Linear Layer Physics

Richard Fitzpatrick

September 24, 2021

1 Introduction

The analysis is based on A. Cole, and R. Fitzpatrick, *Drift-magnetohydrodynamical model of error-field penetration in tokamak plasmas*, Phys. Plasmas **13**, 032503 (2006).

2 Fundamental Quantities

Linear layer width - $\delta_{\rm linear}$

Layer stability index - $\mathit{\Delta}$

Poloidal mode number - m

Toroidal mode number - n

Plasma major radius - R_0

Plasma minor radius - r

Toroidal magnetic field-strength - B_0

Magnetic shear - s

Electrostatic potential - $\mathit{\Phi}$

Poloidal magnetic flux - ψ

Magnitude of electron charge - e

Electron mass - m_e

Electron number density - n_e

Electron temperature - T_e

Ion mass - m_i

Ion number density - n_i

Ion temperature - T_i

Parallel electrical resistivity - $\eta_\parallel$

Perpendicular particle diffusivity - $D_\perp$

Perpendicular energy diffusivity - χ_E

Perpendicular momentum diffusivity - χ_ϕ

3 Derived Quantities

Safety-factor - $q = m/n$

Scale length - $l = R_0\, q/s$

Electron pressure - $p_e = n_e\, T_e$

Ion pressure - $p_i = n_i\, T_i$

Alfvén velocity - $v_A = B_0/\sqrt{\mu_0\, n_e\, m_i}$

Alfvén time - $\tau_A = l/v_A = m\, \tau_H$

Hydromagnetic time - $\tau_H = R_0/(n\, s\, v_A)$

Resistive time - $\tau_R = \mu_0\, r^{\,2}/\eta_\parallel$

Particle/energy confinement time - $\tau_E = r^{\,2}/[D_\perp + (2/3)\, \chi_E]$

Effective momentum confinement time - $\tau_M = (R_0\, q)^{\,2}/\chi_\phi$

Ion sound radius - $\rho_s = \sqrt{m_i\, T_e}/(e\, B_0)$

$\mathbf{E} \times \mathbf{B}$ frequency - $\omega_E = d\Phi/d\psi$

Electron diamagnetic frequency - $\omega_{*\,e} = (T_e/e)\, d\ln p_e/d\psi$

Ion diamagnetic frequency - $\omega_{*\,i} = -(T_i/e)\, d\ln p_i/d\psi$

$\mathbf{E} \times \mathbf{B}$ velocity - $v_E = (r/q)\, \omega_E$

Electron diamagnetic velocity - $v_{*\,e} = (r/q)\, \omega_{*\,e}$

Ion diamagnetic velocity - $v_{*\,i} = (r/q)\, \omega_{*\,i}$

4 Non-dimensional Quantities

Lundquist number - $S = \tau_R/\tau_H$

Magnetic Prandtl number - $P_M = \tau_R/\tau_M$

Energy confinement number - $P_E = \tau_R/\tau_E$

5 Layer Theory Quantities

$$\left(\frac{k}{\eta}\right)^{1/3} = S^{\,1/3}\,\frac{l}{r}$$

$$\tau = -\frac{\omega_{\ast\,i}}{\omega_{\ast\,e}}$$

$$c_\beta^2 = P_E$$

$$P = P_M$$

$$D = \frac{5}{3}\,S^{\,1/3}\,\frac{\rho_s}{r}$$

$$Q_E = S^{\,1/3}\,n\,\omega_E\,\tau_H$$

$$Q_{e,i} = -S^{\,1/3}\,n\,\omega_{\ast\,e,i}\,\tau_H$$

$$\mathit{\Delta} = \frac{\delta}{r}\,\tau_R\,\mathrm{i}\,n\,(\omega_E - \omega_{\ast\,i})$$

$$\hat{\mathit{\Delta}} = \left(\frac{\eta}{k}\right)^{1/3}\,\frac{l}{r}\,\mathit{\Delta}$$

6 Constant-ψ Layer Physics

Layer equation:

$$\frac{d^2 Y}{dp^2} - \left[\frac{-Q_E\,(Q_E-Q_i) + \mathrm{i}\,(Q_E-Q_i)\,(P_M+P_E)\,p^2 + P_M\,P_E\,p^4}{\mathrm{i}\,(Q_E-Q_e) + \{P_E + \mathrm{i}\,(Q_E-Q_i)\,D^2\}\,p^2 + (1+\tau)\,P_M\,D^2\,p^4}\right] p^2\,Y = 0.$$

Suppose that small-p behavior of solution that is well-behaved as $p\rightarrow\infty$ is

$$Y(p) = Y_0\left[1 - c\,p + {\cal O}(p^2)\right].$$

Layer width is

$$\frac{\delta_{\rm linear}}{r} = \frac{\pi\,|c|}{S^{\,1/3}}.$$

Large-p limit of layer equation is

$$\frac{d^2 Y}{dp^2} - C\,p^2\,Y = 0,$$

where

$$C = \frac{P_E}{(1+\tau)\,D^2}.$$

Note that require $\tau > -1$, otherwise matching procedure fails. Solutions are

$$p^{1/2}\,I_{1/4}\left(\frac{C^{\,1/2}\,p^2}{2}\right),\quad p^{1/2}\,K_{1/4}\left(\frac{C^{\,1/2}\,p^2}{2}\right).$$

Should need to integrate to $p\sim C^{\,-1/4}$.