

Rescaling of Plasma Equilibrium

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1 Grad-Shafranov Equation

The Grad-Shafranov equation takes the form

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = -R^2 \frac{dP}{d\psi} - \frac{1}{2} \frac{dT^2}{d\psi}, \quad (1)$$

where

$$j_\phi = -R \frac{dP}{d\psi} - \frac{1}{2} \frac{dT^2}{d\psi}, \quad (2)$$

$$q = \frac{T}{2\pi} \oint_{\psi} \frac{dl}{R |\nabla \psi|}. \quad (3)$$

2 Type I Rescaling

The following rescaling of variables leaves the Grad-Shafranov equation invariant:

$$R_{\text{new}} = R_{\text{old}}, \quad (4)$$

$$Z_{\text{new}} = Z_{\text{old}}, \quad (5)$$

$$\psi_{\text{new}} = a_1 \psi_{\text{old}}, \quad (6)$$

$$T_{\text{new}} = a_1 T_{\text{old}}, \quad (7)$$

$$P_{\text{new}} = a_1^2 P_{\text{old}}. \quad (8)$$

It follows that

$$I_{\phi \text{ new}} = a_1 I_{\phi \text{ old}}, \quad (9)$$

$$q_{\text{new}} = q_{\text{old}}. \quad (10)$$

3 Type II Rescaling

The following rescaling of variable also leaves the Grad-Shafranov equation invariant:

$$R_{\text{new}} = R_{\text{old}}, \quad (11)$$

$$Z_{\text{new}} = Z_{\text{old}}, \quad (12)$$

$$\psi_{\text{new}} = \psi_{\text{old}}, \quad (13)$$

$$T_{\text{new}} = \text{sgn}(T_{\text{old}}) \sqrt{T_{\text{old}}^2 + a_2}, \quad (14)$$

$$P_{\text{new}} = P_{\text{old}}. \quad (15)$$

It follows that

$$I_{\phi \text{ new}} = I_{\phi \text{ old}}, \quad (16)$$

$$q_{\text{new}} = \frac{T_{\text{new}}}{T_{\text{old}}} q_{\text{old}}. \quad (17)$$

4 Program RESCALE

Let

$$\Psi = \frac{\psi - \psi_{\text{axis}}}{\psi_{\text{separatrix}} - \psi_{\text{axis}}}. \quad (18)$$

Program RESCALE first performs a type II rescaling such that

$$a_2 = \left(\frac{q_{95 \text{ target}}^2}{q_{95 \text{ old}}^2} - 1 \right) T_{95 \text{ old}}^2, \quad (19)$$

where

$$q_{95} \equiv q(\Psi = 0.95), \quad (20)$$

$$T_{95} \equiv T(\Psi = 0.95). \quad (21)$$

It follows that

$$\psi_{\text{new}} = \psi_{\text{old}}, \quad (22)$$

$$T_{\text{new}} = T_{\text{old}} \sqrt{1 + \left(\frac{q_{95 \text{ target}}^2}{q_{95 \text{ old}}^2} - 1 \right) \frac{T_{95 \text{ old}}^2}{T_{\text{old}}^2}}, \quad (23)$$

$$P_{\text{new}} = P_{\text{old}}, \quad (24)$$

and

$$I_{\phi \text{ new}} = I_{\phi \text{ old}}, \quad (25)$$

$$q_{\text{new}} = q_{\text{old}} \sqrt{1 + \left(\frac{q_{95 \text{ target}}^2}{q_{95 \text{ old}}^2} - 1 \right) \frac{T_{95 \text{ old}}^2}{T_{\text{old}}^2}}. \quad (26)$$

Program RESCALE then performs a type I rescaling such that

$$a_1 = \frac{T_{1 \text{ old}}}{T_{1 \text{ new}}}, \quad (27)$$

where

$$T_1 \equiv T(\Psi = 1). \quad (28)$$

It follows that

$$\psi_{\text{new new}} = a_1 \psi_{\text{old}}, \quad (29)$$

$$T_{\text{new new}} = a_1 T_{\text{new}} = T_{1 \text{ old}} \sqrt{\frac{T_{\text{old}}^2 + (q_{95 \text{ target}}^2 / q_{95 \text{ old}}^2 - 1) T_{95 \text{ old}}^2}{T_{1 \text{ old}}^2 + (q_{95 \text{ target}}^2 / q_{95 \text{ old}}^2 - 1) T_{95 \text{ old}}^2}}, \quad (30)$$

$$P_{\text{new new}} = a_1^2 P_{\text{old}}, \quad (31)$$

and

$$I_{\phi \text{ new new}} = a_1 I_{\phi \text{ old}}, \quad (32)$$

$$q_{\text{new new}} = q_{\text{new}} = q_{\text{old}} \sqrt{1 + \left(\frac{q_{95 \text{ target}}^2}{q_{95 \text{ old}}^2} - 1 \right) \frac{T_{95 \text{ old}}^2}{T_{\text{old}}^2}}. \quad (33)$$

Note that

$$q_{95 \text{ new new}} = q_{95 \text{ target}}, \quad (34)$$

$$T_{1 \text{ new new}} = T_{1 \text{ old}}. \quad (35)$$