Program NEOCLASSICAL

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1 Plasma Equilibrium

Let

$$\vec{B} = I(\psi)\,\vec{\nabla}\zeta + \vec{\nabla}\zeta \times \vec{\nabla}\psi, \tag{1}$$

$$\vec{E} = -\vec{\nabla}\Phi(\psi) + E_\parallel\,\vec{b}, \tag{2}$$

where $\vec{b} = \vec{B}/B$. Expect $n_a = n_a(\psi)$, $p_a = p_a(\psi)$, and $T_a = T_a(\psi)$. Follows that

$$\mu_0\,\vec{j} \equiv \vec{\nabla} \times \vec{B} = \vec{\nabla}I \times \vec{\nabla}\zeta + R^2\,\vec{\nabla} \cdot \left(\frac{\vec{\nabla}\psi}{R^2}\right)\nabla\zeta. \tag{3}$$

Force balance implies that

$$\vec{j} \times \vec{B} = \vec{\nabla}p, \tag{4}$$

which yields the Grad-Shafranov equation:

$$R^2\,\vec{\nabla} \cdot \left(\frac{\vec{\nabla}\psi}{R^2}\right) = -I\,\frac{dI}{d\psi} - \mu_0\,R^2\,\frac{dp}{d\psi}. \tag{5}$$

Finally,

$$\mu_0\,\vec{j}\cdot\vec{B} = -\left(\frac{I^2}{R^2} + \frac{|\vec{\nabla}\psi|^2}{R^2}\right)\frac{dI}{d\psi} - \mu_0\,I\,\frac{dp}{d\psi}, \tag{6}$$

which implies that

$$\frac{\langle\vec{j}\cdot\vec{B}\rangle}{I} = -\frac{1}{\mu_0}\left(I\left\langle\frac{1}{R^2}\right\rangle + \frac{1}{I}\left\langle\frac{|\vec{\nabla}\psi|^2}{R^2}\right\rangle\right)\frac{dI}{d\psi} - \frac{dp}{d\psi}. \tag{7}$$

2   Fluid Velocities and Heat Fluxes

We can write

$$\vec{u}_j^a = u_{\parallel j}^a \, \vec{b} + \vec{u}_{\perp j}^a, \tag{8}$$

and

$$\vec{u}_0^a = \vec{V}_a, \tag{9}$$

$$\vec{u}_1^a = -\frac{2}{5} \frac{\vec{q}_a}{p_a}, \tag{10}$$

with

$$\vec{u}_{\perp 0}^a = \frac{\vec{E} \times \vec{B}}{B^2} + \frac{\vec{B} \times \vec{\nabla} p_a}{n_a \, e_a \, B^2}, \tag{11}$$

$$\vec{u}_{\perp 1}^a = -\frac{\vec{B} \times \vec{\nabla} T_a}{e_a \, B^2}. \tag{12}$$

It follows that

$$u_{\theta j}^a = \frac{u_{\parallel j}^a}{B} - \frac{V_{\psi j}^a}{B^2}, \tag{13}$$

where

$$u_{\theta j}^a(\psi) = \frac{\vec{u}_j^a \cdot \vec{\nabla}\theta}{\vec{B} \cdot \vec{\nabla}\theta}, \tag{14}$$

$$V_{\psi 0}^a(\psi) = -\left(\frac{\vec{u}_{\perp 0} \cdot \vec{\nabla}\theta}{\vec{B} \cdot \vec{\nabla}\theta} \right) B^2 = -\frac{I \, T_a}{e_a} \left(\frac{d \ln p_a}{d\psi} + \frac{e_a}{T_a} \frac{d\Phi}{d\psi} \right), \tag{15}$$

$$V_{\psi 1}^a(\psi) = -\left(\frac{\vec{u}_{\perp 1} \cdot \vec{\nabla}\theta}{\vec{B} \cdot \vec{\nabla}\theta} \right) B^2 = \frac{I \, T_a}{e_a} \frac{d \ln T_a}{d\psi}. \tag{16}$$

3   Friction Forces

$$\begin{pmatrix} \vec{F}_0^a \\ \vec{F}_1^a \end{pmatrix} = \sum_b \begin{pmatrix} l_{00}^{ab}, & l_{01}^{ab} \\ l_{10}^{ab}, & l_{11}^{ab} \end{pmatrix} \begin{pmatrix} \vec{u}_0^b \\ \vec{u}_1^b \end{pmatrix}, \tag{17}$$

$$l_{jk}^{ab} = \left(\sum_c \frac{n_a \, m_a}{\tau_{ac}} M_{jk}^{ac} \right) \delta_{ab} + \frac{n_a \, m_a}{\tau_{ab}} N_{jk}^{ab}, \tag{18}$$

$$\frac{1}{\tau_{ab}} = \frac{4}{3\sqrt{\pi}} \frac{4\pi \, n_b \, e_a^2 \, e_b^2 \, \ln \Lambda}{(4\pi \, \epsilon_0)^2 \, m_a^2 \, v_{Ta}^3}, \tag{19}$$

$$v_{Ta} = \sqrt{\frac{2 \, T_a}{m_a}}, \tag{20}$$

$$M_{00}^{ab} = -\frac{1 + m_a/m_b}{\left(1 + x_{ab}^2\right)^{3/2}}, \tag{21}$$

$$M_{01}^{ab} = -\frac{3}{2}\frac{1 + m_a/m_b}{\left(1 + x_{ab}^2\right)^{5/2}} = M_{10}^{ab}, \tag{22}$$

$$M_{11}^{ab} = -\frac{13/4 + 4\,x_{ab}^2 + (15/2)\,x_{ab}^4}{\left(1 + x_{ab}^2\right)^{5/2}}, \tag{23}$$

$$N_{00}^{ab} = \frac{1 + m_a/m_b}{\left(1 + x_{ab}^2\right)^{3/2}}, \tag{24}$$

$$N_{01}^{ab} = \frac{3}{2}\frac{T_a}{T_b}\frac{1 + m_b/m_a}{x_{ab}\left(1 + x_{ba}^2\right)^{5/2}}, \tag{25}$$

$$N_{10}^{ab} = \frac{3}{2}\frac{1 + m_a/m_b}{\left(1 + x_{ab}^2\right)^{5/2}}, \tag{26}$$

$$N_{11}^{ab} = \frac{27}{4}\frac{T_a}{T_b}\frac{x_{ab}^2}{\left(1 + x_{ab}^2\right)^{5/2}}, \tag{27}$$

where

$$x_{ab} = \frac{v_{Tb}}{v_{Ta}}. \tag{28}$$

Species are e, i, and I. Let

$$Z_{\text{eff}} = \frac{n_i\,e_i^2 + n_I\,e_I^2}{n_e\,e_e^2}. \tag{29}$$

Follows that

$$\frac{n_e}{n_i} = \frac{Z_I - 1}{Z_I - Z_{\text{eff}}}, \tag{30}$$

$$\frac{n_I}{n_i} = \frac{Z_{\text{eff}} - 1}{Z_I\left(Z_I - Z_{\text{eff}}\right)}, \tag{31}$$

$$\frac{n_I}{n_e} = \frac{Z_{\text{eff}} - 1}{Z_I\left(Z_I - 1\right)}. \tag{32}$$

Thus,

$$M_{00}^{aa} = -\frac{1}{\sqrt{2}}, \tag{33}$$

$$M_{01}^{aa} = -\frac{3}{4\sqrt{2}} = M_{10}^{aa}, \tag{34}$$

$$M_{11}^{aa} = -\frac{59}{16\sqrt{2}}, \tag{35}$$

$$N_{00}^{aa} = \frac{1}{\sqrt{2}}, \tag{36}$$

$$N_{01}^{aa} = \frac{3}{4\sqrt{2}}, \tag{37}$$

$$N_{10}^{aa} = \frac{3}{4\sqrt{2}}, \tag{38}$$

$$N_{11}^{aa} = \frac{27}{16\sqrt{2}}, \tag{39}$$

for $a = e,\ i,\ I$. Now,

$$M_{00}^{iI} = -\frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{3/2}}, \tag{40}$$

$$M_{01}^{iI} = -\frac{3}{2}\frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{5/2}} = M_{10}^{iI}, \tag{41}$$

$$M_{11}^{iI} = -\frac{13/4 + 4\,x_{iI}^2 + (15/2)\,x_{iI}^4}{(1 + x_{iI}^2)^{5/2}}, \tag{42}$$

$$N_{00}^{iI} = \frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{3/2}}, \tag{43}$$

$$N_{01}^{iI} = \frac{3}{2}\frac{T_i}{T_I}\frac{1 + m_I/m_i}{x_{iI}(1 + x_{Ii}^2)^{5/2}}, \tag{44}$$

$$N_{10}^{iI} = \frac{3}{2}\frac{1 + m_i/m_I}{(1 + x_{iI}^2)^{5/2}}, \tag{45}$$

$$N_{11}^{iI} = \frac{27}{4}\frac{T_i}{T_I}\frac{x_{iI}^2}{(1 + x_{iI}^2)^{5/2}}. \tag{46}$$

Furthermore,

$$M_{00}^{Ii} = -\frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{3/2}}, \tag{47}$$

$$M_{01}^{Ii} = -\frac{3}{2}\frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{5/2}} = M_{10}^{Ii}, \tag{48}$$

$$M_{11}^{Ii} = -\frac{13/4 + 4\,x_{Ii}^2 + (15/2)\,x_{Ii}^4}{(1 + x_{Ii}^2)^{5/2}}, \tag{49}$$

$$N_{00}^{Ii} = \frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{3/2}}, \tag{50}$$

$$N_{01}^{Ii} = \frac{3}{2}\frac{T_I}{T_i}\frac{1 + m_i/m_I}{x_{Ii}(1 + x_{iI}^2)^{5/2}}, \tag{51}$$

$$N_{10}^{Ii} = \frac{3}{2}\frac{1 + m_I/m_i}{(1 + x_{Ii}^2)^{5/2}}, \tag{52}$$

$$N_{11}^{Ii} = \frac{27}{4}\frac{T_I}{T_i}\frac{x_{Ii}^2}{(1 + x_{Ii}^2)^{5/2}}. \tag{53}$$

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Also,

$$M_{00}^{ea} = -1, \tag{54}$$

$$M_{01}^{ea} = -\frac{3}{2} = M_{10}^{ea}, \tag{55}$$

$$M_{11}^{ea} = -\frac{13}{4}, \tag{56}$$

$$N_{00}^{ea} = 1, \tag{57}$$

$$N_{01}^{ea} = 0, \tag{58}$$

$$N_{10}^{ea} = \frac{3}{2}, \tag{59}$$

$$N_{11}^{ea} = 0, \tag{60}$$

for $a = i$, I, where we have neglected terms of order $(m_e/m_i)^{1/2}$. Finally,

$$M_{00}^{ae} = 0, \tag{61}$$

$$M_{01}^{ae} = 0 = M_{10}^{ae}, \tag{62}$$

$$M_{11}^{ae} = 0, \tag{63}$$

$$N_{00}^{ae} = 0, \tag{64}$$

$$N_{01}^{ae} = 0, \tag{65}$$

$$N_{10}^{ae} = 0 \tag{66}$$

$$N_{11}^{ae} = 0, \tag{67}$$

for $a = i$, I.

Now,

$$l_{jk}^{ii} = \frac{n_i m_i}{\tau_{ii}} \left[\left(\sum_a \frac{n_a e_a^2}{n_i e_i^2} M_{jk}^{ia} \right) + N_{jk}^{ii} \right] \tag{68}$$

$$= \frac{n_i m_i}{\tau_{ii}} \left(M_{jk}^{ii} + N_{jk}^{ii} + \alpha M_{jk}^{iI} \right), \tag{69}$$

where

$$\alpha = \frac{n_I e_I^2}{n_i e_i^2} = (Z_{\text{eff}} - 1) \left(\frac{Z_I}{Z_I - Z_{\text{eff}}} \right). \tag{70}$$

Thus,

$$l_{00}^{ii} = -\frac{n_i m_i}{\tau_{ii}} \frac{\alpha \left(1 + m_i/m_I \right)}{\left(1 + x_{iI}^2 \right)^{3/2}}, \tag{71}$$

$$l_{01}^{ii} = -\frac{n_i m_i}{\tau_{ii}} \frac{3}{2} \frac{\alpha \left(1 + m_i/m_I \right)}{\left(1 + x_{iI}^2 \right)^{5/2}}, \tag{72}$$

$$l_{10}^{ii} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{3}{2}\,\frac{\alpha\left(1+m_i/m_I\right)}{(1+x_{iI}^2)^{5/2}}, \tag{73}$$

$$l_{11}^{ii} = -\frac{n_i\,m_i}{\tau_{ii}}\left\{\sqrt{2}+\frac{\alpha\left[13/4+4\,x_{iI}^2+(15/2)\,x_{iI}^4\right]}{(1+x_{iI}^2)^{5/2}}\right\}. \tag{74}$$

Now,

$$l_{jk}^{iI} = \frac{n_i\,m_i}{\tau_{ii}}\,\alpha\,N_{jk}^{iI}, \tag{75}$$

so

$$l_{00}^{iI} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{\alpha\left(1+m_i/m_I\right)}{(1+x_{iI}^2)^{3/2}}, \tag{76}$$

$$l_{01}^{iI} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha\left(1+m_I/m_i\right)}{x_{iI}\,(1+x_{Ii}^2)^{5/2}}, \tag{77}$$

$$l_{10}^{iI} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{3}{2}\,\frac{\alpha\left(1+m_i/m_I\right)}{(1+x_{iI}^2)^{5/2}}, \tag{78}$$

$$l_{11}^{iI} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{27}{4}\,\frac{T_i}{T_I}\,\frac{\alpha\,x_{iI}^2}{(1+x_{iI}^2)^{5/2}}. \tag{79}$$

Now,

$$l_{jk}^{II} = \frac{n_I\,m_I}{\tau_{II}}\left[\left(\sum_a \frac{n_a\,e_a^2}{n_I\,e_I^2}\,M_{jk}^{Ia}\right)+N_{jk}^{II}\right] \tag{80}$$

$$= \frac{n_I\,m_I}{\tau_{II}}\left(M_{jk}^{II}+N_{jk}^{II}+\alpha^{-1}\,M_{jk}^{Ii}\right)$$

$$= \frac{n_i\,m_i}{\tau_{ii}}\,\alpha^2\,\frac{m_i}{m_I}\,x_{Ii}^3\left(M_{jk}^{II}+N_{jk}^{II}+\alpha^{-1}\,M_{jk}^{Ii}\right), \tag{81}$$

so

$$l_{00}^{II} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\frac{\alpha\left(1+m_I/m_i\right)}{(1+x_{Ii}^2)^{3/2}}, \tag{82}$$

$$l_{01}^{II} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{\alpha\left(1+m_I/m_i\right)}{(1+x_{Ii}^2)^{5/2}}, \tag{83}$$

$$l_{10}^{II} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{\alpha\left(1+m_I/m_i\right)}{(1+x_{Ii}^2)^{5/2}}, \tag{84}$$

$$l_{11}^{II} = -\frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\left\{\sqrt{2}\,\alpha^2+\frac{\alpha\left[13/4+4\,x_{Ii}^2+(15/2)\,x_{Ii}^4\right]}{(1+x_{Ii}^2)^{5/2}}\right\}. \tag{85}$$

Finally,

$$l_{jk}^{Ii} = \frac{n_i\,m_i}{\tau_{ii}}\,\alpha\,\frac{m_i}{m_I}\,x_{Ii}^3\,N_{jk}^{Ii}, \tag{86}$$

so

$$l_{00}^{Ii} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\,\frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{3/2}},\tag{87}$$

$$l_{01}^{Ii} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{T_I}{T_i}\,\frac{\alpha\,(1+m_i/m_I)}{x_{Ii}\,(1+x_{iI}^2)^{5/2}},\tag{88}$$

$$l_{10}^{Ii} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\,\frac{3}{2}\,\frac{\alpha\,(1+m_I/m_i)}{(1+x_{Ii}^2)^{5/2}},\tag{89}$$

$$l_{11}^{Ii} = \frac{n_i\,m_i}{\tau_{ii}}\,\frac{m_i}{m_I}\,x_{Ii}^3\,\frac{27}{4}\,\frac{T_I}{T_i}\,\frac{\alpha\,x_{Ii}^2}{(1+x_{Ii}^2)^{5/2}}.\tag{90}$$

Thus,

$$(F^i) = \frac{n_i\,m_i}{\tau_{ii}}\,\left\{-\left[F^{ii}\right]\,(u^i) + \left[F^{iI}\right]\,(u^I)\right\},\tag{91}$$

$$(F^I) = \frac{n_i\,m_i}{\tau_{ii}}\,\left\{\left[F^{Ii}\right]\,(u^i) - \left[F^{II}\right]\,(u^I)\right\},\tag{92}$$

where

$$(F^a) = \begin{pmatrix} \vec{F}_0^a \\ \vec{F}_1^a \end{pmatrix},\tag{93}$$

$$(u^a) = \begin{pmatrix} \vec{u}_0^a \\ \vec{u}_1^a \end{pmatrix}.\tag{94}$$

Here,

$$F_{00}^{ii} = \frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{3/2}},\tag{95}$$

$$F_{01}^{ii} = \frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{5/2}},\tag{96}$$

$$F_{10}^{ii} = \frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{5/2}} = F_{01}^{ii},\tag{97}$$

$$F_{11}^{ii} = \sqrt{2} + \frac{\alpha\,[13/4 + 4\,x_{iI}^2 + (15/2)\,x_{iI}^4]}{(1+x_{iI}^2)^{5/2}},\tag{98}$$

$$F_{00}^{iI} = \frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{3/2}} = F_{00}^{ii},\tag{99}$$

$$F_{01}^{iI} = \frac{3}{2}\,\frac{T_i}{T_I}\,\frac{\alpha\,(1+m_I/m_i)}{x_{iI}\,(1+x_{Ii}^2)^{5/2}},\tag{100}$$

$$F_{10}^{iI} = \frac{3}{2}\,\frac{\alpha\,(1+m_i/m_I)}{(1+x_{iI}^2)^{5/2}} = F_{01}^{ii},\tag{101}$$

$$F_{11}^{iI} = \frac{27}{4} \frac{T_i}{T_I} \frac{\alpha\, x_{iI}^2}{(1 + x_{iI}^2)^{5/2}}, \tag{102}$$

$$F_{00}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{\alpha\,(1 + m_I/m_i)}{(1 + x_{Ii}^2)^{3/2}} = \frac{\alpha\,(1 + m_i/m_I)}{(1 + x_{iI}^2)^{3/2}} = F_{00}^{ii}, \tag{103}$$

$$F_{01}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{T_I}{T_i} \frac{\alpha\,(1 + m_I/m_i)}{x_{Ii}\,(1 + x_{Ii}^2)^{5/2}} = \frac{3}{2} \frac{\alpha\,(1 + m_i/m_I)}{(1 + x_{iI}^2)^{5/2}} = F_{01}^{ii}, \tag{104}$$

$$F_{10}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha\,(1 + m_I/m_i)}{(1 + x_{Ii}^2)^{5/2}} = \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha\,(1 + m_I/m_i)}{x_{iI}\,(1 + x_{Ii}^2)^{5/2}} = F_{01}^{iI}, \tag{105}$$

$$F_{11}^{Ii} = \frac{m_i}{m_I} x_{Ii}^3 \frac{27}{4} \frac{T_I}{T_i} \frac{\alpha\, x_{Ii}^2}{(1 + x_{Ii}^2)^{5/2}} = \frac{27}{4} \frac{\alpha\, x_{iI}^2}{(1 + x_{iI}^2)^{5/2}}, \tag{106}$$

$$F_{00}^{II} = \frac{m_i}{m_I} x_{Ii}^3 \frac{\alpha\,(1 + m_I/m_i)}{(1 + x_{Ii}^2)^{3/2}} = \frac{\alpha\,(1 + m_i/m_I)}{(1 + x_{iI}^2)^{3/2}} = F_{00}^{ii}, \tag{107}$$

$$F_{01}^{II} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha\,(1 + m_I/m_i)}{(1 + x_{Ii}^2)^{5/2}} = \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha\,(1 + m_I/m_i)}{x_{iI}\,(1 + x_{Ii}^2)^{5/2}} = F_{01}^{iI}, \tag{108}$$

$$F_{10}^{II} = \frac{m_i}{m_I} x_{Ii}^3 \frac{3}{2} \frac{\alpha\,(1 + m_I/m_i)}{(1 + x_{Ii}^2)^{5/2}} = \frac{3}{2} \frac{T_i}{T_I} \frac{\alpha\,(1 + m_I/m_i)}{x_{iI}\,(1 + x_{Ii}^2)^{5/2}} = F_{01}^{iI}, \tag{109}$$

$$F_{11}^{II} = \frac{m_i}{m_I} x_{Ii}^3 \left\{ \sqrt{2}\,\alpha^2 + \frac{\alpha\,[13/4 + 4\,x_{Ii}^2 + (15/2)\,x_{Ii}^4]}{(1 + x_{Ii}^2)^{5/2}} \right\}$$

$$= \frac{T_i}{T_I} \left\{ \sqrt{2}\,\alpha^2\, x_{Ii} + \frac{\alpha\,[15/2 + 4\,x_{iI}^2 + (13/4)\,x_{iI}^4]}{(1 + x_{iI}^2)^{5/2}} \right\}. \tag{110}$$

Note that $F_{0j}^{ii} = F_{0j}^{Ii}$ and $F_{0j}^{iI} = F_{0j}^{II}$, for $j = 0$, 1, which implies that $(F^I)_0 + (F^i)_0 = 0$. This is a statement of collisional momentum conservation. Furthermore, $F_{j0}^{ii} = F_{j0}^{iI}$ and $F_{j0}^{Ii} = F_{j0}^{II}$.

Now,

$$l_{jk}^{ee} = \frac{n_e\, m_e}{\tau_{ee}} \left[\left(\sum_a \frac{n_a\, e_a^2}{n_e\, e_e^2}\, M_{jk}^{ea} \right) + N_{jk}^{ee} \right] \tag{111}$$

$$= \frac{n_e\, m_e}{\tau_{ee}} \left(M_{jk}^{ee} + N_{jk}^{ee} + Z_{\text{eff}}\, M_{jk}^{ei} \right), \tag{112}$$

where $a = e$, i, I. It follows that

$$l_{00}^{ee} = -\frac{n_e\, m_e}{\tau_{ee}}\, Z_{\text{eff}}, \tag{113}$$

$$l_{01}^{ee} = -\frac{n_e\, m_e}{\tau_{ee}} \frac{3}{2}\, Z_{\text{eff}}, \tag{114}$$

$$l_{10}^{ee} = -\frac{n_e\, m_e}{\tau_{ee}} \frac{3}{2}\, Z_{\text{eff}}, \tag{115}$$

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$$l_{11}^{ee} = -\frac{n_e \, m_e}{\tau_{ee}} \left(\sqrt{2} + \frac{13}{4} \, Z_{\text{eff}} \right). \tag{116}$$

Likewise,

$$l_{jk}^{ea} = \frac{n_e \, m_e}{\tau_{ee}} \, Z_{\text{eff} \, a} \, N_{jk}^{ea}, \tag{117}$$

where

$$Z_{\text{eff} \, a} = \frac{n_a \, e_a^2}{n_e \, e_e^2}, \tag{118}$$

for $a = i$, I. In particular,

$$Z_{\text{eff} \, i} = \frac{Z_I - Z_{\text{eff}}}{Z_I - 1}, \tag{119}$$

$$Z_{\text{eff} \, I} = \frac{Z_I \, (Z_{\text{eff}} - 1)}{Z_I - 1}. \tag{120}$$

Hence,

$$l_{00}^{ei} = \frac{n_e \, m_e}{\tau_{ee}} \, Z_{\text{eff} \, i}, \tag{121}$$

$$l_{01}^{ei} = 0, \tag{122}$$

$$l_{10}^{ei} = \frac{n_e \, m_e}{\tau_{ee}} \, \frac{3}{2} \, Z_{\text{eff} \, i}, \tag{123}$$

$$l_{11}^{ei} = 0, \tag{124}$$

and

$$l_{00}^{eI} = \frac{n_e \, m_e}{\tau_{ee}} \, Z_{\text{eff} \, I}, \tag{125}$$

$$l_{01}^{eI} = 0, \tag{126}$$

$$l_{10}^{eI} = \frac{n_e \, m_e}{\tau_{ee}} \, \frac{3}{2} \, Z_{\text{eff} \, I}, \tag{127}$$

$$l_{11}^{eI} = 0. \tag{128}$$

Thus,

$$(F^e) = \frac{n_e \, m_e}{\tau_{ee}} \left\{ - \left[F^{ee} \right] (u^e) + \left[F^{ei} \right] (u^i) + \left[F^{eI} \right] (u^I) \right\}, \tag{129}$$

where

$$F_{00}^{ee} = Z_{\text{eff}}, \tag{130}$$

$$F_{01}^{ee} = \frac{3}{2} \, Z_{\text{eff}}, \tag{131}$$

$$F_{10}^{ee} = \frac{3}{2} \, Z_{\text{eff}}, \tag{132}$$

$$F_{11}^{ee} = \sqrt{2} + \frac{13}{4} Z_{\text{eff}}, \tag{133}$$

$$F_{00}^{ei} = Z_{\text{eff}\,i}, \tag{134}$$

$$F_{01}^{ei} = 0, \tag{135}$$

$$F_{10}^{ei} = \frac{3}{2} Z_{\text{eff}\,i}, \tag{136}$$

$$F_{11}^{ei} = 0, \tag{137}$$

$$F_{00}^{eI} = Z_{\text{eff}\,I}, \tag{138}$$

$$F_{01}^{eI} = 0, \tag{139}$$

$$F_{10}^{eI} = \frac{3}{2} Z_{\text{eff}\,I}, \tag{140}$$

$$F_{11}^{eI} = 0. \tag{141}$$

Note that $F_{j0}^{ee} = F_{j0}^{ei} + F_{j0}^{eI}$.

4 Neoclassical Viscosities

$$\left\langle \vec{B} \cdot \vec{\nabla} \cdot (\vec{\vec{\Pi}}^{\,a}) \right\rangle = \frac{n_a\, m_a}{\tau_{aa}} [\mu^a]\, (u_\theta^a), \tag{142}$$

where

$$\langle A \rangle = \oint \frac{A(\theta)\, d\theta}{\vec{B} \cdot \vec{\nabla}\theta} \Big/ \oint \frac{d\theta}{\vec{B} \cdot \vec{\nabla}\theta}, \tag{143}$$

$$(\vec{\vec{\Pi}}^{\,a}) = \begin{pmatrix} \vec{\vec{\Pi}}_\parallel^{\,a} \\ \vec{\vec{\Theta}}_\parallel^{\,a} \end{pmatrix}, \tag{144}$$

$$(u_\theta^a) = \begin{pmatrix} u_{\theta\,0}^a \langle B^2 \rangle \\ u_{\theta\,1}^a \langle B^2 \rangle \end{pmatrix}, \tag{145}$$

$$[\mu^a] = \begin{pmatrix} \mu_{00}^a, & \mu_{01}^a \\ \mu_{10}^a, & \mu_{11}^a \end{pmatrix}. \tag{146}$$

Let

$$\mu_{00}^a = K_{00}^a, \tag{147}$$

$$\mu_{01}^a = \frac{5}{2} K_{00}^a - K_{01}^a = \mu_{10}^a, \tag{148}$$

$$\mu_{11}^a = K_{11}^a - 5 K_{01}^a + \frac{25}{4} K_{00}^a, \tag{149}$$

and

$$\omega_{ta} = \frac{v_{Ta}}{L_c}, \tag{150}$$

$$\nu_{*a} = \frac{8}{3\pi} \frac{\langle B^2 \rangle}{\langle (\vec{b} \cdot \vec{\nabla} B)^2 \rangle} \frac{g\,\omega_{ta}}{v_{Ta}^2\,\tau_{aa}}, \tag{151}$$

$$g = \frac{1 - f_c}{f_c}, \tag{152}$$

$$f_c = \frac{3}{4} \langle B^2 \rangle \int_0^{1/B_{\max}} \frac{\lambda\,d\lambda}{\langle \sqrt{1 - \lambda B} \rangle}. \tag{153}$$

We have

$$K_{ij}^a = g\,\frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x}\,x^{4+i+j}\,\hat{\nu}_D^a(x)\,dx}{[x^2 + \nu_{*a}\,\hat{\nu}_D^a(x)]\,[x^2 + (5\pi/8)\,(\omega_{ta}\,\tau_{aa})^{-1}\,\hat{\nu}_T^a(x)]}, \tag{154}$$

and

$$\hat{\nu}_D^a(x) = \frac{3\sqrt{\pi}}{4} \sum_b \frac{n_b e_b^2}{n_a e_a^2} \left[\left(1 - \frac{x_{ab}}{2\,x}\right) \psi\left(\frac{x}{x_{ab}}\right) + \psi'\left(\frac{x}{x_{ab}}\right) \right] \frac{1}{x}, \tag{155}$$

$$\hat{\nu}_\epsilon^a(x) = \frac{3\sqrt{\pi}}{2} \sum_b \frac{n_b e_b^2}{n_a e_a^2} \left[\frac{m_a}{m_b} \psi\left(\frac{x}{x_{ab}}\right) - \psi'\left(\frac{x}{x_{ab}}\right) \right] \frac{1}{x}, \tag{156}$$

$$\hat{\nu}_T^a(x) = 3\,\hat{\nu}_D^a(x) + \hat{\nu}_\epsilon^a(x), \tag{157}$$

$$\psi(x) = \frac{2}{\sqrt{\pi}} \int_0^{x^2} \sqrt{t}\,e^{-t}\,dt = \mathrm{erf}(x) - \frac{2}{\sqrt{\pi}}\,x\,e^{-x^2}, \tag{158}$$

$$\psi'(x) = \frac{2}{\sqrt{\pi}}\,x\,e^{-x^2}. \tag{159}$$

In particular,

$$K_{ij}^i = g\,\frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x}\,x^{2+i+j}\,\hat{\nu}_D^i(x)\,dx}{[x + \nu_{*i}\,\hat{\nu}_D^i(x)]\,[x + (5\pi/8)\,(\omega_{ti}\,\tau_{ii})^{-1}\,\hat{\nu}_T^i(x)]}, \tag{160}$$

$$\hat{\nu}_D^i = \frac{3\sqrt{\pi}}{4} \left[\left(1 - \frac{1}{2\,x}\right) \psi(x) + \psi'(x) \right] \frac{1}{x}$$
$$+ \frac{3\sqrt{\pi}}{4}\,\alpha \left[\left(1 - \frac{x_{iI}}{2\,x}\right) \psi\left(\frac{x}{x_{iI}}\right) + \psi'\left(\frac{x}{x_{iI}}\right) \right] \frac{1}{x}, \tag{161}$$

$$\hat{\nu}_\epsilon^i = \frac{3\sqrt{\pi}}{2} \left[\psi(x) - \psi'(x) \right] \frac{1}{x}$$
$$+ \frac{3\sqrt{\pi}}{2}\,\alpha \left[\frac{m_i}{m_I} \psi\left(\frac{x}{x_{iI}}\right) - \psi'\left(\frac{x}{x_{iI}}\right) \right] \frac{1}{x}, \tag{162}$$

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and

$$K_{ij}^I = g \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x} \, x^{2+i+j} \, \hat{\nu}_D^I(x) \, dx}{[x + \nu_{*I} \, \hat{\nu}_D^I(x)] \, [x + (5\pi/8) \, (\omega_{tI} \, \tau_{II})^{-1} \, \hat{\nu}_T^I(x)]}, \tag{163}$$

$$\hat{\nu}_D^I = \frac{3\sqrt{\pi}}{4} \left[\left(1 - \frac{1}{2\,x}\right) \psi(x) + \psi'(x) \right] \frac{1}{x}$$
$$+ \frac{3\sqrt{\pi}}{4} \, \alpha^{-1} \left[\left(1 - \frac{x_{Ii}}{2\,x}\right) \psi\left(\frac{x}{x_{Ii}}\right) + \psi'\left(\frac{x}{x_{Ii}}\right) \right] \frac{1}{x}, \tag{164}$$

$$\hat{\nu}_\epsilon^I = \frac{3\sqrt{\pi}}{2} \left[\psi(x) - \psi'(x) \right] \frac{1}{x}$$
$$+ \frac{3\sqrt{\pi}}{2} \, \alpha^{-1} \left[\frac{m_I}{m_i} \, \psi\left(\frac{x}{x_{Ii}}\right) - \psi'\left(\frac{x}{x_{Ii}}\right) \right] \frac{1}{x}, \tag{165}$$

and

$$K_{ij}^e = g \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{e^{-x} \, x^{4+i+j} \, \hat{\nu}_D^e(x) \, dx}{[x^2 + \nu_{*e} \, \hat{\nu}_D^e(x)] \, [x^2 + (5\pi/8) \, (\omega_{te} \, \tau_{ee})^{-1} \, \hat{\nu}_T^e(x)]}, \tag{166}$$

$$\hat{\nu}_D^e = \frac{3\sqrt{\pi}}{4} \left[\left(1 - \frac{1}{2\,x}\right) \psi(x) + \psi'(x) \right] + \frac{3\sqrt{\pi}}{4} \, Z_{\text{eff}}, \tag{167}$$

$$\hat{\nu}_\epsilon^e = \frac{3\sqrt{\pi}}{2} \left[\psi(x) - \psi'(x) \right]. \tag{168}$$

5 Parallel Force and Heat Balance Equations

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\Pi}^a \rangle = \langle \vec{B} \cdot \vec{F}_0^a \rangle + \langle \vec{B} \cdot \vec{F}_0^{a\,cx} \rangle + n_a \, e_a \, \langle \vec{E} \cdot \vec{B} \rangle, \tag{169}$$

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\Theta}^a \rangle = \langle \vec{B} \cdot \vec{F}_1^a \rangle + \langle \vec{B} \cdot \vec{F}_1^{a\,cx} \rangle. \tag{170}$$

Now,

$$\langle \vec{B} \cdot (F)^i \rangle = \frac{n_i \, m_i}{\tau_{ii}} \left\{ - \left[F^{ii} \right] (u_\parallel^i) + \left[F^{iI} \right] (u_\parallel^I) \right\}, \tag{171}$$

$$\langle \vec{B} \cdot (F)^I \rangle = \frac{n_i \, m_i}{\tau_{ii}} \left\{ \left[F^{Ii} \right] (u_\parallel^i) - \left[F^{II} \right] (u_\parallel^I) \right\}, \tag{172}$$

$$\langle \vec{B} \cdot (F)^e \rangle = \frac{n_e \, m_e}{\tau_{ee}} \left\{ - \left[F^{ee} \right] (u_\parallel^e) + \left[F^{ei} \right] (u_\parallel^i) + \left[F^{eI} \right] (u_\parallel^I) \right\}, \tag{173}$$

$$\langle \vec{B} \cdot (F^{i\,cx}) \rangle = -n_i \, m_i \, \langle \sigma \, v \rangle_i^{cx} \, [X^i](\tilde{u}_\parallel^i), \tag{174}$$

$$\langle \vec{B} \cdot (F^{I\,cx}) \rangle = 0, \tag{175}$$

$$\langle \vec{B} \cdot (F^{e\,cx}) \rangle = 0, \tag{176}$$

where

$$(F^{a\,cx}) = \begin{pmatrix} \vec{F}_0^{\,a\,cx} \\ \vec{F}_1^{\,a\,cx} \end{pmatrix}, \tag{177}$$

$$(u_\parallel^a) = \begin{pmatrix} \langle u_{\parallel\,0}^a\,B \rangle \\ \langle u_{\parallel\,1}^a\,B \rangle \end{pmatrix}, \tag{178}$$

$$(\tilde{u}_\parallel^a) = \begin{pmatrix} \langle n_n\,u_{\parallel\,0}^a\,B \rangle \\ \langle n_n\,u_{\parallel\,1}^a\,B \rangle \end{pmatrix}, \tag{179}$$

$$[X^i] = \begin{pmatrix} 1, & 0 \\ 0, & E_n/T_i \end{pmatrix}. \tag{180}$$

Here, E_n is the neutral kinetic energy, n_n the neutral number density, and $\langle \sigma\,v \rangle_i^{cx}$ the majority ion charge exchange rate coefficient.

Let

$$(E_\parallel) = \begin{pmatrix} \langle E_\parallel\,B \rangle/\langle B \rangle \\ 0 \end{pmatrix}, \tag{181}$$

$$\Omega_a = \frac{e_a\,\langle B \rangle}{m_a}, \tag{182}$$

$$[\tilde{\mu}^I] = \alpha^2\,\frac{m_i}{m_I}\,x_{Ii}^3\,[\mu^I] = \alpha^2\,\frac{T_i}{T_I}\,x_{Ii}\,[\mu^I]. \tag{183}$$

Follows that

$$\begin{bmatrix} [F^{ii}], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{bmatrix} \begin{bmatrix} (u_\parallel^i) \\ (u_\parallel^I) \end{bmatrix} \simeq -\begin{bmatrix} [\mu^i]\,(u_\theta^i) \\ [\tilde{\mu}^I]\,(u_\theta^I) \end{bmatrix} - \begin{bmatrix} \tau_{ii}\,\langle \sigma\,v \rangle_i^{cx}\,[X^i]\,(\tilde{u}_{\parallel\,i}) \\ 0 \end{bmatrix}, \tag{184}$$

and

$$[\mu^e]\,(u_\theta^e) = -[F^{ee}]\,(u_\parallel^e) + [F^{ei}]\,(u_\parallel^i) + [F^{eI}]\,(u_\parallel^I) + \Omega_e\,\tau_{ee}\,(E_\parallel). \tag{185}$$

Follows from Eq. (13) that

$$(u_\parallel^a) = (u_\theta^a) + (V_\psi^a), \tag{186}$$

and

$$(\tilde{u}_\parallel^a) = \frac{\langle n_n \rangle}{y}\,(u_\theta^a) + \langle n_n \rangle\,(V_\psi^a), \tag{187}$$

where

$$V_\psi^a = \begin{pmatrix} V_{\psi\,0}^a \\ V_{\psi\,1}^a \end{pmatrix}, \tag{188}$$

and

$$y = \frac{\langle n_n \rangle\,\langle B^2 \rangle}{\langle n_n\,B^2 \rangle}. \tag{189}$$

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Thus,

$$
\left[\begin{array}{cc} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{array}\right] \left[\begin{array}{c} (u_\theta^i) \\ (u_\theta^I) \end{array}\right] = - \left[\begin{array}{cc} [F^{ii} + Y^i], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{array}\right] \left[\begin{array}{c} (V_\psi^i) \\ (V_\psi^I) \end{array}\right],
$$

(190)

where

$$
[Y^i] = \tau_{ii} \langle \sigma v \rangle_i^{cx} \langle n_n \rangle [X^i].
$$

(191)

Can write

$$
(V_\psi^a) = (V_*^a) + (V_E),
$$

(192)

where

$$
V_{*0}^a(\psi) = -\frac{I T_a}{e_a} \frac{d \ln p_a}{d\psi},
$$

(193)

$$
V_{*1}^a(\psi) = \frac{I T_a}{e_a} \frac{d \ln T_a}{d\psi},
$$

(194)

$$
V_{E0}(\psi) = -I \frac{d\Phi}{d\psi},
$$

(195)

$$
V_{E1}(\psi) = 0.
$$

(196)

Fact that $F_{j0}^{ii} = F_{j0}^{iI}$ and $F_{j0}^{Ii} = F_{j0}^{II}$ means that we can rewrite Eq. (190) as

$$
\left[\begin{array}{cc} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{array}\right] \left[\begin{array}{c} (u_\theta^i) \\ (u_\theta^I) \end{array}\right] = - \left[\begin{array}{cc} [F^{ii} + Y^i], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{array}\right] \left[\begin{array}{c} (V_*^i) \\ (V_*^I) \end{array}\right]
$$
$$
- \tau_{ii} \langle \sigma v \rangle_i^{cx} \langle n_n \rangle \left[\begin{array}{c} (V_E) \\ (0) \end{array}\right].
$$

(197)

Let

$$
\left[\begin{array}{cc} [L^{ii}], & [L^{iI}] \\ [L^{Ii}], & [L^{II}] \end{array}\right] = \left[\begin{array}{cc} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{array}\right]^{-1} \left[\begin{array}{cc} [F^{ii} + Y^i], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II}] \end{array}\right],
$$

(198)

and

$$
\left[\begin{array}{cc} [G^{ii}], & [G^{iI}] \\ [G^{Ii}], & [G^{II}] \end{array}\right] = \tau_{ii} \langle \sigma v \rangle_i^{cx} \langle n_n \rangle \left[\begin{array}{cc} [F^{ii} + \mu^i + Y^i/y], & -[F^{iI}] \\ -[F^{Ii}], & [F^{II} + \tilde{\mu}^I] \end{array}\right]^{-1}.
$$

(199)

Follows that

$$
(u_\theta^i) = -[G^{ii}] (V_E) - [L^{ii}] (V_*^i) - [L^{iI}] (V_*^I),
$$

(200)

$$
(u_\theta^I) = -[G^{Ii}] (V_E) - [L^{II}] (V_*^I) - [L^{Ii}] (V_*^i),
$$

(201)

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$$(u_\parallel^i) = ([1] - [G^{ii}]) \, (V_E) + (1 - [L^{ii}]) \, (V_*^i) - [L^{iI}] \, (V_*^I), \tag{202}$$

$$(u_\parallel^I) = ([1] - [G^{Ii}]) \, ((V_E) + (1 - [L^{II}]) \, (V_*^I) - [L^{Ii}] \, (V_*^i). \tag{203}$$

Equation (185) becomes

$$
\begin{aligned}
[F^{ee} + \mu^e] \, (u_\theta^e) = & -\left([F^{ee}] - [F^{ei}] - [F^{eI}]\right) (V_E) - \left([F^{ei}] \, [G^{ii}] + [F^{eI}] \, [G^{Ii}]\right) (V_E) \\
& - [F^{ee}] \, (V_*^e) + \left\{[F^{ei}] \, (1 - [L^{ii}]) - [F^{eI}] \, [L^{Ii}]\right\} (V_*^i) \\
& + \left\{[F^{eI}] \, (1 - [L^{II}]) - [F^{ei}] \, [L^{iI}]\right\} (V_*^I) + \Omega_{ee} \, \tau_{ee} \, (E_\parallel).
\end{aligned}
\tag{204}
$$

However, because $F_{j0}^{ee} = F_{j0}^{ei} + F_{j0}^{eI}$, the previous equation becomes

$$
\begin{aligned}
[F^{ee} + \mu^e] \, (u_\theta^e) = & -\left([F^{ei}] \, [G^{ii}] + [F^{eI}] \, [G^{Ii}]\right) (V_E) - [F^{ee}] \, (V_*^e) \\
& + \left\{[F^{ei}] \, (1 - [L^{ii}]) - [F^{eI}] \, [L^{Ii}]\right\} (V_*^i) \\
& + \left\{[F^{eI}] \, (1 - [L^{II}]) - [F^{ei}] \, [L^{iI}]\right\} (V_*^I) + \Omega_{ee} \, \tau_{ee} \, (E_\parallel).
\end{aligned}
\tag{205}
$$

Let

$$[Q^{ee}] = [F^{ee} + \mu^e]^{-1}, \tag{206}$$

$$[G^{ei}] = [Q^{ee}] \, ([F^{ei}] \, [G^{ii}] + [F^{eI}] \, [G^{Ii}]), \tag{207}$$

$$[L^{ee}] = [Q^{ee}] \, [F^{ee}], \tag{208}$$

$$[L^{ei}] = -[Q^{ee}] \left\{[F^{ei}] \, (1 - [L^{ii}]) - [F^{eI}] \, [L^{Ii}]\right\}, \tag{209}$$

$$[L^{eI}] = -[Q^{ee}] \left\{[F^{eI}] \, (1 - [L^{II}]) - [F^{ei}] \, [L^{iI}]\right\}. \tag{210}$$

Follows that

$$
\begin{aligned}
(u_\theta^e) = & -[G^{ei}] \, (V_E) - [L^{ee}] \, (V_*^e) - [L^{ei}] \, (V_*^i) - [L^{eI}] \, (V_*^I) \\
& + \Omega_e \, \tau_{ee} \, [Q^{ee}] \, (E_\parallel),
\end{aligned}
\tag{211}
$$

$$
\begin{aligned}
(u_\parallel^e) = & ([1] - [G^{ei}]) \, (V_E) + (1 - [L^{ee}]) \, (V_*^e) - [L^{ei}] \, (V_*^i) - [L^{eI}] \, (V_*^I) \\
& + \Omega_e \, \tau_{ee} \, [Q^{ee}] \, (E_\parallel),
\end{aligned}
\tag{212}
$$

Finally,

$$
\begin{aligned}
(j_\parallel) \equiv & \sum_a n_a \, e_a \, (u_\parallel^a) \\
= & -\left(n_i \, e_i \, [G^{ii}] + n_I \, e_I \, [G^{Ii}] + n_e \, e_e \, [G^{ei}]\right) (V_E) \\
& + \left\{n_i \, e_i \, (1 - [L^{ii}]) - n_I \, e_I \, [L^{Ii}] - n_e \, e_e \, [L^{ei}]\right\} (V_*^i) \\
& + \left\{n_I \, e_I \, (1 - [L^{II}]) - n_i \, e_i \, [L^{iI}] - n_e \, e_e \, [L^{eI}]\right\} (V_*^I) \\
& + n_e \, e_e \, (1 - [L^{ee}]) \, (V_*^e) + n_e \, e_e \, \Omega_e \, \tau_{ee} \, [Q^{ee}] \, (E_\parallel),
\end{aligned}
\tag{213}
$$

where we have made use of the fact that $n_i \, e_i + n_I \, e_I + n_e \, e_e = 0$.

6 Fluid Velocities

Let

$$\omega_E(\psi) = -\frac{d\Phi}{d\psi}, \tag{214}$$

$$\omega_{*\,a}(\psi) = -\frac{T_a}{e_a}\frac{d\ln p_a}{d\psi}, \tag{215}$$

$$\eta_a(\psi) = \frac{d\ln T_a}{d\ln n_a}. \tag{216}$$

It follows that

$$V^a_{*0}(\psi) = I\,\omega_{*\,a}, \tag{217}$$

$$V^a_{*1}(\psi) = -I\left(\frac{\eta_a}{1+\eta_a}\right)\omega_{*\,a}, \tag{218}$$

$$V_{E0}(\psi) = I\,\omega_E, \tag{219}$$

$$V_{E1}(\psi) = 0. \tag{220}$$

We have $e_e = -e$, $e_i = e$, $e_I = Z_I\,e$. Thus,

$$\frac{V^i_\theta}{B_\theta}\frac{\langle B^2\rangle}{I} = -G^{ii}_{00}\,\omega_E - L^{ii}_{00}\,\omega_{*\,i} - L^{iI}_{00}\,\omega_{*\,I} + L^{ii}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}$$
$$+ L^{iI}_{01}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}, \tag{221}$$

$$\frac{V^I_\theta}{B_\theta}\frac{\langle B^2\rangle}{I} = -G^{Ii}_{00}\,\omega_E - L^{II}_{00}\,\omega_{*\,I} - L^{Ii}_{00}\,\omega_{*\,i} + L^{II}_{01}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}$$
$$+ L^{Ii}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}, \tag{222}$$

$$\frac{\langle V^i_\parallel B\rangle}{I} = (1 - G^{ii}_{00})\,\omega_E + \omega_{*\,i} - L^{ii}_{00}\,\omega_{*\,i} - L^{iI}_{00}\,\omega_{*\,I} + L^{ii}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}$$
$$+ L^{iI}_{01}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}, \tag{223}$$

$$\frac{\langle V^I_\parallel B\rangle}{I} = (1 - G^{Ii}_{00})\,\omega_E + \omega_{*\,I} - L^{II}_{00}\,\omega_{*\,I} - L^{Ii}_{00}\,\omega_{*\,i} + L^{II}_{01}\left(\frac{\eta_I}{1+\eta_I}\right)\omega_{*\,I}$$
$$+ L^{Ii}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}. \tag{224}$$

Also,

$$\frac{V^i_\phi}{R} = \omega_E + \omega_{*\,i} + \frac{I^2}{R^2\langle B^2\rangle}\left[-G^{ii}_{00}\,\omega_E - L^{ii}_{00}\,\omega_{*\,i} - L^{iI}_{00}\,\omega_{*\,I} + L^{ii}_{01}\left(\frac{\eta_i}{1+\eta_i}\right)\omega_{*\,i}\right.$$

$$+ L_{01}^{iI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} \Bigg] , \tag{225}$$

$$\frac{V_\phi^I}{R} = \omega_E + \omega_{*I} + \frac{I^2}{R^2 \langle B^2 \rangle} \Bigg[-G_{00}^{Ii} \omega_E - L_{00}^{II} \omega_{*I} - L_{00}^{Ii} \omega_{*i} + L_{01}^{II} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}$$

$$+ L_{01}^{Ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} \Bigg] . \tag{226}$$

7 Fluid Angular Rotation Velocities

Let

$$\omega_{\theta\,i}(\psi) = \frac{V_\theta^i}{B_\theta} \frac{I}{R^2}, \tag{227}$$

$$\omega_{\theta\,I}(\psi) = \frac{V_\theta^I}{B_\theta} \frac{I}{R^2}, \tag{228}$$

$$\omega_{\phi\,i}(\psi) = \frac{V_\phi^i}{R}, \tag{229}$$

$$\omega_{\phi\,I}(\psi) = \frac{V_\phi^I}{R}, \tag{230}$$

where the right-hand sides are evaluated on the outboard mid-plane. It follows that

$$\omega_{\theta\,i} = K_\theta\, \omega_{\mathrm{nc}\,i}, \tag{231}$$

$$\omega_{\theta\,I} = K_\theta\, \omega_{\mathrm{nc}\,I}, \tag{232}$$

$$\omega_{\phi\,i} = \omega_E + \omega_{*i} + K_\theta\, \omega_{\mathrm{nc}\,i}, \tag{233}$$

$$\omega_{\phi\,I} = \omega_E + \omega_{*I} + K_\theta\, \omega_{\mathrm{nc}\,I}, \tag{234}$$

where

$$K_\theta = \frac{I^2}{R^2 \langle B^2 \rangle}, \tag{235}$$

and

$$\omega_{\mathrm{nc}\,i} = -G_{00}^{ii} \omega_E - L_{00}^{ii} \omega_{*i} + L_{01}^{ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i} - L_{00}^{iI} \omega_{*I} + L_{01}^{iI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}, \tag{236}$$

$$\omega_{\mathrm{nc}\,I} = -G_{00}^{Ii} \omega_E - L_{00}^{II} \omega_{*I} + L_{01}^{II} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I} - L_{00}^{Ii} \omega_{*i} + L_{01}^{Ii} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i}. \tag{237}$$

8 Natural Frequencies

Let

$$\omega_{0\,\mathrm{ln}} = \omega_E + \omega_{*e}, \tag{238}$$

$$\omega_{0\,\mathrm{eb}} = \omega_E, \tag{239}$$

$$\omega_{0\,\mathrm{nl}} = \omega_E + \omega_{*i} + \omega_{\mathrm{nc}\,i}. \tag{240}$$

The corresponding natural frequencies are

$$\varpi_{0\,\mathrm{ln}} = -n\,\omega_{0\,\mathrm{ln}}, \tag{241}$$

$$\varpi_{0\,\mathrm{eb}} = -n\,\omega_{0\,\mathrm{eb}}, \tag{242}$$

$$\varpi_{0\,\mathrm{nl}} = -n\,\omega_{0\,\mathrm{nl}}. \tag{243}$$

9   Current Densities

Can write

$$j_\parallel = j_{\mathrm{bootstrap}} + j_{\mathrm{ohmic}}, \tag{244}$$

where

$$
\begin{aligned}
\frac{\langle j_{\mathrm{bootstrap}}\,B\rangle}{I} = &-\left\{ G_{00}^{ei} - \left(\frac{Z_I - Z_{\mathrm{eff}}}{Z_I - 1}\right) G_{00}^{ii} - \left(\frac{Z_{\mathrm{eff}} - 1}{Z_I - 1}\right) G_{00}^{Ii} \right\} n_e\,e\,\frac{d\Phi}{d\psi} \\
&-\left\{ 1 - L_{00}^{ii} - \left(\frac{Z_{\mathrm{eff}} - 1}{Z_I - Z_{\mathrm{eff}}}\right) L_{00}^{Ii} + \left(\frac{Z_I - 1}{Z_I - Z_{\mathrm{eff}}}\right) L_{00}^{ei} \right\} \frac{dp_i}{d\psi} \\
&-\left\{ 1 - L_{00}^{II} - \left(\frac{Z_I - Z_{\mathrm{eff}}}{Z_{\mathrm{eff}} - 1}\right) L_{00}^{iI} + \left(\frac{Z_I - 1}{Z_{\mathrm{eff}} - 1}\right) L_{00}^{eI} \right\} \frac{dp_I}{d\psi} \\
&- (1 - L_{00}^{ee})\,\frac{dp_e}{d\psi} \\
&-\left\{ L_{01}^{ii} + \left(\frac{Z_{\mathrm{eff}} - 1}{Z_I - Z_{\mathrm{eff}}}\right) L_{01}^{Ii} - \left(\frac{Z_I - 1}{Z_I - Z_{\mathrm{eff}}}\right) L_{01}^{ei} \right\} n_i\,\frac{dT_i}{d\psi} \\
&-\left\{ L_{01}^{II} + \left(\frac{Z_I - Z_{\mathrm{eff}}}{Z_{\mathrm{eff}} - 1}\right) L_{01}^{iI} - \left(\frac{Z_I - 1}{Z_{\mathrm{eff}} - 1}\right) L_{01}^{eI} \right\} n_I\,\frac{dT_I}{d\psi} \\
&- L_{01}^{ee}\,n_e\,\frac{dT_e}{d\psi}
\end{aligned}
\tag{245}
$$

and

$$\langle j_{\mathrm{ohmic}}\,B\rangle = Q_{00}^{ee}\,\sigma_{ee}\,\langle E_\parallel\,B\rangle, \tag{246}$$

where

$$\sigma_{ee} = \frac{n_e\,e^2\,\tau_{ee}}{m_e}. \tag{247}$$

Let

$$\omega_\parallel = \frac{\langle j_\parallel\,B\rangle}{n_e\,e\,I}. \tag{248}$$

It follows that

$$\omega_\parallel = \omega_{\parallel\,\mathrm{bootstrap}} + \omega_{\parallel\,\mathrm{ohmic}}, \tag{249}$$

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where

$$\omega_{\parallel \text{bootstrap}} = \left\{ G_{00}^{ei} - \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) G_{00}^{ii} - \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) G_{00}^{Ii} \right\} \omega_E$$

$$+ \left\{ \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) (1 - L_{00}^{ii}) - \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) L_{00}^{Ii} + L_{00}^{ei} \right\} \omega_{*i}$$

$$+ \left\{ \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) (1 - L_{00}^{II}) - \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) L_{00}^{iI} + L_{00}^{eI} \right\} \omega_{*I}$$

$$- (1 - L_{00}^{ee}) \, \omega_{*e}$$

$$+ \left\{ \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) L_{01}^{ii} + \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) L_{01}^{Ii} - L_{01}^{ei} \right\} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i}$$

$$+ \left\{ \left(\frac{Z_{\text{eff}} - 1}{Z_I - 1} \right) L_{01}^{II} + \left(\frac{Z_I - Z_{\text{eff}}}{Z_I - 1} \right) L_{01}^{iI} - L_{01}^{eI} \right\} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}$$

$$- L_{01}^{ee} \left(\frac{\eta_e}{1 + \eta_e} \right) \omega_{*e}, \tag{250}$$

and

$$\omega_{\parallel \text{ohmic}} = Q_{00}^{ee} \, |\Omega_e| \, \tau_{ee} \, \omega_{\text{loop}}, \tag{251}$$

with

$$\omega_{\text{loop}} = \frac{\langle E_{\parallel} \, B \rangle}{I \, \langle B \rangle} = \frac{V_{\text{loop}}}{2\pi} \frac{\langle B/R \rangle}{I \, \langle B \rangle}. \tag{252}$$

Can also write

$$\omega_{\parallel \text{bootstrap}} = -(\omega_{*e} + \omega_{\text{nc}\,e}) + \frac{n_i}{n_e} (\omega_{*i} + \omega_{\text{nc}\,i}) + \frac{Z_I \, n_I}{n_e} (\omega_{*I} + \omega_{\text{nc}\,I}), \tag{253}$$

where

$$\omega_{\text{nc}\,e} = -G_{00}^{ei} \, \omega_E - L_{00}^{ee} \, \omega_{*e} + L_{01}^{ee} \left(\frac{\eta_e}{1 + \eta_e} \right) \omega_{*e} - L_{00}^{ei} \, \omega_{*i} + L_{01}^{ei} \left(\frac{\eta_i}{1 + \eta_i} \right) \omega_{*i}$$

$$- L_{00}^{eI} \, \omega_{*I} + L_{01}^{eI} \left(\frac{\eta_I}{1 + \eta_I} \right) \omega_{*I}. \tag{254}$$

10 Model Poloidal Flow Damping Term

Our model poloidal flow damping force density in the plasma equation of motion takes the form

$$\frac{m_i \, n_i}{\tau_\theta} \left(V_\theta - V_\theta^{\,nc} \right) \mathbf{e}_\theta, \tag{255}$$

where

$$\frac{1}{\tau_\theta} = \left(\frac{I^2}{\langle |\nabla \psi|^2 \rangle} \right) \frac{\mu_{00}^i}{\tau_{ii}}, \tag{256}$$

and $V_\theta^{\,nc}$ is specified in Eq. (221).