Program FLUX

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September 26, 2021

1 Preliminarly Analysis

1.1 Coordinate Systems

- Cartesian coordinates: X, Y, Z.

- Cylindrical coordinates: $R \equiv (X^2 + Y^2)^{1/2}$, $\phi \equiv \tan^{-1}(Y/X)$, Z, so

$$|\nabla \phi| = \frac{1}{R}. \tag{1}$$

- Flux coordinates: $r(R, Z)$, $\theta(R, Z)$, ϕ, where

$$(\nabla r \cdot \nabla \theta \times \nabla \phi)^{-1} = \mathcal{J}, \tag{2}$$

$$\mathcal{J} = \frac{r\, R^2}{R_0}, \tag{3}$$

where R_0 is a convenient scale major radius, flux-surface label r has units of length, and θ is a poloidal angle. $\theta = 0$ on inboard midplane. $\theta > 0$ above midplane.

1.2 Useful Identities

Easily demonstrated that:

$$\mathbf{A} = A^r\, \mathcal{J}\, \nabla\theta \times \nabla\phi + A^\theta\, \mathcal{J}\, \nabla\phi \times \nabla r + A^\phi\, \mathcal{J}\, \nabla r \times \nabla\theta, \tag{4}$$

$$\mathbf{A} = A_r\, \nabla r + A_\theta\, \nabla\theta + A_\phi\, \nabla\phi, \tag{5}$$

$$\mathbf{A} \cdot \mathbf{B} = A_r\, B^r + A_\theta\, B^\theta + A_\phi\, B^\phi = A^r\, B_r + A^\theta\, B_\theta + A^\phi\, B_\phi, \tag{6}$$

$$(\mathbf{A} \times \mathbf{B})_r = \mathcal{J}\, (A^\theta\, B^\phi - A^\phi\, B^\theta), \tag{7}$$

$$(\mathbf{A} \times \mathbf{B})_\theta = \mathcal{J}\, (A^\phi\, B^r - A^r\, B^\phi), \tag{8}$$

$$(\mathbf{A} \times \mathbf{B})_\phi = \mathcal{J}\, (A^r\, B^\theta - A^\theta\, B^r), \tag{9}$$

$$\mathcal{J}\, (\mathbf{A} \times \mathbf{B})^r = A_\theta\, B_\phi - A_\phi\, B_\theta, \tag{10}$$

$$\mathcal{J}\left(\mathbf{A}\times\mathbf{B}\right)^{\theta}=A_{\phi}\,B_{r}-A_{r}\,B_{\phi}, \tag{11}$$

$$\mathcal{J}\left(\mathbf{A}\times\mathbf{B}\right)^{\phi}=A_{r}\,B_{\theta}-A_{\theta}\,B_{r}, \tag{12}$$

$$\mathcal{J}\,\nabla\cdot\mathbf{A}=\frac{\partial\left(\mathcal{J}\,A^{r}\right)}{\partial r}+\frac{\partial\left(\mathcal{J}\,A^{\theta}\right)}{\partial\theta}+\frac{\partial\left(\mathcal{J}\,A^{\phi}\right)}{\partial\phi}, \tag{13}$$

$$\mathcal{J}\left(\nabla\times\mathbf{A}\right)^{r}=\frac{\partial A_{\phi}}{\partial\theta}-\frac{\partial A_{\theta}}{\partial\phi}, \tag{14}$$

$$\mathcal{J}\left(\nabla\times\mathbf{A}\right)^{\theta}=\frac{\partial A_{r}}{\partial\phi}-\frac{\partial A_{\phi}}{\partial r}, \tag{15}$$

$$\mathcal{J}\left(\nabla\times\mathbf{A}\right)^{\phi}=\frac{\partial A_{\theta}}{\partial r}-\frac{\partial A_{r}}{\partial\theta}. \tag{16}$$

1.3   Equilibrium Magnetic Field

Equilibrium magnetic field:

$$\mathbf{B}=B_{0}\,R_{0}\left[f(r)\,\nabla\phi\times\nabla r+g(r)\,\nabla\phi\right], \tag{17}$$

$$q(r)=\frac{r\,g}{R_{0}\,f}, \tag{18}$$

where B_{0} is a convenient scale magnetic field strength, and g and f are dimensionless. So,

$$B^{r}=0, \tag{19}$$

$$B^{\theta}=B_{0}\,R_{0}^{2}\,\frac{f}{r\,R^{2}}, \tag{20}$$

$$B^{\phi}=B_{0}\,R_{0}^{2}\,\frac{q\,f}{r\,R^{2}}=B_{0}\,B_{0}\,\frac{g}{R^{2}}, \tag{21}$$

$$B_{r}=-B_{0}\,r\,f\,\nabla r\cdot\nabla\theta, \tag{22}$$

$$B_{\theta}=B_{0}\,r\,f\,|\nabla r|^{2}, \tag{23}$$

$$B_{\phi}=B_{0}\,R_{0}^{2}\,\frac{q\,f}{r}=B_{0}\,R_{0}\,g, \tag{24}$$

and

$$\mathbf{B}\cdot\nabla=B_{0}\,R_{0}^{2}\,\frac{f}{r\,R^{2}}\left(\frac{\partial}{\partial\theta}+q\,\frac{\partial}{\partial\phi}\right). \tag{25}$$

1.4   Equilibrium Plasma Current

The relation $\mu_{0}\,\mathbf{J}=\nabla\times\mathbf{B}$ yields

$$\mu_{0}\,\mathcal{J}\,J^{r}=0, \tag{26}$$

$$\mu_{0}\,\mathcal{J}\,J^{\theta}=-B_{0}\,R_{0}\,\frac{dg}{dr}, \tag{27}$$

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$$\mu_0 \, \mathcal{J} \, J^\phi = B_0 \, \frac{\partial}{\partial r} \left(r \, f \, |\nabla r|^2 \right) + B_0 \, \frac{\partial}{\partial \theta} \left(r \, f \, \nabla r \cdot \nabla \theta \right). \tag{28}$$

1.5 Grad-Shafranov Equation

The equilibrium force balance relation

$$\mathbf{J} \times \mathbf{B} = \nabla P \tag{29}$$

yields

$$\mathcal{J} \left(J^\theta \, B^\phi - J^\phi \, B^\theta \right) = \frac{dP}{dr}, \tag{30}$$

which reduces to the Grad-Shafranov equation,

$$\frac{f}{r} \, \frac{\partial}{\partial r} \left(r \, f \, |\nabla r|^2 \right) + \frac{f}{r} \, \frac{\partial}{\partial \theta} (r \, f \, \nabla r \cdot \nabla \theta) + g \, \frac{dg}{dr} + \frac{\mu_0}{B_0^2} \left(\frac{R}{R_0} \right)^2 \frac{dP}{dr} = 0. \tag{31}$$

1.6 Inhomogeneous Tearing Mode Dispersion Relation

1.7 High-q Limit

Now,

$$\nabla \cdot \delta \mathbf{B} = 0 \tag{32}$$

yields

$$\frac{\partial (\mathcal{J} \, \delta B^r)}{\partial r} + \frac{\partial (\mathcal{J} \, \delta B^\theta)}{\partial \theta} + \frac{\partial (\mathcal{J} \, \delta B^\phi)}{\partial \phi} = 0. \tag{33}$$

Furthermore,

$$\delta \mathbf{B} = \delta B_r \, \nabla r + \delta B_\theta \, \nabla \theta + \delta B_\phi \, \nabla \phi. \tag{34}$$

So,

$$\delta B^r = \delta \mathbf{B} \cdot \nabla r = |\nabla r|^2 \, \delta B_r + (\nabla r \cdot \nabla \theta) \, \delta B_\theta, \tag{35}$$

$$\delta B^\theta = \delta \mathbf{B} \cdot \nabla \theta = (\nabla r \cdot \nabla \theta) \, \delta B_r + |\nabla \theta|^2 \, \delta B_\theta, \tag{36}$$

$$\delta B^\phi = \delta \mathbf{B} \cdot \nabla \phi = \frac{\delta B_\phi}{R^2}. \tag{37}$$

Follows that

$$\delta B_r = \left(\frac{1}{|\nabla r|^2} \right) \delta B^r - \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B_\theta, \tag{38}$$

and

$$\delta B^\theta = \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B^r + \left[|\nabla \theta|^2 - \frac{(\nabla r \cdot \nabla \theta)^2}{|\nabla r|^2} \right] \delta B_\theta. \tag{39}$$

But, Eqs. (2) and (3) imply that

$$|\nabla r|^2 \, |\nabla \theta|^2 - (\nabla r \cdot \nabla \theta)^2 = \frac{R_0^2}{r^2 \, R^2}. \tag{40}$$

Hence,

$$\delta B^\theta = \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B^r + \left(\frac{R_0^2}{r^2 \, R^2 \, |\nabla r|^2} \right) \delta B_\theta. \tag{41}$$

Assume that high-q limit is equivalent to $\delta \mathbf{B}$ being curl-free. Follows that

$$\frac{\partial \, \delta B_\phi}{\partial \theta} = \frac{\partial \, \delta B_\theta}{\partial \phi}, \tag{42}$$

$$\frac{\partial \, \delta B_r}{\partial \phi} = \frac{\partial \, \delta B_\phi}{\partial r}, \tag{43}$$

$$\frac{\partial \, \delta B_\theta}{\partial r} = \frac{\partial \, \delta B_r}{\partial \theta}. \tag{44}$$

Previous three equations imply that

$$\delta B_\phi \sim \frac{n}{m} \, \delta B_\theta, \; \frac{n}{m} \, r \, \delta B_r \tag{45}$$

and, hence, that

$$\delta B^\phi \sim \frac{n}{m} \left(\frac{r}{R} \right)^2 \delta B^\theta, \; \frac{n}{m} \left(\frac{r}{R} \right)^2 \frac{\delta B^r}{r}, \tag{46}$$

Consequently, final term on right-hand side of (33) is of order $(n/m)^2 \, (r/R)^2$ smaller than other two terms, and, therefore, negligible. Thus, we get

$$\frac{\partial (\mathcal{J} \, \delta B^r)}{\partial r} + \frac{\partial (\mathcal{J} \, \delta B^\theta)}{\partial \theta} = 0. \tag{47}$$

Follows from (41) and previous equation that

$$r \, \frac{\partial}{\partial r} \left(\frac{r \, R^2 \, \delta B^r}{R_0^2} \right) = - \frac{\partial}{\partial \theta} \left[\left(\frac{r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \left(\frac{r \, R^2 \, \delta B^r}{R_0^2} \right) + \left(\frac{1}{|\nabla r|^2} \right) \delta B_\theta \right]. \tag{48}$$

Follows from (3), (38) and (44) that

$$r \, \frac{\partial \, \delta B_\theta}{\partial r} = \frac{\partial}{\partial \theta} \left[\left(\frac{R_0^2}{R^2 \, |\nabla r|^2} \right) \left(\frac{r \, R^2 \, \delta B^r}{R_0^2} \right) - \left(\frac{r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \delta B_\theta \right]. \tag{49}$$

Let

$$\frac{r \, R^2 \, \delta B^r(r, \theta, \phi)}{R_0^2} = \sum_j \delta \hat{B}_j^r(r) \, \mathrm{e}^{\mathrm{i} \, (m_j \, \theta - n\phi)}, \tag{50}$$

$$\delta B_\theta(r, \theta, \phi) = \sum_j \delta \hat{B}_{\theta \, j}(r) \, \mathrm{e}^{\mathrm{i} \, (m_j \, \theta - n \, \phi)}. \tag{51}$$

Follows that

$$\sum_{j'} r \, \frac{d \, \delta \hat{B}_{j'}^r}{dr} \, \mathrm{e}^{\mathrm{i} \, m_{j'} \, \theta} = - \frac{\partial}{\partial \theta} \sum_{j'} \left[\left(\frac{r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) \mathrm{e}^{\mathrm{i} \, m_{j'} \, \theta} \, \delta \hat{B}_{j'}^r + \left(\frac{1}{|\nabla r|^2} \right) \mathrm{e}^{\mathrm{i} \, m_{j'} \, \theta} \, \delta \hat{B}_{\theta \, j'} \right], \tag{52}$$

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$$\sum_{j'} r \, \frac{d \, \delta \hat{B}_{\theta \, j'}}{dr} \, e^{\,i \, m_{j'} \, \theta} = \frac{\partial}{\partial \theta} \sum_{j'} \left[\left(\frac{R_0^2}{R^2 \, |\nabla r|^2} \right) e^{\,i \, m_{j'} \, \theta} \, \delta \hat{B}_{j'}^r - \left(\frac{r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) e^{\,i \, m_{j'} \, \theta} \, \delta \hat{B}_{\theta \, j'} \right]. \tag{53}$$

Operating with $\oint (\cdots) \, e^{-i \, m_j \, \theta} \, d\theta/(2\pi)$, we get

$$r \, \frac{d \, \delta \hat{B}_j^r}{dr} = -i \, m_j \sum_{j'} \left(-i \, c_{jj'} \, \delta \hat{B}_{j'}^r + a_{jj'} \, \delta \hat{B}_{\theta \, j'} \right), \tag{54}$$

$$r \, \frac{d \, \delta \hat{B}_{\theta \, j}}{dr} = -i \, m_j \sum_{j'} \left(-i \, c_{jj'} \, \delta \hat{B}_{\theta \, j'} - b_{jj'} \, \delta \hat{B}_{j'}^r \right), \tag{55}$$

where

$$a_{jj'} = \oint \frac{1}{|\nabla r|^2} \, e^{-i \, (m_j - m_{j'}) \, \theta} \, \frac{d\theta}{2\pi}, \tag{56}$$

$$b_{jj'} = \oint \frac{R_0^2}{R^2 \, |\nabla r|^2} \, e^{-i \, (m_j - m_{j'}) \, \theta} \, \frac{d\theta}{2\pi}, \tag{57}$$

$$c_{jj'} = \oint \frac{i \, r \, \nabla r \cdot \nabla \theta}{|\nabla r|^2} \, e^{-i \, (m_j - m_{j'}) \, \theta} \, \frac{d\theta}{2\pi}. \tag{58}$$

Let

$$\delta \hat{B}_j^r(r) = i \, \psi_j(r), \tag{59}$$

$$\delta \hat{B}_{\theta \, j}(r) = -\chi_j(r). \tag{60}$$

It follows that

$$r \, \frac{d\psi_j}{dr} = m_j \sum_{j'} \left(-c_{jj'} \, \psi_{j'} + a_{jj'} \, \chi_{j'} \right), \tag{61}$$

$$r \, \frac{d\chi_j}{dr} = m_j \sum_{j'} \left(-c_{jj'} \, \chi_{j'} + b_{jj'} \, \psi_{j'} \right). \tag{62}$$

Hence,

$$\frac{r \, R^2 \, \delta B^r(r, \theta, \phi)}{R_0^2} = i \sum_j \psi_j(r) \, e^{\,i \, (m_j \, \theta - n \phi)}, \tag{63}$$

$$\frac{r \, R^2 \, \delta B^\theta(r, \theta, \phi)}{R_0^2} = -\sum_j \frac{1}{m_j} \, \frac{d\psi_j}{dr} \, e^{\,i \, (m_j \, \theta - n \, \phi)}, \tag{64}$$

$$R^2 \, \delta B^\phi(r, \theta, \phi) = n \sum_j \frac{\chi_j(r)}{m_j} \, e^{\,i \, (m_j \, \theta - n \phi)}. \tag{65}$$

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where use has been made of (42). Also have

$$\delta B_r(r, \theta, \phi) = \mathrm{i} \sum_j \frac{1}{m_j} \frac{d\chi_j}{dr} \, \mathrm{e}^{\mathrm{i}(m_j\,\theta - n\,\phi)}, \tag{66}$$

$$\delta B_\theta(r, \theta, \phi) = -\sum_j \chi_j(r) \, \mathrm{e}^{\mathrm{i}(m_j\,\theta - n\,\phi)}, \tag{67}$$

$$\delta B_\phi(r, \theta, \phi) = n \sum_j \frac{\chi_j(r)}{m_j} \, \mathrm{e}^{\mathrm{i}(m_j\,\theta - n\,\phi)}, \tag{68}$$

We can write

$$\mathcal{J}\,\mu_0\,\delta J^r = \frac{\partial\,\delta B_\phi}{\partial\theta} - \frac{\partial\,\delta B_\theta}{\partial\phi}, \tag{69}$$

$$\mathcal{J}\,\mu_0\,\delta J^\theta = \frac{\partial\,\delta B_r}{\partial\phi} - \frac{\partial\,\delta B_\phi}{\partial r}, \tag{70}$$

$$\mathcal{J}\,\mu_0\,\delta J^\phi = \frac{\partial\,\delta B_\theta}{\partial r} - \frac{\partial\,\delta B_r}{\partial\theta}. \tag{71}$$

Normally, all three components are zero. However, at kth resonant surface (at which $r = r_k$, where $q(r_k) = m_k/n$) ψ_k, $\psi_{j\neq k}$, and $\chi_{j\neq k}$ are continuous, whereas χ_k is discontinuous. Hence,

$$\mathcal{J}\,\mu_0\,\delta J^r(r, \theta, \phi) = 0, \tag{72}$$

$$\mathcal{J}\,\mu_0\,\delta J^\theta(r, \theta, \phi) = -\sum_k \frac{n}{m_k}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\delta(r - r_k)\,\mathrm{e}^{\mathrm{i}(m_k\,\theta - n\,\phi)}, \tag{73}$$

$$\mathcal{J}\,\mu_0\,\delta J^\phi(r, \theta, \phi) = -\sum_k [\chi_k]_{r_{k-}}^{r_{k+}}\,\delta(r - r_k)\,\mathrm{e}^{\mathrm{i}(m_k\,\theta - n\,\phi)}. \tag{74}$$

Now,

$$(\mu_0\,\delta\mathbf{J} \times \delta\mathbf{B})_\theta = \frac{1}{4}\left(\mathcal{J}\,\mu_0\,\delta J^\phi\,\delta B^{r\,*} + \text{c.c.}\right), \tag{75}$$

$$(\mu_0\,\delta\mathbf{J} \times \delta\mathbf{B})_\phi = \frac{1}{4}\left(-\mathcal{J}\,\mu_0\,\delta J^\theta\,\delta B^{r\,*} + \text{c.c.}\right), \tag{76}$$

which implies that

$$\mathcal{J}\,(\mu_0\,\delta\mathbf{J} \times \delta\mathbf{B})_\theta = \frac{R_0}{2}\sum_j\sum_k \mathrm{Re}\left\{\mathrm{i}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\psi_j^*(r_k)\,\mathrm{e}^{\mathrm{i}(m_k - m_j)\,\theta}\right\}\delta(r - r_k), \tag{77}$$

$$\mathcal{J}\,(\mu_0\,\delta\mathbf{J} \times \delta\mathbf{B})_\phi = -\frac{R_0}{2}\sum_j\sum_k \frac{n}{m_k}\,\mathrm{Re}\left\{\mathrm{i}\,[\chi_k]_{r_{k-}}^{r_{k+}}\,\psi_j^*(r_k)\,\mathrm{e}^{\mathrm{i}(m_k - m_j)\,\theta}\right\}\delta(r - r_k). \tag{78}$$

Let

$$\Psi_k = \frac{\psi_k(r_k)}{m_k}, \tag{79}$$

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$$\Delta\Psi_k = [\chi_k]_{r_{k-}}^{r_{k+}}, \tag{80}$$

$$\delta T_{\theta\,k} \equiv \delta\mathbf{T} \cdot \mathbf{e}_\theta = \int_{r_{k-}}^{r_{k+}} \oint \oint (\delta\mathbf{J} \times \delta\mathbf{B})_\theta \, \mathcal{J} \, dr \, d\theta \, d\phi, \tag{81}$$

$$\delta T_{\phi\,k} \equiv \delta\mathbf{T} \cdot \mathbf{e}_\phi = \int_{r_{k-}}^{r_{k+}} \oint \oint (\delta\mathbf{J} \times \delta\mathbf{B})_\phi \, \mathcal{J} \, dr \, d\theta \, d\phi, \tag{82}$$

where $\delta\mathbf{T}(r) \, dr$ is the net electromagnetic torque acting on the plasma between r and $r+dr$. Here, $\mathbf{e}_\theta = (R^2/R_0) \, \nabla\phi \times \nabla r$ and $\mathbf{e}_\phi = R \, \nabla\phi$. Follows that

$$\delta T_{\theta\,k} = -\frac{2\pi^2 \, R_0 \, m_k}{\mu_0} \, \text{Im}\left(\Psi_k^* \, \Delta\Psi_k\right), \tag{83}$$

$$\delta T_{\phi\,k} = \frac{2\pi^2 \, R_0 \, n}{\mu_0} \, \text{Im}\left(\Psi_k^* \, \Delta\Psi_k\right). \tag{84}$$

1.8   Homogeneous Solution

We can write

$$\delta\mathbf{B} = \nabla \times \delta\mathbf{A}. \tag{85}$$

Suppose that $r \, \delta A_r$, δA_θ are negligible with respect to δA_ϕ. In fact, it is easily demonstrated from $\nabla \cdot \delta\mathbf{A} = 0$ that $r \, \delta A_r$, $\delta A_\theta \sim (n/m) \, (r/R)^2 \, \delta A_\phi$. It follows that

$$\mathcal{J} \, \delta B^r \simeq \frac{\partial \, \delta A_\phi}{\partial\theta}, \tag{86}$$

$$\mathcal{J} \, \delta B^\theta \simeq -\frac{\partial \, \delta A_\phi}{\partial r}, \tag{87}$$

$$\mathcal{J} \, \delta B^\phi \simeq 0. \tag{88}$$

The neglected terms in the previous three equations are $(n/m)^2 \, (r/R)^2$ smaller than the dominant terms. The previous three expressions are consistent with (63), (64), and (65) provided

$$\delta A_\phi(r, \theta, \phi) \simeq R_0 \sum_j \frac{\psi_j(r)}{m_j} \, e^{i\,(m_j\,\theta - n\,\phi)}. \tag{89}$$

According to the Biot-Savart law:

$$\delta A_\phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{R \, R' \, \mu_0 \, \delta\mathbf{J}(\mathbf{x}') \cdot \nabla\phi}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}'. \tag{90}$$

Assume that

$$\delta A_\phi(R, \phi, Z) = \delta A_\phi(R, 0, Z) \, e^{-i\,n\,\phi}. \tag{91}$$

Hence, can evaluate integral at $\phi = 0$ without loss of generality. Follows that

$$\mathbf{x} = (R, \, 0, \, Z), \tag{92}$$

$$\mathbf{x}' = (R' \cos \phi', \; R' \sin \phi', \; Z'), \tag{93}$$

and

$$|\mathbf{x} - \mathbf{x}'| = \left[R^2 + R'^2 + (Z - Z')^2 - 2 \, R \, R' \cos \phi' \right]^{1/2}. \tag{94}$$

Now,

$$\delta \mathbf{J}(R', \phi', Z') \cdot \nabla \phi = \delta J^\phi(R', 0, Z') \, \mathrm{e}^{-\mathrm{i} \, n \, \phi'} \cos \phi', \tag{95}$$

so

$$\delta A_\phi(R, 0, Z) = \frac{1}{4\pi} \int_0^\infty \oint \oint \frac{R \, R' \, \mu_0 \, \delta J^\phi(R', 0, Z') \, \mathrm{e}^{-\mathrm{i} \, n \, \phi'} \cos \phi' \, \mathcal{J}' \, dr' \, d\theta' \, d\phi'}{\left[R^2 + R'^2 + (Z - Z')^2 - 2 \, R \, R' \cos \phi' \right]^{1/2}}, \tag{96}$$

which can be written

$$\delta A_\phi(R, 0, Z) = \frac{1}{4\pi} \int_0^\infty \oint R \, R' \, \mu_0 \, \delta J^\phi(R', 0, Z') \, G(R, Z; R', Z') \, \mathcal{J}' \, dr' \, d\theta', \tag{97}$$

where

$$G(R, Z; R', Z') = \oint \frac{\cos \phi' \, \cos(n \, \phi') \, d\phi'}{\left[R^2 + R'^2 + (Z - Z')^2 - 2 \, R \, R' \cos \phi' \right]^{1/2}}, \tag{98}$$

or

$$G(R, Z; R', Z') = \frac{1}{2} \oint \frac{(\cos[(n-1)\,\phi'] + \cos[(n+1)\,\phi']) \, d\phi'}{\left[R^2 + R'^2 + (Z - Z')^2 - 2 \, R \, R' \cos \phi' \right]^{1/2}}. \tag{99}$$

Now,

$$
\begin{aligned}
P_{-1/2}^n(\cosh \eta) &= \frac{(-1)^n}{2\pi} \frac{\Gamma(1/2)}{\Gamma(1/2 - n)} \oint \frac{\cos(n \, \varphi) \, d\varphi}{(\cosh \eta + \sinh \eta \, \cos \varphi)^{1/2}} \\
&= \frac{1}{2\pi} \frac{\Gamma(1/2)}{\Gamma(1/2 - n)} \oint \frac{\cos(n \, \varphi) \, d\varphi}{(\cosh \eta - \sinh \eta \, \cos \varphi)^{1/2}} \\
&= \frac{(-1)^n \, \Gamma(1/2) \, \Gamma(1/2 + n)}{2\pi^2} \oint \frac{\cos(n \, \varphi) \, d\varphi}{(\cosh \eta - \sinh \eta \, \cos \varphi)^{1/2}}.
\end{aligned} \tag{100}
$$

Let

$$\alpha \, \cosh \eta = R^2 + R'^2 + (Z - Z')^2, \tag{101}$$

$$\alpha \, \sinh \eta = 2 \, R \, R'. \tag{102}$$

Hence,

$$\eta = \tanh^{-1} \left[\frac{2 \, R \, R'}{R^2 + R'^2 + (Z - Z')^2} \right], \tag{103}$$

and

$$\alpha = \frac{R^2 + R'^2 + (Z - Z')^2}{\cosh \eta}. \tag{104}$$

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It follows that

$$G(R, Z; R', Z') = \pi \left[\frac{\cosh \eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2}$$

$$\times \left[\frac{\Gamma(3/2 - n)}{\Gamma(1/2)} P_{-1/2}^{n-1}(\cosh \eta) + \frac{\Gamma(-1/2 - n)}{\Gamma(1/2)} P_{-1/2}^{n+1}(\cosh \eta) \right]. \tag{105}$$

However,

$$\Gamma(3/2 - n) = \frac{(-1)^{n+1} \pi (n - 1/2)}{\Gamma(n + 1/2)}, \tag{106}$$

$$\Gamma(-1/2 - n) = \frac{(-1)^{n+1} \pi}{\Gamma(n + 1/2)(n + 1/2)}, \tag{107}$$

so

$$G(R, Z; R', Z') = \frac{(-1)^{n+1} \pi^2}{\Gamma(1/2)\,\Gamma(n + 1/2)} \left[\frac{\cosh \eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2}$$

$$\times \left[(n - 1/2)\, P_{-1/2}^{n-1}(\cosh \eta) + \frac{P_{-1/2}^{n+1}(\cosh \eta)}{n + 1/2} \right]. \tag{108}$$

Now, from (74) and (80),

$$\mathcal{J}\,\mu_0\,\delta J^\phi(r, \theta, 0) = -\sum_k \Delta \Psi_k\, \delta(r - r_k)\, e^{i\,m_k\,\theta}. \tag{109}$$

Furthermore, from (79) and (89),

$$\Psi_k = \frac{1}{R_0} \oint \delta A_\phi(r_k, \theta, 0)\, e^{-i\,m_k\,\theta}\, \frac{d\theta}{2\pi}. \tag{110}$$

Hence, we obtain the homogeneous tearing mode dispersion relation,

$$\Psi_k = \sum_{k'} F_{kk'}\, \Delta \Psi_{k'}, \tag{111}$$

where

$$F_{kk'} = \oint \oint \mathcal{G}(R_k, Z_k; R'_k, Z'_k)\, e^{-i\,(m_k\,\theta - m_{k'}\,\theta')}\, \frac{d\theta}{2\pi}\, \frac{d\theta'}{2\pi}, \tag{112}$$

and

$$\mathcal{G}(R, Z; R', Z') = \frac{(-1)^n \pi^2 R\,R'/R_0}{2\,\Gamma(1/2)\,\Gamma(n + 1/2)} \left[\frac{\cosh \eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2}$$

$$\times \left[(n - 1/2)\, P_{-1/2}^{n-1}(\cosh \eta) + \frac{P_{-1/2}^{n+1}(\cosh \eta)}{n + 1/2} \right]. \tag{113}$$

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Here, R_k, Z_k are the R. Z coordinates of the kth resonant surface in the plane $\phi = 0$. We can also write

$$F_{kk'} = \oint \oint \tilde{\mathcal{G}}(R_k, Z_k; R_k', Z_k') \, e^{-i(m_k \theta - m_{k'} \theta')} \frac{d\Theta}{2\pi} \frac{d\Theta'}{2\pi}, \tag{114}$$

where

$$\tilde{\mathcal{G}}(R, Z; R', Z') = \frac{g \, g'}{q \, q' \, \gamma \, \gamma' \, B \, B' \, R^2 \, R'^2} \frac{(-1)^n \, \pi^2 \, R \, R'/R_0}{2 \, \Gamma(1/2) \, \Gamma(n+1/2)} \left[\frac{\cosh \eta}{R^2 + R'^2 + (Z - Z')^2} \right]^{1/2}$$
$$\times \left[(n - 1/2) \, P_{-1/2}^{n-1}(\cosh \eta) + \frac{P_{-1/2}^{n+1}(\cosh \eta)}{n + 1/2} \right]. \tag{115}$$

Note that

$$\mathcal{G}(R', Z'; R, Z) = \mathcal{G}(R, Z; R', Z'), \tag{116}$$

which implies that

$$F_{k'k} = F_{kk'}^*. \tag{117}$$

1.9 Inhomogeneous Solution

Suppose that there are currents flowing in a number of external poloidal field coils. Let I_l, R_l, and Z_l be the peak current, and coordinates of the lth field coil. The currents are assumed to modulate like $e^{-in\phi}$. It follows that

$$\delta J_{\text{ext}}^{\phi}(R, 0, Z) = \sum_j \frac{I_l}{R_l} \, \delta(R - R_l) \, \delta(Z - Z_l). \tag{118}$$

Hence,

$$\Psi_k = \sum_{k'} F_{kk'} \, \Delta\Psi_k - \sum_l g_{kl} \, \mu_0 \, I_l, \tag{119}$$

where

$$g_{kj} = \frac{1}{2\pi} \oint \mathcal{G}(r_k, \theta; R_j, Z_j) \, e^{-i m_k \theta} \frac{d\theta}{2\pi}. \tag{120}$$

Thus, we obtain the inhomogeneous tearing mode dispersion relation,

$$\Delta\Psi_k = \sum_{k'} E_{kk'} \, \Psi_{k'} + |E_{kk}| \, \chi_k, \tag{121}$$

where $E_{kk'}$ is the inverse of the $F_{kk'}$ matrix, and

$$\chi_k = \frac{1}{|E_{kk}|} \sum_j h_{kl} \, \mu_0 \, I_l, \tag{122}$$

$$h_{kl} = \sum_{k'} E_{kk'} \, g_{k'l}. \tag{123}$$

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1.10 Electromagnetic Torques in Presence of External Currents

Suppose that

$$\Psi_k = B_0 \, R_0 \, \hat{\Psi}_k \, \mathrm{e}^{-\mathrm{i}\,\varphi_k}, \tag{124}$$

$$\chi_k = B_0 \, R_0 \, \hat{\chi}_k \, \mathrm{e}^{-\mathrm{i}\,\zeta_k}, \tag{125}$$

$$E_{kk'} = \hat{E}_{kk'} \, \mathrm{e}^{-\mathrm{i}\,\xi_{kk'}}, \tag{126}$$

where $\hat{\Psi}_k$, $\hat{\chi}_k$, and $\hat{E}_{kk'}$ are real and positive, whereas φ_k, ζ_k, and $\xi_{kk'}$ are real. (Note that all hatted quantities in this report are dimensionless.) It follows from Eqs. (83), (84), and (121) that

$$\delta T_{\theta\,k} = -\left(\frac{2\pi^2 \, B_0^2 \, R_0^3}{\mu_0}\right) m_k \, \delta\hat{T}_k, \tag{127}$$

$$\delta T_{\phi\,k} = \left(\frac{2\pi^2 \, B_0^2 \, R_0^3}{\mu_0}\right) n \, \delta\hat{T}_k, \tag{128}$$

where

$$\delta\hat{T}_k = \sum_{k'=1,K} \hat{E}_{kk'} \, \hat{\Psi}_k \, \hat{\Psi}_{k'} \, \sin(\varphi_k - \varphi_{k'} - \xi_{kk'}) + \hat{E}_{kk} \, \hat{\Psi}_k \, \hat{\chi}_k \, \sin(\varphi_k - \zeta_k). \tag{129}$$

1.11 GPEC Coupling

1.11.1 PRL Derivation

We have

$$\frac{d\psi_p}{dr} = B_0 \, R_0 \, f(r), \tag{130}$$

$$\mathbf{B} \cdot \nabla\phi = \frac{B_0 \, R_0}{R^2} \, g(r), \tag{131}$$

$$\delta B^r = \frac{\mathrm{i}}{r} \left(\frac{R_0}{R}\right)^2 \sum_j \psi_j \, \mathrm{e}^{\mathrm{i}\,(m_j\,\theta - n\,\phi)}. \tag{132}$$

According to PRL **99**, 195003 (2007),

$$\Delta_j \, \mathrm{e}^{\mathrm{i}\,(m_j\,\theta - n\,\phi)} = \left[\frac{\partial}{\partial\psi_p} \frac{\delta\mathbf{B}\cdot\nabla\psi_p}{\mathbf{B}\cdot\nabla\phi}\right]_{r_j} = \left[\frac{\partial}{\partial r} \frac{\delta\mathbf{B}\cdot\nabla r}{\mathbf{B}\cdot\nabla\phi}\right]_{r_j} = \left[\frac{\partial}{\partial r} \frac{\delta B^r}{\mathbf{B}\cdot\nabla\phi}\right]_{r_j}. \tag{133}$$

It follows that

$$\Delta_j = \frac{\mathrm{i}}{r_j} \left(\frac{R_0}{R}\right)^2 \frac{R^2}{B_0 \, R_0 \, g_j} \left[\frac{d\psi_j}{dr}\right]_{r_j}, \tag{134}$$

But,

$$\left[r \, \frac{d\psi_j}{dr}\right]_{r_j} = m_j \, a_{jj} \, [\chi_j]_{r_j} = m_j \, a_{jj} \, \Delta\Psi_j, \tag{135}$$

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and

$$\chi_j = \frac{\Delta\Psi_j}{|E_{jj}|}, \tag{136}$$

which implies that

$$\Delta_j = i \left(\frac{R_0}{r_j}\right)^2 \frac{m_j \, a_{jj}}{g_j} \frac{\Delta\Psi_j}{B_0 \, R_0}, \tag{137}$$

or

$$\frac{\chi_j}{R_0 \, B_0} = -i \left(\frac{r_j}{R_0}\right)^2 \frac{g_j}{m_j \, a_{jj}} \frac{\Delta_j}{|E_{jj}|}. \tag{138}$$

Here, the Δ_j values can be determined from the GPEC code.

1.11.2 PoP Derivation

Now, $d\psi_p/dr = B_0 \, R_0 \, f$. It follows that

$$(\nabla\psi_p \times \nabla\theta \cdot \nabla\phi)^{-1} = \frac{R^2 \, q}{B_0 \, R_0 \, g}. \tag{139}$$

Hence,

$$\mathbf{B} = \nabla\phi \times \nabla\psi_p + B_0 \, R_0 \, g \, \nabla\phi = \nabla\phi \times \nabla\psi_p + q \, \nabla\psi_p \times \nabla\theta. \tag{140}$$

Let $\Psi_p = 2\pi \, \psi_p$ and $d\Psi_p/d\Psi_t = 1/q$. Hence,

$$\frac{d\Psi_p}{dr} = 2\pi \, B_0 \, R_0 \, f, \tag{141}$$

$$\frac{d\Psi_t}{dr} = 2\pi \, B_0 \, r \, g, \tag{142}$$

and [cf. PoP **13**, 102501 (2006), Eq. (41)]

$$2\pi \, \mathbf{B} = q^{-1} \, \nabla\phi \times \nabla\Psi_t + \nabla\Psi_t \times \nabla\theta. \tag{143}$$

We have

$$\delta J^r = 0, \tag{144}$$

$$\mathcal{J} \, \mu_0 \, \delta J^\theta = -\sum_j \frac{\Delta\Psi_j}{q_j} \, e^{i \, (m_j \, \theta - n \, \phi)} \, \delta(r - r_j), \tag{145}$$

$$\mathcal{J} \, \mu_0 \, \delta J^\phi = -\sum_j \Delta\Psi_k \, e^{i \, (m_j \, \theta - n \, \phi)} \, \delta(r - r_j), \tag{146}$$

which implies that

$$\mu_0 \, \delta\mathbf{J} = -2\pi \sum_j \Delta\Psi_j, \, e^{i \, (m_j \, \theta - n \, \phi)} \, \delta(\psi_t - \psi_{t \, j}) \, \mathbf{B}. \tag{147}$$

However, according to PoP **13**, 102501 (2006),

$$\Delta\Psi_j = -\mathrm{i}\,\frac{\mu_0\,J_c\,\Delta_j}{2\pi\,m_j}, \tag{148}$$

where

$$\frac{1}{\mu_0\,J_c} = \left(\oint \frac{B^2}{|\nabla\psi_t|^2}\,\frac{d\theta\,d\phi}{2\pi\,\mathbf{B}\cdot\nabla\phi}\right)_{r_j}. \tag{149}$$

It is easily demonstrated that

$$\frac{1}{\mu_0\,J_c} = \left(\frac{R_0}{r_j}\right)^2 \frac{1}{2\pi\,B_0\,R_0\,g_j}\left[a_{jj} + \left(\frac{r_k}{R_0\,q_j}\right)^2\right]. \tag{150}$$

Hence,

$$\frac{\Delta\Psi_j}{B_0\,R_0} = -\mathrm{i}\,\Delta_j\left(\frac{r_j}{R_0}\right)^2 \frac{g_j}{m_j\,[a_{jj} + (r_k/R_0\,q_j)^2]}, \tag{151}$$

which implies that

$$\frac{\chi_j}{B_0\,R_0} = -\mathrm{i}\,\frac{\Delta_j}{|E_{jj}|}\left(\frac{r_j}{R_0}\right)^2 \frac{g_j}{m_j\,[a_{jj} + (r_k/R_0\,q_j)^2]}. \tag{152}$$

(Note: This is what is actually implemated in EPEC.) As before, the Δ_j values can be determined from the GPEC code.

1.12 Magnetic Island Width

1.12.1 Island Width in r

We have

$$\mathcal{J}\,\delta B^r \simeq \frac{\partial\,\delta A_\phi}{\partial\theta}, \tag{153}$$

$$\mathcal{J}\,\delta B^\theta \simeq -\frac{\partial\,\delta A_\phi}{\partial r}, \tag{154}$$

$$\mathcal{J}\,\delta B^\phi \simeq 0. \tag{155}$$

It follows that

$$\delta\mathbf{B}\cdot\nabla\delta A_\phi = \delta B^r\,\frac{\partial\,\delta A_\phi}{\partial r} + \delta B^\theta\,\frac{\partial\,\delta A_\phi}{\partial\theta} + \delta B^\phi\,\frac{\partial\,\delta A_\phi}{\partial\phi} \simeq 0. \tag{156}$$

We have

$$\delta A_\phi(r,\theta,\phi) \simeq R_0\,\Psi_k\,\mathrm{e}^{\mathrm{i}\,(m_k\,\theta - n\,\phi)} \tag{157}$$

in the vicinity of the kth resonant surface. It follows that

$$\mathbf{B} \cdot \nabla \delta A_\phi = \mathrm{i}\, B_0\, R_0^2\, \frac{f}{r\, R^2}\, (m_k - n\, q)\, R_0\, \Psi_k\, \mathrm{e}^{\mathrm{i}\,(m_k\,\theta - n\,\phi)}. \tag{158}$$

Let us search for a function,

$$F(r, \theta, \phi) = F_0(r) + \delta A_\phi, \tag{159}$$

which is such that

$$(\mathbf{B} + \delta \mathbf{B}) \cdot \nabla F = 0. \tag{160}$$

It follows that

$$\delta B^r \frac{dF_0}{dr} + \mathrm{i}\, B_0\, R_0^2\, \frac{f}{r\, R^2}\, (m_k - n\, q)\, R_0\, \Psi_k\, \mathrm{e}^{\mathrm{i}\,(m_k\,\theta - n\,\phi)} = 0. \tag{161}$$

However,

$$\delta B^r = \frac{R_0^2}{r\, R^2}\, \mathrm{i}\, m_k\, \Psi_k\, \mathrm{e}^{\mathrm{i}\,(m_k\,\theta - n\,\phi)}, \tag{162}$$

so

$$\mathrm{i}\, m_k\, \frac{R_0}{r\, R^2}\, R_0\, \Psi_k\, \frac{dF_0}{dr} + \mathrm{i}\, B_0\, R_0^2\, \frac{f}{r\, R^2}\, (m_k - n\, q)\, R_0\, \Psi_k = 0, \tag{163}$$

which implies that

$$\frac{dF_0}{dr} = -\frac{B_0\, R_0\, f}{m_k}\, (m_k - n\, q), \tag{164}$$

or

$$F_0(r) \simeq \frac{B_0}{2}\left(\frac{g\, s}{q}\right)_{r_k}\, (r - r_k)^2, \tag{165}$$

where $s = r\, q'/q$. Hence,

$$F(r, \theta, \phi) = \frac{B_0}{2}\left(\frac{g\, s}{q}\right)_{r_k}\, (r - r_k)^2 + R_0\, \Psi_k\, \cos(m_k\,\theta - n\,\phi) \tag{166}$$

is a flux surface function in the island region. Thus,

$$\frac{F}{R_0\, |\Psi_k|} = 2\, X^2 + \cos(m_k\,\theta - n\,\phi), \tag{167}$$

where

$$X = \frac{2\,(r - r_k)}{W_k}, \tag{168}$$

and

$$\frac{W_k}{4\, R_0} = \left[\left(\frac{q}{g\, s}\right)_{r_k}\, \frac{|\Psi_k|}{B_0\, R_0}\right]^{1/2}. \tag{169}$$

It follows that W_k is the full radial island width in r. Moreover, W_k has no dependence on θ.

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1.12.2 Island Width in Ψ_N

We have

$$\frac{d\Psi_N}{dr} = \frac{f}{R_0 \, |\psi_c|}, \tag{170}$$

$$\frac{dF_0}{dr} = -\frac{B_0 \, R_0}{m_k} \, f \, (m_k - n \, q). \tag{171}$$

Hence,

$$\frac{dF_0}{d\Psi_N} = -B_0 \, R_0^2 \, |\psi_c| \, (1 - q/q_k). \tag{172}$$

Let

$$q = q_k + q_k' \, (\Psi_N - \Psi_{N\,k}) + \frac{1}{2} \, q_k'' \, (\Psi_N - \Psi_{N\,k})^2 + \frac{1}{6} \, q_k''' \, (\Psi_N - \Psi_{N\,k})^3, \tag{173}$$

where $' \equiv d/d\Psi_N$. Follows that

$$F_0(\Psi_N) = B_0 \, R_0^2 \, |\psi_c| \left\{ \frac{1}{2} \frac{q_k'}{q_k} (\Psi_N - \Psi_{N\,k})^2 + \frac{1}{6} \frac{q_k''}{q_k} (\Psi_N - \Psi_{N\,k})^3 + \frac{1}{24} \frac{q_k'''}{q_k} (\Psi_N - \Psi_{N\,k})^4 \right\}. \tag{174}$$

Hence, the island extends from $\Psi_{N\,k} + X_{k-}$ to $\Psi_{N\,k} + X_{k+}$, where X_{k-} and X_{k+} are the positive and negative roots, respectively, of

$$X_k^2 + A_k^{(2)} \, X_k^3 = \frac{\overline{W}_k^2}{4}, \tag{175}$$

where

$$\overline{W}_k = 4 \left(A_k^{(1)} \, \frac{|\Psi_k|}{R_0 \, B_0} \right)^{1/2}, \tag{176}$$

and

$$A_k^{(1)} = \frac{q_k}{q_k' \, |\psi_c|}, \tag{177}$$

$$A_k^{(2)} = \frac{1}{3} \frac{q_k''}{q_k'}. \tag{178}$$

It follows that

$$X_{k\pm} \simeq \pm \frac{\overline{W}_k}{2} - A_k^{(2)} \, \frac{\overline{W}_k^2}{8}. \tag{179}$$

Alternatively, can say that island extends from $\Psi_{N\,k} + X_{k-} + \Delta\Psi_{N\,k}$ to $\Psi_{N\,k} + X_{k+} + \Delta\Psi_{N\,k}$, where $X_{k\pm} = \pm\Delta_{k\pm} \, x_{k\pm}$, and

$$\Delta_{k+} = \chi_{\max}(\Psi_{N\,k+1} - \Psi_{N\,k}), \tag{180}$$

$$\Delta_{k-} = \chi_{\max}(\Psi_{N\,k} - \Psi_{N\,k-1}), \tag{181}$$

$$F_{k\pm}(x_{k\pm}) = \frac{2\,A_k^{(1)}}{\Delta_{k\pm}^2}\,\frac{|\Psi_k|}{R_0\,B_0},\tag{182}$$

$$F_{k\pm}(x) \equiv -x - \ln(1-x).\tag{183}$$

Here, χ_{max} is the maximum allowable Chirikov parameter. Moreover,

$$\Delta\Psi_{Nk} = -\frac{A_k^{(2)}}{8}\,(X_{k+} - X_{k-})^2.\tag{184}$$

2 Technical Details

2.1 Flux Coordinate System

Let all lengths be normalized to R_0, and all magnetic field-strengths to B_0. We have

$$\mathbf{B} = \nabla\phi \times \nabla\psi_p + g(\psi_p)\,\nabla\phi,\tag{185}$$

and

$$\nabla\psi_p \times \nabla\theta \cdot \nabla\phi = \frac{g}{q\,R^2},\tag{186}$$

where $q = q(\psi_p)$.

Let $\Psi = \psi_p/\psi_c = 1 - \Psi_N$, where ψ_c is the value of ψ_p on the magnetic axis. (It is assumed that $\psi_p = 0$ on the plasma boundary.) The previous equation implies that

$$\frac{d\theta}{dl} = \frac{g}{q}\,\frac{1}{|\psi_c|\,R\,\sqrt{\Psi_R^2 + \Psi_Z^2}},\tag{187}$$

where dl is an element of poloidal path length on a magnetic flux-surface, and $\Psi_R \equiv \partial\Psi/\partial R$, etc. Furthermore,

$$dR = -\frac{\Psi_Z\,dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}},\tag{188}$$

$$dZ = \frac{\Psi_R\,dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}.\tag{189}$$

It follows that

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi\,|\psi_c|}\oint \frac{dl}{R\,\sqrt{\Psi_R^2 + \Psi_Z^2}}.\tag{190}$$

If we define

$$\tan\zeta = \frac{Z - Z_{\mathrm{axis}}}{R_{\mathrm{axis}} - R}\tag{191}$$

then

$$\frac{dR}{d\zeta} = -\Psi_Z\,F,\tag{192}$$

$$\frac{dZ}{d\zeta} = \Psi_R \, F, \tag{193}$$

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi \, |\psi_c|} \oint \frac{F}{R} \, d\zeta, \tag{194}$$

where

$$F = \frac{(R_{\mathrm{axis}} - R)^2 + (Z - Z_{\mathrm{axis}})^2}{-(Z - Z_{\mathrm{axis}}) \, \Psi_Z + (R_{\mathrm{axis}} - R) \, \Psi_R}. \tag{195}$$

It is helpful to define the length-like flux-surface coordinate r, according to

$$\nabla r \times \nabla \theta \cdot \nabla \phi = \frac{1}{r \, R^2}. \tag{196}$$

It follows that

$$r(\Psi) = \left[2 \, |\psi_c| \int_\Psi^1 \frac{q(\Psi')}{g(\Psi')} \, d\Psi' \right]^{1/2}. \tag{197}$$

Let

$$a = r(0). \tag{198}$$

We can calculate $R(r, \theta)$ and $Z(r, \theta)$ from

$$\frac{dR}{d\theta} = -|\psi_c| \frac{q}{g} \, R \, \Psi_Z, \tag{199}$$

$$\frac{dZ}{d\theta} = |\psi_c| \frac{q}{g} \, R \, \Psi_R. \tag{200}$$

Now,

$$r \frac{dr}{d\Psi} = -|\psi_c| \frac{q(r)}{g(r)}. \tag{201}$$

So

$$\nabla r = \frac{dr}{d\Psi} \nabla \Psi = -|\psi_c| \frac{q(r)}{r \, g(r)} \nabla \Psi. \tag{202}$$

Hence,

$$a_{jj} = \left(\oint \frac{1}{|\nabla r|^2} \frac{d\theta}{2\pi} \right)_{r_j} = \left(\frac{r \, g}{|\psi_c| \, q} \right)_{r_j}^2 \oint \frac{1}{\Psi_R^2 + \Psi_Z^2} \frac{d\theta}{2\pi}. \tag{203}$$

Note that

$$\frac{d\Psi_N}{dr} = \frac{r \, g(r)}{|\psi_c| \, q(r)}. \tag{204}$$

Hence, if $\overline{W}_k$ is the full magnetic island width in Ψ_N at the kth resonant surface then

$$\overline{W}_k = \frac{W_k}{R_0} \frac{d\Psi_N(r_k)}{dr}, \tag{205}$$

where W_k is the full island width in r.

2.2 Neoclassical Coordinate System

It is also helpful to define the geometric poloidal angle

$$\mathbf{b} \cdot \nabla \Theta = \gamma(r). \tag{206}$$

It follows that

$$\frac{d\Theta}{dl} = \frac{\gamma \, B \, R}{|\psi_c| \, \sqrt{\Psi_R^2 + \Psi_Z^2}}, \tag{207}$$

where

$$B \, R = \left[g^2 + |\psi_c|^2 \, (\Psi_R^2 + \Psi_Z^2) \right]^{1/2}. \tag{208}$$

Hence,

$$\frac{1}{\gamma(r)} = \frac{1}{2\pi \, |\psi_c|} \oint \frac{B \, R \, dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}} = \frac{1}{2\pi \, |\psi_c|} \oint B \, R \, F \, d\zeta. \tag{209}$$

We can calculate $R(r, \Theta)$ and $Z(r, \Theta)$ from

$$\frac{dR}{d\Theta} = -|\psi_c| \, \frac{\Psi_Z}{\gamma \, B \, R}, \tag{210}$$

$$\frac{dZ}{d\Theta} = |\psi_c| \, \frac{\Psi_R}{\gamma \, B \, R}. \tag{211}$$

Note that

$$\frac{d\Theta}{d\theta} = \left(\frac{\gamma \, q}{g} \right) B \, R^2. \tag{212}$$

Thus,

$$\frac{1}{\gamma} = \frac{q}{g} \oint B \, R^2 \, \frac{d\theta}{2\pi}. \tag{213}$$

Also,

$$\frac{\partial B}{\partial \Theta} = -\frac{B}{R} \frac{\partial R}{\partial \Theta} + \frac{|\psi_c|^2}{B \, R^2} \left[(\Psi_R \, \Psi_{RR} + \Psi_Z \, \Psi_{RZ}) \frac{\partial R}{\partial \Theta} + (\Psi_R \, \Psi_{RZ} + \Psi_Z \, \Psi_{ZZ}) \frac{\partial Z}{\partial \Theta} \right]. \tag{214}$$

2.3 Neoclassical Parameters

The flux-surface average operator has the following properties:

$$\langle 1 \rangle = 1, \tag{215}$$

$$\langle \mathbf{B} \cdot \nabla A \rangle = 0. \tag{216}$$

It follows that

$$\langle A \rangle = \oint R^2 \, A \, \frac{d\theta}{2\pi} \bigg/ \oint R^2 \, \frac{d\theta}{2\pi} = \oint \frac{A}{B} \frac{d\Theta}{2\pi} \bigg/ \oint \frac{1}{B} \frac{d\Theta}{2\pi}. \tag{217}$$

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Let

$$I_0 = \oint \frac{1}{B\,R^2}\,\frac{d\Theta}{2\pi} = \frac{\gamma\,q}{g}, \tag{218}$$

$$I_1 = \oint \frac{1}{B}\,\frac{d\Theta}{2\pi}, \tag{219}$$

$$I_2 = \oint B\,\frac{d\Theta}{2\pi}, \tag{220}$$

$$I_3 = \oint \left(\frac{\partial B}{\partial\Theta}\right)^2 \frac{1}{B}\,\frac{d\Theta}{2\pi}, \tag{221}$$

$$I_{4,k} = \sqrt{\frac{2}{k}}\oint \frac{\sin(k\,\Theta)}{B^2}\,\frac{\partial B}{\partial\Theta}\,\frac{d\Theta}{2\pi} = \oint \frac{\sqrt{2\,k}\,\cos(k\,\Theta)}{B}\,\frac{d\Theta}{2\pi}, \tag{222}$$

$$I_{5,k} = \sqrt{\frac{2}{k}}\oint \frac{\sin(k\,\Theta)}{B^3}\,\frac{\partial B}{\partial\Theta}\,\frac{d\Theta}{2\pi} = \oint \frac{\sqrt{2\,k}\,\cos(k\,\Theta)}{2\,B^2}\,\frac{d\Theta}{2\pi}, \tag{223}$$

$$I_6(\lambda) = \oint \frac{\sqrt{1 - \lambda\,B/B_{\max}}}{B}\,\frac{d\Theta}{2\pi}, \tag{224}$$

$$I_7 = \oint \frac{R^2}{B}\,\frac{d\Theta}{2\pi}, \tag{225}$$

$$I_8 = \oint \frac{1}{B^3\,R^2}\,\frac{d\Theta}{2\pi}. \tag{226}$$

It follows that

$$\langle B\rangle = \frac{1}{I_1}, \tag{227}$$

$$C_1 = \left\langle \frac{1}{R^2}\right\rangle = \frac{\gamma\,q}{I_1\,g}, \tag{228}$$

$$\langle R^2\rangle = \frac{I_7}{I_1}, \tag{229}$$

$$\langle B^2\rangle = \frac{I_2}{I_1}, \tag{230}$$

$$C_2 = g^2\left\langle \frac{1}{B^2\,R^2}\right\rangle = \frac{g^2\,I_8}{I_1}, \tag{231}$$

$$\left\langle \frac{|\nabla r|^2}{R^2}\right\rangle = \frac{\gamma\,q\,a_{jj}}{I_1\,g}, \tag{232}$$

$$\langle (\mathbf{b}\cdot\nabla B)^2\rangle = \gamma^2\,\frac{I_3}{I_1}, \tag{233}$$

$$|\langle \mathbf{B}\cdot\nabla\theta\rangle| = \frac{g}{|q|}\,\frac{I_0}{I_1} = \frac{|\gamma|}{I_1}, \tag{234}$$

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$$\left\langle \sqrt{\frac{2}{k}} \, \sin(k\,\Theta)\,(\mathbf{b}\cdot\nabla\ln B) \right\rangle = \gamma \, \frac{I_{4,k}}{I_1}, \tag{235}$$

$$\left\langle \sqrt{\frac{2}{k}} \, \sin(k\,\Theta)\,\frac{(\mathbf{b}\cdot\nabla\ln B)}{B} \right\rangle = \gamma \, \frac{I_{5,k}}{I_1}. \tag{236}$$

Hence,

$$L_c = \frac{1}{|\gamma|} \frac{I_2^2}{I_1^2 \, I_3} \sum_{k>0} I_{4,k} \, I_{5,k}, \tag{237}$$

$$\omega_{t\,a} \equiv \frac{v_{T\,a}}{L_c} = K_t \, |\gamma| \, v_{T\,a}, \tag{238}$$

$$\nu_{*\,a} \equiv \frac{8}{3\pi} \frac{\langle B^2 \rangle}{\langle (\mathbf{b}\cdot\nabla B)^2 \rangle} \frac{g_t \, \omega_{t\,a}}{v_{T\,a}^2 \, \tau_{aa}} = K_* \, \frac{g_t}{\omega_{t\,a}\,\tau_{aa}}, \tag{239}$$

$$f_c = \frac{3}{4} \frac{I_2}{B_{\max}^2} \int_0^1 \frac{\lambda \, d\lambda}{I_6(\lambda)}, \tag{240}$$

where

$$K_t = \frac{I_1^2 \, I_3}{I_2^2 \, \sum_{k>0} I_{4,k} \, I_{5,k}}, \tag{241}$$

$$K_* = \frac{8}{3\pi} \frac{I_2}{I_3} K_t^2. \tag{242}$$

Also,

$$Q^2 = \frac{q^2}{2\,r^2} \left(\left\langle \frac{1}{R^2} \right\rangle - \frac{1}{\langle R^2 \rangle} \right) \bigg/ \left\langle \frac{|\nabla r|^2}{R^2} \right\rangle = \frac{q^2}{2\,r^2\,a_{jj}} \left(1 - \frac{I_1^2}{I_7} \frac{g}{\gamma\,q} \right), \tag{243}$$

$$\frac{\langle B_T^2 \rangle}{\langle B_p^2 \rangle} = \frac{q^2}{r^2\,a_{jj}}. \tag{244}$$

2.4   Glasser-Greene-Johnson Parameters

Let

$$J_1 = \frac{1}{2\pi\,|\psi_c|} \oint R\,F\,d\zeta, \tag{245}$$

$$J_2 = \frac{1}{2\pi\,|\psi_c|} \oint R\,B^2\,F\,d\zeta, \tag{246}$$

$$J_3 = \frac{1}{2\pi\,|\psi_c|} \oint \frac{R\,F}{B^2}\,d\zeta, \tag{247}$$

$$J_4 = \frac{1}{2\pi\,|\psi_c|^3} \oint \frac{R\,F}{\Psi_R^2 + \Psi_Z^2}\,d\zeta, \tag{248}$$

$$J_5 = \frac{1}{2\pi \, |\psi_c|^3} \oint \frac{R \, B^2 \, F}{\Psi_R^2 + \Psi_Z^2} \, d\zeta, \tag{249}$$

$$J_6 = \frac{1}{2\pi \, |\psi_c|^3} \oint \frac{R \, F}{B^2 \, (\Psi_R^2 + \Psi_Z^2)} \, d\zeta, \tag{250}$$

where

$$B^2 = \frac{g^2 + \psi_c^2 \, (\Psi_R^2 + \Psi_Z^2)}{R^2}. \tag{251}$$

It follows that

$$E = -\frac{dp/dr}{(dq/dr)^2} \left(\frac{dJ_0}{dr} - g \, \frac{dq}{dr} \, \frac{J_1}{J_2} \right) J_5, \tag{252}$$

$$F = \frac{(dp/dr)^2}{(dq/dr)^2} \left[g^2 \, \left(J_5 \, J_6 - J_4^2 \right) + J_3 \, J_5 \right], \tag{253}$$

$$H = \frac{dp/dr}{dq/dr} \left(J_4 - \frac{J_1 \, J_5}{J_2} \right) g. \tag{254}$$

Finally,

$$D_R = E + F + H^2. \tag{255}$$