

Program NEOCLASSICAL

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1 Plasma Equilibrium

Let

$$\vec{B} = I(\psi) \vec{\nabla} \zeta + \vec{\nabla} \zeta \times \vec{\nabla} \psi, \quad (1)$$

$$\vec{E} = -\vec{\nabla} \Phi(\psi) + E_{\parallel} \vec{b}, \quad (2)$$

where $\vec{b} = \vec{B}/B$. Expect $n_a = n_a(\psi)$, $p_a = p_a(\psi)$, and $T_a = T_a(\psi)$. Follows that

$$\mu_0 \vec{j} \equiv \vec{\nabla} \times \vec{B} = \vec{\nabla} I \times \vec{\nabla} \zeta + R^2 \vec{\nabla} \cdot \left(\frac{\vec{\nabla} \psi}{R^2} \right) \nabla \zeta. \quad (3)$$

Force balance implies that

$$\vec{j} \times \vec{B} = \vec{\nabla} p, \quad (4)$$

which yields the Grad-Shafranov equation:

$$R^2 \vec{\nabla} \cdot \left(\frac{\vec{\nabla} \psi}{R^2} \right) = -I \frac{dI}{d\psi} - \mu_0 R^2 \frac{dp}{d\psi}. \quad (5)$$

Finally,

$$\mu_0 \vec{j} \cdot \vec{B} = - \left(\frac{I^2}{R^2} + \frac{|\vec{\nabla} \psi|^2}{R^2} \right) \frac{dI}{d\psi} - \mu_0 I \frac{dp}{d\psi}, \quad (6)$$

which implies that

$$\frac{\langle \vec{j} \cdot \vec{B} \rangle}{I} = - \frac{1}{\mu_0} \left(I \left\langle \frac{1}{R^2} \right\rangle + \frac{1}{I} \left\langle \frac{|\vec{\nabla} \psi|^2}{R^2} \right\rangle \right) \frac{dI}{d\psi} - \frac{dp}{d\psi}. \quad (7)$$

2 Fluid Velocities and Heat Fluxes

We can write

$$\vec{u}_j^a = u_{\parallel j}^a \vec{b} + \vec{u}_{\perp j}^a, \quad (8)$$

and

$$\vec{u}_0^a = \vec{V}_a, \quad (9)$$

$$\vec{u}_1^a = -\frac{2}{5} \frac{\vec{q}_a}{p_a}, \quad (10)$$

with

$$\vec{u}_{\perp 0}^a = \frac{\vec{E} \times \vec{B}}{B^2} + \frac{\vec{B} \times \vec{\nabla} p_a}{n_a e_a B^2}, \quad (11)$$

$$\vec{u}_{\perp 1}^a = -\frac{\vec{B} \times \vec{\nabla} T_a}{e_a B^2}. \quad (12)$$

It follows that

$$u_{\theta j}^a = \frac{u_{\parallel j}^a}{B} - \frac{V_{\psi j}^a}{B^2}, \quad (13)$$

where

$$u_{\theta j}^a(\psi) = \frac{\vec{u}_j^a \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta}, \quad (14)$$

$$V_{\psi 0}^a(\psi) = -\left(\frac{\vec{u}_{\perp 0} \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta} \right) B^2 = -\frac{I T_a}{e_a} \left(\frac{d \ln p_a}{d\psi} + \frac{e_a}{T_a} \frac{d\Phi}{d\psi} \right), \quad (15)$$

$$V_{\psi 1}^a(\psi) = -\left(\frac{\vec{u}_{\perp 1} \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta} \right) B^2 = \frac{I T_a}{e_a} \frac{d \ln T_a}{d\psi}. \quad (16)$$

3 Friction Forces

$$\begin{pmatrix} \vec{F}_0^a \\ \vec{F}_1^a \end{pmatrix} = \sum_b \begin{pmatrix} l_{00}^{ab} & l_{01}^{ab} \\ l_{10}^{ab} & l_{11}^{ab} \end{pmatrix} \begin{pmatrix} \vec{u}_0^b \\ \vec{u}_1^b \end{pmatrix}, \quad (17)$$

$$l_{jk}^{ab} = \left(\sum_c \frac{n_a m_a}{\tau_{ac}} M_{jk}^{ac} \right) \delta_{ab} + \frac{n_a m_a}{\tau_{ab}} N_{jk}^{ab}, \quad (18)$$

$$\frac{1}{\tau_{ab}} = \frac{4}{3\sqrt{\pi}} \frac{4\pi n_b e_a^2 e_b^2 \ln \Lambda}{(4\pi \epsilon_0)^2 m_a^2 v_{Ta}^3}, \quad (19)$$

$$v_{Ta} = \sqrt{\frac{2T_a}{m_a}}, \quad (20)$$

$$M_{00}^{ab} = -\frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{3/2}}, \quad (21)$$

$$M_{01}^{ab} = -\frac{3}{2} \frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{5/2}} = M_{10}^{ab}, \quad (22)$$

$$M_{11}^{ab} = -\frac{13/4 + 4x_{ab}^2 + (15/2)x_{ab}^4}{(1 + x_{ab}^2)^{5/2}}, \quad (23)$$

$$N_{00}^{ab} = \frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{3/2}}, \quad (24)$$

$$N_{01}^{ab} = \frac{3}{2} \frac{T_a}{T_b} \frac{1 + m_b/m_a}{x_{ab}(1 + x_{ba}^2)^{5/2}}, \quad (25)$$

$$N_{10}^{ab} = \frac{3}{2} \frac{1 + m_a/m_b}{(1 + x_{ab}^2)^{5/2}}, \quad (26)$$

$$N_{11}^{ab} = \frac{27}{4} \frac{T_a}{T_b} \frac{x_{ab}^2}{(1 + x_{ab}^2)^{5/2}}, \quad (27)$$

where

$$x_{ab} = \frac{v_{Tb}}{v_{Ta}}. \quad (28)$$

Species are e , i , and I . Let

$$Z_{\text{eff}} = \frac{n_i e_i^2 + n_I e_I^2}{n_e e_e^2}. \quad (29)$$

Follows that

$$\frac{n_e}{n_i} = \frac{Z_I - 1}{Z_I - Z_{\text{eff}}}, \quad (30)$$

$$\frac{n_I}{n_i} = \frac{Z_{\text{eff}} - 1}{Z_I(Z_I - Z_{\text{eff}})}, \quad (31)$$

$$\frac{n_I}{n_e} = \frac{Z_{\text{eff}} - 1}{Z_I(Z_I - 1)}. \quad (32)$$

Thus,

$$M_{00}^{aa} = -\frac{1}{\sqrt{2}}, \quad (33)$$

$$M_{01}^{aa} = -\frac{3}{4\sqrt{2}} = M_{10}^{aa}, \quad (34)$$

$$M_{11}^{aa} = -\frac{59}{16\sqrt{2}}, \quad (35)$$

$$N_{00}^{aa} = \frac{1}{\sqrt{2}}, \quad (36)$$

$$N_{01}^{aa} = \frac{3}{4\sqrt{2}}, \quad (37)$$

$$N_{10}^{aa} = \frac{3}{4\sqrt{2}}, \quad (38)$$

$$N_{11}^{aa} = \frac{27}{16\sqrt{2}}, \quad (39)$$

for $a = e, i, I$. Now,

$$M_{00}^{iI} = -\frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{3/2}}, \quad (40)$$

$$M_{01}^{iI} = -\frac{3}{2} \frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{5/2}} = M_{10}^{iI}, \quad (41)$$

$$M_{11}^{iI} = -\frac{13/4 + 4x_{iI}^2 + (15/2)x_{iI}^4}{(1 + x_{iI}^2)^{5/2}}, \quad (42)$$

$$N_{00}^{iI} = \frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{3/2}}, \quad (43)$$

$$N_{01}^{iI} = \frac{3}{2} \frac{T_i}{T_I} \frac{1 + m_I/m_i}{x_{iI}(1 + x_{iI}^2)^{5/2}}, \quad (44)$$

$$N_{10}^{iI} = \frac{3}{2} \frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{5/2}}, \quad (45)$$

$$N_{11}^{iI} = \frac{27}{4} \frac{T_i}{T_I} \frac{x_{iI}^2}{(1 + x_{iI}^2)^{5/2}}. \quad (46)$$

Furthermore,

$$M_{00}^{Ii} = -\frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{3/2}}, \quad (47)$$

$$M_{01}^{Ii} = -\frac{3}{2} \frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{5/2}} = M_{10}^{Ii}, \quad (48)$$

$$M_{11}^{Ii} = -\frac{13/4 + 4x_{Ii}^2 + (15/2)x_{Ii}^4}{(1 + x_{Ii}^2)^{5/2}}, \quad (49)$$

$$N_{00}^{Ii} = \frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{3/2}}, \quad (50)$$

$$N_{01}^{Ii} = \frac{3}{2} \frac{T_I}{T_i} \frac{1 + m_i/m_I}{x_{Ii}(1 + x_{Ii}^2)^{5/2}}, \quad (51)$$

$$N_{10}^{Ii} = \frac{3}{2} \frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{5/2}}, \quad (52)$$

$$N_{11}^{Ii} = \frac{27}{4} \frac{T_I}{T_i} \frac{x_{Ii}^2}{(1 + x_{Ii}^2)^{5/2}}. \quad (53)$$

Also,

$$M_{00}^{ea} = -1, \quad (54)$$

$$M_{01}^{ea} = -\frac{3}{2} = M_{10}^{ea}, \quad (55)$$

$$M_{11}^{ea} = -\frac{13}{4}, \quad (56)$$

$$N_{00}^{ea} = 1, \quad (57)$$

$$N_{01}^{ea} = 0, \quad (58)$$

$$N_{10}^{ea} = \frac{3}{2}, \quad (59)$$

$$N_{11}^{ea} = 0, \quad (60)$$

for $a = i, I$, where we have neglected terms of order $(m_e/m_i)^{1/2}$. Finally,

$$M_{00}^{ae} = 0, \quad (61)$$

$$M_{01}^{ae} = 0 = M_{10}^{ae}, \quad (62)$$

$$M_{11}^{ae} = 0, \quad (63)$$

$$N_{00}^{ae} = 0, \quad (64)$$

$$N_{01}^{ae} = 0, \quad (65)$$

$$N_{10}^{ae} = 0 \quad (66)$$

$$N_{11}^{ae} = 0, \quad (67)$$

for $a = i, I$.

Now,

$$l_{jk}^{ii} = \frac{n_i m_i}{\tau_{ii}} \left[\left(\sum_a \frac{n_a e_a^2}{n_i e_i^2} M_{jk}^{ia} \right) + N_{jk}^{ii} \right] \quad (68)$$

$$= \frac{n_i m_i}{\tau_{ii}} (M_{jk}^{ii} + N_{jk}^{ii} + \alpha M_{jk}^{iI}), \quad (69)$$

where

$$\alpha = \frac{n_I e_I^2}{n_i e_i^2} = (Z_{\text{eff}} - 1) \left(\frac{Z_I}{Z_I - Z_{\text{eff}}} \right). \quad (70)$$

Thus,

$$l_{00}^{ii} = -\frac{n_i m_i}{\tau_{ii}} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{3/2}}, \quad (71)$$

$$l_{01}^{ii} = -\frac{n_i m_i}{\tau_{ii}} \frac{3}{2} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{5/2}}, \quad (72)$$

$$l_{10}^{ii} = -\frac{n_i m_i}{\tau_{ii}} \frac{3}{2} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{5/2}}, \quad (73)$$

$$l_{11}^{ii} = -\frac{n_i m_i}{\tau_{ii}} \left\{ \sqrt{2} + \frac{\alpha [13/4 + 4 x_{iI}^2 + (15/2) x_{iI}^4]}{(1 + x_{iI}^2)^{5/2}} \right\}. \quad (74)$$

Now,

$$l_{jk}^{iI} = \frac{n_i m_i}{\tau_{ii}} \alpha N_{jk}^{iI}, \quad (75)$$

so

$$l_{00}^{iI} = \frac{n_i m_i}{\tau_{ii}} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{3/2}}, \quad (76)$$

$$l_{01}^{iI} = \frac{n_i m_i}{\tau_{ii}} \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha (1 + m_I/m_i)}{x_{iI} (1 + x_{iI}^2)^{5/2}}, \quad (77)$$

$$l_{10}^{iI} = \frac{n_i m_i}{\tau_{ii}} \frac{3}{2} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{5/2}}, \quad (78)$$

$$l_{11}^{iI} = \frac{n_i m_i}{\tau_{ii}} \frac{27}{4} \frac{T_i}{T_I} \frac{\alpha x_{iI}^2}{(1 + x_{iI}^2)^{5/2}}. \quad (79)$$

Now,

$$l_{jk}^{II} = \frac{n_I m_I}{\tau_{II}} \left[\left(\sum_a \frac{n_a e_a^2}{n_I e_I^2} M_{jk}^{Ia} \right) + N_{jk}^{II} \right] \quad (80)$$

$$\begin{aligned} &= \frac{n_I m_I}{\tau_{II}} (M_{jk}^{II} + N_{jk}^{II} + \alpha^{-1} M_{jk}^{Ii}) \\ &= \frac{n_i m_i}{\tau_{ii}} \alpha^2 \frac{m_i}{m_I} x_{Ii}^3 (M_{jk}^{II} + N_{jk}^{II} + \alpha^{-1} M_{jk}^{Ii}), \end{aligned} \quad (81)$$

so

$$l_{00}^{II} = -\frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \frac{\alpha (1 + m_I/m_i)}{(1 + x_{Ii}^2)^{3/2}}, \quad (82)$$

$$l_{01}^{II} = -\frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha (1 + m_I/m_i)}{(1 + x_{Ii}^2)^{5/2}}, \quad (83)$$

$$l_{10}^{II} = -\frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha (1 + m_I/m_i)}{(1 + x_{Ii}^2)^{5/2}}, \quad (84)$$

$$l_{11}^{II} = -\frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \left\{ \sqrt{2} \alpha^2 + \frac{\alpha [13/4 + 4 x_{Ii}^2 + (15/2) x_{Ii}^4]}{(1 + x_{Ii}^2)^{5/2}} \right\}. \quad (85)$$

Finally,

$$l_{jk}^{Ii} = \frac{n_i m_i}{\tau_{ii}} \alpha \frac{m_i}{m_I} x_{Ii}^3 N_{jk}^{Ii}, \quad (86)$$

so

$$l_{00}^{Ii} = \frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \frac{\alpha (1 + m_I/m_i)}{(1 + x_{Ii}^2)^{3/2}}, \quad (87)$$

$$l_{01}^{Ii} = \frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{T_I}{T_i} \frac{\alpha (1 + m_i/m_I)}{x_{Ii} (1 + x_{iI}^2)^{5/2}}, \quad (88)$$

$$l_{10}^{Ii} = \frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha (1 + m_I/m_i)}{(1 + x_{Ii}^2)^{5/2}}, \quad (89)$$

$$l_{11}^{Ii} = \frac{n_i m_i}{\tau_{ii}} \frac{m_i}{m_I} x_{Ii}^3 \frac{27}{4} \frac{T_I}{T_i} \frac{\alpha x_{Ii}^2}{(1 + x_{Ii}^2)^{5/2}}. \quad (90)$$

Thus,

$$(F^i) = \frac{n_i m_i}{\tau_{ii}} \{ - [F^{ii}] (u^i) + [F^{iI}] (u^I) \}, \quad (91)$$

$$(F^I) = \frac{n_i m_i}{\tau_{ii}} \{ [F^{Ii}] (u^i) - [F^{II}] (u^I) \}, \quad (92)$$

where

$$(F^a) = \begin{pmatrix} \vec{F}_0^a \\ \vec{F}_1^a \end{pmatrix}, \quad (93)$$

$$(u^a) = \begin{pmatrix} \vec{u}_0^a \\ \vec{u}_1^a \end{pmatrix}. \quad (94)$$

Here,

$$F_{00}^{ii} = \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{3/2}}, \quad (95)$$

$$F_{01}^{ii} = \frac{3}{2} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{5/2}}, \quad (96)$$

$$F_{10}^{ii} = \frac{3}{2} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{5/2}} = F_{01}^{ii}, \quad (97)$$

$$F_{11}^{ii} = \sqrt{2} + \frac{\alpha [13/4 + 4 x_{iI}^2 + (15/2) x_{iI}^4]}{(1 + x_{iI}^2)^{5/2}}, \quad (98)$$

$$F_{00}^{iI} = \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{3/2}} = F_{00}^{ii}, \quad (99)$$

$$F_{01}^{iI} = \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha (1 + m_I/m_i)}{x_{iI} (1 + x_{Ii}^2)^{5/2}}, \quad (100)$$

$$F_{10}^{iI} = \frac{3}{2} \frac{\alpha (1 + m_i/m_I)}{(1 + x_{iI}^2)^{5/2}} = F_{01}^{ii}, \quad (101)$$

$$F_{11}^{iI} = \frac{27}{4} \frac{T_i}{T_I} \frac{\alpha x_{iI}^2}{(1+x_{iI}^2)^{5/2}}, \quad (102)$$

$$F_{00}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{\alpha (1+m_I/m_i)}{(1+x_{Ii}^2)^{3/2}} = \frac{\alpha (1+m_i/m_I)}{(1+x_{iI}^2)^{3/2}} = F_{00}^{ii}, \quad (103)$$

$$F_{01}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha (1+m_i/m_I)}{(1+x_{iI}^2)^{5/2}} = \frac{3}{2} \frac{\alpha (1+m_i/m_I)}{(1+x_{iI}^2)^{5/2}} = F_{01}^{ii}, \quad (104)$$

$$F_{10}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha (1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}} = \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha (1+m_I/m_i)}{x_{iI} (1+x_{iI}^2)^{5/2}} = F_{01}^{iI}, \quad (105)$$

$$F_{11}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{27}{4} \frac{T_i}{T_I} \frac{\alpha x_{Ii}^2}{(1+x_{Ii}^2)^{5/2}} = \frac{27}{4} \frac{\alpha x_{iI}^2}{(1+x_{iI}^2)^{5/2}}, \quad (106)$$

$$F_{00}^{II} = \frac{m_i}{m_I} x_{Ii}^3 \frac{\alpha (1+m_I/m_i)}{(1+x_{Ii}^2)^{3/2}} = \frac{\alpha (1+m_i/m_I)}{(1+x_{iI}^2)^{3/2}} = F_{00}^{ii}, \quad (107)$$

$$F_{01}^{II} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha (1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}} = \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha (1+m_I/m_i)}{x_{iI} (1+x_{iI}^2)^{5/2}} = F_{01}^{iI}, \quad (108)$$

$$F_{10}^{II} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha (1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}} = \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha (1+m_I/m_i)}{x_{iI} (1+x_{iI}^2)^{5/2}} = F_{01}^{iI}, \quad (109)$$

$$\begin{aligned} F_{11}^{II} &= \frac{m_i}{m_I} x_{Ii}^3 \left\{ \sqrt{2} \alpha^2 + \frac{\alpha [13/4 + 4 x_{Ii}^2 + (15/2) x_{Ii}^4]}{(1+x_{Ii}^2)^{5/2}} \right\} \\ &= \frac{T_i}{T_I} \left\{ \sqrt{2} \alpha^2 x_{Ii} + \frac{\alpha [15/2 + 4 x_{iI}^2 + (13/4) x_{iI}^4]}{(1+x_{iI}^2)^{5/2}} \right\}. \end{aligned} \quad (110)$$

Note that $F_{0j}^{ii} = F_{0j}^{Ii}$ and $F_{0j}^{iI} = F_{0j}^{II}$, for $j = 0, 1$, which implies that $(F^I)_0 + (F^i)_0 = 0$. This is a statement of collisional momentum conservation. Furthermore, $F_{j0}^{ii} = F_{j0}^{iI}$ and $F_{j0}^{Ii} = F_{j0}^{II}$.

Now,

$$l_{jk}^{ee} = \frac{n_e m_e}{\tau_{ee}} \left[\left(\sum_a \frac{n_a e_a^2}{n_e e_e^2} M_{jk}^{ea} \right) + N_{jk}^{ee} \right] \quad (111)$$

$$= \frac{n_e m_e}{\tau_{ee}} (M_{jk}^{ee} + N_{jk}^{ee} + Z_{\text{eff}} M_{jk}^{ei}), \quad (112)$$

where $a = e, i, I$. It follows that

$$l_{00}^{ee} = -\frac{n_e m_e}{\tau_{ee}} Z_{\text{eff}}, \quad (113)$$

$$l_{01}^{ee} = -\frac{n_e m_e}{\tau_{ee}} \frac{3}{2} Z_{\text{eff}}, \quad (114)$$

$$l_{10}^{ee} = -\frac{n_e m_e}{\tau_{ee}} \frac{3}{2} Z_{\text{eff}}, \quad (115)$$

$$l_{11}^{ee} = -\frac{n_e m_e}{\tau_{ee}} \left(\sqrt{2} + \frac{13}{4} Z_{\text{eff}} \right). \quad (116)$$

Likewise,

$$l_{jk}^{ea} = \frac{n_e m_e}{\tau_{ee}} Z_{\text{eff } a} N_{jk}^{ea}, \quad (117)$$

where

$$Z_{\text{eff } a} = \frac{n_a e_a^2}{n_e e_e^2}, \quad (118)$$

for $a = i, I$. In particular,

$$Z_{\text{eff } i} = \frac{Z_I - Z_{\text{eff}}}{Z_I - 1}, \quad (119)$$

$$Z_{\text{eff } I} = \frac{Z_I (Z_{\text{eff}} - 1)}{Z_I - 1}. \quad (120)$$

Hence,

$$l_{00}^{ei} = \frac{n_e m_e}{\tau_{ee}} Z_{\text{eff } i}, \quad (121)$$

$$l_{01}^{ei} = 0, \quad (122)$$

$$l_{10}^{ei} = \frac{n_e m_e}{\tau_{ee}} \frac{3}{2} Z_{\text{eff } i}, \quad (123)$$

$$l_{11}^{ei} = 0, \quad (124)$$

and

$$l_{00}^{eI} = \frac{n_e m_e}{\tau_{ee}} Z_{\text{eff } I}, \quad (125)$$

$$l_{01}^{eI} = 0, \quad (126)$$

$$l_{10}^{eI} = \frac{n_e m_e}{\tau_{ee}} \frac{3}{2} Z_{\text{eff } I}, \quad (127)$$

$$l_{11}^{eI} = 0. \quad (128)$$

Thus,

$$(F^e) = \frac{n_e m_e}{\tau_{ee}} \left\{ -[F^{ee}] (u^e) + [F^{ei}] (u^i) + [F^{eI}] (u^I) \right\}, \quad (129)$$

where

$$F_{00}^{ee} = Z_{\text{eff}}, \quad (130)$$

$$F_{01}^{ee} = \frac{3}{2} Z_{\text{eff}}, \quad (131)$$

$$F_{10}^{ee} = \frac{3}{2} Z_{\text{eff}}, \quad (132)$$

$$F_{11}^{ee} = \sqrt{2} + \frac{13}{4} Z_{\text{eff}}, \quad (133)$$

$$F_{00}^{ei} = Z_{\text{eff } i}, \quad (134)$$

$$F_{01}^{ei} = 0, \quad (135)$$

$$F_{10}^{ei} = \frac{3}{2} Z_{\text{eff } i}, \quad (136)$$

$$F_{11}^{ei} = 0, \quad (137)$$

$$F_{00}^{eI} = Z_{\text{eff } I}, \quad (138)$$

$$F_{01}^{eI} = 0, \quad (139)$$

$$F_{10}^{eI} = \frac{3}{2} Z_{\text{eff } I}, \quad (140)$$

$$F_{11}^{eI} = 0. \quad (141)$$

Note that $F_{j0}^{ee} = F_{j0}^{ei} + F_{j0}^{eI}$.

4 Neoclassical Viscosities

$$\left\langle \vec{B} \cdot \vec{\nabla} \cdot (\vec{\Pi}^a) \right\rangle = \frac{n_a m_a}{\tau_{aa}} [\mu^a] (u_\theta^a), \quad (142)$$

where

$$\langle A \rangle = \oint \frac{A(\theta) d\theta}{\vec{B} \cdot \vec{\nabla} \theta} \bigg/ \oint \frac{d\theta}{\vec{B} \cdot \vec{\nabla} \theta}, \quad (143)$$

$$(\vec{\Pi}^a) = \begin{pmatrix} \vec{\Pi}_{\parallel}^a \\ \vec{\Theta}_{\parallel}^a \end{pmatrix}, \quad (144)$$

$$(u_\theta^a) = \begin{pmatrix} u_{\theta 0}^a \langle B^2 \rangle \\ u_{\theta 1}^a \langle B^2 \rangle \end{pmatrix}, \quad (145)$$

$$[\mu^a] = \begin{pmatrix} \mu_{00}^a & \mu_{01}^a \\ \mu_{10}^a & \mu_{11}^a \end{pmatrix}. \quad (146)$$

Let

$$\mu_{00}^a = K_{00}^a, \quad (147)$$

$$\mu_{01}^a = \frac{5}{2} K_{00}^a - K_{01}^a = \mu_{10}^a, \quad (148)$$

$$\mu_{11}^a = K_{11}^a - 5 K_{01}^a + \frac{25}{4} K_{00}^a, \quad (149)$$

and

$$\omega_{ta} = \frac{v_{Ta}}{L_c}, \quad (150)$$

$$\nu_{*a} = \frac{8}{3\pi} \frac{\langle B^2 \rangle}{\langle (\vec{b} \cdot \vec{\nabla} B)^2 \rangle} \frac{g \omega_{ta}}{v_{Ta}^2 \tau_{aa}}, \quad (151)$$

$$g = \frac{1 - f_c}{f_c}, \quad (152)$$

$$f_c = \frac{3}{4} \langle B^2 \rangle \int_0^{1/B_{\max}} \frac{\lambda d\lambda}{\langle \sqrt{1 - \lambda B} \rangle}. \quad (153)$$

We have

$$K_{ij}^a = g \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x} x^{4+i+j} \hat{\nu}_D^a(x) dx}{[x^2 + \nu_{*a} \hat{\nu}_D^a(x)] [x^2 + (5\pi/8) (\omega_{ta} \tau_{aa})^{-1} \hat{\nu}_T^a(x)]}, \quad (154)$$

and

$$\hat{\nu}_D^a(x) = \frac{3\sqrt{\pi}}{4} \sum_b \frac{n_b e_b^2}{n_a e_a^2} \left[\left(1 - \frac{x_{ab}}{2x}\right) \psi\left(\frac{x}{x_{ab}}\right) + \psi'\left(\frac{x}{x_{ab}}\right) \right] \frac{1}{x}, \quad (155)$$

$$\hat{\nu}_\epsilon^a(x) = \frac{3\sqrt{\pi}}{2} \sum_b \frac{n_b e_b^2}{n_a e_a^2} \left[\frac{m_a}{m_b} \psi\left(\frac{x}{x_{ab}}\right) - \psi'\left(\frac{x}{x_{ab}}\right) \right] \frac{1}{x}, \quad (156)$$

$$\hat{\nu}_T^a(x) = 3 \hat{\nu}_D^a(x) + \hat{\nu}_\epsilon^a(x), \quad (157)$$

$$\psi(x) = \frac{2}{\sqrt{\pi}} \int_0^{x^2} \sqrt{t} e^{-t} dt = \text{erf}(x) - \frac{2}{\sqrt{\pi}} x e^{-x^2}, \quad (158)$$

$$\psi'(x) = \frac{2}{\sqrt{\pi}} x e^{-x^2}. \quad (159)$$

In particular,

$$K_{ij}^i = g \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x} x^{2+i+j} \hat{\nu}_D^i(x) dx}{[x + \nu_{*i} \hat{\nu}_D^i(x)] [x + (5\pi/8) (\omega_{ti} \tau_{ii})^{-1} \hat{\nu}_T^i(x)]}, \quad (160)$$

$$\begin{aligned} \hat{\nu}_D^i &= \frac{3\sqrt{\pi}}{4} \left[\left(1 - \frac{1}{2x}\right) \psi(x) + \psi'(x) \right] \frac{1}{x} \\ &\quad + \frac{3\sqrt{\pi}}{4} \alpha \left[\left(1 - \frac{x_{iI}}{2x}\right) \psi\left(\frac{x}{x_{iI}}\right) + \psi'\left(\frac{x}{x_{iI}}\right) \right] \frac{1}{x}, \end{aligned} \quad (161)$$

$$\begin{aligned} \hat{\nu}_\epsilon^i &= \frac{3\sqrt{\pi}}{2} [\psi(x) - \psi'(x)] \frac{1}{x} \\ &\quad + \frac{3\sqrt{\pi}}{2} \alpha \left[\frac{m_i}{m_I} \psi\left(\frac{x}{x_{iI}}\right) - \psi'\left(\frac{x}{x_{iI}}\right) \right] \frac{1}{x}, \end{aligned} \quad (162)$$

and

$$K_{ij}^I = g \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x} x^{2+i+j} \hat{\nu}_D^I(x) dx}{[x + \nu_{*I} \hat{\nu}_D^I(x)] [x + (5\pi/8) (\omega_{tI} \tau_{II})^{-1} \hat{\nu}_T^I(x)]}, \quad (163)$$

$$\begin{aligned} \hat{\nu}_D^I &= \frac{3\sqrt{\pi}}{4} \left[\left(1 - \frac{1}{2x}\right) \psi(x) + \psi'(x) \right] \frac{1}{x} \\ &\quad + \frac{3\sqrt{\pi}}{4} \alpha^{-1} \left[\left(1 - \frac{x_{Ii}}{2x}\right) \psi\left(\frac{x}{x_{Ii}}\right) + \psi'\left(\frac{x}{x_{Ii}}\right) \right] \frac{1}{x}, \end{aligned} \quad (164)$$

$$\begin{aligned} \hat{\nu}_\epsilon^I &= \frac{3\sqrt{\pi}}{2} [\psi(x) - \psi'(x)] \frac{1}{x} \\ &\quad + \frac{3\sqrt{\pi}}{2} \alpha^{-1} \left[\frac{m_I}{m_i} \psi\left(\frac{x}{x_{Ii}}\right) - \psi'\left(\frac{x}{x_{Ii}}\right) \right] \frac{1}{x}, \end{aligned} \quad (165)$$

and

$$K_{ij}^e = g \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x} x^{4+i+j} \hat{\nu}_D^e(x) dx}{[x^2 + \nu_{*e} \hat{\nu}_D^e(x)] [x^2 + (5\pi/8) (\omega_{te} \tau_{ee})^{-1} \hat{\nu}_T^e(x)]}, \quad (166)$$

$$\hat{\nu}_D^e = \frac{3\sqrt{\pi}}{4} \left[\left(1 - \frac{1}{2x}\right) \psi(x) + \psi'(x) \right] + \frac{3\sqrt{\pi}}{4} Z_{\text{eff}}, \quad (167)$$

$$\hat{\nu}_\epsilon^e = \frac{3\sqrt{\pi}}{2} [\psi(x) - \psi'(x)]. \quad (168)$$

5 Parallel Force and Heat Balance Equations

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \vec{\Pi}^a \rangle = \langle \vec{B} \cdot \vec{F}_0^a \rangle + \langle \vec{B} \cdot \vec{F}_0^{acx} \rangle + n_a e_a \langle \vec{E} \cdot \vec{B} \rangle, \quad (169)$$

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \vec{\Theta}^a \rangle = \langle \vec{B} \cdot \vec{F}_1^a \rangle + \langle \vec{B} \cdot \vec{F}_1^{acx} \rangle. \quad (170)$$

Now,

$$\langle \vec{B} \cdot (F)^i \rangle = \frac{n_i m_i}{\tau_{ii}} \{ -[F^{ii}] (u_\parallel^i) + [F^{iI}] (u_\parallel^I) \}, \quad (171)$$

$$\langle \vec{B} \cdot (F)^I \rangle = \frac{n_i m_i}{\tau_{ii}} \{ [F^{Ii}] (u_\parallel^i) - [F^{II}] (u_\parallel^I) \}, \quad (172)$$

$$\langle \vec{B} \cdot (F)^e \rangle = \frac{n_e m_e}{\tau_{ee}} \{ -[F^{ee}] (u_\parallel^e) + [F^{ei}] (u_\parallel^i) + [F^{eI}] (u_\parallel^I) \}, \quad (173)$$

$$\langle \vec{B} \cdot (F^{icx}) \rangle = -n_i m_i \langle \sigma v \rangle_i^{cx} [X^i](\tilde{u}_\parallel^i), \quad (174)$$

$$\langle \vec{B} \cdot (F^{Icx}) \rangle = 0, \quad (175)$$

$$\langle \vec{B} \cdot (F^{ecx}) \rangle = 0, \quad (176)$$

where

$$(F^{a\,cx}) = \begin{pmatrix} \vec{F}_0^{a\,cx} \\ \vec{F}_1^{a\,cx} \end{pmatrix}, \quad (177)$$

$$(u_{\parallel}^a) = \begin{pmatrix} \langle u_{\parallel 0}^a B \rangle \\ \langle u_{\parallel 1}^a B \rangle \end{pmatrix}, \quad (178)$$

$$(\tilde{u}_{\parallel}^a) = \begin{pmatrix} \langle n_n u_{\parallel 0}^a B \rangle \\ \langle n_n u_{\parallel 1}^a B \rangle \end{pmatrix}, \quad (179)$$

$$[X^i] = \begin{pmatrix} 1, & 0 \\ 0, & E_n/T_i \end{pmatrix}. \quad (180)$$

Here, E_n is the neutral kinetic energy, n_n the neutral number density, and $\langle \sigma v \rangle_i^{cx}$ the majority ion charge exchange rate coefficient.

Let

$$(E_{\parallel}) = \begin{pmatrix} \langle E_{\parallel} B \rangle / \langle B \rangle \\ 0 \end{pmatrix}, \quad (181)$$

$$\Omega_a = \frac{e_a \langle B \rangle}{m_a}, \quad (182)$$

$$[\tilde{\mu}^I] = \alpha^2 \frac{m_i}{m_I} x_{Ii}^3 [\mu^I] = \alpha^2 \frac{T_i}{T_I} x_{Ii} [\mu^I]. \quad (183)$$

Follows that

$$\begin{bmatrix} [F^{ii}], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{bmatrix} \begin{bmatrix} (u_{\parallel}^i) \\ (u_{\parallel}^I) \end{bmatrix} \simeq - \begin{bmatrix} [\mu^i] (u_{\theta}^i) \\ [\tilde{\mu}^I] (u_{\theta}^I) \end{bmatrix} - \begin{bmatrix} \tau_{ii} \langle \sigma v \rangle_i^{cx} [X^i] (\tilde{u}_{\parallel i}) \\ 0 \end{bmatrix}, \quad (184)$$

and

$$[\mu^e] (u_{\theta}^e) = -[F^{ee}] (u_{\parallel}^e) + [F^{ei}] (u_{\parallel}^i) + [F^{eI}] (u_{\parallel}^I) + \Omega_e \tau_{ee} (E_{\parallel}). \quad (185)$$

Follows from Eq. (13) that

$$(u_{\parallel}^a) = (u_{\theta}^a) + (V_{\psi}^a), \quad (186)$$

and

$$(\tilde{u}_{\parallel}^a) = \frac{\langle n_n \rangle}{y} (u_{\theta}^a) + \langle n_n \rangle (V_{\psi}^a), \quad (187)$$

where

$$V_{\psi}^a = \begin{pmatrix} V_{\psi 0}^a \\ V_{\psi 1}^a \end{pmatrix}, \quad (188)$$

and

$$y = \frac{\langle n_n \rangle \langle B^2 \rangle}{\langle n_n B^2 \rangle}. \quad (189)$$

Thus,

$$\begin{bmatrix} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{bmatrix} \begin{bmatrix} (u_\theta^i) \\ (u_\theta^I) \end{bmatrix} = - \begin{bmatrix} [F^{ii} + Y^i], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{bmatrix} \begin{bmatrix} (V_\psi^i) \\ (V_\psi^I) \end{bmatrix}, \quad (190)$$

where

$$[Y^i] = \tau_{ii} \langle \sigma v \rangle_i^{cx} \langle n_n \rangle [X^i]. \quad (191)$$

Can write

$$(V_\psi^a) = (V_*^a) + (V_E), \quad (192)$$

where

$$V_{*0}^a(\psi) = -\frac{I T_a}{e_a} \frac{d \ln p_a}{d\psi}, \quad (193)$$

$$V_{*1}^a(\psi) = \frac{I T_a}{e_a} \frac{d \ln T_a}{d\psi}, \quad (194)$$

$$V_{E0}(\psi) = -I \frac{d\Phi}{d\psi}, \quad (195)$$

$$V_{E1}(\psi) = 0. \quad (196)$$

Fact that $F_{j0}^{ii} = F_{j0}^{iI}$ and $F_{j0}^{Ii} = F_{j0}^{II}$ means that we can rewrite Eq. (190) as

$$\begin{aligned} \begin{bmatrix} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{bmatrix} \begin{bmatrix} (u_\theta^i) \\ (u_\theta^I) \end{bmatrix} &= - \begin{bmatrix} [F^{ii} + Y^i], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{bmatrix} \begin{bmatrix} (V_*^i) \\ (V_*^I) \end{bmatrix} \\ &\quad - \tau_{ii} \langle \sigma v \rangle_i^{cx} \langle n_n \rangle \begin{bmatrix} (V_E) \\ (0) \end{bmatrix}. \end{aligned} \quad (197)$$

Let

$$\begin{bmatrix} [L^{ii}], & [L^{iI}] \\ [L^{Ii}], & [L^{II}] \end{bmatrix} = \begin{bmatrix} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{bmatrix}^{-1} \begin{bmatrix} [F^{ii} + Y^i], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{bmatrix}, \quad (198)$$

and

$$\begin{bmatrix} [G^{ii}], & [G^{iI}] \\ [G^{Ii}], & [G^{II}] \end{bmatrix} = \tau_{ii} \langle \sigma v \rangle_i^{cx} \langle n_n \rangle \begin{bmatrix} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{bmatrix}^{-1}. \quad (199)$$

Follows that

$$(u_\theta^i) = -[G^{ii}] (V_E) - [L^{ii}] (V_*^i) - [L^{iI}] (V_*^I), \quad (200)$$

$$(u_\theta^I) = -[G^{Ii}] (V_E) - [L^{II}] (V_*^I) - [L^{Ii}] (V_*^i), \quad (201)$$

$$(u_{\parallel}^i) = ([1] - [G^{ii}]) (V_E) + (1 - [L^{ii}]) (V_*^i) - [L^{iI}] (V_*^I), \quad (202)$$

$$(u_{\parallel}^I) = ([1] - [G^{II}]) (V_E) + (1 - [L^{II}]) (V_*^I) - [L^{Ii}] (V_*^i). \quad (203)$$

Equation (185) becomes

$$\begin{aligned} [F^{ee} + \mu^e] (u_{\theta}^e) = & - ([F^{ee}] - [F^{ei}] - [F^{eI}]) (V_E) - ([F^{ei}] [G^{ii}] + [F^{eI}] [G^{Ii}]) (V_E) \\ & - [F^{ee}] (V_*^e) + \{ [F^{ei}] (1 - [L^{ii}]) - [F^{eI}] [L^{Ii}] \} (V_*^i) \\ & + \{ [F^{eI}] (1 - [L^{II}]) - [F^{ei}] [L^{iI}] \} (V_*^I) + \Omega_{ee} \tau_{ee} (E_{\parallel}). \end{aligned} \quad (204)$$

However, because $F_{j0}^{ee} = F_{j0}^{ei} + F_{j0}^{eI}$, the previous equation becomes

$$\begin{aligned} [F^{ee} + \mu^e] (u_{\theta}^e) = & - ([F^{ei}] [G^{ii}] + [F^{eI}] [G^{Ii}]) (V_E) - [F^{ee}] (V_*^e) \\ & + \{ [F^{ei}] (1 - [L^{ii}]) - [F^{eI}] [L^{Ii}] \} (V_*^i) \\ & + \{ [F^{eI}] (1 - [L^{II}]) - [F^{ei}] [L^{iI}] \} (V_*^I) + \Omega_{ee} \tau_{ee} (E_{\parallel}). \end{aligned} \quad (205)$$

Let

$$[Q^{ee}] = [F^{ee} + \mu^e]^{-1}, \quad (206)$$

$$[G^{ei}] = [Q^{ee}] ([F^{ei}] [G^{ii}] + [F^{eI}] [G^{Ii}]), \quad (207)$$

$$[L^{ee}] = [Q^{ee}] [F^{ee}], \quad (208)$$

$$[L^{ei}] = -[Q^{ee}] \{ [F^{ei}] (1 - [L^{ii}]) - [F^{eI}] [L^{Ii}] \}, \quad (209)$$

$$[L^{eI}] = -[Q^{ee}] \{ [F^{eI}] (1 - [L^{II}]) - [F^{ei}] [L^{iI}] \}. \quad (210)$$

Follows that

$$\begin{aligned} (u_{\theta}^e) = & -[G^{ei}] (V_E) - [L^{ee}] (V_*^e) - [L^{ei}] (V_*^i) - [L^{eI}] (V_*^I) \\ & + \Omega_e \tau_{ee} [Q^{ee}] (E_{\parallel}), \end{aligned} \quad (211)$$

$$\begin{aligned} (u_{\parallel}^e) = & ([1] - [G^{ei}]) (V_E) + (1 - [L^{ee}]) (V_*^e) - [L^{ei}] (V_*^i) - [L^{eI}] (V_*^I) \\ & + \Omega_e \tau_{ee} [Q^{ee}] (E_{\parallel}), \end{aligned} \quad (212)$$

Finally,

$$\begin{aligned} (j_{\parallel}) \equiv & \sum_a n_a e_a (u_{\parallel}^a) \\ = & - (n_i e_i [G^{ii}] + n_I e_I [G^{Ii}] + n_e e_e [G^{ei}]) (V_E) \\ & + \{ n_i e_i (1 - [L^{ii}]) - n_I e_I [L^{Ii}] - n_e e_e [L^{ei}] \} (V_*^i) \\ & + \{ n_I e_I (1 - [L^{II}]) - n_i e_i [L^{iI}] - n_e e_e [L^{eI}] \} (V_*^I) \\ & + n_e e_e (1 - [L^{ee}]) (V_*^e) + n_e e_e \Omega_e \tau_{ee} [Q^{ee}] (E_{\parallel}), \end{aligned} \quad (213)$$

where we have made use of the fact that $n_i e_i + n_I e_I + n_e e_e = 0$.

6 Fluid Velocities

Let

$$\omega_E(\psi) = -\frac{d\Phi}{d\psi}, \quad (214)$$

$$\omega_{*a}(\psi) = -\frac{T_a}{e_a} \frac{d \ln p_a}{d\psi}, \quad (215)$$

$$\eta_a(\psi) = \frac{d \ln T_a}{d \ln n_a}. \quad (216)$$

It follows that

$$V_{*0}^a(\psi) = I \omega_{*a}, \quad (217)$$

$$V_{*1}^a(\psi) = -I \left(\frac{\eta_a}{1 + \eta_a} \right) \omega_{*a}, \quad (218)$$

$$V_{E0}(\psi) = I \omega_E, \quad (219)$$

$$V_{E1}(\psi) = 0. \quad (220)$$

We have $e_e = -e$, $e_i = e$, $e_I = Z_I e$. Thus,

$$\begin{aligned} \frac{V_\theta^i}{B_\theta} \frac{\langle B^2 \rangle}{I} &= -G_{00}^{ii} \omega_E - L_{00}^{ii} \omega_{*i} - L_{00}^{iI} \omega_{*I} + L_{01}^{ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} \\ &\quad + L_{01}^{iI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}, \end{aligned} \quad (221)$$

$$\begin{aligned} \frac{V_\theta^I}{B_\theta} \frac{\langle B^2 \rangle}{I} &= -G_{00}^{II} \omega_E - L_{00}^{II} \omega_{*I} - L_{00}^{Ii} \omega_{*i} + L_{01}^{II} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} \\ &\quad + L_{01}^{Ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i}, \end{aligned} \quad (222)$$

$$\begin{aligned} \frac{\langle V_\parallel^i B \rangle}{I} &= (1 - G_{00}^{ii}) \omega_E + \omega_{*i} - L_{00}^{ii} \omega_{*i} - L_{00}^{iI} \omega_{*I} + L_{01}^{ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} \\ &\quad + L_{01}^{iI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}, \end{aligned} \quad (223)$$

$$\begin{aligned} \frac{\langle V_\parallel^I B \rangle}{I} &= (1 - G_{00}^{II}) \omega_E + \omega_{*I} - L_{00}^{II} \omega_{*I} - L_{00}^{Ii} \omega_{*i} + L_{01}^{II} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} \\ &\quad + L_{01}^{Ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i}. \end{aligned} \quad (224)$$

Also,

$$\frac{V_\phi^i}{R} = \omega_E + \omega_{*i} + \frac{I^2}{R^2 \langle B^2 \rangle} \left[-G_{00}^{ii} \omega_E - L_{00}^{ii} \omega_{*i} - L_{00}^{iI} \omega_{*I} + L_{01}^{ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} \right]$$

$$+ L_{01}^{iI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} \Big], \quad (225)$$

$$\begin{aligned} \frac{V_\phi^I}{R} = \omega_E + \omega_{*I} + \frac{I^2}{R^2 \langle B^2 \rangle} \Big[& -G_{00}^{Ii} \omega_E - L_{00}^{II} \omega_{*I} - L_{00}^{Ii} \omega_{*i} + L_{01}^{II} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} \\ & + L_{01}^{Ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} \Big]. \end{aligned} \quad (226)$$

7 Fluid Angular Rotation Velocities

Let

$$\omega_{\theta i}(\psi) = \frac{V_\theta^i}{B_\theta} \frac{I}{R^2}, \quad (227)$$

$$\omega_{\theta I}(\psi) = \frac{V_\theta^I}{B_\theta} \frac{I}{R^2}, \quad (228)$$

$$\omega_{\phi i}(\psi) = \frac{V_\phi^i}{R}, \quad (229)$$

$$\omega_{\phi I}(\psi) = \frac{V_\phi^I}{R}, \quad (230)$$

where the right-hand sides are evaluated on the outboard mid-plane. It follows that

$$\omega_{\theta i} = K_\theta \omega_{\text{nc} i}, \quad (231)$$

$$\omega_{\theta I} = K_\theta \omega_{\text{nc} I}, \quad (232)$$

$$\omega_{\phi i} = \omega_E + \omega_{*i} + K_\theta \omega_{\text{nc} i}, \quad (233)$$

$$\omega_{\phi I} = \omega_E + \omega_{*I} + K_\theta \omega_{\text{nc} I}, \quad (234)$$

where

$$K_\theta = \frac{I^2}{R^2 \langle B^2 \rangle}, \quad (235)$$

and

$$\omega_{\text{nc} i} = -G_{00}^{ii} \omega_E - L_{00}^{ii} \omega_{*i} + L_{01}^{ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} - L_{00}^{iI} \omega_{*I} + L_{01}^{iI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}, \quad (236)$$

$$\omega_{\text{nc} I} = -G_{00}^{II} \omega_E - L_{00}^{II} \omega_{*I} + L_{01}^{II} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} - L_{00}^{Ii} \omega_{*i} + L_{01}^{Ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i}. \quad (237)$$

8 Natural Frequencies

Let

$$\omega_{0\text{ln}} = \omega_E + \omega_{*e}, \quad (238)$$

$$\omega_{0\text{eb}} = \omega_E, \quad (239)$$

$$\omega_{0\text{nl}} = \omega_E + \omega_{*i} + \omega_{\text{nc}i}. \quad (240)$$

The corresponding natural frequencies are

$$\varpi_{0\text{ln}} = -n \omega_{0\text{ln}}, \quad (241)$$

$$\varpi_{0\text{eb}} = -n \omega_{0\text{eb}}, \quad (242)$$

$$\varpi_{0\text{nl}} = -n \omega_{0\text{nl}}. \quad (243)$$

9 Current Densities

Can write

$$j_{\parallel} = j_{\text{bootstrap}} + j_{\text{ohmic}}, \quad (244)$$

where

$$\begin{aligned} \frac{\langle j_{\text{bootstrap}} B \rangle}{I} = & - \left\{ P_{00}^{ei} - \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) G_{00}^{ii} - \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) G_{00}^{Ii} \right\} n_e e \frac{d\Phi}{d\psi} \\ & - \left\{ 1 - L_{00}^{ii} - \left(\frac{Z_{\text{eff}} - 1}{Z_I - Z_{\text{eff}}} \right) L_{00}^{Ii} + \left(\frac{Z_I - 1}{Z_I - Z_{\text{eff}}} \right) L_{00}^{ei} \right\} \frac{dp_i}{d\psi} \\ & - \left\{ 1 - L_{00}^{II} - \left(\frac{Z_I - Z_{\text{eff}}}{Z_{\text{eff}} - 1} \right) L_{00}^{iI} + \left(\frac{Z_I - 1}{Z_{\text{eff}} - 1} \right) L_{00}^{eI} \right\} \frac{dp_I}{d\psi} \\ & - (1 - L_{00}^{ee}) \frac{dp_e}{d\psi} \\ & - \left\{ L_{01}^{ii} + \left(\frac{Z_{\text{eff}} - 1}{Z_I - Z_{\text{eff}}} \right) L_{01}^{Ii} - \left(\frac{Z_I - 1}{Z_I - Z_{\text{eff}}} \right) L_{01}^{ei} \right\} n_i \frac{dT_i}{d\psi} \\ & - \left\{ L_{01}^{II} + \left(\frac{Z_I - Z_{\text{eff}}}{Z_{\text{eff}} - 1} \right) L_{01}^{iI} - \left(\frac{Z_I - 1}{Z_{\text{eff}} - 1} \right) L_{01}^{eI} \right\} n_I \frac{dT_I}{d\psi} \\ & - L_{01}^{ee} n_e \frac{dT_e}{d\psi} \end{aligned} \quad (245)$$

and

$$\langle j_{\text{ohmic}} B \rangle = Q_{00}^{ee} \sigma_{ee} \langle E_{\parallel} B \rangle, \quad (246)$$

where

$$\sigma_{ee} = \frac{n_e e^2 \tau_{ee}}{m_e}. \quad (247)$$

Let

$$\omega_{\parallel} = \frac{\langle j_{\parallel} B \rangle}{n_e e I}. \quad (248)$$

It follows that

$$\omega_{\parallel} = \omega_{\parallel \text{bootstrap}} + \omega_{\parallel \text{ohmic}}, \quad (249)$$

where

$$\begin{aligned}
\omega_{\parallel \text{bootstrap}} = & \left\{ G_{00}^{ei} - \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) G_{00}^{ii} - \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) G_{00}^{Ii} \right\} \omega_E \\
& + \left\{ \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) (1 - L_{00}^{ii}) - \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) L_{00}^{Ii} + L_{00}^{ei} \right\} \omega_{*i} \\
& + \left\{ \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) (1 - L_{00}^{II}) - \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) L_{00}^{iI} + L_{00}^{eI} \right\} \omega_{*I} \\
& - (1 - L_{00}^{ee}) \omega_{*e} \\
& + \left\{ \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) L_{01}^{ii} + \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) L_{01}^{Ii} - L_{01}^{ei} \right\} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} \\
& + \left\{ \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) L_{01}^{II} + \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) L_{01}^{iI} - L_{01}^{eI} \right\} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} \\
& - L_{01}^{ee} \left(\frac{\eta_e}{1 + \eta_e} \right) \omega_{*e}, \tag{250}
\end{aligned}$$

and

$$\omega_{\parallel \text{ohmic}} = Q_{00}^{ee} |\Omega_e| \tau_{ee} \omega_{\text{loop}}, \tag{251}$$

with

$$\omega_{\text{loop}} = \frac{\langle E_{\parallel} B \rangle}{I \langle B \rangle} = \frac{V_{\text{loop}}}{2\pi} \frac{\langle B/R \rangle}{I \langle B \rangle}. \tag{252}$$

Can also write

$$\omega_{\parallel \text{bootstrap}} = -(\omega_{*e} + \omega_{\text{nce}}) + \frac{n_i}{n_e} (\omega_{*i} + \omega_{\text{nci}}) + \frac{Z_I n_I}{n_e} (\omega_{*I} + \omega_{\text{ncI}}), \tag{253}$$

where

$$\begin{aligned}
\omega_{\text{nce}} = & -G_{00}^{ei} \omega_E - L_{00}^{ee} \omega_{*e} + L_{01}^{ee} \left(\frac{\eta_e}{1 + \eta_e} \right) \omega_{*e} - L_{00}^{ei} \omega_{*i} + L_{01}^{ei} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} \\
& - L_{00}^{eI} \omega_{*I} + L_{01}^{eI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}. \tag{254}
\end{aligned}$$

10 Model Poloidal Flow Damping Term

Our model poloidal flow damping force density in the plasma equation of motion takes the form

$$\frac{m_i n_i}{\tau_{\theta}} (V_{\theta} - V_{\theta}^{nc}) \mathbf{e}_{\theta}, \tag{255}$$

where

$$\frac{1}{\tau_{\theta}} = \left(\frac{I^2}{\langle |\nabla \psi|^2 \rangle} \right) \frac{\mu_{00}^i}{\tau_{ii}}, \tag{256}$$

and V_{θ}^{nc} is specified in Eq. (221).