} \right] = \delta T_{\theta k} \delta(r - r_k), \quad (10)$$

$$4\pi^2 R_0^3 \left[ \rho r \frac{\partial \Delta\Omega_{\phi k}}{\partial t} - \frac{\partial}{\partial r} \left( \mu r \frac{\partial \Delta\Omega_{\phi k}}{\partial r} \right) + \frac{\rho}{\tau_\phi} r \Delta\Omega_{\phi k} \right] = \delta T_{\phi k} \delta(r - r_k), \quad (11)$$

and

$$\frac{\partial \Delta\Omega_{\theta k}(0, t)}{\partial r} = \frac{\partial \Delta\Omega_{\phi k}(0, t)}{\partial r} = 0, \quad (12)$$

$$\Delta\Omega_{\theta k}(a, t) = \Delta\Omega_{\phi k}(a, t) = 0. \quad (13)$$

In the presence of poloidal and toroidal flow damping, the modified angular velocity profiles,  $\Delta\Omega_{\theta k}$  and  $\Delta\Omega_{\phi k}$ , are radially localized in the vicinity of the  $k$ th resonant surface. Hence, it is a good approximation to write Eqs. (10) and (11) in the simplified forms

$$4\pi^2 R_0 \left[ (1 + 2q_k^2) \rho_k r^3 \frac{\partial \Delta\Omega_{\theta k}}{\partial t} - \mu_k \frac{\partial}{\partial r} \left( r^3 \frac{\partial \Delta\Omega_{\theta k}}{\partial r} \right) + \frac{\rho_k}{\tau_{\theta k}} r^3 \Delta\Omega_{\theta k} \right] = \delta T_{\theta k} \delta(r - r_k), \quad (14)$$

$$4\pi^2 R_0^3 \left[ \rho_k r \frac{\partial \Delta\Omega_{\phi k}}{\partial t} - \mu_k \frac{\partial}{\partial r} \left( r \frac{\partial \Delta\Omega_{\phi k}}{\partial r} \right) + \frac{\rho_k}{\tau_{\phi k}} r \Delta\Omega_{\phi k} \right] = \delta T_{\phi k} \delta(r - r_k), \quad (15)$$

where  $q_k = q(r_k)$ ,  $\rho_k = \rho(r_k)$ ,  $\mu_k = \mu(r_k)$ ,  $\tau_{\theta k} = \tau_\theta(r_k)$ , and  $\tau_{\phi k} = \tau_\phi(r_k)$ .

### 1.3 Normalized Plasma Equations of Angular Motion

Let

$$\tau_A = \left( \frac{\mu_0 \rho_0 a^2}{B_0^2} \right)^{1/2}, \quad (16)$$

where  $\rho_0 = \rho(0)$ , and

$$\rho(r) \simeq m_i [n_i(r) + n_b(r)] + m_I n_I(r) \quad (17)$$

is the plasma mass density. Furthermore, let  $\hat{r} = r/a$ ,  $\hat{r}_k = r_k/a$ ,  $\hat{a} = a/R_0$ ,  $\hat{t} = t/\tau_A$ ,  $\hat{\rho}_k = \rho_k/\rho_0$ ,  $\tau_{Mk} = \rho_k a^2/\mu_k = a^2/\chi_\phi(r_k)$ ,  $\hat{\tau}_{Mk} = \tau_{Mk}/\tau_A$ ,  $\hat{\tau}_{\theta k} = \tau_{\theta k}/\tau_A$ ,  $\hat{\tau}_{\phi k} = \tau_{\phi k}/\tau_A$ ,  $\hat{\Omega}_\theta = \tau_A \Omega_\theta$ ,  $\hat{\Omega}_{\theta 0} = \tau_A \Omega_{\theta 0}$ ,  $\Delta \hat{\Omega}_{\theta k} = \tau_A \Delta \Omega_{\theta k}$ ,  $\hat{\Omega}_\phi = \tau_A \Omega_\phi$ ,  $\hat{\Omega}_{\phi 0} = \tau_A \Omega_{\phi 0}$ , and  $\Delta \hat{\Omega}_{\phi k} = \tau_A \Delta \Omega_{\phi k}$ . It follows that

$$\hat{\Omega}_\theta(\hat{r}, \hat{t}) = \hat{\Omega}_{\theta 0}(\hat{r}) + \sum_{k=1, K} \Delta \hat{\Omega}_{\theta k}(\hat{r}, \hat{t}), \quad (18)$$

$$\hat{\Omega}_\phi(\hat{r}, \hat{t}) = \hat{\Omega}_{\phi 0}(\hat{r}) + \sum_{k=1, K} \Delta \hat{\Omega}_{\phi k}(\hat{r}, \hat{t}), \quad (19)$$

where

$$\begin{aligned} (1 + 2q_k^2) \hat{r}^3 \frac{\partial \Delta \hat{\Omega}_{\theta k}}{\partial \hat{t}} - \frac{1}{\hat{\tau}_{Mk}} \frac{\partial}{\partial \hat{r}} \left( \hat{r}^3 \frac{\partial \Delta \hat{\Omega}_{\theta k}}{\partial \hat{r}} \right) + \frac{1}{\hat{\tau}_{\theta k}} \hat{r}^3 \Delta \hat{\Omega}_{\theta k} \\ = -\frac{m_k}{2\hat{\rho}_k \hat{a}^2} \delta \hat{T}_k \delta(\hat{r} - \hat{r}_k), \end{aligned} \quad (20)$$

$$\hat{r} \frac{\partial \Delta \hat{\Omega}_{\phi k}}{\partial \hat{t}} - \frac{1}{\hat{\tau}_{Mk}} \frac{\partial}{\partial \hat{r}} \left( \hat{r} \frac{\partial \Delta \hat{\Omega}_{\phi k}}{\partial \hat{r}} \right) + \frac{1}{\hat{\tau}_{\phi k}} \hat{r} \Delta \hat{\Omega}_{\phi k} = \frac{n}{2\hat{\rho}_k} \delta \hat{T}_k \delta(\hat{r} - \hat{r}_k), \quad (21)$$

and

$$\frac{\partial \Delta \hat{\Omega}_{\theta k}(0, \hat{t})}{\partial \hat{r}} = \frac{\partial \Delta \hat{\Omega}_{\phi k}(0, \hat{t})}{\partial \hat{r}} = 0, \quad (22)$$

$$\Delta \hat{\Omega}_{\theta k}(1, \hat{t}) = \Delta \hat{\Omega}_{\phi k}(1, \hat{t}) = 0. \quad (23)$$

### 1.4 Solution of Plasma Angular Equations of Motion

Let

$$\Delta \hat{\Omega}_{\theta k}(\hat{r}, \hat{t}) = -\frac{1}{m_k} \sum_{p=1, P} \alpha_{k,p}(\hat{t}) \frac{y_p(\hat{r})}{y_p(\hat{r}_k)}, \quad (24)$$

$$\Delta \hat{\Omega}_{\phi k}(\hat{r}, \hat{t}) = \frac{1}{n} \sum_{p=1, P} \beta_{k,p}(\hat{t}) \frac{z_p(\hat{r})}{z_p(\hat{r}_k)}, \quad (25)$$

where

$$y_p(\hat{r}) = \frac{J_1(j_{1,p} \hat{r})}{\hat{r}}, \quad (26)$$

$$z_p(\hat{r}) = J_0(j_{0,p} \hat{r}), \quad (27)$$

and  $P \gg 1$ . Here,  $J_m(z)$  is a standard Bessel function, and  $j_{m,p}$  denotes the  $p$ th zero of the  $J_m(z)$  Bessel function. Note that Eqs. (24) and (25) automatically satisfy the boundary conditions Eqs. (22) and (23).

It is easily demonstrated that

$$\frac{d}{d\hat{r}} \left( \hat{r}^3 \frac{dy_p}{d\hat{r}} \right) = -j_{1,p}^2 \hat{r}^3 y_p, \quad (28)$$

$$\frac{d}{d\hat{r}} \left( \hat{r} \frac{dz_p}{d\hat{r}} \right) = -j_{0,p}^2 \hat{r} z_p, \quad (29)$$

and

$$\int_0^1 \hat{r}^3 y_p(\hat{r}) y_q(\hat{r}) d\hat{r} = \frac{1}{2} [J_2(j_{1,p})]^2 \delta_{pq}, \quad (30)$$

$$\int_0^1 \hat{r} z_p(\hat{r}) z_q(\hat{r}) d\hat{r} = \frac{1}{2} [J_1(j_{0,p})]^2 \delta_{pq} \quad (31)$$

Hence, Eqs. (20) and (21) yield

$$(1 + 2q_k^2) \frac{d\alpha_{k,p}}{d\hat{t}} + \left( \frac{j_{1,p}^2}{\hat{\tau}_{Mk}} + \frac{1}{\hat{\tau}_{\theta k}} \right) \alpha_{k,p} = \frac{m_k^2 [y_p(\hat{r}_k)]^2}{\hat{\rho}_k \hat{a}^2 [J_2(j_{1,p})]^2} \delta \hat{T}_k, \quad (32)$$

$$\frac{d\beta_{k,p}}{d\hat{t}} + \left( \frac{j_{0,p}^2}{\hat{\tau}_{Mk}} + \frac{1}{\hat{\tau}_{\phi k}} \right) \beta_{k,p} = \frac{n^2 [z_p(\hat{r}_k)]^2}{\hat{\rho}_k [J_1(j_{0,p})]^2} \delta \hat{T}_k. \quad (33)$$

It follows that

$$\hat{\Omega}_\theta(\hat{r}_k, \hat{t}) = \hat{\Omega}_{\theta 0}(\hat{r}_k) - \sum_{k'=1, K}^{p=1, P} \frac{\alpha_{k',p}(\hat{t})}{m_{k'}} \frac{y_p(\hat{r}_k)}{y_p(\hat{r}_{k'})}, \quad (34)$$

$$\hat{\Omega}_\phi(\hat{r}_k, \hat{t}) = \hat{\Omega}_{\phi 0}(\hat{r}_k) + \sum_{k'=1, K}^{p=1, P} \frac{\beta_{k',p}(\hat{t})}{n} \frac{z_p(\hat{r}_k)}{z_p(\hat{r}_{k'})}. \quad (35)$$

Here,  $k$  indexes the various resonant surfaces in the plasma, whereas  $p$  indexes the various velocity harmonics.

## 2 Resonant Plasma Response Model

Fundamental equation:

$$\left(\hat{W}_k + \hat{\delta}_k\right) \mathcal{S}_k \left(\frac{d}{d\hat{t}} + i \hat{\omega}_k\right) \Psi_k = \sum_{k'=1,K} E_{kk'} \Psi_{k'} + \hat{E}_{kk} \chi_k, \quad (36)$$

where

$$\mathcal{S}_k = \frac{\tau_R(\hat{r}_k)}{\tau_A}, \quad (37)$$

$$\hat{W}_k = \frac{2\mathcal{I}}{\hat{a} \hat{r}_k} \left(\frac{q}{g s}\right)^{1/2}_{\hat{r}_k} |\hat{\Psi}_k|^{1/2}, \quad (38)$$

$$\hat{\delta}_k = \frac{\delta_{\text{SC}}(\hat{r}_k)}{R_0 \hat{a} \hat{r}_k}, \quad (39)$$

with  $\mathcal{I} = 0.8227$ , and

$$\tau_R(r) = \mu_0 r^2 Q_{00}^{ee} \sigma_{ee}, \quad (40)$$

$$\sigma_{ee}(r) = \frac{n_e e^2 \tau_{ee}}{m_e}. \quad (41)$$

Also,

$$\delta_{\text{SC}}(r) = \pi \frac{|n \omega_{*e}|^{1/2} \tau_H}{(\rho_s/r) \tau_R^{1/2}} r, \quad (42)$$

$$\tau_H = \frac{R_0}{B_0 g} \frac{\sqrt{\mu_0 \rho}}{n s}, \quad (43)$$

$$\rho_s = \frac{\sqrt{m_i T_e}}{e B_0 g}. \quad (44)$$

Follows from Eq. (36) that

$$\left(\hat{W}_k + \hat{\delta}_k\right) \mathcal{S}_k \frac{d\hat{\Psi}_k}{d\hat{t}} = \sum_{k'=1,K} \hat{E}_{kk'} \hat{\Psi}_{k'} \cos(\varphi_k - \varphi_{k'} - \xi_{kk'}) + \hat{E}_{kk} \hat{\chi}_k \cos(\varphi_k - \zeta_k), \quad (45)$$

$$\left(\hat{W}_k + \hat{\delta}_k\right) \mathcal{S}_k \hat{\Psi}_k \left(\frac{d\varphi_k}{d\hat{t}} - \hat{\omega}_k\right) = - \sum_{k'=1,K} \hat{E}_{kk'} \hat{\Psi}_{k'} \sin(\varphi_k - \varphi_{k'} - \xi_{kk'}) - \hat{E}_{kk} \hat{\chi}_k \sin(\varphi_k - \zeta_k). \quad (46)$$

Alternatively, let

$$X_k = \hat{\Psi}_k \cos \varphi_k, \quad (47)$$

$$Y_k = \hat{\Psi}_k \sin \varphi_k. \quad (48)$$

So, Eq. (36) gives

$$\begin{aligned} \left( \hat{W}_k + \hat{\delta}_k \right) \mathcal{S}_k \left( \frac{d}{d\hat{t}} + i \hat{\omega}_k \right) (X_k - i Y_k) &= \sum_{k'=1, K} \hat{E}_{kk'} (\cos \xi_{kk'} - i \sin \xi_{kk'}) (X_{k'} - i Y_{k'}) \\ &+ \hat{E}_{kk} \hat{\chi}_k (\cos \zeta_k - i \sin \zeta_k), \end{aligned} \quad (49)$$

or

$$\begin{aligned} &\left( \hat{W}_k + \hat{\delta}_k \right) \mathcal{S}_k \left( \frac{dX_k}{d\hat{t}} - i \frac{dY_k}{d\hat{t}} + i \hat{\omega}_k X_k + \hat{\omega}_k Y_k \right) \\ &= \sum_{k'=1, K} \hat{E}_{kk'} (\cos \xi_{kk'} X_{k'} - i \cos \xi_{kk'} Y_{k'} - i \sin \xi_{kk'} X_{k'} - \sin \xi_{kk'} Y_{k'}) \\ &+ \hat{E}_{kk} \hat{\chi}_k (\cos \zeta_k - i \sin \zeta_k). \end{aligned} \quad (50)$$

Follows that

$$\left( \hat{W}_k + \hat{\delta}_k \right) \mathcal{S}_k \left( \frac{dX_k}{d\hat{t}} + \hat{\omega}_k Y_k \right) = \sum_{k'=1, K} \hat{E}_{kk'} (\cos \xi_{kk'} X_{k'} - \sin \xi_{kk'} Y_{k'}) + \hat{E}_{kk} \hat{\chi}_k \cos \zeta_k, \quad (51)$$

$$\left( \hat{W}_k + \hat{\delta}_k \right) \mathcal{S}_k \left( \frac{dY_k}{d\hat{t}} - \hat{\omega}_k X_k \right) = \sum_{k'=1, K} \hat{E}_{kk'} (\cos \xi_{kk'} Y_{k'} + \sin \xi_{kk'} X_{k'}) + \hat{E}_{kk} \hat{\chi}_k \sin \zeta_k, \quad (52)$$

where

$$\hat{W}_k = \frac{2\mathcal{I}}{\hat{a} \hat{r}_k} \left( \frac{q}{g s} \right)^{1/2} (X_k^2 + Y_k^2)^{1/4}. \quad (53)$$

Note, finally, that

$$\delta \hat{T}_k = \sum_{k' \neq k} \hat{E}_{kk'} \hat{\Psi}_k \hat{\Psi}_{k'} \sin(\varphi_k - \varphi_{k'} - \xi_{kk'}) + \hat{E}_{kk} \hat{\Psi}_k \hat{\chi}_k \sin(\varphi_k - \zeta_k). \quad (54)$$

### 3 Density and Temperature Flattening Widths

Critical electron temperature flattening width (in  $r$ ):

$$W_{T_e k} = \left( \frac{\chi_E}{v_{Te} r} \frac{1}{\epsilon s n} \right)^{1/3}_{r_k} r_k, \quad (55)$$

where  $\epsilon = r/R_0$ . Critical ion temperature flattening width (in  $r$ ):

$$W_{T_i k} = \left( \frac{\chi_E}{v_{Ti} r} \frac{1}{\epsilon s n} \right)^{1/3}_{r_k} r_k. \quad (56)$$

Critical density flattening width (in  $r$ ):

$$W_{n_e k} = \left( \frac{D_{\perp}}{v_{T_i} r} \frac{1}{\epsilon s n} \right)_{r_k}^{1/3} r_k. \quad (57)$$

Critical electron temperature flattening width (in  $\Psi_N$ ):

$$W_{T_e k} = \left( \frac{\chi_E}{v_{T_e} R_0} \frac{f^2 A_k^{(1)}}{n |\psi_c|} \right)_{r_k}^{1/3}. \quad (58)$$

Critical ion temperature flattening width (in  $\Psi_N$ ):

$$W_{T_i k} = \left( \frac{\chi_E}{v_{T_i} R_0} \frac{f^2 A_k^{(1)}}{n |\psi_c|} \right)_{r_k}^{1/3}. \quad (59)$$

Critical density flattening width (in  $\Psi_N$ ):

$$W_{n_e k} = \left( \frac{D_{\perp}}{v_{T_i} R_0} \frac{f^2 A_k^{(1)}}{n |\psi_c|} \right)_{r_k}^{1/3}. \quad (60)$$

Electron temperature, ion temperature, and electron number density profiles flattening widths (in  $\Psi_N$ )

$$\delta_{T_e k} = \frac{2}{\pi} W_k \tanh \left( \frac{W_k}{W_{T_e k}} \right), \quad (61)$$

$$\delta_{T_i k} = \frac{2}{\pi} W_k \tanh \left( \frac{W_k}{W_{T_i k}} \right), \quad (62)$$

$$\delta_{n_e k} = \frac{2}{\pi} W_k \tanh \left( \frac{W_k}{W_{n_e k}} \right), \quad (63)$$

Total pressure change due to islands is

$$\begin{aligned}
\Delta P = \sum_{k=1,K} \left\{ \right. & \left[ \delta_{n_e k} \frac{dn_e}{d\Psi_N} T_e - \delta_{n_e k} W_k^2 \frac{A_k^{(2)}}{8} \left( \frac{dn_e}{d\Psi_N} \frac{dT_e}{d\Psi_N} + \frac{d^2 n_e}{d\Psi_n^2} T_e \right) + \delta_{n_e k}^3 \frac{d^3 n_e}{d\Psi_N^3} \frac{T_e}{24} \right] \\
& + \left[ \delta_{T_e k} n_e \frac{dT_e}{d\Psi_N} - \delta_{T_e k} W_k^2 \frac{A_k^{(2)}}{8} \left( \frac{dn_e}{d\Psi_N} \frac{dT_e}{d\Psi_N} + n_e \frac{d^2 T_e}{d\Psi_n^2} \right) + \delta_{T_e k}^3 \frac{n_e}{24} \frac{d^3 T_e}{d\Psi_N^3} \right] \\
& + \left[ \delta_{n_e k} \frac{dn_i}{d\Psi_N} T_i - \delta_{n_e k} W_k^2 \frac{A_k^{(2)}}{8} \left( \frac{dn_i}{d\Psi_N} \frac{dT_i}{d\Psi_N} + \frac{d^2 n_i}{d\Psi_n^2} T_i \right) + \delta_{n_e k}^3 \frac{d^3 n_i}{d\Psi_N^3} \frac{T_i}{24} \right] \\
& + \left. \left[ \delta_{T_i k} n_i \frac{dT_i}{d\Psi_N} - \delta_{T_i k} W_k^2 \frac{A_k^{(2)}}{8} \left( \frac{dn_i}{d\Psi_N} \frac{dT_i}{d\Psi_N} + n_i \frac{d^2 T_i}{d\Psi_n^2} \right) + \delta_{T_i k}^3 \frac{n_i}{24} \frac{d^3 T_i}{d\Psi_N^3} \right] \right\}_{r_k}.
\end{aligned} \tag{64}$$