Program PHASE

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1 Plasma Angular Velocity Evolution

1.1 Plasma Angular Equations of Motion

We can write

$$\Omega_\theta(r,t) = \Omega_{\theta\,0}(r) + \Delta\Omega_\theta(r,t), \tag{1}$$

$$\Omega_\phi(r,t) = \Omega_{\phi\,0}(r) + \Delta\Omega_\phi(r,t), \tag{2}$$

where $\Omega_\theta(r,t)$ and $\Omega_\phi(r,t)$ are the poloidal and toroidal plasma angular velocity profiles, respectively, whereas $\Omega_{\theta\,0}(r)$ and $\Omega_{\phi\,0}(r)$ are the corresponding unperturbed profiles, and $\Delta\Omega_\theta(r,t)$ and $\Delta\Omega_\phi(r,t)$ are the respective modifications to the profiles induced by the electromagnetic torques exerted at the various resonant surfaces within the plasma. The modifications to the angular velocity profiles are governed by the poloidal and toroidal angular equations of motion of the plasma, which take the respective forms

$$4\pi^{\,2}\,R_0\left[(1+2\,q^{\,2})\,\rho\,r^{\,3}\,\frac{\partial\Delta\Omega_\theta}{\partial t} - \frac{\partial}{\partial r}\!\left(\mu\,r^{\,3}\,\frac{\partial\Delta\Omega_\theta}{\partial r}\right) + \frac{\rho}{\tau_\theta}\,r^{\,3}\,\Delta\Omega_\theta\right] = \sum_{k=1,K}\delta T_{\theta\,k}\,\delta(r-r_k), \tag{3}$$

$$4\pi^{\,2}\,R_0^3\left[\rho\,r\,\frac{\partial\Delta\Omega_\phi}{\partial t} - \frac{\partial}{\partial r}\!\left(\mu\,r\,\frac{\partial\Delta\Omega_\phi}{\partial r}\right) + \frac{\rho}{\tau_\phi}\,r\,\Delta\Omega_\phi\right] = \sum_{k=1,K}\delta T_{\phi\,k}\,\delta(r-r_k), \tag{4}$$

and are subject to the spatial boundary conditions

$$\frac{\partial\Delta\Omega_\theta(0,t)}{\partial r} = \frac{\partial\Delta\Omega_\phi(0,t)}{\partial r} = 0, \tag{5}$$

$$\Delta\Omega_\theta(a,t) = \Delta\Omega_\phi(a,t) = 0. \tag{6}$$

Here, $\mu(r) =$ is the anomalous plasma perpendicular ion viscosity (due to plasma turbulence), whereas $\rho(r)$ is the plasma mass density profile. Furthermore, $1/\tau_\theta(r)$ is the neoclassical poloidal flow-damping rate. Finally, $1/\tau_\phi(r)$ is the neoclassical toroidal flow-damping rate (which is, presently, set to zero in the code).

According to standard neoclassical theory,

$$\frac{1}{\tau_\theta(r)} = \left(\frac{q\,R_0}{r}\right)^2 \frac{\mu_{00}^i}{\tau_{ii}}. \tag{7}$$

1.2 Simplified Plasma Angular Equations of Motion

It is convenient to write

$$\Delta\Omega_\theta(r,t) = \sum_{k=1,K} \Delta\Omega_{\theta\,k}(r,t), \tag{8}$$

$$\Delta\Omega_\phi(r,t) = \sum_{k=1,K} \Delta\Omega_{\phi\,k}(r,t), \tag{9}$$

where

$$4\pi^2 R_0 \left[(1+2\,q^2)\,\rho\,r^3\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial t} - \frac{\partial}{\partial r}\left(\mu\,r^3\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial r}\right) + \frac{\rho}{\tau_\theta}\,r^3\,\Delta\Omega_{\theta\,k}\right] = \delta T_{\theta\,k}\,\delta(r-r_k), \tag{10}$$

$$4\pi^2 R_0^3 \left[\rho\,r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial t} - \frac{\partial}{\partial r}\left(\mu\,r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial r}\right) + \frac{\rho}{\tau_\phi}\,r\,\Delta\Omega_{\phi\,k}\right] = \delta T_{\phi\,k}\,\delta(r-r_k), \tag{11}$$

and

$$\frac{\partial\Delta\Omega_{\theta\,k}(0,t)}{\partial r} = \frac{\partial\Delta\Omega_{\phi\,k}(0,t)}{\partial r} = 0, \tag{12}$$

$$\Delta\Omega_{\theta\,k}(a,t) = \Delta\Omega_{\phi\,k}(a,t) = 0. \tag{13}$$

In the presence of poloidal and toroidal flow damping, the modified angular velocity profiles, $\Delta\Omega_{\theta\,k}$ and $\Delta\Omega_{\phi\,k}$, are radially localized in the vicinity of the kth resonant surface. Hence, it is a good approximation to write Eqs. (10) and (11) in the simplified forms

$$4\pi^2 R_0 \left[(1+2\,q_k^2)\,\rho_k\,r^3\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial t} - \mu_k\,\frac{\partial}{\partial r}\left(r^3\,\frac{\partial\Delta\Omega_{\theta\,k}}{\partial r}\right) + \frac{\rho_k}{\tau_{\theta\,k}}\,r^3\,\Delta\Omega_{\theta\,k}\right] = \delta T_{\theta\,k}\,\delta(r-r_k), \tag{14}$$

$$4\pi^2 R_0^3 \left[\rho_k\,r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial t} - \mu_k\frac{\partial}{\partial r}\left(r\,\frac{\partial\Delta\Omega_{\phi\,k}}{\partial r}\right) + \frac{\rho_k}{\tau_{\phi\,k}}\,r\,\Delta\Omega_{\phi\,k}\right] = \delta T_{\phi\,k}\,\delta(r-r_k), \tag{15}$$

where $q_k = q(r_k)$, $\rho_k = \rho(r_k)$, $\mu_k = \mu(r_k)$, $\tau_{\theta\,k} = \tau_\theta(r_k)$, and $\tau_{\phi\,k} = \tau_\phi(r_k)$.

1.3 Normalized Plasma Equations of Angular Motion

Let

$$\tau_A = \left(\frac{\mu_0\,\rho_0\,a^{\,2}}{B_0^{\,2}}\right)^{1/2}, \tag{16}$$

where $\rho_0 = \rho(0)$, and

$$\rho(r) \simeq m_i\,[n_i(r) + n_b(r)] + m_I\,n_I(r) \tag{17}$$

is the plasma mass density. Furthermore, let $\hat{r} = r/a$, $\hat{r}_k = r_k/a$, $\hat{a} = a/R_0$, $\hat{t} = t/\tau_A$, $\hat{\rho}_k = \rho_k/\rho_0$, $\tau_{M\,k} = \rho_k\,a^{\,2}/\mu_k = a^2/\chi_\phi(r_k)$, $\hat{\tau}_{M\,k} = \tau_{M\,k}/\tau_A$, $\hat{\tau}_{\theta\,k} = \tau_{\theta\,k}/\tau_A$, $\hat{\tau}_{\phi\,k} = \tau_{\phi\,k}/\tau_A$, $\hat{\varOmega}_\theta = \tau_A\,\varOmega_\theta$, $\hat{\varOmega}_{\theta\,0} = \tau_A\,\varOmega_{\theta\,0}$, $\varDelta\hat{\varOmega}_{\theta\,k} = \tau_A\,\varDelta\varOmega_{\theta\,k}$, $\hat{\varOmega}_\phi = \tau_A\,\varOmega_\phi$, $\hat{\varOmega}_{\phi\,0} = \tau_A\,\varOmega_{\phi\,0}$, and $\varDelta\hat{\varOmega}_{\phi\,k} = \tau_A\,\varDelta\varOmega_{\phi\,k}$. It follows that

$$\hat{\varOmega}_\theta(\hat{r},\hat{t}) = \hat{\varOmega}_{\theta\,0}(\hat{r}) + \sum_{k=1,K} \varDelta\hat{\varOmega}_{\theta\,k}(\hat{r},\hat{t}), \tag{18}$$

$$\hat{\varOmega}_\phi(\hat{r},\hat{t}) = \hat{\varOmega}_{\phi\,0}(\hat{r}) + \sum_{k=1,K} \varDelta\hat{\varOmega}_{\phi\,k}(\hat{r},\hat{t}), \tag{19}$$

where

$$\begin{aligned}(1+2\,q_k^2)\,\hat{r}^{\,3}\,\frac{\partial\varDelta\hat{\varOmega}_{\theta\,k}}{\partial\hat{t}} - \frac{1}{\hat{\tau}_{M\,k}}\,\frac{\partial}{\partial\hat{r}}\left(\hat{r}^{\,3}\,\frac{\partial\varDelta\hat{\varOmega}_{\theta\,k}}{\partial\hat{r}}\right) + \frac{1}{\hat{\tau}_{\theta\,k}}\,\hat{r}^{\,3}\,\varDelta\hat{\varOmega}_{\theta\,k}&\\ = -\frac{m_k}{2\,\hat{\rho}_k\,\hat{a}^{\,2}}\,\delta\hat{T}_k\,\delta(\hat{r}-\hat{r}_k),&\end{aligned} \tag{20}$$

$$\hat{r}\,\frac{\partial\varDelta\hat{\varOmega}_{\phi\,k}}{\partial\hat{t}} - \frac{1}{\hat{\tau}_{M\,k}}\,\frac{\partial}{\partial\hat{r}}\left(\hat{r}\,\frac{\partial\varDelta\hat{\varOmega}_{\phi\,k}}{\partial\hat{r}}\right) + \frac{1}{\hat{\tau}_{\phi\,k}}\,\hat{r}\,\varDelta\hat{\varOmega}_{\phi\,k} = \frac{n}{2\,\hat{\rho}_k}\,\delta\hat{T}_k\,\delta(\hat{r}-\hat{r}_k), \tag{21}$$

and

$$\frac{\partial\varDelta\hat{\varOmega}_{\theta\,k}(0,\hat{t})}{\partial\hat{r}} = \frac{\partial\varDelta\hat{\varOmega}_{\phi\,k}(0,\hat{t})}{\partial\hat{r}} = 0, \tag{22}$$

$$\varDelta\hat{\varOmega}_{\theta\,k}(1,\hat{t}) = \varDelta\hat{\varOmega}_{\phi\,k}(1,\hat{t}) = 0. \tag{23}$$

1.4 Solution of Plasma Angular Equations of Motion

Let

$$\varDelta\hat{\varOmega}_{\theta\,k}(\hat{r},\hat{t}) = -\frac{1}{m_k}\sum_{p=1,P}\alpha_{k,p}(\hat{t})\,\frac{y_p(\hat{r})}{y_p(\hat{r}_k)}, \tag{24}$$

$$\varDelta\hat{\varOmega}_{\phi\,k}(\hat{r},\hat{t}) = \frac{1}{n}\sum_{p=1,P}\beta_{k,p}(\hat{t})\,\frac{z_p(\hat{r})}{z_p(\hat{r}_k)}, \tag{25}$$

where

$$y_p(\hat{r}) = \frac{J_1(j_{1,p}\,\hat{r})}{\hat{r}}, \tag{26}$$

$$z_p(\hat{r}) = J_0(j_{0,p}\,\hat{r}), \tag{27}$$

and $P \gg 1$. Here, $J_m(z)$ is a standard Bessel function, and $j_{m,p}$ denotes the pth zero of the $J_m(z)$ Bessel function. Note that Eqs. (24) and (25) automatically satisfy the boundary conditions Eqs. (22) and (23).

It is easily demonstrated that

$$\frac{d}{d\hat{r}}\left(\hat{r}^{\,3}\,\frac{dy_p}{d\hat{r}}\right) = -j_{1,p}^{\,2}\,\hat{r}^{\,3}\,y_p, \tag{28}$$

$$\frac{d}{d\hat{r}}\left(\hat{r}\,\frac{dz_p}{d\hat{r}}\right) = -j_{0,p}^{\,2}\,\hat{r}\,z_p, \tag{29}$$

and

$$\int_0^1 \hat{r}^{\,3}\,y_p(\hat{r})\,y_q(\hat{r})\,d\hat{r} = \frac{1}{2}\,\left[J_2(j_{1,p})\right]^2\,\delta_{pq}, \tag{30}$$

$$\int_0^1 \hat{r}\,z_p(\hat{r})\,z_q(\hat{r})\,d\hat{r} = \frac{1}{2}\,\left[J_1(j_{0,p})\right]^2\,\delta_{pq} \tag{31}$$

Hence, Eqs. (20) and (21) yield

$$(1+2\,q_k^2)\,\frac{d\alpha_{k,p}}{d\hat{t}} + \left(\frac{j_{1,p}^{\,2}}{\hat{\tau}_{M\,k}} + \frac{1}{\hat{\tau}_{\theta\,k}}\right)\alpha_{k,p} = \frac{m_k^{\,2}\,[y_p(\hat{r}_k)]^{\,2}}{\hat{\rho}_k\,\hat{a}^{\,2}\,[J_2(j_{1,p})]^{\,2}}\,\delta\hat{T}_k, \tag{32}$$

$$\frac{d\beta_{k,p}}{d\hat{t}} + \left(\frac{j_{0,p}^{\,2}}{\hat{\tau}_{M\,k}} + \frac{1}{\hat{\tau}_{\phi\,k}}\right)\beta_{k,n} = \frac{n^{\,2}\,[z_p(\hat{r}_k)]^{\,2}}{\hat{\rho}_k\,[J_1(j_{0,p})]^{\,2}}\,\delta\hat{T}_k. \tag{33}$$

It follows that

$$\hat{\Omega}_\theta(\hat{r}_k,\hat{t}) = \hat{\Omega}_{\theta\,0}(\hat{r}_k) - \sum_{k'=1,K}^{p=1,P}\frac{\alpha_{k',p}(\hat{t})}{m_{k'}}\,\frac{y_p(\hat{r}_k)}{y_p(\hat{r}_{k'})}, \tag{34}$$

$$\hat{\Omega}_\phi(\hat{r}_k,\hat{t}) = \hat{\Omega}_{\phi\,0}(\hat{r}_k) + \sum_{k'=1,K}^{p=1,P}\frac{\beta_{k',p}(\hat{t})}{n}\,\frac{z_p(\hat{r}_k)}{z_p(\hat{r}_{k'})}. \tag{35}$$

Here, k indexes the various resonant surfaces in the plasma, whereas p indexes the various velocity harmonics.

2 Resonant Plasma Response Model

Fundamental equation:

$$\left(\hat{W}_k + \hat{\delta}_k\right) \mathcal{S}_k \left(\frac{d}{d\hat{t}} + \mathrm{i}\,\hat{\varpi}_k\right) \Psi_k = \sum_{k'=1,K} E_{kk'}\,\Psi_{k'} + \hat{E}_{kk}\,\chi_k, \tag{36}$$

where

$$\mathcal{S}_k = \frac{\tau_R(\hat{r}_k)}{\tau_A}, \tag{37}$$

$$\hat{W}_k = \frac{2\,\mathcal{I}}{\hat{a}\,\hat{r}_k}\left(\frac{q}{g\,s}\right)^{1/2}_{\hat{r}_k} |\hat{\Psi}_k|^{\,1/2}, \tag{38}$$

$$\hat{\delta}_k = \frac{\delta_{\rm SC}(\hat{r}_k)}{R_0\,\hat{a}\,\hat{r}_k}, \tag{39}$$

with $\mathcal{I} = 0.8227$, and

$$\tau_R(r) = \mu_0\,r^{\,2}\,Q_{00}^{ee}\,\sigma_{ee}, \tag{40}$$

$$\sigma_{ee}(r) = \frac{n_e\,e^{\,2}\,\tau_{ee}}{m_e}. \tag{41}$$

Also,

$$\delta_{\rm SC}(r) = \pi\,\frac{|n\,\omega_{*\,e}|^{\,1/2}\,\tau_H}{(\rho_s/r)\,\tau_R^{1/2}}\,r, \tag{42}$$

$$\tau_H = \frac{R_0}{B_0\,g}\,\frac{\sqrt{\mu_0\,\rho}}{n\,s}, \tag{43}$$

$$\rho_s = \frac{\sqrt{m_i\,T_e}}{e\,B_0\,g}. \tag{44}$$

Follows from Eq. (36) that

$$\left(\hat{W}_k + \hat{\delta}_k\right) \mathcal{S}_k \frac{d\hat{\Psi}_k}{d\hat{t}} = \sum_{k'=1,K} \hat{E}_{kk'}\,\hat{\Psi}_{k'}\,\cos(\varphi_k - \varphi_{k'} - \xi_{kk'}) + \hat{E}_{kk}\,\hat{\chi}_k\,\cos(\varphi_k - \zeta_k), \tag{45}$$

$$\left(\hat{W}_k + \hat{\delta}_k\right) \mathcal{S}_k\,\hat{\Psi}_k \left(\frac{d\varphi_k}{d\hat{t}} - \hat{\varpi}_k\right) = -\sum_{k'=1,K} \hat{E}_{kk'}\,\hat{\Psi}_{k'}\,\sin(\varphi_k - \varphi_{k'} - \xi_{kk'}) - \hat{E}_{kk}\,\hat{\chi}_k\,\sin(\varphi_k - \zeta_k). \tag{46}$$

Alternatively, let

$$X_k = \hat{\Psi}_k\,\cos\varphi_k, \tag{47}$$

$$Y_k = \hat{\Psi}_k\,\sin\varphi_k. \tag{48}$$

So, Eq. (36) gives

$$\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{d}{d\hat{t}} + \mathrm{i}\,\hat{\varpi}_k\right)(X_k - \mathrm{i}\,Y_k) = \sum_{k'=1,K}\hat{E}_{kk'}\,(\cos\xi_{kk'} - \mathrm{i}\,\sin\xi_{kk'})\,(X_{k'} - \mathrm{i}\,Y_{k'}) \\ + \hat{E}_{kk}\,\hat{\chi}_k\,(\cos\zeta_k - \mathrm{i}\,\sin\zeta_k), \tag{49}$$

or

$$\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{dX_k}{d\hat{t}} - \mathrm{i}\,\frac{dY_k}{d\hat{t}} + \mathrm{i}\,\hat{\varpi}_k\,X_k + \hat{\varpi}_k\,Y_k\right) \\ = \sum_{k'=1,K}\hat{E}_{kk'}\,(\cos\xi_{kk'}\,X_{k'} - \mathrm{i}\,\cos\xi_{kk'}\,Y_{k'} - \mathrm{i}\,\sin\xi_{kk'}\,X_{k'} - \sin\xi_{kk'}\,Y_{k'}) \\ + \hat{E}_{kk'}\,\hat{\chi}_k\,(\cos\zeta_k - \mathrm{i}\,\sin\zeta_k). \tag{50}$$

Follows that

$$\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{dX_k}{d\hat{t}} + \hat{\varpi}_k\,Y_k\right) = \sum_{k'=1,K}\hat{E}_{kk'}\,(\cos\xi_{kk'}\,X_{k'} - \sin\xi_{kk'}\,Y_{k'}) + \hat{E}_{kk}\,\hat{\chi}_k\,\cos\zeta_k, \tag{51}$$

$$\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{dY_k}{d\hat{t}} - \hat{\varpi}_k\,X_k\right) = \sum_{k'=1,K}\hat{E}_{kk'}\,(\cos\xi_{kk'}\,Y_{k'} + \sin\xi_{kk'}\,X_{k'}) + \hat{E}_{kk}\,\hat{\chi}_k\,\sin\zeta_k, \tag{52}$$

where

$$\hat{W}_k = \frac{2\,\mathcal{I}}{\hat{a}\,\hat{r}_k}\left(\frac{q}{g\,s}\right)^{1/2}_{\hat{r}_k}(X_k^{\,2} + Y_k^{\,2})^{1/4}. \tag{53}$$

Note, finally, that

$$\delta\hat{T}_k = \sum_{k'\neq k}\hat{E}_{kk'}\,\hat{\Psi}_k\,\hat{\Psi}_{k'}\,\sin(\varphi_k - \varphi_{k'} - \xi_{kk'}) + \hat{E}_{kk'}\,\hat{\Psi}_k\,\hat{\chi}_k\,\sin(\varphi_k - \zeta_k). \tag{54}$$

3 Density and Temperature Flattening Widths

Critical electron temperature flattening width (in r):

$$W_{T_e\,k} = \left(\frac{\chi_E}{v_{T\,e}\,r}\,\frac{1}{\epsilon\,s\,n}\right)^{1/3}_{r_k} r_k, \tag{55}$$

where $\epsilon = r/R_0$. Critical ion temperature flattening width (in r):

$$W_{T_i\,k} = \left(\frac{\chi_E}{v_{T\,i}\,r}\,\frac{1}{\epsilon\,s\,n}\right)^{1/3}_{r_k} r_k. \tag{56}$$

Critical density flattening width (in r):

$$W_{n_e\,k} = \left(\frac{D_\perp}{v_{T\,i}\,r}\,\frac{1}{\epsilon\,s\,n}\right)_{r_k}^{1/3}\,r_k. \tag{57}$$

Critical electron temperature flattening width (in Ψ_N):

$$W_{T_e\,k} = \left(\frac{\chi_E}{v_{T\,e}\,R_0}\,\frac{f^{\,2}\,A_k^{(1)}}{n\,|\psi_c|}\right)_{r_k}^{1/3}. \tag{58}$$

Critical ion temperature flattening width (in Ψ_N):

$$W_{T_i\,k} = \left(\frac{\chi_E}{v_{T\,i}\,R_0}\,\frac{f^{\,2}\,A_k^{(1)}}{n\,|\psi_c|}\right)_{r_k}^{1/3}. \tag{59}$$

Critical density flattening width (in Ψ_N):

$$W_{n_e\,k} = \left(\frac{D_\perp}{v_{T\,i}\,R_0}\,\frac{f^{\,2}\,A_k^{(1)}}{n\,|\psi_c|}\right)_{r_k}^{1/3}. \tag{60}$$

Electron temperature, ion temperature, and electron number density profiles flattening widths (in Ψ_N)

$$\delta_{T_e\,k} = \frac{2}{\pi}\,W_k\,\tanh\left(\frac{W_k}{W_{T_e\,k}}\right), \tag{61}$$

$$\delta_{T_i\,k} = \frac{2}{\pi}\,W_k\,\tanh\left(\frac{W_k}{W_{T_i\,k}}\right), \tag{62}$$

$$\delta_{n_e\,k} = \frac{2}{\pi}\,W_k\,\tanh\left(\frac{W_k}{W_{n_e\,k}}\right), \tag{63}$$

Total pressure change due to islands is

$$\Delta P = \sum_{k=1,K}\left\{\left[\delta_{n_e\,k}\,\frac{dn_e}{d\Psi_N}\,T_e - \delta_{n_e\,k}\,W_k^{\,2}\,\frac{A_k^{(2)}}{8}\left(\frac{dn_e}{d\Psi_N}\,\frac{dT_e}{d\Psi_N} + \frac{d^2 n_e}{d\Psi_n^2}\,T_e\right) + \delta_{n_e\,k}^3\,\frac{d^3 n_e}{d\Psi_N^3}\,\frac{T_e}{24}\right]\right.$$
$$+\left[\delta_{T_e\,k}\,n_e\,\frac{dT_e}{d\Psi_N} - \delta_{T_e\,k}\,W_k^{\,2}\,\frac{A_k^{(2)}}{8}\left(\frac{dn_e}{d\Psi_N}\,\frac{dT_e}{d\Psi_N} + n_e\,\frac{d^2 T_e}{d\Psi_n^2}\right) + \delta_{T_e\,k}^3\,\frac{n_e}{24}\,\frac{d^3 T_e}{d\Psi_N^3}\right]$$
$$+\left[\delta_{n_e\,k}\,\frac{dn_i}{d\Psi_N}\,T_i - \delta_{n_e\,k}\,W_k^{\,2}\,\frac{A_k^{(2)}}{8}\left(\frac{dn_i}{d\Psi_N}\,\frac{dT_i}{d\Psi_N} + \frac{d^2 n_i}{d\Psi_n^2}\,T_i\right) + \delta_{n_e\,k}^3\,\frac{d^3 n_i}{d\Psi_N^3}\,\frac{T_i}{24}\right]$$
$$\left.+\left[\delta_{T_i\,k}\,n_i\,\frac{dT_i}{d\Psi_N} - \delta_{T_i\,k}\,W_k^{\,2}\,\frac{A_k^{(2)}}{8}\left(\frac{dn_i}{d\Psi_N}\,\frac{dT_i}{d\Psi_N} + n_i\,\frac{d^2 T_i}{d\Psi_n^2}\right) + \delta_{T_i\,k}^3\,\frac{n_i}{24}\,\frac{d^3 T_i}{d\Psi_N^3}\right]\right\}_{r_k}. \tag{64}$$

4 Coil Optimization

Assuming an ideal response of the plasma to the applied RMP, characterized by $\Psi_k = 0$ for all k, we have

$$\Delta\Psi_k = |E_{kk}|\left(I_U\,\mathrm{e}^{\,\mathrm{i}\,\Delta_U}\,|\chi_k^U|\,\mathrm{e}^{-\mathrm{i}\,\zeta_k^U} + I_M\,|\chi_k^M|\,\mathrm{e}^{-\mathrm{i}\,\zeta_k^M} + I_L\,\mathrm{e}^{\,\mathrm{i}\,\Delta_L}\,|\chi_k^L|\,\mathrm{e}^{-\mathrm{i}\,\zeta_k^L}\right), \tag{65}$$

where U, M, and L correspond to the upper, middle, and lower coil-sets, respectively, $I_{U,M,L}$ and $\Delta_{U,M,L}$ are the amplitudes and helical phase-shifts of the currents flowing in the upper, middle, and lower coil-sets, respectively, with $\Delta_M = 0$. Finally, $|\chi_k^{U,M,L}|\,\mathrm{e}^{-\mathrm{i}\,\zeta_k^{U,M,L}}$ are the χ_k values when 1 kA flows in the upper, middle, and lower coil-sets, respectively, with the same helical phase as the current flowing in the middle coil set. Let

$$z_k = \frac{\Delta\Psi_k}{|E_{kk}|\,I_M}, \tag{66}$$

$$\lambda_U = \left(\frac{I_U}{I_M}\right)\mathrm{e}^{\,\mathrm{i}\,\Delta_U}, \tag{67}$$

$$\lambda_L = \left(\frac{I_L}{I_M}\right)\mathrm{e}^{\,\mathrm{i}\,\Delta_L}, \tag{68}$$

$$x_k^{U,M,L} = |\chi_k^{U\,M\,L}|\,\mathrm{e}^{-\mathrm{i}\,\zeta_k^{U,M,L}}. \tag{69}$$

It follows that

$$z_k = \lambda_U\,x_k^U + x_k^M + \lambda_L\,x_k^L. \tag{70}$$

Let the $k = 1$ surface correspond to the innermost resonant surface, and let the $k = k'$ surface correspond to the nearest resonant surface to the top of the pedestal. To maximize ELM suppression while minimizing the chance of locked mode formation, we require

$$z_1 = 0, \tag{71}$$

$$z_{k'} = 3\, x^M_{k'}. \tag{72}$$

The latter constraint ensures that the signals from all three coils add in phase, with the same amplitude, at the top of the pedestal. It is easily seen that the choices

$$\lambda_U = \frac{2\, x^M_{k'}\, x^L_1 + x^M_1\, x^L_{k'}}{x^U_{k'}\, x^L_1 - x^U_1\, x^L_{k'}}, \tag{73}$$

$$\lambda_L = \frac{2\, x^M_{k'}\, x^U_1 + x^M_1\, x^U_{k'}}{x^L_{k'}\, x^U_1 - x^L_1\, x^U_{k'}} \tag{74}$$

ensure that (71) and (72) are satisfied.