Program RESCALE

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1 Rescaling Grad-Shafranov Equation

1.1 Grad-Shafranov Equation

Let R, ϕ, Z be a set of cylindrical coordinates. The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\phi\times\nabla\psi + T(\psi)\,\nabla\phi. \tag{1}$$

The Grad-Shafranov equation takes the form

$$\frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\,\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = -R^2\,\frac{dP}{d\psi} - \frac{1}{2}\,\frac{dT^2}{d\psi}, \tag{2}$$

where

$$j_\phi = -R\,\frac{dP}{d\psi} - \frac{1}{2\,R}\,\frac{dT^2}{d\psi}, \tag{3}$$

$$q(\psi) = \frac{T}{2\pi}\oint_\psi \frac{dl}{R\,|\nabla\psi|}. \tag{4}$$

1.2 Type I Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\rm new} = R_{\rm old}, \tag{5}$$

$$Z_{\rm new} = Z_{\rm old}, \tag{6}$$

$$T_{\rm new} = a_1\,T_{\rm old}, \tag{7}$$

$$\psi_{\rm new} = a_1\,\psi_{\rm old}, \tag{8}$$

$$P_{\rm new} = a_1^2\,P_{\rm old}. \tag{9}$$

Note that

$$B_{t,p\,{\rm new}} = a_1\,B_{t,p\,{\rm old}}, \tag{10}$$

$$I_{\phi\,\mathrm{new}} = a_1\, I_{\phi\,\mathrm{old}}, \tag{11}$$

$$\beta_{t,p\,\mathrm{new}} = \beta_{t,p\,\mathrm{old}}, \tag{12}$$

$$q_{\mathrm{new}} = q_{\mathrm{old}}. \tag{13}$$

1.3 Type II Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\mathrm{new}} = a_2\, R_{\mathrm{old}}, \tag{14}$$

$$Z_{\mathrm{new}} = a_2\, Z_{\mathrm{old}}, \tag{15}$$

$$T_{\mathrm{new}} = a_2\, T_{\mathrm{old}}, \tag{16}$$

$$\psi_{\mathrm{new}} = a_2^2\, \psi_{\mathrm{old}}, \tag{17}$$

$$P_{\mathrm{new}} = P_{\mathrm{old}}. \tag{18}$$

Note that

$$B_{t,p\,\mathrm{new}} = B_{t,\,p\,\mathrm{old}}, \tag{19}$$

$$I_{\phi\,\mathrm{new}} = a_2\, I_{\phi\,\mathrm{old}}, \tag{20}$$

$$\beta_{t,p\,\mathrm{new}} = \beta_{t,p\,\mathrm{old}}, \tag{21}$$

$$q_{\mathrm{new}} = q_{\mathrm{old}}. \tag{22}$$

1.4 Type III Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\mathrm{new}} = R_{\mathrm{old}}, \tag{23}$$

$$Z_{\mathrm{new}} = Z_{\mathrm{old}}, \tag{24}$$

$$T_{\mathrm{new}} = T_{\mathrm{old}}, \tag{25}$$

$$\psi_{\mathrm{new}} = \psi_{\mathrm{old}}, \tag{26}$$

$$P_{\mathrm{new}} = P_{\mathrm{old}} + a_3. \tag{27}$$

Note that

$$B_{t,p\,\mathrm{new}} = B_{t,\,p\,\mathrm{old}}, \tag{28}$$

$$I_{\phi\,\mathrm{new}} = I_{\phi\,\mathrm{old}}, \tag{29}$$

$$\beta_{t,p\,\mathrm{new}} = \left(\frac{P_{\mathrm{new}}}{P_{\mathrm{old}}}\right) \beta_{t,p\,\mathrm{old}}, \tag{30}$$

$$q_{\mathrm{new}} = q_{\mathrm{old}}. \tag{31}$$

1.5 Type IV Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\rm new} = R_{\rm old}, \tag{32}$$
$$Z_{\rm new} = Z_{\rm old}, \tag{33}$$
$$T_{\rm new} = {\rm sgn}(T_{\rm old})\sqrt{T_{\rm old}^2 + a_4}, \tag{34}$$
$$\psi_{\rm new} = \psi_{\rm old}, \tag{35}$$
$$P_{\rm new} = P_{\rm old}. \tag{36}$$

Note that

$$B_{t\,{\rm new}} = \left(\frac{T_{\rm new}}{T_{\rm old}}\right) B_{t\,{\rm old}}, \tag{37}$$
$$B_{p\,{\rm new}} = B_{p\,{\rm old}}, \tag{38}$$
$$I_{\phi\,{\rm new}} = I_{\phi\,{\rm old}}, \tag{39}$$
$$\beta_{t\,{\rm new}} = \left(\frac{T_{\rm old}}{T_{\rm new}}\right)^2 \beta_{t\,{\rm old}}, \tag{40}$$
$$\beta_{p\,{\rm new}} = \beta_{p\,{\rm old}}, \tag{41}$$
$$q_{\rm new} = \left(\frac{T_{\rm new}}{T_{\rm old}}\right) q_{\rm old}. \tag{42}$$

2 Rescaling Plasma Equilibrium

2.1 n_e Rescaling

Perform Type I rescaling with $a_1 = a_n^{1/2}$. So,

$$R \to R, \tag{43}$$
$$T \to a_n^{1/2}\, T, \tag{44}$$
$$\psi \to a_n^{1/2}\, \psi, \tag{45}$$
$$P \to a_n\, P. \tag{46}$$

Follows that

$$B_{p,t} \to a_n^{1/2}\, B_{p,t}, \tag{47}$$
$$q_{95} \to q_{95}. \tag{48}$$

Let

$$n_a \to a_n\, n_a, \tag{49}$$

$$T_a \to T_a. \tag{50}$$

Here, $a \equiv e, i, b, I$. Follows that

$$\omega_{*\,a} \to a_n^{-1/2}\, \omega_{*\,a}. \tag{51}$$

Assuming that

$$\omega_{\theta\,a} \to a_n^{-1/2}\, \omega_{\theta\,a}, \tag{52}$$

$$\omega_E \to \omega_E, \tag{53}$$

we require

$$\omega_{\phi\,a} \to \omega_{\phi\,a} + (a_n^{-1/2} - 1)\,(\omega_{*\,a} + \omega_{\theta\,a}). \tag{54}$$

2.2 T_e Rescaling

Perform Type I rescaling with $a_1 = a_T^{1/2}$. So,

$$R \to R, \tag{55}$$

$$T \to a_T^{1/2}\, T, \tag{56}$$

$$\psi \to a_T^{1/2}\, \psi, \tag{57}$$

$$P \to a_T\, P. \tag{58}$$

Follows that

$$B_{p,t} \to a_T^{1/2}\, B_{p,t}, \tag{59}$$

$$q_{95} \to q_{95}. \tag{60}$$

Let

$$n_a \to n_a, \tag{61}$$

$$T_a \to a_T\, T_a. \tag{62}$$

Follows that

$$\omega_{*\,a} \to a_T^{1/2}\, \omega_{*\,a}. \tag{63}$$

Assuming that

$$\omega_{\theta\,a} \to a_T^{1/2}\, \omega_{\theta\,a}, \tag{64}$$

$$\omega_E \to \omega_E, \tag{65}$$

we require

$$\omega_{\phi\,a} \to \omega_{\phi\,a} + (a_T^{1/2} - 1)\,(\omega_{*\,a} + \omega_{\theta\,a}). \tag{66}$$

2.3 R Rescaling

Perform a Type II rescaling with $a_2 = a_R$. It follows that

$$R \to a_R\, R, \tag{67}$$

$$T \to a_R\, T, \tag{68}$$

$$\psi \to a_R^2\, \psi, \tag{69}$$

$$P \to P. \tag{70}$$

Follows that

$$B_{t,p} \to B_{t,p}, \tag{71}$$

$$q_{95} \to q_{95}. \tag{72}$$

Let

$$n_a \to n_a, \tag{73}$$

$$T_a \to T_a, \tag{74}$$

so

$$\omega_{\ast\, a} \to a_R^{-2}\, \omega_{\ast\, a}. \tag{75}$$

Assuming that

$$\omega_{\theta\, a} \to a_R^{-2}\, \omega_{\theta\, a}, \tag{76}$$

$$\omega_E \to \omega_E, \tag{77}$$

we require

$$\omega_{\phi\, a} \to \omega_{\phi\, a} + (a_R^{-2} - 1)\, (\omega_{\ast\, a} + \omega_{\theta\, a}). \tag{78}$$

2.4 P Rescaling

Perform a Type III rescaling with $a_3 = a_P$. Let

$$R \to R, \tag{79}$$

$$P \to P + a_P, \tag{80}$$

$$T \to T, \tag{81}$$

$$\psi \to \psi. \tag{82}$$

$$B_{t,p} \to B_{t,p}, \tag{83}$$

$$q_{95} \to q_{95}. \tag{84}$$

Let

$$n_a \to n_a, \tag{85}$$

$$T_e \to T_e + \left(\frac{a_P}{2\,n_e}\right), \tag{86}$$

$$T_i \to T_i + \left(\frac{a_P}{2\,n_i}\right), \tag{87}$$

$$T_I \to T_I, \tag{88}$$

so

$$\omega_{*\,a} \to \omega_{*\,a}. \tag{89}$$

Assuming that

$$\omega_{\theta\,a} \to \omega_{\theta\,a}, \tag{90}$$

$$\omega_E \to \omega_E, \tag{91}$$

we require

$$\omega_{\phi\,a} \to \omega_{\phi\,a}. \tag{92}$$

2.5 ω_E Rescaling

Do not rescale equilibrium. So,

$$R \to R, \tag{93}$$

$$\psi \to \psi, \tag{94}$$

$$T \to T, \tag{95}$$

$$P \to P. \tag{96}$$

Follows that

$$B_{p,t} \to B_{p,t}, \tag{97}$$

$$q_{95} \to q_{95}. \tag{98}$$

Let

$$n_a \to n_a, \tag{99}$$

$$T_a \to T_a, \tag{100}$$

so

$$\omega_{*\,a} \to \omega_{*\,a}. \tag{101}$$

Assuming that

$$\omega_{\theta\,a} \to \omega_{\theta\,a}, \tag{102}$$

$$\omega_E \to \omega_E + a_E, \tag{103}$$

we require

$$\omega_{\phi\,a} \to \omega_{\phi\,a} + a_E. \tag{104}$$

2.6 χ_ϕ Rescaling

Let everything stay the same, except

$$\chi_{\phi,e,i} \to a_\phi\, \chi_{\phi,e,i}, \tag{105}$$

$$D_\perp \to a_\phi\, D_\perp. \tag{106}$$

2.7 q_{95} Rescaling

Let

$$\Psi_N = \frac{\psi - \psi_{\rm axis}}{\psi_{\rm separatrix} - \psi_{\rm axis}}. \tag{107}$$

First, perform a Type IV rescaling such that

$$a_4 = \left(\frac{q_{95\,{\rm target}}^2}{q_{95\,{\rm old}}^2} - 1\right) T_{95\,{\rm old}}^2, \tag{108}$$

where

$$q_{95} \equiv q(\Psi_N = 0.95), \tag{109}$$

$$T_{95} \equiv T(\Psi_N = 0.95). \tag{110}$$

It follows that

$$\psi_{\rm new} = \psi_{\rm old}, \tag{111}$$

$$T_{\rm new} = T_{\rm old}\sqrt{1 + \left(\frac{q_{95\,{\rm target}}^2}{q_{95\,{\rm old}}^2} - 1\right)\frac{T_{95\,{\rm old}}^2}{T_{\rm old}^2}}, \tag{112}$$

$$P_{\rm new} = P_{\rm old}, \tag{113}$$

and

$$I_{\phi\,{\rm new}} = I_{\phi\,{\rm old}}, \tag{114}$$

$$q_{\rm new} = q_{\rm old}\sqrt{1 + \left(\frac{q_{95\,{\rm target}}^2}{q_{95\,{\rm old}}^2} - 1\right)\frac{T_{95\,{\rm old}}^2}{T_{\rm old}^2}}. \tag{115}$$

Next, perform a Type I rescaling such that

$$a_1 = \frac{T_{1\,\rm old}}{T_{1\,\rm new}}, \tag{116}$$

where

$$T_1 \equiv T(\Psi_N = 1). \tag{117}$$

It follows that

$$\psi_{\rm new\,new} = a_1\,\psi_{\rm old}, \tag{118}$$

$$T_{\rm new\,new} = a_1\,T_{\rm new} = T_{1\,\rm old}\sqrt{\frac{T_{\rm old}^2 + (q_{95\,\rm target}^2/q_{95\,\rm old}^2 - 1)\,T_{95\,\rm old}^2}{T_{1\,\rm old}^2 + (q_{95\,\rm target}^2/q_{95\,\rm old}^2 - 1)\,T_{95\,\rm old}^2}}, \tag{119}$$

$$P_{\rm new\,new} = a_1^2\,P_{\rm old}, \tag{120}$$

and

$$I_{\phi\,\rm new\,new} = a_1\,I_{\phi\,\rm old}, \tag{121}$$

$$q_{\rm new\,new} = q_{\rm new} = q_{\rm old}\sqrt{1 + \left(\frac{q_{95\,\rm target}^2}{q_{95\,\rm old}^2} - 1\right)\frac{T_{95\,\rm old}^2}{T_{\rm old}^2}}. \tag{122}$$

Note that

$$R \rightarrow R, \tag{123}$$

$$T \rightarrow \left[a_1^2 + (1 - a_1^2)\left(\frac{T_1}{T}\right)^2\right]^{1/2} T, \tag{124}$$

$$\psi \rightarrow a_1\,\psi, \tag{125}$$

$$P \rightarrow a_1^2\,P_{\rm old}. \tag{126}$$

It follows that

$$B_t \rightarrow \left[a_1^2 + (1 - a_1^2)\left(\frac{T_1}{T}\right)^2\right]^{1/2} B_t, \tag{127}$$

$$B_p \rightarrow a_1\,B_p, \tag{128}$$

$$q_{95} \rightarrow q_{95\,\rm target}. \tag{129}$$

Let

$$n_a \rightarrow a_1\,n_a, \tag{130}$$

$$T_a \rightarrow a_1\,T_a, \tag{131}$$

so

$$\omega_{*\,a} \to \omega_{*\,a}. \tag{132}$$

Assuming that

$$\omega_{\theta\,a} \to \omega_{\theta\,a}, \tag{133}$$

$$\omega_{E} \to \omega_{E}, \tag{134}$$

we require

$$\omega_{\phi\,a} \to \omega_{\phi\,a}. \tag{135}$$