Tearing Mode Dynamics in Tokamak Plasmas

R. Fitzpatrick [*]
Institute for Fusion Studies,
Department of Physics,
University of Texas at Austin,
Austin TX 78712, USA

November 5, 2021

1 Introduction

2 Plasma Fluid Theory

2.1 Introduction

The aim of this section is to introduce the fundamental plasma fluid theory that underpins the analysis of tearing mode dynamics in tokamak plasmas.

2.2 Fundamental Quantities

Consider an idealized tokamak plasma consisting of an equal number of electrons, with mass m_e and charge $-e$ (here, e denotes the magnitude of the electron charge), and ions, with mass m_i and charge $+e$. We shall employ the symbol

$$T_s = \frac{1}{3}\,m_s\,\langle v_s^2\rangle \tag{1}$$

to denote a *kinetic temperature* measured in units of energy. Here, v is a particle speed, and the angular brackets denote an ensemble average (Reif 1965). The kinetic temperature of species s is a measure of the mean kinetic energy of particles of that species. (Here, s represents either e for electrons, or i for ions.)

Quasi-neutrality demands that

$$n_i \simeq n_e, \tag{2}$$

[*]rfitzp@utexas.edu

where n_s is the *particle number density* (that is, the number of particles per cubic meter) of species s (Fitzpatrick 2015).

We can estimate typical particle speeds in terms of the so-called *thermal speed*,

$$v_{t\,s} = \left(\frac{2\,T_s}{m_s}\right)^{1/2}. \tag{3}$$

In a tokamak plasma, the ambient magnetic field, $\mathbf{B}$, is strong enough to significantly alter charged particle trajectories. Consequently, tokamak plasmas are highly anisotropic, responding differently to forces that are parallel and perpendicular to the direction of $\mathbf{B}$. As is well known, charged particles respond to the Lorentz force by freely streaming in the direction of $\mathbf{B}$, while executing circular *gyro-orbits* in the plane perpendicular to $\mathbf{B}$ (Fitzpatrick 2008). The typical *gyroradius* of a charged particle gyrating in a magnetic field is given by

$$\rho_s = \frac{v_{t\,s}}{\Omega_s}, \tag{4}$$

where

$$\Omega_s = \frac{e\,B}{m_s} \tag{5}$$

is the *gyrofrequency* associated with the gyration.

The electron-ion and ion-ion *collision times* are written

$$\tau_e = \frac{6\sqrt{2}\pi^{3/2}\,\epsilon_0^2\,\sqrt{m_e}\,T_e^{3/2}}{\ln\Lambda_c\,e^4\,n_e}, \tag{6}$$

and

$$\tau_i = \frac{12\pi^{3/2}\,\epsilon_0^2\,\sqrt{m_i}\,T_i^{3/2}}{\ln\Lambda_c\,e^4\,n_e}, \tag{7}$$

respectively (Fitzpatrick 2015). Here, $\ln\Lambda_c \simeq 15$ is the *Coulomb logarithm* (Richardson 2019). Note that τ_e is the typical time required for the cumulative effect of electron-ion collisions to deviate the path of an electron through 90°. Likewise, τ_i is the typical time required for the cumulative effect of ion-ion collisions to deviate the path of an ion through 90°.

2.3 Braginskii Equations

The electron and ion fluid equations in a collisional plasma take the form:

$$\frac{d_e n}{dt} + n\,\nabla\cdot\mathbf{V}_e = 0, \tag{8}$$

$$m_e\,n\,\frac{d_e\mathbf{V}_e}{dt} + \nabla p_e + \nabla\cdot\boldsymbol{\pi}_e + e\,n\,(\mathbf{E} + \mathbf{V}_e\times\mathbf{B}) = \mathbf{F}, \tag{9}$$

$$\frac{3}{2}\frac{d_e p_e}{dt} + \frac{5}{2}\,p_e\,\nabla\cdot\mathbf{V}_e + \boldsymbol{\pi}_e : \nabla\mathbf{V}_e + \nabla\cdot\mathbf{q}_e = w_e, \tag{10}$$

and

$$\frac{d_i n}{dt} + n\,\nabla\cdot\mathbf{V}_i = 0, \tag{11}$$

$$m_i\,n\,\frac{d_i\mathbf{V}_i}{dt} + \nabla p_i + \nabla\cdot\boldsymbol{\pi}_i - e\,n\,(\mathbf{E}+\mathbf{V}_i\times\mathbf{B}) = -\mathbf{F}, \tag{12}$$

$$\frac{3}{2}\frac{d_i p_i}{dt} + \frac{5}{2}\,p_i\,\nabla\cdot\mathbf{V}_i + \boldsymbol{\pi}_i : \nabla\mathbf{V}_i + \nabla\cdot\mathbf{q}_i = w_i, \tag{13}$$

respectively. Equations (8)–(13) are called the *Braginskii equations*, because they were first obtained in a celebrated article by S.I. Braginskii (Braginskii 1965). Here, n is the electron number density, $\mathbf{V}_s$ the species-s flow velocity, $p_s = n\,T_s$ the species-s scalar pressure, $\boldsymbol{\pi}_s$ the species-s viscosity tensor, $\mathbf{F}$ the collisional friction force density, $\mathbf{q}_s$ the species-s heat flux density, w_s the species-s collisional heating rate density, and $\mathbf{E}$ the ambient electric field-strength. Moreover, $d_e/dt \equiv \partial/\partial t + \mathbf{V}_e\cdot\nabla$ and $d_i/dt \equiv \partial/\partial t + \mathbf{V}_i\cdot\nabla$.

A tokamak plasma is *highly magnetized.* In other words,

$$\Omega_i\,\tau_i,\;\;\Omega_e\,\tau_e \gg 1, \tag{14}$$

which implies that the electron and ion gyroradii are much smaller than the corresponding mean-free-paths between 90° collisional scattering events. In this limit, a standard two-Laguerre-polynomial Chapman-Enskog closure scheme (Chapman & Cowling 1953) yields

$$\mathbf{F} = n\,e\left(\frac{\mathbf{j}_\parallel}{\sigma_\parallel} + \frac{\mathbf{j}_\perp}{\sigma_\perp}\right) - 0.71\,n\,\nabla_\parallel T_e - \frac{3\,n}{2\,\Omega_e\,\tau_e}\,\mathbf{b}\times\nabla_\perp T_e, \tag{15}$$

$$w_i = \frac{3\,m_e}{m_i}\frac{n\,(T_e - T_i)}{\tau_e}, \tag{16}$$

$$w_e = -w_i + \frac{\mathbf{j}\cdot\mathbf{F}}{n\,e}. \tag{17}$$

Here, $\mathbf{b} = \mathbf{B}/B$ is a unit vector parallel to the magnetic field, and $\mathbf{j} = n\,e\,(\mathbf{V}_i - \mathbf{V}_e)$ is the plasma current density. Moreover, the *parallel electrical conductivity* is given by

$$\sigma_\parallel = 1.96\,\frac{n\,e^2\,\tau_e}{m_e}. \tag{18}$$

whereas the *perpendicular electrical conductivity* takes the form

$$\sigma_\perp = 0.51\,\sigma_\parallel = \frac{n\,e^2\,\tau_e}{m_e}. \tag{19}$$

Note that $\nabla_\parallel(\cdots) \equiv [\mathbf{b}\cdot\nabla(\cdots)]\,\mathbf{b}$ denotes a gradient parallel to the magnetic field, whereas $\nabla_\perp \equiv \nabla - \nabla_\parallel$ denotes a gradient perpendicular to the magnetic field. Likewise, $\mathbf{j}_\parallel \equiv (\mathbf{b}\cdot\mathbf{j})\,\mathbf{b}$ represents the component of the plasma current density flowing parallel to the magnetic field, whereas $\mathbf{j}_\perp \equiv \mathbf{j} - \mathbf{j}_\parallel$ represents the perpendicular component of the plasma current density.

In a highly magnetized plasma, the electron and ion heat flux densities are written

$$\mathbf{q}_e = -\kappa_{\parallel\,e}\,\nabla_\parallel T_e - \kappa_{\perp\,e}\,\nabla_\perp T_e - \kappa_{\times\,e}\,\mathbf{b}\times\nabla_\perp T_e - 0.71\,\frac{T_e}{e}\,\mathbf{j}_\parallel - \frac{3\,T_e}{2\,\Omega_e\,\tau_e\,e}\,\mathbf{b}\times\mathbf{j}_\perp, \tag{20}$$

$$\mathbf{q}_i = -\kappa_{\parallel\,i}\,\nabla_\parallel T_i - \kappa_{\perp\,i}\,\nabla_\perp T_i + \kappa_{\times\,i}\,\mathbf{b}\times\nabla_\perp T_i, \tag{21}$$

respectively. Here, the *parallel thermal conductivities*, which control the diffusion of heat parallel to magnetic field-lines, are given by

$$\kappa_{\parallel\,e} = 3.2\,\frac{n\,\tau_e\,T_e}{m_e}, \tag{22}$$

$$\kappa_{\parallel\,i} = 3.9\,\frac{n\,\tau_i\,T_i}{m_i}, \tag{23}$$

whereas the *perpendicular thermal conductivities*, which control the diffusion of heat perpendicular to magnetic flux-surfaces, take the form

$$\kappa_{\perp\,e} = 4.7\,\frac{n\,T_e}{m_e\,\Omega_e^2\,\tau_e}, \tag{24}$$

$$\kappa_{\perp\,i} = 2\,\frac{n\,T_i}{m_i\,\Omega_i^2\,\tau_i}. \tag{25}$$

Finally, the *cross thermal conductivities*, which control the flow of heat within magnetic flux-surfaces, are written

$$\kappa_{\times\,e} = \frac{5\,n\,T_e}{2\,m_e\,\Omega_e}, \tag{26}$$

$$\kappa_{\times\,i} = \frac{5\,n\,T_i}{2\,m_i\,\Omega_i}. \tag{27}$$

In order to describe the viscosity tensor in a magnetized plasma, it is helpful to define the *rate-of-strain tensor*

$$W_{\alpha\beta} = \frac{\partial V_\alpha}{\partial r_\beta} + \frac{\partial V_\beta}{\partial r_\alpha} - \frac{2}{3}\,\nabla\cdot\mathbf{V}\,\delta_{\alpha\beta}. \tag{28}$$

Obviously, there is a separate rate-of-strain tensor for the electron and ion fluids. It is easily demonstrated that this tensor is zero if the fluid translates, or rotates as a rigid body, or if it undergoes isotropic compression. Thus, the rate-of-strain tensor measures the deformation of fluid volume elements.

In a highly magnetized plasma, the viscosity tensor is best described as the sum of five component tensors,

$$\boldsymbol{\pi} = \sum_{n=0,4} \boldsymbol{\pi}_n, \tag{29}$$

where

$$\boldsymbol{\pi}_0 = -3\,\eta_0\,\left(\mathbf{bb} - \frac{1}{3}\,\mathbf{I}\right)\left(\mathbf{bb} - \frac{1}{3}\,\mathbf{I}\right) : \nabla\mathbf{V}, \tag{30}$$

with

$$\boldsymbol{\pi}_1 = -\eta_1 \left[\mathbf{I}_\perp \cdot \mathbf{W} \cdot \mathbf{I}_\perp + \frac{1}{2}\,\mathbf{I}_\perp\,(\mathbf{b}\cdot\mathbf{W}\cdot\mathbf{b})\right], \tag{31}$$

and

$$\boldsymbol{\pi}_2 = -4\,\eta_1\,(\mathbf{I}_\perp \cdot \mathbf{W} \cdot \mathbf{b}\mathbf{b} + \mathbf{b}\mathbf{b}\cdot\mathbf{W}\cdot\mathbf{I}_\perp)\,. \tag{32}$$

plus

$$\boldsymbol{\pi}_3 = \frac{\eta_3}{2}\,(\mathbf{b}\times\mathbf{W}\cdot\mathbf{I}_\perp - \mathbf{I}_\perp\cdot\mathbf{W}\times\mathbf{b})\,, \tag{33}$$

and

$$\boldsymbol{\pi}_4 = 2\,\eta_3\,(\mathbf{b}\times\mathbf{W}\cdot\mathbf{b}\mathbf{b} - \mathbf{b}\mathbf{b}\cdot\mathbf{W}\times\mathbf{b})\,. \tag{34}$$

Here, $\mathbf{I}$ is the identity tensor, and $\mathbf{I}_\perp = \mathbf{I} - \mathbf{b}\mathbf{b}$. The previous expressions are valid for both electrons and ions.

The tensor $\boldsymbol{\pi}_0$ describes what is known as *parallel viscosity*. This is a viscosity that controls the variation along magnetic field-lines of the velocity component parallel to field-lines. The parallel viscosity coefficients are given by

$$\eta_{0\,e} = 0.73\,n\,\tau_e\,T_e, \tag{35}$$

$$\eta_{0\,i} = 0.96\,n\,\tau_i\,T_i. \tag{36}$$

The tensors $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ describe what is known as *perpendicular viscosity*. This is a viscosity that controls the variation perpendicular to magnetic field-lines of the velocity components perpendicular to field-lines. The perpendicular viscosity coefficients are given by

$$\eta_{1\,e} = 0.51\,\frac{n\,T_e}{\Omega_e^2\,\tau_e}, \tag{37}$$

$$\eta_{1\,i} = \frac{3\,n\,T_i}{10\,\Omega_i^2\,\tau_i}. \tag{38}$$

Finally, the tensors $\boldsymbol{\pi}_3$ and $\boldsymbol{\pi}_4$ describe what is known as *gyroviscosity*. This is not really viscosity at all, because the associated viscous stresses are always perpendicular to the velocity, implying that there is no dissipation (i.e., viscous heating) associated with this effect. The gyroviscosity coefficients are given by

$$\eta_{3\,e} = -\frac{n\,T_e}{2\,\Omega_e}, \tag{39}$$

$$\eta_{3\,i} = \frac{n\,T_i}{2\,\Omega_i}. \tag{40}$$

2.4 Normalization of Braginskii Equations

As we have just seen, the Braginskii equations contain terms that describe a very wide range of different physical phenomena. For this reason, they are extremely complicated in nature. Fortunately, however, it is not generally necessary to retain all of the terms in these equations when investigating tearing mode dynamics in tokamak plasmas. In this section, we shall construct a systematic normalization scheme for the Braginskii equations that will enable us to determine which terms to keep, and which to discard, when investigating tearing mode dynamics.

It is convenient to split the friction force density, $\mathbf{F}$, into a component, $\mathbf{F}_U$, corresponding to resistivity, and a component, $\mathbf{F}_T$, corresponding to the so-called *thermal force density.* Thus,

$$\mathbf{F} = \mathbf{F}_U + \mathbf{F}_T, \tag{41}$$

where

$$\mathbf{F}_U = n\,e\left(\frac{\mathbf{j}_\parallel}{\sigma_\parallel} + \frac{\mathbf{j}_\perp}{\sigma_\perp}\right), \tag{42}$$

$$\mathbf{F}_T = -0.71\,n\,\nabla_\parallel T_e - \frac{3\,n}{2\,\Omega_e\,\tau_e}\,\mathbf{b}\times\nabla_\perp T_e. \tag{43}$$

Likewise, the electron collisional energy gain term, w_e, is split into a component, $-w_i$, corresponding to the energy lost per unit volume to the ions (in the ion rest frame), a component, w_U, corresponding to work done by the friction force density, $\mathbf{F}_U$, and a component, w_T, corresponding to work done by the thermal force density, $\mathbf{F}_T$. Thus,

$$w_e = -w_i + w_U + w_T, \tag{44}$$

where

$$w_U = \frac{\mathbf{j}\cdot\mathbf{F}_U}{n\,e}, \tag{45}$$

$$w_T = \frac{\mathbf{j}\cdot\mathbf{F}_T}{n\,e}. \tag{46}$$

Finally, it is helpful to split the electron heat flux density, $\mathbf{q}_e$, into a diffusive component, $\mathbf{q}_{Te}$, and a convective component, $\mathbf{q}_{Ue}$. Thus,

$$\mathbf{q}_e = \mathbf{q}_{Te} + \mathbf{q}_{Ue}, \tag{47}$$

where

$$\mathbf{q}_{Te} = -\kappa_{\parallel\,e}\,\nabla_\parallel T_e - \kappa_{\perp\,e}\,\nabla_\perp T_e - \kappa_{\times\,e}\,\mathbf{b}\times\nabla_\perp T_e, \tag{48}$$

$$\mathbf{q}_{Ue} = 0.71\,\frac{T_e}{e}\,\mathbf{j}_\parallel - \frac{3\,T_e}{2\,\Omega_e\,\tau_e\,e}\,\mathbf{b}\times\mathbf{j}_\perp. \tag{49}$$

Let us, first of all, consider the electron fluid equations, which can be written:

$$\frac{d_e n}{dt} + n\,\nabla\cdot\mathbf{V}_e = 0, \tag{50}$$

$$m_e\, n\,\frac{d_e\mathbf{V}_e}{dt}+\nabla p_e+\nabla\cdot\boldsymbol{\pi}_e+e\,n\,(\mathbf{E}+\mathbf{V}_e\times\mathbf{B}) = \mathbf{F}_U+\mathbf{F}_T, \tag{51}$$

$$\frac{3}{2}\frac{d_e p_e}{dt}+\frac{5}{2}\,p_e\,\nabla\cdot\mathbf{V}_e+\boldsymbol{\pi}_e:\nabla\mathbf{V}_e+\nabla\cdot\mathbf{q}_{Te}+\nabla\cdot\mathbf{q}_{Ue} = -w_i+w_U+w_T. \tag{52}$$

Let $\bar{n}$, $\bar{v}_e$, $\bar{\tau}_e$, $\bar{l}_e=\bar{v}_e\,\bar{\tau}_e$, $\bar{B}$, and $\bar{\rho}_e=\bar{v}_e/(e\bar{B}/m_e)$, be typical values of the particle density, the electron thermal velocity, the electron collision time, the electron mean-free-path between collisions, the magnetic field-strength, and the electron gyroradius, respectively. Suppose that the typical spatial variation lengthscale of fluid variables is L. Let

$$\delta_e = \frac{\bar{\rho}_e}{L}, \tag{53}$$

$$\zeta_e = \frac{\bar{\rho}_e}{\bar{l}_e}, \tag{54}$$

$$\mu = \sqrt{\frac{m_e}{m_i}}. \tag{55}$$

All three of these parameters are assumed to be small compared to unity. Finally, the typical electron flow velocity is assumed to be of order $\delta_e\,\bar{v}_e$. This corresponds to the so-called *drift ordering* in which the flow velocity is comparable to the curvature and grad-B particle drift velocities (Fitzpatrick 2015). The drift ordering is appropriate to tearing modes in tokamak plasmas, which are comparatively slowly growing instabilities.

We define the following normalized quantities:

$$\hat{n} = \frac{n}{\bar{n}}, \qquad \hat{v}_e = \frac{v_e}{\bar{v}_e}, \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{L},$$

$$\widehat{\nabla} = L\,\nabla, \qquad \hat{t} = \frac{\delta_e\,\bar{v}_e\,t}{L}, \qquad \widehat{\mathbf{V}}_e = \frac{\mathbf{V}_e}{\delta_e\,\bar{v}_e},$$

$$\widehat{\mathbf{B}} = \frac{\mathbf{B}}{\bar{B}}, \qquad \widehat{\mathbf{E}} = \frac{\mathbf{E}}{\delta_e\,\bar{v}_e\,\bar{B}}, \qquad \widehat{\mathbf{j}} = \frac{\mathbf{j}}{n\,e\,\delta_e\,\bar{v}_e},$$

$$\hat{p}_e = \frac{p_e}{m_e\,\bar{n}\,\bar{v}_e^2}, \qquad \widehat{\boldsymbol{\pi}}_e = \frac{\boldsymbol{\pi}_e}{\delta_e^2\,\zeta_e^{-1}\,m_e\,\bar{n}\,\bar{v}_e^2}, \qquad \widehat{\mathbf{q}}_{Te} = \frac{\mathbf{q}_{Te}}{\delta_e\,\zeta_e^{-1}\,m_e\,\bar{n}\,\bar{v}_e^3},$$

$$\widehat{\mathbf{q}}_{Ue} = \frac{\mathbf{q}_{Ue}}{\delta_e\,m_e\,\bar{n}\,\bar{v}_e^3}, \qquad \widehat{\mathbf{F}}_U = \frac{\mathbf{F}_U}{\zeta_e\,m_e\,\bar{n}\,\bar{v}_e^2/L}, \qquad \widehat{\mathbf{F}}_T = \frac{\mathbf{F}_T}{m_e\,\bar{n}\,\bar{v}_e^2/L},$$

$$\widehat{w}_i = \frac{w_i}{\delta_e^{-1}\,\zeta_e\,\mu^2\,m_e\,\bar{n}\,\bar{v}_e^3/L}, \qquad \widehat{w}_U = \frac{w_U}{\delta_e\,\zeta_e\,m_e\,\bar{n}\,\bar{v}_e^3/L}, \qquad \widehat{w}_T = \frac{w_T}{\delta_e\,m_e\,\bar{n}\,\bar{v}_e^3/L}.$$

The normalization procedure is designed to make all hatted quantities $\mathcal{O}(1)$. The normalization of the electric field is chosen such that the $\mathbf{E}\times\mathbf{B}$ velocity is of similar magnitude to the electron diamagnetic velocity. (See Section 2.5.) Note that the parallel viscosity makes an $\mathcal{O}(1)$ contribution to $\widehat{\boldsymbol{\pi}}_e$, whereas the gyroviscosity makes an $\mathcal{O}(\zeta_e)$ contribution, and the perpendicular viscosity only makes an $\mathcal{O}(\zeta_e^2)$ contribution. Likewise, the parallel thermal conductivity makes an $\mathcal{O}(1)$ contribution to $\widehat{\mathbf{q}}_{Te}$, whereas the cross conductivity makes an $\mathcal{O}(\zeta_e)$ contribution, and the perpendicular conductivity only makes an $\mathcal{O}(\zeta_e^2)$ contribution. Similarly, the parallel components of $\mathbf{F}_T$ and $\mathbf{q}_{Ue}$ are $\mathcal{O}(1)$, whereas the perpendicular components are $\mathcal{O}(\zeta_e)$.

The normalized electron fluid equations take the form:

$$\frac{d_e\hat{n}}{d\hat{t}} + \hat{n}\,\widehat{\nabla}\cdot\widehat{\mathbf{V}}_e = 0, \tag{56}$$

$$\hat{n}\,\frac{d_e\widehat{\mathbf{V}}_e}{d\hat{t}} + \delta_e^{-2}\,\widehat{\nabla}\hat{p}_e + \zeta_e^{-1}\,\widehat{\nabla}\cdot\widehat{\boldsymbol{\pi}}_e + \delta_e^{-2}\,\hat{n}\,(\widehat{\mathbf{E}} + \widehat{\mathbf{V}}_e\times\widehat{\mathbf{B}}) = \delta_e^{-2}\,\zeta_e\,\widehat{\mathbf{F}}_U + \delta_e^{-2}\,\widehat{\mathbf{F}}_T, \tag{57}$$

$$\frac{3}{2}\frac{d_e\hat{p}_e}{d\hat{t}} + \frac{5}{2}\,\hat{p}_e\,\widehat{\nabla}\cdot\widehat{\mathbf{V}}_e + \delta_e^2\,\zeta_e^{-1}\,\widehat{\boldsymbol{\pi}}_e : \widehat{\nabla}\cdot\widehat{\mathbf{V}}_e$$
$$+\zeta_e^{-1}\,\widehat{\nabla}\cdot\widehat{\mathbf{q}}_{Te} + \widehat{\nabla}\cdot\widehat{\mathbf{q}}_{Ue} = -\delta_e^{-2}\,\zeta_e\,\mu^2\,\widehat{w}_i + \zeta_e\,\widehat{w}_U + \widehat{w}_T. \tag{58}$$

The only large or small (compared to unity) quantities in these equations are the parameters δ_e, ζ_e, and μ. Here, $d_e/d\hat{t} \equiv \partial/\partial\hat{t} + \widehat{\mathbf{V}}_e\cdot\widehat{\nabla}$. It is assumed that $T_e \sim T_i$.

Let us now consider the ion fluid equations, which can be written:

$$\frac{d_i n}{dt} + n\,\nabla\cdot\mathbf{V}_i = 0, \tag{59}$$

$$m_i\,n\,\frac{d_i\mathbf{V}_i}{dt} + \nabla p_i + \nabla\cdot\boldsymbol{\pi}_i - e\,n\,(\mathbf{E} + \mathbf{V}_i\times\mathbf{B}) = -\mathbf{F}_U - \mathbf{F}_T, \tag{60}$$

$$\frac{3}{2}\frac{d_i p_i}{dt} + \frac{5}{2}\,p_i\,\nabla\cdot\mathbf{V}_i + \boldsymbol{\pi}_i : \nabla\mathbf{V}_i + \nabla\cdot\mathbf{q}_i = w_i. \tag{61}$$

It is convenient to adopt a normalization scheme for the ion equations which is similar to, but independent of, that employed to normalize the electron equations. Let $\bar{n}$, $\bar{v}_i$, $\bar{l}_i$, $\bar{B}$, and $\bar{\rho}_i = \bar{v}_i/(e\bar{B}/m_i)$, be typical values of the particle density, the ion thermal velocity, the ion mean-free-path between collisions, the magnetic field-strength, and the ion gyroradius, respectively. Suppose that the typical spatial variation lengthscale of fluid quantities is L. Let

$$\delta_i = \frac{\bar{\rho}_i}{L}, \tag{62}$$

$$\zeta_i = \frac{\bar{\rho}_i}{\bar{l}_i}, \tag{63}$$

$$\mu = \sqrt{\frac{m_e}{m_i}}. \tag{64}$$

All three of these parameters are assumed to be small compared to unity. As before, we adopt the drift ordering in which the typical ion flow velocity is assumed to be of order $\delta_i\,\bar{v}_i$.

We define the following normalized quantities:

$$\hat{n} = \frac{n}{\bar{n}}, \qquad \hat{v}_i = \frac{v_i}{\bar{v}_i}, \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{L},$$

$$\widehat{\nabla} = L\,\nabla, \qquad \hat{t} = \frac{\delta_i\,\bar{v}_i\,t}{L}, \qquad \widehat{\mathbf{V}}_i = \frac{\mathbf{V}_i}{\delta_i\,\bar{v}_i},$$

$$\widehat{\mathbf{B}} = \frac{\mathbf{B}}{\bar{B}}, \qquad \widehat{\mathbf{E}} = \frac{\mathbf{E}}{\delta_i\,\bar{v}_i\,\bar{B}}, \qquad \widehat{\mathbf{j}} = \frac{\mathbf{j}}{n\,e\,\delta_i\,\bar{v}_i},$$

$$\hat{p}_i = \frac{p_i}{m_i\,\bar{n}\,\bar{v}_i^2}, \qquad \widehat{\boldsymbol{\pi}}_i = \frac{\boldsymbol{\pi}_i}{\delta_i^2\,\zeta_i^{-1}\,m_i\,\bar{n}\,\bar{v}_i^2}, \qquad \widehat{\mathbf{q}}_i = \frac{\mathbf{q}_i}{\delta_i\,\zeta_i^{-1}\,m_i\,\bar{n}\,\bar{v}_i^3},$$

$$\widehat{\mathbf{F}}_U = \frac{\mathbf{F}_U}{\zeta_i\,\mu\,m_i\,\bar{n}\,\bar{v}_i^2/L}, \qquad \widehat{\mathbf{F}}_T = \frac{\mathbf{F}_T}{m_i\,\bar{n}\,\bar{v}_i^2/L}, \qquad \widehat{w}_i = \frac{w_i}{\delta_i^{-1}\,\zeta_i\,\mu\,m_i\,\bar{n}\,\bar{v}_i^3/L}.$$

As before, the normalization procedure is designed to make all hatted quantities $\mathcal{O}(1)$. The normalization of the electric field is chosen such that the $\mathbf{E}\times\mathbf{B}$ velocity is of similar magnitude to the ion diamagnetic velocity. Note that the parallel viscosity makes an $\mathcal{O}(1)$ contribution to $\widehat{\boldsymbol{\pi}}_i$, whereas the gyroviscosity makes an $\mathcal{O}(\zeta_i)$ contribution, and the perpendicular viscosity only makes an $\mathcal{O}(\zeta_i^2)$ contribution. Likewise, the parallel thermal conductivity makes an $\mathcal{O}(1)$ contribution to $\widehat{\mathbf{q}}_i$, whereas the cross conductivity makes an $\mathcal{O}(\zeta_i)$ contribution, and the perpendicular conductivity only makes an $\mathcal{O}(\zeta_i^2)$ contribution. Similarly, the parallel component of $\mathbf{F}_T$ is $\mathcal{O}(1)$, whereas the perpendicular component is $\mathcal{O}(\zeta_i\,\mu)$.

The normalized ion fluid equations take the form:

$$\frac{d_i\hat{n}}{d\hat{t}} + \hat{n}\,\widehat{\nabla}\cdot\widehat{\mathbf{V}}_i = 0, \tag{65}$$

$$\hat{n}\,\frac{d_i\widehat{\mathbf{V}}_i}{d\hat{t}} + \delta_i^{-2}\,\widehat{\nabla}\hat{p}_i + \zeta_i^{-1}\,\widehat{\nabla}\cdot\widehat{\boldsymbol{\pi}}_i - \delta_i^{-2}\,\hat{n}\,(\widehat{\mathbf{E}} + \widehat{\mathbf{V}}_i\times\widehat{\mathbf{B}}) = -\delta_i^{-2}\,\zeta_i\,\mu\,\widehat{\mathbf{F}}_U - \delta_i^{-2}\,\widehat{\mathbf{F}}_T, \tag{66}$$

$$\frac{3}{2}\frac{d_i\hat{p}_i}{d\hat{t}} + \frac{5}{2}\,\hat{p}_i\,\widehat{\nabla}\cdot\widehat{\mathbf{V}}_i + \delta_i^2\,\zeta_i^{-1}\,\widehat{\boldsymbol{\pi}}_i : \widehat{\nabla}\cdot\widehat{\mathbf{V}}_i + \zeta_i^{-1}\,\widehat{\nabla}\cdot\widehat{\mathbf{q}}_i = \delta_i^{-2}\,\zeta_i\,\mu\,\widehat{w}_i. \tag{67}$$

The only large or small (compared to unity) quantities in these equations are the parameters δ_i, ζ_i, and μ. Here, $d_i/d\hat{t} \equiv \partial/\partial\hat{t} + \widehat{\mathbf{V}}_i\cdot\widehat{\nabla}$.

2.5 Lowest-Order Fluid Equations

If we restore dimensions to the Braginskii equations then we can write them in the form

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\,\mathbf{V}_e) = 0, \tag{68}$$

$$m_e\,n\,\frac{\partial\mathbf{V}_e}{\partial t} + m_e\,n\,(\mathbf{V}_e\cdot\nabla)\mathbf{V}_e + [\delta_e^{-2}]\,\nabla p_e + [\zeta_e^{-1}]\,\nabla\cdot\boldsymbol{\pi}_e$$
$$+[\delta_e^{-2}]\,e\,n\,(\mathbf{E} + \mathbf{V}_e\times\mathbf{B}) = [\delta_e^{-2}\,\zeta_e]\,\mathbf{F}_U + [\delta_e^{-2}]\,\mathbf{F}_T, \tag{69}$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \frac{3}{2}\,\mathbf{V}_e\cdot\nabla p_e + \frac{5}{2}\,p_e\,\nabla\cdot\mathbf{V}_e + [\delta_e^2\,\zeta_e^{-1}]\,\boldsymbol{\pi}_e : \nabla\mathbf{V}_e \tag{70}$$
$$+[\zeta_e^{-1}]\,\nabla\cdot\mathbf{q}_{Te} + \nabla\cdot\mathbf{q}_{Ue} = -[\delta_e^{-2}\,\zeta_e\,\mu^2]\,w_i + [\zeta_e]\,w_U + w_T,$$

and

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\,\mathbf{V}_i) = 0, \tag{71}$$

$$m_i\,n\,\frac{\partial \mathbf{V}_i}{\partial t} + m_i\,n\,(\mathbf{V}_i\cdot\nabla)\mathbf{V}_i + [\delta_i^{-2}]\,\nabla p_i + [\zeta_i^{-1}]\,\nabla\cdot\boldsymbol{\pi}_i$$
$$-[\delta_i^{-2}]\,e\,n\,(\mathbf{E}+\mathbf{V}_i\times\mathbf{B}) = -[\delta_i^{-2}\,\zeta_i\,\mu]\,\mathbf{F}_U - [\delta_i^{-2}]\,\mathbf{F}_T, \tag{72}$$

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \frac{3}{2}\,\mathbf{V}_i\cdot\nabla p_i + \frac{5}{2}\,p_i\,\nabla\cdot\mathbf{V}_i + [\delta_i^2\,\zeta_i^{-1}]\,\boldsymbol{\pi}_i : \nabla\mathbf{V}_i$$
$$+[\zeta_i^{-1}]\,\nabla\cdot\mathbf{q}_i = [\delta_i^{-2}\,\zeta_i\,\mu]\,w_i. \tag{73}$$

The factors in square brackets are there to remind us that the terms they precede are smaller or larger than the other terms in the equations (by the corresponding factors within the brackets).

If we assume that $\delta_e \sim \zeta_e \sim \mu \ll 1$ then the dominant terms in the electron energy conservation equation (70) yield

$$\nabla\cdot\mathbf{q}_{T_e} \simeq \nabla\cdot(-\kappa_{\parallel\,e}\,\nabla_\parallel T_e) \simeq 0, \tag{74}$$

which implies that

$$\mathbf{B}\cdot\nabla T_e = 0. \tag{75}$$

In other words, the parallel electron heat conductivity in a tokamak plasma is usually sufficiently large that it forces the electron temperature to be constant on magnetic flux-surfaces. Likewise, if we assume that $\delta_i \sim \zeta_i \sim \mu \ll 1$ then the dominant terms in the ion energy conservation equation (73) yield

$$\nabla\cdot\mathbf{q}_{T_i} \simeq \nabla\cdot(-\kappa_{\parallel\,i}\,\nabla_\parallel T_i) \simeq 0, \tag{76}$$

which implies that

$$\mathbf{B}\cdot\nabla T_i = 0. \tag{77}$$

In other words, the parallel ion heat conductivity in a tokamak plasma is usually sufficiently large that it forces the ion temperature to be constant on magnetic flux-surfaces.

The dominant terms in the electron and ion momentum conservation equations, (69) and (72), respectively, yield

$$\nabla p_e + n\,e\,(\mathbf{E}+\mathbf{V}_e\times\mathbf{B}) \simeq 0, \tag{78}$$

$$\nabla p_i - n\,e\,(\mathbf{E}+\mathbf{V}_i\times\mathbf{B}) \simeq 0. \tag{79}$$

The sum of the preceding two equations gives

$$\mathbf{j}\times\mathbf{B} \simeq \nabla p, \tag{80}$$

where $p = p_e + p_i$. In other words, a conventional tokamak plasma exists in a state in which the magnetic force density exactly balances the total scalar pressure force density. It follows from the previous equation that

$$\mathbf{B}\cdot\nabla p \simeq 0. \tag{81}$$

In other words, the total pressure in a conventional tokamak plasma is constant on magnetic flux-surfaces. However, given that $p = n\,(T_e + T_i)$, and making use of Eqs. (75) and (77), we deduce that

$$\mathbf{B}\cdot\nabla n \simeq 0. \tag{82}$$

In other words, the electron number density in a conventional tokamak plasma is also constant on magnetic flux-surfaces.

Taking the difference of the electron and ion particle conservation equations, (68) and (71), respectively, we obtain

$$\nabla\cdot\mathbf{j} = 0. \tag{83}$$

Equations (80) and (83) can be combined to give

$$(\mathbf{B}\cdot\nabla)\,\mathbf{j} - (\mathbf{j}\cdot\nabla)\,\mathbf{B} \simeq \mathbf{0}. \tag{84}$$

Here, we have made use of the fact that $\nabla\cdot\mathbf{B} = 0$.

Finally, Equations (78) and (79) imply that

$$\mathbf{V}_{\perp\,e} \simeq \mathbf{V}_E + \mathbf{V}_{*\,e}, \tag{85}$$

$$\mathbf{V}_{\perp\,i} \simeq \mathbf{V}_E + \mathbf{V}_{*\,i}, \tag{86}$$

where

$$\mathbf{V}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^{\,2}} \tag{87}$$

is the *E-cross-B drift velocity* (otherwise known as the $\mathbf{E}\times\mathbf{B}$ velocity), and

$$\mathbf{V}_{*\,e} = \frac{\nabla p_e\times\mathbf{B}}{e\,n\,B^{\,2}}, \tag{88}$$

$$\mathbf{V}_{*\,i} = -\frac{\nabla p_i\times\mathbf{B}}{e\,n\,B^{\,2}}, \tag{89}$$

are termed the *electron diamagnetic velocity* and the *ion diamagnetic velocity*, respectively (Fitzpatrick 2015).

3 Cylindrical Tearing Mode Theory

3.1 Introduction

The aim of this section is to describe the simplest theory of tearing mode dynamics in tokamak plasmas, according to which the plasma equilibrium is approximated as a periodic cylinder.

3.2 Tokamak Equilibrium

Consider a large aspect-ratio, toroidal, tokamak plasma equilibrium whose magnetic flux-surfaces map out (almost) concentric circles in the poloidal plane. Such an equilibrium can be approximated as a periodic cylinder (Wesson 1978). Let us employ a conventional set of right-handed cylindrical coordinates, r, θ, z. The equilibrium magnetic flux-surfaces lie on surfaces of constant r. The system is assumed to be periodic in the z ('toroidal') direction, with periodicity length $2\pi\,R_0$, where R_0 is the simulated *major radius* of the plasma. The *safety-factor* profile takes the form

$$q(r) = \frac{r\,B_z}{R_0\,B_\theta(r)}, \tag{90}$$

where B_z is the constant 'toroidal' magnetic field-strength, and $B_\theta(r)$ is the poloidal magnetic field-strength. It is assumed that $q \sim \mathcal{O}(1)$. The equilibrium 'toroidal' current density satisfies

$$\mu_0\,j_z(r) = \frac{(r\,B_\theta)'}{r}, \tag{91}$$

where $'$ denotes d/dr. Finally, the standard large aspect-ratio tokamak orderings,

$$\frac{r}{R_0} \ll 1, \quad \frac{B_\theta}{B_z} \ll 1, \tag{92}$$

are adopted (Fitzpatrick 1993).

3.3 Perturbed Magnetic Field

Consider a tearing mode perturbation that has m periods in the poloidal direction, and n periods in the toroidal direction, where $m/n \sim \mathcal{O}(1)$. We can write the perturbed magnetic field in the divergence-free form

$$\delta\mathbf{B} \simeq \nabla\times(\psi\,\mathbf{e}_z), \tag{93}$$

where

$$\psi(r,\theta,\varphi,t) = \psi(r,t)\,\exp[\mathrm{i}\,(m\,\theta - n\,\varphi)]. \tag{94}$$

Here, $\varphi = z/R_0$ is a simulated toroidal angle. The perturbed current density becomes

$$\mu_0\,\delta\mathbf{j} \simeq -\nabla^2\psi\,\mathbf{e}_z. \tag{95}$$

It is assumed that $|\delta\mathbf{B}| \ll |\mathbf{B}|$.

3.4 Cylindrical Tearing Mode Equation

To lowest order, the tearing perturbation is governed by the linearized version of Equation (84), which yields

$$(\delta\mathbf{B}\cdot\nabla)\,\mathbf{j} + (\mathbf{B}\cdot\nabla)\,\delta\mathbf{j} - (\delta\mathbf{j}\cdot\nabla)\,\mathbf{B} - (\mathbf{j}\cdot\nabla)\,\delta\mathbf{B} \simeq \mathbf{0}. \tag{96}$$

Making use of Equations (90)–(95), the z-component of the previous equation gives the *cylindrical tearing mode equation* (Wesson 1978; Fitzpatrick 1993),

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{m^2}{r^2}\,\psi-\frac{J_z'\,\psi}{r\,(1/q-n/m)}\simeq 0, \tag{97}$$

where

$$J_z(r)=\frac{R_0\,\mu_0\,j_z(r)}{B_z}. \tag{98}$$

3.5 Solution in Presence of Perfectly-Conducting Wall

Suppose that the plasma occupies the region $0\leq r\leq a$, where a is the plasma *minor radius*. It follows that $J_z(r)=0$ for $r>a$. Let the plasma be surrounded by a concentric, rigid, radially-thin, perfectly-conducting, wall of radius $r_w>a$. (By definition, $a\ll R_0$ in a large aspect-ratio tokamak. In most circumstances, the wall represents the metal vacuum vessel that surrounds the plasma.) An appropriate physical solution of the cylindrical tearing mode equation takes the separable form

$$\psi(r,t)=\Psi_s(t)\,\hat{\psi}_s(r), \tag{99}$$

where the real function $\hat{\psi}_s(r)$ is a solution of

$$\frac{d^2\hat{\psi}_s}{dr^2}+\frac{1}{r}\frac{d\hat{\psi}_s}{dr}-\frac{m^2}{r^2}\,\hat{\psi}_s-\frac{J_z'\,\hat{\psi}_s}{r\,(1/q-n/m)}\simeq 0 \tag{100}$$

that satisfies

$$\hat{\psi}_s(0)=0, \tag{101}$$

$$\hat{\psi}_s(r_s)=1, \tag{102}$$

$$\hat{\psi}_s(r\geq r_w)=0. \tag{103}$$

Note that Equation (100) is singular at the so-called *resonant* magnetic flux-surface, radius $r=r_s$, at which

$$q(r_s)=\frac{m}{n}. \tag{104}$$

At the resonant surface, $\mathbf{k}\cdot\mathbf{B}=0$, where $\mathbf{k}=(0,\,m/r,\,-n/R_0)$ is the wavevector of the tearing perturbation. The value of $\psi(r,t)$ at the resonant surface—namely, $\Psi_s(t)$—is known as the *reconnected magnetic flux* (Fitzpatrick 1993). Note that Ψ_s is, in general, a complex quantity. Let $\rho=(r-r_s)/r_s$. The solution of Equation (100) in the vicinity of the resonant surface is

$$\hat{\psi}_s(\rho)=1+\Delta_{s+}\,\rho+\alpha_s\,\rho\,\ln|\rho|+\mathcal{O}(\rho^2) \tag{105}$$

for $\rho>0$, and

$$\hat{\psi}_s(\rho)=1+\Delta_{s-}\,\rho+\alpha_s\,\rho\,\ln|\rho|+\mathcal{O}(\rho^2) \tag{106}$$

for $\rho < 0$. Here,

$$\alpha_s = -\left(\frac{r\,q\,J_z'}{s}\right)_{r=r_s}, \tag{107}$$

$$s(r) = \frac{r\,q'}{q}. \tag{108}$$

Moreover, the real parameters Δ_{s+} and Δ_{s-} are fully determined by Equation (100) and the boundary conditions (101)–(103). Note that, in general, ψ is continuous across the resonant surface (in accordance with Maxwell's equations), whereas $\partial\psi/\partial r$ is discontinuous. The discontinuity in $\partial\psi/\partial r$ implies the presence of a radially-thin, helical, current sheet at the resonant surface. This current sheet can only be resolved by retaining more terms in the perturbed plasma equation of motion, (96). (See Section 5.)

The complex quantity

$$\Delta\Psi_s = \left[r\,\frac{\partial\psi}{\partial r}\right]_{r_{s-}}^{r_{s+}} \tag{109}$$

parameterizes the amplitude and phase of the current sheet flowing at the resonant surface. Matching the solutions in the so-called *inner region* (i.e., the region of the plasma in the immediate vicinity of the resonant surface) and the so-called *outer region* (i.e., everywhere in the plasma other than the inner region) with the help of Equations (99), (105), and (106), we obtain

$$\Delta\Psi_s = E_{ss}\,\Psi_s, \tag{110}$$

where $E_{ss} = \Delta_{s+} - \Delta_{s-}$ is a real quantity that is known as the *tearing stability index* (Furth, et al. 1963).

3.6 Solution in Presence of Resistive Wall

Suppose, now, that the wall at $r = r_w$ possesses a finite electrical resistivity, but is surrounded by a second perfectly-conducting wall located at radius $r_c > r_w$. The solution in the outer region can be written

$$\psi(r,t) = \Psi_s(t)\,\hat{\psi}_s(r) + \Psi_w(t)\,\hat{\psi}_w(r), \tag{111}$$

where the real function $\hat{\psi}_s(r)$ is specified in the previous section, and the real function $\hat{\psi}_w(r)$ is a solution of

$$\frac{d^2\hat{\psi}_w}{dr^2} + \frac{1}{r}\,\frac{d\hat{\psi}_w}{dr} - \frac{m^2}{r^2}\,\hat{\psi}_w - \frac{J_z'\,\hat{\psi}_w}{r\,(1/q - n/m)} \simeq 0 \tag{112}$$

that satisfies

$$\hat{\psi}_w(r \leq r_s) = 0, \tag{113}$$

$$\hat{\psi}_w(r_w) = 1, \tag{114}$$

$$\hat{\psi}_w(r \geq r_c) = 0. \tag{115}$$

It is easily seen that

$$\hat{\psi}_w(r_w < r < r_c) = \frac{(r/r_c)^{-m} - (r/r_c)^m}{(r_w/r_c)^{-m} - (r_w/r_c)^m}. \tag{116}$$

In general, ψ is continuous across the wall (in accordance with Maxwell's equations), whereas $\partial\psi/\partial r$ is discontinuous. The discontinuity in $\partial\psi/\partial r$ is caused by a helical current sheet induced in the wall. The complex quantity $\Psi_w(t)$ determines the amplitude and phase of the perturbed magnetic flux that penetrates the wall. The complex quantity

$$\Delta\Psi_w = \left[r\,\frac{\partial\psi}{\partial r}\right]_{r_{w-}}^{r_{w+}} \tag{117}$$

parameterizes the amplitude and phase of the helical current sheet flowing in the wall. Simultaneously matching the outer solution (111) across the resonant surface and the wall yields

$$\Delta\Psi_s = E_{ss}\,\Psi_s + E_{sw}\,\Psi_w, \tag{118}$$

$$\Delta\Psi_w = E_{ws}\,\Psi_s + E_{ww}\,\Psi_w \tag{119}$$

(Fitzpatrick 1993). Here,

$$E_{ww} = \left[r\,\frac{d\hat{\psi}_w}{dr}\right]_{r_{w-}}^{r_{w+}}, \tag{120}$$

$$E_{sw} = \left[r\,\frac{d\hat{\psi}_w}{dr}\right]_{r=r_{s+}}, \tag{121}$$

$$E_{ws} = -\left[r\,\frac{d\hat{\psi}_s}{dr}\right]_{r=r_{w-}} \tag{122}$$

are real quantities determined by the solutions of Equations (100) and (112) in the outer region.

Equations (100) and (112) can be combined to give

$$\frac{d}{dr}\left(\hat{\psi}_s\,r\,\frac{d\hat{\psi}_w}{dr} - \hat{\psi}_w\,r\,\frac{d\hat{\psi}_s}{dr}\right) = 0. \tag{123}$$

If we integrate the previous equation from $r = r_{s+}$ to $r = r_{w-}$, making use of Equations (102), (103), (113), (114), (121), and (122), then we obtain

$$E_{sw} = E_{ws} \tag{124}$$

(Fitzpatrick 1993).

3.7 Resistive Wall Physics

It is clear from Equation (93) that

$$\delta\mathbf{A} \simeq \psi\,\mathbf{e}_z, \tag{125}$$

where $\delta\mathbf{A}$ is the perturbed magnetic vector potential. Hence, the perturbed electric field can be written

$$\delta\mathbf{E} \simeq -\nabla\delta\Phi - \frac{\partial\psi}{\partial t}\,\mathbf{e}_z, \tag{126}$$

where $\delta\Phi$ is the perturbed electric scalar potential. Assuming that $|\nabla\delta\Phi| \sim |\partial\psi/\partial t|$, the z-component of the previous equation yields

$$\delta E_z \simeq -\frac{\partial\psi}{\partial t}, \tag{127}$$

where use has been made of the large aspect-ratio orderings (92). Within the wall, we can write

$$\delta E_z \simeq -\frac{d\Psi_w}{dt}. \tag{128}$$

Here, we are making use of the so-called *thin wall approximation*, according to which ψ is assumed to only vary weakly in r across the wall. Ohm's law implies that

$$\delta j_z = \frac{\delta E_z}{\eta_w} = -\frac{1}{\eta_w}\,\frac{d\Psi_w}{dt} \tag{129}$$

within the wall, where η_w is the wall resistivity. Equations (95) and (117) yield

$$\Delta\Psi_w = -\mu_0\int_{r_{w-}}^{r_{w+}} r\,\delta j_z\,dr. \tag{130}$$

Suppose that the radial thickness of the wall is $\delta_w \ll r_w$. The previous two equations give

$$\Delta\Psi_w = \tau_w\,\frac{d\Psi_w}{dt}, \tag{131}$$

where

$$\tau_w = \frac{\mu_0\,r_w\,\delta_w}{\eta_w} \tag{132}$$

is the so-called *time constant* of the wall (Nave & Wesson 1990; Fitzpatrick 1993). The thin wall approximation is valid as long as

$$\frac{r_w}{\delta_w} \gg \tau_w\left|\frac{d\ln\Psi_w}{dt}\right|. \tag{133}$$

3.8 Resonant Layer Physics

By analogy with Equation (131), we can write

$$\Delta\Psi_s = \tau_s\left(\frac{d\Psi_s}{dt} + \mathrm{i}\,\varpi_s\,\Psi_s\right). \tag{134}$$

Here, τ_s is the *reconnection time* (i.e., the typical timescale on which magnetic reconnection takes place in the resonant layer surrounding the resonant surface), whereas ϖ_s is the so-called *natural frequency*. The natural frequency is the phase velocity of the tearing mode when the wall is perfectly conducting. The natural frequency is non-zero because the reconnected flux in the resonant layer is convected by the local plasma flow (Fitzpatrick 1993).

3.9 Solution in Presence of External Magnetic Field-Coil

Suppose that the perfectly-conducting wall at $r = r_c$ is replaced by a radially-thin, magnetic field-coil that carries a helical current possessing m periods in the poloidal direction, and n periods in the toroidal direction. Let the current density in the field-coil take the form

$$\delta j_z = \frac{I_c(t)}{r_c}\,\delta(r-r_c)\,\mathrm{e}^{\,\mathrm{i}\,(m\,\theta-n\,\varphi)}. \tag{135}$$

Here, the complex quantity $I_c(t)$ parameterizes the amplitude and phase of the helical current flowing in the field-coil.

The solution in the outer region can be written

$$\psi(r,t) = \Psi_s(t)\,\hat{\psi}_s(r) + \Psi_w(t)\,\hat{\psi}_w(r) + \Psi_c\,\hat{\psi}_c(r), \tag{136}$$

where the real functions $\hat{\psi}_s(r)$ and $\hat{\psi}_w(r)$ are specified in Sections 3.5 and 3.6, respectively. Moreover, the real function $\hat{\psi}_c(r)$ is a solution of

$$\frac{d^2\hat{\psi}_c}{dr^2} + \frac{1}{r}\frac{d\hat{\psi}_c}{dr} - \frac{m^2}{r^2}\,\hat{\psi}_c \simeq 0 \tag{137}$$

that satisfies

$$\hat{\psi}_c(r\leq r_w) = 0, \tag{138}$$

$$\hat{\psi}_c(r_c) = 1, \tag{139}$$

$$\hat{\psi}_c(\infty) = 0. \tag{140}$$

It is easily seen that

$$\hat{\psi}_c(r\leq r_w) = 0, \tag{141}$$

$$\hat{\psi}_c(r_w < r \leq r_c) = \frac{(r/r_w)^m - (r/r_w)^{-m}}{(r_c/r_w)^m - (r_c/r_w)^{-m}}, \tag{142}$$

$$\hat{\psi}_c(r > r_c) = \left(\frac{r}{r_c}\right)^{-m}. \tag{143}$$

In general, ψ is continuous across the field-coil (in accordance with Maxwell's equations), whereas $\partial\psi/\partial r$ is discontinuous. The discontinuity in $\partial\psi/\partial r$ is caused by the helical currents flowing in the coil. The complex quantity $\Psi_c(t)$ determines the amplitude and phase of the perturbed magnetic flux at the coil. The complex quantity

$$\Delta\Psi_c = \left[r\,\frac{\partial\psi}{\partial r}\right]_{r_{c-}}^{r_{c+}} \tag{144}$$

parameterizes the amplitude and phase of the helical current sheet flowing in the coil. It follows from Equations (94), (95), and (135) that

$$\Delta\Psi_c = -\mu_0\, I_c. \tag{145}$$

Simultaneously matching the outer solution across the resonant surface, the wall, and the field-coil, we obtain

$$\Delta\Psi_s = E_{ss}\,\Psi_s + E_{sw}\,\Psi_w, \tag{146}$$

$$\Delta\Psi_w = E_{ws}\,\Psi_s + E_{ww}\,\Psi_w + E_{wc}\,\Psi_c, \tag{147}$$

$$\Delta\Psi_c = E_{cw}\,\Psi_w + E_{cc}\,\Psi_c. \tag{148}$$

Here,

$$E_{cc} = \left[r\,\frac{d\hat{\psi}_c}{dr}\right]_{r_{c-}}^{r_{c+}} = -\frac{2m}{1-(r_w/r_c)^{2m}}, \tag{149}$$

$$E_{wc} = \left[r\,\frac{d\hat{\psi}_c}{dr}\right]_{r_{w+}} = \frac{2m\,(r_w/r_c)^m}{1-(r_w/r_c)^{2m}}, \tag{150}$$

$$E_{cw} = -\left[r\,\frac{d\hat{\psi}_w}{dr}\right]_{r_{c-}} = \frac{2m\,(r_w/r_c)^m}{1-(r_w/r_c)^{2m}}, \tag{151}$$

where use has been made of Equations (116), (142), and (143).

3.10 Electromagnetic Torques

The flux-surface integrated poloidal and toroidal electromagnetic torque densities acting on the plasma can be written

$$T_\theta(r) = \langle r\,\mathbf{j}\times\mathbf{B}\cdot\mathbf{e}_\theta\rangle \tag{152}$$

$$T_\varphi(r) = \langle R_0\,\mathbf{j}\times\mathbf{B}\cdot\mathbf{e}_z\rangle\,, \tag{153}$$

respectively, where

$$\langle \cdots \rangle \equiv \oint \oint r\, R_0\, (\cdots)\, d\theta\, d\varphi \tag{154}$$

is a flux-surface integration operator. However, according to Equation (80),

$$\mathbf{j} \times \mathbf{B} \simeq \nabla p. \tag{155}$$

Given that the scalar pressure is a single-valued function of θ and φ, it immediately follows that $T_\theta = T_\varphi = 0$ throughout the plasma (Fitzpatrick 1993). The only exception to this rule occurs in the immediate vicinity of the resonant surface, where Equation (155) breaks down. It follows that we can write

$$T_\theta(r) = T_{\theta\, s}\, \delta(r - r_s), \tag{156}$$

$$T_\varphi(r) = T_{\varphi\, s}\, \delta(r - r_s), \tag{157}$$

where

$$T_{\theta\, s} = \frac{1}{4} \int_{r_{s-}}^{r_{s+}} \oint \oint R_0\, r^{\,2}\, (\delta j_z\, \delta B_r^\ast + \delta j_z^\ast\, \delta B_r)\, dr\, d\theta\, d\varphi, \tag{158}$$

$$T_{\varphi\, s} = -\frac{1}{4} \int_{r_{s-}}^{r_{s+}} \oint \oint R_0^{\,2}\, r\, (\delta j_\theta\, \delta B_r^\ast + \delta j_\theta^\ast\, \delta B_r)\, dr\, d\theta\, d\varphi \tag{159}$$

are the net poloidal and toroidal torques, respectively, acting at the resonant surface. Note that the zeroth-order (in perturbed quantities) torques are zero because $B_r = 0$. Furthermore, the linear (in perturbed quantities) torques average to zero over the flux-surface. Hence, the largest non-zero torques are *quadratic* in perturbed quantities. According to Equation (83),

$$\nabla \cdot \delta\mathbf{j} = \mathrm{i} \left(\frac{m}{r}\, \delta j_\theta - \frac{n}{R_0}\, \delta j_z \right) = 0, \tag{160}$$

where $\delta\mathbf{j}$ is the density of the current sheet flowing at the resonant surface, and we have made use of the fact that δj_r is negligible in a radially-thin current sheet (Fitzpatrick 1993). It follows from the previous three equations that

$$T_{\varphi\, s} = -\frac{n}{m}\, T_{\theta\, s}. \tag{161}$$

Finally, Equations (93), (95), (158), and (161) imply that

$$T_{\theta\, s} = -\frac{2\pi^2\, R_0\, m}{\mu_0}\, \mathrm{Im}(\mathit{\Delta\Psi}_s\, \mathit{\Psi}_s^{\ast}), \tag{162}$$

$$T_{\varphi\, s} = \frac{2\pi^2\, R_0\, n}{\mu_0}\, \mathrm{Im}(\mathit{\Delta\Psi}_s\, \mathit{\Psi}_s^{\ast}) \tag{163}$$

(Fitzpatrick 1993).

By analogy with the previous analysis, the net poloidal and toroidal electromagnetic torques acting on the resistive wall are

$$T_{\theta\,w} = -\frac{2\pi^2\,R_0\,m}{\mu_0}\,\mathrm{Im}(\mathit{\Delta\Psi}_w\,\Psi_w^{\,*}), \tag{164}$$

$$T_{\varphi\,w} = \frac{2\pi^2\,R_0\,n}{\mu_0}\,\mathrm{Im}(\mathit{\Delta\Psi}_w\,\Psi_w^{\,*}), \tag{165}$$

respectively. Finally, the net poloidal and toroidal electromagnetic torques acting on the magnetic field-coil are

$$T_{\theta\,c} = -\frac{2\pi^2\,R_0\,m}{\mu_0}\,\mathrm{Im}(\mathit{\Delta\Psi}_c\,\Psi_c^{\,*}), \tag{166}$$

$$T_{\varphi\,c} = \frac{2\pi^2\,R_0\,n}{\mu_0}\,\mathrm{Im}(\mathit{\Delta\Psi}_c\,\Psi_c^{\,*}), \tag{167}$$

respectively.

It follows from Equations (146)–(148) that

$$\mathrm{Im}(\mathit{\Delta\Psi}_s\,\Psi_s^{\,*}) = E_{sw}\,\mathrm{Im}(\Psi_w\,\Psi_s^{\,*}), \tag{168}$$

$$\mathrm{Im}(\mathit{\Delta\Psi}_w\,\Psi_w^{\,*}) = E_{ws}\,\mathrm{Im}(\Psi_s\,\Psi_w^{\,*}) + E_{wc}\,\mathrm{Im}(\Psi_c\,\Psi_w^{\,*}), \tag{169}$$

$$\mathrm{Im}(\mathit{\Delta\Psi}_c\,\Psi_c^{\,*}) = E_{cw}\,\mathrm{Im}(\Psi_w\,\Psi_c^{\,*}). \tag{170}$$

Thus,

$$\begin{aligned}\mathrm{Im}(\mathit{\Delta\Psi}_s\,\Psi_s^{\,*}) + \mathrm{Im}(\mathit{\Delta\Psi}_w\,\Psi_w^{\,*}) + \mathrm{Im}(\mathit{\Delta\Psi}_c\,\Psi_c^{\,*}) &= (E_{sw}-E_{ws})\,\mathrm{Im}(\Psi_w\,\Psi_s^{\,*})\\ &\quad + (E_{wc}-E_{cw})\,\mathrm{Im}(\Psi_c\,\Psi_w^{\,*}).\end{aligned} \tag{171}$$

However, according to Equations (124), (150), and (151), $E_{sw}=E_{ws}$ and $E_{wc}=E_{cw}$. We deduce that

$$\mathrm{Im}(\mathit{\Delta\Psi}_s\,\Psi_s^{\,*}) + \mathrm{Im}(\mathit{\Delta\Psi}_w\,\Psi_w^{\,*}) + \mathrm{Im}(\mathit{\Delta\Psi}_c\,\Psi_c^{\,*}) = 0. \tag{172}$$

Hence, Equations (162)–(167) yield

$$T_{\theta\,s} + T_{\theta\,w} + T_{\theta\,c} = 0, \tag{173}$$

$$T_{\varphi\,s} + T_{\varphi\,w} + T_{\varphi\,c} = 0. \tag{174}$$

In other words, the plasma/wall/field-coil system exerts zero net poloidal electromagnetic torque, and zero net toroidal electromagnetic torque, on itself.

3.11 Plasma Angular Equations of Motion

The sum of the electron fluid equation of motion, (69), and the ion fluid equation of motion, (72), yields the plasma equation of motion,

$$\rho\,\frac{\partial\mathbf{V}_i}{\partial t} + \rho\,(\mathbf{V}_i\cdot\nabla)\mathbf{V}_i + [\delta_i^{-2}]\,\nabla p + [\zeta_i^{-1}]\,\nabla\cdot\boldsymbol{\pi}_i - [\delta_i^{-2}]\,\mathbf{j}\times\mathbf{B} = 0, \tag{175}$$

where $\rho = m_i\, n$ is the plasma mass density. Here, we have neglected electron inertial terms with respect to ion inertial terms (because the former are order m_e/m_i smaller than the latter). We have also neglected electron viscosity terms with respect to ion viscosity terms (because the former are order $\sqrt{m_e/m_i}$ smaller than the latter). (See Sections 2.3 and 2.4.) Recall that the factors in square brackets are there to remind us that the terms they precede are larger than the other terms in the equation (by the corresponding factors within the brackets).

Taking the scalar product of $r\,\mathbf{e}_\theta$ with the plasma equation of motion, (175), and integrating around magnetic flux-surfaces, we obtain

$$\left\langle \rho\,\frac{\partial(r\,V_{\theta\,i})}{\partial t}\right\rangle + \left\langle \rho\left[V_{r\,i}\,\frac{\partial(r\,V_{\theta\,i})}{\partial r} + V_{z\,i}\,\frac{\partial(r\,V_{\theta\,i})}{\partial z}\right]\right\rangle + [\zeta_i^{-1}]\,\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle = [\delta_i^{-2}]\,T_{\theta\,s}\,\delta(r-r_s), \tag{176}$$

where use has been made of Equations (152) and (156). Note that the dominant ∇p term has averaged to zero. Moreover, because $T_{\theta\,s}$ is quadratic in perturbed quantities (see Section 3.10), it is plausible that the right-hand side of the previous equation is no longer a dominant term [i.e., it is not $\mathcal{O}(\delta_i^{-2})$ larger than the other terms in the equation]. To be more exact, if $|\delta\mathbf{B}|/|\mathbf{B}| \sim \delta_i$ then the right-hand side ceases to be a dominant term in the equation.

The largest contribution to $\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle$ comes from the ion parallel viscosity tensor acting on the zeroth-order (in perturbed quantities) ion flow. (See Section 2.3.) However, this contribution averages to zero in a cylindrical plasma, in which B_z is constant on magnetic flux-surfaces. Of course, in reality, in a large aspect-ratio tokamak plasma, B_z varies slightly around magnetic flux-surfaces, being larger on the inboard side of the torus than on the outboard side (Wesson 2011). This variation gives rise to a residual contribution to $\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle$. It is plausible that this contribution is not a dominant term in Equation (176) [i.e., it is not $\mathcal{O}(\zeta_i^{-1})$ larger than the other terms in the equation]. To be more exact, if the residual contribution is $\mathcal{O}(\zeta_i)$ times smaller than the non-averaged contribution then $\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle$ ceases to be a dominant term in the equation.

According to the previous discussion, it is plausible that all terms appearing in Equation (176) are of approximately equal magnitude when the inertial and viscous terms are evaluated with the zeroth-order ion flow, which can be written

$$\mathbf{V}_i = r\,\Omega_\theta(r,t)\,\mathbf{e}_\theta + R_0\,\Omega_\varphi(r,t)\,\mathbf{e}_z. \tag{177}$$

Here, $\Omega_\theta(r,t)$ and $\Omega_\varphi(r,t)$ are the ion poloidal and toroidal angular velocity profiles, respectively. When $\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle$ is evaluated with the ion parallel viscosity tensor, and the zeroth-order ion flow, it gives rise to a term that acts to relax the ion poloidal angular velocity profile toward a so-called *neoclassical profile* determined by ion density and temperature gradients (Stix 1973; Hirshman & Sigmar 1981). In principle, the contribution of the ion perpendicular viscosity tensor to $\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle$ is $\mathcal{O}(\zeta_i^2)$ smaller than the contribution of the ion parallel viscosity tensor. (See Section 2.3.) In practice, the magnitude of the ion perpendicular viscosity tensor is greatly enhanced above the value predicted by the Braginskii equations by the action of small-scale plasma turbulence (Wesson 2011). Hence, it is reasonable to include the contribution of the enhanced perpendicular viscosity tensor to $\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle$. Our model form for $\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle$ is written

$$\langle r\,\nabla\cdot\boldsymbol{\pi}_i\cdot\mathbf{e}_\theta\rangle = 4\pi^2\,R_0\left[\frac{\rho}{\tau_\theta}\,r^{\,3}\,(\varOmega_\theta-\varOmega_{\theta\,\rm nc})-\frac{\partial}{\partial r}\left(\mu_\perp\,r^{\,3}\,\frac{\partial\varOmega_\theta}{\partial r}\right)\right]. \tag{178}$$

Here, $\rho(r)$ is the equilibrium plasma mass density profile, $\tau_\theta(r)$ is the *poloidal flow-damping time* (which is generally of order τ_i), $\varOmega_{\theta\,\rm nc}(r)$ is the neoclassical poloidal angular velocity profile, and $\mu_\perp(r)$ is a phenomenological perpendicular viscosity due to plasma turbulence.

When evaluated with the zeroth-order ion flow, Equation (176) reduces to

$$4\pi^2\,R_0\left[\rho\,r^{\,3}\,\frac{\partial\varOmega_\theta}{\partial t}+\frac{\rho}{\tau_\theta}\,r^{\,3}\,(\varOmega_\theta-\varOmega_{\theta\,\rm nc})-\frac{\partial}{\partial r}\left(\mu_\perp\,r^{\,3}\,\frac{\partial\varOmega_\theta}{\partial r}\right)\right]=T_{\theta\,s}\,\delta(r-r_s)+S_\theta, \tag{179}$$

where use has been made of Equation (178). Here, we have added a source term, $S_\theta(r)$, to the equation in order to account for the equilibrium poloidal flow. Let $\varOmega_{\theta\,0}(r)$ be the unperturbed (by the tearing mode) ion poloidal angular velocity profile. It follows that

$$S_\theta = 4\pi^2\,R_0\left[\frac{\rho}{\tau_\theta}\,r^{\,3}\,(\varOmega_{\theta\,0}-\varOmega_{\theta\,\rm nc})-\frac{d}{dr}\left(\mu_\perp\,r^{\,3}\,\frac{d\varOmega_{\theta\,0}}{dr}\right)\right]. \tag{180}$$

Hence, writing

$$\varOmega_\theta(r,t)=\varOmega_{\theta\,0}(r)+\Delta\varOmega_\theta(r,t), \tag{181}$$

where $\Delta\varOmega_\theta(r,t)$ is the modification to the ion poloidal angular velocity profile induced by the poloidal electromagnetic torque that develops at the resonant surface, Equation (179) yields

$$4\pi^2\,R_0\left[\rho\,r^{\,3}\,\frac{\partial\Delta\varOmega_\theta}{\partial t}+\frac{\rho}{\tau_\theta}\,r^{\,3}\,\Delta\varOmega_\theta-\frac{\partial}{\partial r}\left(\mu_\perp\,r^{\,3}\,\frac{\partial\Delta\varOmega_\theta}{\partial r}\right)\right]=T_{\theta\,s}\,\delta(r-r_s). \tag{182}$$

There is one further refinement that we can make to the previous equation. It turns out that neoclassical poloidal flow damping gives rise to an enhancement of poloidal ion inertia by a factor of $1+2\,q^{\,2}$ (Hirshman 1978). Hence, our final version of the perturbed plasma poloidal angular equation of motion becomes

$$4\pi^2\,R_0\left[(1+2\,q^{\,2})\,\rho\,r^{\,3}\,\frac{\partial\Delta\varOmega_\theta}{\partial t}+\frac{\rho}{\tau_\theta}\,r^{\,3}\,\Delta\varOmega_\theta-\frac{\partial}{\partial r}\left(\mu_\perp\,r^{\,3}\,\frac{\partial\Delta\varOmega_\theta}{\partial r}\right)\right]=T_{\theta\,s}\,\delta(r-r_s). \tag{183}$$

Taking the scalar product of $R_0\,\mathbf{e}_z$ with the plasma equation of motion, (175), and integrating around magnetic flux-surfaces, similar arguments to those just employed for the ion poloidal flow allow us to write the zeroth-order ion toroidal angular velocity profile in the form

$$\varOmega_\varphi(r,t)=\varOmega_{\varphi\,0}(r)+\Delta\varOmega_\varphi(r,t), \tag{184}$$

where $\varOmega_{\varphi\,0}(r)$ is the unperturbed (by the tearing mode) ion toroidal angular velocity profile, and $\Delta\varOmega_\theta(r,t)$ is the modification to this profile induced by the toroidal electromagnetic torque that develops at the resonant surface. The perturbed ion toroidal angular equation of motion is written

$$4\pi^2 R_0^3 \left[\rho\, r\, \frac{\partial \Delta\Omega_\varphi}{\partial t} - \frac{\partial}{\partial r}\left(\mu_\perp\, r\, \frac{\partial \Delta\Omega_\varphi}{\partial r}\right)\right] = T_{\varphi\, s}\, \delta(r - r_s). \tag{185}$$

Note that the ion parallel viscosity tensor does not give rise to damping of the toroidal flow profile. Furthermore, there is no neoclassical enhancement of the plasma toroidal ion inertia.

Equations (183) and (185) are subject to the boundary conditions

$$\frac{\partial \Delta\Omega_\theta(0,t)}{\partial r} = \frac{\partial \Delta\Omega_\varphi(0,t)}{\partial r} = 0, \tag{186}$$

$$\Delta\Omega_\theta(a,t) = \Delta\Omega_\varphi(a,t) = 0. \tag{187}$$

The boundary conditions (186) merely ensure that the ion angular velocities remain finite at the magnetic axis. On the other hand, the boundary conditions (187) are a consequence of the action of *charge exchange* with electrically neutral particles emitted isotropically from the wall in the edge regions of the plasma (Brau, et al. 1983; Fitzpatrick 1993; Monier-Garbet, et al. 1997). Charge exchange with neutrals gives rise to dominant damping torques acting at the edge of the plasma that relax the edge ion angular velocities toward particular values. Moreover, the electromagnetic torques that develop at the resonant surface are not large enough, compared with the charge-exchange torques, to significantly modify the edge ion angular velocities (Fitzpatrick 1993).

3.12 Solution of Plasma Angular Equations of Motion

In general, the perturbed angular velocity profiles, $\Delta\Omega_\theta(r,t)$ and $\Delta\Omega_\varphi(r,t)$, are localized in the vicinity of the resonant surface (Fitzpatrick 1993). Hence, it is reasonable to express the perturbed angular equations of motion, (183) and (185), in the simplified forms

$$4\pi^2 R_0 \left[(1 + 2\, q_s^2)\, \rho_s\, r^3\, \frac{\partial \Delta\Omega_\theta}{\partial t} + \frac{\rho_s}{\tau_{\theta\, s}}\, r^3\, \Delta\Omega_\theta - \mu_s \frac{\partial}{\partial r}\left(r^3\, \frac{\partial \Delta\Omega_\theta}{\partial r}\right)\right] = T_{\theta\, s}\, \delta(r - r_s), \tag{188}$$

$$4\pi^2 R_0^3 \left[\rho_s\, r\, \frac{\partial \Delta\Omega_\varphi}{\partial t} - \mu_s \frac{\partial}{\partial r}\left(r\, \frac{\partial \Delta\Omega_\varphi}{\partial r}\right)\right] = T_{\varphi\, s}\, \delta(r - r_s), \tag{189}$$

where $q_s = q(r_s)$, $\rho_s = \rho(r_s)$, $\tau_{\theta\, s} = \tau_\theta(r_s)$, and $\mu_s = \mu_\perp(r_s)$.

Let us write

$$\Delta\Omega_\theta(r,t) = -\frac{1}{m} \sum_{p=1,\infty} \alpha_p(t)\, \frac{y_p(r/a)}{y_p(r_s/a)}, \tag{190}$$

$$\Delta\Omega_\varphi(r,t) = \frac{1}{n} \sum_{p=1,\infty} \beta_p(t)\, \frac{z_p(r/a)}{z_p(r_s/a)}, \tag{191}$$

where

$$y_p(r) = \frac{J_1(j_{1\,p}\,r/a)}{r/a}, \tag{192}$$

$$z_p(r) = J_0(j_{0\,p}\,r/a). \tag{193}$$

Here, $J_m(z)$ is a Bessel function, and $j_{m,p}$ denotes its pth zero. Note that Equations (190)–(193) automatically satisfy the boundary conditions (186) and (187).

It is easily demonstrated that

$$\frac{d}{dr}\left(r^3\,\frac{dy_p}{dr}\right) = -\frac{j_{1\,p}^2\,r^3\,y_p}{a^2}, \tag{194}$$

$$\frac{d}{dr}\left(r\,\frac{dz_p}{dr}\right) = -\frac{j_{0\,p}^2\,r\,z_p}{a^2}, \tag{195}$$

and

$$\int_0^a r^3\,y_p(r)\,y_q(r)\,dr = \frac{a^4}{2}\,[J_2(j_{1\,p})]^2\,\delta_{p,q}, \tag{196}$$

$$\int_0^a r\,z_p(r)\,z_q(r)\,dr = \frac{a^2}{2}\,[J_1(j_{0\,p})]^2\,\delta_{p,q}. \tag{197}$$

Equations (188) and (189) yield

$$(1+2\,q_k^2)\,\frac{d\alpha_p}{dt} + \left(\frac{1}{\tau_{\theta\,s}} + \frac{j_{1,p}^2}{\tau_M}\right)\alpha_p = \frac{m^2\,[y_p(r_s/a)]^2}{\tau_A^2\,\epsilon_a^2\,[J_2(j_{1,p})]^2}\,\mathrm{Im}(\Delta\hat{\Psi}_s\,\hat{\Psi}_s^{\,*}), \tag{198}$$

$$\frac{d\beta_p}{dt} + \frac{j_{0,p}^2}{\tau_M}\,\beta_p = \frac{n^2\,[z_p(r_s/a)]^2}{\tau_A^2\,[J_1(j_{0,p})]^2}\,\mathrm{Im}(\Delta\hat{\Psi}_s\,\hat{\Psi}_s^{\,*}). \tag{199}$$

Here, $\tau_M = \rho_s\,a^2/\mu_s$ is the *momentum confinement time*, $\tau_A = (\mu_0\,\rho_s\,a^2/B_z^2)^{1/2}$ is the *Alfvén time*, $\epsilon_a = a/R_0 \ll 1$ is the *inverse aspect-ratio* of the plasma, $\Delta\Psi_s = R_0\,B_z\,\Delta\hat{\Psi}_s$, and $\Psi_s = R_0\,B_z\,\hat{\Psi}_s$. Moreover, use has been made of Equations (162), (163), and (190)–(197).

3.13 No-Slip Constraint

The reconnected magnetic flux at the resonant surface is assumed to convected by the local plasma flow. (See Section 3.8.) Hence, changes in the plasma flow at the resonant surface induced by the electromagnetic torques that develop in the vicinity of this surface will modify the convection velocity. Consequently, the natural frequency of the reconnected flux can be written

$$\varpi_s(t) = \varpi_{s\,0} + (\mathbf{k}\cdot\Delta\mathbf{V}_i)_{r=r_s} = \varpi_{s\,0} + m\,\Delta\Omega_\theta(r_s,t) - n\,\Delta\Omega_\varphi(r_s,t), \tag{200}$$

where $\varpi_{s\,0}$ is the unperturbed natural frequency. The previous expression is known as the *no-slip constraint* (Fitzpatrick 1993), and has been verified experimentally (Vahala, et al. 1980). It follows from Equations (190) and (191) that

$$\varpi_s(t) = \varpi_{s\,0} - \sum_{p=1,\infty}\left[\alpha_p(t) + \beta_p(t)\right]. \tag{201}$$

3.14 Final Equations

Equations (124), (131), (134), (145)–(151), (198), (199), and (201) can be combined to give the following complete set of equations that determine the time evolution of the reconnected magnetic flux at the resonant surface:

$$\tau_s\left(\frac{d\hat{\Psi}_s}{dt}+{\rm i}\,\varpi_s\,\hat{\Psi}_s\right)=E_{ss}\,\hat{\Psi}_s+E_{sw}\,\hat{\Psi}_w, \tag{202}$$

$$\tau_w\,\frac{d\hat{\Psi}_w}{dt}=E_{sw}\,\hat{\Psi}_s+\tilde{E}_{ww}\,\hat{\Psi}_w+\hat{I}_c, \tag{203}$$

$$\varpi_s(t)=\varpi_{s\,0}-\sum_{p=1,\infty}\left[\alpha_p(t)+\beta_p(t)\right], \tag{204}$$

$$(1+2\,q_k^2)\,\frac{d\alpha_p}{dt}+\left(\frac{1}{\tau_{\theta\,s}}+\frac{j_{1,p}^{\,2}}{\tau_M}\right)\alpha_p=\frac{m^2\,[y_p(r_s/a)]^{\,2}}{\tau_A^{\,2}\,\epsilon_a^{\,2}\,[J_2(j_{1,p})]^{\,2}}\,E_{sw}\,{\rm Im}(\hat{\Psi}_w\,\hat{\Psi}_s^{\,\ast}), \tag{205}$$

$$\frac{d\beta_p}{dt}+\frac{j_{0,p}^{\,2}}{\tau_M}\,\beta_p=\frac{n^2\,[z_p(r_s/a)]^{\,2}}{\tau_A^{\,2}\,[J_1(j_{0,p})]^{\,2}}\,E_{sw}\,{\rm Im}(\hat{\Psi}_w\,\hat{\Psi}_s^{\,\ast}). \tag{206}$$

Here,

$$\hat{I}_c=\frac{\mu_0\,I_c}{R_0\,B_z}\left(\frac{r_w}{r_c}\right)^m, \tag{207}$$

$$\tilde{E}_{ww}=E_{ww}+\frac{2m\,(r_w/r_c)^{2m}}{1-(r_w/r_c)^{2m}}. \tag{208}$$

4 Reduced Drift-MHD Resonant Response Model

4.1 Introduction

Equations (202)–(206) contain two quantities—namely, the reconnection time, τ_s, and the unperturbed natural frequency, $\varpi_{s\,0}$—that can only be determined by solving for the plasma response in the radially thin resonant layer surrounding the resonant surface. Unfortunately, determining the resonant plasma response is not a straightforward task. The problem is that the small characteristic lengthscale in the resonant region (i.e., the resonant layer width) causes the fundamental ordering parameters δ_e and δ_i (see Section 2.4) to take values that are of order unity. Consequently, most of the simplifications described in Section 2.5 are not applicable in the resonant layer. In particular, many of the terms in the Braginskii equations (see Section 2.3) that are negligible in the outer region (e.g., plasma resistivity) have to be taken into account in the resonant layer.

A tearing mode is a modified shear-Alfvén wave (Hazeltine & Meiss 1985), and has very little connection with a compressible-Alfvén wave. For instance, the resonance condition, $\mathbf{k}\cdot\mathbf{B}=0$, which determines the position of the resonant surface, is identical to the resonance condition for a shear-Alfvén wave, but differs substantially from that of a compressible-Alfvén wave (Fitzpatrick 2015). Hence, one very effective way of simplifying the resonant layer equations is to remove the physics of a compressible-Alfvén wave from them altogether, in the process converting a drift-magnetohydrodynamical (MHD) response model into a so-called *reduced* drift-MHD response model (Strauss 1976; Hazeltine, et al. 1985; Furuya, et al. 2003). The aim of this section is to describe the reduction process (which essentially boils down to arguing that the divergence of the plasma flow is small), and to derive the reduced drift-MHD equations that will subsequently be used to model the resonant response of the plasma.

4.2 Drift-MHD Equations

It is helpful to define the following parameters:

$$n_0 = n(r_s), \tag{209}$$

$$p_0 = p(r_s), \tag{210}$$

$$\eta_e = \left.\frac{d\ln T_e}{d\ln n}\right|_{r=r_s}, \tag{211}$$

$$\eta_i = \left.\frac{d\ln T_i}{d\ln n}\right|_{r=r_s}, \tag{212}$$

$$\tau = \left(\frac{T_e}{T_i}\right)_{r=r_s}\left(\frac{1+\eta_e}{1+\eta_i}\right), \tag{213}$$

where $n(r)$, $p(r)$, $T_e(r)$, and $T_i(r)$ refer to density, pressure, and temperature profiles that are unperturbed by the tearing mode.

Let us assume, for the sake of simplicity, that the perturbed electron and ion temperature profiles in the vicinity of the resonant layer are functions of the perturbed electron number density profile. In other words, $T_e = T_e(n)$ and $T_i = T_i(n)$. It follows that the perturbed total pressure profile is also a function of the perturbed electron number density profile: that is, $p = p(n)$. Let us define

$$\mathbf{V}_* = \frac{\mathbf{b}\times\nabla p}{e\,n_0\,B_z}. \tag{214}$$

To lowest order, the electron and ion diamagnetic velocities take the respective forms

$$\mathbf{V}_{*\,e} = -\left(\frac{\tau}{1+\tau}\right)\mathbf{V}_*, \tag{215}$$

$$\mathbf{V}_{*\,i} = \left(\frac{1}{1+\tau}\right)\mathbf{V}_*. \tag{216}$$

(See Section 2.5.) The so-called *MHD velocity*, which is the velocity of a fictional 'MHD fluid', is defined

$$\mathbf{V} = \mathbf{V}_E + V_{\parallel\,i}\,\mathbf{b}, \tag{217}$$

where $\mathbf{V}_E$ is the $\mathbf{E}\times\mathbf{B}$ velocity, and $V_{\parallel\,i}$ is the parallel component of the ion fluid velocity. The lowest-order electron and ion fluid velocities take the respective forms:

$$\mathbf{V}_e = \mathbf{V} + \mathbf{V}_{*\,i} - \frac{\mathbf{j}}{n_0\,e}, \tag{218}$$

$$\mathbf{V}_i = \mathbf{V} + \mathbf{V}_{*\,i}. \tag{219}$$

(See Section 2.5.)

Following Hazeltine & Meiss 1992, if we neglect electron inertia and electron viscosity in the electron equation of motion, (9), then we obtain the *generalized Ohm's law*:

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} + \frac{1}{e\,n_0}\left[\nabla p + \frac{1}{1+\tau}\left(\frac{0.71\,\eta_e\,\tau}{1+\eta_e} - 1\right)(\mathbf{b}\cdot\nabla p)\,\mathbf{b} - \mathbf{j}\times\mathbf{b}\right] \simeq \eta_\parallel\,\mathbf{j}_\parallel + \eta_\perp\,\mathbf{j}_\perp. \tag{220}$$

Here, $\eta_\parallel = 1/\sigma_\parallel$ and $\eta_\perp = 1/\sigma_\perp$ are the parallel and perpendicular plasma resistivities, respectively. (See Section 2.3.) We have neglected the cross term in Equation (15) because it is a factor $(\Omega_e\,\tau_e)^{-1}$ smaller than ∇p.

Next, we can add the electron and ion equations of motion, (9) and (12), again neglecting electron inertia and electron viscosity, to give the *plasma equation of motion*:

$$n_0\,m_i\left[\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} + \frac{1}{1+\tau}\,(\mathbf{V}_*\cdot\nabla)\mathbf{V}_E\right] + \nabla p + \nabla\cdot\boldsymbol{\pi}_i - \mathbf{j}\times\mathbf{B} \simeq 0. \tag{221}$$

Here, $\nabla\cdot\boldsymbol{\pi}_i$ includes the contribution from the anomalous (i.e., enhanced due to the action of small-scale plasma turbulence) perpendicular viscosity tensor. The contribution from the gyroviscosity tensor, which is the same size as the ion inertial terms, has already been incorporated into the equation (Hazeltine & Meiss 1992). For the moment, we are neglecting the contribution of the parallel viscosity tensor.

Finally, we can combine the electron and ion energy evolution equations, (10) and (13), to obtain a *plasma energy evolution equation*:

$$\frac{\partial p}{\partial t} + \mathbf{V}\cdot\nabla p + \Gamma\,p\,\nabla\cdot\mathbf{V} + \frac{2}{3}\,\nabla\cdot\mathbf{q} \simeq 0, \tag{222}$$

where $\Gamma = 5/3$. Here, we have neglected a number of terms (e.g., the ohmic heating terms) because they are, at least, factors $(\Omega_e\,\tau_e)^{-1}$ or $(\Omega_i\,\tau_i)^{-1}$ smaller than $\nabla\cdot\mathbf{q}$. We have also neglected the viscous heating term because we expect the dominant contribution, which comes from the parallel viscosity, to be much smaller than its nominal magnitude in a large aspect-ratio tokamak plasma.

Equations (220)–(222) are known collectively as the *drift-MHD equations*, and contain both shear-Alfvén and compressible-Alfvén wave dynamics. The system of equations is completed by Maxwell's equations:

$$\nabla\cdot\mathbf{B} = 0, \tag{223}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{224}$$

$$\mu_0\,\mathbf{j} = \nabla\times\mathbf{B}. \tag{225}$$

4.3 Normalization Scheme

Let l be a typical variation lengthscale in the resonant layer. The *Alfvén velocity*, which is the typical phase velocity of compressible Alfvén waves, is defined

$$V_A = \frac{B_z}{\sqrt{\mu_0\,n_0\,m_i}}. \tag{226}$$

It is helpful to define the *collisionless ion skin-depth*:

$$d_i = \left(\frac{m_i}{n_0\,e^2\,\mu_0}\right)^{1/2}. \tag{227}$$

Note that $d_i = c/\omega_{p\,i}$, where c is the velocity of light, and $\omega_{p\,i}$ is the ion plasma frequency (Fitzpatrick 2015).

It is convenient to adopt the following normalization scheme that renders all quantities in the drift-MHD equations dimensionless: $\hat{\nabla} = l\,\nabla$, $\hat{d}_i = d_i/l$, $\hat{t} = t/(l/V_A)$, $\hat{\mathbf{B}} = \mathbf{B}/B_z$, $\hat{\mathbf{E}} = \mathbf{E}/(B_z\,V_A)$, $\hat{\mathbf{j}} = \mathbf{j}/(B_z/\mu_0\,l)$, $\hat{\eta}_{\parallel,\perp} = \eta_{\parallel,\perp}/(\mu_0\,V_A\,l)$, $\hat{\mathbf{V}} = \mathbf{V}/V_A$, $\hat{\mathbf{V}}_{*,i} = \mathbf{V}_{*,i}/V_A$, $\hat{V}_{\parallel\,i} = V_{\parallel\,i}/V_A$, $\hat{p} = p/(B_z^2/\mu_0)$, $\hat{p}_0 = p_0/(B_z^2/\mu_0)$, $\hat{\boldsymbol{\pi}}_i = \boldsymbol{\pi}_i/(B_z^2/\mu_0)$, $\hat{\mathbf{q}} = \mathbf{q}/(B_z^2\,V_A/\mu_0)$. Equations (220)–(225) yield

$$\hat{\mathbf{E}} + \hat{\mathbf{V}}\times\hat{\mathbf{B}} + \hat{d}_i\left[\hat{\nabla}\hat{p} + \frac{1}{1+\tau}\left(\frac{0.71\,\eta_e\,\tau}{1+\eta_e} - 1\right)(\mathbf{b}\cdot\hat{\nabla}\hat{p})\,\mathbf{b} + \hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_e - \hat{\mathbf{j}}\times\hat{\mathbf{b}}\right] \simeq \hat{\eta}_\parallel\,\hat{\mathbf{j}}_\parallel + \hat{\eta}_\perp\,\hat{\mathbf{j}}_\perp, \tag{228}$$

$$\frac{\partial\hat{\mathbf{V}}}{\partial\hat{t}} + (\hat{\mathbf{V}}\cdot\hat{\nabla})\hat{\mathbf{V}} + \frac{1}{1+\tau}\,(\hat{\mathbf{V}}_*\cdot\hat{\nabla})\hat{\mathbf{V}}_E + \hat{\nabla}\hat{p} + \hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i - \hat{\mathbf{j}}\times\hat{\mathbf{B}} \simeq 0, \tag{229}$$

$$\frac{\partial\hat{p}}{\partial\hat{t}} + \hat{\mathbf{V}}\cdot\hat{\nabla}\hat{p} + \Gamma\,\hat{p}\,\hat{\nabla}\cdot\hat{\mathbf{V}} + \frac{2}{3}\,\hat{\nabla}\cdot\hat{\mathbf{q}} \simeq 0, \tag{230}$$

and

$$\hat{\nabla}\cdot\hat{\mathbf{B}} = 0, \tag{231}$$

$$\hat{\nabla}\times\hat{\mathbf{E}} = -\frac{\partial\hat{\mathbf{B}}}{\partial\hat{t}}, \tag{232}$$

$$\hat{\mathbf{j}} = \hat{\nabla}\times\hat{\mathbf{B}}. \tag{233}$$

Finally,

$$\hat{\mathbf{V}}_E = \hat{\mathbf{E}}\times\mathbf{b}, \tag{234}$$

$$\hat{\mathbf{V}} = \hat{\mathbf{V}}_E + \hat{V}_{\parallel\,i}\,\mathbf{b}, \tag{235}$$

$$\hat{\mathbf{V}}_* = \hat{d}_i\,\mathbf{b}\times\hat{\nabla}\hat{p}, \tag{236}$$

$$\hat{\mathbf{V}}_i = \hat{\mathbf{V}} + \frac{\hat{\mathbf{V}}_{*\,i}}{1+\tau}. \tag{237}$$

4.4 Reduction Process

All variables in the resonant layer are assumed to be functions of $\hat{x} = (r - r_s)/l$ and $\zeta = m\,\theta - n\,\varphi$ only. Let $\mathbf{n} = (0,\,\epsilon/q_s,\,1)$, where $\epsilon = r/R_0$ and $q_s = m/n$. It follows that $\mathbf{n}\cdot\nabla A = 0$ for any $A(\hat{x},\zeta)$. Let us write (Fitzpatrick & Waelbroeck 2009)

$$\hat{\mathbf{B}} \simeq (1 + \delta b)\,\mathbf{n} + \hat{\nabla}\psi \times \mathbf{n}, \tag{238}$$

which automatically satisfies Equation (231). (Recall that we are assuming that $|r - r_s| \ll r_s$ inside the resonant layer.) Here, $\delta b(\hat{x},\zeta)$ and $\psi(\hat{x},\zeta)$ are assumed to be small compared to unity, which merely indicates that the perturbed magnetic field due to the tearing mode is small compared to the equilibrium magnetic field. We can write

$$\hat{\mathbf{E}} \simeq \left(\hat{E}_\parallel - \frac{\partial\psi}{\partial\hat{t}}\right)\mathbf{n} + \hat{\nabla}\left(\frac{\partial\chi}{\partial\hat{t}}\right) \times \mathbf{n} + \hat{\nabla}\phi, \tag{239}$$

which automatically satisfies Equation (232) provided that

$$\hat{\nabla}^2\chi = \delta b. \tag{240}$$

Here, $\hat{E}_\parallel = E_\parallel/(B_z\,V_A)$ represents the constant inductive electric field that drives the equilibrium plasma current in the vicinity of the resonant surface. Moreover, $\partial/\partial\hat{t}$ and $\phi(\hat{x},\zeta)$ are assumed to be first order in small quantities, whereas $\hat{E}_\parallel$ is assumed to be second order. It follows that

$$\hat{\mathbf{V}}_E \simeq \hat{\nabla}\phi \times \mathbf{n}, \tag{241}$$

$$\hat{\mathbf{V}} \simeq \hat{V}_{\parallel\,i}\,\mathbf{n} + \hat{\nabla}\phi \times \mathbf{n} + \hat{\nabla}\mathit{\Xi}, \tag{242}$$

where $\hat{V}_{\parallel\,i}(\hat{x},\zeta)$ is assumed to be first order in small quantities, whereas $\mathit{\Xi}(\hat{x},\zeta)$ is assumed to be second order. We can write

$$\hat{p} \simeq \frac{\beta}{\mathit{\Gamma}} + \delta p, \tag{243}$$

where

$$\beta = \mathit{\Gamma}\,\hat{p}_0 = \frac{\mathit{\Gamma}\,p_0\,\mu_0}{B_z^2}. \tag{244}$$

Here, $\delta p(\hat{x},\zeta)$ is assumed to be first order in small quantities. It follows that

$$\hat{\mathbf{V}}_* \simeq \hat{d}_i\,\mathbf{n} \times \hat{\nabla}\delta p, \tag{245}$$

$$\hat{\mathbf{V}}_i \simeq \hat{V}_{\parallel\,i}\,\mathbf{n} + \hat{\nabla}\left(\phi - \frac{\hat{d}_i\,\delta p}{1+\tau}\right) \times \mathbf{n}. \tag{246}$$

The normalized drift-MHD equations, (228)–(230), yield

$$\hat{d}_i\,\hat{\nabla}(\delta p + \delta b)$$

$$+\left(\hat{E}_\parallel - \frac{\partial\psi}{\partial\hat{t}} - [\psi,\phi] + \frac{\hat{d}_i\,[\psi,\delta p]}{1+\tau}\left(1 - \frac{0.71\,\eta_e\,\tau}{1+\eta_e}\right) + \hat{d}_i\,[\psi,\delta b] + \hat{\eta}_\parallel\,J\right)\mathbf{n}$$

$$+\hat{\nabla}\left(\frac{\partial\chi}{\partial\hat{t}} + \varXi - \hat{\eta}_\perp\,\delta b\right)\times\mathbf{n} + \frac{\hat{d}_i}{2}\,\hat{\nabla}\delta b^{\,2} - b_z\,\hat{\nabla}\phi + \hat{V}_{\parallel\,i}\,\hat{\nabla}\psi + \hat{d}_i\,J\,\hat{\nabla}\psi \simeq 0, \tag{247}$$

$$\hat{\nabla}(\delta p + \delta b) + \left(\frac{\partial\hat{V}_{\parallel\,i}}{\partial\hat{t}} + [\hat{V}_{\parallel\,i},\phi] + [\psi,\delta b]\right)\mathbf{n}$$

$$+\hat{\nabla}\left(\frac{\partial\phi}{\partial\hat{t}} - \frac{\hat{d}_i\,[\phi,\delta p]}{2\,(1+\tau)}\right)\times\mathbf{n} + \frac{1}{2}\,\hat{\nabla}\left(\hat{V}_\perp^{\,2} + \frac{\hat{\mathbf{V}}_*\cdot\hat{\mathbf{V}}_E}{1+\tau} + \delta b^{\,2}\right)$$

$$-\hat{\nabla}^2\phi\,\hat{\nabla}\phi + \frac{\hat{d}_i}{2\,(1+\tau)}\left(\hat{\nabla}^2\phi\,\hat{\nabla}\delta p + \hat{\nabla}^2\delta p\,\hat{\nabla}\phi\right) + J\,\hat{\nabla}\psi + \hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i \simeq 0, \tag{248}$$

$$\frac{\partial\delta p}{\partial\hat{t}} + [\delta p,\phi] + \beta\,\hat{\nabla}^2\varXi + \frac{2}{3}\,\hat{\nabla}\cdot\hat{\mathbf{q}} \simeq 0, \tag{249}$$

where use has been made of Equation (233), and

$$[A,B] \equiv \hat{\nabla}A\times\hat{\nabla}B\cdot\mathbf{n}, \tag{250}$$

$$J = -\frac{2}{q_s\,\hat{R}_0} + \hat{\nabla}^2\psi, \tag{251}$$

with $R_0 = R_0/l$. Here, $\hat{d}_i$ and β are assumed to be zeroth order in small quantities, J, $1/\hat{R}_0$, $\hat{\eta}_\parallel$, and $\hat{\eta}_\perp$ are assumed to be first order, while $\hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i$ and $\hat{\nabla}\cdot\hat{\mathbf{q}}$ are assumed to be second order.

To first order in small quantities, both Equations (247) and (248) yield

$$\delta b \simeq -\delta p, \tag{252}$$

which is simply an expression of lowest-order equilibrium force balance. The scalar product of Equation (247) with $\mathbf{n}$ gives

$$\frac{\partial\psi}{\partial\hat{t}} \simeq [\phi,\psi] + \hat{d}_i\left(\frac{\tau}{1+\tau}\right)\left(1 + \frac{0.71\,\eta_e}{1+\eta_e}\right)[\delta p,\psi] + \hat{\eta}_\parallel\,J + \hat{E}_\parallel. \tag{253}$$

The scalar product of Equation (248) with $\mathbf{n}$ yields

$$\frac{\partial\hat{V}_{\parallel\,i}}{\partial\hat{t}} \simeq [\phi,\hat{V}_{\parallel\,i}] - [\delta p,\psi] - \mathbf{n}\cdot\hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i. \tag{254}$$

The scalar product of the curl of Equation (247) with $\mathbf{n}$ gives

$$\frac{\partial\delta p}{\partial\hat{t}} \simeq [\phi,\delta p] + \hat{\nabla}^2\varXi - [\hat{V}_{\parallel\,i},\psi] - \hat{d}_i\,[J,\psi] + \hat{\eta}_\perp\,\hat{\nabla}^{\,2}\delta p. \tag{255}$$

Eliminating $\hat{\nabla}^2 \Xi$ between Equation (249) and the previous equation, we obtain

$$\frac{\partial \delta p}{\partial \hat{t}} \simeq [\phi, \delta p] - c_\beta^2\, [\hat{V}_{\parallel\, i}, \psi] - c_\beta^2\, \hat{d}_i\, [J, \psi] - \frac{2}{3}\, (1 - c_\beta^2)\, \hat{\nabla}\cdot\hat{\mathbf{q}} + c_\beta^2\, \hat{\eta}_\perp \hat{\nabla}^2 \delta p, \tag{256}$$

where

$$c_\beta = \sqrt{\frac{\beta}{1+\beta}}. \tag{257}$$

Finally, the scalar product of the curl of Equation (248) with $\mathbf{n}$ gives

$$\begin{aligned}\frac{\partial U}{\partial \hat{t}} &\simeq [\phi, U] + \frac{\hat{d}_i}{2\,(1+\tau)} \left(\hat{\nabla}^2[\phi, \delta p] + [\hat{\nabla}^2\phi, \delta p] + [\hat{\nabla}^2 \delta p, \phi]\right) + [J, \psi] \\ &\quad + \mathbf{n}\cdot\hat{\nabla}\times\hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i, \end{aligned}\tag{258}$$

where

$$U = \hat{\nabla}^2 \phi. \tag{259}$$

We can write

$$\begin{aligned}\hat{\nabla}\cdot\hat{\mathbf{q}} &\simeq -\left(\frac{\eta_e}{1+\eta_e}\, \frac{\tau}{1+\tau}\, \hat{\chi}_{\parallel\, e} + \frac{\eta_i}{1+\eta_i}\, \frac{1}{1+\tau}\, \hat{\chi}_{\parallel\, i}\right) [[\delta p, \psi], \psi] \\ &\quad - \left(\frac{\eta_e}{1+\eta_e}\, \frac{\tau}{1+\tau}\, \hat{\chi}_{\perp\, e} + \frac{\eta_i}{1+\eta_i}\, \frac{1}{1+\tau}\, \hat{\chi}_{\perp\, i}\right) \nabla_\perp^2 \delta p, \end{aligned}\tag{260}$$

where $\hat{\nabla}_\perp^2 \equiv \hat{\nabla}\cdot\hat{\nabla}_\perp$, $\chi_{\parallel\, e,i} = \kappa_{\parallel\, e,i}/n_0$, $\chi_{\perp\, e,i} = \kappa_{\perp\, e,i}/n_0$, $\hat{\chi}_{\parallel\, e,i} = \chi_{\parallel\, e,i}/(l\, V_A)$, $\hat{\chi}_{\perp\, e,i} = \chi_{\perp\, e,i}/(l\, V_A)$, and use has been made of Equations (20), (21), (211)–(213), and (238). We can also write

$$\hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i \simeq -\hat{\chi}_\varphi\, \hat{\nabla}^2 \hat{\mathbf{V}}_i, \tag{261}$$

where $\chi_\varphi = \mu_\perp/(n_0\, m_i)$, $\hat{\chi}_\varphi = \chi_\varphi/(l\, V_A)$, and $\mu_\perp$ is the anomalous ion perpendicular viscosity. Note that the previous equation is entirely phenomenological in nature. It is easily seen that

$$\mathbf{n}\cdot\hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i \simeq -\hat{\chi}_\varphi\, \hat{\nabla}^2 \hat{V}_{\parallel\, i}, \tag{262}$$

$$\mathbf{n}\cdot\hat{\nabla}\times\hat{\nabla}\cdot\hat{\boldsymbol{\pi}}_i \simeq \hat{\chi}_\varphi\, \hat{\nabla}^4 \left(\phi - \frac{\hat{d}_i\, \delta p}{1+\tau}\right). \tag{263}$$

Let us define $V = \hat{V}_{\parallel\, i}$ and $N = \delta p / c_\beta$. Equations (251), (253)–(254), (256), (258)–(260), (262), and (263) yield the following closed set of equations (Fitzpatrick & Waelbroeck 2005):

$$\frac{\partial \psi}{\partial \hat{t}} \simeq [\phi, \psi] + \hat{d}_\beta \left(\frac{\tau}{1+\tau}\right) \left(1 + \frac{0.71\, \eta_e}{1+\eta_e}\right) [N, \psi] + \hat{\eta}_\parallel\, J + \hat{E}_\parallel, \tag{264}$$

$$\frac{\partial U}{\partial \hat{t}} \simeq [\phi, U] + \frac{\hat{d}_\beta}{2\,(1+\tau)} \left(\hat{\nabla}^2[\phi, N] + [\hat{\nabla}^2\phi, N] + [\hat{\nabla}^2 N, \phi]\right) + [J, \psi]$$

$$+\hat{\chi}_\varphi\,\hat{\nabla}^4\left(\phi-\frac{\hat{d}_\beta\,N}{1+\tau}\right), \tag{265}$$

$$\frac{\partial N}{\partial\hat{t}}\simeq[\phi,N]-c_\beta\,[V,\psi]-\hat{d}_\beta\,[J,\psi]+\hat{D}_\parallel\,[[N,\psi],\psi]+\hat{D}_\perp\,\hat{\nabla}_\perp^2 N, \tag{266}$$

$$\frac{\partial V}{\partial\hat{t}}\simeq[\phi,V]-c_\beta\,[N,\psi]+\hat{\chi}_\varphi\,\hat{\nabla}^2 V, \tag{267}$$

$$J=-\frac{2}{q_s\,\hat{R}_0}+\hat{\nabla}^2\psi, \tag{268}$$

$$U=\hat{\nabla}^2\phi. \tag{269}$$

Here,

$$d_\beta=c_\beta\,d_i=\left(\frac{\mathit{\Gamma}}{1+\beta}\right)^{1/2}\rho_s, \tag{270}$$

$$\rho_s=\frac{v_t}{\mathit{\Omega}_i}, \tag{271}$$

$$v_t=\left(\frac{T_e+T_i}{m_i}\right)^{1/2}_{r_s}, \tag{272}$$

and $\hat{d}_\beta=d_\beta/l$. The quantity ρ_s is usually referred to as the *ion sound radius*. Moreover,

$$\hat{D}_\parallel=\frac{2}{3}\,(1-c_\beta^2)\left(\frac{\eta_e}{1+\eta_e}\,\frac{\tau}{1+\tau}\,\hat{\chi}_{\parallel\,e}+\frac{\eta_i}{1+\eta_i}\,\frac{1}{1+\tau}\,\hat{\chi}_{\parallel\,i}\right), \tag{273}$$

$$\hat{D}_\perp=c_\beta^2\,\hat{\eta}_\perp+\frac{2}{3}\,(1-c_\beta^2)\left(\frac{\eta_e}{1+\eta_e}\,\frac{\tau}{1+\tau}\,\hat{\chi}_{\perp\,e}+\frac{\eta_i}{1+\eta_i}\,\frac{1}{1+\tau}\,\hat{\chi}_{\perp\,i}\right). \tag{274}$$

4.5 Discussion

Equations (264)–(269) constitute our reduced, four-field, drift-MHD, resonant, plasma response model. Our model is very similar to the original four-field model derived by Hazeltine, et al. (Hazeltine, et al. 1985). Equation (264) is the *generalized Ohm's law* that governs the time evolution of the helical magnetic flux, ψ. Equation (265) is the *parallel ion vorticity equation* that governs the time evolution of the parallel ion vorticity, U. Equation (266) is the *electron number density continuity equation* that governs the time evolution of the electron number density, N. Finally, Equation (267) is the *ion parallel equation of motion* that governs the time evolution of the parallel ion velocity, V.

If we neglect $\hat{d}_\beta$ in our reduced drift-MHD model, which is equivalent to saying that the ion sound radius is much smaller than the radial extent of the resonant layer, then our four-field resonant response model decouples into two independent two-field models. The first two-field model takes the form:

$$\frac{\partial\psi}{\partial\hat{t}}\simeq[\phi,\psi]+\hat{\eta}_\parallel\,J+\hat{E}_\parallel, \tag{275}$$

$$\frac{\partial U}{\partial \hat{t}} \simeq [\phi, U] + [J, \psi] + \hat{\chi}_\perp \hat{\nabla}^4 \phi. \tag{276}$$

Of course, this is a *reduced MHD model* that describes shear-Alfvén dynamics perpendicular to magnetic field-lines (Strauss 1976). The second two-field model takes the form:

$$\frac{\partial N}{\partial \hat{t}} \simeq [\phi, N] - c_\beta [V, \psi] + \hat{D}_\parallel [[N, \psi], \psi] + \hat{D}_\perp \hat{\nabla}_\perp^2 N, \tag{277}$$

$$\frac{\partial V}{\partial \hat{t}} \simeq [\phi, V] - c_\beta [N, \psi] + \hat{\chi}_\phi \hat{\nabla}^2 V, \tag{278}$$

where ψ is assumed to take its unperturbed value. This model describes ion sound wave dynamics parallel to magnetic field-lines. It should be noted that the ion sound radius is *not* negligibly small compared to the radial extents of resonant layers in conventional tokamak plasmas. Hence, the 'single-fluid' reduced MHD model, (275)–(276), provides a very poor description of the response of resonant layers in such plasmas.

Finally, the crucial element of the reduction process, by which our original drift-MHD response model is converted into a reduced drift-MHD model, is the set of ordering assumptions that render $\hat{\nabla}\cdot\hat{\mathbf{V}} = \hat{\nabla}^2 \varXi$ a second-order quantity while leaving $\hat{\mathbf{V}}$ a first-order quantity. This ordering implies that the MHD fluid is *incompressible* to lowest order. It should be noted, however, that the small, but finite, compressibility of the MHD fluid has a significant influence on the form of the continuity equation (266).

5 Linear Resonant Response Model

5.1 Introduction

The aim of this section is to employ the reduced, drift-MHD, four-field, resonant, plasma response model derived in the previous section to determine the *linear* response of the resonant layer to external perturbations.

It is convenient to set the normalization scale-length, l, in our four-field response model equal to the minor radius of the resonant surface, r_s. Recall that the four fields in question—namely, ψ, ϕ, N, and V—have the following definitions:

$$\nabla\psi = \frac{\mathbf{n}\times\mathbf{B}}{r_s\, B_z}, \tag{279}$$

$$\nabla\phi = \frac{\mathbf{n}\times\mathbf{V}}{r_s\, V_A}, \tag{280}$$

$$N = \frac{p - p_0}{c_\beta\, B_z^2/\mu_0}, \tag{281}$$

$$V = \frac{\mathbf{n}\cdot\mathbf{V}}{V_A}, \tag{282}$$

where $\mathbf{V}$ is the MHD velocity defined in Equation (217), p is the total plasma pressure, p_0 is defined in Equation (210), V_A is the Alfvén speed defined in Equation (226), and c_β is defined in Equations (244) and (257). Our model also employs the auxiliary fields $J = -2\,\epsilon_s/q_s + r_s^2\,\nabla^2\psi$ and $U = r_s^2\,\nabla^2\phi$. Here, $\epsilon_s = r_s/R_0$.

5.2 Plasma Equilibrium

The unperturbed plasma equilibrium is such that

$$\mathbf{B} = (0,\, B_\theta(r),\, B_z), \tag{283}$$

$$p = p(r), \tag{284}$$

$$\mathbf{V} = (0,\, V_E(r),\, V_z(r)), \tag{285}$$

where

$$V_E(r) \simeq \frac{E_r}{B_z} \tag{286}$$

is the (dominant θ-component of the) $\mathbf{E}\times\mathbf{B}$ velocity. Now, the resonant layer is assumed to have a radial thickness that is much smaller than r_s. (See Section 5.3.) Hence, we only need to evaluate plasma equilibrium quantities in the immediate vicinity of the resonant surface. Equations (279)–(285) suggest that

$$\psi = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s}, \tag{287}$$

$$\phi = -\hat{V}_E\,\hat{x}, \tag{288}$$

$$N = \frac{\hat{V}_*\,\hat{x}}{\hat{d}_\beta}, \tag{289}$$

$$V = \hat{V}_\parallel, \tag{290}$$

where $\hat{x} = (r-r_s)/r_s$, $\hat{L}_s = L_s/r_s$, $L_s = (R_0\,q/s)_{r_s}$ is the *magnetic shear length*, the *magnetic shear*, $s(r)$, is defined in Equation (108), $\hat{V}_E = V_E(r_s)/V_A$, $\hat{V}_* = V_*(r_s)/V_A$,

$$V_*(r) = \frac{1}{e\,n_0\,B_z}\,\frac{dp}{dr} \tag{291}$$

is the (dominant θ-component of the) diamagnetic velocity, $\hat{d}_\beta = d_\beta/r_s$, d_β is the modified ion sound radius defined in Equation (270), and $\hat{V}_\parallel = V_z(r_s)/V_A$. We also have

$$J = -\left(\frac{2}{s_s} - 1\right)\frac{1}{\hat{L}_s}, \tag{292}$$

$$U = 0, \tag{293}$$

$$\hat{E}_\parallel = \left(\frac{2}{s_s} - 1\right)\frac{\hat{\eta}_\parallel(r_s)}{\hat{L}_s}, \tag{294}$$

where $s_s = s(r_s)$. Note that we are neglecting the radial shear of the equilibrium MHD velocity because, in conventional tokamak plasmas, such shear does not significantly affect the linear response of a resonant layer, due to the fact that the shear is comparatively weak (i.e., $|dV/dr| \ll V_A/a$) combined with the fact that a linear resonant layer is very narrow (Chen & Morrison 1990).

5.3 Derivation of Linear Layer Equations

Suppose that the system is perturbed by a tearing mode with m periods in the poloidal direction, and n periods in the toroidal direction. In accordance with Equations (287)–(290), and (292)–(293), we can write

$$\psi(\hat{x},\zeta,t) = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s} + \tilde{\psi}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{295}$$

$$\phi(\hat{x},\zeta,t) = -\hat{V}_E\,\hat{x} + \tilde{\phi}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{296}$$

$$N(\hat{x},\zeta,t) = \frac{\hat{V}_*\,\hat{x}}{\hat{d}_\beta} - \frac{1}{\hat{d}_\beta}\left(\frac{1+\tau}{\tau}\right)\tilde{N}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{297}$$

$$V(\hat{x},\zeta,t) = \hat{V}_\parallel + \frac{c_\beta}{\hat{d}_\beta}\left(\frac{1+\tau}{\tau}\right)\tilde{V}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{298}$$

$$J(\hat{x},\zeta,t) = -\left(\frac{2}{s_s} - 1\right)\frac{1}{\hat{L}_s} + \hat{\nabla}^2\tilde{\psi}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{299}$$

$$U(\hat{x},\zeta,t) = \hat{\nabla}^2\tilde{\phi}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{300}$$

where $\zeta = m\,\theta - n\,\varphi$, $\hat{t} = V_A\,t/r_s$, $\hat{\omega} = r_s\,\omega/V_A$, and ω is the frequency of the tearing mode in the laboratory frame. Substituting Equations (294)–(300) into the four-field model, (264)–(269), and only retaining terms that are first order in perturbed quantities, we obtain the following set of linear layer equations:

$$-{\rm i}\,[\omega - \omega_E - (1+\lambda_e)\,\omega_{*\,e}]\,\tau_H\,\tilde{\psi} = -{\rm i}\,\hat{x}\,[\tilde{\phi} - (1+\lambda_e)\,\tilde{N}) + S^{-1}\,\hat{\nabla}^2\tilde{\psi}, \tag{301}$$

$$-{\rm i}\,(\omega - \omega_E - \omega_{*\,i})\,\tau_H\,\hat{\nabla}^2\tilde{\psi} = -{\rm i}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\varphi\,\hat{\nabla}^4\left(\tilde{\phi} + \frac{\tilde{N}}{\tau}\right), \tag{302}$$

$$\begin{aligned}-{\rm i}\,(\omega - \omega_E)\,\tau_H\,\tilde{N} &= -{\rm i}\,\omega_{*\,e}\,\tau_H\,\tilde{\phi} - {\rm i}\left(\frac{\tau}{1+\tau}\right)\hat{d}_\beta^{\,2}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} - {\rm i}\,c_\beta^{\,2}\,\hat{x}\,\tilde{V} \\ &\quad + S^{-1}\,P_\parallel\left(m\,\omega_{*\,e}\,\tau_H\,\hat{x}\,\tilde{\psi} - \frac{m}{\hat{L}_s}\,\hat{x}^2\,\tilde{N}\right) + S^{-1}\,P_\perp\,\hat{\nabla}^2\tilde{N},\end{aligned} \tag{303}$$

$$-{\rm i}\,(\omega - \omega_E)\,\tau_H\,\tilde{V} = {\rm i}\,\omega_{*\,e}\,\tau_H\,\tilde{\psi} - {\rm i}\,\hat{x}\,\tilde{N} + S^{-1}\,P_\varphi\,\hat{\nabla}^2\tilde{V}. \tag{304}$$

Here, $\tau_H = L_s/(m\,V_A)$ is the *hydromagnetic time*, $\omega_E = (m/r_s)\,V_E(r_s)$ the *E-cross-B frequency*, $\omega_{\ast\,e} = -[\tau/(1+\tau)]\,(m/r_s)\,V_\ast(r_s)$ the *electron diamagnetic frequency*, $\lambda_e = 0.71\,[\eta_e/(1+\eta_e)]\,[\tau/(1+\tau)]$, $\omega_{\ast\,i} = [1/(1+\tau)]\,(m/r_s)\,V_\ast(r_s)$ the *ion diamagnetic frequency*, $S = \tau_R/\tau_H$ the *Lundquist number*, $\tau_R = \mu_0\,r_s^{\,2}/\eta_\parallel(r_s)$ the *resistive diffusion time*, $P_\varphi = \tau_R/\tau_\varphi$ the *magnetic Prandtl number*, $\tau_\varphi = r_s^{\,2}/\chi_\varphi(r_s)$ the *toroidal momentum confinement time*, $P_\perp = \tau_R/\tau_\perp$, $\tau_\perp = r_s^{\,2}/D_\perp(r_s)$ the *particle/energy confinement time*, $P_\parallel = \tau_R/\tau_\parallel$, and $\tau_\parallel = r_s^{\,2}/D_\parallel(r_s)$.

In conventional tokamak plasmas, the Lundquist number, S, which is the ratio of the resistive diffusion and plasma inertia terms in the plasma Ohm's law, is very much greater than unity. In fact, S typically exceeds 10^8 (Wesson 2011). However, a resonant layer is characterized by a balance between plasma inertia and resistive diffusion (Furth, et al. 1963). Such a balance is only possible if the layer is very narrow in the radial direction (because a narrow layer enhances radial derivatives, and, thereby, enhances resistive diffusion). Let us define the stretched radial variable (Ara, et al. 1978)

$$X = S^{1/3}\,\hat{x}. \tag{305}$$

Assuming that $X \sim \mathcal{O}(1)$ in the layer (i.e., assuming that the layer thickness is of order $S^{-1/3}\,r_s$), and making use of the fact that $S \gg 1$, we deduce that $\hat{\nabla}^2 \simeq \hat{\nabla}_\perp^2 \simeq S^{\,2/3}\,d^2/dX^2$. Hence, the linear layer equations, (301)–(304), reduce to (Cole & Fitzpatrick 2006)

$$-{\rm i}\,[Q - Q_E - (1+\lambda_e)\,Q_e]\,\tilde{\psi} = -{\rm i}\,X\left[\tilde{\phi} - (1+\lambda_e)\,\tilde{N}\right] + \frac{d^2\tilde{\psi}}{dX^2}, \tag{306}$$

$$-{\rm i}\,(Q - Q_E - Q_i)\,\frac{d^2\tilde{\phi}}{dX^2} = -{\rm i}\,X\,\frac{d^2\tilde{\psi}}{dX^2} + P_\varphi\,\frac{d^4}{dX^4}\left(\tilde{\phi} + \frac{\tilde{N}}{\tau}\right), \tag{307}$$

$$-{\rm i}\,(Q - Q_E)\tilde{N} = -{\rm i}\,Q_e\,\tilde{\phi} - {\rm i}\,D^2\,X\,\frac{d^2\tilde{\psi}}{dX^2} - {\rm i}\,c_\beta^2\,X\,\tilde{V} + P_\perp\,\frac{d^2\tilde{N}}{dX^2}, \tag{308}$$

$$-{\rm i}\,(Q - Q_E)\,\tilde{V} = {\rm i}\,Q_e\,\tilde{\psi} - {\rm i}\,X\,\tilde{N} + P_\varphi\,\frac{d^2\tilde{V}}{dX^2}. \tag{309}$$

Here, $Q = S^{\,1/3}\,\omega\,\tau_H$, $Q_E = S^{\,1/3}\,\omega_E\,\tau_H$, $Q_{e,i} = S^{\,1/3}\,\omega_{\ast\,e,i}\,\tau_H$, and $D = S^{\,1/3}\,[\tau/(1+\tau)]^{1/2}\,\hat{d}_\beta$. Note that the terms involving parallel transport have dropped out of Equation (308). The neglect of these terms is justified provided that the layer width is much less than the critical value,

$$\delta_c = \left(\frac{1}{\epsilon\,s\,n}\sqrt{\frac{D_\perp}{D_\parallel}}\right)_{r_s}^{1/2} r_s, \tag{310}$$

below which parallel transport is unable to constrain the electron number density (and, by implication, the electron and ion temperatures) to be constant along magnetic field-lines (Fitzpatrick 1995). This condition is well satisfied in conventional tokamak plasmas. Note that all of the terms appearing in the normalized layer equations, (306)–(309), are assumed to be roughly the same order of magnitude.

5.4 Asymptotic Matching

The normalized linear layer equations, (306)–(309), possess independent *tearing parity* solutions, characterized by the symmetry $\tilde{\psi}(-X) = \tilde{\psi}(X)$, $\tilde{\phi}(-X) = -\tilde{\phi}(X)$, $\tilde{N}(-X) = -\tilde{N}(X)$, $\tilde{V}(-X) = \tilde{V}(X)$, and *twisting parity* solutions, characterized by the symmetry $\tilde{\psi}(-X) = -\tilde{\psi}(X)$, $\tilde{\phi}(-X) = \tilde{\phi}(X)$, $\tilde{N}(-X) = \tilde{N}(X)$, $\tilde{V}(-X) = -\tilde{V}(X)$ (Fitzpatrick & Hender 1994). However, only the tearing parity solutions can be asymptotically matched to tearing mode solutions in the outer region. If we assume that the asymptotic behavior of the tearing parity layer solutions is

$$\tilde{\psi} \to \psi_0 \left[1 + \frac{\hat{\Delta}}{2}\,|X| + \mathcal{O}(X^2)\right] \tag{311}$$

as $|X| \to \infty$, where ψ_0 is an arbitrary constant, then asymptotic matching to the outer solution yields

$$\Delta\Psi_s = S^{1/3}\,\hat{\Delta}\,\Psi_s. \tag{312}$$

[See Equations (109) and (134).]

5.5 Fourier Transformation

Equations (306)–(309) are most conveniently solved in Fourier transform space (Cole & Fitzpatrick 2006). Let

$$\bar{\phi}(p) = \int_{-\infty}^{\infty} \tilde{\phi}(X)\,\mathrm{e}^{-\mathrm{i}\,p\,X}\,dX, \tag{313}$$

et cetera. The Fourier transformed linear layer equations become

$$-\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e]\,\bar{\psi} = \frac{d}{dp}\left[\bar{\phi} - (1+\lambda_e)\,\bar{N}\right] - p^2\,\bar{\psi}, \tag{314}$$

$$-\mathrm{i}\,(Q - Q_E - Q_i)\,p^2\,\bar{\phi} = \frac{d(p^2\,\bar{\psi})}{dp} - P_\varphi\,p^4\left(\bar{\phi} + \frac{\bar{N}}{\tau}\right), \tag{315}$$

$$-\mathrm{i}\,(Q - Q_E)\bar{N} = -\mathrm{i}\,Q_e\,\bar{\phi} - D^2\,\frac{d(p^2\,\bar{\psi})}{dp} + c_\beta^2\,\frac{d\bar{V}}{dp} - P_\perp\,p^2\bar{N}, \tag{316}$$

$$-\mathrm{i}\,(Q - Q_E)\,\bar{V} = \mathrm{i}\,Q_e\,\bar{\psi} + \frac{d\bar{N}}{dp} - P_\varphi\,p^2\,\bar{V}, \tag{317}$$

where, for a tearing parity solution,

$$\bar{\phi}(p) \to \bar{\phi}_0\left[\frac{\hat{\Delta}}{\pi\,p} + 1 + \mathcal{O}(p)\right] \tag{318}$$

as $p \to 0$.

Let us ignore the term $c_\beta^2\,d\bar{V}/dp$ in Equation (316). This approximation can be justified *a posteriori*. It is equivalent to the neglect of the contribution of ion parallel dynamics to the linear plasma response in the resonant layer, and effectively decouples Equation (317) from Equations (314)–(316) (Waelbroeck, et al. 2012). Equations (314)–(317) reduce to (Cole & Fitzpatrick 2006)

$$\frac{d}{dp}\left[G(p)\,\frac{dY}{dp}\right]-H(p)\,p^2\,Y=0, \tag{319}$$

where

$$G(p)=\frac{p^2}{-\mathrm{i}\,[Q-Q_E-(1+\lambda_e)\,Q_e]+p^2}, \tag{320}$$

$$H(p)=\frac{A(p)}{B(p)}, \tag{321}$$

$$A(p)=-(Q-Q_E)\,(Q-Q_E-Q_i)-\mathrm{i}\,(Q-Q_E-Q_i)\,(P_\varphi+P_\perp)\,p^2+P_\varphi\,P_\perp\,p^4, \tag{322}$$

$$B(p)=-\mathrm{i}\,[Q-Q_E-(1+\lambda_e)\,Q_e]+\{P_\perp-\mathrm{i}\,(1+\lambda_e)\,(Q-Q_E-Q_i)\,D^2\}p^2 \\ +[(1+\lambda_e)+1/\tau]\,P_\varphi\,D^2\,p^4, \tag{323}$$

and $Y=\bar{\phi}-(1+\lambda_e)\,\bar{N}$. The boundary conditions are that $Y(p)$ is bounded as $p\rightarrow\infty$, and

$$Y(p)\rightarrow Y_0\left[\frac{\hat{\Delta}}{\pi\,p}+1+\mathcal{O}(p)\right] \tag{324}$$

as $p\rightarrow 0$. In the following, we shall assume that $|Q|\sim|Q_E|\sim|Q_e|\sim|Q_i|$, $P_\varphi\sim P_\perp\sim P$, $\tau\sim\mathcal{O}(1)$, and $\lambda_e\sim\mathcal{O}(1)$, for the sake of simplicity.

5.6 Constant-ψ Approximation

Let us suppose that there are two layers in p space. In the small-p layer, suppose that Equation (319) reduces to

$$\frac{d}{dp}\left[\frac{p^2}{-\mathrm{i}\,[Q-Q_E-(1+\lambda_e)\,Q_e]+p^2}\,\frac{dY}{dp}\right]\simeq 0 \tag{325}$$

when $p\sim Q^{1/2}$. Integrating directly, we find that

$$Y(p)\simeq Y_0\left[\frac{\hat{\Delta}}{\pi}\left(\frac{1}{p}+\frac{p}{\mathrm{i}\,[Q-Q_E-(1+\lambda_e)\,Q_e]}\right)+1+\mathcal{O}(p^2)\right] \tag{326}$$

for $p\lesssim\mathcal{O}(Q^{1/2})$, where use has been made of Equation (324). The two-layer approximation, which is equivalent to the well-known *constant-ψ approximation* (Furth, et al. 1963), is valid provided

$$\frac{Q^3\,(1+P)}{1+Q\,D^2}\ll 1. \tag{327}$$

In the large-p layer, for $p \gg \mathcal{O}(Q^{1/2})$, we obtain

$$\frac{d^2Y}{dp^2} - H(p)\,p^2\,Y \simeq 0, \tag{328}$$

with Y bounded as $p \to \infty$. Asymptotic matching to the small-p layer yields the boundary condition

$$Y(p) \simeq Y_0 \left\{1 + \frac{\hat{\Delta}}{\pi}\,\frac{p}{\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e]} + \mathcal{O}(p^2)\right\} \tag{329}$$

as $p \to 0$.

In the various constant-ψ linear response regimes considered in Section 5.7, Equation (328) reduces to an equation of the form

$$\frac{d^2Y}{dp^2} - C\,p^m\,Y \simeq 0, \tag{330}$$

where m is real and non-negative, and C is a complex constant. Let $Y = \sqrt{p}\,Z$ and $q = \sqrt{C}\,p^n/n$, where $n = (m+2)/2$. The previous equation transforms into a modified Bessel equation of general order,

$$q^2\,\frac{d^2Z}{dq^2} + q\,\frac{dZ}{dq} - (q^2+\nu^2)\,Z = 0, \tag{331}$$

where $\nu = 1/(m+2)$. The solution that is bounded as $q \to \infty$ has the small-q expansion

$$K_\nu(z) = \frac{1}{\Gamma(1-\nu)}\left(\frac{2}{q}\right)^\nu - \frac{1}{\Gamma(1+\nu)}\left(\frac{q}{2}\right)^\nu + \mathcal{O}(q^{2-\nu}), \tag{332}$$

where $\Gamma(z)$ is a gamma function. A comparison of this expression with Equation (329) reveals that

$$\hat{\Delta} = \pi\,(-\mathrm{i}\;[Q - Q_E - (1+\lambda_e)\,Q_e])\,\frac{\Gamma(1-\nu)}{\Gamma(\nu)}\,\nu^{2\nu-1}\,C^\nu. \tag{333}$$

Note, finally, that $p_\ast \sim |C|^{-\nu}$, where $p_\ast$ denotes the width of the large-p layer in p space.

5.7 Constant-ψ Linear Response Regimes

Suppose that $Q \gg P\,p^2$ and $D^2\,p^2 \ll 1$. It follows that $\nu = 1/4$ and

$$C = \frac{[-\mathrm{i}\,(Q - Q_E)]\,[-\mathrm{i}\,(Q - Q_E - Q_i)]}{-\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e]}. \tag{334}$$

Hence, we deduce that

$$\hat{\Delta} = \frac{2\pi\,\Gamma(3/4)}{\Gamma(1/4)}\,(-\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])^{3/4}\,[-\mathrm{i}\,(Q - Q_E)]^{1/4}\,[-\mathrm{i}\,(Q - Q_E - Q_i)]^{1/4}. \tag{335}$$

This response regime is known as the *resistive-inertial regime*, because the layer response is dominated by resistivity and ion inertia (Furth, et al. 1963; Ara, et al. 1976; Fitzpatrick 1998). The characteristic layer width is $p_\ast \sim Q^{-1/4}$, which implies that the regime is valid when $P \ll Q^{3/2}$, $Q \gg D^4$, and $Q \ll 1$.

Suppose that $Q \ll P\,p^2$ and $D^2\,p^2 \ll 1$. It follows that $\nu = 1/6$ and $C = P_\varphi$. Hence, we deduce that

$$\hat{\Delta} = \frac{6^{2/3}\,\pi\,\Gamma(5/6)}{\Gamma(1/6)}\,(-\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])\,P_\varphi^{1/6}. \tag{336}$$

This response regime is known as the *visco-resistive regime*, because the layer response is dominated by resistivity and viscosity. (Fitzpatrick 1998). The characteristic layer width is $p_\ast \sim P^{-1/6}$, which implies that the regime is valid when $P \gg Q^{3/2}$, $P \gg D^6$, and $P \ll Q^{-3}$.

Suppose that $Q \gg P\,p^2$ and $D^2\,p^2 \gg 1$. It follows that $\nu = 1/2$ and

$$C = \frac{-\mathrm{i}\,(Q - Q_E)}{(1+\lambda_e)\,D^2}. \tag{337}$$

Hence, we deduce that

$$\hat{\Delta} = \frac{\pi\,(-\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])\,[-\mathrm{i}\,(Q - Q_E)]^{1/2}}{\sqrt{1+\lambda_e}\,D}. \tag{338}$$

This response regime is known as the *semi-collisional regime* (Drake & Lee 1977; Waelbroeck 2003). The characteristic layer width is $p_\ast \sim D\,Q^{-1/2}$. which implies that the regime is valid when $Q \ll D^4$, $P \ll Q^2/D^2$, and $Q \ll D$.

Suppose, finally, that $Q \ll P\,p^2$ and $D^2\,p^2 \gg 1$. It follows that $\nu = 1/4$ and

$$C = \frac{P_\perp}{[(1+\lambda_e) + 1/\tau]\,D^2}. \tag{339}$$

Hence, we deduce that

$$\hat{\Delta} = \frac{2\pi\,\Gamma(3/4)}{\Gamma(1/4)}\,\frac{(-\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])\,P_\perp^{1/4}}{[(1+\lambda_e) + 1/\tau]^{1/4}\,D^{1/2}}. \tag{340}$$

This response regime is known as the *Hall-resistive regime* (Cole & Fitzpatrick 2006). The characteristic layer width is $p_\ast \sim D^{1/2}\,P^{-1/4}$, which implies that the regime is valid when $P \gg Q^2/D^2$, $P \ll D^6$, and $P \ll D^2/Q^2$.

5.8 Nonconstant-ψ Regimes

Suppose that $p \ll Q^{-1/2}$. In this limit, Equation (319) reduces to

$$\frac{d}{dp}\left(p^2\,\frac{dY}{dp}\right) - (-\mathrm{i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])\,H(p)\,p^2\,Y = 0. \tag{341}$$

In the various nonconstant-ψ regimes considered in this section, the previous equation takes the form

$$\frac{d}{dp}\left(p^2\,\frac{dY}{dp}\right) - C\,p^{m+2}\,Y = 0, \tag{342}$$

where m is real and non-negative, and C is a complex constant. Let $U = p\,Y$. The previous equation yields

$$\frac{d^2U}{dp^2} - C\,p^m\,U = 0. \tag{343}$$

This equation is identical in form to Equation (330), which we have already solved. Indeed, the solution that is bounded as $p \to \infty$ has the small-p expansion (332), where $z = \sqrt{C}\,p^n/n$, $n = (m+2)/2$, and $\nu = 1/(m+2)$. The layer width in p-space again scales as $p_\ast \sim |C|^{-\nu}$. Matching to Equation (324) yields

$$\hat{\Delta} = -\pi\,\frac{\Gamma(\nu)}{\Gamma(1-\nu)}\,\nu^{1-2\nu}\,C^{-\nu}. \tag{344}$$

Suppose that $Q \gg P\,p^2$ and $D^2\,p^2 \ll 1$. It follows that $\nu = 1/2$ and

$$C = [-{\rm i}\,(Q - Q_E)]\,[-{\rm i}\,(Q - Q_E - Q_i)]. \tag{345}$$

Hence, we deduce that

$$\hat{\Delta} = -\frac{\pi}{[-{\rm i}\,(Q - Q_E)]^{1/2}\,[-{\rm i}\,(Q - Q_E - Q_i)]^{1/2}}. \tag{346}$$

This response regime is known as the *inertial regime*, because the layer response is dominated by ion inertia (Ara, et al. 1976; Fitzpatrick 1998). Note that the plasma response in the inertial regime is equivalent to that of two closely spaced Alfvén resonances that straddle the resonant surface (Boozer 1996). The characteristic layer width is $p_\ast \sim Q^{-1}$, which implies that the regime is valid when $P \ll Q^3$, $Q \gg D$, and $Q \gg 1$.

Suppose that $Q \ll P\,p^2$ and $D^2\,p^2 \ll 1$. It follows that $\nu = 1/4$ and

$$C = (-{\rm i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])\,P_\varphi. \tag{347}$$

Hence, we deduce that

$$\hat{\Delta} = -\frac{\pi}{2}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}\,(-{\rm i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])^{-1/4}\,P_\varphi^{-1/4}. \tag{348}$$

This response regime is known as the *visco-inertial regime*, because the layer response is dominated by viscosity and ion inertia (Fitzpatrick 1998). The characteristic layer width is $p_\ast \sim Q^{-1/4}\,P^{-1/4}$, which implies that the regime is valid when $P \gg Q^3$, $P \gg D^4/Q$, and $P \gg Q^{-3}$.

Suppose, finally, that $Q \ll P\,p^2$ and $D^2\,p^2 \gg 1$. It follows that $\nu = 1/2$ and

$$C = \frac{(-{\rm i}\,[Q - Q_E - (1+\lambda_e)\,Q_e])\,P_\perp}{[(1+\lambda_e) + 1/\tau]\,D^2}. \tag{349}$$

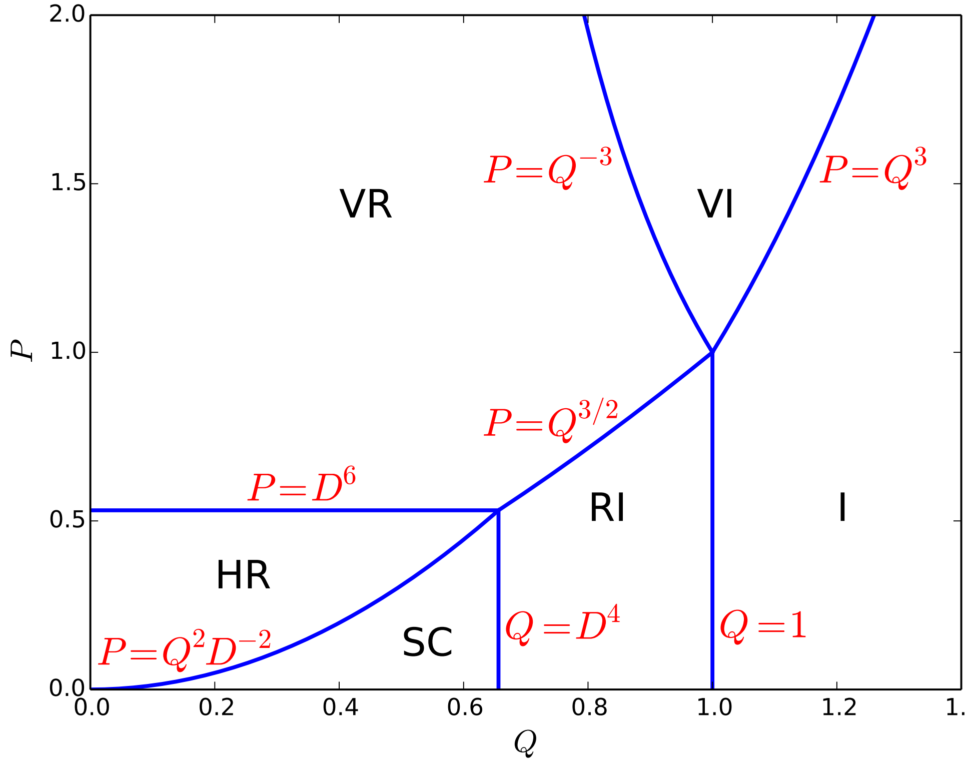

Figure 1: Linear resonant plasma response regimes in Q-P space for the case $D = 0.9$. The various regimes are the Hall-resisitive (HR), the semi-collisional (SC), the resistive-inertial (RI), the visco-resistive (VR), the visco-inertial (VI), and the inertial (I).

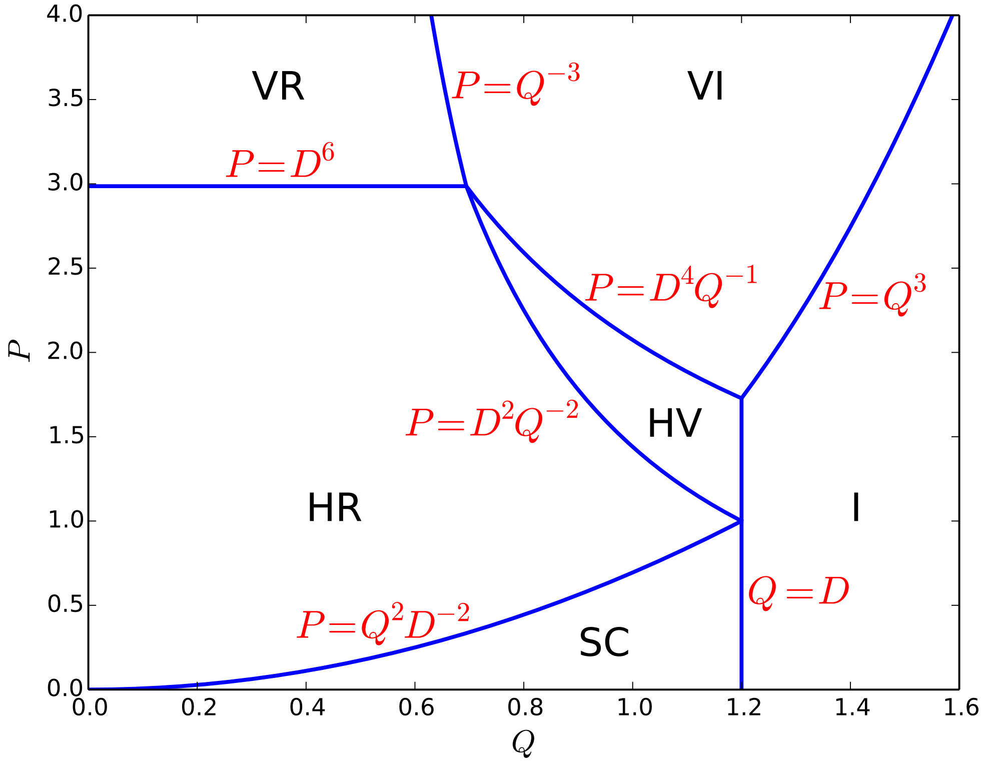

Figure 2: Linear resonant plasma response regimes in Q-P space for the case $D = 1.2$. The various regimes are the Hall-resisitive (HR), the semi-collisional (SC), the Hall-viscous (HV), the visco-resistive (VR), the visco-inertial (VI), and the inertial (I).

Hence, we deduce that

$$\hat{\Delta} = -\frac{\pi\,[(1+\lambda_e)+1/\tau]^{1/2}\,D}{(-\mathrm{i}\,[Q-Q_E-(1+\lambda_e)\,Q_e)])^{1/2}\,P_\perp^{1/2}}. \tag{350}$$

This response regime is known as the *Hall-viscous regime*. The characteristic layer width is $p_* \sim D\,Q^{-1/2}\,P^{-1/2}$, which implies that the regime is valid when $Q \ll D$, $P \ll D^4/Q$, and $P \gg D^2/Q^2$.

Figures 1 and 2 show the extents of the various different response regimes in the Q-P plane for the cases $D < 1$ and $D > 1$, respectively.

6 Linear Tearing Mode Stability

6.1 Introduction

The aim of this section is to combine the analysis of Sections 3 and 5 to determine the linear stability of tearing modes in tokamak plasmas. In this study, we shall assume that the wall is perfectly conducting.

6.2 Linear Dispersion Relation

Let us write

$$\omega = \gamma + \mathrm{i}\,\omega_r, \tag{351}$$

where γ is the growth-rate of the tearing mode, whereas ω_r is the real phase velocity of the mode in the laboratory frame. We shall assume that $|\gamma| \ll |\omega_r|$. This assumption can easily be verified *a posteriori*. Asymptotic matching between the solution in the outer region and that in the thin linear layer surrounding the resonant surface yields a linear tearing mode dispersion relation of the form

$$S^{1/3}\,\hat{\Delta}(\gamma,\omega_r) = E_{ss}, \tag{352}$$

where use has been made of Equations (110) and (312). Here, E_{ss} is the real tearing stability index, whereas the complex layer matching parameter, $\hat{\Delta}$, is defined in Equation (311).

In a conventional tokamak plasma, $E_{ss} \sim \mathcal{O}(1)$,[1] and $S^{1/3} \gg 1$. Hence, the previous dispersion relation can only be satisfied if ω_r takes a value that renders $|\hat{\Delta}| \ll 1$. It is clear from Equations (335), (336), (338), and (340) that this goal can be achieved in all of the constant-ψ linear response regimes if

$$\omega_r = \omega_{\perp\,e}, \tag{353}$$

where

$$\omega_{\perp\,e} = \omega_E + (1+\lambda_e)\,\omega_{\ast\,e}. \tag{354}$$

According to Equations (353) and (354), the tearing mode co-rotates with the electron fluid at the resonant surface.[2]

6.3 Determination of Growth-Rates

Reusing the analysis of Sections 5.6 and 5.7, there are two layers in p space (i.e., Fourier space). The small-p layer is of width $|\hat{\gamma}|^{1/2}$, where $\hat{\gamma} = S^{1/3}\,\gamma\,\tau_H$. Given that we are effectively assuming that $|\hat{\gamma}| \ll 1$, the condition for the separation of the layer solution into two layers

[1] Note that we are neglecting $m = 1$ modes, for which $|E_{ss}| \gg 1$, because these are not really tearing modes (Wesson 1978).

[2] In fact, it rotates slightly faster than the electron fluid because of the influence of the thermal force, $\mathbf{F}_T$ [see Equation (43)], which generates the factor λ_e in Equation (354).

(i.e., that the width of the small-p layer is less than that of the large-p layer) is always satisfied. The large-p layer is governed by the equation

$$\frac{d^2Y}{dp^2} - \frac{A(p)}{C(p)}\,Y = 0, \tag{355}$$

where

$$A(p) = (1+\lambda_e)\left(\frac{\tau}{1+\tau}\right)\left(1+\frac{\lambda_e\,\tau}{1+\tau}\right)Q_\ast^2 + {\rm i}\left(1+\frac{\lambda_e\,\tau}{1+\tau}\right)Q_\ast\,(P_\varphi+P_\perp)\,p^2 + P_\varphi\,P_\perp\,p^4, \tag{356}$$

$$C(p) = P_\perp - {\rm i}\,(1+\lambda_e)\left(1+\frac{\lambda_e\,\tau}{1+\tau}\right)Q_\ast\,D^2 + \left(\frac{1+\tau}{\tau}\right)\left(1+\frac{\lambda_e\,\tau}{1+\tau}\right)D^2\,P_\varphi\,p^2. \tag{357}$$

Here, $Q_\ast = S^{1/3}\,\omega_\ast\,\tau_H$, where

$$\omega_\ast = -\frac{m}{r_s}\frac{1}{e\,n_0\,B_z}\left.\frac{dp}{dr}\right|_{r_s}. \tag{358}$$

In the following, $\omega_\ast$ is assumed to be positive. The boundary conditions on Equation (355) are that Y is bounded as $p\to\infty$, and

$$Y(p) = Y_0\left[1+\frac{\hat{\Delta}\,p}{\pi\,\hat{\gamma}} + {\cal O}(p^2)\right] \tag{359}$$

as $p\to 0$.

In the various constant-ψ linear response regimes considered in this section, Equation (355) reduces to an equation of the form

$$\frac{d^2Y}{dp^2} - C\,p^m\,Y = 0, \tag{360}$$

where m is real and non-negative, and C is a complex constant. As described in Section 5.6, the solution of this equation that is bounded as $p\to\infty$ can be matched to the small-p asymptotic form (359) to give

$$\Delta = \pi\,\hat{\gamma}\,\frac{\Gamma(1-\nu)}{\Gamma(\nu)}\,\nu^{2\nu-1}\,C^\nu, \tag{361}$$

where $\nu = 1/(m+2)$. The width of the large-p layer in p space is $C^{-\nu}$.

As before, we shall assume that. $P_\varphi\sim P_\perp\sim P$, $\lambda_e\sim{\cal O}(1)$, and $\tau\sim{\cal O}(2)$, for the sake of simplicity. Suppose that $Q_\ast\gg P\,p^2$ and $P\gg Q_\ast\,D^2$. It follows that $\nu=1/2$ and

$$C = (1+\lambda_e)\left(\frac{\tau}{1+\tau}\right)\left(1+\frac{\lambda_e\,\tau}{1+\tau}\right)\frac{Q_\ast}{P_\perp^{1/2}}. \tag{362}$$

Hence,

$$\mathit{\Delta} = \pi\,\hat{\gamma}\left[(1+\lambda_e)\left(\frac{\tau}{1+\tau}\right)\left(1+\frac{\lambda_e\,\tau}{1+\tau}\right)\right]^{1/2}\,\frac{Q_\ast}{P_\perp^{1/2}}, \tag{363}$$

and $p_\ast \sim P^{1/2}/Q_\ast$. This so-called *resisrtive-inertial regime* is valid when $P \ll Q_\ast^{3/2}$ and $P \gg Q_\ast\,D^2$. Making use of Equation (352), the corresponding tearing mode growth-rate is

$$\gamma = \frac{E_{ss}}{\pi\left[(1+\lambda_e)\left(\frac{\tau}{1+\tau}\right)\left(1+\frac{\lambda_e\,\tau}{1+\tau}\right)\right]^{1/2}}\,\frac{1}{\omega_\ast\,\tau_H\,\tau_R^{1/2}\,\tau_\perp^{1/2}}. \tag{364}$$

Suppose that $Q_\ast \ll P\,p^2$ and $D^2\,p^2 \ll 1$. It follows that $\nu = 1/6$ and $C = P_\varphi$. Hence,

$$\mathit{\Delta} = \frac{6^{2/3}\,\pi\,\hat{\gamma}\,\mathit{\Gamma}(5/6)}{\Gamma(1/6)}\,P_\varphi^{1/6}, \tag{365}$$

and $p_\ast \sim P^{-1/6}$. This so-called *visco-resistive regime* is valid when $P \gg Q_\ast^{3/2}$ and $P \gg D^6$. The corresponding tearing mode growth-rate is

$$\gamma = \frac{E_{ss}}{6^{2/3}\,\pi\,[\mathit{\Gamma}(5/6)/\mathit{\Gamma}(1/6)]}\,\frac{\tau_\varphi^{1/6}}{\tau_R^{5/6}\,\tau_H^{1/3}}. \tag{366}$$

Suppose that $Q_\ast \gg P\,p^2$ and $P \ll Q_\ast\,D^2$. It follows that $\nu = 1/2$ and

$$C = \mathrm{i}\left(\frac{\tau}{1+\tau}\right)\frac{Q_\ast}{D^2}. \tag{367}$$

Hence,

$$\mathit{\Delta} = \pi\,\hat{\gamma}\left(\frac{\tau}{1+\tau}\right)^{1/2}\,\frac{Q_\ast^{1/2}}{D}, \tag{368}$$

and $p_\ast \sim D/Q_\ast^{1/2}$. This so-called *semi-collisional regime* is valid when $P \ll Q_\ast^2/D^2$ and $P \ll Q_\ast\,D^2$. The corresponding tearing mode growth-rate is

$$\gamma = \frac{E_{ss}\,\mathrm{e}^{-\mathrm{i}\,\pi/4}}{\pi}\,\frac{d_\beta/r_s}{\omega_\ast^{1/2}\,\tau_H\,\tau_R^{1/2}}. \tag{369}$$

Suppose, finally, that $Q_\ast \ll P\,p^2$ and $D^2\,p^2 \gg 1$. It follows that $\nu = 1/4$ and

$$C = \frac{P_\perp\,[\tau/(1+\tau)]}{[1+\lambda_e\,\tau/(1+\tau)]\,D^2}. \tag{370}$$

Hence,

$$\hat{\Delta} = 2\pi\,\hat{\gamma}\,\frac{\mathit{\Gamma}(3/4)}{\Gamma(1/4)}\,\frac{[\tau/(1+\tau)]^{1/4}\,P_\perp^{1/4}}{[1+\lambda_e\,\tau/(1+\tau)]^{1/4}\,D^{1/2}}, \tag{371}$$

and $p_\ast \sim D^{1/2}/P^{1/4}$. This so-called *Hall-resistive regime* is valid when $P \gg Q_\ast^2/D^2$ and $P \ll D^6$. The corresponding tearing mode growth rate is

$$\gamma = \frac{E_{ss}}{2\pi\,[\mathit{\Gamma}(3/4)/\mathit{\Gamma}(1/4)]\,[1+\lambda_e\,\tau/(1+\tau)]^{-1/4}}\,\frac{(d_\beta/r_s)^{1/4}\,\tau_\perp^{1/4}}{\tau_H^{1/3}\,\tau_R^{11/12}}. \tag{372}$$

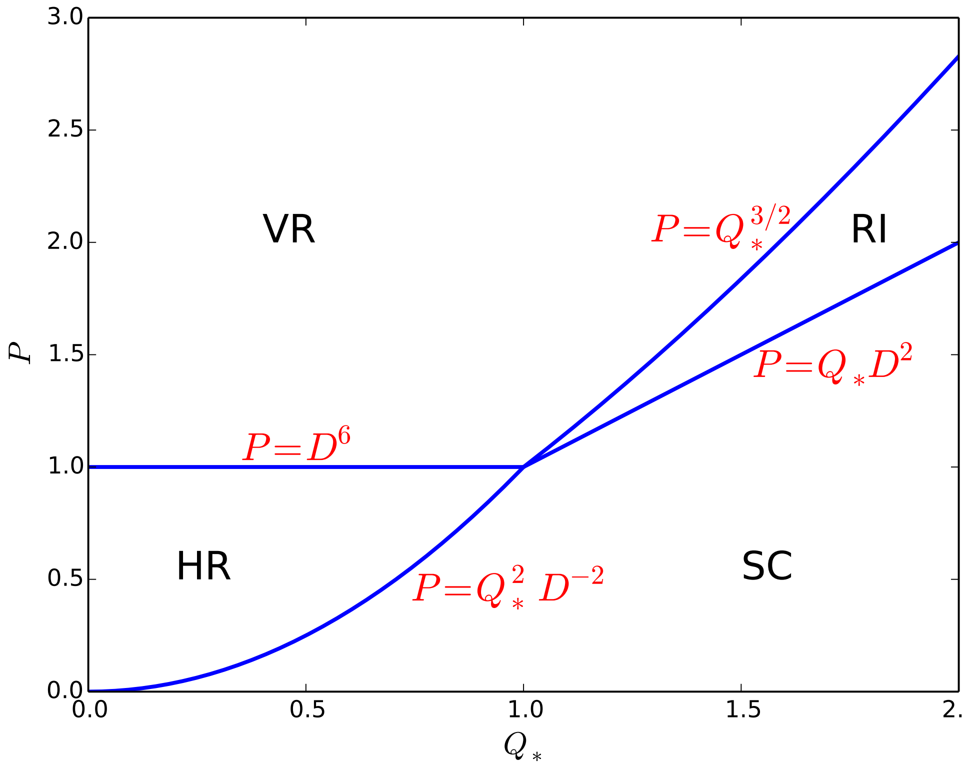

Figure 3: Linear tearing mode growth-rate regimes in Q_*-P space for the case $D = 1$. The various regimes are the Hall-resisitive (HR), the semi-collisional (SC), the visco-resistive (VR), and the resistive-inertia (RI).

6.4 Discussion

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