The EPEC Model

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I. PLASMA RESPONSE IN OUTER REGION

A. Coordinates

Let R, ϕ, and Z be right-handed cylindrical coordinates whose symmetry axis corresponds to the toroidal symmetry axis of the plasma. Let r, θ, and ϕ be right-handed flux coordinates whose Jacobian is $\mathcal{J} \equiv (\nabla r \times \nabla\theta \cdot \nabla\phi)^{-1} = r\,R^2/R_0$. Here, R_0 is a convenient scale major radius, r is a magnetic flux-surface label with dimensions of length, and θ is an axisymmetric angular coordinate that increases by 2π radians for every poloidal circuit of the magnetic axis. Let $r = 0$ correspond to the magnetic axis, and let $r = r_{100}$ correspond to the LCFS. Let $\theta = 0$ correspond to the inboard midplane, and let $0 < \theta < \pi$ correspond to the region above the midplane.

B. Equilibrium Magnetic Field

The equilibrium magnetic field is written $\mathbf{B} = R_0\,B_0\,[f(r)\,\nabla\phi \times \nabla r + g(r)\,\nabla\phi]$, where B_0 is a convenient scale toroidal magnetic field-strength, and $q(r) = r\,g/(R_0\,f)$ is the safety-factor profile. The equilibrium poloidal magnetic flux (divided by 2π), $\Psi_p(r)$, satisfies $d\Psi_p/dr = R_0\,B_0\,f(r)$, where, by convention, $\Psi_p(r_{100}) = 0$. The normalized poloidal magnetic flux, $\Psi_N(r)$, is defined such that $\Psi_N(r) = 1 - \Psi_p(r)/\Psi_p(0)$. Hence, $\Psi_N(0) = 0$ and $\Psi_N(r_{100}) = 1$. Finally, if $\Psi_N(r_{95}) = 0.95$ then $q_{95} \equiv q(r_{95})$.

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C. Perturbed Magnetic Field

Consider the response of the plasma to an RMP with $n > 0$ periods in the toroidal direction. We can write the components of the perturbed magnetic field in the form

$$\frac{r\,R^2\,\delta\mathbf{B}\cdot\nabla r}{R_0^2} = \mathrm{i}\sum_j \psi_j(r)\,\mathrm{e}^{\,\mathrm{i}\,(m_j\theta-n\,\phi)}, \tag{1}$$

$$\mathcal{J}\,\delta\mathbf{B}\cdot\nabla\phi\times\nabla r = -\sum_j \Xi_j(r)\,\mathrm{e}^{\,\mathrm{i}\,(m_j\theta-n\,\phi)}, \tag{2}$$

$$R^2\,\delta\mathbf{B}\cdot\nabla\phi = n\sum_j \frac{\Xi_j(r)}{m_j}\,\mathrm{e}^{\,\mathrm{i}\,(m_j\,\theta-n\,\phi)}, \tag{3}$$

where the sum is over all relevant poloidal harmonics of the perturbed magnetic field.

Let there be K resonant magnetic flux-surfaces in the plasma, labelled 1 through K. Consider the kth resonant surface, $r = r_k$, at which $n\,q(r_k) = m_k$, where m_k is a positive integer. Let $\Psi_k = \psi_k(r_k)/m_k$, and $\Delta\Psi_k = [\Xi_k]_{r_{k-}}^{r_{k+}}$. Here, Ψ_k is the (complex) reconnected helical magnetic flux (divided by $2\pi\,R_0$) at the kth resonant surface, whereas $\Delta\Psi_k$ (which has the same dimensions as Ψ_k) is a (complex) measure of the strength of the current sheet (consisting of filaments running parallel to the local equilibrium magnetic field) at the same resonant surface.

D. Toroidal Tearing Mode Dispersion Relation

In the presence of the RMP, the Ψ_k and the $\Delta\Psi_k$ values are related according to the inhomogeneous toroidal tearing mode dispersion relation, which takes the form

$$\Delta\Psi_k = \sum_{k'=1,K} E_{kk'}\,\Psi_{k'} + |E_{kk}|\,\chi_k. \tag{4}$$

Here, $E_{kk'}$ (for k, $k' = 1$, K) is the dimensionless, Hermitian, toroidal tearing mode stability matrix, whereas the χ_k (for $k = 1, K$) parameterize the current sheets driven at the various resonant surfaces when the plasma responds to the applied RMP in accordance with the equations of linearized, marginally-stable, ideal-MHD.

The EPEC model determines the elements of the $E_{kk'}$ matrix using a high-q approximation. In fact, if $F_{kk'}$ is the inverse of the $E_{kk'}$ matrix then

$$F_{kk'} = \oint\oint G(R_k, Z_k; R_{k'}, Z_{k'})\,\mathrm{e}^{-\mathrm{i}\,(m_k\,\theta_k - m_{k'}\,\theta_{k'})}\,\frac{d\theta_k}{2\pi}\,\frac{d\theta_{k'}}{2\pi}, \tag{5}$$

and

$$G(R_k, Z_k; R_{k'}, Z_{k'}) = \frac{(-1)^n\,\pi^2\,R_k\,R_{k'}/R_0}{2\,\Gamma(1/2)\,\Gamma(n+1/2)}\left[\frac{\cosh\eta_{kk'}}{R_k^2+R_{k'}^2+(Z_k-Z_{k'})^2}\right]^{1/2}$$
$$\times\left[(n-1/2)\,P^{n-1}_{-1/2}(\cosh\eta_{kk'})+\frac{P^{n+1}_{-1/2}(\cosh\eta_{kk'})}{n+1/2}\right], \tag{6}$$

with

$$\eta_{kk'} = \tanh^{-1}\left[\frac{2\,R_k\,R_{k'}}{R_k^2+R_{k'}^2+(Z_k-Z_{k'})^2}\right]. \tag{7}$$

Here, the double integral in Eq. (5) is taken around the kth resonant surface (cylindrical coordinates R_k, 0, Z_k; flux coordinates r_k, θ_k, 0, with r_k constant; resonant poloidal mode number m_k) and the k'th resonant surface (cylindrical coordinates $R_{k'}$, 0, $Z_{k'}$; flux coordinates $r_{k'}$, $\theta_{k'}$, 0, with $r_{k'}$ constant; resonant poloidal mode number $m_{k'}$). Finally, the $\Gamma(z)$ and $P_\mu^\nu(z)$ are gamma functions and associated Legendre functions, respectively.

The (complex) χ_k parameters are determined from the GPEC code. To be more exact, the GPEC code calculates the (complex) dimensionless $\Delta_{m_k\,n}$ parameters which measure the strengths of the ideal current sheets that develop at the various resonant magnetic flux-surfaces in the plasma in response to the applied RMP. The $\Delta_{m_k\,n}$ parameters are related to the χ_k parameters according to

$$\frac{\chi_k}{R_0\,B_0} = -{\rm i}\,\frac{\Delta_{m_k\,n}}{|E_{kk}|}\left(\frac{r_k}{R_0}\right)^2\frac{g(r_k)}{m_k\,[a_{kk}(r_k)+(r_k/R_0\,q_k)^2]}, \tag{8}$$

where $q_k = m_k/n$, and $a_{kk}(r) = \oint |\nabla r|^{-2}\,d\theta/(2\pi)$.

II. NEOCLASSICAL PHYSICS

A. Plasma Species

The plasma is assumed to consist of three (charged) species; namely, electrons (e), majority ions (i), and impurity ions (I). The charges of the three species are $e_e = -e$, $e_i = e$, and $e_I = Z_I\,e$, respectively, where e is the magnitude of the electron charge. Quasi-neutrality demands that $n_e = n_i + Z_I\,n_I$, where $n_a(r)$ is the species-a number density. Let $\alpha_I(r) = Z_I\,(Z_{\rm eff}-1)/(Z_I - Z_{\rm eff})$, where $Z_{\rm eff}(r) = (n_i + Z_I^2\,n_I)/n_e$ is the effective ion charge number. It follows that $n_i/n_e = (Z_I - Z_{\rm eff})/(Z_I-1)$ and $n_I/n_e = (Z_{\rm eff}-1)/[Z_I\,(Z_I-1)]$. Finally, let $Z_{{\rm eff}\,i} = (Z_I - Z_{\rm eff})/(Z_I-1)$, and $Z_{{\rm eff}\,I} = Z_I\,(Z_{\rm eff}-1)/(Z_I-1)$. Note that $Z_{\rm eff} = Z_{{\rm eff}\,i} + Z_{{\rm eff}\,I}$.

B. Collisionality Parameters

Consider an equilibrium magnetic flux-surface whose label is r. Let

$$\frac{1}{\gamma(r)} = \frac{q}{g} \oint \frac{B\,R^2}{B_0\,R_0^2}\,\frac{d\theta}{2\pi}, \tag{9}$$

where $B = |\mathbf{B}|$. It is helpful to define a new poloidal angle Θ such that

$$\frac{d\Theta}{d\theta} = \frac{\gamma\,q}{g}\,\frac{B\,R^2}{B_0\,R_0^2}. \tag{10}$$

Let

$$I_1 = \oint \frac{B_0}{B}\,\frac{d\Theta}{2\pi}, \tag{11}$$

$$I_2 = \oint \frac{B}{B_0}\,\frac{d\Theta}{2\pi}, \tag{12}$$

$$I_3 = \oint \left(\frac{\partial B}{\partial\Theta}\right)^2 \frac{1}{B_0\,B}\,\frac{d\Theta}{2\pi}, \tag{13}$$

$$I_{4,j} = \sqrt{2\,j}\oint \frac{\cos(j\,\Theta)}{B/B_0}\,\frac{d\Theta}{2\pi}, \tag{14}$$

$$I_{5,j} = \sqrt{2\,j}\oint \frac{\cos(j\,\Theta)}{2\,(B/B_0)^2}\,\frac{d\Theta}{2\pi}, \tag{15}$$

$$I_6(\lambda) = \oint \frac{\sqrt{1-\lambda\,B/B_{\rm max}}}{B/B_0}\,\frac{d\Theta}{2\pi}, \tag{16}$$

where $B_{\rm max}$ is the maximum value of B on the magnetic flux-surface, and j a positive integer. The species-a transit frequency is written $\omega_{t\,a}(r) = K_t\,\gamma\,v_{T\,a}$, where

$$K_t(r) = \frac{I_1^2\,I_3}{I_2^2\,\sum_{j=1,\infty} I_{4,j}\,I_{5,j}}, \tag{17}$$

and $v_{T\,a} = \sqrt{2\,T_a/m_a}$. Here, m_a is the species-a mass, and $T_a(r)$ the species-a temperature (in energy units). The fraction of circulating particles is

$$f_c(r) = \frac{3\,I_2}{4}\,\frac{B_0^2}{B_{\rm max}^2}\int_0^1 \frac{\lambda\,d\lambda}{I_6(\lambda)}. \tag{18}$$

Finally, the dimensionless species-a collisionality parameter is written $\nu_{*\,a}(r) = K_*\,g_t/(\omega_{t\,a}\,\tau_{aa})$, where $g_t(r) = f_c/(1-f_c)$,

$$K_*(r) = \frac{3}{8\pi}\,\frac{I_2}{I_3}\,K_t^2, \tag{19}$$

$$\frac{1}{\tau_{aa}(r)} = \frac{4}{3\sqrt{\pi}}\,\frac{4\pi\,n_a\,e_a^4\,\ln\Lambda}{(4\pi\,\epsilon_0)^2\,m_a^2\,v_{T\,a}^3}. \tag{20}$$

Here, the Coulomb logarithm, $\ln\Lambda$, is assumed to take the same large constant value (i.e., $\ln\Lambda \simeq 17$), independent of species.

C. Collisional Friction Matrices

Let $x_{ab} = v_{T\,b}/v_{T\,a}$. In the following, all quantities that are of order $(m_e/m_i)^{1/2}$, $(m_e/m_I)^{1/2}$, or smaller, are neglected with respect to unity. The 2×2 dimensionless ion collisional friction matrices, $[F^{\,ii}](r)$, $[F^{\,iI}](r)$, $[F^{\,Ii}](r)$, and $[F^{\,II}](r)$, are defined to have the following elements:

$$F^{\,ii}_{00} = \frac{\alpha_I\,(1+m_i/m_I)}{(1+x_{iI}^{\,2})^{\,3/2}}, \tag{21}$$

$$F^{\,ii}_{01} = \frac{3}{2}\,\frac{\alpha_I\,(1+m_i/m_I)}{(1+x_{iI}^{\,2})^{\,5/2}}, \tag{22}$$

$$F^{\,ii}_{11} = \sqrt{2} + \frac{\alpha_I\,[13/4 + 4\,x_{iI}^{\,2} + (15/2)\,x_{iI}^{\,4}]}{(1+x_{iI}^{\,2})^{\,5/2}}, \tag{23}$$

$$F^{\,iI}_{01} = \frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha_I\,(1+m_I/m_i)}{x_{iI}\,(1+x_{Ii}^{\,2})^{\,5/2}}, \tag{24}$$

$$F^{\,iI}_{11} = \frac{27}{4}\,\frac{T_i}{T_I}\,\frac{\alpha_I\,x_{iI}^{\,2}}{(1+x_{iI}^{\,2})^{\,5/2}}, \tag{25}$$

$$F^{\,Ii}_{11} = \frac{27}{4}\,\frac{\alpha_I\,x_{iI}^{\,2}}{(1+x_{iI}^{\,2})^{\,5/2}}, \tag{26}$$

$$F^{\,II}_{11} = \frac{T_i}{T_I}\left\{\sqrt{2}\,\alpha_I^{\,2}\,x_{Ii} + \frac{\alpha_I\,[15/2 + 4\,x_{iI}^{\,2} + (13/4)\,x_{iI}^{\,4}]}{(1+x_{iI}^{\,2})^{\,5/2}}\right\}, \tag{27}$$

$F^{\,ii}_{10} = F^{\,ii}_{01}$, $F^{\,iI}_{00} = F^{\,ii}_{00}$, $F^{\,iI}_{10} = F^{\,ii}_{01}$, $F^{\,Ii}_{00} = F^{\,ii}_{00}$, $F^{\,Ii}_{01} = F^{\,ii}_{01}$, $F^{\,Ii}_{10} = F^{\,iI}_{01}$, $F^{\,II}_{00} = F^{\,ii}_{00}$, $F^{\,II}_{01} = F^{\,iI}_{01}$, $F^{\,II}_{10} = F^{\,iI}_{01}$.

The 2×2 dimensionless electron collisional friction matrices, $[F^{\,ee}](r)$, $[F^{\,ei}](r)$, and $[F^{\,eI}](r)$, are defined to have the following elements: $F^{\,ee}_{00} = Z_{\rm eff}$, $F^{\,ee}_{01} = (3/2)\,Z_{\rm eff}$, $F^{\,ee}_{10} = F^{\,ee}_{01}$, $F^{\,ee}_{11} = \sqrt{2} + (13/4)\,Z_{\rm eff}$, $F^{\,ei}_{00} = Z_{{\rm eff}\,i}$, $F^{\,ei}_{01} = F^{\,ei}_{11} = 0$, $F^{\,ei}_{10} = (3/2)\,Z_{{\rm eff}\,i}$, $F^{\,eI}_{00} = Z_{{\rm eff}\,I}$, $F^{\,eI}_{01} = F^{\,eI}_{11} = 0$, $F^{\,eI}_{10} = (3/2)\,Z_{{\rm eff}\,I}$.

D. Neoclassical Viscosity Matrices

The 2×2 dimensionless species-a neoclassical viscosity matrix, $[\mu^{\,a}](r)$, is defined to have the following elements: $\mu^{\,a}_{00} = K^{\,a}_{00}$, $\mu^{\,a}_{01} = (5/2)\,K^{\,a}_{00} - K^{\,a}_{01}$, $\mu^{\,a}_{10} = \mu^{\,a}_{01}$, $\mu^{\,a}_{11} = K^{\,a}_{11} - 5\,K^{\,a}_{01} + (25/4)\,K^{\,a}_{00}$. Here,

$$K^{\,e}_{jk} = g_t\,\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{e^{-x}\,x^{\,4+j+k}\,\nu_D^{\,e}(x)\,dx}{[x^{\,2} + \nu_{\ast\,e}\,\nu_D^{\,e}(x)]\,[x^{\,2} + (5\pi/8)\,(\omega_{t\,e}\,\tau_{ee})^{-1}\,\nu_T^{\,e}(x)]}, \tag{28}$$

$$\nu_D^e = \frac{3\sqrt{\pi}}{4}\left[\left(1-\frac{1}{2\,x}\right)\psi(x)+\psi'(x)\right]+\frac{3\sqrt{\pi}}{4}\,Z_{\rm eff}, \tag{29}$$

$$\nu_\epsilon^e = \frac{3\sqrt{\pi}}{2}\left[\psi(x)-\psi'(x)\right], \tag{30}$$

$$\nu_T^a(x) = 3\,\nu_D^a(x)+\nu_\epsilon^{\,a}(x), \tag{31}$$

and

$$\psi(x) = \frac{2}{\sqrt{\pi}}\int_0^x {\rm e}^{-t^{\,2}}\,dt-\frac{2}{\sqrt{\pi}}\,x\,{\rm e}^{-x^{\,2}}, \tag{32}$$

$$\psi'(x) = \frac{2}{\sqrt{\pi}}\,x\,{\rm e}^{-x^{\,2}}. \tag{33}$$

Furthermore,

$$K_{jk}^{\,i} = g_t\,\frac{4}{3\sqrt{\pi}}\int_0^\infty\frac{e^{-x}\,x^{\,2+j+k}\,\nu_D^{\,i}(x)\,dx}{[x+\nu_{*\,i}\,\nu_D^{\,i}(x)]\,[x+(5\pi/8)\,(\omega_{t\,i}\,\tau_{ii})^{-1}\,\nu_T^{\,i}(x)]}, \tag{34}$$

$$\nu_D^{\,i} = \frac{3\sqrt{\pi}}{4}\left[\left(1-\frac{1}{2\,x}\right)\psi(x)+\psi'(x)\right]\frac{1}{x} + \frac{3\sqrt{\pi}}{4}\,\alpha_I\left[\left(1-\frac{x_{iI}}{2\,x}\right)\psi\!\left(\frac{x}{x_{iI}}\right)+\psi'\!\left(\frac{x}{x_{iI}}\right)\right]\frac{1}{x}, \tag{35}$$

$$\nu_\epsilon^{\,i} = \frac{3\sqrt{\pi}}{2}\left[\psi(x)-\psi'(x)\right]\frac{1}{x} + \frac{3\sqrt{\pi}}{2}\,\alpha_I\left[\frac{m_i}{m_I}\,\psi\!\left(\frac{x}{x_{iI}}\right)-\psi'\!\left(\frac{x}{x_{iI}}\right)\right]\frac{1}{x}, \tag{36}$$

and, finally,

$$K_{jk}^{\,I} = g_t\,\frac{4}{3\sqrt{\pi}}\int_0^\infty\frac{e^{-x}\,x^{\,2+j+k}\,\nu_D^{\,I}(x)\,dx}{[x+\nu_{*\,I}\,\nu_D^{\,I}(x)]\,[x+(5\pi/8)\,(\omega_{t\,I}\,\tau_{II})^{-1}\,\nu_T^{\,I}(x)]}, \tag{37}$$

$$\nu_D^{\,I} = \frac{3\sqrt{\pi}}{4}\left[\left(1-\frac{1}{2\,x}\right)\psi(x)+\psi'(x)\right]\frac{1}{x} + \frac{3\sqrt{\pi}}{4}\,\frac{1}{\alpha_I}\left[\left(1-\frac{x_{Ii}}{2\,x}\right)\psi\!\left(\frac{x}{x_{Ii}}\right)+\psi'\!\left(\frac{x}{x_{Ii}}\right)\right]\frac{1}{x}, \tag{38}$$

$$\nu_\epsilon^{\,I} = \frac{3\sqrt{\pi}}{2}\left[\psi(x)-\psi'(x)\right]\frac{1}{x} + \frac{3\sqrt{\pi}}{2}\,\frac{1}{\alpha_I}\left[\frac{m_I}{m_i}\,\psi\!\left(\frac{x}{x_{Ii}}\right)-\psi'\!\left(\frac{x}{x_{Ii}}\right)\right]\frac{1}{x}. \tag{39}$$

Note that our expressions for the neoclassical viscosity matrices interpolate in the most accurate manner possible between the three standard neoclassical collisionality regimes (i.e., the banana, plateau, and Pfirsch-Schüter regimes).

E. Parallel Force and Heat Balance

Let $[\tilde{\mu}^{\,I}] = \alpha_I^2\,(T_i/T_I)\,x_{Ii}\,[\mu^{\,I}]$. The requirement of equilibrium force and heat balance parallel to the magnetic field leads us to define four 2×2 dimensionless ion matrices, $[L^{\,ii}](r)$, $[L^{\,iI}](r)$, $[L^{\,Ii}](r)$, and $[L^{\,II}](r)$, where

$$\begin{pmatrix} [L^{\,ii}], & [L^{\,iI}] \\ [L^{\,Ii}], & [L^{\,II}] \end{pmatrix} = \begin{pmatrix} [F^{\,ii}+\mu^{\,i}+Y^{\,in}/y_n], & -[F^{\,iI}] \\ -[F^{\,Ii}], & [F^{\,II}+\tilde{\mu}^{\,I}] \end{pmatrix}^{-1} \begin{pmatrix} [F^{\,ii}+Y^{\,in}], & -[F^{\,iI}] \\ -[F^{\,Ii}], & [F^{\,II}] \end{pmatrix}, \tag{40}$$

the additional four 2×2 dimensionless ion matrices, $[G^{\,ii}](r)$, $[G^{\,iI}](r)$, $[G^{\,Ii}](r)$, and $[G^{\,II}](r)$, where

$$\begin{pmatrix} [G^{\,ii}], & [G^{\,iI}] \\ [G^{\,Ii}], & [G^{\,II}] \end{pmatrix} = \tau_{ii}\,\langle\sigma\,v\rangle_i^{\rm cx}\,\langle n_n\rangle \begin{pmatrix} [F^{\,ii}+\mu^{\,i}+Y^{\,in}/y_n], & -[F^{\,iI}] \\ -[F^{\,Ii}], & [F^{\,II}+\tilde{\mu}^{\,I}] \end{pmatrix}^{-1}, \tag{41}$$

and the 2×2 dimensionless electron matrices, $[Q^{\,ee}](r)$, $[G^{\,ei}](r)$, $[L^{\,ee}](r)$, $[L^{\,ei}](r)$, and $[L^{\,eI}](r)$, where

$$[Q^{\,ee}] = [F^{\,ee}+\mu^{\,e}]^{-1}, \tag{42}$$

$$[G^{\,ei}] = [Q^{\,ee}]\,([F^{\,ei}]\,[G^{\,ii}] + [F^{\,eI}]\,[G^{\,Ii}]), \tag{43}$$

$$[L^{\,ee}] = [Q^{\,ee}]\,[F^{\,ee}], \tag{44}$$

$$[L^{\,ei}] = [Q^{\,ee}]\left\{[F^{\,ei}]\,[L^{\,ii}] - [F^{\,ei}] + [F^{\,eI}]\,[L^{\,Ii}]\right\}, \tag{45}$$

$$[L^{\,eI}] = [Q^{\,ee}]\left\{[F^{\,eI}]\,[L^{\,II}] - [F^{\,eI}] + [F^{\,ei}]\,[L^{\,iI}]\right\}. \tag{46}$$

Here,

$$[Y^{\,in}] = \tau_{ii}\,\langle\sigma\,v\rangle_i^{\rm cx}\,\langle n_n\rangle \begin{bmatrix} 1, & 0 \\ 0, & E_n/T_i \end{bmatrix}, \tag{47}$$

$$y_n = \frac{\langle n_n\rangle\,\langle B^{\,2}\rangle}{\langle n_n\,B^{\,2}\rangle}, \tag{48}$$

where

$$\langle A\rangle(r) \equiv \oint \frac{A(r,\Theta)\,d\Theta}{B(r,\Theta)} \bigg/ \oint \frac{d\Theta}{B(r,\Theta)}. \tag{49}$$

Moreover, $\langle\sigma\,v\rangle_i^{\rm cx}$ is the flux-surface averaged rate constant for charge exchange reactions between neutrals and majority ions, $n_n(r,\Theta)$ the neutral particle number density, and E_n/T_i

the ratio of the incoming neutral energy to the majority ion energy. The parameter y_n takes into account the fact that the incoming neutrals at the edge of an H-mode tokamak plasma are usually concentrated at the X-point (i.e., $y_n > 1$).

F. Neoclassical Frequencies

The neoclassical frequencies of the three plasma species have the following definitions:

$$\omega_{\mathrm{nc}\,i}(r) = -G^{ii}_{00}\,\omega_E - \left[L^{ii}_{00} - L^{ii}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\right]\omega_{*\,i} - \left[L^{iI}_{00} - L^{iI}_{01}\left(\frac{\eta_I}{1+\eta_I}\right)\right]\omega_{*\,I}, \tag{50}$$

$$\omega_{\mathrm{nc}\,I}(r) = -G^{Ii}_{00}\,\omega_E - \left[L^{II}_{00} - L^{II}_{01}\left(\frac{\eta_I}{1+\eta_I}\right)\right]\omega_{*\,I} - \left[L^{Ii}_{00} - L^{Ii}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\right]\omega_{*\,i}, \tag{51}$$

$$\omega_{\mathrm{nc}\,e}(r) = -G^{ei}_{00}\,\omega_E - \left[L^{ee}_{00} - L^{ee}_{01}\left(\frac{\eta_e}{1+\eta_e}\right)\right]\omega_{*\,e} - \left[L^{ei}_{00} - L^{ei}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\right]\omega_{*\,i} - \left[L^{eI}_{00} - L^{eI}_{01}\left(\frac{\eta_I}{1+\eta_I}\right)\right]\omega_{*\,I}. \tag{52}$$

Here,

$$\omega_E(r) = -\frac{d\Phi}{d\Psi_p}, \tag{53}$$

$$\omega_{*\,a}(r) = -\frac{T_a}{e_a}\,\frac{d\ln p_a}{d\Psi_p}, \tag{54}$$

$$\eta_a(r) = \frac{d\ln T_a}{d\ln n_a}. \tag{55}$$

Moreover, $p_a(r) = n_a\,T_a$, and $\Phi(r)$ is the equilibrium electric scalar potential.

G. Impurity Ion Angular Rotation Velocities

Let

$$\omega_{\theta\,I}(r) = \frac{\mathbf{V}^I\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\theta}\,\frac{R_0\,B_0\,g}{R^2}, \tag{56}$$

$$\omega_{\phi\,I}(r) = \mathbf{V}^I\cdot\nabla\phi, \tag{57}$$

where $\mathbf{V}^I$ is the impurity ion fluid velocity, and the right-hand sides are evaluated on the outboard mid-plane. According to neoclassical theory,

$$\omega_{\theta\,I} = K_\theta\,\omega_{\mathrm{nc}\,I}, \tag{58}$$

$$\omega_{\phi\,I} = \omega_E + \omega_{*\,I} + \omega_{\theta\,I}, \tag{59}$$

where

$$K_\theta(r) = \frac{R_0^2\,B_0^2\,g^2}{R^2\,\langle B^2\rangle}. \tag{60}$$

III. PLASMA RESPONSE IN INNER REGION

A. Evolution Equations

Let $\Psi_k = R_0\,B_0\,\hat{\Psi}_k\,{\rm e}^{-{\rm i}\,\varphi_k}$, $\chi_k = R_0\,B_0\,\hat{\chi}_k\,{\rm e}^{-{\rm i}\,\zeta_k}$, and $E_{kk'} = \hat{E}_{kk'}\,{\rm e}^{-{\rm i}\,\xi_{kk'}}$, where $\hat{\Psi}_k > 0$, φ_k, $\hat{\chi}_k > 0$, ζ_k, $\hat{E}'_{kk} > 0$, and $\xi_{kk'}$ are all real quantities. Furthermore, let $X_k = \hat{\Psi}_k\,\cos\varphi_k$ and $Y_k = \hat{\Psi}_k\,\sin\varphi_k$. The resonant plasma response model at the kth resonant surface takes the form

$$\begin{aligned}\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{dX_k}{d\hat{t}} + \hat{\varpi}_k\,Y_k\right) &= \sum_{k'=1,K}\hat{E}_{kk'}\,(\cos\xi_{kk'}\,X_{k'} - \sin\xi_{kk'}\,Y_{k'}) + \hat{E}_{kk}\,\hat{\chi}_k\,\cos\zeta_k\\
&\quad+\frac{\alpha_{b\,e\,k}}{\hat{W}_k}\left(\frac{\hat{W}_k^2}{\hat{W}_{T_e\,k}\,\hat{W}_{n_e\,k} + \hat{\rho}_{\theta_e}^2 + W_k^2}\right)X_k + \frac{\alpha_{b\,i\,k}}{\hat{W}_k}\left(\frac{\hat{W}_k^2}{\hat{W}_{T_i\,k}\,\hat{W}_{n_e\,k} + \hat{\rho}_{\theta_i}^2 + \hat{W}_k^2}\right)X_k\\
&\quad+\frac{\alpha_{c\,k}}{\hat{W}_k}\left(\frac{\hat{W}_k^2}{\hat{W}_{n_e\,k}^2 + \hat{W}_k^2}\right)X_k + \frac{\alpha_{p\,k}}{\hat{W}_k^3}\left(\frac{\hat{W}_k^2}{\hat{W}_{T_i\,k}\,\hat{W}_{n_e\,k} + \hat{W}_k^2}\right)^2 X_k,\end{aligned} \tag{61}$$

$$\begin{aligned}\left(\hat{W}_k + \hat{\delta}_k\right)\mathcal{S}_k\left(\frac{dY_k}{d\hat{t}} - \hat{\varpi}_k\,X_k\right) &= \sum_{k'=1,K}\hat{E}_{kk'}\,(\cos\xi_{kk'}\,Y_{k'} + \sin\xi_{kk'}\,X_{k'}) + \hat{E}_{kk}\,\hat{\chi}_k\,\sin\zeta_k\\
&\quad+\frac{\alpha_{b\,e\,k}}{\hat{W}_k}\left(\frac{\hat{W}_k^2}{\hat{W}_{T_e\,k}\,\hat{W}_{n_e\,k} + \hat{\rho}_{\theta_e}^2 + \hat{W}_k^2}\right)Y_k + \frac{\alpha_{b\,i\,k}}{\hat{W}_k}\left(\frac{\hat{W}_k^2}{\hat{W}_{T_i\,k}\,\hat{W}_{n_e\,k} + \hat{\rho}_{\theta_i}^2 + \hat{W}_k^2}\right)Y_k\\
&\quad+\frac{\alpha_{c\,k}}{\hat{W}_k}\left(\frac{\hat{W}_k^2}{\hat{W}_{n_e\,k}^2 + \hat{W}_k^2}\right)Y_k + \frac{\alpha_{p\,k}}{\hat{W}_k^3}\left(\frac{\hat{W}_k^2}{\hat{W}_{T_i\,k}\,\hat{W}_{n_e\,k} + \hat{W}_k^2}\right)^2 Y_k,\end{aligned} \tag{62}$$

where

$$\mathcal{S}_k = \frac{\tau_R(r_k)}{\tau_A}, \tag{63}$$

$$\tau_A = \left[\frac{\mu_0\,\rho(0)\,r_{100}^2}{B_0^2}\right]^{1/2}, \tag{64}$$

$$\tau_R(r) = \mu_0\,r^2\,\sigma_{ee}\,Q_{00}^{ee}, \tag{65}$$

$$\sigma_{ee}(r) = \frac{n_e\,e^2\,\tau_{ee}}{m_e}, \tag{66}$$

$$\hat{W}_k = \frac{2\,\mathcal{I}}{\epsilon_{100}\,\hat{r}_k}\left(\frac{q}{g\,s}\right)_{r_k}^{1/2}(X_k^{\,2}+Y_k^{\,2})^{\,1/4}, \tag{67}$$

$$\hat{W}_{T_e\,k} = 2\,\mathcal{I}\left(\frac{\chi_e\,R_0}{v_{T\,e}\,r^{\,2}\,s\,n}\right)_{r_k}^{1/3}, \tag{68}$$

$$\hat{W}_{T_i\,k} = 2\,\mathcal{I}\left(\frac{\chi_i\,R_0}{v_{T\,e}\,r^{\,2}\,s\,n}\right)_{r_k}^{1/3}, \tag{69}$$

$$\hat{W}_{n_e\,k} = 2\,\mathcal{I}\left(\frac{D_\perp\,R_0}{v_{T\,e}\,r^{\,2}\,s\,n}\right)_{r_k}^{1/3}, \tag{70}$$

$$s(r) = \frac{d\ln q}{d\ln r}, \tag{71}$$

$$\hat{\rho}_{\theta\,e} = \left(\frac{2\,\mathcal{I}\,v_{T\,e}\,m_e\,q\,R_0}{e\,B_0\,g\,r^{\,2}}\right)_{r_k}, \tag{72}$$

$$\hat{\rho}_{\theta\,i} = \left(\frac{2\,\mathcal{I}\,v_{T\,i}\,m_i\,q\,R_0}{e\,B_0\,g\,r^{\,2}}\right)_{r_k}, \tag{73}$$

$$\hat{\delta}_k = \frac{\delta_{\rm linear}(r_k)}{R_0\,\epsilon_{100}\,\hat{r}_k}, \tag{74}$$

$$\alpha_{b\,e\,k} = -2\,\mathcal{I}\,I_g\left(\frac{\omega_{\ast\,e}+\omega_{{\rm nc}\,e}}{\omega_\beta}\right)_{r_k}, \tag{75}$$

$$\alpha_{b\,i\,k} = 2\,\mathcal{I}\,I_g\left[\frac{(n_i/n_e)\,(\omega_{\ast\,i}+\omega_{{\rm nc}\,i})+(Z_I\,n_I/n_e)\,(\omega_{\ast\,I}+\omega_{{\rm nc}\,I})}{\omega_\beta}\right]_{r_k}, \tag{76}$$

$$\alpha_{c\,k} = 2\,\mathcal{I}\,I_g\,D_R(r_k), \tag{77}$$

$$\alpha_{p\,k} = 8\,\mathcal{I}^{\,3}\,I_p\left[\frac{(\omega_{\ast\,i}+\omega_{{\rm nc}\,i})\,\omega_{{\rm nc}\,i}}{\omega_\beta\,\omega_\Omega}\right]_{r_k}, \tag{78}$$

$$\omega_\beta(r) = \frac{s\,g\,B_0}{\mu_0\,n_e\,e\,R_0^{2}\,q}, \tag{79}$$

$$\omega_\Omega(r) = \frac{e\,g\,B_0\,s\,q}{m_i}. \tag{80}$$

Here, $\mathcal{I} = 0.8227$, $I_g = 1.58$, $I_p = 1.38$, $\epsilon_{100} = r_{100}/R_0$, $\hat{r} = r/r_{100}$, $\hat{r}_k = r_k/r_{100}$, and $\hat{t} = t/\tau_A$. Furthermore, $\delta_{\rm linear}(r)$ is the linear layer width, whereas $\rho(r)$ is the plasma mass density profile. Moreover, $\chi_e(r)$, $\chi_i(r)$, and $D_\perp(r)$ are the perpendicular electron energy, ion energy, and particle diffusivity profile, respectively.

The EPEC resonant plasma response model interpolates smoothly between the linear regime and the nonlinear Rutherford regime. The linear regime corresponds to $\hat{W}_k \ll \hat{\delta}_k$, whereas the nonlinear region corresponds to $\hat{W}_k \gg \hat{\delta}_k$.

B. Linear Layer Width

Let

$$\tau_H(r) = \frac{R_0}{B_0\,g}\,\frac{\sqrt{\mu_0\,\rho}}{n\,s}, \tag{81}$$

$$\tau_E(r) = \frac{r^{\,2}}{D_\perp + (2/3)\,\chi_e}, \tag{82}$$

$$\tau_M(r) = \frac{R_0^2\,q^{\,2}}{\chi_\phi}, \tag{83}$$

$$\rho_s(r) = \frac{\sqrt{m_i\,T_e}}{e\,B_0\,g}, \tag{84}$$

$$\tau(r) = -\frac{\omega_{\ast\,i}}{\omega_{\ast\,e}}, \tag{85}$$

$$S(r) = \frac{\tau_R}{\tau_H}, \tag{86}$$

$$P_M(r) = \frac{\tau_R}{\tau_M}, \tag{87}$$

$$P_E(r) = \frac{\tau_R}{\tau_E}, \tag{88}$$

$$D(r) = \frac{5}{3}\,S^{\,1/3}\,\frac{\rho_s}{r}, \tag{89}$$

$$Q_E(r) = S^{\,1/3}\,n\,\omega_E\,\tau_H, \tag{90}$$

$$Q_{e,i}(r) = -S^{\,1/3}\,n\,\omega_{\ast\,e,i}\,\tau_H \tag{91}$$

The constant-ψ linear layer width is determined from the solution of

$$\frac{d^2\,Y}{dp^2} - \left[\frac{-Q_E\,(Q_E-Q_i) + \mathrm{i}\,(Q_E-Q_i)\,(P_M+P_E)\,p^{\,2} + P_M\,P_E\,p^{\,4}}{\mathrm{i}\,(Q_E-Q_e) + \{P_E + \mathrm{i}\,(Q_E-Q_i)\,D^{\,2}\}\,p^{\,2} + (1+\tau)\,P_M\,D^{\,2}\,p^{\,4}}\right] p^{\,2}\,Y = 0. \tag{92}$$

If the small-p behavior of solution of the previous equation that is well-behaved as $p\to\infty$ is

$$Y(p) = Y_0\left[1 - c\,p + {\cal O}(p^{\,2})\right]. \tag{93}$$

then the linear layer width is

$$\frac{\delta_{\rm linear}}{r} = \frac{\pi\,|c|}{S^{\,1/3}}. \tag{94}$$

IV. PLASMA ANGULAR VELOCITY EVOLUTION

A. Evolution Equations

The quantity $\hat{\varpi}_k$ that appears in Eqs. (61) and (62) evolves in time according to

$$\hat{\varpi}_k(\hat{t}) = \hat{\varpi}_{k\,0} - \sum_{k'=1,K}^{p=1,\infty} \frac{m_k}{m_{k'}}\,\frac{y_p(\hat{r}_k)}{y_p(\hat{r}_{k'})}\,\alpha_{k',p}(\hat{t}) - \sum_{k'=1,K}^{p=1,\infty} \frac{z_p(\hat{r}_k)}{z_p(\hat{r}_{k'})}\,\beta_{k',p}(\hat{t}), \tag{95}$$

Here, $\hat{\varpi}_{k\,0} = \varpi_{k\,0}\,\tau_A$, $y_p(\hat{r}) = J_1(j_{1,p}\,\hat{r})/\hat{r}$, and $z_p(\hat{r}) = J_0(j_{0,p}\,\hat{r})$. Moreover, $\varpi_{k\,0}$ is the so-called "natural frequency" (in the absence of the RMP) at the kth resonant surface; this quantity is defined as the helical phase velocity of a naturally unstable island chain, resonant at the surface, in the absence of an RMP (or any other island chains). Furthermore, $J_m(z)$ is a standard Bessel function, and $j_{m,p}$ denotes the pth zero of this function. The time evolution equations for the $\alpha_{k,p}$ and $\beta_{k,p}$ parameters specify how the plasma poloidal and toroidal angular velocity profiles are modified by the electromagnetic torques that develop within the plasma, in response to the applied RMP, and how these modifications affect the natural frequencies. The evolution equations take the form

$$(1+2\,Q_k^2)\,\frac{d\alpha_{k,p}}{d\hat{t}} + \left(\frac{j_{1,p}^{\,2}}{\hat{\tau}_{M\,k}} + \frac{1}{\hat{\tau}_{\theta\,k}} + \frac{1}{\hat{\tau}_{{\rm cx}\,k}}\right)\alpha_{k,p} = \frac{m_k^2\,[y_p(\hat{r}_k)]^{\,2}}{\hat{\rho}_k\,\epsilon_{100}^2\,[J_2(j_{1,p})]^{\,2}}\,\delta\hat{T}_k, \tag{96}$$

$$\frac{d\beta_{k,p}}{d\hat{t}} + \left(\frac{j_{0,p}^{\,2}}{\hat{\tau}_{M\,k}} + \frac{1}{\hat{\tau}_{{\rm cx}\,k}}\right)\beta_{k,p} = \frac{n^{\,2}\,[z_p(\hat{r}_k)]^{\,2}}{\hat{\rho}_k\,[J_1(j_{0,p})]^{\,2}}\,\delta\hat{T}_k, \tag{97}$$

where

$$\delta\hat{T}_k = \sum_{k'=1,K} \hat{E}_{kk'}\,\hat{\Psi}_k\,\hat{\Psi}_{k'}\,\sin(\varphi_k-\varphi_{k'}-\xi_{kk'}) + \hat{E}_{kk}\,\hat{\Psi}_k\,\hat{\chi}_k\,\sin(\varphi_k-\zeta_k). \tag{98}$$

Here, $Q_k = Q(r_k)$, $\hat{\rho}_k = \rho(r_k)/\rho(0)$, $\hat{\tau}_{M\,k} = r_{100}^2/[\chi_\phi(r_k)\,\tau_A]$, $\hat{\tau}_{\theta\,k} = \tau_\theta(r_k)/\tau_A$, $\hat{\tau}_{{\rm cx}\,k} = \tau_{\rm cx}(r_k)/\tau_A$. Moreover, $\chi_\phi(r)$ is the perpendicular toroidal momentum diffusivity profile, whereas

$$\tau_\theta(r) = \frac{\tau_{ii}}{\mu_{00}^i}\bigg/\left(1+\frac{q^{\,2}\,R_0^2}{r^{\,2}\,a_{kk}}\right). \tag{99}$$

is the poloidal flow damping timescale, and

$$\tau_{\rm cx}(r) = \frac{1}{\langle n_n\rangle\,\langle\sigma\,v\rangle_i^{\rm cx}} \tag{100}$$

is the charge exchange damping timescale. Furthermore,

$$Q^{\,2}(r) = \frac{q^{\,2}\,R_0^2}{2\,r^{\,2}}\left(\left\langle\frac{1}{R^{\,2}}\right\rangle - \frac{1}{\langle R^{\,2}\rangle}\right)\bigg/\left\langle\frac{|\nabla r|^{\,2}}{R^{\,2}}\right\rangle. \tag{101}$$

B. Natural Frequencies

According to linear tearing mode theory, in the absence of the RMP, the natural frequency of the tearing mode resonant at the kth resonant surface is given by

$$\varpi_{\mathrm{linear}\,k} = -n\,(\omega_E + \omega_{\ast\,e})_{r_k}. \tag{102}$$

According to nonlinear tearing mode theory, in the absence of the RMP, the natural frequency of the tearing mode resonant at the kth resonant surface is given by

$$\varpi_{\mathrm{nonlinear}\,k} = -n\,(\omega_E + \omega_{\ast\,i} + \omega_{\mathrm{nc}\,i})_{r_k}. \tag{103}$$

Given that our resonant plasma response model interpolates smoothly between the linear regime ($\hat{W}_k \ll \hat{\delta}_k$) and the nonlinear regime ($\hat{W}_k \gg \hat{\delta}_k$), it makes sense to adopt a model for the natural frequency that does the same. Hence, the EPEC model for the natural frequency is

$$\varpi_{k\,0} = \frac{\hat{\delta}_k\,\varpi_{\mathrm{linear}\,k} + \hat{W}_k\,\varpi_{\mathrm{nonlinear}\,k}}{\hat{\delta}_k + \hat{W}_k}. \tag{104}$$

V. PRESSURE REDUCTION

The full width (in Ψ_N) of the magnetic separatrix of the island chain induced at the kth resonant surface is

$$W_k = 4\left[\frac{\hat{\Psi}_k}{\sigma_k\,|\hat{\Psi}_p(0)|}\right]^{1/2}, \tag{105}$$

where $\sigma_k = \sigma(r_k)$, $\sigma(r) = d\ln q/d\Psi_N$, and $\hat{\Psi}_p(r) = \Psi_p/(R_0^2\,B_0)$. However, the halfway point between the inner and outer extremities of the separatrix is located at the flux-surface where $\Psi_N = \Psi_N(r_k) - A_k\,W_k^2/8$ and $A_k = [(1/3)\,(d^2q/d\Psi_N^2)/(dq/d\Psi_N)]_{r_k}$. On average, the presence of a magnetic island chain of full width W_k at the kth resonant surface causes the electron temperature, ion temperature, and electron number density profiles to be flattened in annular regions, centered on the halfway points between the inner and outer radii of the island separatrices, of widths (in Ψ_N)

$$\delta_{T_e\,k} = \frac{2}{\pi}\,W_k\,\tanh\left(\frac{W_k}{W_{T_e\,k}}\right), \tag{106}$$

$$\delta_{T_i\,k} = \frac{2}{\pi}\,W_k\,\tanh\left(\frac{W_k}{W_{T_i\,k}}\right), \tag{107}$$

$$\delta_{n_e\,k} = \frac{2}{\pi}\,W_k\,\tanh\left(\frac{W_k}{W_{n_e\,k}}\right), \tag{108}$$

respectively, where

$$W_{T_e\,k} = \left(\frac{\chi_e}{v_{T\,e}\,R_0}\,\frac{f^{\,2}}{n\,\sigma\,|\hat{\Psi}_p(0)|^{\,2}}\right)^{1/3}_{r_k}, \tag{109}$$

$$W_{T_i\,k} = \left(\frac{\chi_i}{v_{T\,i}\,R_0}\,\frac{f^{\,2}}{n\,\sigma\,|\hat{\Psi}_p(0)|^{\,2}}\right)^{1/3}_{r_k}, \tag{110}$$

$$W_{n_e\,k} = \left(\frac{D_\perp}{v_{T\,i}\,R_0}\,\frac{f^{\,2}}{n\,\sigma\,|\hat{\Psi}_p(0)|^{\,2}}\right)^{1/3}_{r_k}. \tag{111}$$

The total change in the plasma pressure interior to the jth resonant surface due to the temperature and density flattening at the various resonant surfaces (assuming that the pressure at the LCFS is fixed) is

$$\begin{aligned}\varDelta P_j = \sum_{k=j,K}\Bigg\{&\left[\delta_{n_e\,k}\,\frac{dn_e}{d\Psi_N}\,T_e - \delta_{n_e\,k}\,W_k^2\,\frac{A_k}{8}\left(\frac{dn_e}{d\Psi_N}\,\frac{dT_e}{d\Psi_N} + \frac{d^{\,2}n_e}{d\Psi_n^2}\,T_e\right) + \delta_{n_e\,k}^3\,\frac{d^{\,3}n_e}{d\Psi_N^3}\,\frac{T_e}{24}\right]\\
&+\left[\delta_{T_e\,k}\,n_e\,\frac{dT_e}{d\Psi_N} - \delta_{T_e\,k}\,W_k^2\,\frac{A_k}{8}\left(\frac{dn_e}{d\Psi_N}\,\frac{dT_e}{d\Psi_N} + n_e\,\frac{d^{\,2}T_e}{d\Psi_n^2}\right) + \delta_{T_e\,k}^3\,\frac{n_e}{24}\,\frac{d^{\,3}T_e}{d\Psi_N^3}\right]\\
&+\left[\delta_{n_e\,k}\,\frac{dn_i}{d\Psi_N}\,T_i - \delta_{n_e\,k}\,W_k^2\,\frac{A_k}{8}\left(\frac{dn_i}{d\Psi_N}\,\frac{dT_i}{d\Psi_N} + \frac{d^{\,2}n_i}{d\Psi_n^2}\,T_i\right) + \delta_{n_e\,k}^3\,\frac{d^{\,3}n_i}{d\Psi_N^3}\,\frac{T_i}{24}\right]\\
&+\left[\delta_{T_i\,k}\,n_i\,\frac{dT_i}{d\Psi_N} - \delta_{T_i\,k}\,W_k^2\,\frac{A_k}{8}\left(\frac{dn_i}{d\Psi_N}\,\frac{dT_i}{d\Psi_N} + n_i\,\frac{d^{\,2}T_i}{d\Psi_n^2}\right) + \delta_{T_i\,k}^3\,\frac{n_i}{24}\,\frac{d^{\,3}T_i}{d\Psi_N^3}\right]\Bigg\}_{r_k}.\end{aligned} \tag{112}$$

where we have neglected the impurity ion pressure.