Program FLUX

1 Coordinate Systems

- Cartesian coordinates: X, Y, Z.
- Cylindrical coordinates: $R \equiv (X^2 + Y^2)^{1/2}$, $\phi \equiv \tan^{-1}(Y/X)$, Z, so

$$|\nabla\phi| = \frac{1}{R}. \tag{1}$$

- Flux coordinates: $r(R,Z)$, $\theta(R,Z)$, ϕ, where

$$(\nabla r \cdot \nabla\theta \times \nabla\phi)^{-1} = \mathcal{J}, \tag{2}$$

$$\mathcal{J} = \frac{r\,R^2}{R_0}, \tag{3}$$

where R_0 is a convenient scale major radius.

2 Useful Identities

Easily demonstrated that:

$$\mathbf{A} = A^r\,\mathcal{J}\,\nabla\theta \times \nabla\phi + A^\theta\,\mathcal{J}\,\nabla\phi \times \nabla r + A^\phi\,\mathcal{J}\,\nabla r \times \nabla\theta, \tag{4}$$

$$\mathbf{A} = A_r\,\nabla r + A_\theta\,\nabla\theta + A_\phi\,\nabla\phi, \tag{5}$$

$$\mathbf{A}\cdot\mathbf{B} = A_r\,B^r + A_\theta\,B^\theta + A_\phi\,B^\phi = A^r\,B_r + A^\theta\,B_\theta + A^\phi\,B_\phi, \tag{6}$$

$$(\mathbf{A}\times\mathbf{B})_r = \mathcal{J}\,(A^\theta\,B^\phi - A^\phi\,B^\theta), \tag{7}$$

$$(\mathbf{A}\times\mathbf{B})_\theta = \mathcal{J}\,(A^\phi\,B^r - A^r\,B^\phi), \tag{8}$$

$$(\mathbf{A}\times\mathbf{B})_\phi = \mathcal{J}\,(A^r\,B^\theta - A^\theta\,B^r), \tag{9}$$

$$\mathcal{J}\,(\mathbf{A}\times\mathbf{B})^r = A_\theta\,B_\phi - A_\phi\,B_\theta, \tag{10}$$

$$\mathcal{J}\,(\mathbf{A}\times\mathbf{B})^\theta = A_\phi\,B_r - A_r\,B_\phi, \tag{11}$$

$$\mathcal{J}\,(\mathbf{A}\times\mathbf{B})^\phi = A_r\,B_\theta - A_\theta\,B_r, \tag{12}$$

$$\mathcal{J}\,\nabla\cdot\mathbf{A} = \frac{\partial\,(\mathcal{J}\,A^r)}{\partial r} + \frac{\partial\,(\mathcal{J}\,A^\theta)}{\partial\theta} + \frac{\partial\,(\mathcal{J}\,A^\phi)}{\partial\phi}, \tag{13}$$

$$\mathcal{J}\,(\nabla\times\mathbf{A})^r = \frac{\partial A_\phi}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}, \tag{14}$$

$$\mathcal{J}\,(\nabla\times\mathbf{A})^\theta = \frac{\partial A_r}{\partial\phi} - \frac{\partial A_\phi}{\partial r}, \tag{15}$$

$$\mathcal{J}\,(\nabla\times\mathbf{A})^\phi = \frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}. \tag{16}$$

3 Equilibrium Magnetic Field

Equilibrium magnetic field:

$$\mathbf{B} = B_0\,R_0\,[f(r)\,\nabla\phi\times\nabla r + g(r)\,\nabla\phi]\,, \tag{17}$$

$$q(r) = \frac{r\,g}{R_0\,f}, \tag{18}$$

where B_0 is a convenient scale magnetic field strength. So,

$$B^r = 0, \tag{19}$$

$$B^\theta = B_0\,R_0^2\,\frac{f}{r\,R^2}, \tag{20}$$

$$B^\phi = B_0\,R_0^2\,\frac{q\,f}{r\,R^2} = B_0\,B_0\,\frac{g}{R^2}, \tag{21}$$

$$B_r = -B_0\,r\,f\,\nabla r\cdot\nabla\theta, \tag{22}$$

$$B_\theta = B_0\,r\,f\,|\nabla r|^2, \tag{23}$$

$$B_\phi = B_0\,R_0^2\,\frac{q\,f}{r} = B_0\,R_0\,g, \tag{24}$$

and

$$\mathbf{B}\cdot\nabla = B_0\,R_0^2\,\frac{f}{r\,R^2}\left(\frac{\partial}{\partial\theta} + q\,\frac{\partial}{\partial\phi}\right). \tag{25}$$

4 Equilibrium Current

The relation $\mu_0\,\mathbf{J} = \nabla\times\mathbf{B}$ yields

$$\mu_0\,\mathcal{J}\,J^r = 0, \tag{26}$$

$$\mu_0\,\mathcal{J}\,J^\theta = -B_0\,R_0\,\frac{dg}{dr}, \tag{27}$$

$$\mu_0\,\mathcal{J}\,J^\phi = B_0\,\frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + B_0\,\frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\,. \tag{28}$$

5 Grad-Shafranov Equation

The force balance relation

$$\mathbf{J}\times\mathbf{B} = \nabla P \tag{29}$$

yields

$$\mathcal{J}\,(J^{\,\theta}\,B^{\,\phi} - J^{\,\phi}\,B^{\,\theta}) = \frac{dP}{dr}, \tag{30}$$

which reduces to

$$\frac{f}{r}\,\frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^{\,2}\big) + \frac{f}{r}\,\frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta) + g\,\frac{dg}{dr} + \frac{\mu_0}{B_0^2}\left(\frac{R}{R_0}\right)^2\frac{dP}{dr} = 0. \tag{31}$$

6 High-q Limit

Now,

$$\nabla\cdot\delta\mathbf{B} = 0 \tag{32}$$

yields

$$\frac{\partial(\mathcal{J}\,\delta B^{\,r})}{\partial r} + \frac{\partial(\mathcal{J}\,\delta B^{\,\theta})}{\partial\theta} + \frac{\partial(\mathcal{J}\,\delta B^{\,\phi})}{\partial\phi} = 0. \tag{33}$$

Furthermore,

$$\delta\mathbf{B} = \delta B_r\,\nabla r + \delta B_\theta\,\nabla\theta + \delta B_\phi\,\nabla\phi. \tag{34}$$

So,

$$\delta B^{\,r} = \delta\mathbf{B}\cdot\nabla r = |\nabla r|^{\,2}\,\delta B_r + (\nabla r\cdot\nabla\theta)\,\delta B_\theta, \tag{35}$$

$$\delta B^{\,\theta} = \delta\mathbf{B}\cdot\nabla\theta = (\nabla r\cdot\nabla\theta)\,\delta B_r + |\nabla\theta|^{\,2}\,\delta B_\theta, \tag{36}$$

$$\delta B^{\,\phi} = \delta\mathbf{B}\cdot\nabla\phi = \frac{\delta B_\phi}{R^{\,2}}. \tag{37}$$

Follows that

$$\delta B_r = \left(\frac{1}{|\nabla r|^{\,2}}\right)\delta B^{\,r} - \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\right)\delta B_\theta, \tag{38}$$

and

$$\delta B^{\,\theta} = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\right)\delta B^{\,r} + \left[|\nabla\theta|^{\,2} - \frac{(\nabla r\cdot\nabla\theta)^{\,2}}{|\nabla r|^{\,2}}\right]\delta B_\theta. \tag{39}$$

But, Eqs. (2) and (3) imply that

$$|\nabla r|^{\,2}\,|\nabla\theta|^{\,2} - (\nabla r\cdot\nabla\theta)^{\,2} = \frac{R_0^2}{r^{\,2}\,R^{\,2}}. \tag{40}$$

Hence,

$$\delta B^{\,\theta} = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\right)\delta B^{\,r} + \left(\frac{R_0^2}{r^{\,2}\,R^{\,2}\,|\nabla r|^{\,2}}\right)\delta B_\theta. \tag{41}$$

Assume that high-q limit is equivalent to $\delta\mathbf{B}$ being curl-free. Follows that

$$\frac{\partial\,\delta B_\phi}{\partial\theta} = \frac{\partial\,\delta B_\theta}{\partial\phi}, \tag{42}$$

$$\frac{\partial\,\delta B_r}{\partial\phi} = \frac{\partial\,\delta B_\phi}{\partial r}, \tag{43}$$

$$\frac{\partial\,\delta B_\theta}{\partial r} = \frac{\partial\,\delta B_r}{\partial\theta}. \tag{44}$$

Previous three equations imply that

$$\delta B_\phi \sim \frac{n}{m}\,\delta B_\theta,\; \frac{n}{m}\,r\,\delta B_r \tag{45}$$

and, hence, that

$$\delta B^{\,\phi} \sim \frac{n}{m}\left(\frac{r}{R}\right)^2 \delta B^{\,\theta},\; \frac{n}{m}\left(\frac{r}{R}\right)^2 \frac{\delta B^{\,r}}{r}, \tag{46}$$

Consequently, final term on right-hand side of (33) is of order $(n/m)^2\,(r/R)^2$ smaller than other two terms, and, therefore, negligible. Thus, we get

$$\frac{\partial(\mathcal{J}\,\delta B^{\,r})}{\partial r} + \frac{\partial(\mathcal{J}\,\delta B^{\,\theta})}{\partial\theta} = 0. \tag{47}$$

Follows from (41) and previous equation that

$$r\,\frac{\partial}{\partial r}\left(\frac{r\,R^2\,\delta B^{\,r}}{R_0^2}\right) = -\frac{\partial}{\partial\theta}\left[\left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\right)\left(\frac{r\,R^2\,\delta B^{\,r}}{R_0^2}\right) + \left(\frac{1}{|\nabla r|^{\,2}}\right)\delta B_\theta\right]. \tag{48}$$

Follows from (3), (38) and (44) that

$$r\,\frac{\partial\,\delta B_\theta}{\partial r} = \frac{\partial}{\partial\theta}\left[\left(\frac{R_0^2}{R^2\,|\nabla r|^{\,2}}\right)\left(\frac{r\,R^2\,\delta B^{\,r}}{R_0^2}\right) - \left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\right)\delta B_\theta\right]. \tag{49}$$

Let

$$\frac{r\,R^2\,\delta B^{\,r}(r,\theta,\phi)}{R_0^2} = \sum_j \delta\hat{B}^{\,r}_j(r)\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta - n\phi)}, \tag{50}$$

$$\delta B_\theta(r,\theta,\phi) = \sum_j \delta\hat{B}_{\theta\,j}(r)\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta - n\,\phi)}. \tag{51}$$

Follows that

$$\sum_{j'} r\,\frac{d\,\delta\hat{B}^{\,r}_{j'}}{dr}\,\mathrm{e}^{\,\mathrm{i}\,m_{j'}\,\theta} = -\frac{\partial}{\partial\theta}\sum_{j'}\left[\left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\right)\mathrm{e}^{\,\mathrm{i}\,m_{j'}\,\theta}\,\delta\hat{B}^{\,r}_{j'} + \left(\frac{1}{|\nabla r|^{\,2}}\right)\mathrm{e}^{\,\mathrm{i}\,m_{j'}\,\theta}\,\delta\hat{B}_{\theta\,j'}\right], \tag{52}$$

$$\sum_{j'} r\,\frac{d\,\delta\hat{B}_{\theta\,j'}}{dr}\,\mathrm{e}^{\,\mathrm{i}\,m_{j'}\,\theta} = \frac{\partial}{\partial\theta}\sum_{j'}\left[\left(\frac{R_0^2}{R^2\,|\nabla r|^{\,2}}\right)\mathrm{e}^{\,\mathrm{i}\,m_{j'}\,\theta}\,\delta\hat{B}^{\,r}_{j'} - \left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\right)\mathrm{e}^{\,\mathrm{i}\,m_{j'}\,\theta}\,\delta\hat{B}_{\theta\,j'}\right]. \tag{53}$$

Operating with $\oint(\cdots)\,\mathrm{e}^{-\mathrm{i}\,m_j\,\theta}\,d\theta/(2\pi)$, we get

$$r\,\frac{d\,\delta\hat{B}^{\,r}_j}{dr} = -\mathrm{i}\,m_j\sum_{j'}\left(-\mathrm{i}\,c_{jj'}\,\delta\hat{B}^{\,r}_{j'} + a_{jj'}\,\delta\hat{B}_{\theta\,j'}\right), \tag{54}$$

$$r\,\frac{d\,\delta\hat{B}_{\theta\,j}}{dr} = -\mathrm{i}\,m_j\sum_{j'}\left(-\mathrm{i}\,c_{jj'}\,\delta\hat{B}_{\theta\,j'} - b_{jj'}\,\delta\hat{B}^{\,r}_{j'}\right), \tag{55}$$

where

$$a_{jj'} = \oint\frac{1}{|\nabla r|^{\,2}}\,\mathrm{e}^{-\mathrm{i}\,(m_j-m_{j'})\,\theta}\,\frac{d\theta}{2\pi}, \tag{56}$$

$$b_{jj'} = \oint\frac{R_0^2}{R^{\,2}\,|\nabla r|^{\,2}}\,\mathrm{e}^{-\mathrm{i}\,(m_j-m_{j'})\,\theta}\,\frac{d\theta}{2\pi}, \tag{57}$$

$$c_{jj'} = \oint\frac{\mathrm{i}\,r\,\nabla r\cdot\nabla\theta}{|\nabla r|^{\,2}}\,\mathrm{e}^{-\mathrm{i}\,(m_j-m_{j'})\,\theta}\,\frac{d\theta}{2\pi}. \tag{58}$$

Let

$$\delta\hat{B}^{\,r}_j(r) = \mathrm{i}\,\psi_j(r), \tag{59}$$

$$\delta\hat{B}_{\theta\,j}(r) = -\chi_j(r). \tag{60}$$

It follows that

$$r\,\frac{d\psi_j}{dr} = m_j\sum_{j'}\left(-c_{jj'}\,\psi_{j'} + a_{jj'}\,\chi_{j'}\right), \tag{61}$$

$$r\,\frac{d\chi_j}{dr} = m_j\sum_{j'}\left(-c_{jj'}\,\chi_{j'} + b_{jj'}\,\psi_{j'}\right). \tag{62}$$

Hence,

$$\frac{r\,R^{\,2}\,\delta B^{\,r}(r,\theta,\phi)}{R_0^2} = \mathrm{i}\sum_j\psi_j(r)\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\phi)}, \tag{63}$$

$$\frac{r\,R^{\,2}\,\delta B^{\,\theta}(r,\theta,\phi)}{R_0^2} = -\sum_j\frac{1}{m_j}\,\frac{d\psi_j}{dr}\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}, \tag{64}$$

$$R^{\,2}\,\delta B^{\,\phi}(r,\theta,\phi) = n\sum_j\frac{\chi_j(r)}{m_j}\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}. \tag{65}$$

where use has been made of (42). Also have

$$\delta B_r(r,\theta,\phi) = \mathrm{i}\sum_j\frac{1}{m_j}\,\frac{d\chi_j}{dr}\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}, \tag{66}$$

$$\delta B_\theta(r,\theta,\phi) = -\sum_j \chi_j(r)\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}, \tag{67}$$

$$\delta B_\phi(r,\theta,\phi) = n\sum_j \frac{\chi_j(r)}{m_j}\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}, \tag{68}$$

We can write

$$\mathcal{J}\,\mu_0\,\delta J^{\,r} = \frac{\partial\,\delta B_\phi}{\partial\theta} - \frac{\partial\,\delta B_\theta}{\partial\phi}, \tag{69}$$

$$\mathcal{J}\,\mu_0\,\delta J^{\,\theta} = \frac{\partial\,\delta B_r}{\partial\phi} - \frac{\partial\,\delta B_\phi}{\partial r}, \tag{70}$$

$$\mathcal{J}\,\mu_0\,\delta J^{\,\phi} = \frac{\partial\,\delta B_\theta}{\partial r} - \frac{\partial\,\delta B_r}{\partial\theta}. \tag{71}$$

Normally, all three components are zero. However, at kth rational surface ψ_k, $\psi_{j\neq k}$, and $\chi_{j\neq k}$ are continuous, whereas χ_k is discontinuous. Hence,

$$\mathcal{J}\,\mu_0\,\delta J^{\,r}(r,\theta,\phi) = 0, \tag{72}$$

$$\mathcal{J}\,\mu_0\,\delta J^{\,\theta}(r,\theta,\phi) = -\sum_k \frac{n}{m_k}\,\left[\chi_k\right]_{r_{k-}}^{r_{k+}}\,\delta(r-r_k)\,\mathrm{e}^{\,\mathrm{i}\,(m_k\,\theta-n\,\phi)}, \tag{73}$$

$$\mathcal{J}\,\mu_0\,\delta J^{\,\phi}(r,\theta,\phi) = -\sum_k \left[\chi_k\right]_{r_{k-}}^{r_{k+}}\,\delta(r-r_k)\,\mathrm{e}^{\,\mathrm{i}\,(m_k\,\theta-n\,\phi)}. \tag{74}$$

Now,

$$(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\theta = \frac{1}{4}\left(\mathcal{J}\,\mu_0\,\delta J^{\,\phi}\,\delta B^{\,r\,*} + \text{c.c.}\right), \tag{75}$$

$$(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\phi = \frac{1}{4}\left(-\mathcal{J}\,\mu_0\,\delta J^{\,\theta}\,\delta B^{\,r\,*} + \text{c.c.}\right), \tag{76}$$

which implies that

$$\mathcal{J}\,(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\theta = \frac{R_0}{2}\sum_j\sum_k \mathrm{Re}\left\{\mathrm{i}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\psi_j^{\,*}(r_k)\,\mathrm{e}^{\,\mathrm{i}\,(m_k-m_j)\,\theta}\right\}\delta(r-r_k), \tag{77}$$

$$\mathcal{J}\,(\mu_0\,\delta\mathbf{J}\times\delta\mathbf{B})_\phi = -\frac{R_0}{2}\sum_j\sum_k \frac{n}{m_k}\,\mathrm{Re}\left\{\mathrm{i}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\psi_j^{\,*}(r_k)\,\mathrm{e}^{\,\mathrm{i}\,(m_k-m_j)\,\theta}\right\}\delta(r-r_k). \tag{78}$$

Let

$$\Psi_k = \frac{\psi_k(r_k)}{m_k}, \tag{79}$$

$$\Delta\Psi_k = \left[\chi_k\right]_{r_{k-}}^{r_{k+}}, \tag{80}$$

$$\delta T_{\theta\,k} \equiv \delta\mathbf{T}\cdot\mathbf{e}_\theta = \int_{r_{k-}}^{r_{k+}}\oint\oint(\delta\mathbf{J}\times\delta\mathbf{B})_\theta\,\mathcal{J}\,dr\,d\theta\,d\phi, \tag{81}$$

$$\delta T_{\phi\,k} \equiv \delta\mathbf{T}\cdot\mathbf{e}_\phi = \int_{r_{k-}}^{r_{k+}}\oint\oint(\delta\mathbf{J}\times\delta\mathbf{B})_\phi\,\mathcal{J}\,dr\,d\theta\,d\phi, \tag{82}$$

where $\delta\mathbf{T}(r)\,dr$ is the net electromagnetic torque acting on the plasma between r and $r+dr$. Here, $\mathbf{e}_\theta = (R^2/R_0)\,\nabla\phi\times\nabla r$ and $\mathbf{e}_\phi = R\,\nabla\phi$. Follows that

$$\delta T_{\theta\,k} = -\frac{2\pi^2\,R_0\,m_k}{\mu_0}\,\mathrm{Im}\left(\Psi_k^*\,\Delta\Psi_k\right), \tag{83}$$

$$\delta T_{\phi\,k} = \frac{2\pi^2\,R_0\,n}{\mu_0}\,\mathrm{Im}\left(\Psi_k^*\,\Delta\Psi_k\right). \tag{84}$$

7 Homogeneous Solution

We can write

$$\delta\mathbf{B} = \nabla\times\delta\mathbf{A}. \tag{85}$$

Suppose that $r\,\delta A_r$, δA_θ are negligible with respect to δA_ϕ. In fact, it is easily demonstrated from $\nabla\cdot\delta\mathbf{A} = 0$ that $r\,\delta A_r$, $\delta A_\theta \sim (n/m)\,(r/R)^2\,\delta A_\phi$. It follows that

$$\mathcal{J}\,\delta B^{\,r} \simeq \frac{\partial\,\delta A_\phi}{\partial\theta}, \tag{86}$$

$$\mathcal{J}\,\delta B^{\,\theta} \simeq -\frac{\partial\,\delta A_\phi}{\partial r}, \tag{87}$$

$$\mathcal{J}\,\delta B^{\,\phi} \simeq 0. \tag{88}$$

The neglected terms in the previous three equations are $(n/m)^2\,(r/R)^2$ smaller than the dominant terms. The previous three expressions are consistent with (63), (64), and (65) provided

$$\delta A_\phi(r,\theta,\phi) \simeq R_0\sum_j\frac{\psi_j(r)}{m_j}\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}. \tag{89}$$

According to the Biot-Savart law:

$$\delta A_{\,\phi}(\mathbf{x}) = \frac{1}{4\pi}\int\frac{R\,R'\,\mu_0\,\delta\mathbf{J}(\mathbf{x}')\cdot\nabla\phi}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'. \tag{90}$$

Assume that

$$\delta A_\phi(R,\phi,Z) = \delta A_\phi(R,0,Z)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi}. \tag{91}$$

Hence, can evaluate integral at $\phi = 0$ without loss of generality. Follows that

$$\mathbf{x} = (R,\,0,\,Z), \tag{92}$$

$$\mathbf{x}' = (R'\,\cos\phi',\ R'\,\sin\phi',\ Z'), \tag{93}$$

and

$$|\mathbf{x}-\mathbf{x}'| = \left[R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}-2\,R\,R'\,\cos\phi'\right]^{\,1/2}. \tag{94}$$

Now,

$$\delta\mathbf{J}(R',\phi',Z')\cdot\nabla\phi = \delta J^{\,\phi}(R',0,Z')\,{\rm e}^{-{\rm i}\,n\,\phi'}\,\cos\phi', \tag{95}$$

so

$$\delta A_\phi(r,\theta,0) = \frac{1}{4\pi}\int_0^\infty\oint\oint\frac{R\,R'\,\mu_0\,\delta J^{\,\phi}(r',\theta',0)\,{\rm e}^{-{\rm i}\,n\,\phi'}\,\cos\phi'\,\mathcal{J}'\,dr'\,d\theta'\,d\phi'}{[R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}-2\,R\,R'\,\cos\phi']^{\,1/2}}, \tag{96}$$

which can be written

$$\delta A_\phi(r,\theta,0) = \frac{1}{4\pi}\int_0^\infty\oint R\,R'\,\mu_0\,\delta J^{\,\phi}(r',\theta',0)\,G(r,\theta;r',\theta')\,\mathcal{J}'\,dr'\,d\theta', \tag{97}$$

where

$$G(r,\theta;r',\theta') = \oint\frac{\cos\phi'\,\cos(n\,\phi')\,d\phi'}{[R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}-2\,R\,R'\,\cos\phi']^{\,1/2}}, \tag{98}$$

or

$$G(r,\theta;r',\theta') = \frac{1}{2}\oint\frac{(\cos[(n-1)\,\phi']+\cos[(n+1)\,\phi'])\;d\phi'}{[R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}-2\,R\,R'\,\cos\phi']^{\,1/2}}. \tag{99}$$

Now (GR 8.85.1),

$$\begin{aligned}P^{\,n}_{-1/2}(\cosh\eta) &= \frac{(-1)^{\,n}}{2\pi}\,\frac{\Gamma(1/2)}{\Gamma(1/2-n)}\oint\frac{\cos(n\,\varphi)\,d\varphi}{(\cosh\eta+\sinh\eta\,\cos\varphi)^{\,1/2}}\\ &= \frac{1}{2\pi}\,\frac{\Gamma(1/2)}{\Gamma(1/2-n)}\oint\frac{\cos(n\,\varphi)\,d\varphi}{(\cosh\eta-\sinh\eta\,\cos\varphi)^{\,1/2}}\\ &= \frac{(-1)^{\,n}\,\Gamma(1/2)\,\Gamma(1/2+n)}{2\pi^{\,2}}\oint\frac{\cos(n\,\varphi)\,d\varphi}{(\cosh\eta-\sinh\eta\,\cos\varphi)^{\,1/2}}.\end{aligned} \tag{100}$$

It follows that

$$\alpha\,\cosh\eta = R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}, \tag{101}$$

$$\alpha\,\sinh\eta = 2\,R\,R'. \tag{102}$$

Hence,

$$\eta = \tanh^{-1}\left[\frac{2\,R\,R'}{R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}}\right], \tag{103}$$

and

$$\alpha = \frac{R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}}{\cosh\eta}. \tag{104}$$

It follows that

$$G(r,\theta;r',\theta') = \pi\left[\frac{\cosh\eta}{R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}}\right]^{1/2}$$

$$\times\left[\frac{\Gamma(3/2-n)}{\Gamma(1/2)}\,P_{-1/2}^{\,n-1}(\cosh\eta)+\frac{\Gamma(-1/2-n)}{\Gamma(1/2)}\,P_{-1/2}^{\,n+1}(\cosh\eta)\right]. \tag{105}$$

However,

$$\Gamma(3/2-n)=\frac{(-1)^{\,n+1}\,\pi\,(n-1/2)}{\Gamma(n+1/2)}, \tag{106}$$

$$\Gamma(-1/2-n)=\frac{(-1)^{\,n+1}\,\pi}{\Gamma(n+1/2)\,(n+1/2)}, \tag{107}$$

so

$$G(r,\theta;r',\theta')=\frac{(-1)^{\,n+1}\,\pi^{\,2}}{\Gamma(1/2)\,\Gamma(n+1/2)}\left[\frac{\cosh\eta}{R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}}\right]^{1/2}$$
$$\times\left[(n-1/2)\,P_{-1/2}^{\,n-1}(\cosh\eta)+\frac{P_{-1/2}^{\,n+1}(\cosh\eta)}{n+1/2}\right]. \tag{108}$$

Now, from (74) and (80),

$$\mathcal{J}\,\mu_0\,\delta J^{\,\phi}(r,\theta,0)=-\sum_k\varDelta\varPsi_k\,\delta(r-r_k)\,{\rm e}^{\,{\rm i}\,m_k\,\theta}. \tag{109}$$

Furthermore, from (79) and (89),

$$\varPsi_k=\frac{1}{R_0}\oint\delta A_\phi(r_k,\theta,0)\,{\rm e}^{-{\rm i}\,m_k\,\theta}\,\frac{d\theta}{2\pi}. \tag{110}$$

Hence,

$$\varPsi_k=\sum_{k'}F_{kk'}\,\varDelta\varPsi_{k'}, \tag{111}$$

where

$$F_{kk'}=\oint\oint\mathcal{G}(r_k,\theta;r_{k'},\theta')\,{\rm e}^{-{\rm i}\,(m_k\,\theta-m_{k'}\,\theta')}\,\frac{d\theta}{2\pi}\,\frac{d\theta'}{2\pi}, \tag{112}$$

and

$$\mathcal{G}(r,\theta;r',\theta')=\frac{(-1)^{\,n}\,\pi^{\,2}\,R\,R'/R_0}{2\,\Gamma(1/2)\,\Gamma(n+1/2)}\left[\frac{\cosh\eta}{R^{\,2}+R'^{\,2}+(Z-Z')^{\,2}}\right]^{1/2}$$
$$\times\left[(n-1/2)\,P_{-1/2}^{\,n-1}(\cosh\eta)+\frac{P_{-1/2}^{\,n+1}(\cosh\eta)}{n+1/2}\right]. \tag{113}$$

Note that

$$\mathcal{G}(r',\theta';r,\theta)=\mathcal{G}(r,\theta;r',\theta'), \tag{114}$$

which implies that

$$F_{k'k}=F_{kk'}^{\,\ast}. \tag{115}$$

8 Inomogeneous Solution

Suppose that there are currents flowing in a number of external poloidal field coils. Let I_l, R_l, and Z_l be the peak current, and coordinates of the lth field coil. The currents are assumed to modulate like $\mathrm{e}^{-\mathrm{i}\,n\,\phi}$. It follows that

$$\delta J^{\phi}_{\rm ext}(R,0,Z) = \sum_j \frac{I_l}{R_l}\,\delta(R-R_l)\,\delta(Z-Z_l). \tag{116}$$

Hence,

$$\Psi_k = \sum_{k'} F_{kk'}\,\Delta\Psi_k - \sum_l g_{kl}\,\mu_0\,I_l, \tag{117}$$

where

$$g_{kj} = \frac{1}{2\pi}\oint \mathcal{G}(r_k,\theta;R_j,Z_j)\,\mathrm{e}^{-\mathrm{i}\,m_k\,\theta}\,\frac{d\theta}{2\pi}. \tag{118}$$

It follows that

$$\Delta\Psi_k = \sum_{k'} E_{kk'}\,\Psi_{k'} + |E_{kk}|\,\chi_k \tag{119}$$

where $E_{kk'}$ is the inverse of the $F_{kk'}$ matrix, and

$$\chi_k = \frac{1}{|E_{kk}|}\sum_j h_{kl}\,\mu_0\,I_l, \tag{120}$$

$$h_{kl} = \sum_{k'} E_{kk'}\,g_{k'l}. \tag{121}$$

9 Island Width

We have

$$\mathcal{J}\,\delta B^{\,r} \simeq \frac{\partial\,\delta A_\phi}{\partial\theta}, \tag{122}$$

$$\mathcal{J}\,\delta B^{\,\theta} \simeq -\frac{\partial\,\delta A_\phi}{\partial r}, \tag{123}$$

$$\mathcal{J}\,\delta B^{\,\phi} \simeq 0. \tag{124}$$

It follows that

$$\delta\mathbf{B}\cdot\nabla\delta A_\phi = \delta B^{\,r}\,\frac{\partial\,\delta A_\phi}{\partial r} + \delta B^{\,\theta}\,\frac{\partial\,\delta A_\phi}{\partial\theta} + \delta B^{\,\phi}\,\frac{\partial\,\delta A_\phi}{\partial\phi} \simeq 0. \tag{125}$$

Suppose that

$$\delta A_\phi(r,\theta,\phi) \simeq \delta\hat{A}_\phi\,\mathrm{e}^{\,\mathrm{i}\,(m_k\,\theta-n\,\phi)} \tag{126}$$

in the vicinity of the kth rational surface. It follows that

$$\mathbf{B}\cdot\nabla\delta A_\phi = \mathrm{i}\,B_0\,R_0^2\,\frac{f}{r\,R^2}\,(m_k-n\,q)\,\delta\hat{A}_\phi\,\mathrm{e}^{\,\mathrm{i}\,(m_k\,\theta-n\,\phi)}. \tag{127}$$

Let us search for a function,

$$F(r,\theta,\phi) = F_0(r)+\delta A_\phi, \tag{128}$$

which is such that

$$(\mathbf{B}+\delta\mathbf{B})\cdot\nabla F = 0. \tag{129}$$

It follows that

$$\delta B^{\,r}\,\frac{dF_0}{dr}+\mathrm{i}\,B_0\,R_0^2\,\frac{f}{r\,R^2}\,(m_k-n\,q)\,\delta\hat{A}_\phi\,\mathrm{e}^{\,\mathrm{i}\,(m_k\,\theta-n\,\phi)} = 0, \tag{130}$$

or

$$\mathrm{i}\,m_k\,\frac{R_0}{r\,R^2}\,\delta\hat{A}_\phi\,\frac{dF_0}{dr}+\mathrm{i}\,B_0\,R_0^2\,\frac{f}{r\,R^2}\,(m_k-n\,q)\,\delta\hat{A}_\phi = 0, \tag{131}$$

which implies that

$$\frac{dF_0}{dr} = -\frac{B_0\,R_0\,f}{m_k}\,(m_k-n\,q), \tag{132}$$

or

$$F_0(r)\simeq\frac{B_0}{2}\left(\frac{g\,s}{q}\right)_{r_k}(r-r_k)^{\,2}. \tag{133}$$

Hence,

$$F(r,\theta,\phi) = \frac{B_0}{2}\left(\frac{g\,s}{q}\right)_{r_k}(r-r_k)^{\,2}+R_0\,\Psi_k\,\cos(m_k\,\theta-n\,\phi) \tag{134}$$

is a flux surface function in the island region. Thus,

$$\frac{F}{R_0\,|\Psi_k|} = 2\,X^{\,2}+\cos(m_k\,\theta-n\,\phi), \tag{135}$$

where

$$X = \frac{2\,(r-r_k)}{W_k}, \tag{136}$$

and

$$\frac{W_k}{4\,R_0} = \left[\left(\frac{q}{g\,s}\right)_{r_k}\frac{|\Psi_k|}{B_0\,R_0}\right]^{1/2}. \tag{137}$$

It follows that W_k is the full radial island width (which is constant in r).

10 Flux Coordinate System

Let all lengths be normalized to R_0, and all magnetic field-strengths to B_0. We have

$$\mathbf{B} = \nabla\phi \times \nabla\psi_p + g(\psi_p)\,\nabla\phi, \tag{138}$$

and

$$\nabla\psi_p \times \nabla\theta \cdot \nabla\phi = \frac{g}{q\,R^2}, \tag{139}$$

where $q = q(\psi_p)$.

Let $\Psi = \psi_p/\psi_c = 1 - \Psi_N$, where ψ_c is the value of ψ_p on the magnetic axis. (It is assumed that $\psi_p = 0$ on the plasma boundary.) The previous equation implies that

$$\frac{d\theta}{dl} = \frac{g}{q}\,\frac{1}{|\psi_c|\,R\,\sqrt{\Psi_R^2 + \Psi_Z^2}}, \tag{140}$$

where dl is an element of poloidal path length on a magnetic flux-surface, and $\Psi_R \equiv \partial\Psi/\partial R$, etc. Furthermore,

$$dR = -\frac{\Psi_Z\,dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}, \tag{141}$$

$$dZ = \frac{\Psi_R\,dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}. \tag{142}$$

It follows that

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi\,|\psi_c|}\oint \frac{dl}{R\,\sqrt{\Psi_R^2 + \Psi_Z^2}}. \tag{143}$$

If we define

$$\tan\zeta = \frac{Z - Z_{\rm axis}}{R_{\rm axis} - R} \tag{144}$$

then

$$\frac{dR}{d\zeta} = -\Psi_Z\,F, \tag{145}$$

$$\frac{dZ}{d\zeta} = \Psi_R\,F, \tag{146}$$

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi\,|\psi_c|}\oint \frac{F}{R}\,d\zeta, \tag{147}$$

where

$$F = \frac{(R_{\rm axis} - R)^2 + (Z - Z_{\rm axis})^2}{-(Z - Z_{\rm axis})\,\Psi_Z + (R_{\rm axis} - R)\,\Psi_R}. \tag{148}$$

Note that

$$\frac{d\theta}{d\zeta} = \frac{g}{q\,|\psi_c|}\,\frac{F}{R}. \tag{149}$$

Hence, [from (193)]

$$\langle A\rangle = \oint R^{\,2}\,A\,d\theta\bigg/\oint R^{\,2}\,d\theta = \oint F\,R\,A\,d\zeta\bigg/\oint F\,R\,d\zeta. \tag{150}$$

In particular, because

$$B^{\,2} = \frac{\psi_c^{\,2}\,|\nabla\Psi|^{\,2}}{R^2}+\frac{g^{\,2}}{R^2} \tag{151}$$

it follows that

$$\langle B^{\,2}\rangle = \oint \frac{F}{R}\left[\psi_c^{\,2}\left(\Psi_R^2+\Psi_Z^2\right)+g^{\,2}\right]d\zeta\bigg/\oint F\,R\,d\zeta. \tag{152}$$

It is helpful to define the length-like flux-surface coordinate r, according to

$$\nabla r\times\nabla\theta\cdot\nabla\phi = \frac{1}{r\,R^{\,2}}. \tag{153}$$

It follows that

$$r(\Psi) = \left[2\,|\psi_c|\int_{\Psi}^{1}\frac{q(\Psi')}{g(\Psi')}\,d\Psi'\right]^{1/2}. \tag{154}$$

We can calculate $R(r,\theta)$ and $Z(r,\theta)$ from

$$\frac{dR}{d\theta} = -|\psi_c|\,\frac{q}{g}\,R\,\Psi_Z, \tag{155}$$

$$\frac{dZ}{d\theta} = |\psi_c|\,\frac{q}{g}\,R\,\Psi_R. \tag{156}$$

Now,

$$r\,\frac{dr}{d\Psi} = -|\psi_c|\,\frac{q(r)}{g(r)}. \tag{157}$$

So

$$\nabla r = \frac{dr}{d\Psi}\,\nabla\Psi = -|\psi_c|\,\frac{q(r)}{r\,g(r)}\,\nabla\Psi. \tag{158}$$

Hence,

$$a_{jj} = \left(\oint\frac{1}{|\nabla r|^{\,2}}\,\frac{d\theta}{2\pi}\right)_{r_j} = \left(\frac{r\,g}{|\psi_c|\,q}\right)^2_{r_j}\oint\frac{1}{\Psi_R^2+\Psi_Z^2}\,\frac{d\theta}{2\pi}. \tag{159}$$

Note that

$$\frac{d\Psi_N}{dr} = \frac{r\,g(r)}{|\psi_c|\,q(r)}. \tag{160}$$

Hence, if $\overline{W}_k$ is the full magnetic island width in Ψ_N at the kth rational surface then

$$\overline{W}_k = \frac{W_k}{R_0}\,\frac{d\Psi_N}{dr}. \tag{161}$$

11 GPEC Coupling

We have

$$\frac{d\psi_p}{dr} = B_0\,R_0\,f(r), \tag{162}$$

$$\mathbf{B}\cdot\nabla\phi = \frac{B_0\,R_0}{R^2}\,g(r), \tag{163}$$

$$\delta B^{\,r} = \frac{\mathrm{i}}{r}\left(\frac{R_0}{R}\right)^2 \sum_j \psi_j\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}. \tag{164}$$

According to PRL **99**, 195003 (2007),

$$\Delta_j\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)} = \left[\frac{\partial}{\partial\psi_p}\,\frac{\delta\mathbf{B}\cdot\nabla\psi_p}{\mathbf{B}\cdot\nabla\phi}\right]_{r_j} = \left[\frac{\partial}{\partial r}\,\frac{\delta\mathbf{B}\cdot\nabla r}{\mathbf{B}\cdot\nabla\phi}\right]_{r_j} = \left[\frac{\partial}{\partial r}\,\frac{\delta B^{\,r}}{\mathbf{B}\cdot\nabla\phi}\right]_{r_j}. \tag{165}$$

It follows that

$$\Delta_j = \frac{\mathrm{i}}{r_j}\left(\frac{R_0}{R}\right)^2 \frac{R^2}{B_0\,R_0\,g_j}\left[\frac{d\psi_j}{dr}\right]_{r_j}, \tag{166}$$

But,

$$\left[r\,\frac{d\psi_j}{dr}\right]_{r_j} = m_j\,a_{jj}\,[\chi_j]_{r_j} = m_j\,a_{jj}\,\Delta\Psi_j, \tag{167}$$

and

$$\chi_j = \frac{\Delta\Psi_j}{|E_{jj}|}, \tag{168}$$

which implies that

$$\Delta_j = \mathrm{i}\left(\frac{R_0}{r_j}\right)^2 \frac{m_j\,a_{jj}}{g_j}\,\frac{\Delta\Psi_j}{B_0\,R_0}, \tag{169}$$

$$\frac{\chi_j}{R_0\,B_0} = -\mathrm{i}\left(\frac{r_j}{R_0}\right)^2 \frac{g_j}{m_j\,a_{jj}}\,\frac{\Delta_j}{|E_{jj}|}. \tag{170}$$

Now, $d\psi_p/dr = B_0\,R_0\,f$. It follows that

$$(\nabla\psi_p\times\nabla\theta\cdot\nabla\phi)^{-1} = \frac{R^2\,q}{B_0\,R_0\,g}. \tag{171}$$

Hence,

$$\mathbf{B} = \nabla\phi\times\nabla\psi_p + B_0\,R_0\,g\,\nabla\phi = \nabla\phi\times\nabla\psi_p + q\,\nabla\psi_p\times\nabla\theta. \tag{172}$$

Let $\Psi_p = 2\pi\,\psi_p$ and $d\Psi_p/d\Psi_t = 1/q$. Hence,

$$\frac{d\Psi_p}{dr} = 2\pi\,B_0\,R_0\,f, \tag{173}$$

$$\frac{d\Psi_t}{dr} = 2\pi\, B_0\, r\, g, \tag{174}$$

and [cf. PoP **13**, 102501 (2006), Eq. (41)]

$$2\pi\, \mathbf{B} = q^{-1}\, \nabla\phi\times\nabla\Psi_t + \nabla\Psi_t\times\nabla\theta. \tag{175}$$

We have

$$\delta J^{\,r} = 0, \tag{176}$$

$$\mathcal{J}\,\mu_0\,\delta J^{\,\theta} = -\sum_j \frac{\Delta\Psi_j}{q_j}\,{\rm e}^{\,{\rm i}\,(m_j\,\theta-n\,\phi)}\,\delta(r-r_j), \tag{177}$$

$$\mathcal{J}\,\mu_0\,\delta J^{\,\phi} = -\sum_j \Delta\Psi_k\,{\rm e}^{\,{\rm i}\,(m_j\,\theta-n\,\phi)}\,\delta(r-r_j), \tag{178}$$

which implies that

$$\mu_0\,\delta\mathbf{J} = -2\pi\sum_j \Delta\Psi_j,{\rm e}^{\,{\rm i}\,(m_j\,\theta-n\,\phi)}\,\delta(\psi_t-\psi_{t\,j})\,\mathbf{B}. \tag{179}$$

However, according to PoP **13**, 102501 (2006),

$$\Delta\Psi_j = -{\rm i}\,\frac{\mu_0\,J_c\,\Delta_j}{2\pi\,m_j}, \tag{180}$$

where

$$\frac{1}{\mu_0\,J_c} = \left(\oint\frac{B^{\,2}}{|\nabla\psi_t|^{\,2}}\,\frac{d\theta\,d\phi}{2\pi\,\mathbf{B}\cdot\nabla\phi}\right)_{r_j}. \tag{181}$$

It is easily demonstrated that

$$\frac{1}{\mu_0\,J_c} = \left(\frac{R_0}{r_j}\right)^2\frac{1}{2\pi\,B_0\,R_0\,g_j}\left[a_{jj}+\left(\frac{r_k}{R_0\,q_j}\right)^2\right]. \tag{182}$$

Hence,

$$\frac{\Delta\Psi_j}{B_0\,R_0} = -{\rm i}\,\Delta_j\left(\frac{r_j}{R_0}\right)^2\frac{g_j}{m_j\,[a_{jj}+(r_k/R_0\,q_j)^2]}. \tag{183}$$

12 Neoclassical Coordinate System

It is also helpful to define the geometric poloidal angle

$$\mathbf{b}\cdot\nabla\Theta = \gamma(r). \tag{184}$$

It follows that

$$\frac{d\Theta}{dl} = \frac{\gamma\,B\,R}{|\psi_c|\,\sqrt{\Psi_R^{\,2}+\Psi_Z^{\,2}}}, \tag{185}$$

where

$$B\,R = \left[g^{\,2} + |\psi_c|^{\,2}\,(\Psi_R^{\,2}+\Psi_Z^{\,2})\right]^{1/2}. \tag{186}$$

Hence,

$$\frac{1}{\gamma(r)} = \frac{1}{2\pi\,|\psi_c|}\oint\frac{B\,R\,dl}{\sqrt{\Psi_R^{\,2}+\Psi_Z^{\,2}}} = \frac{1}{2\pi\,|\psi_c|}\oint B\,R\,F\,d\zeta. \tag{187}$$

We can calculate $R(r,\Theta)$ and $Z(r,\Theta)$ from

$$\frac{dR}{d\Theta} = -|\psi_c|\,\frac{\Psi_Z}{\gamma\,B\,R}, \tag{188}$$

$$\frac{dZ}{d\Theta} = |\psi_c|\,\frac{\Psi_R}{\gamma\,B\,R}. \tag{189}$$

Note that

$$\frac{d\Theta}{d\theta} = \left(\frac{\gamma\,q}{g}\right)B\,R^{\,2}. \tag{190}$$

Thus,

$$\frac{1}{\gamma} = \frac{q}{g}\oint B\,R^{\,2}\,\frac{d\theta}{2\pi}. \tag{191}$$

13 Neoclassical Parameters

The flux-surface average operator has the following properties:

$$\langle 1\rangle = 1, \tag{192}$$

$$\langle \mathbf{B}\cdot\nabla A\rangle = 0. \tag{193}$$

It follows that

$$\langle A\rangle = \oint R^{\,2}\,A\,\frac{d\theta}{2\pi}\bigg/\oint R^{\,2}\,\frac{d\theta}{2\pi} = \oint\frac{A}{B}\,\frac{d\Theta}{2\pi}\bigg/\oint\frac{1}{B}\,\frac{d\Theta}{2\pi}. \tag{194}$$

Let

$$I_0 = \oint\frac{1}{B\,R^{\,2}}\,\frac{d\Theta}{2\pi} = \frac{\gamma\,q}{g}, \tag{195}$$

$$I_1 = \oint\frac{1}{B}\,\frac{d\Theta}{2\pi}, \tag{196}$$

$$I_2 = \oint B\,\frac{d\Theta}{2\pi}, \tag{197}$$

$$I_3 = \oint\left(\frac{\partial B}{\partial\Theta}\right)^{2}\frac{1}{B}\,\frac{d\Theta}{2\pi}, \tag{198}$$

$$I_{4,k} = \sqrt{\frac{2}{k}}\oint\frac{\sin(k\,\Theta)}{B^{\,2}}\,\frac{\partial B}{\partial\Theta}\,\frac{d\Theta}{2\pi} = \oint\frac{\sqrt{2\,k}\,\cos(k\,\Theta)}{B}\,\frac{d\Theta}{2\pi}, \tag{199}$$

$$I_{5,k} = \sqrt{\frac{2}{k}} \oint \frac{\sin(k\,\Theta)}{B^{\,3}}\,\frac{\partial B}{\partial\Theta}\,\frac{d\Theta}{2\pi} = \oint \frac{\sqrt{2\,k}\,\cos(k\,\Theta)}{2\,B^{\,2}}\,\frac{d\Theta}{2\pi}, \tag{200}$$

$$I_6(\lambda) = \oint \frac{\sqrt{1-\lambda\,B/B_{\rm max}}}{B}\,\frac{d\Theta}{2\pi}. \tag{201}$$

It follows that

$$\langle B^{\,2}\rangle = \frac{I_2}{I_1}, \tag{202}$$

$$\langle(\mathbf{b}\cdot\nabla B)^{\,2}\rangle = \gamma^{\,2}\,\frac{I_3}{I_1}, \tag{203}$$

$$|\langle\mathbf{B}\cdot\nabla\theta\rangle| = \frac{g}{|q|}\,\frac{I_0}{I_1} = \frac{|\gamma|}{I_1}, \tag{204}$$

$$\langle\sin(k\,\Theta)\,(\mathbf{b}\cdot\nabla\ln B)\rangle = \gamma\,\frac{I_{4,k}}{I_1}, \tag{205}$$

$$\left\langle\sin(k\,\Theta)\,\frac{(\mathbf{b}\cdot\nabla\ln B)}{B}\right\rangle = \gamma\,\frac{I_{5,k}}{I_1}. \tag{206}$$

Hence,

$$L_c = \frac{1}{|\gamma|}\,\frac{I_2^2}{I_1^2\,I_3}\sum_{k>0}\frac{2}{k}\,I_{5,k}\,I_{6,k}, \tag{207}$$

$$\omega_{t\,a} \equiv \frac{v_{T\,a}}{L_c} = K_t\,|\gamma|\,v_{T\,a}, \tag{208}$$

$$\nu_{*\,a} \equiv \frac{8}{3\pi}\,\frac{\langle B^{\,2}\rangle}{\langle(\mathbf{b}\cdot\nabla B)^{\,2}\rangle}\,\frac{g_t\,\omega_{t\,a}}{v_{T\,a}^2\,\tau_{aa}} = K_*\,\frac{g_t}{\omega_{t\,a}\,\tau_{aa}}, \tag{209}$$

$$f_c = \frac{3}{4}\frac{I_2}{B_{\rm max}^2}\int_0^1\frac{\lambda\,d\lambda}{I_6(\lambda)}, \tag{210}$$

where

$$K_t = \frac{I_1^{\,2}\,I_3}{I_2^2\,\sum_{k>0} I_{4,k}\,I_{5,k}}, \tag{211}$$

$$K_* = \frac{8}{3\pi}\,\frac{I_2}{I_3}\,K_t^{\,2}. \tag{212}$$