

# Program FLUX

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## 1 Coordinate Systems

- Cartesian coordinates:  $X, Y, Z$ .
- Cylindrical coordinates:  $R \equiv (X^2 + Y^2)^{1/2}$ ,  $\phi \equiv \tan^{-1}(Y/X)$ ,  $Z$ , so

$$|\nabla\phi| = \frac{1}{R}. \quad (1)$$

- Flux coordinates:  $r(R, Z)$ ,  $\theta(R, Z)$ ,  $\phi$ , where

$$(\nabla r \cdot \nabla \theta \times \nabla \phi)^{-1} = \mathcal{J}, \quad (2)$$

$$\mathcal{J} = \frac{r R^2}{R_0}, \quad (3)$$

where  $R_0$  is a convenient scale major radius.

## 2 Useful Identities

Easily demonstrated that:

$$\mathbf{A} = A^r \mathcal{J} \nabla \theta \times \nabla \phi + A^\theta \mathcal{J} \nabla \phi \times \nabla r + A^\phi \mathcal{J} \nabla r \times \nabla \theta, \quad (4)$$

$$\mathbf{A} = A_r \nabla r + A_\theta \nabla \theta + A_\phi \nabla \phi, \quad (5)$$

$$\mathbf{A} \cdot \mathbf{B} = A_r B^r + A_\theta B^\theta + A_\phi B^\phi = A^r B_r + A^\theta B_\theta + A^\phi B_\phi, \quad (6)$$

$$(\mathbf{A} \times \mathbf{B})_r = \mathcal{J} (A^\theta B^\phi - A^\phi B^\theta), \quad (7)$$

$$(\mathbf{A} \times \mathbf{B})_\theta = \mathcal{J} (A^\phi B^r - A^r B^\phi), \quad (8)$$

$$(\mathbf{A} \times \mathbf{B})_\phi = \mathcal{J} (A^r B^\theta - A^\theta B^r), \quad (9)$$

$$\mathcal{J} (\mathbf{A} \times \mathbf{B})^r = A_\theta B_\phi - A_\phi B_\theta, \quad (10)$$

$$\mathcal{J} (\mathbf{A} \times \mathbf{B})^\theta = A_\phi B_r - A_r B_\phi, \quad (11)$$

$$\mathcal{J}(\mathbf{A} \times \mathbf{B})^\phi = A_r B_\theta - A_\theta B_r, \quad (12)$$

$$\mathcal{J} \nabla \cdot \mathbf{A} = \frac{\partial (\mathcal{J} A^r)}{\partial r} + \frac{\partial (\mathcal{J} A^\theta)}{\partial \theta} + \frac{\partial (\mathcal{J} A^\phi)}{\partial \phi}, \quad (13)$$

$$\mathcal{J}(\nabla \times \mathbf{A})^r = \frac{\partial A_\phi}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}, \quad (14)$$

$$\mathcal{J}(\nabla \times \mathbf{A})^\theta = \frac{\partial A_r}{\partial \phi} - \frac{\partial A_\phi}{\partial r}, \quad (15)$$

$$\mathcal{J}(\nabla \times \mathbf{A})^\phi = \frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}. \quad (16)$$

### 3 Equilibrium Magnetic Field

Equilibrium magnetic field:

$$\mathbf{B} = B_0 R_0 [f(r) \nabla \phi \times \nabla r + g(r) \nabla \phi], \quad (17)$$

$$q(r) = \frac{r g}{R_0 f}, \quad (18)$$

where  $B_0$  is a convenient scale magnetic field strength. So,

$$B^r = 0, \quad (19)$$

$$B^\theta = B_0 R_0^2 \frac{f}{r R^2}, \quad (20)$$

$$B^\phi = B_0 R_0^2 \frac{q f}{r R^2} = B_0 B_0 \frac{g}{R^2}, \quad (21)$$

$$B_r = -B_0 r f \nabla r \cdot \nabla \theta, \quad (22)$$

$$B_\theta = B_0 r f |\nabla r|^2, \quad (23)$$

$$B_\phi = B_0 R_0^2 \frac{q f}{r} = B_0 R_0 g, \quad (24)$$

and

$$\mathbf{B} \cdot \nabla = B_0 R_0^2 \frac{f}{r R^2} \left( \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \phi} \right). \quad (25)$$

### 4 Equilibrium Current

The relation  $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$  yields

$$\mu_0 \mathcal{J} J^r = 0, \quad (26)$$

$$\mu_0 \mathcal{J} J^\theta = -B_0 R_0 \frac{dg}{dr}, \quad (27)$$

$$\mu_0 \mathcal{J} J^\phi = B_0 \frac{\partial}{\partial r} (r f |\nabla r|^2) + B_0 \frac{\partial}{\partial \theta} (r f \nabla r \cdot \nabla \theta). \quad (28)$$

## 5 Grad-Shafranov Equation

The force balance relation

$$\mathbf{J} \times \mathbf{B} = \nabla P \quad (29)$$

yields

$$\mathcal{J} (J^\theta B^\phi - J^\phi B^\theta) = \frac{dP}{dr}, \quad (30)$$

which reduces to

$$\frac{f}{r} \frac{\partial}{\partial r} (r f |\nabla r|^2) + \frac{f}{r} \frac{\partial}{\partial \theta} (r f \nabla r \cdot \nabla \theta) + g \frac{dg}{dr} + \frac{\mu_0}{B_0^2} \left( \frac{R}{R_0} \right)^2 \frac{dP}{dr} = 0. \quad (31)$$

## 6 High- $q$ Limit

Now,

$$\nabla \cdot \delta \mathbf{B} = 0 \quad (32)$$

yields

$$\frac{\partial(\mathcal{J} \delta B^r)}{\partial r} + \frac{\partial(\mathcal{J} \delta B^\theta)}{\partial \theta} + \frac{\partial(\mathcal{J} \delta B^\phi)}{\partial \phi} = 0. \quad (33)$$

Furthermore,

$$\delta \mathbf{B} = \delta B_r \nabla r + \delta B_\theta \nabla \theta + \delta B_\phi \nabla \phi. \quad (34)$$

So,

$$\delta B^r = \delta \mathbf{B} \cdot \nabla r = |\nabla r|^2 \delta B_r + (\nabla r \cdot \nabla \theta) \delta B_\theta, \quad (35)$$

$$\delta B^\theta = \delta \mathbf{B} \cdot \nabla \theta = (\nabla r \cdot \nabla \theta) \delta B_r + |\nabla \theta|^2 \delta B_\theta, \quad (36)$$

$$\delta B^\phi = \delta \mathbf{B} \cdot \nabla \phi = \frac{\delta B_\phi}{R^2}. \quad (37)$$

Follows that

$$\delta B_r = \left( \frac{1}{|\nabla r|^2} \right) \delta B^r - \left( \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B_\theta, \quad (38)$$

and

$$\delta B^\theta = \left( \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B^r + \left[ |\nabla \theta|^2 - \frac{(\nabla r \cdot \nabla \theta)^2}{|\nabla r|^2} \right] \delta B_\theta. \quad (39)$$

But, Eqs. (2) and (3) imply that

$$|\nabla r|^2 |\nabla \theta|^2 - (\nabla r \cdot \nabla \theta)^2 = \frac{R_0^2}{r^2 R^2}. \quad (40)$$

Hence,

$$\delta B^\theta = \left( \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B^r + \left( \frac{R_0^2}{r^2 R^2 |\nabla r|^2} \right) \delta B_\theta. \quad (41)$$

Assume that high- $q$  limit is equivalent to  $\delta \mathbf{B}$  being curl-free. Follows that

$$\frac{\partial \delta B_\phi}{\partial \theta} = \frac{\partial \delta B_\theta}{\partial \phi}, \quad (42)$$

$$\frac{\partial \delta B_r}{\partial \phi} = \frac{\partial \delta B_\phi}{\partial r}, \quad (43)$$

$$\frac{\partial \delta B_\theta}{\partial r} = \frac{\partial \delta B_r}{\partial \theta}. \quad (44)$$

Previous three equations imply that

$$\delta B_\phi \sim \frac{n}{m} \delta B_\theta, \quad \frac{n}{m} r \delta B_r \quad (45)$$

and, hence, that

$$\delta B^\phi \sim \frac{n}{m} \left( \frac{r}{R} \right)^2 \delta B^\theta, \quad \frac{n}{m} \left( \frac{r}{R} \right)^2 \frac{\delta B^r}{r}, \quad (46)$$

Consequently, final term on right-hand side of (33) is of order  $(n/m)^2 (r/R)^2$  smaller than other two terms, and, therefore, negligible. Thus, we get

$$\frac{\partial(\mathcal{J} \delta B^r)}{\partial r} + \frac{\partial(\mathcal{J} \delta B^\theta)}{\partial \theta} = 0. \quad (47)$$

Follows from (41) and previous equation that

$$r \frac{\partial}{\partial r} \left( \frac{r R^2 \delta B^r}{R_0^2} \right) = - \frac{\partial}{\partial \theta} \left[ \left( \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \left( \frac{r R^2 \delta B^r}{R_0^2} \right) + \left( \frac{1}{|\nabla r|^2} \right) \delta B_\theta \right]. \quad (48)$$

Follows from (3), (38) and (44) that

$$r \frac{\partial \delta B_\theta}{\partial r} = \frac{\partial}{\partial \theta} \left[ \left( \frac{R_0^2}{R^2 |\nabla r|^2} \right) \left( \frac{r R^2 \delta B^r}{R_0^2} \right) - \left( \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B_\theta \right]. \quad (49)$$

Let

$$\frac{r R^2 \delta B^r(r, \theta, \phi)}{R_0^2} = \sum_j \delta \hat{B}_j^r(r) e^{i(m_j \theta - n \phi)}, \quad (50)$$

$$\delta B_\theta(r, \theta, \phi) = \sum_j \delta \hat{B}_{\theta j}(r) e^{i(m_j \theta - n \phi)}. \quad (51)$$

Follows that

$$\sum_{j'} r \frac{d \delta \hat{B}_{j'}^r}{dr} e^{i m_{j'} \theta} = - \frac{\partial}{\partial \theta} \sum_{j'} \left[ \left( \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) e^{i m_{j'} \theta} \delta \hat{B}_{j'}^r + \left( \frac{1}{|\nabla r|^2} \right) e^{i m_{j'} \theta} \delta \hat{B}_{\theta j'} \right], \quad (52)$$

$$\sum_{j'} r \frac{d \delta \hat{B}_{\theta j'}}{dr} e^{i m_{j'} \theta} = \frac{\partial}{\partial \theta} \sum_{j'} \left[ \left( \frac{R_0^2}{R^2 |\nabla r|^2} \right) e^{i m_{j'} \theta} \delta \hat{B}_{j'}^r - \left( \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) e^{i m_{j'} \theta} \delta \hat{B}_{\theta j'} \right]. \quad (53)$$

Operating with  $\oint (\dots) e^{-i m_j \theta} d\theta / (2\pi)$ , we get

$$r \frac{d \delta \hat{B}_j^r}{dr} = -i m_j \sum_{j'} \left( -i c_{jj'} \delta \hat{B}_{j'}^r + a_{jj'} \delta \hat{B}_{\theta j'} \right), \quad (54)$$

$$r \frac{d \delta \hat{B}_{\theta j}}{dr} = -i m_j \sum_{j'} \left( -i c_{jj'} \delta \hat{B}_{\theta j'} - b_{jj'} \delta \hat{B}_{j'}^r \right), \quad (55)$$

where

$$a_{jj'} = \oint \frac{1}{|\nabla r|^2} e^{-i(m_j - m_{j'})\theta} \frac{d\theta}{2\pi}, \quad (56)$$

$$b_{jj'} = \oint \frac{R_0^2}{R^2 |\nabla r|^2} e^{-i(m_j - m_{j'})\theta} \frac{d\theta}{2\pi}, \quad (57)$$

$$c_{jj'} = \oint \frac{i r \nabla r \cdot \nabla \theta}{|\nabla r|^2} e^{-i(m_j - m_{j'})\theta} \frac{d\theta}{2\pi}. \quad (58)$$

Let

$$\delta \hat{B}_j^r(r) = i \psi_j(r), \quad (59)$$

$$\delta \hat{B}_{\theta j}(r) = -\chi_j(r). \quad (60)$$

It follows that

$$r \frac{d \psi_j}{dr} = m_j \sum_{j'} (-c_{jj'} \psi_{j'} + a_{jj'} \chi_{j'}), \quad (61)$$

$$r \frac{d \chi_j}{dr} = m_j \sum_{j'} (-c_{jj'} \chi_{j'} + b_{jj'} \psi_{j'}). \quad (62)$$

Hence,

$$\frac{r R^2 \delta B^r(r, \theta, \phi)}{R_0^2} = i \sum_j \psi_j(r) e^{i(m_j \theta - n \phi)}, \quad (63)$$

$$\frac{r R^2 \delta B^\theta(r, \theta, \phi)}{R_0^2} = - \sum_j \frac{1}{m_j} \frac{d \psi_j}{dr} e^{i(m_j \theta - n \phi)}, \quad (64)$$

$$R^2 \delta B^\phi(r, \theta, \phi) = n \sum_j \frac{\chi_j(r)}{m_j} e^{i(m_j \theta - n \phi)}. \quad (65)$$

where use has been made of (42). Also have

$$\delta B_r(r, \theta, \phi) = i \sum_j \frac{1}{m_j} \frac{d\chi_j}{dr} e^{i(m_j \theta - n \phi)}, \quad (66)$$

$$\delta B_\theta(r, \theta, \phi) = - \sum_j \chi_j(r) e^{i(m_j \theta - n \phi)}, \quad (67)$$

$$\delta B_\phi(r, \theta, \phi) = n \sum_j \frac{\chi_j(r)}{m_j} e^{i(m_j \theta - n \phi)}, \quad (68)$$

We can write

$$\mathcal{J} \mu_0 \delta J^r = \frac{\partial \delta B_\phi}{\partial \theta} - \frac{\partial \delta B_\theta}{\partial \phi}, \quad (69)$$

$$\mathcal{J} \mu_0 \delta J^\theta = \frac{\partial \delta B_r}{\partial \phi} - \frac{\partial \delta B_\phi}{\partial r}, \quad (70)$$

$$\mathcal{J} \mu_0 \delta J^\phi = \frac{\partial \delta B_\theta}{\partial r} - \frac{\partial \delta B_r}{\partial \theta}. \quad (71)$$

Normally, all three components are zero. However, at  $k$ th rational surface  $\psi_k$ ,  $\psi_{j \neq k}$ , and  $\chi_{j \neq k}$  are continuous, whereas  $\chi_k$  is discontinuous. Hence,

$$\mathcal{J} \mu_0 \delta J^r(r, \theta, \phi) = 0, \quad (72)$$

$$\mathcal{J} \mu_0 \delta J^\theta(r, \theta, \phi) = - \sum_k \frac{n}{m_k} [\chi_k]_{r_{k-}}^{r_{k+}} \delta(r - r_k) e^{i(m_k \theta - n \phi)}, \quad (73)$$

$$\mathcal{J} \mu_0 \delta J^\phi(r, \theta, \phi) = - \sum_k [\chi_k]_{r_{k-}}^{r_{k+}} \delta(r - r_k) e^{i(m_k \theta - n \phi)}. \quad (74)$$

Now,

$$(\mu_0 \delta \mathbf{J} \times \delta \mathbf{B})_\theta = \frac{1}{4} (\mathcal{J} \mu_0 \delta J^\phi \delta B^{r*} + \text{c.c.}), \quad (75)$$

$$(\mu_0 \delta \mathbf{J} \times \delta \mathbf{B})_\phi = \frac{1}{4} (-\mathcal{J} \mu_0 \delta J^\theta \delta B^{r*} + \text{c.c.}), \quad (76)$$

which implies that

$$\mathcal{J} (\mu_0 \delta \mathbf{J} \times \delta \mathbf{B})_\theta = \frac{R_0}{2} \sum_j \sum_k \text{Re} \left\{ i [\chi_k]_{r_{k-}}^{r_{k+}} \psi_j^*(r_k) e^{i(m_k - m_j) \theta} \right\} \delta(r - r_k), \quad (77)$$

$$\mathcal{J} (\mu_0 \delta \mathbf{J} \times \delta \mathbf{B})_\phi = -\frac{R_0}{2} \sum_j \sum_k \frac{n}{m_k} \text{Re} \left\{ i [\chi_k]_{r_{k-}}^{r_{k+}} \psi_j^*(r_k) e^{i(m_k - m_j) \theta} \right\} \delta(r - r_k). \quad (78)$$

Let

$$\Psi_k = \frac{\psi_k(r_k)}{m_k}, \quad (79)$$

$$\Delta\Psi_k = [\chi_k]_{r_{k-}}^{r_{k+}}, \quad (80)$$

$$\delta T_{\theta k} \equiv \delta \mathbf{T} \cdot \mathbf{e}_\theta = \int_{r_{k-}}^{r_{k+}} \oint \oint (\delta \mathbf{J} \times \delta \mathbf{B})_\theta \mathcal{J} dr d\theta d\phi, \quad (81)$$

$$\delta T_{\phi k} \equiv \delta \mathbf{T} \cdot \mathbf{e}_\phi = \int_{r_{k-}}^{r_{k+}} \oint \oint (\delta \mathbf{J} \times \delta \mathbf{B})_\phi \mathcal{J} dr d\theta d\phi, \quad (82)$$

where  $\delta \mathbf{T}(r) dr$  is the net electromagnetic torque acting on the plasma between  $r$  and  $r + dr$ . Here,  $\mathbf{e}_\theta = (R^2/R_0) \nabla \phi \times \nabla r$  and  $\mathbf{e}_\phi = R \nabla \phi$ . Follows that

$$\delta T_{\theta k} = -\frac{2\pi^2 R_0 m_k}{\mu_0} \text{Im}(\Psi_k^* \Delta\Psi_k), \quad (83)$$

$$\delta T_{\phi k} = \frac{2\pi^2 R_0 n}{\mu_0} \text{Im}(\Psi_k^* \Delta\Psi_k). \quad (84)$$

## 7 Homogeneous Solution

We can write

$$\delta \mathbf{B} = \nabla \times \delta \mathbf{A}. \quad (85)$$

Suppose that  $r \delta A_r$ ,  $\delta A_\theta$  are negligible with respect to  $\delta A_\phi$ . In fact, it is easily demonstrated from  $\nabla \cdot \delta \mathbf{A} = 0$  that  $r \delta A_r$ ,  $\delta A_\theta \sim (n/m)(r/R)^2 \delta A_\phi$ . It follows that

$$\mathcal{J} \delta B^r \simeq \frac{\partial \delta A_\phi}{\partial \theta}, \quad (86)$$

$$\mathcal{J} \delta B^\theta \simeq -\frac{\partial \delta A_\phi}{\partial r}, \quad (87)$$

$$\mathcal{J} \delta B^\phi \simeq 0. \quad (88)$$

The neglected terms in the previous three equations are  $(n/m)^2 (r/R)^2$  smaller than the dominant terms. The previous three expressions are consistent with (63), (64), and (65) provided

$$\delta A_\phi(r, \theta, \phi) \simeq R_0 \sum_j \frac{\psi_j(r)}{m_j} e^{i(m_j \theta - n \phi)}. \quad (89)$$

According to the Biot-Savart law:

$$\delta A_\phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{R R' \mu_0 \delta \mathbf{J}(\mathbf{x}') \cdot \nabla \phi}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}'. \quad (90)$$

Assume that

$$\delta A_\phi(R, \phi, Z) = \delta A_\phi(R, 0, Z) e^{-in\phi}. \quad (91)$$

Hence, can evaluate integral at  $\phi = 0$  without loss of generality. Follows that

$$\mathbf{x} = (R, 0, Z), \quad (92)$$

$$\mathbf{x}' = (R' \cos \phi', R' \sin \phi', Z'), \quad (93)$$

and

$$|\mathbf{x} - \mathbf{x}'| = [R^2 + R'^2 + (Z - Z')^2 - 2 R R' \cos \phi']^{1/2}. \quad (94)$$

Now,

$$\delta \mathbf{J}(R', \phi', Z') \cdot \nabla \phi = \delta J^\phi(R', 0, Z') e^{-i n \phi'} \cos \phi', \quad (95)$$

so

$$\delta A_\phi(r, \theta, 0) = \frac{1}{4\pi} \int_0^\infty \oint \oint \frac{R R' \mu_0 \delta J^\phi(r', \theta', 0) e^{-i n \phi'} \cos \phi' \mathcal{J}' dr' d\theta' d\phi'}{[R^2 + R'^2 + (Z - Z')^2 - 2 R R' \cos \phi']^{1/2}}, \quad (96)$$

which can be written

$$\delta A_\phi(r, \theta, 0) = \frac{1}{4\pi} \int_0^\infty \oint R R' \mu_0 \delta J^\phi(r', \theta', 0) G(r, \theta; r', \theta') \mathcal{J}' dr' d\theta', \quad (97)$$

where

$$G(r, \theta; r', \theta') = \oint \frac{\cos \phi' \cos(n \phi') d\phi'}{[R^2 + R'^2 + (Z - Z')^2 - 2 R R' \cos \phi']^{1/2}}, \quad (98)$$

or

$$G(r, \theta; r', \theta') = \frac{1}{2} \oint \frac{(\cos[(n-1)\phi'] + \cos[(n+1)\phi']) d\phi'}{[R^2 + R'^2 + (Z - Z')^2 - 2 R R' \cos \phi']^{1/2}}. \quad (99)$$

Now (GR 8.85.1),

$$\begin{aligned} P_{-1/2}^n(\cosh \eta) &= \frac{(-1)^n}{2\pi} \frac{\Gamma(1/2)}{\Gamma(1/2 - n)} \oint \frac{\cos(n \varphi) d\varphi}{(\cosh \eta + \sinh \eta \cos \varphi)^{1/2}} \\ &= \frac{1}{2\pi} \frac{\Gamma(1/2)}{\Gamma(1/2 - n)} \oint \frac{\cos(n \varphi) d\varphi}{(\cosh \eta - \sinh \eta \cos \varphi)^{1/2}} \\ &= \frac{(-1)^n \Gamma(1/2) \Gamma(1/2 + n)}{2\pi^2} \oint \frac{\cos(n \varphi) d\varphi}{(\cosh \eta - \sinh \eta \cos \varphi)^{1/2}}. \end{aligned} \quad (100)$$

It follows that

$$\alpha \cosh \eta = R^2 + R'^2 + (Z - Z')^2, \quad (101)$$

$$\alpha \sinh \eta = 2 R R'. \quad (102)$$

Hence,

$$\eta = \tanh^{-1} \left[ \frac{2 R R'}{R^2 + R'^2 + (Z - Z')^2} \right], \quad (103)$$

and

$$\alpha = \frac{R^2 + R'^2 + (Z - Z')^2}{\cosh \eta}. \quad (104)$$

It follows that

$$G(r, \theta; r', \theta') = \pi \left[ \frac{\cosh \eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2}$$



$$\times \left[ \frac{\Gamma(3/2 - n)}{\Gamma(1/2)} P_{-1/2}^{n-1}(\cosh \eta) + \frac{\Gamma(-1/2 - n)}{\Gamma(1/2)} P_{-1/2}^{n+1}(\cosh \eta) \right]. \quad (105)$$

However,

$$\Gamma(3/2 - n) = \frac{(-1)^{n+1} \pi (n - 1/2)}{\Gamma(n + 1/2)}, \quad (106)$$

$$\Gamma(-1/2 - n) = \frac{(-1)^{n+1} \pi}{\Gamma(n + 1/2) (n + 1/2)}, \quad (107)$$

so

$$\begin{aligned} G(r, \theta; r', \theta') &= \frac{(-1)^{n+1} \pi^2}{\Gamma(1/2) \Gamma(n + 1/2)} \left[ \frac{\cosh \eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2} \\ &\times \left[ (n - 1/2) P_{-1/2}^{n-1}(\cosh \eta) + \frac{P_{-1/2}^{n+1}(\cosh \eta)}{n + 1/2} \right]. \end{aligned} \quad (108)$$

Now, from (74) and (80),

$$\mathcal{J} \mu_0 \delta J^\phi(r, \theta, 0) = - \sum_k \Delta \Psi_k \delta(r - r_k) e^{i m_k \theta}. \quad (109)$$

Furthermore, from (79) and (89),

$$\Psi_k = \frac{1}{R_0} \oint \delta A_\phi(r_k, \theta, 0) e^{-i m_k \theta} \frac{d\theta}{2\pi}. \quad (110)$$

Hence,

$$\Psi_k = \sum_{k'} F_{kk'} \Delta \Psi_{k'}, \quad (111)$$

where

$$F_{kk'} = \oint \oint \mathcal{G}(r_k, \theta; r_{k'}, \theta') e^{-i(m_k \theta - m_{k'} \theta')} \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}, \quad (112)$$

and

$$\begin{aligned} \mathcal{G}(r, \theta; r', \theta') &= \frac{(-1)^n \pi^2 R R' / R_0}{2 \Gamma(1/2) \Gamma(n + 1/2)} \left[ \frac{\cosh \eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2} \\ &\times \left[ (n - 1/2) P_{-1/2}^{n-1}(\cosh \eta) + \frac{P_{-1/2}^{n+1}(\cosh \eta)}{n + 1/2} \right]. \end{aligned} \quad (113)$$

Note that

$$\mathcal{G}(r', \theta'; r, \theta) = \mathcal{G}(r, \theta; r', \theta'), \quad (114)$$

which implies that

$$F_{k'k} = F_{kk'}^*. \quad (115)$$

## 8 Inhomogeneous Solution

Suppose that there are currents flowing in a number of external poloidal field coils. Let  $I_l$ ,  $R_l$ , and  $Z_l$  be the peak current, and coordinates of the  $l$ th field coil. The currents are assumed to modulate like  $e^{-in\phi}$ . It follows that

$$\delta J_{\text{ext}}^\phi(R, 0, Z) = \sum_j \frac{I_l}{R_l} \delta(R - R_l) \delta(Z - Z_l). \quad (116)$$

Hence,

$$\Psi_k = \sum_{k'} F_{kk'} \Delta \Psi_k - \sum_l g_{kl} \mu_0 I_l, \quad (117)$$

where

$$g_{kj} = \frac{1}{2\pi} \oint \mathcal{G}(r_k, \theta; R_j, Z_j) e^{-im_k \theta} \frac{d\theta}{2\pi}. \quad (118)$$

It follows that

$$\Delta \Psi_k = \sum_{k'} E_{kk'} \Psi_{k'} + |E_{kk}| \chi_k \quad (119)$$

where  $E_{kk'}$  is the inverse of the  $F_{kk'}$  matrix, and

$$\chi_k = \frac{1}{|E_{kk}|} \sum_j h_{kl} \mu_0 I_l, \quad (120)$$

$$h_{kl} = \sum_{k'} E_{kk'} g_{k'l}. \quad (121)$$

## 9 Island Width

We have

$$\mathcal{J} \delta B^r \simeq \frac{\partial \delta A_\phi}{\partial \theta}, \quad (122)$$

$$\mathcal{J} \delta B^\theta \simeq -\frac{\partial \delta A_\phi}{\partial r}, \quad (123)$$

$$\mathcal{J} \delta B^\phi \simeq 0. \quad (124)$$

It follows that

$$\delta \mathbf{B} \cdot \nabla \delta A_\phi = \delta B^r \frac{\partial \delta A_\phi}{\partial r} + \delta B^\theta \frac{\partial \delta A_\phi}{\partial \theta} + \delta B^\phi \frac{\partial \delta A_\phi}{\partial \phi} \simeq 0. \quad (125)$$

Suppose that

$$\delta A_\phi(r, \theta, \phi) \simeq \delta \hat{A}_\phi e^{i(m_k \theta - n \phi)} \quad (126)$$

in the vicinity of the  $k$ th rational surface. It follows that

$$\mathbf{B} \cdot \nabla \delta A_\phi = i B_0 R_0^2 \frac{f}{r R^2} (m_k - n q) \delta \hat{A}_\phi e^{i(m_k \theta - n \phi)}. \quad (127)$$

Let us search for a function,

$$F(r, \theta, \phi) = F_0(r) + \delta A_\phi, \quad (128)$$

which is such that

$$(\mathbf{B} + \delta \mathbf{B}) \cdot \nabla F = 0. \quad (129)$$

It follows that

$$\delta B^r \frac{dF_0}{dr} + i B_0 R_0^2 \frac{f}{r R^2} (m_k - n q) \delta \hat{A}_\phi e^{i(m_k \theta - n \phi)} = 0, \quad (130)$$

or

$$i m_k \frac{R_0}{r R^2} \delta \hat{A}_\phi \frac{dF_0}{dr} + i B_0 R_0^2 \frac{f}{r R^2} (m_k - n q) \delta \hat{A}_\phi = 0, \quad (131)$$

which implies that

$$\frac{dF_0}{dr} = -\frac{B_0 R_0 f}{m_k} (m_k - n q), \quad (132)$$

or

$$F_0(r) \simeq \frac{B_0}{2} \left( \frac{g s}{q} \right)_{r_k} (r - r_k)^2. \quad (133)$$

Hence,

$$F(r, \theta, \phi) = \frac{B_0}{2} \left( \frac{g s}{q} \right)_{r_k} (r - r_k)^2 + R_0 \Psi_k \cos(m_k \theta - n \phi) \quad (134)$$

is a flux surface function in the island region. Thus,

$$\frac{F}{R_0 |\Psi_k|} = 2 X^2 + \cos(m_k \theta - n \phi), \quad (135)$$

where

$$X = \frac{2(r - r_k)}{W_k}, \quad (136)$$

and

$$\frac{W_k}{4 R_0} = \left[ \left( \frac{q}{g s} \right)_{r_k} \frac{|\Psi_k|}{B_0 R_0} \right]^{1/2}. \quad (137)$$

It follows that  $W_k$  is the full radial island width (which is constant in  $r$ ).

## 10 Flux Coordinate System

Let all lengths be normalized to  $R_0$ , and all magnetic field-strengths to  $B_0$ . We have

$$\mathbf{B} = \nabla\phi \times \nabla\psi_p + g(\psi_p) \nabla\phi, \quad (138)$$

and

$$\nabla\psi_p \times \nabla\theta \cdot \nabla\phi = \frac{g}{q R^2}, \quad (139)$$

where  $q = q(\psi_p)$ .

Let  $\Psi = \psi_p/\psi_c = 1 - \Psi_N$ , where  $\psi_c$  is the value of  $\psi_p$  on the magnetic axis. (It is assumed that  $\psi_p = 0$  on the plasma boundary.) The previous equation implies that

$$\frac{d\theta}{dl} = \frac{g}{q |\psi_c| R \sqrt{\Psi_R^2 + \Psi_Z^2}}, \quad (140)$$

where  $dl$  is an element of poloidal path length on a magnetic flux-surface, and  $\Psi_R \equiv \partial\Psi/\partial R$ , etc. Furthermore,

$$dR = -\frac{\Psi_Z dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}, \quad (141)$$

$$dZ = \frac{\Psi_R dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}. \quad (142)$$

It follows that

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi |\psi_c|} \oint \frac{dl}{R \sqrt{\Psi_R^2 + \Psi_Z^2}}. \quad (143)$$

If we define

$$\tan \zeta = \frac{Z - Z_{\text{axis}}}{R_{\text{axis}} - R} \quad (144)$$

then

$$\frac{dR}{d\zeta} = -\Psi_Z F, \quad (145)$$

$$\frac{dZ}{d\zeta} = \Psi_R F, \quad (146)$$

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi |\psi_c|} \oint \frac{F}{R} d\zeta, \quad (147)$$

where

$$F = \frac{(R_{\text{axis}} - R)^2 + (Z - Z_{\text{axis}})^2}{-(Z - Z_{\text{axis}}) \Psi_Z + (R_{\text{axis}} - R) \Psi_R}. \quad (148)$$

Note that

$$\frac{d\theta}{d\zeta} = \frac{g}{q |\psi_c|} \frac{F}{R}. \quad (149)$$

Hence, [from (194)]

$$\langle A \rangle = \oint R^2 A d\theta \Big/ \oint R^2 d\theta = \oint F R A d\zeta \Big/ \oint F R d\zeta. \quad (150)$$

In particular, because

$$B^2 = \frac{\psi_c^2 |\nabla \Psi|^2}{R^2} + \frac{g^2}{R^2} \quad (151)$$

it follows that

$$\langle B^2 \rangle = \oint \frac{F}{R} [\psi_c^2 (\Psi_R^2 + \Psi_Z^2) + g^2] d\zeta \Big/ \oint F R d\zeta. \quad (152)$$

It is helpful to define the length-like flux-surface coordinate  $r$ , according to

$$\nabla r \times \nabla \theta \cdot \nabla \phi = \frac{1}{r R^2}. \quad (153)$$

It follows that

$$r(\Psi) = \left[ 2 |\psi_c| \int_{\Psi}^1 \frac{q(\Psi')}{g(\Psi')} d\Psi' \right]^{1/2}. \quad (154)$$

We can calculate  $R(r, \theta)$  and  $Z(r, \theta)$  from

$$\frac{dR}{d\theta} = -|\psi_c| \frac{q}{g} R \Psi_Z, \quad (155)$$

$$\frac{dZ}{d\theta} = |\psi_c| \frac{q}{g} R \Psi_R. \quad (156)$$

Now,

$$r \frac{dr}{d\Psi} = -|\psi_c| \frac{q(r)}{g(r)}. \quad (157)$$

So

$$\nabla r = \frac{dr}{d\Psi} \nabla \Psi = -|\psi_c| \frac{q(r)}{r g(r)} \nabla \Psi. \quad (158)$$

Hence,

$$a_{jj} = \left( \oint \frac{1}{|\nabla r|^2} \frac{d\theta}{2\pi} \right)_{r_j} = \left( \frac{r g}{|\psi_c| q} \right)_{r_j}^2 \oint \frac{1}{\Psi_R^2 + \Psi_Z^2} \frac{d\theta}{2\pi}. \quad (159)$$

Note that

$$\frac{d\Psi_N}{dr} = \frac{r g(r)}{|\psi_c| q(r)}. \quad (160)$$

Hence, if  $\overline{W}_k$  is the full magnetic island width in  $\Psi_N$  at the  $k$ th rational surface then

$$\overline{W}_k = \frac{W_k}{R_0} \frac{d\Psi_N}{dr}. \quad (161)$$

## 11 GPEC Coupling

We have

$$\frac{d\psi_p}{dr} = B_0 R_0 f(r), \quad (162)$$

$$\mathbf{B} \cdot \nabla \phi = \frac{B_0 R_0}{R^2} g(r), \quad (163)$$

$$\delta B^r = \frac{i}{r} \left( \frac{R_0}{R} \right)^2 \sum_j \psi_j e^{i(m_j \theta - n \phi)}. \quad (164)$$

According to PRL **99**, 195003 (2007),

$$\Delta_j e^{i(m_j \theta - n \phi)} = \left[ \frac{\partial}{\partial \psi_p} \frac{\delta \mathbf{B} \cdot \nabla \psi_p}{\mathbf{B} \cdot \nabla \phi} \right]_{r_j} = \left[ \frac{\partial}{\partial r} \frac{\delta \mathbf{B} \cdot \nabla r}{\mathbf{B} \cdot \nabla \phi} \right]_{r_j} = \left[ \frac{\partial}{\partial r} \frac{\delta B^r}{\mathbf{B} \cdot \nabla \phi} \right]_{r_j}. \quad (165)$$

It follows that

$$\Delta_j = \frac{i}{r_j} \left( \frac{R_0}{R} \right)^2 \frac{R^2}{B_0 R_0 g_j} \left[ \frac{d\psi_j}{dr} \right]_{r_j}, \quad (166)$$

But,

$$\left[ r \frac{d\psi_j}{dr} \right]_{r_j} = m_j a_{jj} [\chi_j]_{r_j} = m_j a_{jj} \Delta \Psi_j, \quad (167)$$

and

$$\chi_j = \frac{\Delta \Psi_j}{|E_{jj}|}, \quad (168)$$

which implies that

$$\Delta_j = i \left( \frac{R_0}{r_j} \right)^2 \frac{m_j a_{jj}}{g_j} \frac{\Delta \Psi_j}{B_0 R_0}, \quad (169)$$

$$\frac{\chi_j}{R_0 B_0} = -i \left( \frac{r_j}{R_0} \right)^2 \frac{g_j}{m_j a_{jj}} \frac{\Delta_j}{|E_{jj}|}. \quad (170)$$

Now,  $d\psi_p/dr = B_0 R_0 f$ . It follows that

$$(\nabla \psi_p \times \nabla \theta \cdot \nabla \phi)^{-1} = \frac{R^2 q}{B_0 R_0 g}. \quad (171)$$

Hence,

$$\mathbf{B} = \nabla \phi \times \nabla \psi_p + B_0 R_0 g \nabla \phi = \nabla \phi \times \nabla \psi_p + q \nabla \psi_p \times \nabla \theta. \quad (172)$$

Let  $\Psi_p = 2\pi \psi_p$  and  $d\Psi_p/d\Psi_t = 1/q$ . Hence,

$$\frac{d\Psi_p}{dr} = 2\pi B_0 R_0 f, \quad (173)$$

$$\frac{d\Psi_t}{dr} = 2\pi B_0 r g, \quad (174)$$

and [cf. PoP **13**, 102501 (2006), Eq. (41)]

$$2\pi \mathbf{B} = q^{-1} \nabla \phi \times \nabla \Psi_t + \nabla \Psi_t \times \nabla \theta. \quad (175)$$

We have

$$\delta J^r = 0, \quad (176)$$

$$\mathcal{J} \mu_0 \delta J^\theta = - \sum_j \frac{\Delta \Psi_j}{q_j} e^{i(m_j \theta - n \phi)} \delta(r - r_j), \quad (177)$$

$$\mathcal{J} \mu_0 \delta J^\phi = - \sum_j \Delta \Psi_k e^{i(m_j \theta - n \phi)} \delta(r - r_j), \quad (178)$$

which implies that

$$\mu_0 \delta \mathbf{J} = -2\pi \sum_j \Delta \Psi_j e^{i(m_j \theta - n \phi)} \delta(\psi_t - \psi_{tj}) \mathbf{B}. \quad (179)$$

However, according to PoP **13**, 102501 (2006),

$$\Delta \Psi_j = -i \frac{\mu_0 J_c \Delta_j}{2\pi m_j}, \quad (180)$$

where

$$\frac{1}{\mu_0 J_c} = \left( \oint \frac{B^2}{|\nabla \psi_t|^2} \frac{d\theta d\phi}{2\pi \mathbf{B} \cdot \nabla \phi} \right)_{r_j}. \quad (181)$$

It is easily demonstrated that

$$\frac{1}{\mu_0 J_c} = \left( \frac{R_0}{r_j} \right)^2 \frac{1}{2\pi B_0 R_0 g_j} \left[ a_{jj} + \left( \frac{r_k}{R_0 q_j} \right)^2 \right]. \quad (182)$$

Hence,

$$\frac{\Delta \Psi_j}{B_0 R_0} = -i \Delta_j \left( \frac{r_j}{R_0} \right)^2 \frac{g_j}{m_j [a_{jj} + (r_k/R_0 q_j)^2]}. \quad (183)$$

## 12 Neoclassical Coordinate System

It is also helpful to define the geometric poloidal angle

$$\mathbf{b} \cdot \nabla \Theta = \gamma(r). \quad (184)$$

It follows that

$$\frac{d\Theta}{dl} = \frac{\gamma B R}{|\psi_c| \sqrt{\Psi_R^2 + \Psi_Z^2}}, \quad (185)$$

where

$$B R = [g^2 + |\psi_c|^2 (\Psi_R^2 + \Psi_Z^2)]^{1/2}. \quad (186)$$

Hence,

$$\frac{1}{\gamma(r)} = \frac{1}{2\pi |\psi_c|} \oint \frac{B R dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}} = \frac{1}{2\pi |\psi_c|} \oint B R F d\zeta. \quad (187)$$

We can calculate  $R(r, \Theta)$  and  $Z(r, \Theta)$  from

$$\frac{dR}{d\Theta} = -|\psi_c| \frac{\Psi_Z}{\gamma B R}, \quad (188)$$

$$\frac{dZ}{d\Theta} = |\psi_c| \frac{\Psi_R}{\gamma B R}. \quad (189)$$

Note that

$$\frac{d\Theta}{d\theta} = \left( \frac{\gamma q}{g} \right) B R^2. \quad (190)$$

Thus,

$$\frac{1}{\gamma} = \frac{q}{g} \oint B R^2 \frac{d\theta}{2\pi}. \quad (191)$$

### 13 Neoclassical Parameters

The flux-surface average operator has the following properties:

$$\langle 1 \rangle = 1, \quad (192)$$

$$\langle \mathbf{B} \cdot \nabla A \rangle = 0. \quad (193)$$

It follows that

$$\langle A \rangle = \oint R^2 A \frac{d\theta}{2\pi} \Big/ \oint R^2 \frac{d\theta}{2\pi} = \oint \frac{A}{B} \frac{d\Theta}{2\pi} \Big/ \oint \frac{1}{B} \frac{d\Theta}{2\pi}. \quad (194)$$

Let

$$I_0 = \oint \frac{1}{B R^2} \frac{d\Theta}{2\pi} = \frac{\gamma q}{g}, \quad (195)$$

$$I_1 = \oint \frac{1}{B} \frac{d\Theta}{2\pi}, \quad (196)$$

$$I_2 = \oint B \frac{d\Theta}{2\pi}, \quad (197)$$

$$I_3 = \oint \left( \frac{\partial B}{\partial \Theta} \right)^2 \frac{1}{B} \frac{d\Theta}{2\pi}, \quad (198)$$

$$I_{4,k} = \sqrt{\frac{2}{k}} \oint \frac{\sin(k\Theta)}{B^2} \frac{\partial B}{\partial \Theta} \frac{d\Theta}{2\pi} = \oint \frac{\sqrt{2k} \cos(k\Theta)}{B} \frac{d\Theta}{2\pi}, \quad (199)$$



$$I_{5,k} = \sqrt{\frac{2}{k}} \oint \frac{\sin(k\Theta)}{B^3} \frac{\partial B}{\partial \Theta} \frac{d\Theta}{2\pi} = \oint \frac{\sqrt{2k} \cos(k\Theta)}{2B^2} \frac{d\Theta}{2\pi}, \quad (200)$$

$$I_6(\lambda) = \oint \frac{\sqrt{1 - \lambda B/B_{\max}}}{B} \frac{d\Theta}{2\pi}. \quad (201)$$

It follows that

$$\langle B^2 \rangle = \frac{I_2}{I_1}, \quad (202)$$

$$\langle (\mathbf{b} \cdot \nabla B)^2 \rangle = \gamma^2 \frac{I_3}{I_1}, \quad (203)$$

$$|\langle \mathbf{B} \cdot \nabla \theta \rangle| = \frac{g}{|q|} \frac{I_0}{I_1} = \frac{|\gamma|}{I_1}, \quad (204)$$

$$\left\langle \sqrt{\frac{2}{k}} \sin(k\Theta) (\mathbf{b} \cdot \nabla \ln B) \right\rangle = \gamma \frac{I_{4,k}}{I_1}, \quad (205)$$

$$\left\langle \sqrt{\frac{2}{k}} \sin(k\Theta) \frac{(\mathbf{b} \cdot \nabla \ln B)}{B} \right\rangle = \gamma \frac{I_{5,k}}{I_1}. \quad (206)$$

Hence,

$$L_c = \frac{1}{|\gamma|} \frac{I_2^2}{I_1^2 I_3} \sum_{k>0} I_{5,k} I_{6,k}, \quad (207)$$

$$\omega_{ta} \equiv \frac{v_{Ta}}{L_c} = K_t |\gamma| v_{Ta}, \quad (208)$$

$$\nu_{*a} \equiv \frac{8}{3\pi} \frac{\langle B^2 \rangle}{\langle (\mathbf{b} \cdot \nabla B)^2 \rangle} \frac{g_t \omega_{ta}}{v_{Ta}^2 \tau_{aa}} = K_* \frac{g_t}{\omega_{ta} \tau_{aa}}, \quad (209)$$

$$f_c = \frac{3}{4} \frac{I_2}{B_{\max}^2} \int_0^1 \frac{\lambda d\lambda}{I_6(\lambda)}, \quad (210)$$

where

$$K_t = \frac{I_1^2 I_3}{I_2^2 \sum_{k>0} I_{4,k} I_{5,k}}, \quad (211)$$

$$K_* = \frac{8}{3\pi} \frac{I_2}{I_3} K_t^2. \quad (212)$$