Program RESCALE

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1 Rescaling Grad-Shafranov Equation

1.1 Grad-Shafranov Equation

Let R, ϕ, Z be a set of cylindrical coordinates. The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\phi \times \nabla\psi + T(\psi)\,\nabla\phi. \tag{1}$$

The Grad-Shafranov equation takes the form

$$\frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = -R^2\frac{dP}{d\psi} - \frac{1}{2}\frac{dT^2}{d\psi}, \tag{2}$$

where

$$j_\phi = -R\frac{dP}{d\psi} - \frac{1}{2\,R}\frac{dT^2}{d\psi}, \tag{3}$$

$$q(\psi) = \frac{T}{2\pi}\oint_\psi \frac{dl}{R\,|\nabla\psi|}. \tag{4}$$

1.2 Type I Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\mathrm{new}} = R_{\mathrm{old}}, \tag{5}$$

$$Z_{\mathrm{new}} = Z_{\mathrm{old}}, \tag{6}$$

$$T_{\mathrm{new}} = a_1\,T_{\mathrm{old}}, \tag{7}$$

$$\psi_{\mathrm{new}} = a_1\,\psi_{\mathrm{old}}, \tag{8}$$

$$P_{\mathrm{new}} = a_1^2\,P_{\mathrm{old}}. \tag{9}$$

Note that

$$B_{t,p\,\mathrm{new}} = a_1\,B_{t,p\,\mathrm{old}}, \tag{10}$$

$$I_{\phi\,\text{new}} = a_1\,I_{\phi\,\text{old}}, \tag{11}$$

$$\beta_{t,p\,\text{new}} = \beta_{t,p\,\text{old}}, \tag{12}$$

$$q_{\text{new}} = q_{\text{old}}. \tag{13}$$

1.3 Type II Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\text{new}} = a_2\,R_{\text{old}}, \tag{14}$$

$$Z_{\text{new}} = a_2\,Z_{\text{old}}, \tag{15}$$

$$T_{\text{new}} = a_2\,T_{\text{old}}, \tag{16}$$

$$\psi_{\text{new}} = a_2^2\,\psi_{\text{old}}, \tag{17}$$

$$P_{\text{new}} = P_{\text{old}}. \tag{18}$$

Note that

$$B_{t,p\,\text{new}} = B_{t,p\,\text{old}}, \tag{19}$$

$$I_{\phi\,\text{new}} = a_2\,I_{\phi\,\text{old}}, \tag{20}$$

$$\beta_{t,p\,\text{new}} = \beta_{t,p\,\text{old}}, \tag{21}$$

$$q_{\text{new}} = q_{\text{old}}. \tag{22}$$

1.4 Type III Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\text{new}} = R_{\text{old}}, \tag{23}$$

$$Z_{\text{new}} = Z_{\text{old}}, \tag{24}$$

$$T_{\text{new}} = T_{\text{old}}, \tag{25}$$

$$\psi_{\text{new}} = \psi_{\text{old}}, \tag{26}$$

$$P_{\text{new}} = P_{\text{old}} + a_3. \tag{27}$$

Note that

$$B_{t,p\,\text{new}} = B_{t,p\,\text{old}}, \tag{28}$$

$$I_{\phi\,\text{new}} = I_{\phi\,\text{old}}, \tag{29}$$

$$\beta_{t,p\,\text{new}} = \left(\frac{P_{\text{new}}}{P_{\text{old}}}\right)\beta_{t,p\,\text{old}}, \tag{30}$$

$$q_{\text{new}} = q_{\text{old}}. \tag{31}$$

1.5 Type IV Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\text{new}} = R_{\text{old}}, \tag{32}$$

$$Z_{\text{new}} = Z_{\text{old}}, \tag{33}$$

$$T_{\text{new}} = \text{sgn}(T_{\text{old}})\sqrt{T_{\text{old}}^2 + a_4}, \tag{34}$$

$$\psi_{\text{new}} = \psi_{\text{old}}, \tag{35}$$

$$P_{\text{new}} = P_{\text{old}}. \tag{36}$$

Note that

$$B_{t\,\text{new}} = \left(\frac{T_{\text{new}}}{T_{\text{old}}}\right) B_{t\,\text{old}}, \tag{37}$$

$$B_{p\,\text{new}} = B_{p\,\text{old}}, \tag{38}$$

$$I_{\phi\,\text{new}} = I_{\phi\,\text{old}}, \tag{39}$$

$$\beta_{t\,\text{new}} = \left(\frac{T_{\text{old}}}{T_{\text{new}}}\right)^2 \beta_{t\,\text{old}}, \tag{40}$$

$$\beta_{p\,\text{new}} = \beta_{p\,\text{old}}, \tag{41}$$

$$q_{\text{new}} = \left(\frac{T_{\text{new}}}{T_{\text{old}}}\right) q_{\text{old}}. \tag{42}$$

2 Rescaling Plasma Equilibrium

2.1 Type 1 Rescaling

Perform Type I rescaling with $a_1 = a_n^{1/2}$. It follows that

$$R \to R, \tag{43}$$

$$T \to a_n^{1/2}\, T, \tag{44}$$

$$\psi \to a_n^{1/2}\, \psi, \tag{45}$$

$$P \to a_n\, P, \tag{46}$$

and

$$B_{p,t} \to a_n^{1/2}\, B_{p,t}, \tag{47}$$

$$q_{95} \to q_{95}. \tag{48}$$

Let

$$n_a \rightarrow a_n\, n_a, \tag{49}$$

$$T_a \rightarrow T_a. \tag{50}$$

Here, $a \equiv e, i, b, I$. It follows that

$$\omega_{*a} \rightarrow a_n^{-1/2}\, \omega_{*a}. \tag{51}$$

Assuming that

$$\omega_{\theta\, a} \rightarrow a_n^{-1/2}\, \omega_{\theta\, a}, \tag{52}$$

$$\omega_E \rightarrow a_n^{-1/2}\, \omega_E, \tag{53}$$

we require

$$\omega_{\phi\, a} \rightarrow a_n^{-1/2}\, \omega_{\phi\, a}. \tag{54}$$

2.2 Type 2 Rescaling

Perform Type I rescaling with $a_1 = a_T^{1/2}$. It follows that

$$R \rightarrow R, \tag{55}$$

$$T \rightarrow a_T^{1/2}\, T, \tag{56}$$

$$\psi \rightarrow a_T^{1/2}\, \psi, \tag{57}$$

$$P \rightarrow a_T\, P, \tag{58}$$

and

$$B_{p,t} \rightarrow a_T^{1/2}\, B_{p,t}, \tag{59}$$

$$q_{95} \rightarrow q_{95}. \tag{60}$$

Let

$$n_a \rightarrow n_a, \tag{61}$$

$$T_a \rightarrow a_T\, T_a, \tag{62}$$

so

$$\omega_{*a} \rightarrow a_T^{1/2}\, \omega_{*a}. \tag{63}$$

Assuming that

$$\omega_{\theta\, a} \rightarrow a_T^{1/2}\, \omega_{\theta\, a}, \tag{64}$$

$$\omega_E \rightarrow a_T^{1/2}\, \omega_E, \tag{65}$$

we require

$$\omega_{\phi\, a} \rightarrow a_T^{1/2}\, \omega_{\phi\, a}. \tag{66}$$

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2.3 Type 3 Rescaling

Perform a Type II rescaling with $a_2 = a_R$. It follows that

$$R \to a_R R, \tag{67}$$
$$T \to a_R T, \tag{68}$$
$$\psi \to a_R^2 \psi, \tag{69}$$
$$P \to P, \tag{70}$$

and

$$B_{t,p} \to B_{t,p}, \tag{71}$$
$$q_{95} \to q_{95}. \tag{72}$$

Let

$$n_a \to n_a, \tag{73}$$
$$T_a \to T_a, \tag{74}$$

so

$$\omega_{*a} \to a_R^{-2} \omega_{*a}. \tag{75}$$

Assuming that

$$\omega_{\theta a} \to a_R^{-2} \omega_{\theta a}, \tag{76}$$
$$\omega_E \to a_R^{-2} \omega_E, \tag{77}$$

we require

$$\omega_{\phi a} \to a_R^{-2} \omega_{\phi a}. \tag{78}$$

2.4 Type 4 Rescaling

Let everything stay the same, except

$$\chi_{\phi,e,i} \to a_\phi \chi_{\phi,e,i}, \tag{79}$$
$$D_\perp \to a_\phi D_\perp. \tag{80}$$

2.5 Type 5 Rescaling

Perform a Type III rescaling with $a_3 = a_P$. It follows that

$$R \to R, \tag{81}$$
$$T \to T, \tag{82}$$

$$\psi \to \psi, \tag{83}$$

$$P \to P + a_P, \tag{84}$$

and

$$B_{t,p} \to B_{t,p}, \tag{85}$$

$$q_{95} \to q_{95}. \tag{86}$$

Let

$$n_a \to n_a, \tag{87}$$

$$T_e \to T_e + \left(\frac{a_P}{2\,n_e} \right), \tag{88}$$

$$T_i \to T_i + \left(\frac{a_P}{2\,n_i} \right), \tag{89}$$

$$T_I \to T_I, \tag{90}$$

so

$$\omega_{*a} \to \omega_{*a}. \tag{91}$$

Assuming that

$$\omega_{\theta\,a} \to \omega_{\theta\,a}, \tag{92}$$

$$\omega_E \to \omega_E, \tag{93}$$

we require

$$\omega_{\phi\,a} \to \omega_{\phi\,a}. \tag{94}$$

2.6   Type 6 Rescaling

Do not rescale equilibrium. It follows that

$$R \to R, \tag{95}$$

$$\psi \to \psi, \tag{96}$$

$$T \to T, \tag{97}$$

$$P \to P, \tag{98}$$

and

$$B_{p,t} \to B_{p,t}, \tag{99}$$

$$q_{95} \to q_{95}. \tag{100}$$

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Let

$$n_a \to n_a, \tag{101}$$

$$T_a \to T_a, \tag{102}$$

so

$$\omega_{*a} \to \omega_{*a}. \tag{103}$$

Assuming that

$$\omega_{\theta a} \to \omega_{\theta a}, \tag{104}$$

$$\omega_E \to \omega_E + a_E, \tag{105}$$

we require

$$\omega_{\phi a} \to \omega_{\phi a} + a_E. \tag{106}$$

2.7 Type 7 Rescaling

Let

$$\Psi_N = \frac{\psi - \psi_{\text{axis}}}{\psi_{\text{separatrix}} - \psi_{\text{axis}}}. \tag{107}$$

First, perform a Type IV rescaling such that

$$a_4 = \left(\frac{q_{95\,\text{target}}^2}{q_{95\,\text{old}}^2} - 1 \right) T_{95\,\text{old}}^2, \tag{108}$$

where

$$q_{95} \equiv q(\Psi_N = 0.95), \tag{109}$$

$$T_{95} \equiv T(\Psi_N = 0.95). \tag{110}$$

It follows that

$$\psi_{\text{new}} = \psi_{\text{old}}, \tag{111}$$

$$T_{\text{new}} = T_{\text{old}} \sqrt{ 1 + \left(\frac{q_{95\,\text{target}}^2}{q_{95\,\text{old}}^2} - 1 \right) \frac{T_{95\,\text{old}}^2}{T_{\text{old}}^2} }, \tag{112}$$

$$P_{\text{new}} = P_{\text{old}}, \tag{113}$$

and

$$I_{\phi\,\text{new}} = I_{\phi\,\text{old}}, \tag{114}$$

$$q_{\text{new}} = q_{\text{old}} \sqrt{ 1 + \left(\frac{q_{95\,\text{target}}^2}{q_{95\,\text{old}}^2} - 1 \right) \frac{T_{95\,\text{old}}^2}{T_{\text{old}}^2} }. \tag{115}$$

Next, perform a Type I rescaling such that

$$a_1 = \frac{T_{1\,\mathrm{old}}}{T_{1\,\mathrm{new}}}, \tag{116}$$

where

$$T_1 \equiv T(\Psi_N = 1). \tag{117}$$

It follows that

$$\psi_{\mathrm{new\,new}} = a_1 \, \psi_{\mathrm{old}}, \tag{118}$$

$$T_{\mathrm{new\,new}} = a_1 \, T_{\mathrm{new}} = T_{1\,\mathrm{old}} \sqrt{\frac{T_{\mathrm{old}}^2 + (q_{95\,\mathrm{target}}^2/q_{95\,\mathrm{old}}^2 - 1)\, T_{95\,\mathrm{old}}^2}{T_{1\,\mathrm{old}}^2 + (q_{95\,\mathrm{target}}^2/q_{95\,\mathrm{old}}^2 - 1)\, T_{95\,\mathrm{old}}^2}}, \tag{119}$$

$$P_{\mathrm{new\,new}} = a_1^2 \, P_{\mathrm{old}}, \tag{120}$$

and

$$I_{\phi\,\mathrm{new\,new}} = a_1 \, I_{\phi\,\mathrm{old}}, \tag{121}$$

$$q_{\mathrm{new\,new}} = q_{\mathrm{new}} = q_{\mathrm{old}} \sqrt{1 + \left(\frac{q_{95\,\mathrm{target}}^2}{q_{95\,\mathrm{old}}^2} - 1\right) \frac{T_{95\,\mathrm{old}}^2}{T_{\mathrm{old}}^2}}. \tag{122}$$

Note that

$$R \to R, \tag{123}$$

$$T \to \left[a_1^2 + (1 - a_1^2)\left(\frac{T_1}{T}\right)^2\right]^{1/2} T, \tag{124}$$

$$\psi \to a_1 \, \psi, \tag{125}$$

$$P \to a_1^2 \, P_{\mathrm{old}}. \tag{126}$$

It follows that

$$B_t \to \left[a_1^2 + (1 - a_1^2)\left(\frac{T_1}{T}\right)^2\right]^{1/2} B_t, \tag{127}$$

$$B_p \to a_1 \, B_p, \tag{128}$$

$$q_{95} \to q_{95\,\mathrm{target}}. \tag{129}$$

Let

$$n_a \to a_1 \, n_a, \tag{130}$$

$$T_a \to a_1 \, T_a, \tag{131}$$

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so
$$\omega_{*a} \to \omega_{*a}. \tag{132}$$

Assuming that
$$\omega_{\theta a} \to \omega_{\theta a}, \tag{133}$$
$$\omega_E \to \omega_E, \tag{134}$$

we require
$$\omega_{\phi a} \to \omega_{\phi a}. \tag{135}$$