EUCLID'S ELEMENTS OF GEOMETRY

The Greek text of J.L. Heiberg (1883–1885)

from *Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus B.G. Teubneri, 1883–1885*

edited, and provided with a modern English translation, by

Richard Fitzpatrick

First edition - 2007
Revised and corrected - 2008
Second edition - 2014

Contents

Introduction

Euclid's Elements is by far the most famous mathematical work of classical antiquity, and also has the distinction of being the world's oldest continuously used mathematical textbook. Little is known about the author, beyond the fact that he lived in Alexandria around 300 BCE. The main subjects of the work are geometry, proportion, and number theory.

Most of the theorems appearing in the Elements were not discovered by Euclid himself, but were the work of earlier Greek mathematicians such as Pythagoras (and his school), Hippocrates of Chios, Theaetetus of Athens, and Eudoxus of Cnidos. However, Euclid is generally credited with arranging these theorems in a logical manner, so as to demonstrate (admittedly, not always with the rigour demanded by modern mathematics) that they necessarily follow from five simple axioms. Euclid is also credited with devising a number of particularly ingenious proofs of previously discovered theorems: e.g., Theorem 48 in Book 1.

The geometrical constructions employed in the Elements are restricted to those which can be achieved using a straight-rule and a compass. Furthermore, empirical proofs by means of measurement are strictly forbidden: i.e., any comparison of two magnitudes is restricted to saying that the magnitudes are either equal, or that one is greater than the other.

The Elements consists of thirteen books. Book 1 outlines the fundamental propositions of plane geometry, including the three cases in which triangles are congruent, various theorems involving parallel lines, the theorem regarding the sum of the angles in a triangle, and the Pythagorean theorem. Book 2 is commonly said to deal with "geometric algebra", since most of the theorems contained within it have simple algebraic interpretations. Book 3 investigates circles and their properties, and includes theorems on tangents and inscribed angles. Book 4 is concerned with regular polygons inscribed in, and circumscribed around, circles. Book 5 develops the arithmetic theory of proportion. Book 6 applies the theory of proportion to plane geometry, and contains theorems on similar figures. Book 7 deals with elementary number theory: e.g., prime numbers, greatest common denominators, etc. Book 8 is concerned with geometric series. Book 9 contains various applications of results in the previous two books, and includes theorems on the infinitude of prime numbers, as well as the sum of a geometric series. Book 10 attempts to classify incommensurable (i.e., irrational) magnitudes using the so-called "method of exhaustion", an ancient precursor to integration. Book 11 deals with the fundamental propositions of three-dimensional geometry. Book 12 calculates the relative volumes of cones, pyramids, cylinders, and spheres using the method of exhaustion. Finally, Book 13 investigates the five so-called Platonic solids.

This edition of Euclid's Elements presents the definitive Greek text—i.e., that edited by J.L. Heiberg (1883–1885)—accompanied by a modern English translation, as well as a Greek-English lexicon. Neither the spurious books 14 and 15, nor the extensive scholia which have been added to the Elements over the centuries, are included. The aim of the translation is to make the mathematical argument as clear and unambiguous as possible, whilst still adhering closely to the meaning of the original Greek. Text within square parenthesis (in both Greek and English) indicates material identified by Heiberg as being later interpolations to the original text (some particularly obvious or unhelpful interpolations have been omitted altogether). Text within round parenthesis (in English) indicates material which is implied, but not actually present, in the Greek text.

My thanks to Mariusz Wodzicki (Berkeley) for typesetting advice, and to Sam Watson & Jonathan Fenno (U. Mississippi), and Gregory Wong (UCSD) for pointing out a number of errors in Book 1.

In the second edition of this work I have opted for more compact (and nicer looking) Greek and English fonts. I have also made some minor improvements and corrections to my translation.

ELEMENTS BOOK 1

Fundamentals of Plane Geometry Involving Straight-Lines

Ὅροι.

α′. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

β′. Γραμμὴ δὲ μῆκος ἀπλατές.

γ′. Γραμμῆς δὲ πέρατα σημεῖα.

δ′. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται.

ε′. Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

ϛ′. Ἐπιφανείας δὲ πέρατα γραμμαί.

ζ′. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ᾽ ἑαυτῆς εὐθείαις κεῖται.

η′. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾽ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

θ′. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

ι′. Ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾽ ἣν ἐφέστηκεν.

ια′. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

ιβ′. Ὀξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

ιγ′. Ὅρος ἐστίν, ὅ τινός ἐστι πέρας.

ιδ′. Σχῆμά ἐστι τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον.

ιε′. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ᾽ ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

ιϛ′. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

ιζ′. Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾽ ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.

ιη′. Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ᾽ αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

ιθ′. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

κ′. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

κα′ Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

κβ′. Τὼν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν

Definitions

1. A point is that of which there is no part.

2. And a line is a length without breadth.

3. And the extremities of a line are points.

4. A straight-line is (any) one which lies evenly with points on itself.

5. And a surface is that which has length and breadth only.

6. And the extremities of a surface are lines.

7. A plane surface is (any) one which lies evenly with the straight-lines on itself.

8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.

11. An obtuse angle is one greater than a right-angle.

12. And an acute angle (is) one less than a right-angle.

13. A boundary is that which is the extremity of something.

14. A figure is that which is contained by some boundary or boundaries.

15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

16. And the point is called the center of the circle.

17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.[†]

18. And a semi-circle is the figure contained by the diameter and the circumference cut off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.

19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that

ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

κγ΄. Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

† This should really be counted as a postulate, rather than as part of a definition.

Αἰτήματα.

α΄. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β΄. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

γ΄. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ΄. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε΄. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

1. Let it be postulated† (that it is possible) to draw a straight-line from any point to any point;

2. And to produce a finite straight-line continuously in a straight-line;

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, (then) the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

† Literally, "let it have been postulated". The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative *ἠιτήσθω* means "let it be postulated" in the sense "let it stand as postulated", but not "let the postulate be now brought forward". This peculiar Greek idiom is used throughout the Elements.

‡ This postulate effectively specifies that we are dealing with the geometry of flat, rather than curved, space.

Κοιναὶ ἔννοιαι.

α΄. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

β΄. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.

γ΄. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

δ΄. Καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἀλλήλα ἴσα ἀλλήλοις ἐστίν.

ε΄. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστιν].

Common Notions

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things (then) the wholes are equal.

3. And if equal things are subtracted from equal things (then) the remainders are equal.†

4. And things coinciding with one another are equal to one another.

5. And the whole [is] greater than the part.

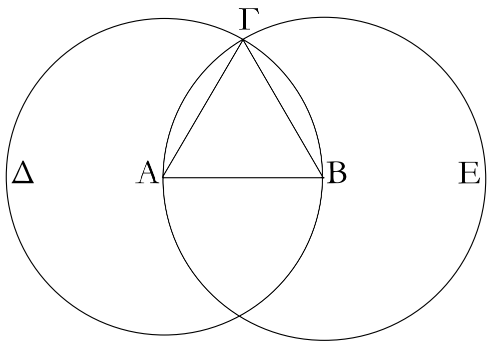
† As an obvious extension of C.N.s 2 & 3—if equal things are added or subtracted from the two sides of an inequality then the inequality remains an inequality of the same type.

α´.

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.

Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ.

Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.



Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ᾽ ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπί τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

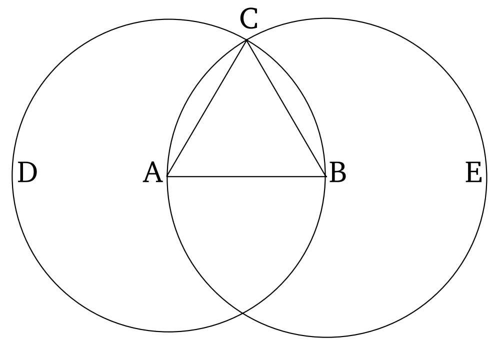
Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ. ὅπερ ἔδει ποιῆσαι.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.

Let *AB* be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line *AB*.



Let the circle *BCD* with center *A* and radius *AB* be drawn[†] [Post. 3], and again let the circle *ACE* with center *B* and radius *BA* be drawn [Post. 3]. And let the straight-lines *CA* and *CB* be joined from the point *C*, where the circles cut one another,[‡] to the points *A* and *B* (respectively) [Post. 1].

And since the point *A* is the center of the circle *CDB*, *AC* is equal to *AB* [Def. 1.15]. Again, since the point *B* is the center of the circle *CAE*, *BC* is equal to *BA* [Def. 1.15]. But *CA* was also shown (to be) equal to *AB*. Thus, *CA* and *CB* are each equal to *AB*. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, *CA* is also equal to *CB*. Thus, the three (straight-lines) *CA*, *AB*, and *BC* are equal to one another.

Thus, the triangle *ABC* is equilateral, and has been constructed on the given finite straight-line *AB*. (Which is) the very thing it was required to do.

[†] Literally, "have been drawn". The use of this peculiar Greek idiom will pass without further comment.

[‡] The assumption that the circles do indeed cut one another should be counted as an additional postulate. There is also an implicit assumption that two straight-lines cannot share a common segment.

β´.

Πρὸς τῷ δοθέντι σημείῳ τῇ δοθείσῃ εὐθείᾳ ἴσην εὐθεῖαν θέσθαι.

Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ· δεῖ δὴ πρὸς τῷ Α σημείῳ τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ ἴσην εὐθεῖαν θέσθαι.

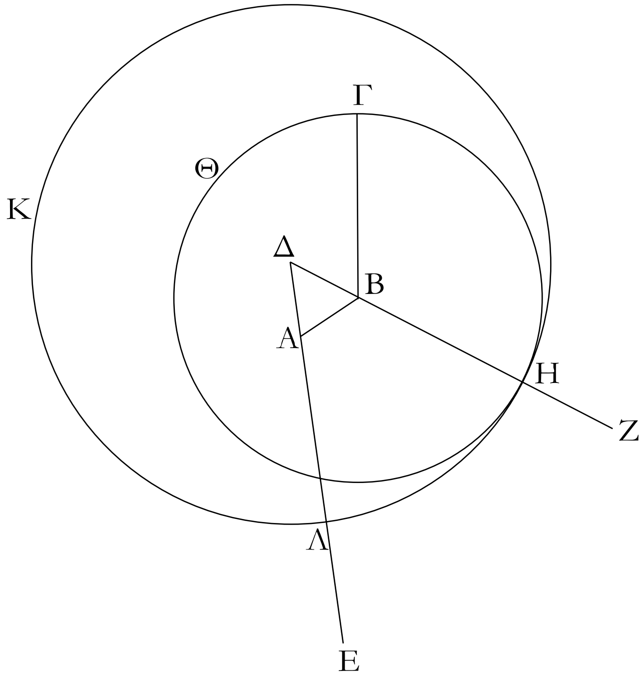
Ἐπεζεύχθω γὰρ ἀπὸ τοῦ Α σημείου ἐπί τὸ Β σημεῖον εὐθεῖα ἡ ΑΒ, καὶ συνεστάτω ἐπ᾽ αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΔΑΒ, καὶ ἐκβεβλήσθωσαν ἐπ᾽ εὐθείας ταῖς ΔΑ, ΔΒ εὐθεῖαι αἱ ΑΕ, ΒΖ, καὶ κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΓ κύκλος γεγράφθω ὁ ΓΗΘ, καὶ πάλιν κέντρῳ τῷ Δ καὶ διαστήματι τῷ ΔΗ κύκλος γεγράφθω ὁ ΗΚΛ.

Proposition 2[†]

To place a straight-line equal to a given straight-line at a given point (as an extremity).

Let *A* be the given point, and *BC* the given straight-line. So it is required to place a straight-line at point *A* equal to the given straight-line *BC*.

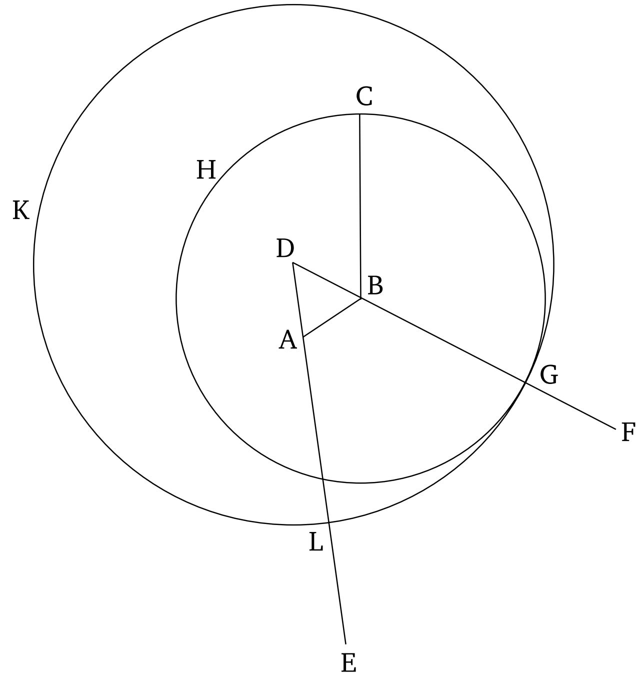
For let the straight-line *AB* be joined from point *A* to point *B* [Post. 1], and let the equilateral triangle *DAB* be constructed upon it [Prop. 1.1]. And let the straight-lines *AE* and *BF* be produced in a straight-line with *DA* and *DB* (respectively) [Post. 2]. And let the circle *CGH* with center *B* and radius *BC* be drawn [Post. 3], and again let the circle *GKL* with center *D*



Ἐπεὶ οὖν τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΗΘ, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΗ. πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΗΚΛ κύκλου, ἴση ἐστὶν ἡ ΔΛ τῇ ΔΗ, ὧν ἡ ΔΑ τῇ ΔΒ ἴση ἐστίν. λοιπὴ ἄρα ἡ ΑΛ λοιπῇ τῇ ΒΗ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ΒΓ τῇ ΒΗ ἴση· ἑκατέρα ἄρα τῶν ΑΛ, ΒΓ τῇ ΒΗ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΑΛ ἄρα τῇ ΒΓ ἐστιν ἴση.

Πρὸς ἄρα τῷ δοθέντι σημείῳ τῷ Α τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ ἴση εὐθεῖα κεῖται ἡ ΑΛ· ὅπερ ἔδει ποιῆσαι.

and radius *DG* be drawn [Post. 3].



Therefore, since the point *B* is the center of (the circle) *CGH*, *BC* is equal to *BG* [Def. 1.15]. Again, since the point *D* is the center of the circle *GKL*, *DL* is equal to *DG* [Def. 1.15]. And within these, *DA* is equal to *DB*. Thus, the remainder *AL* is equal to the remainder *BG* [C.N. 3]. But *BC* was also shown (to be) equal to *BG*. Thus, *AL* and *BC* are each equal to *BG*. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, *AL* is also equal to *BC*.

Thus, the straight-line *AL*, equal to the given straight-line *BC*, has been placed at the given point *A*. (Which is) the very thing it was required to do.

† This proposition admits of a number of different cases, depending on the relative positions of the point *A* and the line *BC*. In such situations, Euclid invariably only considers one particular case—usually, the most difficult—and leaves the remaining cases as exercises for the reader.

γ΄.

Δύο δοθεισῶν εὐθειῶν ἀνίσων ἀπὸ τῆς μείζονος τῇ ἐλάσσονι ἴσην εὐθεῖαν ἀφελεῖν.

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι ἄνισοι αἱ ΑΒ, Γ, ὧν μείζων ἔστω ἡ ΑΒ· δεῖ δὴ ἀπὸ τῆς μείζονος τῆς ΑΒ τῇ ἐλάσσονι τῇ Γ ἴσην εὐθεῖαν ἀφελεῖν.

Κείσθω πρὸς τῷ Α σημείῳ τῇ Γ εὐθείᾳ ἴση ἡ ΑΔ· καὶ κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΔ κύκλος γεγράφθω ὁ ΔΕΖ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΔΕΖ κύκλου, ἴση ἐστὶν ἡ ΑΕ τῇ ΑΔ· ἀλλὰ καὶ ἡ Γ τῇ ΑΔ ἐστιν ἴση. ἑκατέρα ἄρα τῶν ΑΕ, Γ τῇ ΑΔ ἐστιν ἴση· ὥστε καὶ ἡ ΑΕ τῇ Γ ἐστιν ἴση.

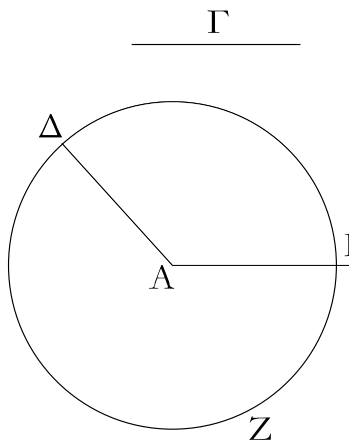
Proposition 3

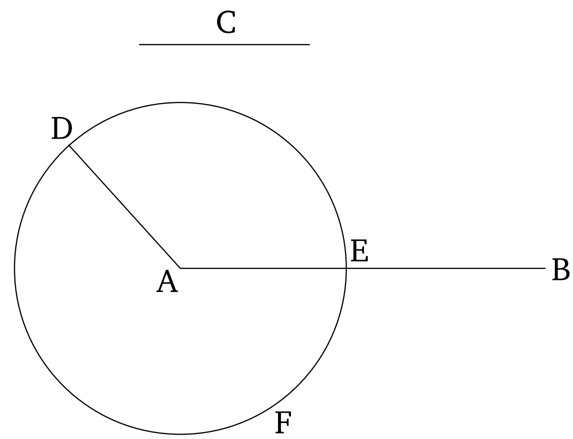
For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.

Let *AB* and *C* be the two given unequal straight-lines, of which let the greater be *AB*. So it is required to cut off a straight-line equal to the lesser *C* from the greater *AB*.

Let the line *AD*, equal to the straight-line *C*, be placed at point *A* [Prop. 1.2]. And let the circle *DEF* be drawn with center *A* and radius *AD* [Post. 3].

And since point *A* is the center of circle *DEF*, *AE* is equal to *AD* [Def. 1.15]. But, *C* is also equal to *AD*. Thus, *AE* and *C* are each equal to *AD*. So *AE* is also equal to *C* [C.N. 1].





Δύο ἄρα δοθεισῶν εὐθειῶν ἀνίσων τῶν ΑΒ, Γ ἀπὸ τῆς μείζονος τῆς ΑΒ τῇ ἐλάσσονι τῇ Γ ἴση ἀφῄρηται ἡ ΑΕ· ὅπερ ἔδει ποιῆσαι.

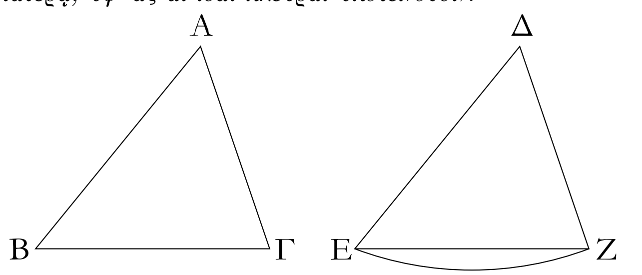
Thus, for two given unequal straight-lines, AB and C, the (straight-line) AE, equal to the lesser C, has been cut off from the greater AB. (Which is) the very thing it was required to do.

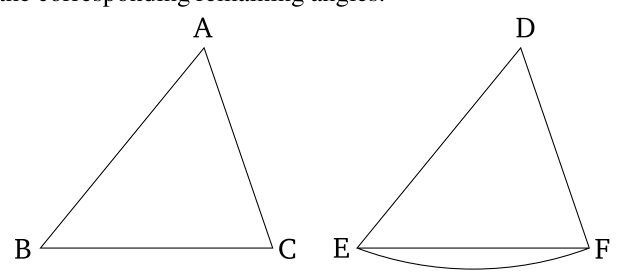
δʹ.

Proposition 4

Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῇ βάσει ἴσην ἕξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν.

If two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, (then) they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.





Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς ΑΒ, ΑΓ ταῖς δυσὶ πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ καὶ γωνίαν τὴν ὑπὸ ΒΑΓ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴσην. λέγω, ὅτι καὶ βάσις ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστίν, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ, ἡ δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ.

Ἐφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Δ σημεῖον τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΔΕ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Ε διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῇ ΔΕ· ἐφαρμοσάσης δὴ τῆς ΑΒ ἐπὶ τὴν ΔΕ ἐφαρμόσει καὶ ἡ ΑΓ εὐθεῖα ἐπὶ τὴν ΔΖ διὰ τὸ ἴσην εἶναι τὴν ὑπὸ ΒΑΓ γωνίαν τῇ ὑπὸ ΕΔΖ· ὥστε καὶ τὸ Γ σημεῖον ἐπὶ τὸ Ζ σημεῖον ἐφαρμόσει διὰ τὸ ἴσην πάλιν εἶναι τὴν ΑΓ τῇ ΔΖ. ἀλλὰ μὴν καὶ τὸ Β ἐπὶ τὸ Ε ἐφηρμόκει·

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is) AB to DE, and AC to DF. And (let) the angle BAC (be) equal to the angle EDF. I say that the base BC is also equal to the base EF, and triangle ABC will be equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (That is) ABC to DEF, and ACB to DFE.

For if triangle ABC is applied to triangle DEF,[†] the point A being placed on the point D, and the straight-line AB on DE, (then) the point B will also coincide with E, on account of AB being equal to DE. So (because of) AB coinciding with DE, the straight-line AC will also coincide with DF, on account of the angle BAC being equal to EDF. So the point C will also coincide with the point F, again on account of AC being equal to DF. But, point B certainly also coincided with point E, so that the base BC will coincide with the base EF. For if *ὥστε βάσις ἡ ΒΓ ἐπὶ βάσιν τὴν ΕΖ ἐφαρμόσει. εἰ γὰρ τοῦ μὲν Β ἐπὶ τὸ Ε ἐφαρμόσαντος τοῦ δὲ Γ ἐπὶ τὸ Ζ ἡ ΒΓ βάσις ἐπὶ τὴν ΕΖ οὐκ ἐφαρμόσει, δύο εὐθεῖαι χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. ἐφαρμόσει ἄρα ἡ ΒΓ βάσις ἐπὶ τὴν ΕΖ καὶ ἴση αὐτῇ ἔσται· ὥστε καὶ ὅλον τὸ ΑΒΓ τρίγωνον ἐπὶ ὅλον τὸ ΔΕΖ τρίγωνον ἐφαρμόσει καὶ ἴσον αὐτῷ ἔσται, καὶ αἱ λοιπαὶ γωνίαι ἐπὶ τὰς λοιπὰς γωνίας ἐφαρμόσουσι καὶ ἴσαι αὐταῖς ἔσονται, ἡ μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ ἡ δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ.*

Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῇ βάσει ἴσην ἕξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

B coincides with *E*, and *C* with *F*, and the base *BC* does not coincide with *EF*, (then) two straight-lines will encompass an area. The very thing is impossible [Post. 1].‡ Thus, the base *BC* will coincide with *EF*, and will be equal to it [C.N. 4]. So the whole triangle *ABC* will coincide with the whole triangle *DEF*, and will be equal to it [C.N. 4]. And the remaining angles will coincide with the remaining angles, and will be equal to them [C.N. 4]. (That is) *ABC* to *DEF*, and *ACB* to *DFE* [C.N. 4].

Thus, if two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, (then) they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (Which is) the very thing it was required to show.

† The application of one figure to another should be counted as an additional postulate.

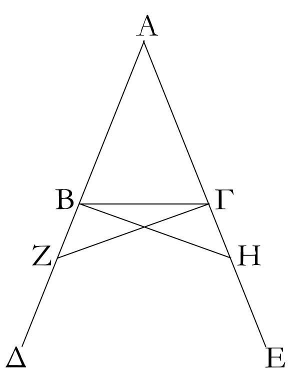
‡ Since Post. 1 implicitly assumes that the straight-line joining two given points is unique.

ε′.

Τῶν ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται.

Ἔστω τρίγωνον ἰσοσκελὲς τὸ ΑΒΓ ἴσην ἔχον τὴν ΑΒ πλευρὰν τῇ ΑΓ πλευρᾷ, καὶ προσεκβεβλήσθωσαν ἐπ᾽ εὐθείας ταῖς ΑΒ, ΑΓ εὐθεῖαι αἱ ΒΔ, ΓΕ· λέγω, ὅτι ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΑΓΒ ἴση ἐστίν, ἡ δὲ ὑπὸ ΓΒΔ τῇ ὑπὸ ΒΓΕ.

Εἰλήφθω γὰρ ἐπὶ τῆς ΒΔ τυχὸν σημεῖον τὸ Ζ, καὶ ἀφῃρήσθω ἀπὸ τῆς μείζονος τῆς ΑΕ τῇ ἐλάσσονι τῇ ΑΖ ἴση ἡ ΑΗ, καὶ ἐπεζεύχθωσαν αἱ ΖΓ, ΗΒ εὐθεῖαι.



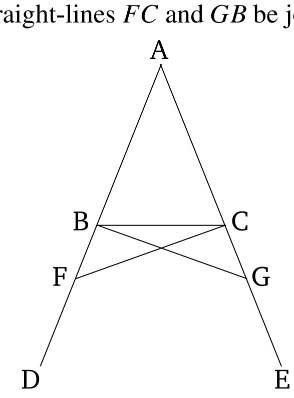
Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΖ τῇ ΑΗ ἡ δὲ ΑΒ τῇ ΑΓ, δύο δὴ αἱ ΖΑ, ΑΓ δυσὶ ταῖς ΗΑ, ΑΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίαν κοινὴν περιέχουσι τὴν ὑπὸ ΖΑΗ· βάσις ἄρα ἡ ΖΓ βάσει τῇ ΗΒ ἴση ἐστίν, καὶ τὸ ΑΖΓ τρίγωνον τῷ ΑΗΒ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΑΓΖ τῇ ὑπὸ ΑΒΗ, ἡ δὲ ὑπὸ ΑΖΓ

Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced (then) the angles under the base will be equal to one another.

Let *ABC* be an isosceles triangle having the side *AB* equal to the side *AC*, and let the straight-lines *BD* and *CE* be produced in a straight-line with *AB* and *AC* (respectively) [Post. 2]. I say that the angle *ABC* is equal to *ACB*, and (angle) *CBD* to *BCE*.

For let the point *F* be taken at random on *BD*, and let *AG* be cut off from the greater *AE*, equal to the lesser *AF* [Prop. 1.3]. Also, let the straight-lines *FC* and *GB* be joined [Post. 1].



In fact, since *AF* is equal to *AG*, and *AB* to *AC*, the two (straight-lines) *FA*, *AC* are equal to the two (straight-lines) *GA*, *AB*, respectively. They also encompass a common angle, *FAG*. Thus, the base *FC* is equal to the base *GB*, and the triangle *AFC* will be equal to the triangle *AGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) *ACF* to *ABG*, τῇ ὑπὸ ΑΗΒ. καὶ ἐπεὶ ὅλη ἡ ΑΖ ὅλῃ τῇ ΑΗ ἐστιν ἴση, ὧν ἡ ΑΒ τῇ ΑΓ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΒΖ λοιπῇ τῇ ΓΗ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ΖΓ τῇ ΗΒ ἴση· δύο δὴ αἱ ΒΖ, ΖΓ δυσὶ ταῖς ΓΗ, ΗΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΒΖΓ γωνίᾳ τῃ ὑπὸ ΓΗΒ ἴση, καὶ βάσις αὐτῶν κοινὴ ἡ ΒΓ· καὶ τὸ ΒΖΓ ἄρα τρίγωνον τῷ ΓΗΒ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ᾿ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΖΒΓ τῇ ὑπὸ ΗΓΒ ἡ δὲ ὑπὸ ΒΓΖ τῇ ὑπὸ ΓΒΗ. ἐπεὶ οὖν ὅλη ἡ ὑπὸ ΑΒΗ γωνία ὅλῃ τῇ ὑπὸ ΑΓΖ γωνίᾳ ἐδείχθη ἴση, ὧν ἡ ὑπὸ ΓΒΗ τῇ ὑπὸ ΒΓΖ ἴση, λοιπὴ ἄρα ἡ ὑπὸ ΑΒΓ λοιπῇ τῇ ὑπὸ ΑΓΒ ἐστιν ἴση· καί εἰσι πρὸς τῇ βάσει τοῦ ΑΒΓ τριγώνου. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΒΓ τῇ ὑπὸ ΗΓΒ ἴση· καί εἰσιν ὑπὸ τὴν βάσιν.

Τῶν ἄρα ἰσοσκελῶν τριγώνων αἱ τρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι.

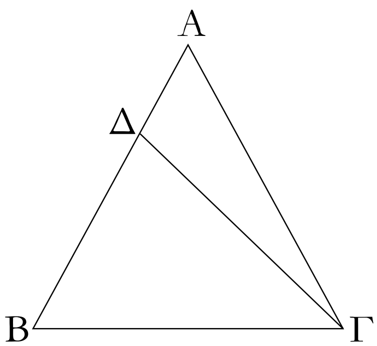
and *AFC* to *AGB*. And since the whole of *AF* is equal to the whole of *AG*, within which *AB* is equal to *AC*, the remainder *BF* is thus equal to the remainder *CG* [C.N. 3]. But *FC* was also shown (to be) equal to *GB*. So the two (straight-lines) *BF*, *FC* are equal to the two (straight-lines) *CG*, *GB*, respectively, and the angle *BFC* (is) equal to the angle *CGB*, and the base *BC* is common to them. Thus, the triangle *BFC* will be equal to the triangle *CGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, *FBC* is equal to *GCB*, and *BCF* to *CBG*. Therefore, since the whole angle *ABG* was shown (to be) equal to the whole angle *ACF*, within which *CBG* is equal to *BCF*, the remainder *ABC* is thus equal to the remainder *ACB* [C.N. 3]. And they are at the base of triangle *ABC*. And *FBC* was also shown (to be) equal to *GCB*. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced (then) the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

ς′.

Ἐὰν τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ὦσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι πλευραὶ ἴσαι ἀλλήλαις ἔσονται.

Ἔστω τρίγωνον τὸ ΑΒΓ ἴσην ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν τῇ ὑπὸ ΑΓΒ γωνίᾳ· λέγω, ὅτι καὶ πλευρὰ ἡ ΑΒ πλευρᾷ τῇ ΑΓ ἐστιν ἴση.



Εἰ γὰρ ἄνισός ἐστιν ἡ ΑΒ τῇ ΑΓ, ἡ ἑτέρα αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ΑΒ, καὶ ἀφῃρήσθω ἀπὸ τῆς μείζονος τῆς ΑΒ τῇ ἐλάττονι τῇ ΑΓ ἴση ἡ ΔΒ, καὶ ἐπεζεύχθω ἡ ΔΓ.

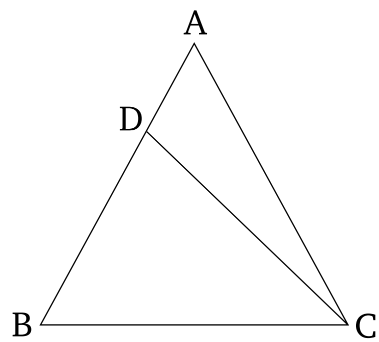
Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΒ τῇ ΑΓ κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΔΒ, ΒΓ δύο ταῖς ΑΓ, ΓΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΔΒΓ γωνίᾳ τῇ ὑπὸ ΑΓΒ ἐστιν ἴση· βάσις ἄρα ἡ ΔΓ βάσει τῇ ΑΒ ἴση ἐστίν, καὶ τὸ ΔΒΓ τρίγωνον τῷ ΑΓΒ τριγώνῳ ἴσον ἔσται, τὸ ἔλασσον τῷ μείζονι· ὅπερ ἄτοπον· οὐκ ἄρα ἄνισός ἐστιν ἡ ΑΒ τῇ ΑΓ· ἴση ἄρα.

Ἐὰν ἄρα τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ὦσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι πλευραὶ ἴσαι ἀλλήλαις

Proposition 6

If a triangle has two angles equal to one another (then) the sides subtending the equal angles will also be equal to one another.

Let *ABC* be a triangle having the angle *ABC* equal to the angle *ACB*. I say that side *AB* is also equal to side *AC*.



For if *AB* is unequal to *AC* (then) one of them is greater. Let *AB* be greater. And let *DB*, equal to the lesser *AC*, be cut off from the greater *AB* [Prop. 1.3]. And let *DC* be joined [Post. 1].

Therefore, since *DB* is equal to *AC*, and *BC* (is) common, the two sides *DB*, *BC* are equal to the two sides *AC*, *CB*, respectively, and the angle *DBC* is equal to the angle *ACB*. Thus, the base *DC* is equal to the base *AB*, and the triangle *DBC* will be equal to the triangle *ACB* [Prop. 1.4], the lesser to the greater. The very notion (is) absurd [C.N. 5]. Thus, *AB* is not unequal to *AC*. Thus, (it is) equal.†

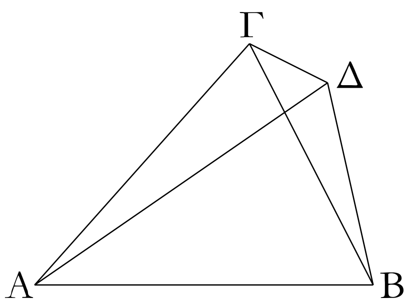
ἔσονται· ὅπερ ἔδει δεῖξαι.

Thus, if a triangle has two angles equal to one another (then) the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show.

† Here, use is made of the previously unmentioned common notion that if two quantities are not unequal then they must be equal. Later on, use is made of the closely related common notion that if two quantities are not greater than or less than one another, respectively, then they must be equal to one another.

ζ'.

Ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ οὐ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις.



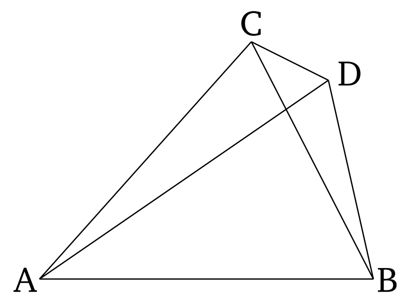
Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς ΑΒ δύο ταῖς αὐταῖς εὐθείαις ταῖς ΑΓ, ΓΒ ἄλλαι δύο εὐθεῖαι αἱ ΑΔ, ΔΒ ἴσαι ἑκατέρα ἑκατέρᾳ συνεστάτωσαν πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ τῷ τε Γ καὶ Δ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι, ὥστε ἴσην εἶναι τὴν μὲν ΓΑ τῇ ΔΑ τὸ αὐτὸ πέρας ἔχουσαν αὐτῇ τὸ Α, τὴν δὲ ΓΒ τῇ ΔΒ τὸ αὐτὸ πέρας ἔχουσαν αὐτῇ τὸ Β, καὶ ἐπεζεύχθω ἡ ΓΔ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΑΓ τῇ ΑΔ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΓΔ τῇ ὑπὸ ΑΔΓ· μείζων ἄρα ἡ ὑπὸ ΑΔΓ τῆς ὑπὸ ΔΓΒ· πολλῷ ἄρα ἡ ὑπὸ ΓΔΒ μείζων ἐστί τῆς ὑπὸ ΔΓΒ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΓΒ τῇ ΔΒ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΓΔΒ γωνίᾳ τῇ ὑπὸ ΔΓΒ. ἐδείχθη δὲ αὐτῆς καὶ πολλῷ μείζων· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις· ὅπερ ἔδει δεῖξαι.

Proposition 7

On the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines.



For, if possible, let the two straight-lines *AC*, *CB*, equal to two other straight-lines *AD*, *DB*, respectively, be constructed on the same straight-line *AB*, meeting at different points, *C* and *D*, on the same side (of *AB*), and having the same ends (on *AB*). So *CA* is equal to *DA*, having the same end *A* as it, and *CB* is equal to *DB*, having the same end *B* as it. And let *CD* be joined [Post. 1].

Therefore, since *AC* is equal to *AD*, the angle *ACD* is also equal to angle *ADC* [Prop. 1.5]. Thus, *ADC* (is) greater than *DCB* [C.N. 5]. Thus, *CDB* is much greater than *DCB* [C.N. 5]. Again, since *CB* is equal to *DB*, the angle *CDB* is also equal to angle *DCB* [Prop. 1.5]. But it was shown that the former (angle) is also much greater (than the latter). The very thing is impossible.

Thus, on the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines. (Which is) the very thing it was required to show.

η'.

Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, ἔχῃ δὲ καὶ τὴν βάσιν τῇ βάσει ἴσην, καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.

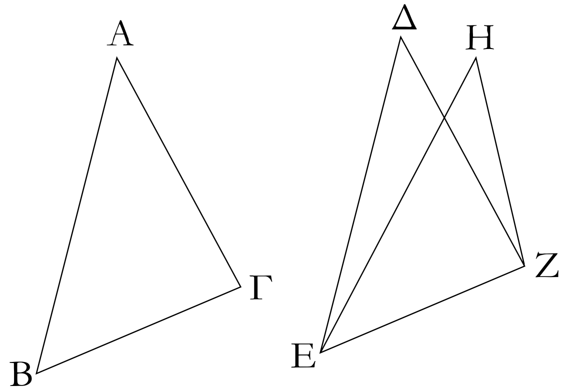
Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς

Proposition 8

If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, (then) they will also have equal the angles encompassed by the equal straight-lines.

Let *ABC* and *DEF* be two triangles having the two sides

ΑΒ, ΑΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ· ἐχέτω δὲ καὶ βάσιν τὴν ΒΓ βάσει τῇ ΕΖ ἴσην· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ΒΑΓ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση.



Ἐφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Β σημείου ἐπὶ τὸ Ε σημεῖον τῆς δὲ ΒΓ εὐθείας ἐπὶ τὴν ΕΖ ἐφαρμόσει καὶ τὸ Γ σημεῖον ἐπὶ τὸ Ζ διὰ τὸ ἴσην εἶναι τὴν ΒΓ τῇ ΕΖ· ἐφαρμοσάσης δὴ τῆς ΒΓ ἐπὶ τὴν ΕΖ ἐφαρμόσουσι καὶ αἱ ΒΑ, ΓΑ ἐπὶ τὰς ΕΔ, ΔΖ. εἰ γὰρ βάσις μὲν ἡ ΒΓ ἐπὶ βάσιν τὴν ΕΖ ἐφαρμόσει, αἱ δὲ ΒΑ, ΑΓ πλευραὶ ἐπὶ τὰς ΕΔ, ΔΖ οὐκ ἐφαρμόσουσιν ἀλλὰ παραλλάξουσιν ὡς αἱ ΕΗ, ΗΖ, συσταθήσονται ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι. οὐ συνίστανται δέ· οὐκ ἄρα ἐφαρμοζομένης τῆς ΒΓ βάσεως ἐπὶ τὴν ΕΖ βάσιν οὐκ ἐφαρμόσουσι καὶ αἱ ΒΑ, ΑΓ πλευραὶ ἐπὶ τὰς ΕΔ, ΔΖ. ἐφαρμόσουσιν ἄρα· ὥστε καὶ γωνία ἡ ὑπὸ ΒΑΓ ἐπὶ γωνίαν τὴν ὑπὸ ΕΔΖ ἐφαρμόσει καὶ ἴση αὐτῇ ἔσται.

Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν βάσιν τῇ βάσει ἴσην ἔχῃ, καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην· ὅπερ ἔδει δεῖξαι.

θʹ.

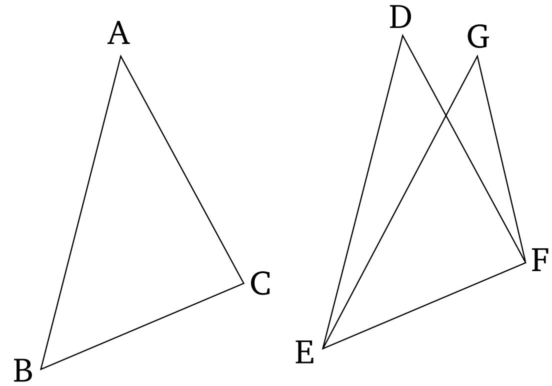
Τὴν δοθεῖσαν γωνίαν εὐθύγραμμον δίχα τεμεῖν.

Ἔστω ἡ δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΒΑΓ. δεῖ δὴ αὐτὴν δίχα τεμεῖν.

Εἰλήφθω ἐπὶ τῆς ΑΒ τυχὸν σημεῖον τὸ Δ, καὶ ἀφῃρήσθω ἀπὸ τῆς ΑΓ τῇ ΑΔ ἴση ἡ ΑΕ, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ συνεστάτω ἐπὶ τῆς ΔΕ τρίγωνον ἰσόπλευρον τὸ ΔΕΖ, καὶ ἐπεζεύχθω ἡ ΑΖ· λέγω, ὅτι ἡ ὑπὸ ΒΑΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΖ εὐθείας.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΔ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΖ, δύο δὴ αἱ ΔΑ, ΑΖ δυσὶ ταῖς ΕΑ, ΑΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. καὶ βάσις ἡ ΔΖ βάσει τῇ ΕΖ ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ ΔΑΖ

AB and *AC* equal to the two sides *DE* and *DF*, respectively. (That is) *AB* to *DE*, and *AC* to *DF*. Let them also have the base *BC* equal to the base *EF*. I say that the angle *BAC* is also equal to the angle *EDF*.



For if triangle *ABC* is applied to triangle *DEF*, the point *B* being placed on point *E*, and the straight-line *BC* on *EF*, (then) point *C* will also coincide with *F*, on account of *BC* being equal to *EF*. So (because of) *BC* coinciding with *EF*, (the sides) *BA* and *CA* will also coincide with *ED* and *DF* (respectively). For if base *BC* coincides with base *EF*, but the sides *AB* and *AC* do not coincide with *ED* and *DF* (respectively), but miss like *EG* and *GF* (in the above figure), then we will have constructed upon the same straight-line two other straight-lines equal, respectively, to two (given) straight-lines, and (meeting) at a different point on the same side (of the straight-line), but having the same ends. But (such straight-lines) cannot be constructed [Prop. 1.7]. Thus, the base *BC* being applied to the base *EF*, the sides *BA* and *AC* cannot not coincide with *ED* and *DF* (respectively). Thus, they will coincide. So the angle *BAC* will also coincide with angle *EDF*, and will be equal to it [C.N. 4].

Thus, if two triangles have two sides equal to two side, respectively, and have the base equal to the base, (then) they will also have equal the angles encompassed by the equal straight-lines. (Which is) the very thing it was required to show.

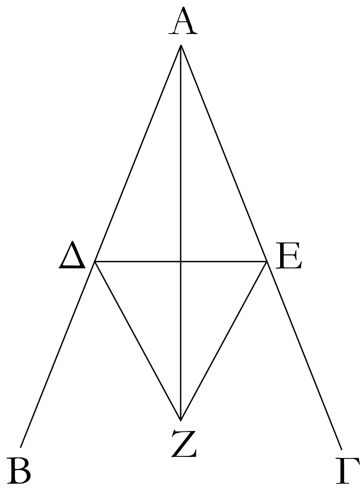
Proposition 9

To cut a given rectilinear angle in half.

Let *BAC* be the given rectilinear angle. So it is required to cut it in half.

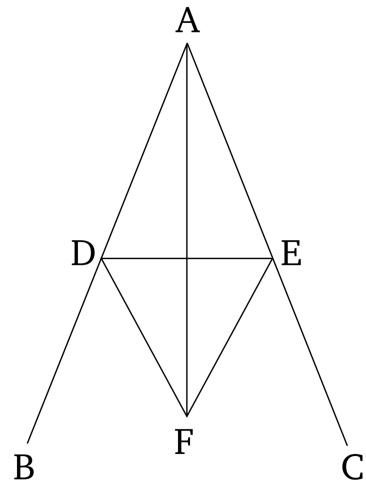
Let the point *D* be taken at random on *AB*, and let *AE*, equal to *AD*, be cut off from *AC* [Prop. 1.3], and let *DE* be joined. And let the equilateral triangle *DEF* be constructed upon *DE* [Prop. 1.1], and let *AF* be joined. I say that the angle *BAC* has been cut in half by the straight-line *AF*.

For since *AD* is equal to *AE*, and *AF* is common, the two (straight-lines) *DA*, *AF* are equal to the two (straight-lines) *EA*, *AF*, respectively. And the base *DF* is equal to the base *γωνίᾳ τῇ ὑπὸ ΕΑΖ ἴση ἐστίν.*



Ἡ ἄρα δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΒΑΓ δίχα τέτμηται ὑπὸ τῆς ΑΖ εὐθείας· ὅπερ ἔδει ποιῆσαι.

EF. Thus, angle DAF is equal to angle EAF [Prop. 1.8].



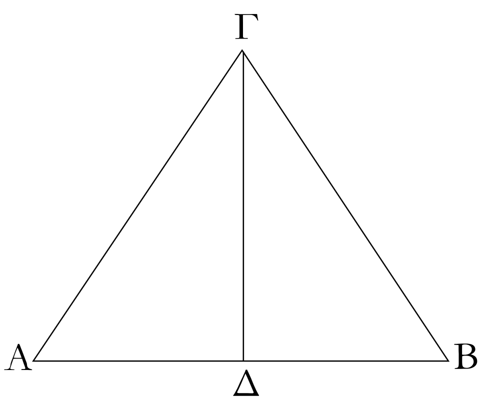
Thus, the given rectilinear angle BAC has been cut in half by the straight-line AF. (Which is) the very thing it was required to do.

ι'.

Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην δίχα τεμεῖν.

Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ εὐθεῖαν πεπερασμένην δίχα τεμεῖν.

Συνεστάτω ἐπ᾽ αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΑΒΓ, καὶ τετμήσθω ἡ ὑπὸ ΑΓΒ γωνία δίχα τῇ ΓΔ εὐθείᾳ· λέγω, ὅτι ἡ ΑΒ εὐθεῖα δίχα τέτμηται κατὰ τὸ Δ σημεῖον.



Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, κοινὴ δὲ ἡ ΓΔ, δύο δὴ αἱ ΑΓ, ΓΔ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΓΔ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση ἐστίν· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΔ ἴση ἐστίν.

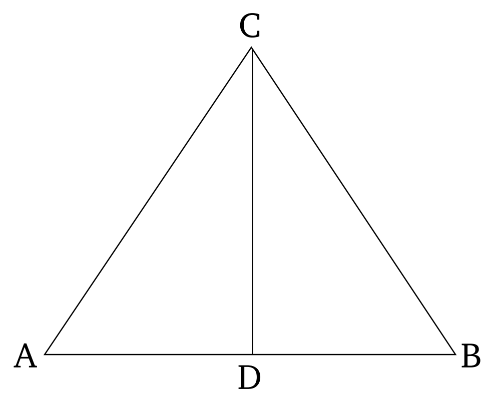
Ἡ ἄρα δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ δίχα τέτμηται κατὰ τὸ Δ· ὅπερ ἔδει ποιῆσαι.

Proposition 10

To cut a given finite straight-line in half.

Let AB be the given finite straight-line. So it is required to cut the finite straight-line AB in half.

Let the equilateral triangle ABC be constructed upon (AB) [Prop. 1.1], and let the angle ACB be cut in half by the straight-line CD [Prop. 1.9]. I say that the straight-line AB has been cut in half at point D.



For since AC is equal to CB, and CD (is) common, the two (straight-lines) AC, CD are equal to the two (straight-lines) BC, CD, respectively. And the angle ACD is equal to the angle BCD. Thus, the base AD is equal to the base BD [Prop. 1.4].

Thus, the given finite straight-line AB has been cut in half at (point) D. (Which is) the very thing it was required to do.

ια'.

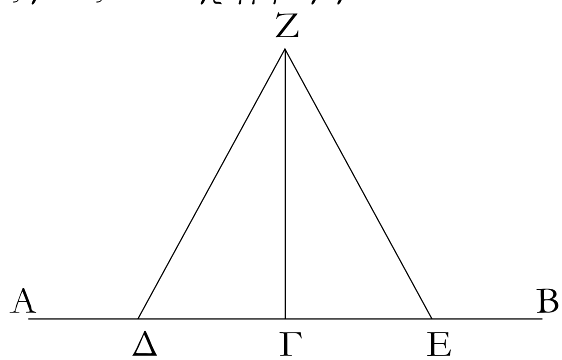
Τῇ δοθείσῃ εὐθείᾳ ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου

Proposition 11

To draw a straight-line at right-angles to a given straight-

πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ τὸ δὲ δοθὲν σημεῖον ἐπ᾽ αὐτῆς τὸ Γ· δεῖ δὴ ἀπὸ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν.



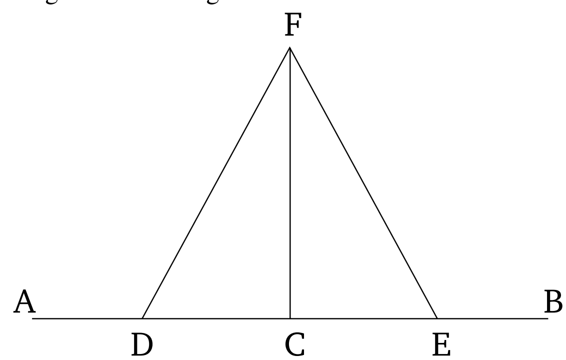
Εἰλήφθω ἐπὶ τῆς ΑΓ τυχὸν σημεῖον τὸ Δ, καὶ κείσθω τῇ ΓΔ ἴση ἡ ΓΕ, καὶ συνεστάτω ἐπὶ τῆς ΔΕ τρίγωνον ἰσόπλευρον τὸ ΖΔΕ, καὶ ἐπεζεύχθω ἡ ΖΓ· λέγω, ὅτι τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἦκται ἡ ΖΓ.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΔΓ τῇ ΓΕ, κοινὴ δὲ ἡ ΓΖ, δύο δὴ αἱ ΔΓ, ΓΖ δυσὶ ταῖς ΕΓ, ΓΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΔΖ βάσει τῇ ΖΕ ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ ΔΓΖ γωνίᾳ τῇ ὑπὸ ΕΓΖ ἴση ἐστίν· καί εἰσιν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΔΓΖ, ΖΓΕ.

Τῇ ἄρα δοθείσῃ εὐθείᾳ τῇ ΑΒ ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἦκται ἡ ΓΖ· ὅπερ ἔδει ποιῆσαι.

line from a given point on it.

Let AB be the given straight-line, and C the given point on it. So it is required to draw a straight-line from the point C at right-angles to the straight-line AB.



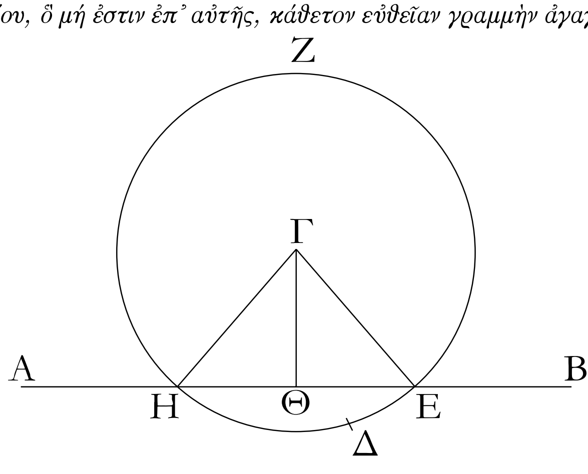
Let the point D be be taken at random on AC, and let CE be made equal to CD [Prop. 1.3], and let the equilateral triangle FDE be constructed on DE [Prop. 1.1], and let FC be joined. I say that the straight-line FC has been drawn at right-angles to the given straight-line AB from the given point C on it.

For since DC is equal to CE, and CF is common, the two (straight-lines) DC, CF are equal to the two (straight-lines), EC, CF, respectively. And the base DF is equal to the base FE. Thus, the angle DCF is equal to the angle ECF [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, each of the (angles) DCF and FCE is a right-angle.

Thus, the straight-line CF has been drawn at right-angles to the given straight-line AB from the given point C on it. (Which is) the very thing it was required to do.

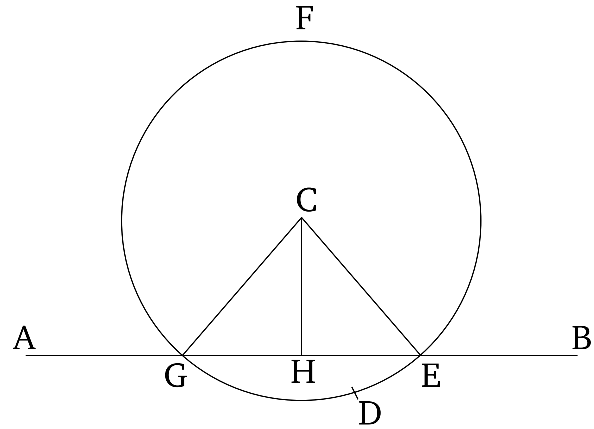
ιβ΄.

Ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον ἀπὸ τοῦ δοθέντος σημείου, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.



Proposition 12

To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.



Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἄπειρος ἡ ΑΒ τὸ δὲ δοθὲν σημεῖον, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, τὸ Γ· δεῖ δὴ ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν ΑΒ ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω γὰρ ἐπὶ τὰ ἕτερα μέρη τῆς ΑΒ εὐθείας τυχὸν σημεῖον τὸ Δ, καὶ κέντρῳ μὲν τῷ Γ διαστήματι δὲ τῷ ΓΔ κύκλος γεγράφθω ὁ ΕΖΗ, καὶ τετμήσθω ἡ ΕΗ εὐθεῖα δίχα κατὰ τὸ Θ, καὶ ἐπεζεύχθωσαν αἱ ΓΗ, ΓΘ, ΓΕ εὐθεῖαι· λέγω, ὅτι ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν ΑΒ ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετος ἦκται ἡ ΓΘ.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΗΘ τῇ ΘΕ, κοινὴ δὲ ἡ ΘΓ, δύο δὴ αἱ ΗΘ, ΘΓ δύο ταῖς ΕΘ, ΘΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΓΗ βάσει τῇ ΓΕ ἐστιν ἴση· γωνία ἄρα ἡ ὑπὸ ΓΘΗ γωνίᾳ τῇ ὑπὸ ΕΘΓ ἐστιν ἴση. καί εἰσιν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται ἐφ᾽ ἣν ἐφέστηκεν.

Ἐπὶ τὴν δοθεῖσαν ἄρα εὐθεῖαν ἄπειρον τὴν ΑΒ ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ, ὃ μή ἐστιν ἐπ᾽ αὐτῆς, κάθετος ἦκται ἡ ΓΘ· ὅπερ ἔδει ποιῆσαι.

Let *AB* be the given infinite straight-line and *C* the given point, which is not on it. So it is required to draw a straight-line perpendicular to the given infinite straight-line *AB* from the given point *C*, which is not on it.

For let point *D* be taken at random on the other side (to *C*) of the straight-line *AB*, and let the circle *EFG* be drawn with center *C* and radius *CD* [Post. 3], and let the straight-line *EG* be cut in half at (point) *H* [Prop. 1.10], and let the straight-lines *CG*, *CH*, and *CE* be joined. I say that the (straight-line) *CH* has been drawn perpendicular to the given infinite straight-line *AB* from the given point *C*, which is not on it.

For since *GH* is equal to *HE*, and *HC* (is) common, the two (straight-lines) *GH*, *HC* are equal to the two (straight-lines) *EH*, *HC*, respectively, and the base *CG* is equal to the base *CE*. Thus, the angle *CHG* is equal to the angle *EHC* [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands [Def. 1.10].

Thus, the (straight-line) *CH* has been drawn perpendicular to the given infinite straight-line *AB* from the given point *C*, which is not on it. (Which is) the very thing it was required to do.

ιγ'.

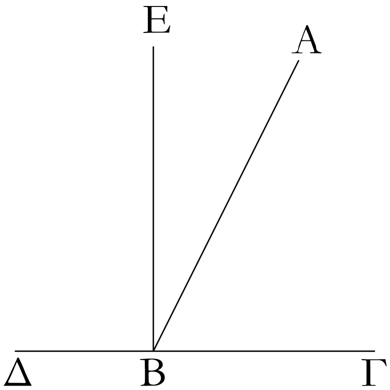
Ἐὰν εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα γωνίας ποιῇ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσει.

Εὐθεῖα γάρ τις ἡ ΑΒ ἐπ᾽ εὐθεῖαν τὴν ΓΔ σταθεῖσα γωνίας ποιείτω τὰς ὑπὸ ΓΒΑ, ΑΒΔ· λὲγω, ὅτι αἱ ὑπὸ ΓΒΑ, ΑΒΔ γωνίαι ἤτοι δύο ὀρθαί εἰσιν ἢ δυσὶν ὀρθαῖς ἴσαι.

Proposition 13

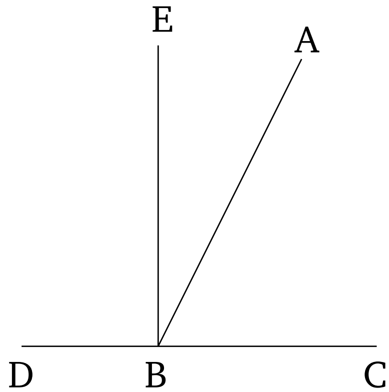
If a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles.

For let some straight-line *AB* stood on the straight-line *CD* make the angles *CBA* and *ABD*. I say that the angles *CBA* and *ABD* are certainly either two right-angles, or (have a sum) equal to two right-angles.



Εἰ μὲν οὖν ἴση ἐστὶν ἡ ὑπὸ ΓΒΑ τῇ ὑπὸ ΑΒΔ, δύο ὀρθαί εἰσιν. εἰ δὲ οὔ, ἤχθω ἀπὸ τοῦ Β σημείου τῇ ΓΔ [εὐθείᾳ] πρὸς ὀρθὰς ἡ ΒΕ· αἱ ἄρα ὑπὸ ΓΒΕ, ΕΒΔ δύο ὀρθαί εἰσιν· καὶ ἐπεὶ ἡ ὑπὸ ΓΒΕ δυσὶ ταῖς ὑπὸ ΓΒΑ, ΑΒΕ ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ ΕΒΔ· αἱ ἄρα ὑπὸ ΓΒΕ, ΕΒΔ τρισὶ ταῖς ὑπὸ ΓΒΑ, ΑΒΕ, ΕΒΔ ἴσαι εἰσίν. πάλιν, ἐπεὶ ἡ ὑπὸ ΔΒΑ δυσὶ ταῖς ὑπὸ ΔΒΕ, ΕΒΑ ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· αἱ ἄρα ὑπὸ ΔΒΑ, ΑΒΓ τρισὶ ταῖς ὑπὸ ΔΒΕ, ΕΒΑ, ΑΒΓ ἴσαι εἰσίν. ἐδείχθησαν δὲ καὶ αἱ ὑπὸ ΓΒΕ, ΕΒΔ τρισὶ ταῖς αὐταῖς ἴσαι· τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ αἱ ὑπὸ ΓΒΕ, ΕΒΔ ἄρα ταῖς ὑπὸ ΔΒΑ, ΑΒΓ ἴσαι εἰσίν· ἀλλὰ αἱ ὑπὸ ΓΒΕ, ΕΒΔ δύο ὀρθαί εἰσιν· καὶ αἱ ὑπὸ ΔΒΑ, ΑΒΓ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἐὰν ἄρα εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα γωνίας ποιῇ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσει· ὅπερ ἔδει δεῖξαι.



In fact, if *CBA* is equal to *ABD* (then) they are two right-angles [Def. 1.10]. But, if not, let *BE* be drawn from the point *B* at right-angles to [the straight-line] *CD* [Prop. 1.11]. Thus, *CBE* and *EBD* are two right-angles. And since *CBE* is equal to the two (angles) *CBA* and *ABE*, let *EBD* be added to both. Thus, the (sum of the angles) *CBE* and *EBD* is equal to the (sum of the) three (angles) *CBA*, *ABE*, and *EBD* [C.N. 2]. Again, since *DBA* is equal to the two (angles) *DBE* and *EBA*, let *ABC* be added to both. Thus, the (sum of the angles) *DBA* and *ABC* is equal to the (sum of the) three (angles) *DBE*, *EBA*, and *ABC* [C.N. 2]. But (the sum of) *CBE* and *EBD* was also shown (to be) equal to the (sum of the) same three (angles). And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, (the sum of) *CBE* and *EBD* is also equal to (the sum of) *DBA* and *ABC*. But, (the sum of) *CBE* and *EBD* is two right-angles. Thus, (the sum of) *ABD* and *ABC* is also equal to two right-angles.

Thus, if a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles. (Which is) the very thing it was required to show.

ιδ΄.

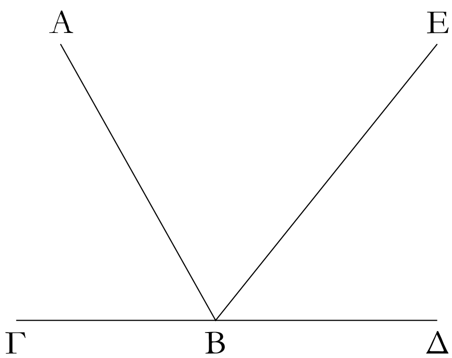
Ἐὰν πρός τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ δύο εὐθεῖαι μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ᾽ εὐθείας ἔσονται ἀλλήλαις αἱ εὐθεῖαι.

Πρὸς γάρ τινι εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β δύο εὐθεῖαι αἱ ΒΓ, ΒΔ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας τὰς ὑπὸ ΑΒΓ, ΑΒΔ δύο ὀρθαῖς ἴσας ποιείτωσαν· λέγω, ὅτι ἐπ᾽ εὐθείας ἐστὶ τῇ ΓΒ ἡ ΒΔ.

Proposition 14

If two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, (then) the two straight-lines will be straight-on (with respect) to one another.

For let two straight-lines *BC* and *BD*, not lying on the same side, make adjacent angles *ABC* and *ABD* (whose sum is) equal to two right-angles with some straight-line *AB*, at the point *B* on it. I say that *BD* is straight-on with respect to *CB*.



Εἰ γὰρ μὴ ἔστι τῇ ΒΓ ἐπ᾽ εὐθείας ἡ ΒΔ, ἔστω τῇ ΓΒ ἐπ᾽ εὐθείας ἡ ΒΕ.

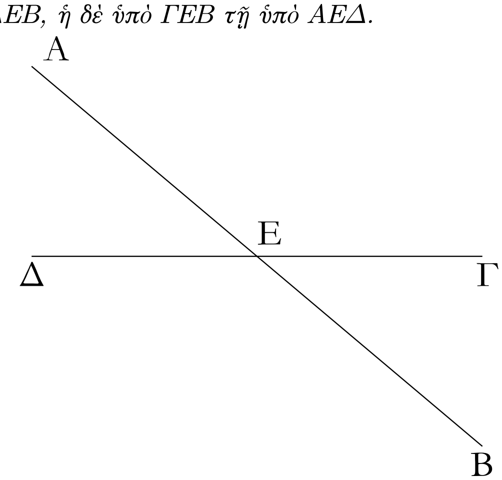
Ἐπεὶ οὖν εὐθεῖα ἡ ΑΒ ἐπ᾽ εὐθεῖαν τὴν ΓΒΕ ἐφέστηκεν, αἱ ἄρα ὑπὸ ΑΒΓ, ΑΒΕ γωνίαι δύο ὀρθαῖς ἴσαι εἰσίν· εἰσὶ δὲ καὶ αἱ ὑπὸ ΑΒΓ, ΑΒΔ δύο ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΓΒΑ, ΑΒΕ ταῖς ὑπὸ ΓΒΑ, ΑΒΔ ἴσαι εἰσίν. κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΓΒΑ· λοιπὴ ἄρα ἡ ὑπὸ ΑΒΕ λοιπῇ τῇ ὑπὸ ΑΒΔ ἐστιν ἴση, ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐπ᾽ εὐθείας ἐστὶν ἡ ΒΕ τῇ ΓΒ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλη τις πλὴν τῆς ΒΔ· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΓΒ τῇ ΒΔ.

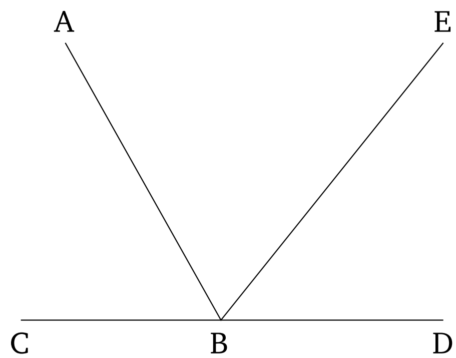
Ἐὰν ἄρα πρός τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ δύο εὐθεῖαι μὴ ἐπὶ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ᾽ εὐθείας ἔσονται ἀλλήλαις αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

ιε΄.

Ἐὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφὴν γωνίας ἴσας ἀλλήλαις ποιοῦσιν.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον· λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΕΓ γωνία τῇ ὑπὸ ΔΕΒ, ἡ δὲ ὑπὸ ΓΕΒ τῇ ὑπὸ ΑΕΔ.





For if BD is not straight-on to BC (then) let BE be straight-on to CB.

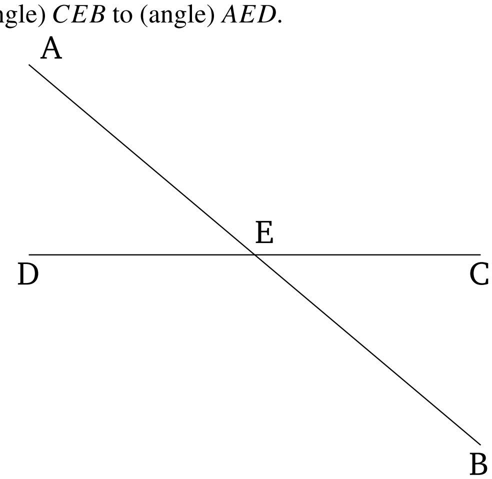
Therefore, since the straight-line AB stands on the straight-line CBE, the (sum of the) angles ABC and ABE is thus equal to two right-angles [Prop. 1.13]. But (the sum of) ABC and ABD is also equal to two right-angles. Thus, (the sum of angles) CBA and ABE is equal to (the sum of angles) CBA and ABD [C.N. 1]. Let (angle) CBA be subtracted from both. Thus, the remainder ABE is equal to the remainder ABD [C.N. 3], the lesser to the greater. The very thing is impossible. Thus, BE is not straight-on with respect to CB. Similarly, we can show that neither (is) any other (straight-line) than BD. Thus, CB is straight-on with respect to BD.

Thus, if two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, (then) the two straight-lines will be straight-on (with respect) to one another. (Which is) the very thing it was required to show.

Proposition 15

If two straight-lines cut one another (then) they make the vertically opposite angles equal to one another.

For let the two straight-lines AB and CD cut one another at the point E. I say that angle AEC is equal to (angle) DEB, and (angle) CEB to (angle) AED.



Ἐπεὶ γὰρ εὐθεῖα ἡ ΑΕ ἐπ᾿ εὐθεῖαν τὴν ΓΔ ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ ΓΕΑ, ΑΕΔ, αἱ ἄρα ὑπὸ ΓΕΑ, ΑΕΔ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. πάλιν, ἐπεὶ εὐθεῖα ἡ ΔΕ ἐπ᾿ εὐθεῖαν τὴν ΑΒ ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ ΑΕΔ, ΔΕΒ, αἱ ἄρα ὑπὸ ΑΕΔ, ΔΕΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ἐδείχθησαν δὲ καὶ αἱ ὑπὸ ΓΕΑ, ΑΕΔ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΓΕΑ, ΑΕΔ ταῖς ὑπὸ ΑΕΔ, ΔΕΒ ἴσαι εἰσίν. κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΑΕΔ· λοιπὴ ἄρα ἡ ὑπὸ ΓΕΑ λοιπῇ τῇ ὑπὸ ΒΕΔ ἴση ἐστίν· ὁμοίως δὴ δειχθήσεται, ὅτι καὶ αἱ ὑπὸ ΓΕΒ, ΔΕΑ ἴσαι εἰσίν.

Ἐὰν ἄρα δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφὴν γωνίας ἴσας ἀλλήλαις ποιοῦσιν· ὅπερ ἔδει δεῖξαι.

For since the straight-line AE stands on the straight-line CD, making the angles CEA and AED, the (sum of the) angles CEA and AED is thus equal to two right-angles [Prop. 1.13]. Again, since the straight-line DE stands on the straight-line AB, making the angles AED and DEB, the (sum of the) angles AED and DEB is thus equal to two right-angles [Prop. 1.13]. But (the sum of) CEA and AED was also shown (to be) equal to two right-angles. Thus, (the sum of) CEA and AED is equal to (the sum of) AED and DEB [C.N. 1]. Let AED be subtracted from both. Thus, the remainder CEA is equal to the remainder BED [C.N. 3]. Similarly, it can be shown that CEB and DEA are also equal.

Thus, if two straight-lines cut one another (then) they make the vertically opposite angles equal to one another. (Which is) the very thing it was required to show.

ις΄.

Proposition 16

Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν μείζων ἐστίν.

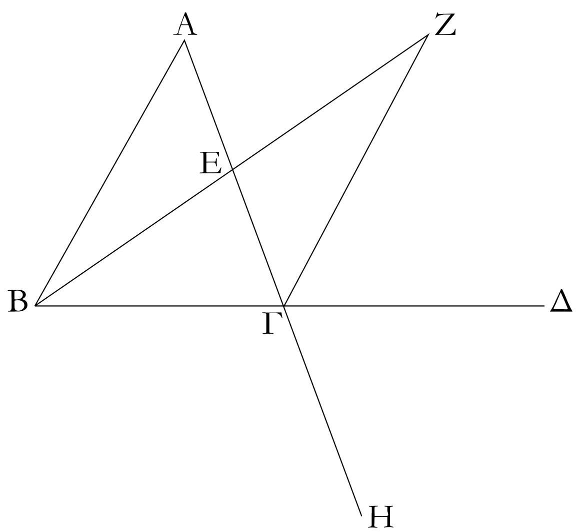
Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ μείζων ἐστὶν ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον τῶν ὑπὸ ΓΒΑ, ΒΑΓ γωνιῶν.

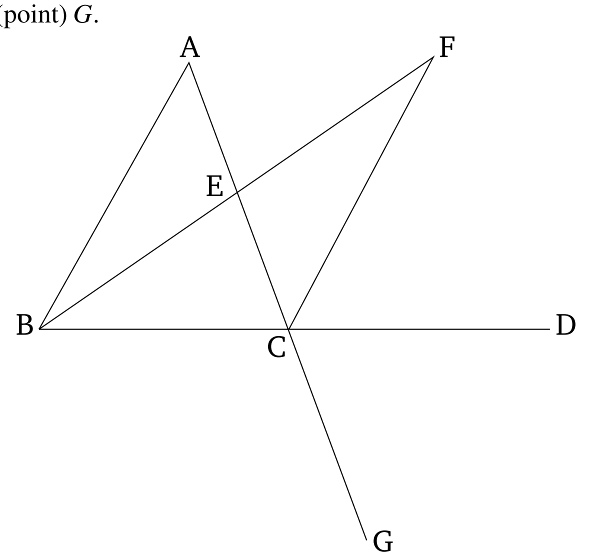
Τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Ε, καὶ ἐπιζευχθεῖσα ἡ ΒΕ ἐκβεβλήσθω ἐπ᾿ εὐθείας ἐπὶ τὸ Ζ, καὶ κείσθω τῇ ΒΕ ἴση ἡ ΕΖ, καὶ ἐπεζεύχθω ἡ ΖΓ, καὶ διήχθω ἡ ΑΓ ἐπὶ τὸ Η.

For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

Let ABC be a triangle, and let one of its sides BC be produced to D. I say that the external angle ACD is greater than each of the internal and opposite angles, CBA and BAC.

Let the (straight-line) AC be cut in half at (point) E [Prop. 1.10]. And BE being joined, let it be produced in a straight-line to (point) F.† And let EF be made equal to BE [Prop. 1.3], and let FC be joined, and let AC be drawn through to (point) G.





Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΕ τῇ ΕΓ, ἡ δὲ ΒΕ τῇ ΕΖ, δύο δὴ αἱ ΑΕ, ΕΒ δυσὶ ταῖς ΓΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΕΒ γωνίᾳ τῇ ὑπὸ ΖΕΓ ἴση ἐστίν· κατὰ κορυφὴν γάρ· βάσις ἄρα ἡ ΑΒ βάσει τῇ ΖΓ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΕΓ τριγώνῳ ἐστὶν ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, ὑφ᾿ ἃς

Therefore, since AE is equal to EC, and BE to EF, the two (straight-lines) AE, EB are equal to the two (straight-lines) CE, EF, respectively. Also, angle AEB is equal to angle FEC, for (they are) vertically opposite [Prop. 1.15]. Thus, the base AB is equal to the base FC, and the triangle ABE is equal to the triangle FEC, and the remaining angles subtended by the *αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΕ τῇ ὑπὸ ΕΓΖ. μείζων δέ ἐστιν ἡ ὑπὸ ΕΓΔ τῆς ὑπὸ ΕΓΖ· μείζων ἄρα ἡ ὑπὸ ΑΓΔ τῆς ὑπὸ ΒΑΕ. Ὁμοίως δὴ τῆς ΒΓ τετμημένης δίχα δειχθήσεται καὶ ἡ ὑπὸ ΒΓΗ, τουτέστιν ἡ ὑπὸ ΑΓΔ, μείζων καὶ τῆς ὑπὸ ΑΒΓ.*

Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

equal sides are equal to the corresponding remaining angles [Prop. 1.4]. Thus, *BAE* is equal to *ECF*. But *ECD* is greater than *ECF*. Thus, *ACD* is greater than *BAE*. Similarly, by having cut *BC* in half, it can be shown (that) *BCG*—that is to say, *ACD*—(is) also greater than *ABC*.

Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles. (Which is) the very thing it was required to show.

† The implicit assumption that the point *F* lies in the interior of the angle *ABC* should be counted as an additional postulate.

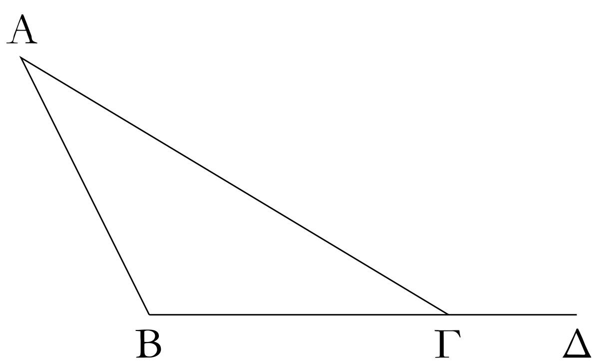
ιζ'.

Παντὸς τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονές εἰσι πάντῃ μεταλαμβανόμεναι.

Ἔστω τρίγωνον τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάττονές εἰσι πάντῃ μεταλαμβανόμεναι.

Ἐκβεβλήσθω γὰρ ἡ ΒΓ ἐπὶ τὸ Δ.

Καὶ ἐπεὶ τριγώνου τοῦ ΑΒΓ ἐκτός ἐστι γωνία ἡ ὑπὸ ΑΓΔ, μείζων ἐστὶ τῆς ἐντὸς καὶ ἀπεναντίον τῆς ὑπὸ ΑΒΓ. κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τῶν ὑπὸ ΑΒΓ, ΒΓΑ μείζονές εἰσιν. ἀλλ' αἱ ὑπὸ ΑΓΔ, ΑΓΒ δύο ὀρθαῖς ἴσαι εἰσίν· αἱ ἄρα ὑπὸ ΑΒΓ, ΒΓΑ δύο ὀρθῶν ἐλάσσονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ὑπὸ ΒΑΓ, ΑΓΒ δύο ὀρθῶν ἐλάσσονές εἰσι καὶ ἔτι αἱ ὑπὸ ΓΑΒ, ΑΒΓ.



Παντὸς ἄρα τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

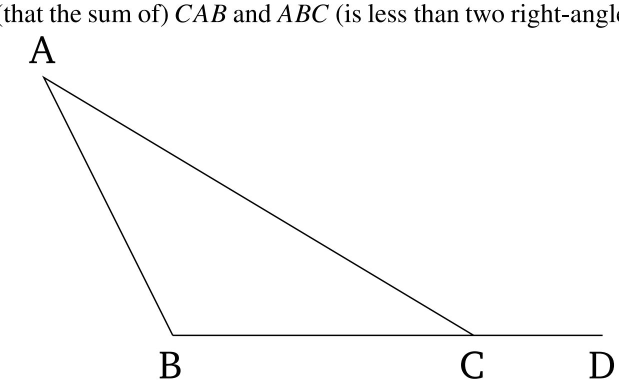
Proposition 17

For any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles.

Let *ABC* be a triangle. I say that (the sum of) two angles of triangle *ABC* taken together in any (possible way) is less than two right-angles.

For let *BC* be produced to *D*.

And since the angle *ACD* is external to triangle *ABC*, it is greater than the internal and opposite angle *ABC* [Prop. 1.16]. Let *ACB* be added to both. Thus, the (sum of the angles) *ACD* and *ACB* is greater than the (sum of the angles) *ABC* and *BCA*. But, (the sum of) *ACD* and *ACB* is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) *ABC* and *BCA* is less than two right-angles. Similarly, we can show that (the sum of) *BAC* and *ACB* is also less than two right-angles, and further (that the sum of) *CAB* and *ABC* (is less than two right-angles).



Thus, for any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles. (Which is) the very thing it was required to show.

ιη'.

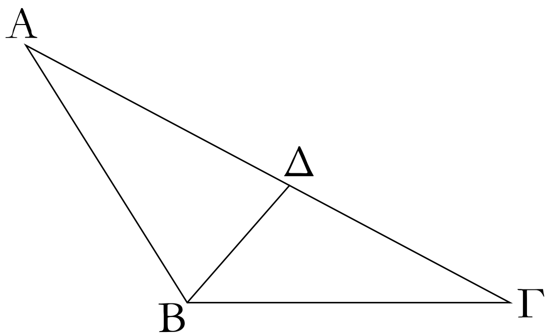
Παντὸς τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει.

Ἔστω γὰρ τρίγωνον τὸ ΑΒΓ μείζονα ἔχον τὴν ΑΓ πλευρὰν τῆς ΑΒ· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ΑΒΓ μείζων ἐστὶ τῆς ὑπὸ ΒΓΑ.

Proposition 18

In any triangle, the greater side subtends the greater angle.

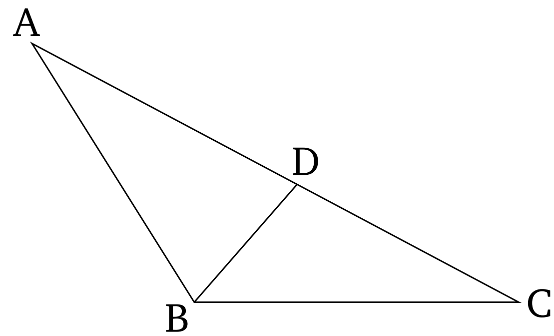
For let *ABC* be a triangle having side *AC* greater than *AB*. I say that angle *ABC* is also greater than *BCA*.



Ἐπεὶ γὰρ μείζων ἐστὶν ἡ ΑΓ τῆς ΑΒ, κείσθω τῇ ΑΒ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΒΔ.

Καὶ ἐπεὶ τριγώνου τοῦ ΒΓΔ ἐκτός ἐστι γωνία ἡ ὑπὸ ΑΔΒ, μείζων ἐστὶ τῆς ἐντὸς καὶ ἀπεναντίον τῆς ὑπὸ ΔΓΒ· ἴση δὲ ἡ ὑπὸ ΑΔΒ τῇ ὑπὸ ΑΒΔ, ἐπεὶ καὶ πλευρὰ ἡ ΑΒ τῇ ΑΔ ἐστιν ἴση· μείζων ἄρα καὶ ἡ ὑπὸ ΑΒΔ τῆς ὑπὸ ΑΓΒ· πολλῷ ἄρα ἡ ὑπὸ ΑΒΓ μείζων ἐστὶ τῆς ὑπὸ ΑΓΒ.

Παντὸς ἄρα τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει· ὅπερ ἔδει δεῖξαι.



For since *AC* is greater than *AB*, let *AD* be made equal to *AB* [Prop. 1.3], and let *BD* be joined.

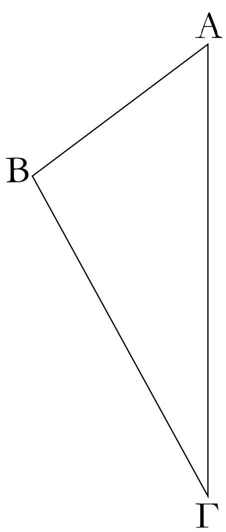
And since angle *ADB* is external to triangle *BCD*, it is greater than the internal and opposite (angle) *DCB* [Prop. 1.16]. But *ADB* (is) equal to *ABD*, since side *AB* is also equal to side *AD* [Prop. 1.5]. Thus, *ABD* is also greater than *ACB*. Thus, *ABC* is much greater than *ACB*.

Thus, in any triangle, the greater side subtends the greater angle. (Which is) the very thing it was required to show.

ιθ΄.

Παντὸς τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει.

Ἔστω τρίγωνον τὸ ΑΒΓ μείζονα ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν τῆς ὑπὸ ΒΓΑ· λέγω, ὅτι καὶ πλευρὰ ἡ ΑΓ πλευρᾶς τῆς ΑΒ μείζων ἐστίν.

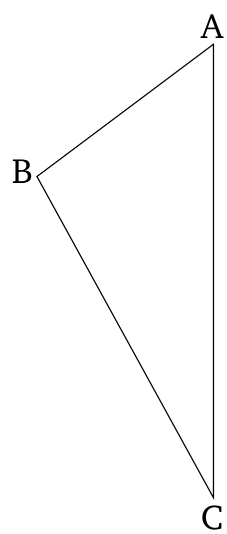


Εἰ γὰρ μή, ἤτοι ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ ἢ ἐλάσσων· ἴση μὲν οὖν οὐκ ἔστιν ἡ ΑΓ τῇ ΑΒ· ἴση γὰρ ἂν ἦν καὶ γωνία ἡ ὑπὸ ΑΒΓ τῇ ὑπὸ ΑΓΒ· οὐκ ἔστι δέ· οὐκ ἄρα ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ. οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ ΑΓ τῆς ΑΒ· ἐλάσσων γὰρ ἂν ἦν καὶ γωνία ἡ ὑπὸ ΑΒΓ τῆς ὑπὸ ΑΓΒ· οὐκ ἔστι δέ· οὐκ ἄρα ἐλάσσων ἐστὶν ἡ ΑΓ τῆς ΑΒ. ἐδείχθη δέ, ὅτι οὐδὲ ἴση ἐστίν. μείζων ἄρα ἐστὶν ἡ ΑΓ τῆς ΑΒ.

Proposition 19

In any triangle, the greater angle is subtended by the greater side.

Let *ABC* be a triangle having the angle *ABC* greater than *BCA*. I say that side *AC* is also greater than side *AB*.



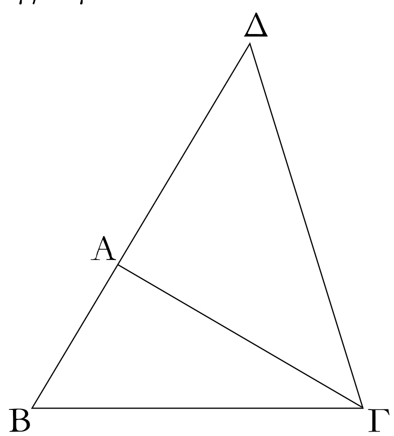
For if not, *AC* is certainly either equal to, or less than, *AB*. In fact, *AC* is not equal to *AB*. For then angle *ABC* would also be equal to *ACB* [Prop. 1.5]. But it is not. Thus, *AC* is not equal to *AB*. Neither, indeed, is *AC* less than *AB*. For then angle *ABC* would also be less than *ACB* [Prop. 1.18]. But it is not. Thus, *AC* is not less than *AB*. But it was shown that (*AC*) is not equal (to *AB*) either. Thus, *AC* is greater than *AB*.

Παντὸς ἄρα τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· ὅπερ ἔδει δεῖξαι.

Thus, in any triangle, the greater angle is subtended by the greater side. (Which is) the very thing it was required to show.

κʹ.

Παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.



Ἔστω γὰρ τρίγωνον τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, αἱ μὲν ΒΑ, ΑΓ τῆς ΒΓ, αἱ δὲ ΑΒ, ΒΓ τῆς ΑΓ, αἱ δὲ ΒΓ, ΓΑ τῆς ΑΒ.

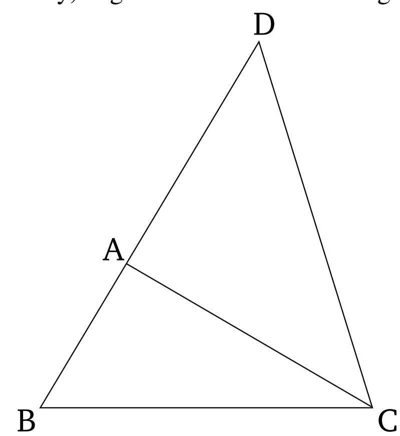
Διήχθω γὰρ ἡ ΒΑ ἐπὶ τὸ Δ σημεῖον, καὶ κείσθω τῇ ΓΑ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΓ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΑ τῇ ΑΓ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΔΓ τῇ ὑπὸ ΑΓΔ· μείζων ἄρα ἡ ὑπὸ ΒΓΔ τῆς ὑπὸ ΑΔΓ· καὶ ἐπεὶ τρίγωνόν ἐστι τὸ ΔΓΒ μείζονα ἔχον τὴν ὑπὸ ΒΓΔ γωνίαν τῆς ὑπὸ ΒΔΓ, ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, ἡ ΔΒ ἄρα τῆς ΒΓ ἐστι μείζων. ἴση δὲ ἡ ΔΑ τῇ ΑΓ· μείζονες ἄρα αἱ ΒΑ, ΑΓ τῆς ΒΓ· ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν ΑΒ, ΒΓ τῆς ΓΑ μείζονές εἰσιν, αἱ δὲ ΒΓ, ΓΑ τῆς ΑΒ.

Παντὸς ἄρα τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

Proposition 20

In any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side).



For let *ABC* be a triangle. I say that in triangle *ABC* (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (So), (the sum of) *BA* and *AC* (is greater) than *BC*, (the sum of) *AB* and *BC* than *AC*, and (the sum of) *BC* and *CA* than *AB*.

For let *BA* be drawn through to point *D*, and let *AD* be made equal to *CA* [Prop. 1.3], and let *DC* be joined.

Therefore, since *DA* is equal to *AC*, the angle *ADC* is also equal to *ACD* [Prop. 1.5]. Thus, *BCD* is greater than *ADC*. And since *DCB* is a triangle having the angle *BCD* greater than *BDC*, and the greater angle subtends the greater side [Prop. 1.19], *DB* is thus greater than *BC*. But *DA* is equal to *AC*. Thus, (the sum of) *BA* and *AC* is greater than *BC*. Similarly, we can show that (the sum of) *AB* and *BC* is also greater than *CA*, and (the sum of) *BC* and *CA* than *AB*.

Thus, in any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (Which is) the very thing it was required to show.

καʹ.

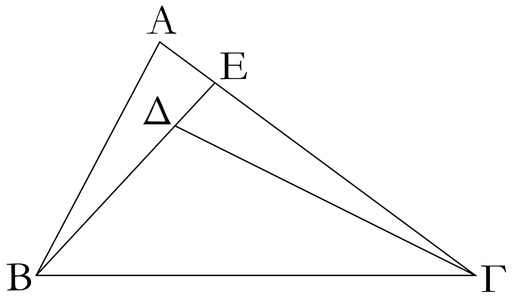
Ἐὰν τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μὲν ἔσονται, μείζονα δὲ γωνίαν περιέξουσιν.

Τριγώνου γὰρ τοῦ ΑΒΓ ἐπὶ μιᾶς τῶν πλευρῶν τῆς ΒΓ ἀπὸ τῶν περάτων τῶν Β, Γ δύο εὐθεῖαι ἐντὸς συνεστάτωσαν αἱ ΒΔ, ΔΓ· λέγω, ὅτι αἱ ΒΔ, ΔΓ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τῶν ΒΑ, ΑΓ ἐλάσσονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσι τὴν ὑπὸ ΒΔΓ τῆς ὑπὸ ΒΑΓ.

Proposition 21

If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle.

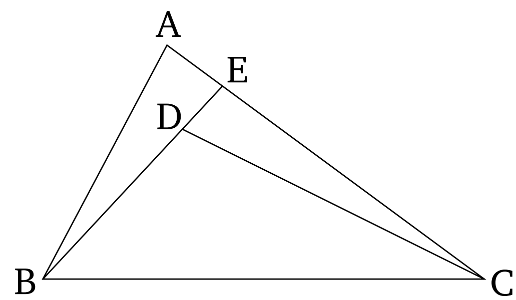
For let the two internal straight-lines *BD* and *DC* be constructed on one of the sides *BC* of the triangle *ABC*, from its ends *B* and *C* (respectively). I say that *BD* and *DC* are less than the (sum of the) two remaining sides of the triangle *BA* and *AC*, but encompass an angle *BDC* greater than *BAC*.



Διήχθω γὰρ ἡ ΒΔ ἐπὶ τὸ Ε. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσιν, τοῦ ΑΒΕ ἄρα τριγώνου αἱ δύο πλευραὶ αἱ ΑΒ, ΑΕ τῆς ΒΕ μείζονές εἰσιν· κοινὴ προσκείσθω ἡ ΕΓ· αἱ ἄρα ΒΑ, ΑΓ τῶν ΒΕ, ΕΓ μείζονές εἰσιν. πάλιν, ἐπεὶ τοῦ ΓΕΔ τριγώνου αἱ δύο πλευραὶ αἱ ΓΕ, ΕΔ τῆς ΓΔ μείζονές εἰσιν, κοινὴ προσκείσθω ἡ ΔΒ· αἱ ΓΕ, ΕΒ ἄρα τῶν ΓΔ, ΔΒ μείζονές εἰσιν. ἀλλὰ τῶν ΒΕ, ΕΓ μείζονες ἐδείχθησαν αἱ ΒΑ, ΑΓ· πολλῷ ἄρα αἱ ΒΑ, ΑΓ τῶν ΒΔ, ΔΓ μείζονές εἰσιν.

Πάλιν, ἐπεὶ παντὸς τριγώνου ἡ ἐκτὸς γωνία τῆς ἐντὸς καὶ ἀπεναντίον μείζων ἐστίν, τοῦ ΓΔΕ ἄρα τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΓΕΔ. διὰ ταὐτὰ τοίνυν καὶ τοῦ ΑΒΕ τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ ΓΕΒ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. ἀλλὰ τῆς ὑπὸ ΓΕΒ μείζων ἐδείχθη ἡ ὑπὸ ΒΔΓ· πολλῷ ἄρα ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ.

Ἐὰν ἄρα τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι.



For let *BD* be drawn through to *E*. And since in any triangle (the sum of any) two sides is greater than the remaining (side) [Prop. 1.20], in triangle *ABE* the (sum of the) two sides *AB* and *AE* is thus greater than *BE*. Let *EC* be added to both. Thus, (the sum of) *BA* and *AC* is greater than (the sum of) *BE* and *EC*. Again, since in triangle *CED* the (sum of the) two sides *CE* and *ED* is greater than *CD*, let *DB* be added to both. Thus, (the sum of) *CE* and *EB* is greater than (the sum of) *CD* and *DB*. But, (the sum of) *BA* and *AC* was shown (to be) greater than (the sum of) *BE* and *EC*. Thus, (the sum of) *BA* and *AC* is much greater than (the sum of) *BD* and *DC*.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle *CDE* the external angle *BDC* is thus greater than *CED*. Accordingly, for the same (reason), the external angle *CEB* of the triangle *ABE* is also greater than *BAC*. But, *BDC* was shown (to be) greater than *CEB*. Thus, *BDC* is much greater than *BAC*.

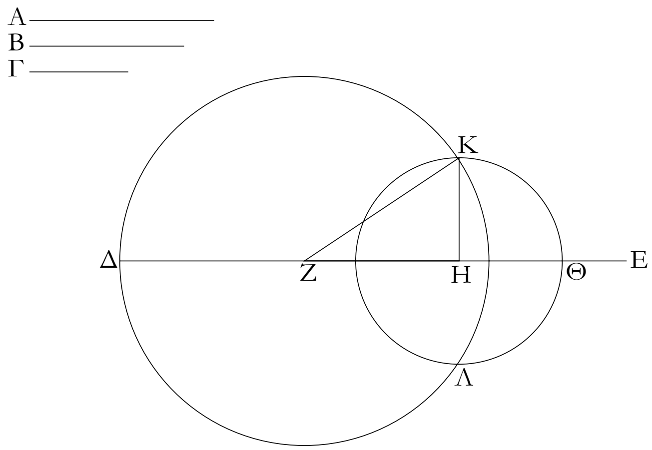
Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

κβ΄.

Ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντη μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντη μεταλαμβανομένας].

Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].

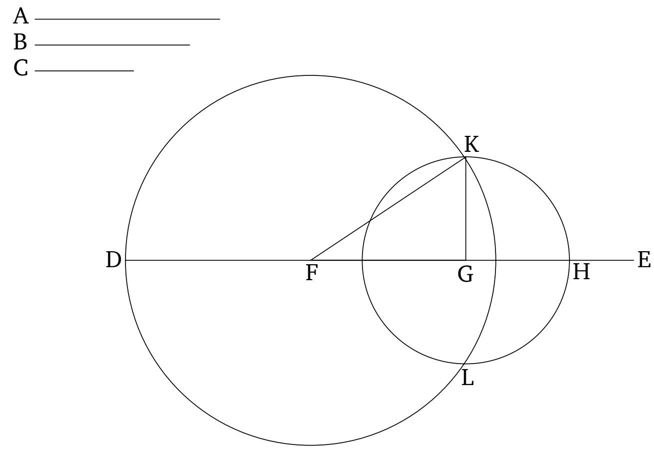


Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι.

Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.

Ἐπεὶ γὰρ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστιν ἴση. καὶ ἡ ΚΖ ἄρα τῇ Α ἐστιν ἴση. πάλιν, ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΛΚΘ κύκλου, ἴση ἐστὶν ἡ ΗΘ τῇ ΗΚ· ἀλλὰ ἡ ΗΘ τῇ Γ ἐστιν ἴση· καὶ ἡ ΚΗ ἄρα τῇ Γ ἐστιν ἴση. ἐστὶ δὲ καὶ ἡ ΖΗ τῇ Β ἴση· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΚΖ, ΖΗ, ΗΚ τρισὶ ταῖς Α, Β, Γ ἴσαι εἰσίν.

Ἐκ τριῶν ἄρα εὐθειῶν τῶν ΚΖ, ΖΗ, ΗΚ, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις εὐθείαις ταῖς Α, Β, Γ, τρίγωνον συνέσταται τὸ ΚΖΗ· ὅπερ ἔδει ποιῆσαι.



Let A, B, and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C, (the sum of) A and C than B, and also (the sum of) B and C than A. So it is required to construct a triangle from (straight-lines) equal to A, B, and C.

Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 1.3]. And let the circle DKL be drawn with center F and radius FD. Again, let the circle KLH be drawn with center G and radius GH. And let KF and KG be joined. I say that the triangle KFG has been constructed from three straight-lines equal to A, B, and C.

For since point F is the center of the circle DKL, FD is equal to FK. But, FD is equal to A. Thus, KF is also equal to A. Again, since point G is the center of the circle LKH, GH is equal to GK. But, GH is equal to C. Thus, KG is also equal to C. And FG is also equal to B. Thus, the three straight-lines KF, FG, and GK are equal to A, B, and C (respectively).

Thus, the triangle KFG has been constructed from the three straight-lines KF, FG, and GK, which are equal to the three given straight-lines A, B, and C (respectively). (Which is) the very thing it was required to do.

κγ΄.

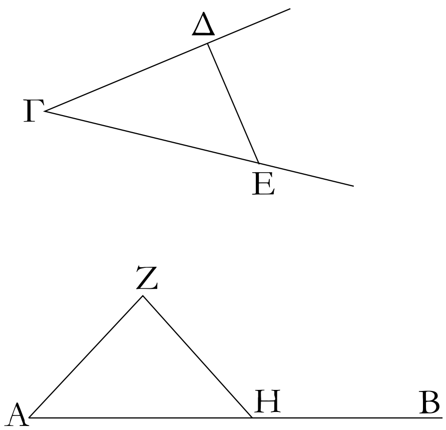
Πρὸς τῇ δοθείσῃ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.

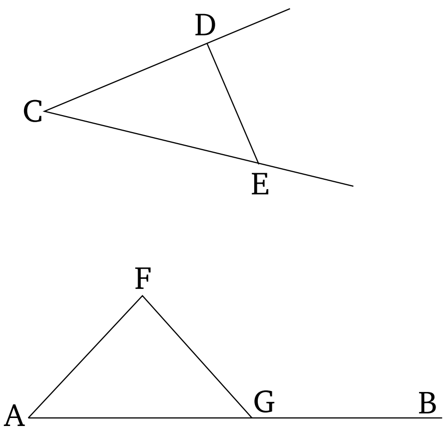
Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ πρὸς αὐτῇ σημεῖον τὸ Α, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΔΓΕ· δεῖ δὴ πρὸς τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ ὑπὸ ΔΓΕ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.

Proposition 23

To construct a rectilinear angle equal to a given rectilinear angle at a (given) point on a given straight-line.

Let AB be the given straight-line, A the (given) point on it, and DCE the given rectilinear angle. So it is required to construct a rectilinear angle equal to the given rectilinear angle DCE at the (given) point A on the given straight-line AB.





Εἰλήφθω ἐφ᾽ ἑκατέρας τῶν ΓΔ, ΓΕ τυχόντα σημεῖα τὰ Δ, Ε, καὶ ἐπεζεύχθω ἡ ΔΕ· καὶ ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς ΓΔ, ΔΕ, ΓΕ, τρίγωνον συνεστάτω τὸ ΑΖΗ, ὥστε ἴσην εἶναι τὴν μὲν ΓΔ τῇ ΑΖ, τὴν δὲ ΓΕ τῇ ΑΗ, καὶ ἔτι τὴν ΔΕ τῇ ΖΗ.

Ἐπεὶ οὖν δύο αἱ ΔΓ, ΓΕ δύο ταῖς ΖΑ, ΑΗ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ ΔΕ βάσει τῇ ΖΗ ἴση, γωνία ἄρα ἡ ὑπὸ ΔΓΕ γωνίᾳ τῇ ὑπὸ ΖΑΗ ἐστιν ἴση.

Πρὸς ἄρα τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ ὑπὸ ΔΓΕ ἴση γωνία εὐθύγραμμος συνέσταται ἡ ὑπὸ ΖΑΗ· ὅπερ ἔδει ποιῆσαι.

Let the points D and E be taken at random on each of the (straight-lines) CD and CE (respectively), and let DE be joined. And let the triangle AFG be constructed from three straight-lines which are equal to CD, DE, and CE, such that CD is equal to AF, CE to AG, and further DE to FG [Prop. 1.22].

Therefore, since the two (straight-lines) DC, CE are equal to the two (straight-lines) FA, AG, respectively, and the base DE is equal to the base FG, the angle DCE is thus equal to the angle FAG [Prop. 1.8].

Thus, the rectilinear angle FAG, equal to the given rectilinear angle DCE, has been constructed at the (given) point A on the given straight-line AB. (Which is) the very thing it was required to do.

κδ΄.

Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἕξει.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς ΑΒ, ΑΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ, ἡ δὲ πρὸς τῷ Α γωνία τῆς πρὸς τῷ Δ γωνίας μείζων ἔστω· λέγω, ὅτι καὶ βάσις ἡ ΒΓ βάσεως τῆς ΕΖ μείζων ἐστίν.

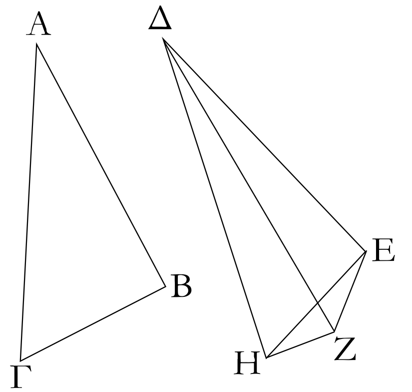
Ἐπεὶ γὰρ μείζων ἡ ὑπὸ ΒΑΓ γωνία τῆς ὑπὸ ΕΔΖ γωνίας, συνεστάτω πρὸς τῇ ΔΕ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Δ τῇ ὑπὸ ΒΑΓ γωνίᾳ ἴση ἡ ὑπὸ ΕΔΗ, καὶ κείσθω ὁποτέρᾳ τῶν ΑΓ, ΔΖ ἴση ἡ ΔΗ, καὶ ἐπεζεύχθωσαν αἱ ΕΗ, ΖΗ.

Proposition 24

If two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), (then the former triangle) will also have a base greater than the base (of the latter).

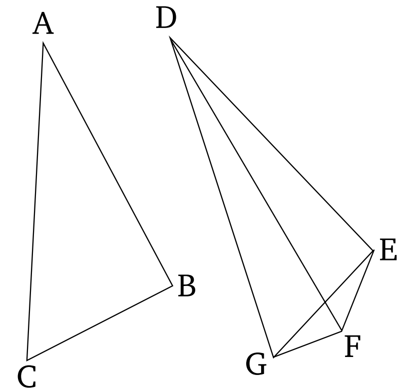
Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is), AB (equal) to DE, and AC to DF. Let them also have the angle at A greater than the angle at D. I say that the base BC is also greater than the base EF.

For since angle BAC is greater than angle EDF, let (angle) EDG, equal to angle BAC, be constructed at the point D on the straight-line DE [Prop. 1.23]. And let DG be made equal to either of AC or DF [Prop. 1.3], and let EG and FG be joined.



Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΒ τῇ ΔΕ, ἡ δὲ ΑΓ τῇ ΔΗ, δύο δὴ αἱ ΒΑ, ΑΓ δυσὶ ταῖς ΕΔ, ΔΗ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΒΑΓ γωνίᾳ τῇ ὑπὸ ΕΔΗ ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΗ ἐστιν ἴση. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΔΖ τῇ ΔΗ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΔΗΖ γωνία τῇ ὑπὸ ΔΖΗ· μείζων ἄρα ἡ ὑπὸ ΔΖΗ τῆς ὑπὸ ΕΗΖ· πολλῷ ἄρα μείζων ἐστὶν ἡ ὑπὸ ΕΖΗ τῆς ὑπὸ ΕΗΖ. καὶ ἐπεὶ τρίγωνόν ἐστι τὸ ΕΖΗ μείζονα ἔχον τὴν ὑπὸ ΕΖΗ γωνίαν τῆς ὑπὸ ΕΗΖ, ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, μείζων ἄρα καὶ πλευρὰ ἡ ΕΗ τῆς ΕΖ. ἴση δὲ ἡ ΕΗ τῇ ΒΓ· μείζων ἄρα καὶ ἡ ΒΓ τῆς ΕΖ.

Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἕξει· ὅπερ ἔδει δεῖξαι.



Therefore, since AB is equal to DE and AC to DG, the two (straight-lines) BA, AC are equal to the two (straight-lines) ED, DG, respectively. Also the angle BAC is equal to the angle EDG. Thus, the base BC is equal to the base EG [Prop. 1.4]. Again, since DF is equal to DG, angle DGF is also equal to angle DFG [Prop. 1.5]. Thus, DFG (is) greater than EGF. Thus, EFG is much greater than EGF. And since triangle EFG has angle EFG greater than EGF, and the greater angle is subtended by the greater side [Prop. 1.19], side EG (is) thus also greater than EF. But EG (is) equal to BC. Thus, BC (is) also greater than EF.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), (then the former triangle) will also have a base greater than the base (of the latter). (Which is) the very thing it was required to show.

κε′.

Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ βάσιν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο πλευρὰς τὰς ΑΒ, ΑΓ ταῖς δύο πλευραῖς ταῖς ΔΕ, ΔΖ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ, τὴν δὲ ΑΓ τῇ ΔΖ· βάσις δὲ ἡ ΒΓ βάσεως τῆς ΕΖ μείζων ἔστω· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ΒΑΓ γωνίας τῆς ὑπὸ ΕΔΖ μείζων ἐστίν.

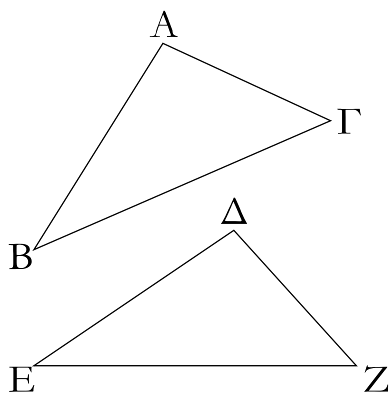
Εἰ γὰρ μή, ἤτοι ἴση ἐστὶν αὐτῇ ἢ ἐλάσσων· ἴση μὲν οὖν οὐκ ἔστιν ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ· ἴση γὰρ ἂν ἦν καὶ βάσις ἡ ΒΓ βάσει τῇ ΕΖ· οὐκ ἔστι δέ. οὐκ ἄρα ἴση ἐστὶ γωνία ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ· οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ ὑπὸ ΒΑΓ τῆς ὑπὸ ΕΔΖ· ἐλάσσων γὰρ ἂν ἦν καὶ βάσις ἡ ΒΓ βάσεως τῆς ΕΖ· οὐκ ἔστι δέ· οὐκ ἄρα ἐλάσσων ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῆς ὑπὸ ΕΔΖ. ἐδείχθη δέ, ὅτι οὐδὲ ἴση· μείζων ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ τῆς ὑπὸ ΕΔΖ.

Proposition 25

If two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), (then the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter).

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively (That is), AB (equal) to DE, and AC to DF. And let the base BC be greater than the base EF. I say that angle BAC is also greater than EDF.

For if not, (BAC) is certainly either equal to, or less than, (EDF). In fact, BAC is not equal to EDF. For then the base BC would also be equal to the base EF [Prop. 1.4]. But it is not. Thus, angle BAC is not equal to EDF. Neither, indeed, is BAC less than EDF. For then the base BC would also be less than the base EF [Prop. 1.24]. But it is not. Thus, angle BAC is not less than EDF. But it was shown that (BAC is) not equal (to EDF) either. Thus, BAC is greater than EDF.



Ἐὰν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ βασίν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἕξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην· ὅπερ ἔδει δεῖξαι.

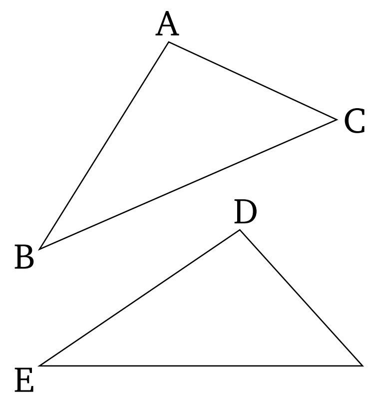
κϛ΄.

Ἐὰν δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην ἤτοι τὴν πρὸς ταῖς ἴσαις γωνίαις ἢ τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει [ἑκατέραν ἑκατέρᾳ] καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς δύο γωνίας τὰς ὑπὸ ΑΒΓ, ΒΓΑ δυσὶ ταῖς ὑπὸ ΔΕΖ, ΕΖΔ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ· ἐχέτω δὲ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην, πρότερον τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΒΓ τῇ ΕΖ· λέγω, ὅτι καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει ἑκατέραν ἑκατέρᾳ, τὴν μὲν ΑΒ τῇ ΔΕ τὴν δὲ ΑΓ τῇ ΔΖ, καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ, τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ.

Εἰ γὰρ ἄνισός ἐστιν ἡ ΑΒ τῇ ΔΕ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ΑΒ, καὶ κείσθω τῇ ΔΕ ἴση ἡ ΒΗ, καὶ ἐπεζεύχθω ἡ ΗΓ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΒΗ τῇ ΔΕ, ἡ δὲ ΒΓ τῇ ΕΖ, δύο δὴ αἱ ΒΗ, ΒΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΗΒΓ γωνίᾳ τῇ ὑπὸ ΔΕΖ ἴση ἐστίν· βάσις ἄρα ἡ ΗΓ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ τὸ ΗΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΗΓΒ γωνία τῇ ὑπὸ ΔΖΕ. ἀλλὰ ἡ ὑπὸ ΔΖΕ τῇ ὑπὸ ΒΓΑ ὑπόκειται ἴση· καὶ ἡ ὑπὸ ΒΓΗ ἄρα τῇ ὑπὸ ΒΓΑ ἴση ἐστίν, ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ΑΒ τῇ ΔΕ. ἴση ἄρα. ἔστι δὲ καὶ ἡ ΒΓ τῇ ΕΖ ἴση· δύο δὴ αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση· βάσις ἄρα ἡ ΑΓ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ λοιπὴ γωνία



Thus, if two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), (then the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter). (Which is) the very thing it was required to show.

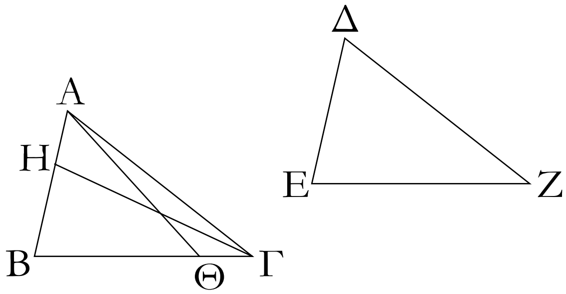
Proposition 26

If two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—(then the triangles) will also have the remaining sides equal to the [corresponding] remaining sides, and the remaining angle (equal) to the remaining angle.

Let ABC and DEF be two triangles having the two angles ABC and BCA equal to the two (angles) DEF and EFD, respectively. (That is) ABC (equal) to DEF, and BCA to EFD. And let them also have one side equal to one side. First of all, the (side) by the equal angles. (That is) BC (equal) to EF. I say that they will have the remaining sides equal to the corresponding remaining sides. (That is) AB (equal) to DE, and AC to DF. And (they will have) the remaining angle (equal) to the remaining angle. (That is) BAC (equal) to EDF.

For if AB is unequal to DE (then) one of them is greater. Let AB be greater, and let BG be made equal to DE [Prop. 1.3], and let GC be joined.

Therefore, since BG is equal to DE, and BC to EF, the two (straight-lines) GB, BC† are equal to the two (straight-lines) DE, EF, respectively. And angle GBC is equal to angle DEF. Thus, the base GC is equal to the base DF, and triangle GBC is equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, GCB (is equal) to DFE. But, DFE was assumed (to be) equal to BCA. Thus, BCG is also equal to BCA, the lesser to the greater. The very thing (is) impossible. Thus, AB is not unequal to DE. Thus, (it is) equal. And BC is also equal to EF. So the two (straight-lines) AB, BC are equal to the two (straight-lines) DE, EF, respectively. And *ἡ ὑπὸ ΒΑΓ τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴση ἐστίν.*

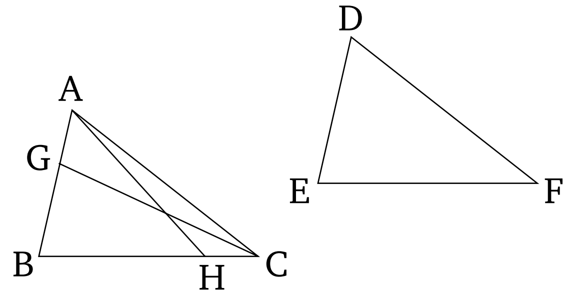


Ἀλλὰ δὴ πάλιν ἔστωσαν αἱ ὑπὸ τὰς ἴσας γωνίας πλευραὶ ὑποτείνουσαι ἴσαι, ὡς ἡ ΑΒ τῇ ΔΕ· λέγω πάλιν, ὅτι καὶ αἱ λοιπαὶ πλευραὶ ταῖς λοιπαῖς πλευραῖς ἴσαι ἔσονται, ἡ μὲν ΑΓ τῇ ΔΖ, ἡ δὲ ΒΓ τῇ ΕΖ καὶ ἔτι ἡ λοιπὴ γωνία ἡ ὑπὸ ΒΑΓ τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴση ἐστίν.

Εἰ γὰρ ἄνισός ἐστιν ἡ ΒΓ τῇ ΕΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων, εἰ δυνατόν, ἡ ΒΓ, καὶ κείσθω τῇ ΕΖ ἴση ἡ ΒΘ, καὶ ἐπεζεύχθω ἡ ΑΘ. καὶ ἐπὲι ἴση ἐστὶν ἡ μὲν ΒΘ τῇ ΕΖ ἡ δὲ ΑΒ τῇ ΔΕ, δύο δὴ αἱ ΑΒ, ΒΘ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκαρέρᾳ· καὶ γωνίας ἴσας περιέχουσιν· βάσις ἄρα ἡ ΑΘ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ τὸ ΑΒΘ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ᾽ ἃς αἱ ἴσας πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΘΑ γωνία τῇ ὑπὸ ΕΖΔ. ἀλλὰ ἡ ὑπὸ ΕΖΔ τῇ ὑπὸ ΒΓΑ ἐστιν ἴση· τριγώνου δὴ τοῦ ΑΘΓ ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΘΑ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΒΓΑ· ὅπερ ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ΒΓ τῇ ΕΖ· ἴση ἄρα. ἐστὶ δὲ καὶ ἡ ΑΒ τῇ ΔΕ ἴση. δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίας ἴσας περιέχουσι· βάσις ἄρα ἡ ΑΓ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον καὶ λοιπὴ γωνία ἡ ὑπὸ ΒΑΓ τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴση.

Ἐὰν ἄρα δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην ἤτοι τὴν πρὸς ταῖς ἴσαις γωνίαις, ἢ τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

angle ABC is equal to angle DEF. Thus, the base AC is equal to the base DF, and the remaining angle BAC is equal to the remaining angle EDF [Prop. 1.4].



But, again, let the sides subtending the equal angles be equal: for instance, (let) AB (be equal) to DE. Again, I say that the remaining sides will be equal to the remaining sides. (That is) AC (equal) to DF, and BC to EF. Furthermore, the remaining angle BAC is equal to the remaining angle EDF.

For if BC is unequal to EF (then) one of them is greater. If possible, let BC be greater. And let BH be made equal to EF [Prop. 1.3], and let AH be joined. And since BH is equal to EF, and AB to DE, the two (straight-lines) AB, BH are equal to the two (straight-lines) DE, EF, respectively. And the angles they encompass (are also equal). Thus, the base AH is equal to the base DF, and the triangle ABH is equal to the triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, angle BHA is equal to EFD. But, EFD is equal to BCA. So, in triangle AHC, the external angle BHA is equal to the internal and opposite angle BCA. The very thing (is) impossible [Prop. 1.16]. Thus, BC is not unequal to EF. Thus, (it is) equal. And AB is also equal to DE. So the two (straight-lines) AB, BC are equal to the two (straight-lines) DE, EF, respectively. And they encompass equal angles. Thus, the base AC is equal to the base DF, and triangle ABC (is) equal to triangle DEF, and the remaining angle BAC (is) equal to the remaining angle EDF [Prop. 1.4].

Thus, if two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—(then the triangles) will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle (equal) to the remaining angle. (Which is) the very thing it was required to show.

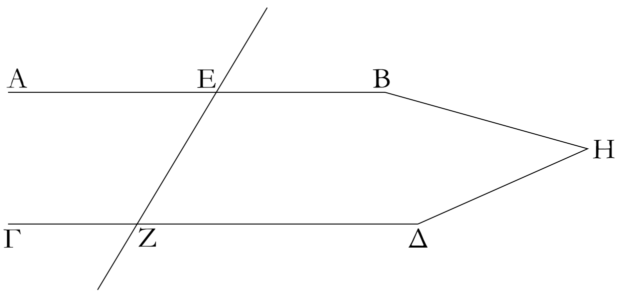
† The Greek text has "BG, BC", which is obviously a mistake.

κζ'.

Ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῇ, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.

Proposition 27

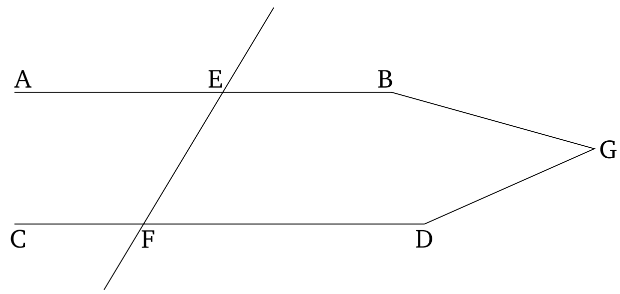
If a straight-line falling across two straight-lines makes the alternate angles equal to one another (then) the (two) straight-lines will be parallel to one another.



Εἰς γὰρ δύο εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΑΕΖ, ΕΖΔ ἴσας ἀλλήλαις ποιείτω· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Εἰ γὰρ μή, ἐκβαλλόμεναι αἱ ΑΒ, ΓΔ συμπεσοῦνται ἤτοι ἐπὶ τὰ Β, Δ μέρη ἢ ἐπὶ τὰ Α, Γ. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν ἐπὶ τὰ Β, Δ μέρη κατὰ τὸ Η. τριγώνου δὴ τοῦ ΗΕΖ ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΕΖ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΖΗ· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα αἱ ΑΒ, ΔΓ ἐκβαλλόμεναι συμπεσοῦνται ἐπὶ τὰ Β, Δ μέρη. ὁμοίως δὴ δειχθήσεται, ὅτι οὐδὲ ἐπὶ τὰ Α, Γ· αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Ἐὰν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῇ, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.



For let the straight-line EF, falling across the two straight-lines AB and CD, make the alternate angles AEF and EFD equal to one another. I say that AB and CD are parallel.

For if not, being produced, AB and CD will certainly meet together: either in the direction of B and D, or (in the direction) of A and C [Def. 1.23]. Let them be produced, and let them meet together in the direction of B and D at (point) G. So, for the triangle GEF, the external angle AEF is equal to the interior and opposite (angle) EFG. The very thing is impossible [Prop. 1.16]. Thus, being produced, AB and CD will not meet together in the direction of B and D. Similarly, it can be shown that neither (will they meet together) in (the direction of) A and C. But (straight-lines) meeting in neither direction are parallel [Def. 1.23]. Thus, AB and CD are parallel.

Thus, if a straight-line falling across two straight-lines makes the alternate angles equal to one another (then) the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

κη′.

Ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῇ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην ποιῇ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.

Εἰς γὰρ δύο εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ ΕΗΒ τῇ ἐντὸς καὶ ἀπεναντίον γωνίᾳ τῇ ὑπὸ ΗΘΔ ἴσην ποιείτω ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ ΒΗΘ, ΗΘΔ δυσὶν ὀρθαῖς ἴσας· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ ΕΗΒ τῇ ὑπὸ ΗΘΔ, ἀλλὰ ἡ ὑπὸ ΕΗΒ τῇ ὑπὸ ΑΗΘ ἐστιν ἴση, καὶ ἡ ὑπὸ ΑΗΘ ἄρα τῇ ὑπὸ ΗΘΔ ἐστιν ἴση· καί εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Πάλιν, ἐπεὶ αἱ ὑπὸ ΒΗΘ, ΗΘΔ δύο ὀρθαῖς ἴσαι εἰσίν, εἰσὶ δὲ καὶ αἱ ὑπὸ ΑΗΘ, ΒΗΘ δυσὶν ὀρθαῖς ἴσαι, αἱ ἄρα ὑπὸ ΑΗΘ, ΒΗΘ ταῖς ὑπὸ ΒΗΘ, ΗΘΔ ἴσαι εἰσίν· κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΒΗΘ· λοιπὴ ἄρα ἡ ὑπὸ ΑΗΘ λοιπῇ τῇ ὑπὸ ΗΘΔ ἐστιν ἴση· καί εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

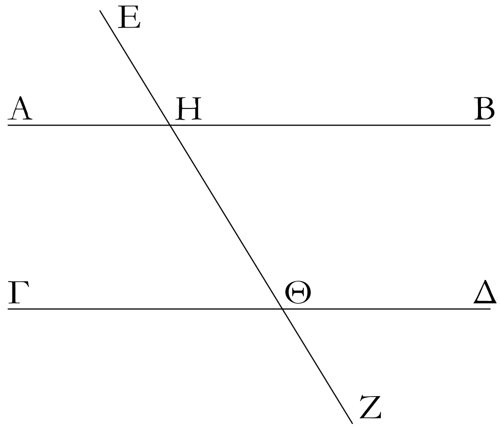
Proposition 28

If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, (then) the (two) straight-lines will be parallel to one another.

For let EF, falling across the two straight-lines AB and CD, make the external angle EGB equal to the internal and opposite angle GHD, or the (sum of the) internal (angles) on the same side, BGH and GHD, equal to two right-angles. I say that AB is parallel to CD.

For since (in the first case) EGB is equal to GHD, but EGB is equal to AGH [Prop. 1.15], AGH is thus also equal to GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

Again, since (in the second case, the sum of) BGH and GHD is equal to two right-angles, and (the sum of) AGH and BGH is also equal to two right-angles [Prop. 1.13], (the sum of) AGH and BGH is thus equal to (the sum of) BGH and GHD. Let BGH be subtracted from both. Thus, the remainder AGH is equal to the remainder GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].



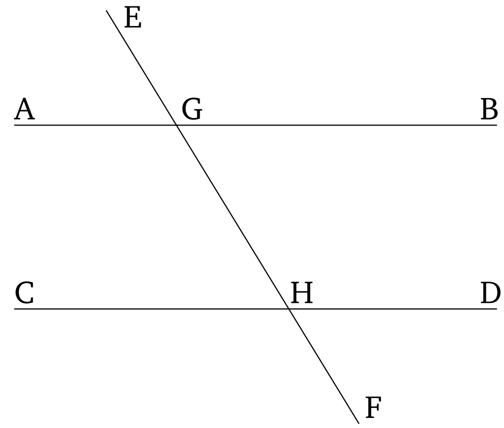
Ἐὰν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῇ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην ποιῇ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

κθ΄.

Ἡ εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τάς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῇ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας.

Εἰς γὰρ παραλλήλους εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπιπτέτω ἡ ΕΖ· λέγω, ὅτι τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΑΗΘ, ΗΘΔ ἴσας ποιεῖ καὶ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ ΕΗΒ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΗΘΔ ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ ΒΗΘ, ΗΘΔ δυσὶν ὀρθαῖς ἴσας.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ΑΗΘ τῇ ὑπὸ ΗΘΔ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΑΗΘ· κοινὴ προσκείσθω ἡ ὑπὸ ΒΗΘ· αἱ ἄρα ὑπὸ ΑΗΘ, ΒΗΘ τῶν ὑπὸ ΒΗΘ, ΗΘΔ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ ΑΗΘ, ΒΗΘ δυσὶν ὀρθαῖς ἴσαι εἰσίν. [καὶ] αἱ ἄρα ὑπὸ ΒΗΘ, ΗΘΔ δύο ὀρθῶν ἐλάσσονές εἰσιν. αἱ δὲ ἀπ᾽ ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν· αἱ ἄρα ΑΒ, ΓΔ ἐκβαλλόμεναι εἰς ἄπειρον συμπεσοῦνται· οὐ συμπίπτουσι δὲ διὰ τὸ παραλλήλους αὐτὰς ὑποκεῖσθαι· οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΑΗΘ τῇ ὑπὸ ΗΘΔ· ἴση ἄρα. ἀλλὰ ἡ ὑπὸ ΑΗΘ τῇ ὑπὸ ΕΗΒ ἐστιν ἴση· καὶ ἡ ὑπὸ ΕΗΒ ἄρα τῇ ὑπὸ ΗΘΔ ἐστιν ἴση· κοινὴ προσκείσθω ἡ ὑπὸ ΒΗΘ· αἱ ἄρα ὑπὸ ΕΗΒ, ΒΗΘ ταῖς ὑπὸ ΒΗΘ, ΗΘΔ ἴσαι εἰσίν. ἀλλὰ αἱ ὑπὸ ΕΗΒ, ΒΗΘ δύο ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΒΗΘ, ΗΘΔ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν.



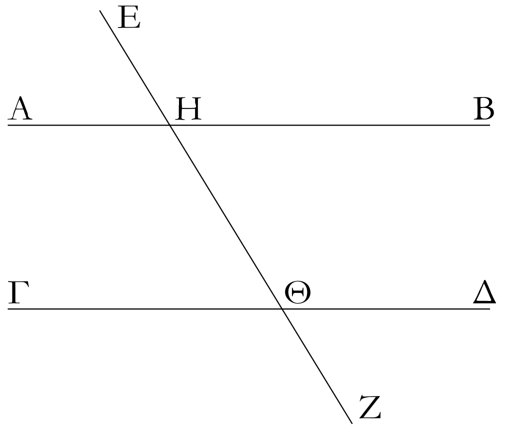
Thus, if a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, (then) the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Proposition 29

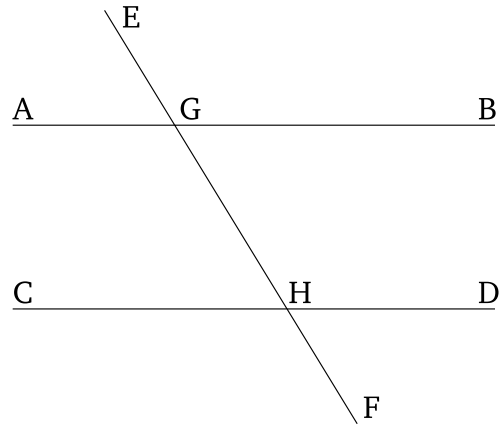
A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.

For let the straight-line *EF* fall across the parallel straight-lines *AB* and *CD*. I say that it makes the alternate angles, *AGH* and *GHD*, equal, the external angle *EGB* equal to the internal and opposite (angle) *GHD*, and the (sum of the) internal (angles) on the same side, *BGH* and *GHD*, equal to two right-angles.

For if *AGH* is unequal to *GHD* (then) one of them is greater. Let *AGH* be greater. Let *BGH* be added to both. Thus, (the sum of) *AGH* and *BGH* is greater than (the sum of) *BGH* and *GHD*. But, (the sum of) *AGH* and *BGH* is equal to two right-angles [Prop 1.13]. Thus, (the sum of) *BGH* and *GHD* is [also] less than two right-angles. But (straight-lines) being produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, *AB* and *CD*, being produced to infinity, will meet together. But they do not meet, on account of them (initially) being assumed parallel (to one another) [Def. 1.23]. Thus, *AGH* is not unequal to *GHD*. Thus, (it is) equal. But, *AGH* is equal to *EGB* [Prop. 1.15]. And *EGB* is thus also equal to *GHD*. Let *BGH* be added to both. Thus, (the sum of) *EGB* and *BGH* is equal to (the sum of) *BGH* and *GHD*. But, (the sum of) *EGB* and *BGH* is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) *BGH* and *GHD* is also equal to two right-angles.



Ἡ ἄρα εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τάς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῇ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας· ὅπερ ἔδει δεῖξαι.



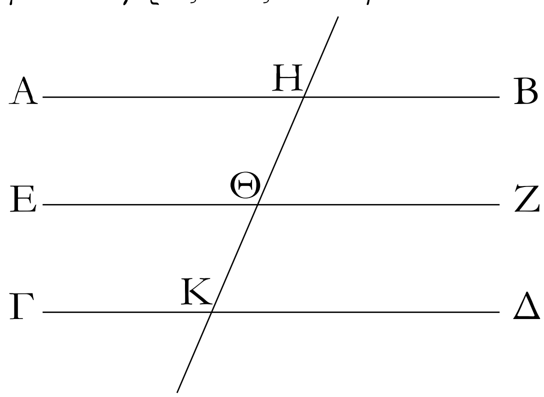
Thus, a straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles. (Which is) the very thing it was required to show.

λ´.

Αἱ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι.

Ἔστω ἑκατέρα τῶν ΑΒ, ΓΔ τῇ ΕΖ παράλληλος· λέγω, ὅτι καὶ ἡ ΑΒ τῇ ΓΔ ἐστι παράλληλος.

Ἐμπιπτέτω γὰρ εἰς αὐτὰς εὐθεῖα ἡ ΗΚ.



Καὶ ἐπεὶ εἰς παραλλήλους εὐθείας τὰς ΑΒ, ΕΖ εὐθεῖα ἐμπέπτωκεν ἡ ΗΚ, ἴση ἄρα ἡ ὑπὸ ΑΗΚ τῇ ὑπὸ ΗΘΖ. πάλιν, ἐπεὶ εἰς παραλλήλους εὐθείας τὰς ΕΖ, ΓΔ εὐθεῖα ἐμπέπτωκεν ἡ ΗΚ, ἴση ἐστὶν ἡ ὑπὸ ΗΘΖ τῇ ὑπὸ ΗΚΔ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΗΚ τῇ ὑπὸ ΗΘΖ ἴση. καὶ ἡ ὑπὸ ΑΗΚ ἄρα τῇ ὑπὸ ΗΚΔ ἐστιν ἴση· καί εἰσιν ἐναλλάξ. παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

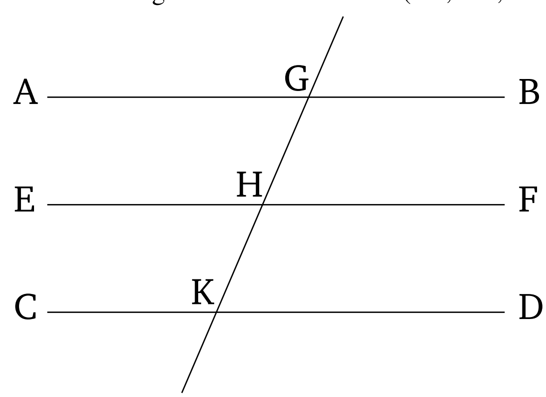
[Αἱ ἄρα τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι·] ὅπερ ἔδει δεῖξαι.

Proposition 30

(Straight-lines) parallel to the same straight-line are also parallel to one another.

Let each of the (straight-lines) AB and CD be parallel to EF. I say that AB is also parallel to CD.

For let the straight-line GK fall across (AB, CD, and EF).



And since the straight-line GK has fallen across the parallel straight-lines AB and EF, (angle) AGK (is) thus equal to GHF [Prop. 1.29]. Again, since the straight-line GK has fallen across the parallel straight-lines EF and CD, (angle) GHF is equal to GKD [Prop. 1.29]. But AGK was also shown (to be) equal to GHF. Thus, AGK is also equal to GKD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

[Thus, (straight-lines) parallel to the same straight-line are also parallel to one another.] (Which is) the very thing it was required to show.

λα´.

Proposition 31

Διὰ τοῦ δοθέντος σημείου τῇ δοθείσῃ εὐθείᾳ παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

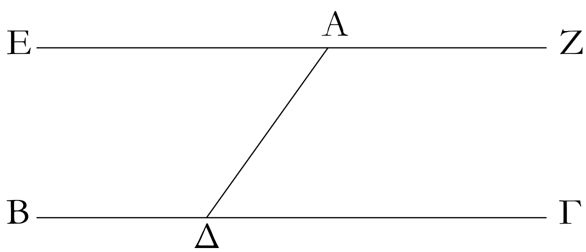
Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ· δεῖ δὴ διὰ τοῦ Α σημείου τῇ ΒΓ εὐθείᾳ παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

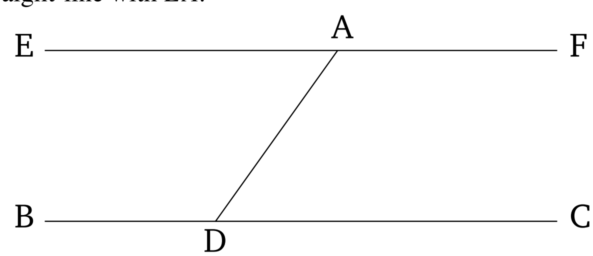
Εἰλήφθω ἐπὶ τῆς ΒΓ τυχὸν σημεῖον τὸ Δ, καὶ ἐπεζεύχθω ἡ ΑΔ· καὶ συνεστάτω πρὸς τῇ ΔΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΑΔΓ γωνίᾳ ἴση ἡ ὑπὸ ΔΑΕ· καὶ ἐκβεβλήσθω ἐπ᾿ εὐθείας τῇ ΕΑ εὐθεῖα ἡ ΑΖ.

To draw a straight-line parallel to a given straight-line, through a given point.

Let *A* be the given point, and *BC* the given straight-line. So it is required to draw a straight-line parallel to the straight-line *BC*, through the point *A*.

Let the point *D* be taken a random on *BC*, and let *AD* be joined. And let (angle) *DAE*, equal to angle *ADC*, be constructed on the straight-line *DA* at the point *A* on it [Prop. 1.23]. And let the straight-line *AF* be produced in a straight-line with *EA*.





Καὶ ἐπεὶ εἰς δύο εὐθείας τὰς ΒΓ, ΕΖ εὐθεῖα ἐμπίπτουσα ἡ ΑΔ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΕΑΔ, ΑΔΓ ἴσας ἀλλήλαις πεποίηκεν, παράλληλος ἄρα ἐστὶν ἡ ΕΑΖ τῇ ΒΓ.

Διὰ τοῦ δοθέντος ἄρα σημείου τοῦ Α τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ παράλληλος εὐθεῖα γραμμὴ ἦκται ἡ ΕΑΖ· ὅπερ ἔδει ποιῆσαι.

And since the straight-line *AD*, (in) falling across the two straight-lines *BC* and *EF*, has made the alternate angles *EAD* and *ADC* equal to one another, *EAF* is thus parallel to *BC* [Prop. 1.27].

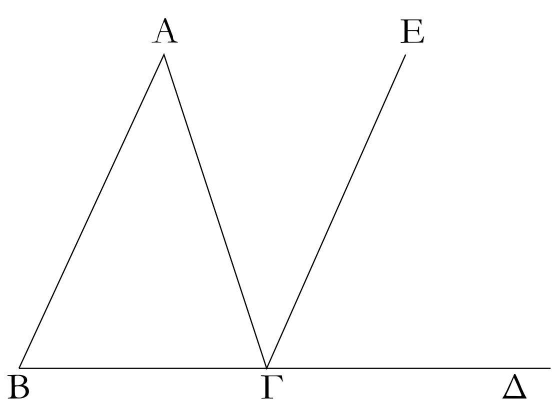
Thus, the straight-line *EAF* has been drawn parallel to the given straight-line *BC*, through the given point *A*. (Which is) the very thing it was required to do.

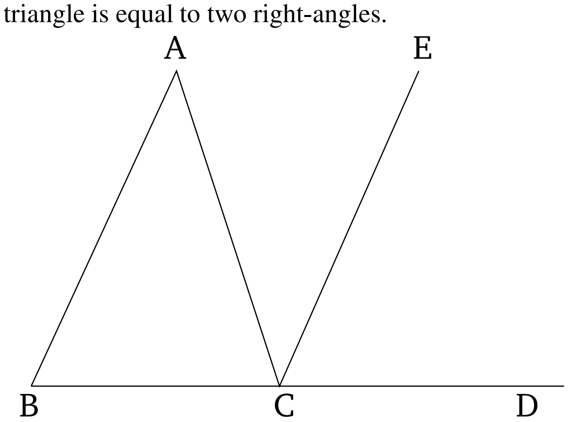
λβ΄.

Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Proposition 32

In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.





Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ ΒΓ ἐπὶ τὸ Δ· λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΓΔ ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΓΑΒ, ΑΒΓ, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἤχθω γὰρ διὰ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ παράλληλος ἡ ΓΕ.

Let *ABC* be a triangle, and let one of its sides *BC* be produced to *D*. I say that the external angle *ACD* is equal to the (sum of the) two internal and opposite angles *CAB* and *ABC*, and the (sum of the) three internal angles of the triangle—*ABC*, *BCA*, and *CAB*—is equal to two right-angles.

For let *CE* be drawn through point *C* parallel to the straight-line *AB* [Prop. 1.31].

Καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΕ ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΕ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΕΓΔ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΑΒΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΔ γωνία ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ ΒΑΓ, ΑΒΓ.

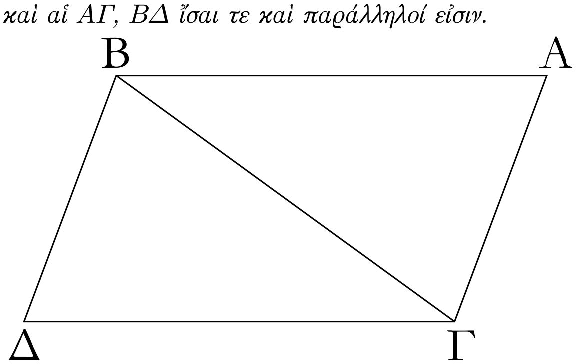
Κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τρισὶ ταῖς ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ ἴσαι εἰσίν. ἀλλ' αἱ ὑπὸ ΑΓΔ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΒ, ΓΒΑ, ΓΑΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἔδει δεῖξαι.

λγ'.

Αἱ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν.

Ἔστωσαν ἴσαι τε καὶ παράλληλοι αἱ ΑΒ, ΓΔ, καὶ ἐπιζευγνύτωσαν αὐτὰς ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι αἱ ΑΓ, ΒΔ· λέγω, ὅτι καὶ αἱ ΑΓ, ΒΔ ἴσαι τε καὶ παράλληλοί εἰσιν.



Ἐπεζεύχθω ἡ ΒΓ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΓΔ κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση· βάσις ἄρα ἡ ΑΓ βάσει τῇ ΒΔ ἐστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΓΒΔ. καὶ ἐπεὶ εἰς δύο εὐθείας τὰς ΑΓ, ΒΔ εὐθεῖα ἐμπίπτουσα ἡ ΒΓ τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις πεποίηκεν, παράλληλος ἄρα ἐστὶν ἡ ΑΓ τῇ ΒΔ. ἐδείχθη δὲ αὐτῇ καὶ ἴση.

Αἱ ἄρα τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν· ὅπερ ἔδει δεῖξαι.

And since *AB* is parallel to *CE*, and *AC* has fallen across them, the alternate angles *BAC* and *ACE* are equal to one another [Prop. 1.29]. Again, since *AB* is parallel to *CE*, and the straight-line *BD* has fallen across them, the external angle *ECD* is equal to the internal and opposite (angle) *ABC* [Prop. 1.29]. But *ACE* was also shown (to be) equal to *BAC*. Thus, the whole angle *ACD* is equal to the (sum of the) two internal and opposite (angles) *BAC* and *ABC*.

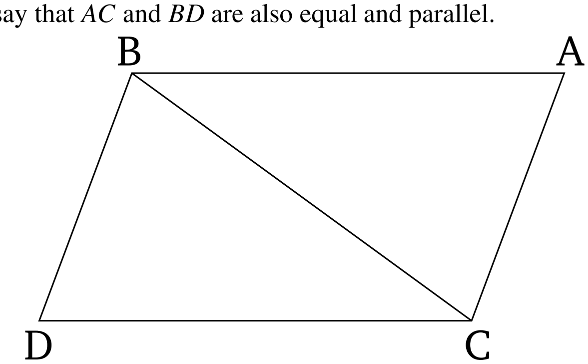
Let *ACB* be added to both. Thus, (the sum of) *ACD* and *ACB* is equal to the (sum of the) three (angles) *ABC*, *BCA*, and *CAB*. But, (the sum of) *ACD* and *ACB* is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) *ACB*, *CBA*, and *CAB* is also equal to two right-angles.

Thus, in any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 33

Straight-lines joining equal and parallel (straight-lines) on the same sides are themselves also equal and parallel.

Let *AB* and *CD* be equal and parallel (straight-lines), and let the straight-lines *AC* and *BD* join them on the same sides. I say that *AC* and *BD* are also equal and parallel.



Let *BC* be joined. And since *AB* is parallel to *CD*, and *BC* has fallen across them, the alternate angles *ABC* and *BCD* are equal to one another [Prop. 1.29]. And since *AB* is equal to *CD*, and *BC* is common, the two (straight-lines) *AB*, *BC* are equal to the two (straight-lines) *DC*, *CB*.† And the angle *ABC* is equal to the angle *BCD*. Thus, the base *AC* is equal to the base *BD*, and triangle *ABC* is equal to triangle *DCB*‡, and the remaining angles will be equal to the corresponding remaining angles subtended by the equal sides [Prop. 1.4]. Thus, angle *ACB* is equal to *CBD*. Also, since the straight-line *BC*, (in) falling across the two straight-lines *AC* and *BD*, has made the alternate angles (*ACB* and *CBD*) equal to one another, *AC* is thus parallel to *BD* [Prop. 1.27]. And (*AC*) was also shown (to be) equal to (*BD*).

Thus, straight-lines joining equal and parallel (straight-lines) on the same sides are themselves also equal and parallel. (Which is) the very thing it was required to show.

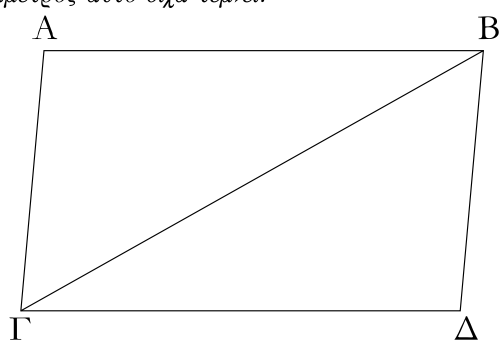
† The Greek text has "*BC*, *CD*", which is obviously a mistake.

‡ The Greek text has "*DCB*", which is obviously a mistake.

λδ΄.

Τῶν παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ ἡ διάμετρος αὐτὰ δίχα τέμνει.

Ἔστω παραλληλόγραμμον χωρίον τὸ ΑΓΔΒ, διάμετρος δὲ αὐτοῦ ἡ ΒΓ· λέγω, ὅτι τοῦ ΑΓΔΒ παραλληλογράμμου αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ ἡ ΒΓ διάμετρος αὐτὸ δίχα τέμνει.



Ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. πάλιν ἐπεὶ παράλληλός ἐστιν ἡ ΑΓ τῇ ΒΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΓΒ, ΓΒΔ ἴσαι ἀλλήλαις εἰσίν. δύο δὴ τρίγωνά ἐστι τὰ ΑΒΓ, ΒΓΔ τὰς δύο γωνίας τὰς ὑπὸ ΑΒΓ, ΒΓΑ δυσὶ ταῖς ὑπὸ ΒΓΔ, ΓΒΔ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις κοινὴν αὐτῶν τὴν ΒΓ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς ἴσας ἕξει ἑκατέραν ἑκατέρᾳ καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ἴση ἄρα ἡ μὲν ΑΒ πλευρὰ τῇ ΓΔ, ἡ δὲ ΑΓ τῇ ΒΔ, καὶ ἔτι ἴση ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΓΔΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΒΓΔ, ἡ δὲ ὑπὸ ΓΒΔ τῇ ὑπὸ ΑΓΒ, ὅλη ἄρα ἡ ὑπὸ ΑΒΔ ὅλῃ τῇ ὑπὸ ΑΓΔ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΒ ἴση.

Τῶν ἄρα παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

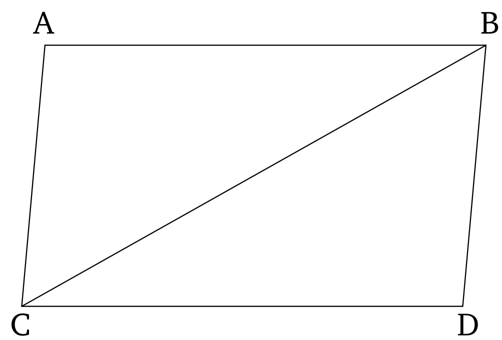
Λέγω δή, ὅτι καὶ ἡ διάμετρος αὐτὰ δίχα τέμνει. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΒ τῇ ΓΔ, κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΓΔ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση. καὶ βάσις ἄρα ἡ ΑΓ τῇ ΔΒ ἴση. καὶ τὸ ΑΒΓ [ἄρα] τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν.

Ἡ ἄρα ΒΓ διάμετρος δίχα τέμνει τὸ ΑΒΓΔ παραλληλόγραμμον· ὅπερ ἔδει δεῖξαι.

Proposition 34

In parallelogramic figures, the opposite sides and angles are equal to one another, and a diagonal cuts them in half.

Let *ACDB* be a parallelogramic figure, and *BC* its diagonal. I say that for parallelogram *ACDB*, the opposite sides and angles are equal to one another, and the diagonal *BC* cuts it in half.



For since *AB* is parallel to *CD*, and the straight-line *BC* has fallen across them, the alternate angles *ABC* and *BCD* are equal to one another [Prop. 1.29]. Again, since *AC* is parallel to *BD*, and *BC* has fallen across them, the alternate angles *ACB* and *CBD* are equal to one another [Prop. 1.29]. So *ABC* and *BCD* are two triangles having the two angles *ABC* and *BCA* equal to the two (angles) *BCD* and *CBD*, respectively, and one side equal to one side—the (one) by the equal angles and common to them, (namely) *BC*. Thus, they will also have the remaining sides equal to the corresponding remaining (sides), and the remaining angle (equal) to the remaining angle [Prop. 1.26]. Thus, side *AB* is equal to *CD*, and *AC* to *BD*. Furthermore, angle *BAC* is equal to *CDB*. And since angle *ABC* is equal to *BCD*, and *CBD* to *ACB*, the whole (angle) *ABD* is thus equal to the whole (angle) *ACD*. And *BAC* was also shown (to be) equal to *CDB*.

Thus, in parallelogramic figures, the opposite sides and angles are equal to one another.

And, I also say that a diagonal cuts them in half. For since *AB* is equal to *CD*, and *BC* (is) common, the two (straight-lines) *AB*, *BC* are equal to the two (straight-lines) *DC*, *CB*†, respectively. And angle *ABC* is equal to angle *BCD*. Thus, the base *AC* (is) also equal to *DB*, and triangle *ABC* is equal to triangle *BCD* [Prop. 1.4].

Thus, the diagonal *BC* cuts the parallelogram *ACDB*‡ in half. (Which is) the very thing it was required to show.

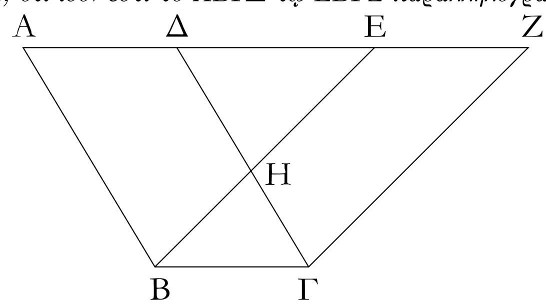
† The Greek text has “*CD*, *BC*”, which is obviously a mistake.

‡ The Greek text has “*ABCD*”, which is obviously a mistake.

λε΄.

Τὰ παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω παραλληλόγραμμα τὰ ΑΒΓΔ, ΕΒΓΖ ἐπὶ τῆς αὐτῆς βάσεως τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΖ, ΒΓ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔ τῷ ΕΒΓΖ παραλληλογράμμῳ.



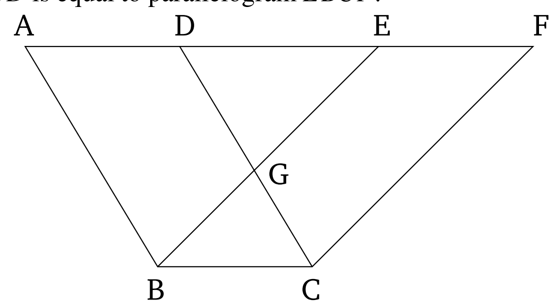
Ἐπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΑΒΓΔ, ἴση ἐστὶν ἡ ΑΔ τῇ ΒΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΕΖ τῇ ΒΓ ἐστιν ἴση· ὥστε καὶ ἡ ΑΔ τῇ ΕΖ ἐστιν ἴση· καὶ κοινὴ ἡ ΔΕ· ὅλη ἄρα ἡ ΑΕ ὅλῃ τῇ ΔΖ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΑΒ τῇ ΔΓ ἴση· δύο δὴ αἱ ΕΑ, ΑΒ δύο ταῖς ΖΔ, ΔΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΖΔΓ γωνίᾳ τῇ ὑπὸ ΕΑΒ ἐστιν ἴση ἡ ἐκτὸς τῇ ἐντός· βάσις ἄρα ἡ ΕΒ βάσει τῇ ΖΓ ἴση ἐστίν, καὶ τὸ ΕΑΒ τρίγωνον τῷ ΔΖΓ τριγώνῳ ἴσον ἔσται· κοινὸν ἀφῃρήσθω τὸ ΔΗΕ· λοιπὸν ἄρα τὸ ΑΒΗΔ τραπέζιον λοιπῷ τῷ ΕΗΓΖ τραπεζίῳ ἐστὶν ἴσον· κοινὸν προσκείσθω τὸ ΗΒΓ τρίγωνον· ὅλον ἄρα τὸ ΑΒΓΔ παραλληλόγραμμον ὅλῳ τῷ ΕΒΓΖ παραλληλογράμμῳ ἴσον ἐστίν.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 35

Parallelograms which are on the same base and between the same parallels are equal† to one another.

Let *ABCD* and *EBCF* be parallelograms on the same base *BC*, and between the same parallels *AF* and *BC*. I say that *ABCD* is equal to parallelogram *EBCF*.



For since *ABCD* is a parallelogram, *AD* is equal to *BC* [Prop. 1.34]. So, for the same (reasons), *EF* is also equal to *BC*. So *AD* is also equal to *EF*. And *DE* is common. Thus, the whole (straight-line) *AE* is equal to the whole (straight-line) *DF*. And *AB* is also equal to *DC*. So the two (straight-lines) *EA*, *AB* are equal to the two (straight-lines) *FD*, *DC*, respectively. And angle *FDC* is equal to angle *EAB*, the external to the internal [Prop. 1.29]. Thus, the base *EB* is equal to the base *FC*, and triangle *EAB* will be equal to triangle *DFC* [Prop. 1.4]. Let *DGE* be taken away from both. Thus, the remaining trapezium *ABGD* is equal to the remaining trapezium *EGCF*. Let triangle *GBC* be added to both. Thus, the whole parallelogram *ABCD* is equal to the whole parallelogram *EBCF*.

Thus, parallelograms which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

† Here, for the first time, “equal” means “equal in area”, rather than “congruent”.

λϛ΄.

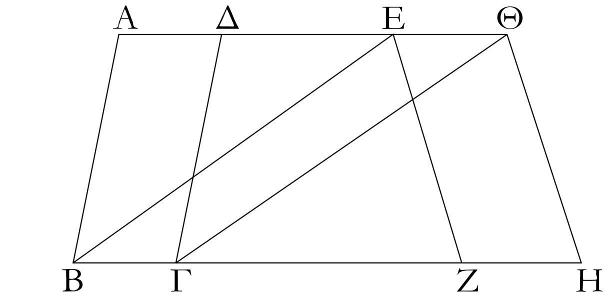
Τὰ παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω παραλληλόγραμμα τὰ ΑΒΓΔ, ΕΖΗΘ ἐπὶ ἴσων βάσεων ὄντα τῶν ΒΓ, ΖΗ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΘ, ΒΗ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΖΗΘ.

Proposition 36

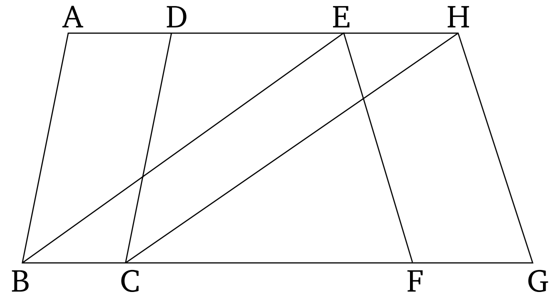
Parallelograms which are on equal bases and between the same parallels are equal to one another.

Let *ABCD* and *EFGH* be parallelograms which are on the equal bases *BC* and *FG*, and (are) between the same parallels *AH* and *BG*. I say that the parallelogram *ABCD* is equal to *EFGH*.



Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῇ ΖΗ, ἀλλὰ ἡ ΖΗ τῇ ΕΘ ἐστιν ἴση, καὶ ἡ ΒΓ ἄρα τῇ ΕΘ ἐστιν ἴση. εἰσὶ δὲ καὶ παράλληλοι. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΕΒ, ΘΓ· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι ἴσαι τε καὶ παράλληλοί εἰσι [καὶ αἱ ΕΒ, ΘΓ ἄρα ἴσαι τέ εἰσι καὶ παράλληλοι]. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΕΒΓΘ. καί ἐστιν ἴσον τῷ ΑΒΓΔ· βάσιν τε γὰρ αὐτῷ τὴν αὐτὴν ἔχει τὴν ΒΓ, καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστὶν αὐτῷ ταῖς ΒΓ, ΑΘ. δὶα τὰ αὐτὰ δὴ καὶ τὸ ΕΖΗΘ τῷ αὐτῷ τῷ ΕΒΓΘ ἐστιν ἴσον· ὥστε καὶ τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΖΗΘ ἐστιν ἴσον.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.



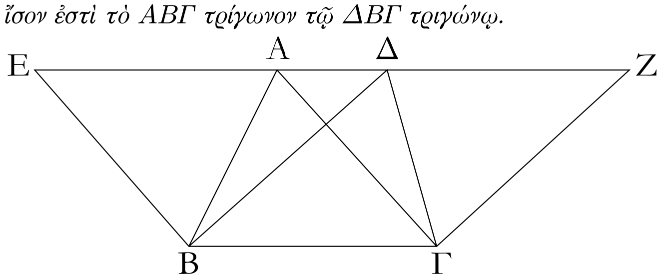
For let *BE* and *CH* be joined. And since *BC* is equal to *FG*, but *FG* is equal to *EH* [Prop. 1.34], *BC* is thus equal to *EH*. And they are also parallel, and *EB* and *HC* join them. But (straight-lines) joining equal and parallel (straight-lines) on the same sides are (themselves) equal and parallel [Prop. 1.33] [thus, *EB* and *HC* are also equal and parallel]. Thus, *EBCH* is a parallelogram [Prop. 1.34], and is equal to *ABCD*. For it has the same base, *BC*, (as *ABCD*), and is between the same parallels, *BC* and *AH*, (as *ABCD*) [Prop. 1.35]. So, for the same (reasons), *EFGH* is also equal to the same (parallelogram) *EBCH* [Prop. 1.34]. So that the parallelogram *ABCD* is also equal to *EFGH*.

Thus, parallelograms which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

λζ΄.

Τὰ τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω τρίγωνα τὰ ΑΒΓ, ΔΒΓ ἐπὶ τῆς αὐτῆς βάσεως τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΔ, ΒΓ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΒΓ τριγώνῳ.

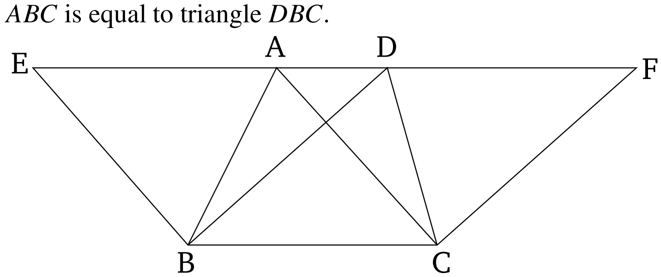


Ἐκβεβλήσθω ἡ ΑΔ ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ τὰ Ε, Ζ, καὶ διὰ μὲν τοῦ Β τῇ ΓΑ παράλληλος ἤχθω ἡ ΒΕ, δὶα δὲ τοῦ Γ τῇ ΒΔ παράλληλος ἤχθω ἡ ΓΖ. παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΕΒΓΑ, ΔΒΓΖ· καί εἰσιν ἴσα· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΕΖ· καί ἐστι τοῦ μὲν ΕΒΓΑ παραλληλογράμμου ἥμισυ τὸ ΑΒΓ τρίγωνον· ἡ γὰρ ΑΒ διάμετρος αὐτὸ δίχα τέμνει· τοῦ δὲ ΔΒΓΖ παραλληλογράμμου ἥμισυ τὸ ΔΒΓ τρίγωνον· ἡ γὰρ ΔΓ διάμετρος αὐτὸ δίχα τέμνει. [τὰ δὲ τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΒΓ τριγώνῳ.

Proposition 37

Triangles which are on the same base and between the same parallels are equal to one another.

Let *ABC* and *DBC* be triangles on the same base *BC*, and between the same parallels *AD* and *BC*. I say that triangle *ABC* is equal to triangle *DBC*.



Let *AD* be produced in both directions to *E* and *F*, and let the (straight-line) *BE* be drawn through *B* parallel to *CA* [Prop. 1.31], and let the (straight-line) *CF* be drawn through *C* parallel to *BD* [Prop. 1.31]. Thus, *EBCA* and *DBCF* are both parallelograms, and are equal. For they are on the same base *BC*, and between the same parallels *BC* and *EF* [Prop. 1.35]. And the triangle *ABC* is half of the parallelogram *EBCA*. For the diagonal *AB* cuts the latter in half [Prop. 1.34]. And the triangle *DBC* (is) half of the parallelogram *DBCF*. For the diagonal *DC* cuts the latter in half [Prop. 1.34]. [And the halves of equal things are equal to one another.][†] Thus, triangle *ABC*

Τὰ ἄρα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

is equal to triangle *DBC*.

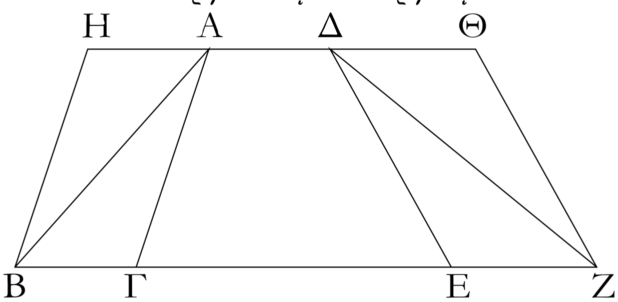
Thus, triangles which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

† This is an additional common notion.

λη΄.

Τὰ τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

Ἔστω τρίγωνα τὰ ΑΒΓ, ΔΕΖ ἐπὶ ἴσων βάσεων τῶν ΒΓ, ΕΖ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΖ, ΑΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.



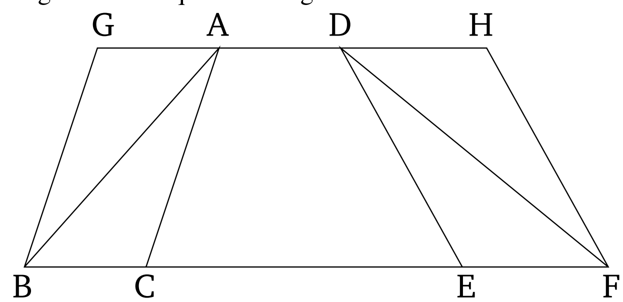
Ἐκβεβλήσθω γὰρ ἡ ΑΔ ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ τὰ Η, Θ, καὶ διὰ μὲν τοῦ Β τῇ ΓΑ παράλληλος ἤχθω ἡ ΒΗ, δια δὲ τοῦ Ζ τῇ ΔΕ παράλληλος ἤχθω ἡ ΖΘ. παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΗΒΓΑ, ΔΕΖΘ· καὶ ἴσον τὸ ΗΒΓΑ τῷ ΔΕΖΘ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΓ, ΕΖ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΖ, ΗΘ· καί ἐστι τοῦ μὲν ΗΒΓΑ παραλληλογράμμου ἥμισυ τὸ ΑΒΓ τρίγωνον. ἡ γὰρ ΑΒ διάμετρος αὐτὸ δίχα τέμνει· τοῦ δὲ ΔΕΖΘ παραλληλογράμμου ἥμισυ τὸ ΖΕΔ τρίγωνον· ἡ γὰρ ΔΖ δίαμετρος αὐτὸ δίχα τέμνει [τὰ δὲ τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Τὰ ἄρα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 38

Triangles which are on equal bases and between the same parallels are equal to one another.

Let *ABC* and *DEF* be triangles on the equal bases *BC* and *EF*, and between the same parallels *BF* and *AD*. I say that triangle *ABC* is equal to triangle *DEF*.



For let *AD* be produced in both directions to *G* and *H*, and let the (straight-line) *BG* be drawn through *B* parallel to *CA* [Prop. 1.31], and let the (straight-line) *FH* be drawn through *F* parallel to *DE* [Prop. 1.31]. Thus, *GBCA* and *DEFH* are each parallelograms. And *GBCA* is equal to *DEFH*. For they are on the equal bases *BC* and *EF*, and between the same parallels *BF* and *GH* [Prop. 1.36]. And triangle *ABC* is half of the parallelogram *GBCA*. For the diagonal *AB* cuts the latter in half [Prop. 1.34]. And triangle *FED* (is) half of parallelogram *DEFH*. For the diagonal *DF* cuts the latter in half. [And the halves of equal things are equal to one another.] Thus, triangle *ABC* is equal to triangle *DEF*.

Thus, triangles which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

λθ΄.

Τὰ ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἔστω ἴσα τρίγωνα τὰ ΑΒΓ, ΔΒΓ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη τῆς ΒΓ· λέγω, ὅτι καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἐπεζεύχθω γὰρ ἡ ΑΔ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΔ τῇ ΒΓ.

Εἰ γὰρ μή, ἤχθω διὰ τοῦ Α σημείου τῇ ΒΓ εὐθείᾳ παράλληλος ἡ ΑΕ, καὶ ἐπεζεύχθω ἡ ΕΓ. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν αὐτῷ τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις. ἀλλὰ τὸ ΑΒΓ

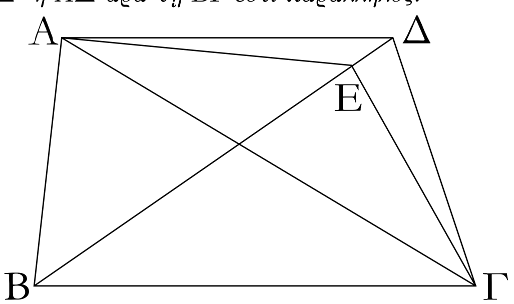
Proposition 39

Equal triangles which are on the same base, and on the same side, are also between the same parallels.

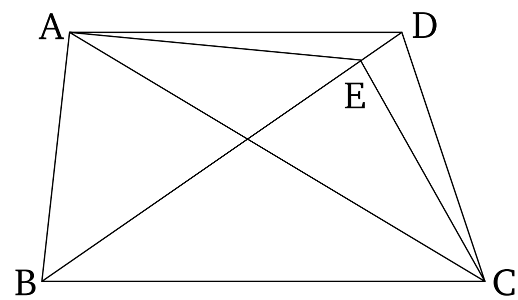
Let *ABC* and *DBC* be equal triangles which are on the same base *BC*, and on the same side (of it). I say that they are also between the same parallels.

For let *AD* be joined. I say that *AD* and *BC* are parallel.

For, if not, let *AE* be drawn through point A parallel to the straight-line *BC* [Prop. 1.31], and let *EC* be joined. Thus, triangle *ABC* is equal to triangle *EBC*. For it is on the same base as it, *BC*, and between the same parallels [Prop. 1.37]. But *ABC* is equal to *DBC*. Thus, *DBC* is also equal to *EBC*, *τῷ ΔΒΓ ἐστιν ἴσον· καὶ τὸ ΔΒΓ ἄρα τῷ ΕΒΓ ἴσον ἐστὶ τὸ μεῖζον τῷ ἐλάσσονι· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα παράλληλός ἐστιν ἡ ΑΕ τῇ ΒΓ. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾿ ἄλλη τις πλὴν τῆς ΑΔ· ἡ ΑΔ ἄρα τῇ ΒΓ ἐστι παράλληλος.*



the greater to the lesser. The very thing is impossible. Thus, AE is not parallel to BC. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BC.



Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

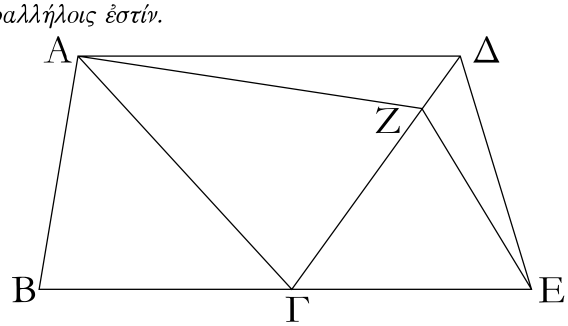
Thus, equal triangles which are on the same base, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

μ΄.

Proposition 40[†]

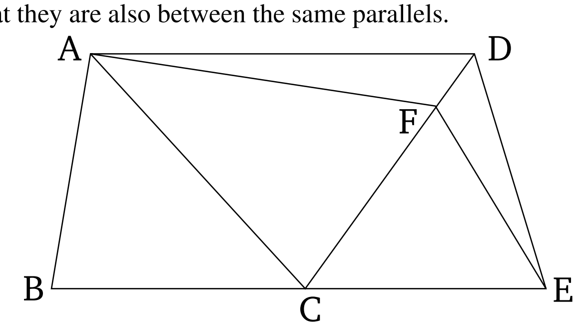
Τὰ ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἔστω ἴσα τρίγωνα τὰ ΑΒΓ, ΓΔΕ ἐπὶ ἴσων βάσεων τῶν ΒΓ, ΓΕ καὶ ἐπὶ τὰ αὐτὰ μέρη. λέγω, ὅτι καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.



Equal triangles which are on equal bases, and on the same side, are also between the same parallels.

Let ABC and CDE be equal triangles on the equal bases BC and CE (respectively), and on the same side (of BE). I say that they are also between the same parallels.



Ἐπεζεύχθω γὰρ ἡ ΑΔ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΔ τῇ ΒΕ.

Εἰ γὰρ μή, ἤχθω διὰ τοῦ Α τῇ ΒΕ παράλληλος ἡ ΑΖ, καὶ ἐπεζεύχθω ἡ ΖΕ. ἴσον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΖΓΕ τριγώνῳ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΓ, ΓΕ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΕ, ΑΖ. ἀλλὰ τὸ ΑΒΓ τρίγωνον ἴσον ἐστὶ τῷ ΔΓΕ [τριγώνῳ]· καὶ τὸ ΔΓΕ ἄρα [τρίγωνον] ἴσον ἐστὶ τῷ ΖΓΕ τριγώνῳ τὸ μεῖζον τῷ ἐλάσσονι· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα παράλληλος ἡ ΑΖ τῇ ΒΕ. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾿ ἄλλη τις πλὴν τῆς ΑΔ· ἡ ΑΔ ἄρα τῇ ΒΕ ἐστι παράλληλος.

Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

For let AD be joined. I say that AD is parallel to BE.

For if not, let AF be drawn through A parallel to BE [Prop. 1.31], and let FE be joined. Thus, triangle ABC is equal to triangle FCE. For they are on equal bases, BC and CE, and between the same parallels, BE and AF [Prop. 1.38]. But, triangle ABC is equal to [triangle] DCE. Thus, [triangle] DCE is also equal to triangle FCE, the greater to the lesser. The very thing is impossible. Thus, AF is not parallel to BE. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BE.

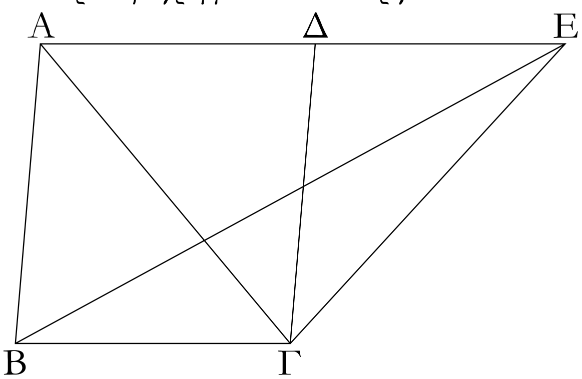
Thus, equal triangles which are on equal bases, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

[†] This whole proposition is regarded by Heiberg as a relatively early interpolation to the original text.

μα'.

Ἐὰν παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστί τὸ παραλληλόγραμμον τοῦ τριγώνου.

Παραλληλόγραμμον γὰρ τὸ ΑΒΓΔ τριγώνῳ τῷ ΕΒΓ βάσιν τε ἐχέτω τὴν αὐτὴν τὴν ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἔστω ταῖς ΒΓ, ΑΕ· λέγω, ὅτι διπλάσιόν ἐστι τὸ ΑΒΓΔ παραλληλόγραμμον τοῦ ΒΕΓ τριγώνου.



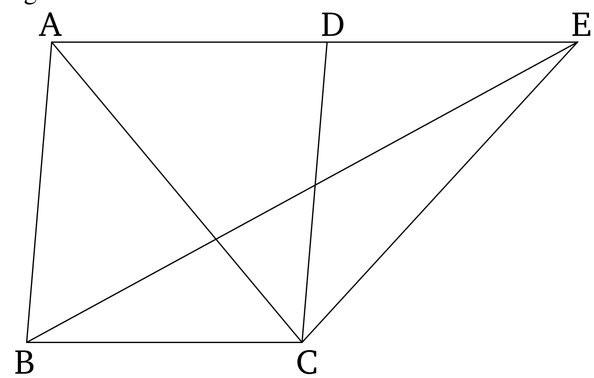
Ἐπεζεύχθω γὰρ ἡ ΑΓ. ἴσον δή ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΕΒΓ τριγώνῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς ἐστιν αὐτῷ τῆς ΒΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΑΕ. ἀλλὰ τὸ ΑΒΓΔ παραλληλόγραμμον διπλάσιόν ἐστι τοῦ ΑΒΓ τριγώνου· ἡ γὰρ ΑΓ διάμετρος αὐτὸ δίχα τέμνει· ὥστε τὸ ΑΒΓΔ παραλληλόγραμμον καὶ τοῦ ΕΒΓ τριγώνου ἐστὶ διπλάσιον.

Ἐὰν ἄρα παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστί τὸ παραλληλόγραμμον τοῦ τριγώνου· ὅπερ ἔδει δεῖξαι.

Proposition 41

If a parallelogram has the same base as a triangle, and is between the same parallels, (then) the parallelogram is double (the area) of the triangle.

For let parallelogram $ABCD$ have the same base BC as triangle EBC, and let it be between the same parallels, BC and AE. I say that parallelogram $ABCD$ is double (the area) of triangle BEC.



For let AC be joined. So triangle ABC is equal to triangle EBC. For it is on the same base, BC, (as EBC), and between the same parallels, BC and AE [Prop. 1.37]. But, parallelogram $ABCD$ is double (the area) of triangle ABC. For the diagonal AC cuts the former in half [Prop. 1.34]. So parallelogram $ABCD$ is also double (the area) of triangle EBC.

Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, (then) the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

μβ'.

Τῷ δοθέντι τριγώνῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἔστω τὸ μὲν δοθὲν τρίγωνον τὸ ΑΒΓ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ· δεῖ δὴ τῷ ΑΒΓ τριγώνῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ Δ γωνίᾳ εὐθυγράμμῳ.

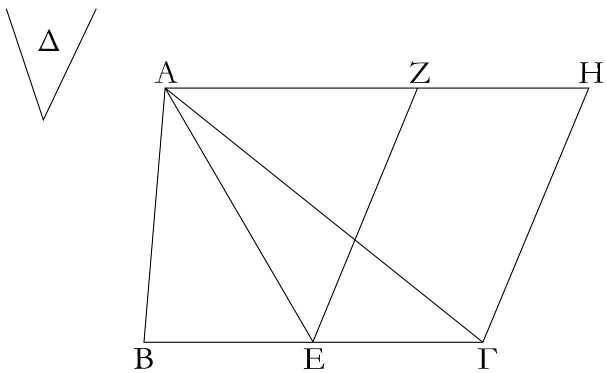
Τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΑΕ, καὶ συνεστάτω πρὸς τῇ ΕΓ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ε τῇ Δ γωνίᾳ ἴση ἡ ὑπὸ ΓΕΖ, καὶ διὰ μὲν τοῦ Α τῇ ΕΓ παράλληλος ἤχθω ἡ ΑΗ, διὰ δὲ τοῦ Γ τῇ ΕΖ παράλληλος ἤχθω ἡ ΓΗ· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΖΕΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΓ, ἴσον ἐστὶ καὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΕΓ τριγώνῳ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΒΕ, ΕΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΑΗ· διπλάσιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τοῦ ΑΕΓ τριγώνου. ἔστι δὲ καὶ τὸ ΖΕΓΗ παραλληλόγραμμον διπλάσιον τοῦ ΑΕΓ τριγώνου· βάσιν τε γὰρ αὐτῷ τὴν αὐτὴν ἔχει καὶ ἐν ταῖς αὐταῖς ἐστιν αὐτῷ παραλλήλοις· ἴσον ἄρα ἐστὶ τὸ ΖΕΓΗ παραλληλόγραμμον τῷ ΑΒΓ τριγώνῳ. καὶ ἔχει τὴν ὑπὸ ΓΕΖ γωνίαν ἴσην τῇ δοθείσῃ

Proposition 42

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Let ABC be the given triangle, and D the given rectilinear angle. So it is required to construct a parallelogram equal to triangle ABC in the rectilinear angle D.

Let BC be cut in half at E [Prop. 1.10], and let AE be joined. And let (angle) CEF, equal to angle D, be constructed at the point E on the straight-line EC [Prop. 1.23]. And let AG be drawn through A parallel to EC [Prop. 1.31], and let CG be drawn through C parallel to EF [Prop. 1.31]. Thus, $FECG$ is a parallelogram. And since BE is equal to EC, triangle ABE is also equal to triangle AEC. For they are on the equal bases, BE and EC, and between the same parallels, BC and AG [Prop. 1.38]. Thus, triangle ABC is double (the area) of triangle AEC. And parallelogram $FECG$ is also double (the area) of triangle AEC. For it has the same base (as AEC), and is between the same parallels (as AEC) [Prop. 1.41]. Thus, parallelogram $FECG$ is equal to triangle ABC. ($FECG$) also *τῇ Δ.*

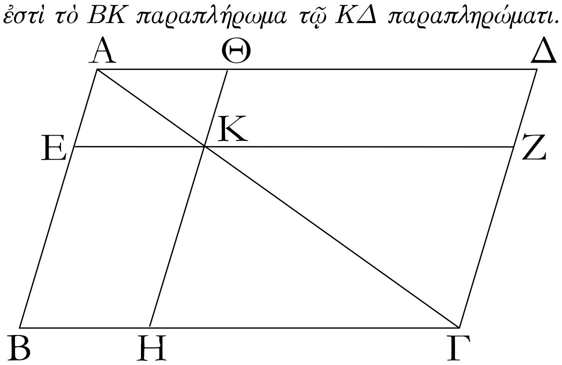


Τῷ ἄρα δοθέντι τριγώνῳ τῷ ΑΒΓ ἴσον παραλληλόγραμμον συνέσταται τὸ ΖΕΓΗ ἐν γωνίᾳ τῇ ὑπὸ ΓΕΖ, ἥτις ἐστὶν ἴση τῇ Δ· ὅπερ ἔδει ποιῆσαι.

μγʹ.

Παντὸς παραλληλογράμμου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν.

Ἔστω παραλληλόγραμμον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, περὶ δὲ τὴν ΑΓ παραλληλόγραμμα μὲν ἔστω τὰ ΕΘ, ΖΗ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΒΚ, ΚΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΒΚ παραπλήρωμα τῷ ΚΔ παραπληρώματι.



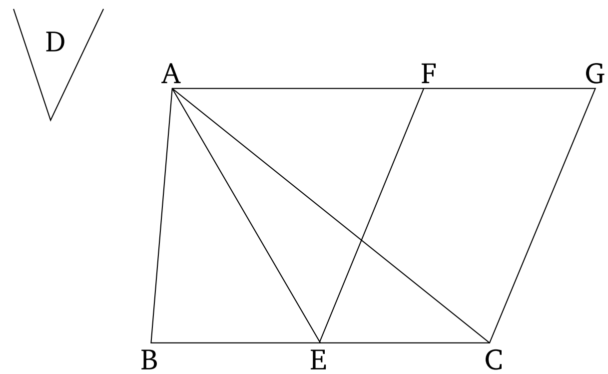
Ἐπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΓΔ τριγώνῳ. πάλιν, ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΕΘ, διάμετρος δὲ αὐτοῦ ἐστιν ἡ ΑΚ, ἴσον ἐστὶ τὸ ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΚΖΓ τρίγωνον τῷ ΚΗΓ ἐστιν ἴσον. ἐπεὶ οὖν τὸ μὲν ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ ἐστὶν ἴσον, τὸ δὲ ΚΖΓ τῷ ΚΗΓ, τὸ ΑΕΚ τρίγωνον μετὰ τοῦ ΚΗΓ ἴσον ἐστὶ τῷ ΑΘΚ τριγώνῳ μετὰ τοῦ ΚΖΓ· ἔστι δὲ καὶ ὅλον τὸ ΑΒΓ τρίγωνον ὅλῳ τῷ ΑΔΓ ἴσον· λοιπὸν ἄρα τὸ ΒΚ παραπλήρωμα λοιπῷ τῷ ΚΔ παραπληρώματί ἐστιν ἴσον.

Παντὸς ἄρα παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μδʹ.

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι τριγώνῳ ἴσον πα-

has the angle CEF equal to the given (angle) D.

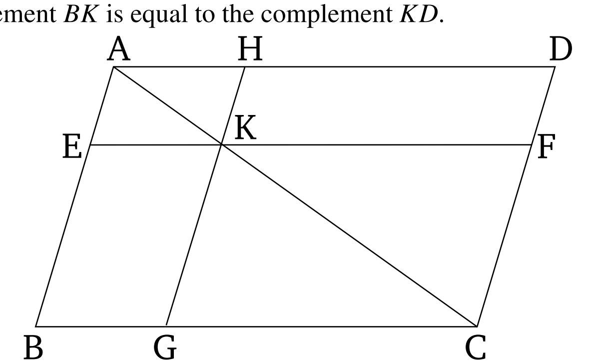


Thus, parallelogram $FECG$, equal to the given triangle ABC, has been constructed in the angle CEF, which is equal to D. (Which is) the very thing it was required to do.

Proposition 43

For any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.

Let $ABCD$ be a parallelogram, and AC its diagonal. And let EH and FG be the parallelograms about AC, and BK and KD the so-called complements (about AC). I say that the complement BK is equal to the complement KD.



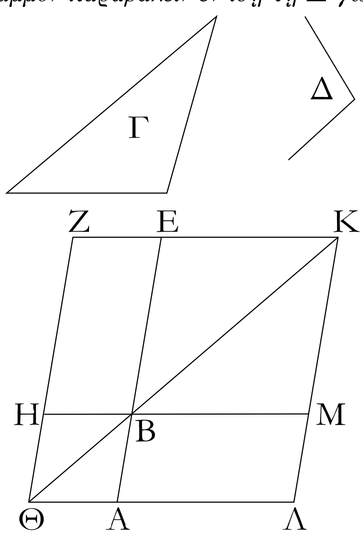
For since $ABCD$ is a parallelogram, and AC its diagonal, triangle ABC is equal to triangle ACD [Prop. 1.34]. Again, since EH is a parallelogram, and AK is its diagonal, triangle AEK is equal to triangle AHK [Prop. 1.34]. So, for the same (reasons), triangle KFC is also equal to (triangle) KGC. Therefore, since triangle AEK is equal to triangle AHK, and KFC to KGC, triangle AEK plus KGC is equal to triangle AHK plus KFC. And the whole triangle ABC is also equal to the whole (triangle) ADC. Thus, the remaining complement BK is equal to the remaining complement KD.

Thus, for any parallelogramic figure, the complements of the parallelograms about the diagonal are equal to one another. (Which is) the very thing it was required to show.

Proposition 44

To apply a parallelogram equal to a given triangle to a *ραλληλόγραμμον παραβαλεῖν ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.*

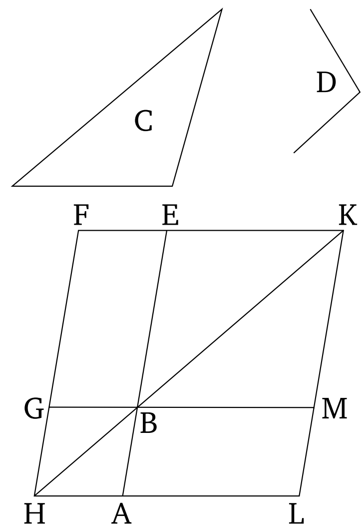
Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν τρίγωνον τὸ Γ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ· δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν ἴσῃ τῇ Δ γωνίᾳ.



Συνεστάτω τῷ Γ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΒΕΖΗ ἐν γωνίᾳ τῇ ὑπὸ ΕΒΗ, ἥ ἐστιν ἴση τῇ Δ· καὶ κείσθω ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΒΕ τῇ ΑΒ, καὶ διήχθω ἡ ΖΗ ἐπὶ τὸ Θ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΗ, ΕΖ παράλληλος ἤχθω ἡ ΑΘ, καὶ ἐπεζεύχθω ἡ ΘΒ. καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΘ, ΕΖ εὐθεῖα ἐνέπεσεν ἡ ΘΖ, αἱ ἄρα ὑπὸ ΑΘΖ, ΘΖΕ γωνίαι δυσὶν ὀρθαῖς εἰσιν ἴσαι. αἱ ἄρα ὑπὸ ΒΘΗ, ΗΖΕ δύο ὀρθῶν ἐλάσσονές εἰσιν· αἱ δὲ ἀπὸ ἐλασσόνων ἢ δύο ὀρθῶν εἰς ἄπειρον ἐκβαλλόμεναι συμπίπτουσιν· αἱ ΘΒ, ΖΕ ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Κ, καὶ διὰ τοῦ Κ σημείου ὁποτέρᾳ τῶν ΕΑ, ΖΘ παράλληλος ἤχθω ἡ ΚΛ, καὶ ἐκβεβλήσθωσαν αἱ ΘΑ, ΗΒ ἐπὶ τὰ Λ, Μ ση-

given straight-line in a given rectilinear angle.

Let *AB* be the given straight-line, *C* the given triangle, and *D* the given rectilinear angle. So it is required to apply a parallelogram equal to the given triangle *C* to the given straight-line *AB* in an angle equal to (angle) *D*.



Let the parallelogram *BEFG*, equal to the triangle *C*, be constructed in the angle *EBG*, which is equal to *D* [Prop. 1.42]. And let it be placed so that *BE* is straight-on to *AB*.† And let *FG* be drawn through to *H*, and let *AH* be drawn through *A* parallel to either of *BG* or *EF* [Prop. 1.31], and let *HB* be joined. And since the straight-line *HF* falls across the parallels *AH* and *EF*, the (sum of the) angles *AHF* and *HFE* is thus equal to two right-angles [Prop. 1.29]. Thus, (the sum of) *BHG* and *GFE* is less than two right-angles. And (straight-lines) produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced, *HB* and *FE* will meet together. Let them be produced, and let them meet together at *K*. And let *KL* be drawn *μεῖα. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΛΚΖ, διάμετρος δὲ αὐτοῦ ἡ ΘΚ, περὶ δὲ τὴν ΘΚ παραλληλόγραμμα μὲν τὰ ΑΗ, ΜΕ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΛΒ, ΒΖ· ἴσον ἄρα ἐστὶ τὸ ΛΒ τῷ ΒΖ. ἀλλὰ τὸ ΒΖ τῷ Γ τριγώνῳ ἐστὶν ἴσον· καὶ τὸ ΛΒ ἄρα τῷ Γ ἐστιν ἴσον. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΗΒΕ γωνία τῇ ὑπὸ ΑΒΜ, ἀλλὰ ἡ ὑπὸ ΗΒΕ τῇ Δ ἐστιν ἴση, καὶ ἡ ὑπὸ ΑΒΜ ἄρα τῇ Δ γωνίᾳ ἐστὶν ἴση.*

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΛΒ ἐν γωνίᾳ τῇ ὑπὸ ΑΒΜ, ἥ ἐστιν ἴση τῇ Δ· ὅπερ ἔδει ποιῆσαι.

through point K parallel to either of EA or FH [Prop. 1.31]. And let HA and GB be produced to points L and M (respectively). Thus, $HLKF$ is a parallelogram, and HK its diagonal. And AG and ME (are) parallelograms, and LB and BF the so-called complements, about HK. Thus, LB is equal to BF [Prop. 1.43]. But, BF is equal to triangle C. Thus, LB is also equal to C. Also, since angle GBE is equal to ABM [Prop. 1.15], but GBE is equal to D, ABM is thus also equal to angle D.

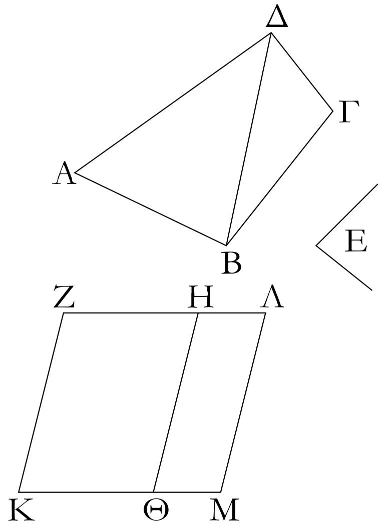
Thus, the parallelogram LB, equal to the given triangle C, has been applied to the given straight-line AB in the angle ABM, which is equal to D. (Which is) the very thing it was required to do.

† This can be achieved using Props. 1.3, 1.23, and 1.31.

μεʹ.

Τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἔστω τὸ μὲν δοθὲν εὐθύγραμμον τὸ ΑΒΓΔ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Ε· δεῖ δὴ τῷ ΑΒΓΔ εὐθυγράμμῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ τῇ Ε.

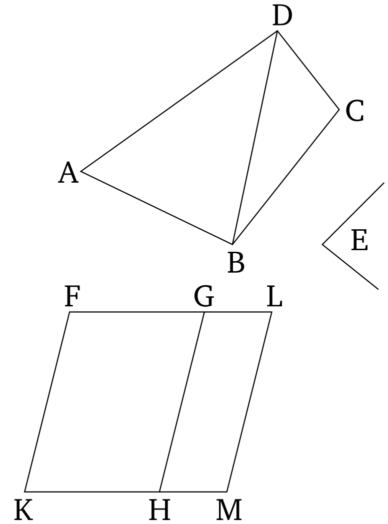


Ἐπεζεύχθω ἡ ΔΒ, καὶ συνεστάτω τῷ ΑΒΔ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΖΘ ἐν τῇ ὑπὸ ΘΚΖ γωνίᾳ, ἥ ἐστιν ἴση τῇ Ε· καὶ παραβεβλήσθω παρὰ τὴν ΗΘ εὐθεῖαν τῷ ΔΒΓ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΗΜ ἐν τῇ ὑπὸ ΗΘΜ γωνίᾳ, ἥ ἐστιν ἴση τῇ Ε. καὶ ἐπεὶ ἡ Ε γωνία ἑκατέρᾳ τῶν ὑπὸ ΘΚΖ, ΗΘΜ ἐστιν ἴση, καὶ ἡ ὑπὸ ΘΚΖ ἄρα τῇ ὑπὸ ΗΘΜ ἐστιν ἴση. κοινὴ προσκείσθω ἡ ὑπὸ ΚΘΗ· αἱ ἄρα ὑπὸ ΖΚΘ, ΚΘΗ ταῖς ὑπὸ ΚΘΗ, ΗΘΜ ἴσαι εἰσίν. ἀλλ᾽ αἱ ὑπὸ ΖΚΘ, ΚΘΗ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΚΘΗ, ΗΘΜ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν. πρὸς δή τινι εὐθείᾳ τῇ ΗΘ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ δύο εὐθεῖαι αἱ ΚΘ, ΘΜ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δύο

Proposition 45

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Let $ABCD$ be the given rectilinear figure,† and E the given rectilinear angle. So it is required to construct a parallelogram equal to the rectilinear figure $ABCD$ in the given angle E.



Let DB be joined, and let the parallelogram FH, equal to the triangle ABD, be constructed in the angle HKF, which is equal to E [Prop. 1.42]. And let the parallelogram GM, equal to the triangle DBC, be applied to the straight-line GH in the angle GHM, which is equal to E [Prop. 1.44]. And since angle E is equal to each of (angles) HKF and GHM, (angle) HKF is thus also equal to GHM. Let KHG be added to both. Thus, (the sum of) FKH and KHG is equal to (the sum of) KHG and GHM. But, (the sum of) FKH and KHG is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) KHG and GHM is also equal to two right-angles. So two straight-lines, KH and HM, not lying on the same side, make adjacent angles *ὀρθαῖς ἴσας ποιοῦσιν· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΚΘ τῇ ΘΜ· καὶ ἐπεὶ εἰς παραλλήλους τὰς ΚΜ, ΖΗ εὐθεῖα ἐνέπεσεν ἡ ΘΗ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΜΘΗ, ΘΗΖ ἴσαι ἀλλήλαις εἰσίν. κοινὴ προσκείσθω ἡ ὑπὸ ΘΗΛ· αἱ ἄρα ὑπὸ ΜΘΗ, ΘΗΛ ταῖς ὑπὸ ΘΗΖ, ΘΗΛ ἴσαι εἰσιν. ἀλλ᾽ αἱ ὑπὸ ΜΘΗ, ΘΗΛ δύο ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΘΗΖ, ΘΗΛ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΛ. καὶ ἐπεὶ ἡ ΖΚ τῇ ΘΗ ἴση τε καὶ παράλληλός ἐστιν, ἀλλὰ καὶ ἡ ΘΗ τῇ ΜΛ, καὶ ἡ ΚΖ ἄρα τῇ ΜΛ ἴση τε καὶ παράλληλός ἐστιν· καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΚΜ, ΖΛ· καὶ αἱ ΚΜ, ΖΛ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΚΖΛΜ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν ΑΒΔ τρίγωνον τῷ ΖΘ παραλληλογράμμῳ, τὸ δὲ ΔΒΓ τῷ ΗΜ, ὅλον ἄρα τὸ ΑΒΓΔ εὐθύγραμμον ὅλῳ τῷ ΚΖΛΜ παραλληλογράμμῳ ἐστὶν ἴσον.*

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ ΑΒΓΔ ἴσον παραλληλόγραμμον συνέσταται τὸ ΚΖΛΜ ἐν γωνίᾳ τῇ ὑπὸ ΖΚΜ, ἥ ἐστιν ἴση τῇ δοθείσῃ τῇ Ε· ὅπερ ἔδει ποιῆσαι.

with some straight-line *GH*, at the point *H* on it, (whose sum is) equal to two right-angles. Thus, *KH* is straight-on to *HM* [Prop. 1.14]. And since the straight-line *HG* falls across the parallels *KM* and *FG*, the alternate angles *mhG* and *HGF* are equal to one another [Prop. 1.29]. Let *HGL* have been added to both. Thus, (the sum of) *mhG* and *HGL* is equal to (the sum of) *HGF* and *HGL*. But, (the sum of) *mhG* and *HGL* is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) *HGF* and *HGL* is also equal to two right-angles. Thus, *FG* is straight-on to *GL* [Prop. 1.14]. And since *FK* is equal and parallel to *HG* [Prop. 1.34], but also *HG* to *ML* [Prop. 1.34], *KF* is thus also equal and parallel to *ML* [Prop. 1.30]. And the straight-lines *KM* and *FL* join them. Thus, *KM* and *FL* are equal and parallel as well [Prop. 1.33]. Thus, *KFLM* is a parallelogram. And since triangle *ABD* is equal to parallelogram *FH*, and *DBC* to *GM*, the whole rectilinear figure *ABCD* is thus equal to the whole parallelogram *KFLM*.

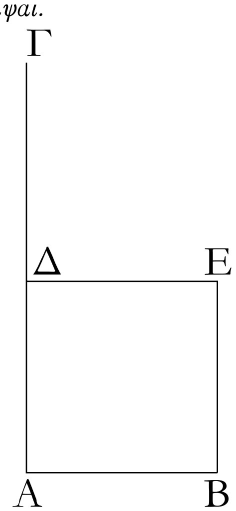
Thus, the parallelogram *KFLM*, equal to the given rectilinear figure *ABCD*, has been constructed in the angle *FKM*, which is equal to the given (angle) *E*. (Which is) the very thing it was required to do.

† The proof is only given for a four-sided figure. However, the extension to many-sided figures is trivial.

μϛ´.

Ἀπὸ τῆς δοθείσης εὐθείας τετράγωνον ἀναγράψαι.

Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ· δεῖ δὴ ἀπὸ τῆς ΑΒ εὐθείας τετράγωνον ἀναγράψαι.

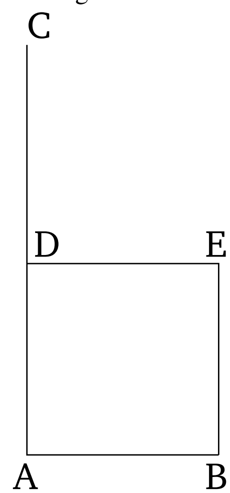


Ἤχθω τῇ ΑΒ εὐθείᾳ ἀπὸ τοῦ πρὸς αὐτῇ σημείου τοῦ Α πρὸς ὀρθὰς ἡ ΑΓ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΑΔ· καὶ διὰ μὲν τοῦ Δ σημείου τῇ ΑΒ παράλληλος ἤχθω ἡ ΔΕ, διὰ δὲ τοῦ Β σημείου τῇ ΑΔ παράλληλος ἤχθω ἡ ΒΕ. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΑΔΕΒ· ἴση ἄρα ἐστὶν ἡ μὲν ΑΒ τῇ ΔΕ, ἡ δὲ ΑΔ τῇ ΒΕ. ἀλλὰ ἡ ΑΒ τῇ ΑΔ ἐστιν ἴση· αἱ τέσσαρες ἄρα αἱ ΒΑ, ΑΔ, ΔΕ, ΕΒ ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ τὸ Α-

Proposition 46

To describe a square on a given straight-line.

Let *AB* be the given straight-line. So it is required to describe a square on the straight-line *AB*.



Let *AC* be drawn at right-angles to the straight-line *AB* from the point *A* on it [Prop. 1.11], and let *AD* be made equal to *AB* [Prop. 1.3]. And let *DE* be drawn through point *D* parallel to *AB* [Prop. 1.31], and let *BE* be drawn through point *B* parallel to *AD* [Prop. 1.31]. Thus, *ADEB* is a parallelogram. Therefore, *AB* is equal to *DE*, and *AD* to *BE* [Prop. 1.34]. But, *AB* is equal to *AD*. Thus, the four (sides) *BA*, *AD*, *DE*, and

ΔΕΒ παραλληλόγραμμον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ εἰς παραλλήλους τὰς ΑΒ, ΔΕ εὐθεῖα ἐνέπεσεν ἡ ΑΔ, αἱ ἄρα ὑπὸ ΒΑΔ, ΑΔΕ γωνίαι δύο ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΒΑΔ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΑΔΕ. τῶν δὲ παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραί τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὀρθὴ ἄρα καὶ ἑκατέρα τῶν ἀπεναντίον τῶν ὑπὸ ΑΒΕ, ΒΕΔ γωνιῶν· ὀρθογώνιον ἄρα ἐστὶ τὸ ΑΔΕΒ. ἐδείχθη δὲ καὶ ἰσόπλευρον.

Τετράγωνον ἄρα ἐστίν· καί ἐστιν ἀπὸ τῆς ΑΒ εὐθείας ἀναγεγραμμένον· ὅπερ ἔδει ποιῆσαι.

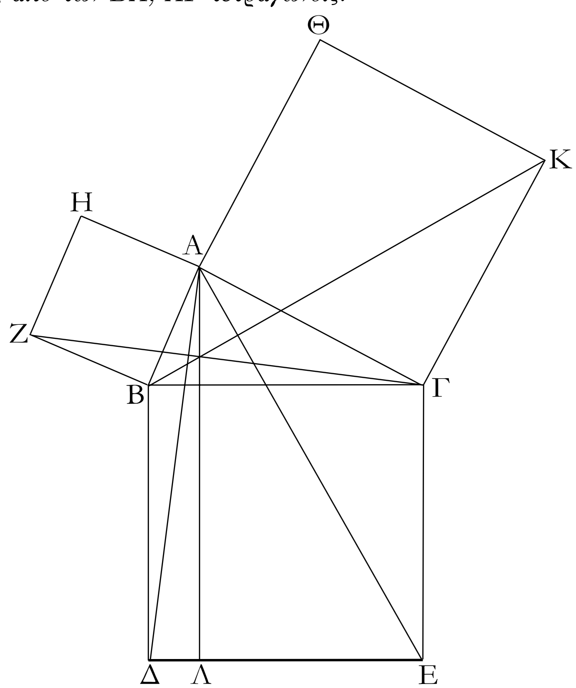
EB are equal to one another. Thus, the parallelogram *ADEB* is equilateral. So I say that (it is) also right-angled. For since the straight-line *AD* falls across the parallels *AB* and *DE*, the (sum of the) angles *BAD* and *ADE* is equal to two right-angles [Prop. 1.29]. But *BAD* (is a) right-angle. Thus, *ADE* (is) also a right-angle. And for parallelogramic figures, the opposite sides and angles are equal to one another [Prop. 1.34]. Thus, each of the opposite angles *ABE* and *BED* (are) also right-angles. Thus, *ADEB* is right-angled. And it was also shown (to be) equilateral.

Thus, (*ADEB*) is a square [Def. 1.22]. And it is described on the straight-line *AB*. (Which is) the very thing it was required to do.

μζ'.

Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς ΒΓ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις.

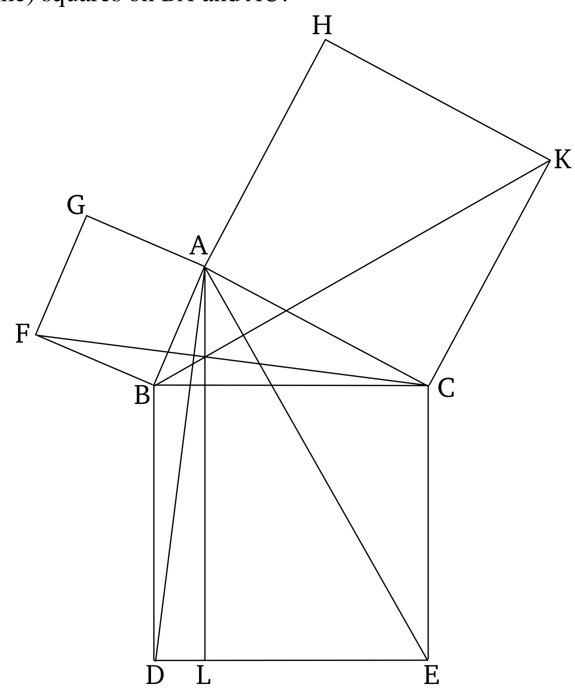


Ἀναγεγράφθω γὰρ ἀπὸ μὲν τῆς ΒΓ τετράγωνον τὸ ΒΔΕΓ, ἀπὸ δὲ τῶν ΒΑ, ΑΓ τὰ ΗΒ, ΘΓ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΔ, ΓΕ παράλληλος ἤχθω ἡ ΑΛ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΖΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἑκατέρα τῶν ὑπὸ ΒΑΓ, ΒΑΗ γωνιῶν, πρὸς δή τινι εὐθείᾳ τῇ ΒΑ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α δύο εὐθεῖαι αἱ ΑΓ, ΑΗ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΓΑ τῇ ΑΗ. διὰ τὰ αὐτὰ δὴ καὶ ἡ

Proposition 47

In right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides containing the right-angle.

Let *ABC* be a right-angled triangle having the angle *BAC* a right-angle. I say that the square on *BC* is equal to the (sum of the) squares on *BA* and *AC*.



For let the square *BDEC* be described on *BC*, and (the squares) *GB* and *HC* on *AB* and *AC* (respectively) [Prop. 1.46]. And let *AL* be drawn through point *A* parallel to either of *BD* or *CE* [Prop. 1.31]. And let *AD* and *FC* be joined. And since angles *BAC* and *BAG* are each right-angles, then two straight-lines *AC* and *AG*, not lying on the same side, make the adjacent angles with some straight-line *BA*, at the point *A* on it, (whose sum is) equal to two right-angles. Thus, ΒΑ τῇ ΑΘ ἐστιν ἐπ᾽ εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῇ ὑπὸ ΖΒΑ· ὀρθὴ γὰρ ἑκατέρα· κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· ὅλη ἄρα ἡ ὑπὸ ΔΒΑ ὅλῃ τῇ ὑπὸ ΖΒΓ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΓ, ἡ δὲ ΖΒ τῇ ΒΑ, δύο δὴ αἱ ΔΒ, ΒΑ δύο ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΔΒΑ γωνίᾳ τῇ ὑπὸ ΖΒΓ ἴση· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΖΓ [ἐστιν] ἴση, καὶ τὸ ΑΒΔ τρίγωνον τῷ ΖΒΓ τριγώνῳ ἐστὶν ἴσον· καί [ἐστι] τοῦ μὲν ΑΒΔ τριγώνου διπλάσιον τὸ ΒΛ παραλληλόγραμμον· βάσιν τε γὰρ τὴν αὐτὴν ἔχουσι τὴν ΒΔ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΒΔ, ΑΛ· τοῦ δὲ ΖΒΓ τριγώνου διπλάσιον τὸ ΗΒ τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν ΖΒ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΖΒ, ΗΓ. [τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἐστίν·] ἴσον ἄρα ἐστὶ καὶ τὸ ΒΛ παραλληλόγραμμον τῷ ΗΒ τετραγώνῳ. ὁμοίως δὴ ἐπιζευγνυμένων τῶν ΑΕ, ΒΚ δειχθήσεται καὶ τὸ ΓΛ παραλληλόγραμμον ἴσον τῷ ΘΓ τετραγώνῳ· ὅλον ἄρα τὸ ΒΔΕΓ τετράγωνον δυσὶ τοῖς ΗΒ, ΘΓ τετραγώνοις ἴσον ἐστίν. καί ἐστι τὸ μὲν ΒΔΕΓ τετράγωνον ἀπὸ τῆς ΒΓ ἀναγραφέν, τὰ δὲ ΗΒ, ΘΓ ἀπὸ τῶν ΒΑ, ΑΓ. τὸ ἄρα ἀπὸ τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις.

Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχουσῶν πλευρῶν τετραγώνοις· ὅπερ ἔδει δεῖξαι.

CA is straight-on to *AG* [Prop. 1.14]. So, for the same (reasons), *BA* is also straight-on to *AH*. And since angle *DBC* is equal to *FBA*, for (they are) both right-angles, let *ABC* be added to both. Thus, the whole (angle) *DBA* is equal to the whole (angle) *FBC*. And since *DB* is equal to *BC*, and *FB* to *BA*, the two (straight-lines) *DB*, *BA* are equal to the two (straight-lines) *CB*, *BF*,[†] respectively. And angle *DBA* (is) equal to angle *FBC*. Thus, the base *AD* [is] equal to the base *FC*, and the triangle *ABD* is equal to the triangle *FBC* [Prop. 1.4]. And parallelogram *BL* [is] double (the area) of triangle *ABD*. For they have the same base, *BD*, and are between the same parallels, *BD* and *AL* [Prop. 1.41]. And square *GB* is double (the area) of triangle *FBC*. For again they have the same base, *FB*, and are between the same parallels, *FB* and *GC* [Prop. 1.41]. [And the doubles of equal things are equal to one another.][‡] Thus, the parallelogram *BL* is also equal to the square *GB*. So, similarly, *AE* and *BK* being joined, the parallelogram *CL* can be shown (to be) equal to the square *HC*. Thus, the whole square *BDEC* is equal to the (sum of the) two squares *GB* and *HC*. And the square *BDEC* is described on *BC*, and the (squares) *GB* and *HC* on *BA* and *AC* (respectively). Thus, the square on the side *BC* is equal to the (sum of the) squares on the sides *BA* and *AC*.

Thus, in right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-[angle]. (Which is) the very thing it was required to show.

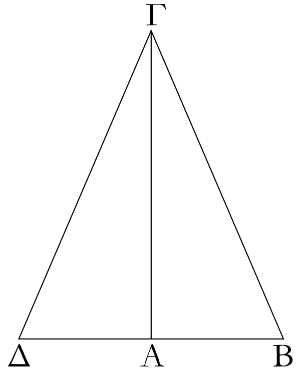
[†] The Greek text has "*FB*, *BC*", which is obviously a mistake.

[‡] This is an additional common notion.

μη΄.

Ἐὰν τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ᾖ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν.

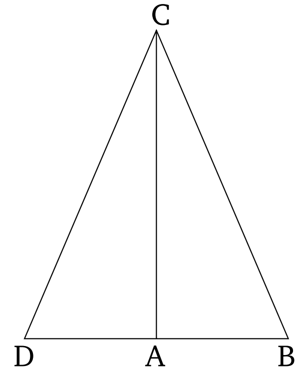
Τριγώνου γὰρ τοῦ ΑΒΓ τὸ ἀπὸ μιᾶς τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἔστω τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις· λέγω, ὅτι ὀρθή ἐστιν ἡ ὑπὸ ΒΑΓ γωνία.



Proposition 48

If the square on one of the sides of a triangle is equal to the (sum of the) squares on the two remaining sides of the triangle (then) the angle contained by the two remaining sides of the triangle is a right-angle.

For let the square on one of the sides, *BC*, of triangle *ABC* be equal to the (sum of the) squares on the sides *BA* and *AC*. I say that angle *BAC* is a right-angle.



Ἤχθω γὰρ ἀπὸ τοῦ Α σημείου τῇ ΑΓ εὐθείᾳ πρὸς ὀρθὰς ἡ ΑΔ καὶ κείσθω τῇ ΒΑ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΓ. ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΔΑ τετράγωνον τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΑΓ τετράγωνον· τὰ ἄρα ἀπὸ τῶν ΔΑ, ΑΓ τετράγωνα ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΔΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΔΓ· ὀρθὴ γάρ ἐστιν ἡ ὑπὸ ΔΑΓ γωνία· τοῖς δὲ ἀπὸ τῶν ΒΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΓ· ὑπόκειται γάρ· τὸ ἄρα ἀπὸ τῆς ΔΓ τετράγωνον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· ὥστε καὶ πλευρὰ ἡ ΔΓ τῇ ΒΓ ἐστιν ἴση· καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δὴ αἱ ΔΑ, ΑΓ δύο ταῖς ΒΑ, ΑΓ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΓ βάσει τῇ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνίᾳ τῇ ὑπὸ ΒΑΓ [ἐστιν] ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΔΑΓ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΒΑΓ.

Ἐὰν ἄρα τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ᾖ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν· ὅπερ ἔδει δεῖξαι.

For let *AD* be drawn from point *A* at right-angles to the straight-line *AC* [Prop. 1.11], and let *AD* be made equal to *BA* [Prop. 1.3], and let *DC* be joined. Since *DA* is equal to *AB*, the square on *DA* is thus also equal to the square on *AB*.† Let the square on *AC* be added to both. Thus, the (sum of the) squares on *DA* and *AC* is equal to the (sum of the) squares on *BA* and *AC*. But, the (square) on *DC* is equal to the (sum of the squares) on *DA* and *AC*. For angle *DAC* is a right-angle [Prop. 1.47]. But, the (square) on *BC* is equal to (sum of the squares) on *BA* and *AC*. For (that) was assumed. Thus, the square on *DC* is equal to the square on *BC*. So side *DC* is also equal to (side) *BC*. And since *DA* is equal to *AB*, and *AC* (is) common, the two (straight-lines) *DA*, *AC* are equal to the two (straight-lines) *BA*, *AC*. And the base *DC* is equal to the base *BC*. Thus, angle *DAC* [is] equal to angle *BAC* [Prop. 1.8]. But *DAC* is a right-angle. Thus, *BAC* is also a right-angle.

Thus, if the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining two sides of the triangle (then) the angle contained by the remaining two sides of the triangle is a right-angle. (Which is) the very thing it was required to show.

† Here, use is made of the additional common notion that the squares of equal things are themselves equal. Later on, the inverse notion is used.

ELEMENTS BOOK 2

Fundamentals of Geometric Algebra

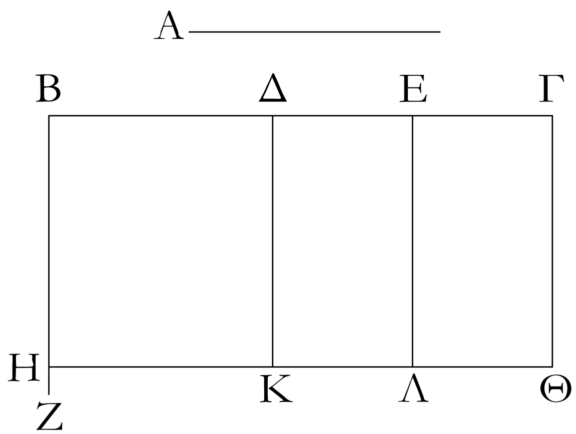
Ὅροι.

α΄. Πᾶν παραλληλόγραμμον ὀρθογώνιον περιέχεσθαι λέγεται ὑπὸ δύο τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν εὐθειῶν.

β΄. Παντὸς δὲ παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον αὐτοῦ παραλληλογράμμων ἓν ὁποιονοῦν σὺν τοῖς δυσὶ παραπληρώμασι γνώμων καλείσθω.

α΄.

Ἐὰν ὦσι δύο εὐθεῖαι, τμηθῇ δὲ ἡ ἑτέρα αὐτῶν εἰς ὁσαδηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπό τε τῆς ἀτμήτου καὶ ἑκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις.



Ἔστωσαν δύο εὐθεῖαι αἱ Α, ΒΓ, καὶ τετμήσθω ἡ ΒΓ, ὡς ἔτυχεν, κατὰ τὰ Δ, Ε σημεῖα· λέγω, ὅτι τὸ ὑπὸ τῶν Α, ΒΓ περιεχομένον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν Α, ΒΔ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ὑπὸ τῶν Α, ΔΕ καὶ ἔτι τῷ ὑπὸ τῶν Α, ΕΓ.

Ἤχθω γὰρ ἀπὸ τοῦ Β τῇ ΒΓ πρὸς ὀρθὰς ἡ ΒΖ, καὶ κείσθω τῇ Α ἴση ἡ ΒΗ, καὶ διὰ μὲν τοῦ Η τῇ ΒΓ παράλληλος ἤχθω ἡ ΗΘ, διὰ δὲ τῶν Δ, Ε, Γ τῇ ΒΗ παράλληλοι ἤχθωσαν αἱ ΔΚ, ΕΛ, ΓΘ.

Ἴσον δή ἐστι τὸ ΒΘ τοῖς ΒΚ, ΔΛ, ΕΘ. καί ἐστι τὸ μὲν ΒΘ τὸ ὑπὸ τῶν Α, ΒΓ· περιέχεται μὲν γὰρ ὑπὸ τῶν ΗΒ, ΒΓ, ἴση δὲ ἡ ΒΗ τῇ Α· τὸ δὲ ΒΚ τὸ ὑπὸ τῶν Α, ΒΔ· περιέχεται μὲν γὰρ ὑπὸ τῶν ΗΒ, ΒΔ, ἴση δὲ ἡ ΒΗ τῇ Α. τὸ δὲ ΔΛ τὸ ὑπὸ τῶν Α, ΔΕ· ἴση γὰρ ἡ ΔΚ, τουτέστιν ἡ ΒΗ, τῇ Α. καὶ ἔτι ὁμοίως τὸ ΕΘ τὸ ὑπὸ τῶν Α, ΕΓ· τὸ ἄρα ὑπὸ τῶν Α, ΒΓ ἴσον ἐστὶ τῷ τε ὑπὸ Α, ΒΔ καὶ τῷ ὑπὸ Α, ΔΕ καὶ ἔτι τῷ ὑπὸ Α, ΕΓ.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι, τμηθῇ δὲ ἡ ἑτέρα αὐτῶν εἰς ὁσαδηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπό τε τῆς ἀτμήτου καὶ ἑκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις· ὅπερ ἔδει δεῖξαι.

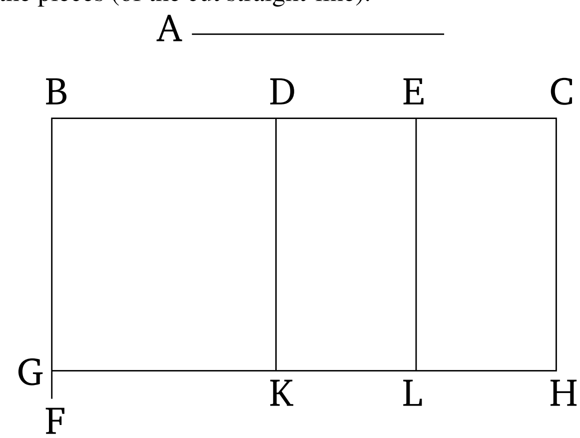
Definitions

1. Any rectangular parallelogram is said to be contained by the two straight-lines containing the right-angle.

2. And in any parallelogrammic figure, let any one whatsoever of the parallelograms about its diagonal, (taken) with its two complements, be called a gnomon.

Proposition 1†

If there are two straight-lines, and one of them is cut into any number of pieces whatsoever, (then) the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line).



Let *A* and *BC* be the two straight-lines, and let *BC* be cut, at random, at points *D* and *E*. I say that the rectangle contained by *A* and *BC* is equal to the rectangle(s) contained by *A* and *BD*, by *A* and *DE*, and, finally, by *A* and *EC*.

For let *BF* be drawn from point *B*, at right-angles to *BC* [Prop. 1.11], and let *BG* be made equal to *A* [Prop. 1.3], and let *GH* be drawn through (point) *G*, parallel to *BC* [Prop. 1.31], and let *DK*, *EL*, and *CH* be drawn through (points) *D*, *E*, and *C* (respectively), parallel to *BG* [Prop. 1.31].

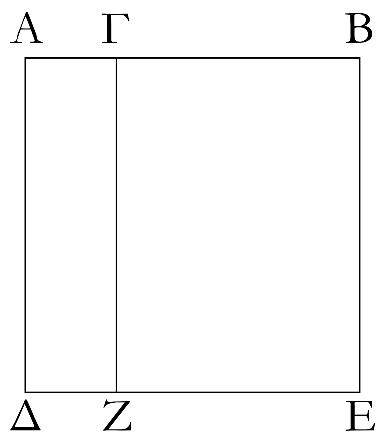
So the (rectangle) *BH* is equal to the (rectangles) *BK*, *DL*, and *EH*. And *BH* is the (rectangle contained) by *A* and *BC*. For it is contained by *GB* and *BC*, and *BG* (is) equal to *A*. And *BK* (is) the (rectangle contained) by *A* and *BD*. For it is contained by *GB* and *BD*, and *BG* (is) equal to *A*. And *DL* (is) the (rectangle contained) by *A* and *DE*. For *DK*, that is to say *BG* [Prop. 1.34], (is) equal to *A*. Similarly, *EH* (is) also the (rectangle contained) by *A* and *EC*. Thus, the (rectangle contained) by *A* and *BC* is equal to the (rectangles contained) by *A* and *BD*, by *A* and *DE*, and, finally, by *A* and *EC*.

Thus, if there are two straight-lines, and one of them is cut into any number of pieces whatsoever, (then) the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line). (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $a\,(b + c + d + \cdots) = a\,b + a\,c + a\,d + \cdots$.

β′.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ.



Εὐθεῖα γὰρ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ὑπὸ ΒΑ, ΑΓ περιεχομένου ὀρθογωνίου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ.

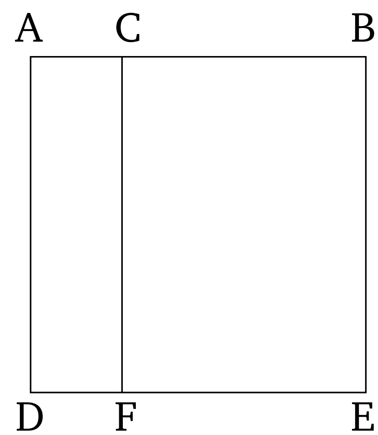
Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ ἤχθω διὰ τοῦ Γ ὁποτέρᾳ τῶν ΑΔ, ΒΕ παράλληλος ἡ ΓΖ.

Ἴσον δή ἐστι τὸ ΑΕ τοῖς ΑΖ, ΓΕ. καί ἐστι τὸ μὲν ΑΕ τὸ ἀπὸ τῆς ΑΒ τετράγωνον, τὸ δὲ ΑΖ τὸ ὑπὸ τῶν ΒΑ, ΑΓ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν ΔΑ, ΑΓ, ἴση δὲ ἡ ΑΔ τῇ ΑΒ· τὸ δὲ ΓΕ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ἴση γὰρ ἡ ΒΕ τῇ ΑΒ. τὸ ἄρα ὑπὸ τῶν ΒΑ, ΑΓ μετὰ τοῦ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

Proposition 2†

If a straight-line is cut at random, (then) the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole.



For let the straight-line AB be cut, at random, at point C. I say that the rectangle contained by AB and BC, plus the rectangle contained by BA and AC, is equal to the square on AB.

For let the square $ADEB$ be described on AB [Prop. 1.46], and let CF be drawn through C, parallel to either of AD or BE [Prop. 1.31].

So the (square) AE is equal to the (rectangles) AF and CE. And AE is the square on AB. And AF (is) the rectangle contained by the (straight-lines) BA and AC. For it is contained by DA and AC, and AD (is) equal to AB. And CE (is) the (rectangle contained) by AB and BC. For BE (is) equal to AB. Thus, the (rectangle contained) by BA and AC, plus the (rectangle contained) by AB and BC, is equal to the square on AB.

Thus, if a straight-line is cut at random, (then) the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole. (Which is) the very thing it was required to show.

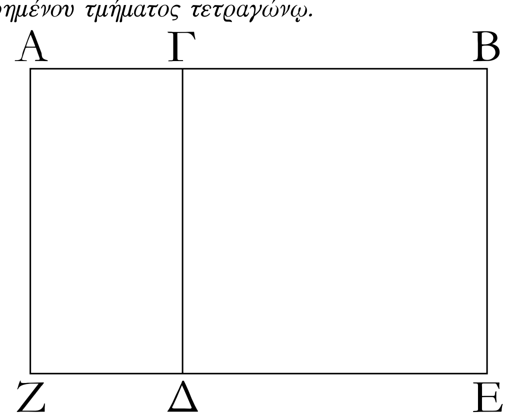
† This proposition is a geometric version of the algebraic identity: $a\,b + a\,c = a^2$ if $a = b + c$.

γ′.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ

Proposition 3†

If a straight-line is cut at random, (then) the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both *προειρημένου τμήματος τετραγώνῳ.*

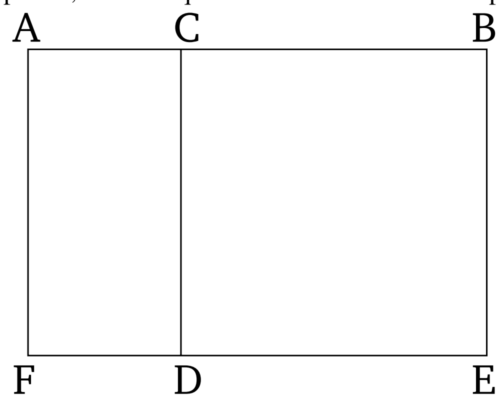


Εὐθεῖα γὰρ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΒΓ τετραγώνου.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΒ τετράγωνον τὸ ΓΔΕΒ, καὶ διήχθω ἡ ΕΔ ἐπὶ τὸ Ζ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΔ, ΒΕ παράλληλος ἤχθω ἡ ΑΖ. ἴσον δή ἐστι τὸ ΑΕ τοῖς ΑΔ, ΓΕ· καί ἐστι τὸ μὲν ΑΕ τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν ΑΒ, ΒΕ, ἴση δὲ ἡ ΒΕ τῇ ΒΓ· τὸ δὲ ΑΔ τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἴση γὰρ ἡ ΔΓ τῇ ΓΒ· τὸ δὲ ΔΒ τὸ ἀπὸ τῆς ΓΒ τετράγωνον· τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΒΓ τετραγώνου.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ προειρημένου τμήματος τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

of) the pieces, and the square on the aforementioned piece.



For let the straight-line AB be cut, at random, at (point) C. I say that the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

For let the square $CDEB$ be described on CB [Prop. 1.46], and let ED be drawn through to F, and let AF be drawn through A, parallel to either of CD or BE [Prop. 1.31]. So the (rectangle) AE is equal to the (rectangle) AD and the (square) CE. And AE is the rectangle contained by AB and BC. For it is contained by AB and BE, and BE (is) equal to BC. And AD (is) the (rectangle contained) by AC and CB. For DC (is) equal to CB. And DB (is) the square on CB. Thus, the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

Thus, if a straight-line is cut at random, (then) the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both of) the pieces, and the square on the aforementioned piece. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(a + b)\,a = a\,b + a^2$.

δ΄.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ.

Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ. λέγω, ὅτι τὸ ἀπὸ τῆς ΑΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ ἐπεζεύχθω ἡ ΒΔ, καὶ διὰ μὲν τοῦ Γ ὁποτέρᾳ τῶν ΑΔ, ΕΒ παράλληλος ἤχθω ἡ ΓΖ, διὰ δὲ τοῦ Η ὁποτέρᾳ τῶν ΑΒ, ΔΕ παράλληλος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΓΖ τῇ ΑΔ, καὶ εἰς αὐτὰς ἐμπέπτωκεν ἡ ΒΔ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΓΗΒ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΑΔΒ. ἀλλ᾽ ἡ ὑπὸ ΑΔΒ τῇ ὑπὸ ΑΒΔ ἐστιν ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΒΑ τῇ ΑΔ ἐστιν ἴση· καὶ ἡ ὑπὸ ΓΗΒ ἄρα γωνιά τῇ ὑπὸ

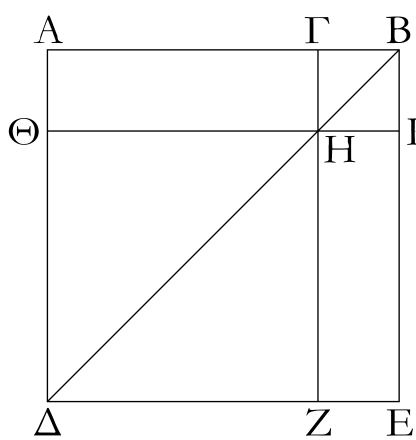
Proposition 4†

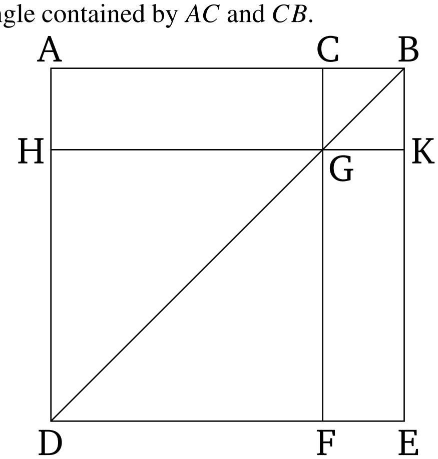
If a straight-line is cut at random, (then) the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces.

For let the straight-line AB be cut, at random, at (point) C. I say that the square on AB is equal to the (sum of the) squares on AC and CB, and twice the rectangle contained by AC and CB.

For let the square $ADEB$ be described on AB [Prop. 1.46], and let BD be joined, and let CF be drawn through C, parallel to either of AD or EB [Prop. 1.31], and let HK be drawn through G, parallel to either of AB or DE [Prop. 1.31]. And since CF is parallel to AD, and BD has fallen across them, the external angle CGB is equal to the internal and opposite (angle) ADB [Prop. 1.29]. But, ADB is equal to ABD, ΗΒΓ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΒΓ πλευρᾷ τῇ ΓΗ ἐστιν ἴση· ἀλλ᾿ ἡ μὲν ΓΒ τῇ ΗΚ ἐστιν ἴση, ἡ δὲ ΓΗ τῇ ΚΒ· καὶ ἡ ΗΚ ἄρα τῇ ΚΒ ἐστιν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΓΗΚΒ. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΓΗ τῇ ΒΚ [καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΓΒ], αἱ ἄρα ὑπὸ ΚΒΓ, ΗΓΒ γωνίαι δύο ὀρθαῖς εἰσιν ἴσαι. ὀρθὴ δὲ ἡ ὑπὸ ΚΒΓ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΒΓΗ· ὥστε καὶ αἱ ἀπεναντίον αἱ ὑπὸ ΓΗΚ, ΗΚΒ ὀρθαί εἰσιν. ὀρθογώνιον ἄρα ἐστὶ τὸ ΓΗΚΒ· ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν· καί ἐστιν ἀπὸ τῆς ΓΒ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΘΖ τετράγωνόν ἐστιν· καί ἐστιν ἀπὸ τῆς ΘΗ, τουτέστιν [ἀπὸ] τῆς ΑΓ· τὰ ἄρα ΘΖ, ΚΓ τετράγωνα ἀπὸ τῶν ΑΓ, ΓΒ εἰσιν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΗ τῷ ΗΕ, καί ἐστι τὸ ΑΗ τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἴση γὰρ ἡ ΗΓ τῇ ΓΒ· καὶ τὸ ΗΕ ἄρα ἴσον ἐστὶ τῷ ὑπὸ ΑΓ, ΓΒ· τὰ ἄρα ΑΗ, ΗΕ ἴσα ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. ἔστι δὲ καὶ τὰ ΘΖ, ΓΚ τετράγωνα ἀπὸ τῶν ΑΓ, ΓΒ· τὰ ἄρα τέσσαρα τὰ ΘΖ, ΓΚ, ΑΗ, ΗΕ ἴσα ἐστὶ τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ. ἀλλὰ τὰ ΘΖ, ΓΚ, ΑΗ, ΗΕ ὅλον ἐστὶ τὸ ΑΔΕΒ, ὅ ἐστιν ἀπὸ τῆς ΑΒ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΑΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ περιεχομένῳ ὀρθογωνίῳ.

since the side *BA* is also equal to *AD* [Prop. 1.5]. Thus, angle *CGB* is also equal to *GBC*. So the side *BC* is equal to the side *CG* [Prop. 1.6]. But, *CB* is equal to *GK*, and *CG* to *KB* [Prop. 1.34]. Thus, *GK* is also equal to *KB*. Thus, *CGKB* is equilateral. So I say that (it is) also right-angled. For since *CG* is parallel to *BK* [and the straight-line *CB* has fallen across them], the angles *KBC* and *GCB* are thus equal to two right-angles [Prop. 1.29]. But *KBC* (is) a right-angle. Thus, *BCG* (is) also a right-angle. So the opposite (angles) *CGK* and *GKB* are also right-angles [Prop. 1.34]. Thus, *CGKB* is right-angled. And it was also shown (to be) equilateral. Thus, it is a square. And it is on *CB*. So, for the same (reasons), *HF* is also a square. And it is on *HG*, that is to say [on] *AC* [Prop. 1.34]. Thus, the squares *HF* and *KC* are on *AC* and *CB* (respectively). And the (rectangle) *AG* is equal to the (rectangle) *GE* [Prop. 1.43]. And *AG* is the (rectangle contained) by *AC* and *CB*. For *GC* (is) equal to *CB*. Thus, *GE* is also equal to the (rectangle contained) by *AC* and *CB*. Thus, the (rectangles) *AG* and *GE* are equal to twice the (rectangle contained) by *AC* and *CB*. And *HF* and *CK* are the squares on *AC* and *CB* (respectively). Thus, the four (figures) *HF*, *CK*, *AG*, and *GE* are equal to the (sum of the) squares on *AC* and *BC*, and twice the rectangle contained by *AC* and *CB*. But, the (figures) *HF*, *CK*, *AG*, and *GE* are (equivalent to) the whole of *ADEB*, which is the square on *AB*. Thus, the square on *AB* is equal to the (sum of the) squares on *AC* and *CB*, and twice the rectangle contained by *AC* and *CB*.





Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ· ὅπερ ἔδει δεῖξαι.

Thus, if a straight-line is cut at random, (then) the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces. (Which is) the very thing it was required to show.

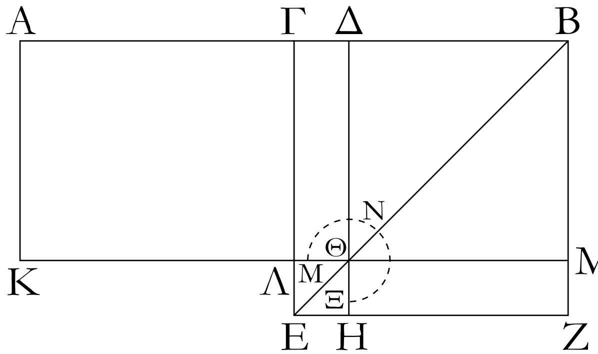
† This proposition is a geometric version of the algebraic identity: $(a+b)^2 = a^2 + b^2 + 2\,a\,b$.

ε′.

Proposition 5‡

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν

If a straight-line is cut into equal and unequal (pieces, then) ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ.

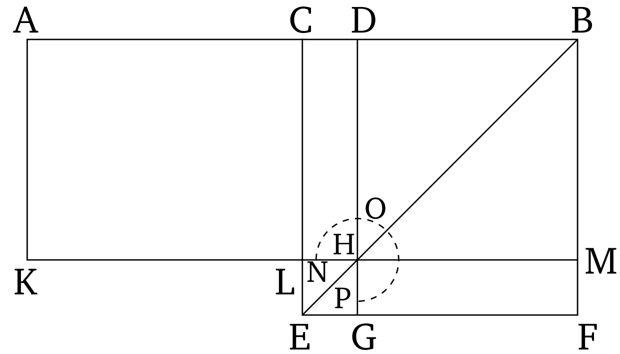


Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΒ τετράγωνον τὸ ΓΕΖΒ, καὶ ἐπεζεύχθω ἡ ΒΕ, καὶ διὰ μὲν τοῦ Δ ὁποτέρᾳ τῶν ΓΕ, ΒΖ παράλληλος ἤχθω ἡ ΔΗ, διὰ δὲ τοῦ Θ ὁποτέρᾳ τῶν ΑΒ, ΕΖ παράλληλος πάλιν ἤχθω ἡ ΚΜ, καὶ πάλιν διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΛ, ΒΜ παράλληλος ἤχθω ἡ ΑΚ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΓΘ παραπλήρωμα τῷ ΘΖ παραπληρώματι, κοινὸν προσκείσθω τὸ ΔΜ· ὅλον ἄρα τὸ ΓΜ ὅλῳ τῷ ΔΖ ἴσον ἐστίν. ἀλλὰ τὸ ΓΜ τῷ ΑΛ ἴσον ἐστίν, ἐπεὶ καὶ ἡ ΑΓ τῇ ΓΒ ἐστιν ἴση· καὶ τὸ ΑΛ ἄρα τῷ ΔΖ ἴσον ἐστίν. κοινὸν προσκείσθω τὸ ΓΘ· ὅλον ἄρα τὸ ΑΘ τῷ ΜΝΞ[†] γνώμονι ἴσον ἐστίν. ἀλλὰ τὸ ΑΘ τὸ ὑπὸ τῶν ΑΔ, ΔΒ ἐστιν· ἴση γὰρ ἡ ΔΘ τῇ ΔΒ· καὶ ὁ ΜΝΞ ἄρα γνώμων ἴσος ἐστὶ τῷ ὑπὸ ΑΔ, ΔΒ. κοινὸν προσκείσθω τὸ ΛΗ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΓΔ· ὁ ἄρα ΜΝΞ γνώμων καὶ τὸ ΛΗ ἴσα ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ. ἀλλὰ ὁ ΜΝΞ γνώμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΒ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ. ὅπερ ἔδει δεῖξαι.

the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line).



For let any straight-line AB be cut—equally at C, and unequally at D. I say that the rectangle contained by AD and DB, plus the square on CD, is equal to the square on CB.

For let the square $CEFB$ be described on CB [Prop. 1.46], and let BE be joined, and let DG be drawn through D, parallel to either of CE or BF [Prop. 1.31], and again let KM be drawn through H, parallel to either of AB or EF [Prop. 1.31], and again let AK be drawn through A, parallel to either of CL or BM [Prop. 1.31]. And since the complement CH is equal to the complement HF [Prop. 1.43], let the (square) DM be added to both. Thus, the whole (rectangle) CM is equal to the whole (rectangle) DF. But, (rectangle) CM is equal to (rectangle) AL, since AC is also equal to CB [Prop. 1.36]. Thus, (rectangle) AL is also equal to (rectangle) DF. Let (rectangle) CH be added to both. Thus, the whole (rectangle) AH is equal to the gnomon NOP. But, AH is the (rectangle contained) by AD and DB. For DH (is) equal to DB. Thus, the gnomon NOP is also equal to the (rectangle contained) by AD and DB. Let LG, which is equal to the (square) on CD, be added to both. Thus, the gnomon NOP and the (square) LG are equal to the rectangle contained by AD and DB, and the square on CD. But, the gnomon NOP and the (square) LG is (equivalent to) the whole square $CEFB$, which is on CB. Thus, the rectangle contained by AD and DB, plus the square on CD, is equal to the square on CB.

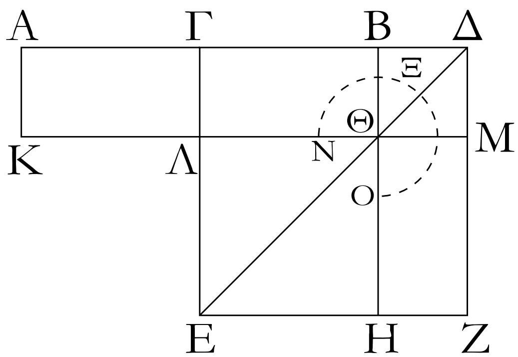
Thus, if a straight-line is cut into equal and unequal (pieces, then) the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line). (Which is) the very thing it was required to show.

[†] Note the (presumably mistaken) double use of the label M in the Greek text.

[‡] This proposition is a geometric version of the algebraic identity: $a\,b + [(a+b)/2 - b]^2 = [(a+b)/2]^2$.

ϛ΄.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ᾿ εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τῆς προσκειμένης περιεχόμενον ὀρθόγώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνῳ.



Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω δίχα κατὰ τὸ Γ σημεῖον, προσκείσθω δέ τις αὐτῇ εὐθεῖα ἐπ᾿ εὐθείας ἡ ΒΔ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ.

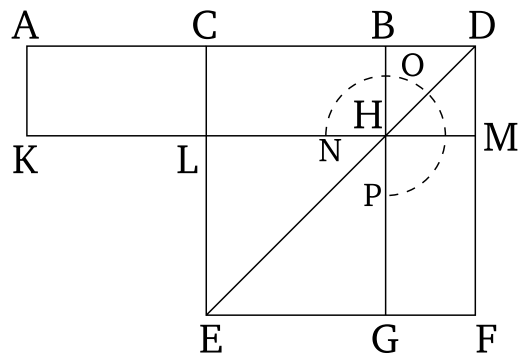
Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΔ τετράγωνον τὸ ΓΕΖΔ, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ διὰ μὲν τοῦ Β σημείου ὁποτέρᾳ τῶν ΕΓ, ΔΖ παράλληλος ἤχθω ἡ ΒΗ, διὰ δὲ τοῦ Θ σημείου ὁποτέρᾳ τῶν ΑΒ, ΕΖ παράλληλος ἤχθω ἡ ΚΜ, καὶ ἔτι διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΛ, ΔΜ παράλληλος ἤχθω ἡ ΑΚ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, ἴσον ἐστὶ καὶ τὸ ΑΛ τῷ ΓΘ. ἀλλὰ τὸ ΓΘ τῷ ΘΖ ἴσον ἐστίν. καὶ τὸ ΑΛ ἄρα τῷ ΘΖ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΓΜ· ὅλον ἄρα τὸ ΑΜ τῷ ΝΞΟ γνώμονί ἐστιν ἴσον. ἀλλὰ τὸ ΑΜ ἐστι τὸ ὑπὸ τῶν ΑΔ, ΔΒ· ἴση γάρ ἐστιν ἡ ΔΜ τῇ ΔΒ· καὶ ὁ ΝΞΟ ἄρα γνώμων ἴσος ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ [περιεχομένῳ ὀρθογωνίῳ]. κοινὸν προσκείσθω τὸ ΛΗ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ΝΞΟ γνώμονι καὶ τῷ ΛΗ. ἀλλὰ ὁ ΝΞΟ γνώμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΔ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΓΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ᾿ εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τῆς προσκειμένης περιεχόμενον ὀρθόγώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

Proposition 6†

If a straight-line is cut in half, and any straight-line added to it straight-on, (then) the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added.



For let any straight-line *AB* be cut in half at point *C*, and let any straight-line *BD* be added to it straight-on. I say that the rectangle contained by *AD* and *DB*, plus the square on *CB*, is equal to the square on *CD*.

For let the square *CEFD* be described on *CD* [Prop. 1.46], and let *DE* be joined, and let *BG* be drawn through point *B*, parallel to either of *EC* or *DF* [Prop. 1.31], and let *KM* be drawn through point *H*, parallel to either of *AB* or *EF* [Prop. 1.31], and finally let *AK* be drawn through *A*, parallel to either of *CL* or *DM* [Prop. 1.31].

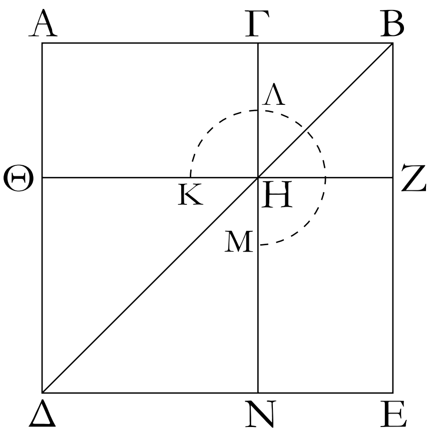
Therefore, since *AC* is equal to *CB*, (rectangle) *AL* is also equal to (rectangle) *CH* [Prop. 1.36]. But, (rectangle) *CH* is equal to (rectangle) *HF* [Prop. 1.43]. Thus, (rectangle) *AL* is also equal to (rectangle) *HF*. Let (rectangle) *CM* be added to both. Thus, the whole (rectangle) *AM* is equal to the gnomon *NOP*. But, *AM* is the (rectangle contained) by *AD* and *DB*. For *DM* is equal to *DB*. Thus, gnomon *NOP* is also equal to the [rectangle contained] by *AD* and *DB*. Let *LG*, which is equal to the square on *BC*, be added to both. Thus, the rectangle contained by *AD* and *DB*, plus the square on *CB*, is equal to the gnomon *NOP* and the (square) *LG*. But the gnomon *NOP* and the (square) *LG* is (equivalent to) the whole square *CEFD*, which is on *CD*. Thus, the rectangle contained by *AD* and *DB*, plus the square on *CB*, is equal to the square on *CD*.

Thus, if a straight-line is cut in half, and any straight-line added to it straight-on, (then) the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(2\,a+b)\,b+a^2=(a+b)^2$.

ζ'.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ᾿ ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.



Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΓΑ τετραγώνῳ.

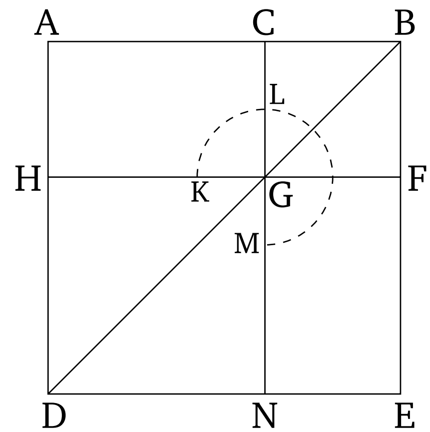
Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ· καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ ΗΕ, κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ ὅλῳ τῷ ΓΕ ἴσον ἐστίν· τὰ ἄρα ΑΖ, ΓΕ διπλάσιά ἐστι τοῦ ΑΖ. ἀλλὰ τὰ ΑΖ, ΓΕ ὁ ΚΛΜ ἐστι γνώμων καὶ τὸ ΓΖ τετράγωνον· ὁ ΚΛΜ ἄρα γνώμων καὶ τὸ ΓΖ διπλάσιά ἐστι τοῦ ΑΖ. ἔστι δὲ τοῦ ΑΖ διπλάσιον καὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἴση γὰρ ἡ ΒΖ τῇ ΒΓ· ὁ ἄρα ΚΛΜ γνώμων καὶ τὸ ΓΖ τετράγωνον ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. κοινὸν προσκείσθω τὸ ΔΗ, ὅ ἐστιν ἀπὸ τῆς ΑΓ τετράγωνον· ὁ ἄρα ΚΛΜ γνώμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ. ἀλλὰ ὁ ΚΛΜ γνώμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ὅλον ἐστὶ τὸ ΑΔΕΒ καὶ τὸ ΓΖ, ἅ ἐστιν ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα· τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα ἴσα ἐστὶ τῷ [τε] δὶς ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ᾿ ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

Proposition 7†

If a straight-line is cut at random, (then) the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece.



For let any straight-line *AB* be cut, at random, at point *C*. I say that the (sum of the) squares on *AB* and *BC* is equal to twice the rectangle contained by *AB* and *BC*, and the square on *CA*.

For let the square *ADEB* be described on *AB* [Prop. 1.46], and let the (rest of) the figure be drawn.

Therefore, since (rectangle) *AG* is equal to (rectangle) *GE* [Prop. 1.43], let the (square) *CF* be added to both. Thus, the whole (rectangle) *AF* is equal to the whole (rectangle) *CE*. Thus, (rectangle) *AF* plus (rectangle) *CE* is double (rectangle) *AF*. But, (rectangle) *AF* plus (rectangle) *CE* is the gnomon *KLM*, and the square *CF*. Thus, the gnomon *KLM*, and the square *CF*, is double the (rectangle) *AF*. But double the (rectangle) *AF* is also twice the (rectangle contained) by *AB* and *BC*. For *BF* (is) equal to *BC*. Thus, the gnomon *KLM*, and the square *CF*, are equal to twice the (rectangle contained) by *AB* and *BC*. Let *DG*, which is the square on *AC*, be added to both. Thus, the gnomon *KLM*, and the squares *BG* and *GD*, are equal to twice the rectangle contained by *AB* and *BC*, and the square on *AC*. But, the gnomon *KLM* and the squares *BG* and *GD* is (equivalent to) the whole of *ADEB* and *CF*, which are the squares on *AB* and *BC* (respectively). Thus, the (sum of the) squares on *AB* and *BC* is equal to twice the rectangle contained by *AB* and *BC*, and the square on *AC*.

Thus, if a straight-line is cut at random, (then) the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece. (Which is) the very thing it was required to show.

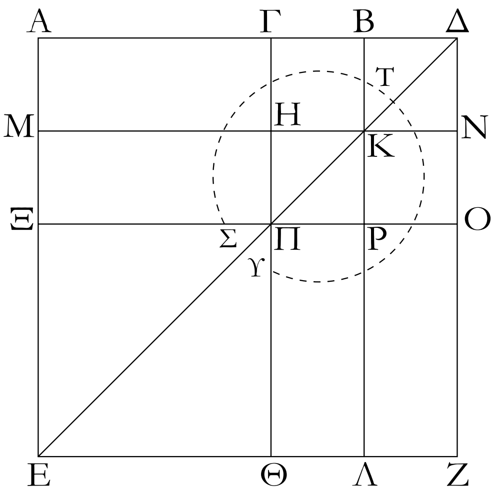
† This proposition is a geometric version of the algebraic identity: $(a+b)^2 + a^2 = 2\,(a+b)\,a + b^2$.

η΄.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ τετράκις ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσον ἐστὶ τῷ ἀπό τε τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ τετράκις ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, ΒΓ ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Ἐκβεβλήσθω γὰρ ἐπ᾽ εὐθείας [τῇ ΑΒ εὐθεῖα] ἡ ΒΔ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΒΔ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΔ τετράγωνον τὸ ΑΕΖΔ, καὶ καταγεγράφθω διπλοῦν τὸ σχῆμα.



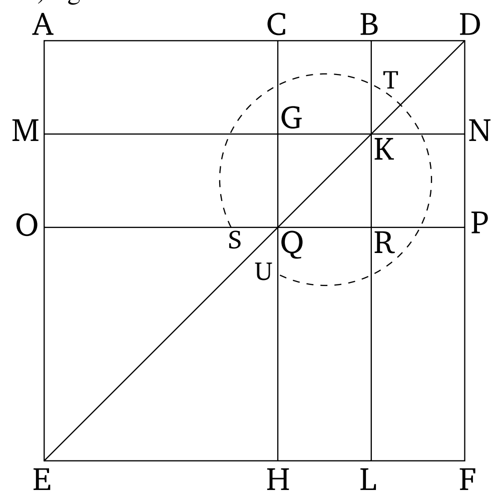
Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΓΒ τῇ ΒΔ, ἀλλὰ ἡ μὲν ΓΒ τῇ ΗΚ ἐστιν ἴση, ἡ δὲ ΒΔ τῇ ΚΝ, καὶ ἡ ΗΚ ἄρα τῇ ΚΝ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΠΡ τῇ ΡΟ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῇ ΒΔ, ἡ δὲ ΗΚ τῇ ΚΝ, ἴσον ἄρα ἐστὶ καὶ τὸ μὲν ΓΚ τῷ ΚΔ, τὸ δὲ ΗΡ τῷ ΡΝ. ἀλλὰ τὸ ΓΚ τῷ ΡΝ ἐστιν ἴσον· παραπληρώματα γὰρ τοῦ ΓΟ παραλληλογράμμου· καὶ τὸ ΚΔ ἄρα τῷ ΗΡ ἴσον ἐστίν· τὰ τέσσαρα ἄρα τὰ ΔΚ, ΓΚ, ΗΡ, ΡΝ ἴσα ἀλλήλοις ἐστίν. τὰ τέσσαρα ἄρα τετραπλάσιά ἐστι τοῦ ΓΚ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΓΒ τῇ ΒΔ, ἀλλὰ ἡ μὲν ΒΔ τῇ ΒΚ, τουτέστι τῇ ΓΗ ἴση, ἡ δὲ ΓΒ τῇ ΗΚ, τουτέστι τῇ ΗΠ, ἐστιν ἴση, καὶ ἡ ΓΗ ἄρα τῇ ΗΠ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΓΗ τῇ ΗΠ, ἡ δὲ ΠΡ τῇ ΡΟ, ἴσον ἐστὶ καὶ τὸ μὲν ΑΗ τῷ ΜΠ, τὸ δὲ ΠΛ τῷ ΡΖ. ἀλλὰ τὸ ΜΠ τῷ ΠΛ ἐστιν ἴσον· παραπληρώματα γὰρ τοῦ ΜΛ παραλληλογράμμου· καὶ τὸ ΑΗ ἄρα τῷ ΡΖ ἴσον ἐστίν· τὰ τέσσαρα ἄρα τὰ ΑΗ, ΜΠ,

Proposition 8†

If a straight-line is cut at random, (then) four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line).

For let any straight-line AB be cut, at random, at point C. I say that four times the rectangle contained by AB and BC, plus the square on AC, is equal to the square described on AB and BC, as on one (complete straight-line).

For let BD be produced in a straight-line [with the straight-line AB], and let BD be made equal to CB [Prop. 1.3], and let the square $AEFD$ be described on AD [Prop. 1.46], and let the (rest of the) figure be drawn double.



Therefore, since CB is equal to BD, but CB is equal to GK [Prop. 1.34], and BD to KN [Prop. 1.34], GK is thus also equal to KN. So, for the same (reasons), QR is equal to RP. And since BC is equal to BD, and GK to KN, (square) CK is thus also equal to (square) KD, and (square) GR to (square) RN [Prop. 1.36]. But, (square) CK is equal to (square) RN. For (they are) complements in the parallelogram CP [Prop. 1.43]. Thus, (square) KD is also equal to (square) GR. Thus, the four (squares) DK, CK, GR, and RN are equal to one another. Thus, the four (taken together) are quadruple (square) CK. Again, since CB is equal to BD, but BD (is) equal to BK—that is to say, CG—and CB is equal to GK—that is to say, GQ—CG is thus also equal to GQ. And since CG is equal to GQ, and QR to RP, (rectangle) AG is also equal to (rectangle) MQ, and (rectangle) QL to (rectangle) RF [Prop. 1.36]. But, (rectan-

ΠΛ, ΡΖ ἴσα ἀλλήλοις ἐστίν· τὰ τέσσαρα ἄρα τοῦ ΑΗ ἐστι τετραπλάσια. ἐδείχθη δὲ καὶ τὰ τέσσαρα τὰ ΓΚ, ΚΔ, ΗΡ, ΡΝ τοῦ ΓΚ τετραπλάσια· τὰ ἄρα ὀκτώ, ἃ περιέχει τὸν ΣΤΥ γνώμονα, τετραπλάσιά ἐστι τοῦ ΑΚ. καὶ ἐπεὶ τὸ ΑΚ τὸ ὑπὸ τῶν ΑΒ, ΒΔ ἐστιν· ἴση γὰρ ἡ ΒΚ τῇ ΒΔ· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ τετραπλάσιόν ἐστι τοῦ ΑΚ. ἐδείχθη δὲ τοῦ ΑΚ τετραπλάσιος καὶ ὁ ΣΤΥ γνώμων· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ ἴσον ἐστὶ τῷ ΣΤΥ γνώμονι. κοινὸν προσκείσθω τὸ ΞΘ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ΣΤΥ γνώμονι καὶ τῷ ΞΘ. ἀλλὰ ὁ ΣΤΥ γνώμων καὶ τὸ ΞΘ ὅλον ἐστὶ τὸ ΑΕΖΔ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΑΔ· τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΔ μετὰ τοῦ ἀπὸ ΑΓ ἴσον ἐστὶ τῷ ἀπὸ ΑΔ τετραγώνῳ· ἴση δὲ ἡ ΒΔ τῇ ΒΓ. τὸ ἄρα τετράκις ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΔ, τουτέστι τῷ ἀπὸ τῆς ΑΒ καὶ ΒΓ ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ τετράκις ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσον ἐστὶ τῷ ἀπό τε τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

gle) MQ is equal to (rectangle) QL. For (they are) complements in the parallelogram ML [Prop. 1.43]. Thus, (rectangle) AG is also equal to (rectangle) RF. Thus, the four (rectangles) AG, MQ, QL, and RF are equal to one another. Thus, the four (taken together) are quadruple (rectangle) AG. And it was also shown that the four (squares) CK, KD, GR, and RN (taken together are) quadruple (square) CK. Thus, the eight (figures taken together), which comprise the gnomon STU, are quadruple (rectangle) AK. And since AK is the (rectangle contained) by AB and BD, for BK (is) equal to BD, four times the (rectangle contained) by AB and BD is quadruple (rectangle) AK. But the gnomon STU was also shown (to be equal to) quadruple (rectangle) AK. Thus, four times the (rectangle contained) by AB and BD is equal to the gnomon STU. Let OH, which is equal to the square on AC, be added to both. Thus, four times the rectangle contained by AB and BD, plus the square on AC, is equal to the gnomon STU, and the (square) OH. But, the gnomon STU and the (square) OH is (equivalent to) the whole square $AEFD$, which is on AD. Thus, four times the (rectangle contained) by AB and BD, plus the (square) on AC, is equal to the square on AD. And BD (is) equal to BC. Thus, four times the rectangle contained by AB and BC, plus the square on AC, is equal to the (square) on AD, that is to say the square described on AB and BC, as on one (complete straight-line).

Thus, if a straight-line is cut at random, (then) four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line). (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $4\,(a+b)\,a+b^2=[(a+b)+a]^2$.

θ'.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἤχθω γὰρ ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΕ, καὶ κείσθω ἴση ἑκατέρᾳ τῶν ΑΓ, ΓΒ, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ, καὶ διὰ μὲν τοῦ Δ τῇ ΕΓ παράλληλος ἤχθω ἡ ΔΖ, διὰ δὲ τοῦ Ζ τῇ ΑΒ ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΑΖ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΕΑΓ γωνία τῇ ὑπὸ ΑΕΓ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ πρὸς τῷ Γ, λοιπαὶ ἄρα αἱ ὑπὸ ΕΑΓ, ΑΕΓ μιᾷ ὀρθῇ ἴσαι εἰσίν· καί εἰσιν ἴσαι· ἡμίσεια ἄρα ὀρθῆς ἐστιν ἑκατέρα τῶν ὑπὸ ΓΕΑ, ΓΑΕ. δία τὰ αὐτὰ δὴ

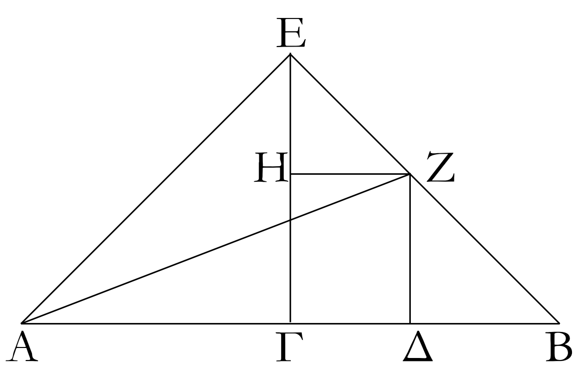
Proposition 9†

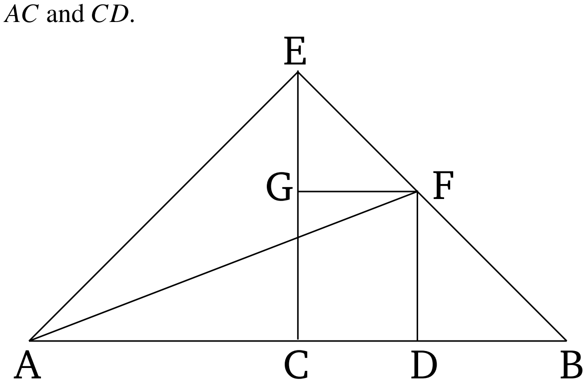
If a straight-line is cut into equal and unequal (pieces, then) the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces.

For let any straight-line AB be cut—equally at C, and unequally at D. I say that the (sum of the) squares on AD and DB is double the (sum of the squares) on AC and CD.

For let CE be drawn from (point) C, at right-angles to AB [Prop. 1.11], and let it be made equal to each of AC and CB [Prop. 1.3], and let EA and EB be joined. And let DF be drawn through (point) D, parallel to EC [Prop. 1.31], and (let) FG (be drawn) through (point) F, (parallel) to AB [Prop. 1.31]. And let AF be joined. And since AC is equal to CE, the angle EAC is also equal to the (angle) AEC [Prop. 1.5]. And since the *καὶ ἑκατέρα τῶν ὑπὸ ΓΕΒ, ΕΒΓ ἡμίσειά ἐστιν ὀρθῆς· ὅλη ἄρα ἡ ὑπὸ ΑΕΒ ὀρθή ἐστιν. καὶ ἐπεὶ ἡ ὑπὸ ΗΕΖ ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ΕΗΖ· ἴση γάρ ἐστι τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ἡ ὑπὸ ΕΖΗ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα [ἐστὶν] ἡ ὑπὸ ΗΕΖ γωνία τῇ ὑπὸ ΕΖΗ· ὥστε καὶ πλευρὰ ἡ ΕΗ τῇ ΗΖ ἐστιν ἴση. πάλιν ἐπεὶ ἡ πρὸς τῷ Β γωνία ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ΖΔΒ· ἴση γὰρ πάλιν ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΓΒ· λοιπὴ ἄρα ἡ ὑπὸ ΒΖΔ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα ἡ πρὸς τῷ Β γωνία τῇ ὑπὸ ΔΖΒ· ὥστε καὶ πλευρὰ ἡ ΖΔ πλευρᾷ τῇ ΔΒ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴσον ἐστὶ καὶ τὸ ἀπὸ ΑΓ τῷ ἀπὸ ΓΕ· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΕ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ ΑΓ. τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ τετράγωνον· ὀρθὴ γὰρ ἡ ὑπὸ ΑΓΕ γωνία· τὸ ἄρα ἀπὸ τῆς ΕΑ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΕΗ τῇ ΗΖ, ἴσον καὶ τὸ ἀπὸ τῆς ΕΗ τῷ ἀπὸ τῆς ΗΖ· τὰ ἄρα ἀπὸ τῶν ΕΗ, ΗΖ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΗΖ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΗ, ΗΖ τετραγώνοις ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΖ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΕΖ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΗΖ. ἴση δὲ ἡ ΗΖ τῇ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΖ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΖ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΖ τετράγωνον· ὀρθὴ γάρ ἐστιν ἡ ὑπὸ ΑΕΖ γωνία· τὸ ἄρα ἀπὸ τῆς ΑΖ τετράγωνον διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΖ ἴσα τὰ ἀπὸ τῶν ΑΔ, ΔΖ· ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΖ διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. ἴση δὲ ἡ ΔΖ τῇ ΔΒ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.*

(angle) at C is a right-angle, the (sum of the) remaining angles (of triangle AEC), EAC and AEC, is thus equal to one right-angle [Prop. 1.32]. And they are equal. Thus, (angles) CEA and CAE are each half a right-angle. So, for the same (reasons), (angles) CEB and EBC are also each half a right-angle. Thus, the whole (angle) AEB is a right-angle. And since GEF is half a right-angle, and EGF (is) a right-angle—for it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) EFG is thus half a right-angle [Prop. 1.32]. Thus, angle GEF [is] equal to EFG. So the side EG is also equal to the (side) GF [Prop. 1.6]. Again, since the angle at B is half a right-angle, and (angle) FDB (is) a right-angle—for again it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) BFD is half a right-angle [Prop. 1.32]. Thus, the angle at B (is) equal to DFB. So the side FD is also equal to the side DB [Prop. 1.6]. And since AC is equal to CE, the (square) on AC (is) also equal to the (square) on CE. Thus, the (sum of the) squares on AC and CE is double the (square) on AC. And the square on EA is equal to the (sum of the) squares on AC and CE. For angle ACE (is) a right-angle [Prop. 1.47]. Thus, the (square) on EA is double the (square) on AC. Again, since EG is equal to GF, the (square) on EG (is) also equal to the (square) on GF. Thus, the (sum of the squares) on EG and GF is double the square on GF. And the square on EF is equal to the (sum of the) squares on EG and GF [Prop. 1.47]. Thus, the square on EF is double the (square) on GF. And GF (is) equal to CD [Prop. 1.34]. Thus, the (square) on EF is double the (square) on CD. And the (square) on EA is also double the (square) on AC. Thus, the (sum of the) squares on AE and EF is double the (sum of the) squares on AC and CD. And the square on AF is equal to the (sum of the squares) on AE and EF. For the angle AEF is a right-angle [Prop. 1.47]. Thus, the square on AF is double the (sum of the squares) on AC and CD. And the (sum of the squares) on AD and DF (is) equal to the (square) on AF. For the angle at D is a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AD and DF is double the (sum of the) squares on AC and CD. And DF (is) equal to DB. Thus, the (sum of the) squares on AD and DB is double the (sum of the) squares on AC and CD.





Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου· ὅπερ ἔδει δεῖξαι.

Thus, if a straight-line is cut into equal and unequal (pieces, then) the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces. (Which is) the very thing it was required to show.

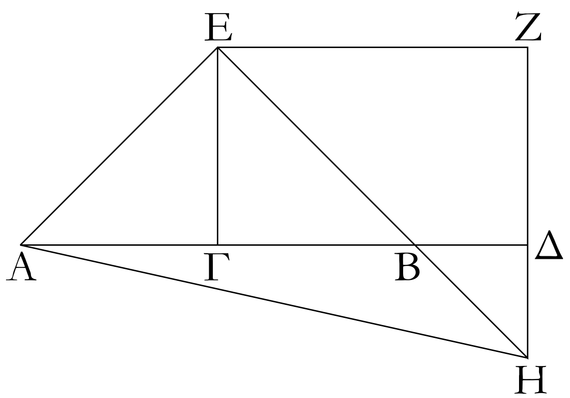
† This proposition is a geometric version of the algebraic identity: $a^2 + b^2 = 2\left[([a+b]/2)^2 + ([a+b]/2 - b)^2\right]$.

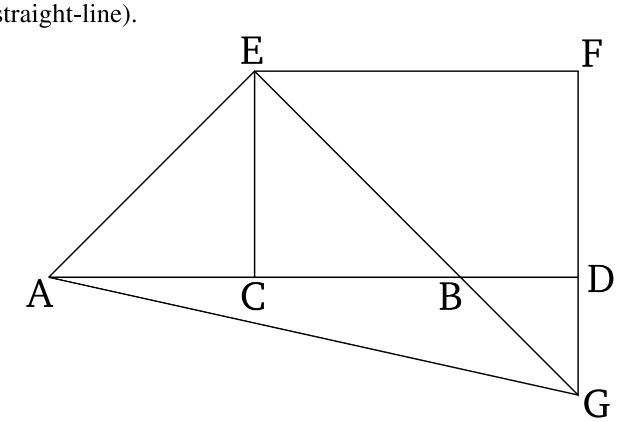
ι′.

Proposition 10†

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ᾽ εὐθείας, τὸ ἀπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τὸ ἀπὸ τῆς προσκειμένης τὰ συναμφότερα τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης ὡς ἀπὸ μιᾶς ἀναγραφέντος τετραγώνου.

If a straight-line is cut in half, and any straight-line added to it straight-on, (then) the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line).





Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω δίχα κατὰ τὸ Γ, προσκείσθω δέ τις αὐτῇ εὐθεῖα ἐπ᾽ εὐθείας ἡ ΒΔ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἤχθω γὰρ ἀπὸ τοῦ Γ σημείου τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΕ, καὶ κείσθω ἴση ἑκατέρᾳ τῶν ΑΓ, ΓΒ, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ· καὶ διὰ μὲν τοῦ Ε τῇ ΑΔ παράλληλος ἤχθω ἡ ΕΖ, διὰ δὲ τοῦ Δ τῇ ΓΕ παράλληλος ἤχθω ἡ ΖΔ. καὶ ἐπεὶ εἰς παραλλήλους εὐθείας τὰς ΕΓ, ΖΔ εὐθεῖά τις ἐνέπεσεν ἡ ΕΖ, αἱ ὑπὸ ΓΕΖ, ΕΖΔ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν· αἱ ἄρα ὑπὸ ΖΕΒ, ΕΖΔ δύο ὀρθῶν ἐλάσσονές εἰσιν· αἱ δὲ ἀπ᾽ ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι συμπίπτουσιν· αἱ ἄρα ΕΒ, ΖΔ ἐκβαλλόμεναι ἐπὶ τὰ Β, Δ μέρη συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΑΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΕΑΓ τῇ ὑπὸ ΑΕΓ· καὶ ὀρθὴ ἡ πρὸς τῷ Γ· ἡμίσεια ἄρα ὀρθῆς [ἐστιν] ἑκατέρα τῶν ὑπὸ ΕΑΓ, ΑΕΓ. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΕΒ, ΕΒΓ ἡμίσειά ἐστιν ὀρθῆς· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ. καὶ ἐπεὶ ἡμίσεια ὀρθῆς ἐστιν ἡ ὑπὸ ΕΒΓ, ἡμίσεια ἄρα ὀρθῆς καὶ ἡ ὑπὸ ΔΒΗ. ἔστι δὲ καὶ ἡ ὑπὸ

For let any straight-line *AB* be cut in half at (point) *C*, and let any straight-line *BD* be added to it straight-on. I say that the (sum of the) squares on *AD* and *DB* is double the (sum of the) squares on *AC* and *CD*.

For let *CE* be drawn from point *C*, at right-angles to *AB* [Prop. 1.11], and let it be made equal to each of *AC* and *CB* [Prop. 1.3], and let *EA* and *EB* be joined. And let *EF* be drawn through *E*, parallel to *AD* [Prop. 1.31], and let *FD* be drawn through *D*, parallel to *CE* [Prop. 1.31]. And since some straight-line *EF* falls across the parallel straight-lines *EC* and *FD*, the (internal angles) *CEF* and *EFD* are thus equal to two right-angles [Prop. 1.29]. Thus, *FEB* and *EFD* are less than two right-angles. And (straight-lines) produced from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced in the direction of *B* and *D*, the (straight-lines) *EB* and *FD* will meet. Let them be produced, and let them meet together at *G*, and let *AG* be joined. And since *AC* is equal to *CE*, angle *EAC* is also equal to (angle) *AEC* [Prop. 1.5]. And the (angle) at *C* (is) a right-angle. Thus, *EAC* and *AEC* [are] each half a right-angle [Prop. 1.32].

ΒΔΗ ὀρθή· ἴση γάρ ἐστι τῇ ὑπὸ ΔΓΕ· ἐναλλὰξ γάρ· λοιπὴ ἄρα ἡ ὑπὸ ΔΗΒ ἡμίσειά ἐστιν ὀρθῆς· ἡ ἄρα ὑπὸ ΔΗΒ τῇ ὑπὸ ΔΒΗ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΒΔ πλευρᾷ τῇ ΗΔ ἐστιν ἴση. πάλιν, ἐπεὶ ἡ ὑπὸ ΕΗΖ ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ πρὸς τῷ Ζ· ἴση γάρ ἐστι τῇ ἀπεναντίον τῇ πρὸς τῷ Γ· λοιπὴ ἄρα ἡ ὑπὸ ΖΕΗ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα ἡ ὑπὸ ΕΗΖ γωνία τῇ ὑπὸ ΖΕΗ· ὥστε καὶ πλευρὰ ἡ ΗΖ πλευρᾷ τῇ ΕΖ ἐστιν ἴση. καὶ ἐπεὶ [ἴση ἐστὶν ἡ ΕΓ τῇ ΓΑ], ἴσον ἐστὶ [καὶ] τὸ ἀπὸ τῆς ΕΓ τετράγωνον τῷ ἀπὸ τῆς ΓΑ τετραγώνῳ· τὰ ἄρα ἀπὸ τῶν ΕΓ, ΓΑ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΓ, ΓΑ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ· τὸ ἄρα ἀπὸ τῆς ΕΑ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ τετραγώνου. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΖΗ τῇ ΕΖ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΖΕ· τὰ ἄρα ἀπὸ τῶν ΗΖ, ΖΕ διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΕΖ. τοῖς δὲ ἀπὸ τῶν ΗΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΗ· τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΖ. ἴση δὲ ἡ ΕΖ τῇ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΗ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΗ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΗ τετραγώνοις ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΗ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΑΗ διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΗ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΔ, ΔΗ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΗ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ [τετραγώνων]. ἴση δὲ ἡ ΔΗ τῇ ΔΒ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΒ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ᾽ εὐθείας, τὸ ἀπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τὸ ἀπὸ τῆς προσκειμένης τὰ συναμφότερα τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης ὡς ἀπὸ μιᾶς ἀναγραφέντος τετραγώνου· ὅπερ ἔδει δεῖξαι.

So, for the same (reasons), CEB and EBC are also each half a right-angle. Thus, (angle) AEB is a right-angle. And since EBC is half a right-angle, DBG (is) thus also half a right-angle [Prop. 1.15]. And BDG is also a right-angle. For it is equal to DCE. For (they are) alternate (angles) [Prop. 1.29]. Thus, the remaining (angle) DGB is half a right-angle. Thus, DGB is equal to DBG. So side BD is also equal to side GD [Prop. 1.6]. Again, since EGF is half a right-angle, and the (angle) at F (is) a right-angle, for it is equal to the opposite (angle) at C [Prop. 1.34], the remaining (angle) FEG is thus half a right-angle. Thus, angle EGF (is) equal to FEG. So the side GF is also equal to the side EF [Prop. 1.6]. And since [EC is equal to CA] the square on EC is [also] equal to the square on CA. Thus, the (sum of the) squares on EC and CA is double the square on CA. And the (square) on EA is equal to the (sum of the squares) on EC and CA [Prop. 1.47]. Thus, the square on EA is double the square on AC. Again, since FG is equal to EF, the (square) on FG is also equal to the (square) on FE. Thus, the (sum of the squares) on GF and FE is double the (square) on EF. And the (square) on EG is equal to the (sum of the squares) on GF and FE [Prop. 1.47]. Thus, the (square) on EG is double the (square) on EF. And EF (is) equal to CD [Prop. 1.34]. Thus, the square on EG is double the (square) on CD. But it was also shown that the (square) on EA (is) double the (square) on AC. Thus, the (sum of the) squares on AE and EG is double the (sum of the) squares on AC and CD. And the square on AG is equal to the (sum of the) squares on AE and EG [Prop. 1.47]. Thus, the (square) on AG is double the (sum of the squares) on AC and CD. And the (sum of the squares) on AD and DG is equal to the (square) on AG [Prop. 1.47]. Thus, the (sum of the) [squares] on AD and DG is double the (sum of the) [squares] on AC and CD. And DG (is) equal to DB. Thus, the (sum of the) [squares] on AD and DB is double the (sum of the) squares on AC and CD.

Thus, if a straight-line is cut in half, and any straight-line added to it straight-on, (then) the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line). (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(2\,a+b)^2+b^2=2\left[a^2+(a+b)^2\right]$.

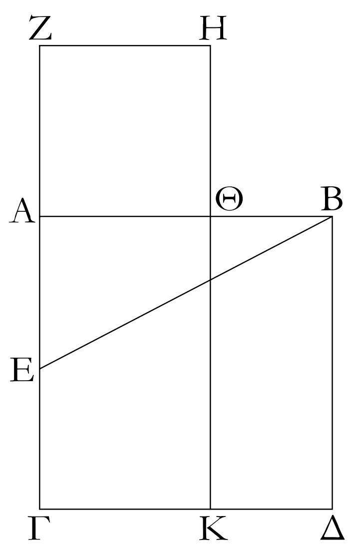
ια'.

Τὴν δοθεῖσαν εὐθεῖαν τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

Proposition 11†

To cut a given straight-line such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the square on the remaining piece.

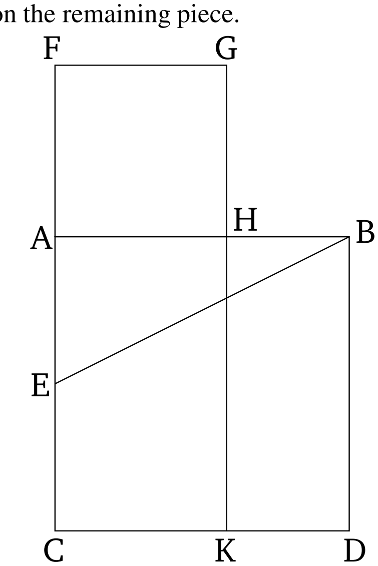
Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.



Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΒΔΓ, καὶ τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Ε σημεῖον, καὶ ἐπεζεύχθω ἡ ΒΕ, καὶ διήχθω ἡ ΓΑ ἐπὶ τὸ Ζ, καὶ κείσθω τῇ ΒΕ ἴση ἡ ΕΖ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΖ τετράγωνον τὸ ΖΘ, καὶ διήχθω ἡ ΗΘ ἐπὶ τὸ Κ· λέγω, ὅτι ἡ ΑΒ τέτμηται κατὰ τὸ Θ, ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ποιεῖν τῷ ἀπὸ τῆς ΑΘ τετραγώνῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΑΓ τέτμηται δίχα κατὰ τὸ Ε, πρόσκειται δὲ αὐτῇ ἡ ΖΑ, τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΑΕ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ τετραγώνῳ. ἴση δὲ ἡ ΕΖ τῇ ΕΒ· τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τῷ ἀπὸ ΕΒ. ἀλλὰ τῷ ἀπὸ ΕΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΒΑ, ΑΕ· ὀρθὴ γὰρ

Let *AB* be the given straight-line. So it is required to cut *AB* such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the square on the remaining piece.



For let the square *ABDC* be described on *AB* [Prop. 1.46], and let *AC* be cut in half at point *E* [Prop. 1.10], and let *BE* be joined. And let *CA* be drawn through to (point) *F*, and let *EF* be made equal to *BE* [Prop. 1.3]. And let the square *FH* be described on *AF* [Prop. 1.46], and let *GH* be drawn through to (point) *K*. I say that *AB* has been cut at *H* such as to make the rectangle contained by *AB* and *BH* equal to the square on *AH*.

For since the straight-line *AC* has been cut in half at *E*, and *FA* has been added to it, the rectangle contained by *CF* and *FA*, plus the square on *AE*, is thus equal to the square on *EF* [Prop. 2.6]. And *EF* (is) equal to *EB*. Thus, the (rectangle contained) by *CF* and *FA*, plus the (square) on *AE*, is equal to the (square) on *EB*. But, the (sum of the squares) on *BA* and *AE* is equal to the (square) on *EB*. For the angle at *A* (is) ἡ πρὸς τῷ Α γωνία· τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΕ. κοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς ΑΕ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΓΖ, ΖΑ τὸ ΖΚ· ἴση γὰρ ἡ ΑΖ τῇ ΖΗ· τὸ δὲ ἀπὸ τῆς ΑΒ τὸ ΑΔ· τὸ ἄρα ΖΚ ἴσον ἐστὶ τῷ ΑΔ. κοινὸν ἀρηρήσθω τὸ ΑΚ· λοιπὸν ἄρα τὸ ΖΘ τῷ ΘΔ ἴσον ἐστίν. καί ἐστι τὸ μὲν ΘΔ τὸ ὑπὸ τῶν ΑΒ, ΒΘ· ἴση γὰρ ἡ ΑΒ τῇ ΒΔ· τὸ δὲ ΖΘ τὸ ἀπὸ τῆς ΑΘ· τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ ΘΑ τετραγώνῳ.

Ἡ ἄρα δοθεῖσα εὐθεῖα ἡ ΑΒ τέτμηται κατὰ τὸ Θ ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ποιεῖν τῷ ἀπὸ τῆς ΘΑ τετραγώνῳ· ὅπερ ἔδει ποιῆσαι.

a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by *CF* and *FA*, plus the (square) on *AE*, is equal to the (sum of the squares) on *BA* and *AE*. Let the square on *AE* have been subtracted from both. Thus, the remaining rectangle contained by *CF* and *FA* is equal to the square on *AB*. And *FK* is the (rectangle contained) by *CF* and *FA*. For *AF* (is) equal to *FG*. And *AD* (is) the (square) on *AB*. Thus, the (rectangle) *FK* is equal to the (square) *AD*. Let (rectangle) *AK* be subtracted from both. Thus, the remaining (square) *FH* is equal to the (rectangle) *HD*. And *HD* is the (rectangle contained) by *AB* and *BH*. For *AB* (is) equal to *BD*. And *FH* (is) the (square) on *AH*. Thus, the rectangle contained by *AB* and *BH* is equal to the square on *HA*.

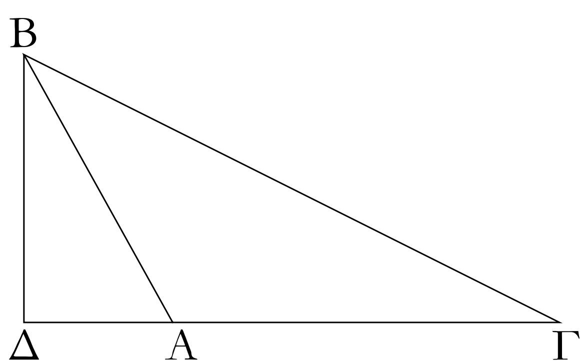
Thus, the given straight-line *AB* has been cut at (point) *H* such as to make the rectangle contained by *AB* and *BH* equal to the square on *HA*. (Which is) the very thing it was required to do.

† This manner of cutting a straight-line—so that the ratio of the whole to the larger piece is equal to the ratio of the larger to the smaller piece—is sometimes called the "Golden Section".

ιβ'.

Ἐν τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ἀμβλεῖαν γωνίαν, ἐφ᾽ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῇ ἀμβλείᾳ γωνίᾳ.

Ἔστω ἀμβλυγώνιον τρίγωνον τὸ ΑΒΓ ἀμβλεῖαν ἔχον τὴν ὑπὸ ΒΑΓ, καὶ ἤχθω ἀπὸ τοῦ Β σημείου ἐπὶ τὴν ΓΑ ἐκβληθεῖσαν κάθετος ἡ ΒΔ. λέγω, ὅτι τὸ ἀπὸ τῆς ΒΓ τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ.

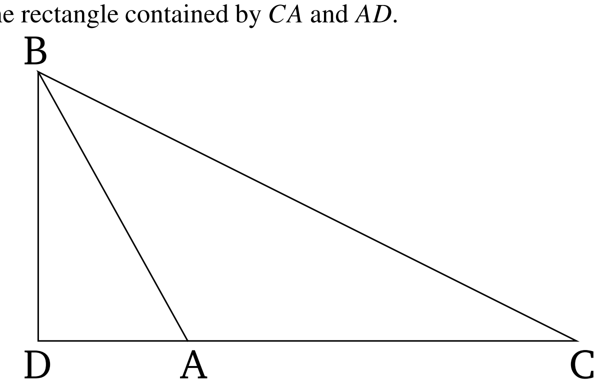


Ἐπεὶ γὰρ εὐθεῖα ἡ ΓΔ τέτμηται, ὡς ἔτυχεν, κατὰ τὸ Α σημεῖον, τὸ ἄρα ἀπὸ τῆς ΔΓ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΓΑ, ΑΔ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔΒ· τὰ ἄρα ἀπὸ τῶν ΓΔ, ΔΒ ἴσα ἐστὶ τοῖς τε ἀπὸ τῶν ΓΑ, ΑΔ, ΔΒ τετραγώνοις

Proposition 12†

In obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle.

Let *ABC* be an obtuse-angled triangle, having the angle *BAC* obtuse. And let *BD* be drawn from point *B*, perpendicular to *CA* produced [Prop. 1.12]. I say that the square on *BC* is greater than the (sum of the) squares on *BA* and *AC* by twice the rectangle contained by *CA* and *AD*.



For since the straight-line *CD* has been cut, at random, at point *A*, the (square) on *DC* is thus equal to the (sum of the) squares on *CA* and *AD*, and twice the rectangle contained by *CA* and *AD* [Prop. 2.4]. Let the (square) on *DB* be added to both. Thus, the (sum of the squares) on *CD* and *DB* is equal *καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ [περιεχομένῳ ὀρθογωνίῳ]. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΓΔ, ΔΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΓΒ· ὀρθὴ γὰρ ἡ προς τῷ Δ γωνία· τοῖς δὲ ἀπὸ τῶν ΑΔ, ΔΒ ἴσον τὸ ἀπὸ τῆς ΑΒ· τὸ ἄρα ἀπὸ τῆς ΓΒ τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΓΑ, ΑΒ τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ· ὥστε τὸ ἀπὸ τῆς ΓΒ τετράγωνον τῶν ἀπὸ τῶν ΓΑ, ΑΒ τετραγώνων μεῖζόν ἐστι τῷ δὶς ὑπὸ τῶν ΓΑ, ΑΔ περιεχομένῳ ὀρθογωνίῳ.*

Ἐν ἄρα τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ἀμβλεῖαν γωνίαν, ἐφ᾽ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῇ ἀμβλείᾳ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

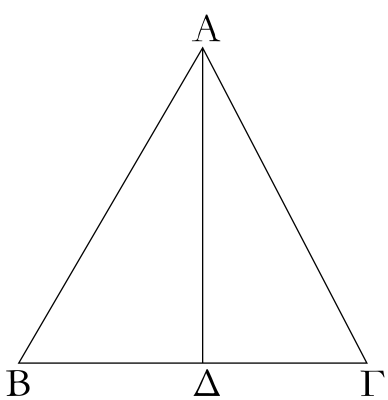
to the (sum of the) squares on CA, AD, and DB, and twice the [rectangle contained] by CA and AD. But, the (square) on CB is equal to the (sum of the squares) on CD and DB. For the angle at D (is) a right-angle [Prop. 1.47]. And the (square) on AB (is) equal to the (sum of the squares) on AD and DB [Prop. 1.47]. Thus, the square on CB is equal to the (sum of the) squares on CA and AB, and twice the rectangle contained by CA and AD. So the square on CB is greater than the (sum of the) squares on CA and AB by twice the rectangle contained by CA and AD.

Thus, in obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle. (Which is) the very thing it was required to show.

† This proposition is equivalent to the well-known cosine formula: $BC^2 = AB^2 + AC^2 - 2\,AB\,AC\,\cos BAC$, since $\cos BAC = -AD/AB$.

ιγ'.

Ἐν τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἔλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ὀξεῖαν γωνίαν, ἐφ᾽ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῇ ὀξείᾳ γωνίᾳ.

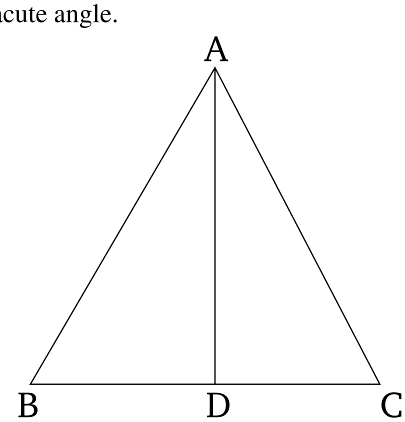


Ἔστω ὀξυγώνιον τρίγωνον τὸ ΑΒΓ ὀξεῖαν ἔχον τὴν πρὸς τῷ Β γωνίαν, καὶ ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ· λέγω, ὅτι τὸ ἀπὸ τῆς ΑΓ τετράγωνον ἔλαττόν ἐστι τῶν ἀπὸ τῶν ΓΒ, ΒΑ τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΓΒ τέτμηται, ὡς ἔτυχεν, κατὰ τὸ Δ, τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΔ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΔΓ τετραγώνῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔΑ τετράγωνον·

Proposition 13†

In acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle.



Let ABC be an acute-angled triangle, having the angle at (point) B acute. And let AD be drawn from point A, perpendicular to BC [Prop. 1.12]. I say that the square on AC is less than the (sum of the) squares on CB and BA by twice the rectangle contained by CB and BD.

For since the straight-line CB has been cut, at random, at (point) D, the (sum of the) squares on CB and BD is thus equal to twice the rectangle contained by CB and BD, and the square on DC [Prop. 2.7]. Let the square on DA be added to both.

τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΔ, ΔΑ τετράγωνα ἴσα ἐστὶ τῷ τε δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ καὶ τοῖς ἀπὸ τῶν ΑΔ, ΔΓ τετραγώνιος. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΒΔ, ΔΑ ἴσον τὸ ἀπὸ τῆς ΑΒ· ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τοῖς δὲ ἀπὸ τῶν ΑΔ, ΔΓ ἴσον τὸ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΓΒ, ΒΑ ἴσα ἐστὶ τῷ τε ἀπὸ τῆς ΑΓ καὶ τῷ δὶς ὑπὸ τῶν ΓΒ, ΒΔ· ὥστε μόνον τὸ ἀπὸ τῆς ΑΓ ἔλαττόν ἐστι τῶν ἀπὸ τῶν ΓΒ, ΒΑ τετραγώνων τῷ δὶς ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογωνίῳ.

Ἐν ἄρα τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξεῖαν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἔλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δὶς ὑπό τε μιᾶς τῶν περὶ τὴν ὀξεῖαν γωνίαν, ἐφ᾽ ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῇ ὀξείᾳ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

Thus, the (sum of the) squares on *CB*, *BD*, and *DA* is equal to twice the rectangle contained by *CB* and *BD*, and the (sum of the) squares on *AD* and *DC*. But, the (square) on *AB* (is) equal to the (sum of the squares) on *BD* and *DA*. For the angle at (point) *D* is a right-angle [Prop. 1.47]. And the (square) on *AC* (is) equal to the (sum of the squares) on *AD* and *DC* [Prop. 1.47]. Thus, the (sum of the squares) on *CB* and *BA* is equal to the (square) on *AC*, and twice the (rectangle contained) by *CB* and *BD*. So the (square) on *AC* alone is less than the (sum of the) squares on *CB* and *BA* by twice the rectangle contained by *CB* and *BD*.

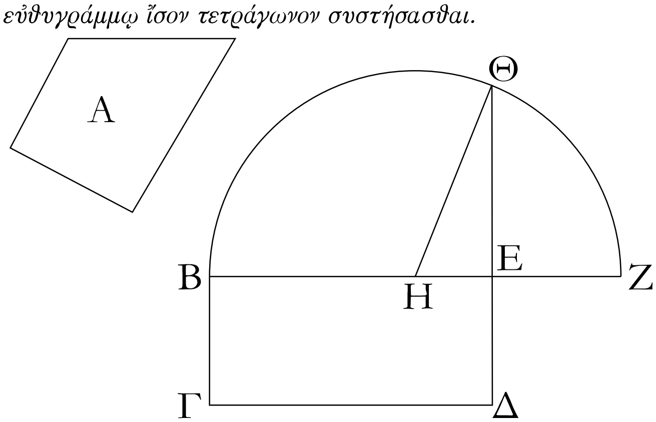
Thus, in acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle. (Which is) the very thing it was required to show.

† This proposition is equivalent to the well-known cosine formula: $AC^2 = AB^2 + BC^2 - 2\,AB\,BC\,\cos ABC$, since $\cos ABC = BD/AB$.

ιδ′.

Τῷ δοθέντι εὐθυγράμμῳ ἴσον τετράγωνον συστήσασθαι.

Ἔστω τὸ δοθὲν εὐθύγραμμον τὸ Α· δεῖ δὴ τῷ Α εὐθυγράμμῳ ἴσον τετράγωνον συστήσασθαι.



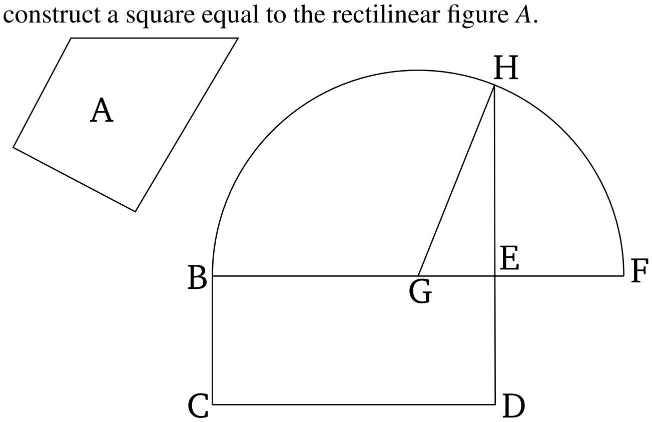
Συνεστάτω γὰρ τῷ Α ἐυθυγράμμῳ ἴσον παραλληλόγραμμον ὀρθογώνιον τὸ ΒΔ· εἰ μὲν οὖν ἴση ἐστὶν ἡ ΒΕ τῇ ΕΔ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. συνέσταται γὰρ τῷ Α εὐθυγράμμῳ ἴσον τετράγωνον τὸ ΒΔ· εἰ δὲ οὔ, μία τῶν ΒΕ, ΕΔ μείζων ἐστίν. ἔστω μείζων ἡ ΒΕ, καὶ ἐκβεβλήσθω ἐπὶ τὸ Ζ, καὶ κείσθω τῇ ΕΔ ἴση ἡ ΕΖ, καὶ τετμήσθω ἡ ΒΖ δίχα κατὰ τὸ Η, καὶ κέντρῳ τῷ Η, διαστήματι δὲ ἑνὶ τῶν ΗΒ, ΗΖ ἡμικύκλιον γεγράφθω τὸ ΒΘΖ, καὶ ἐκβεβλήσθω ἡ ΔΕ ἐπὶ τὸ Θ, καὶ ἐπεζεύχθω ἡ ΗΘ.

Ἐπεὶ οὖν εὐθεῖα ἡ ΒΖ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Η, εἰς δὲ ἄνισα κατὰ τὸ Ε, τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΗ τετραγώνου ἴσον ἐστὶ τῷ

Proposition 14

To construct a square equal to a given rectilinear figure.

Let *A* be the given rectilinear figure. So it is required to construct a square equal to the rectilinear figure *A*.



For let the right-angled parallelogram *BD*, equal to the rectilinear figure *A*, be constructed [Prop. 1.45]. Therefore, if *BE* is equal to *ED* then that (which) was prescribed has taken place. For the square *BD*, equal to the rectilinear figure *A*, has been constructed. And if not, (then) one of the (straight-lines) *BE* or *ED* is greater (than the other). Let *BE* be greater, and let it be produced to *F*, and let *EF* be made equal to *ED* [Prop. 1.3]. And let *BF* be cut in half at (point) *G* [Prop. 1.10]. And, with center *G*, and radius one of the (straight-lines) *GB* or *GF*, let the semi-circle *BHF* be drawn. And let *DE* be produced to *H*, and let *GH* be joined.

Therefore, since the straight-line *BF* has been cut—

ἀπὸ τῆς ΗΖ τετραγώνῳ. ἴση δὲ ἡ ΗΖ τῇ ΗΘ· τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ μετὰ τοῦ ἀπὸ τῆς ΗΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΘ. τῷ δὲ ἀπὸ τῆς ΗΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΘΕ, ΕΗ τετράγωνα· τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ μετὰ τοῦ ἀπὸ ΗΕ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΘΕ, ΕΗ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΗΕ τετράγωνον· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΒΕ, ΕΖ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ τετραγώνῳ. ἀλλὰ τὸ ὑπὸ τῶν ΒΕ, ΕΖ τὸ ΒΔ ἐστιν· ἴση γὰρ ἡ ΕΖ τῇ ΕΔ· τὸ ἄρα ΒΔ παραλληλόγραμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΘΕ τετραγώνῳ. ἴσον δὲ τὸ ΒΔ τῷ Α εὐθυγράμμῳ. καὶ τὸ Α ἄρα εὐθύγραμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ ἀναγραφησομένῳ τετραγώνῳ.

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ Α ἴσον τετράγωνον συνέσταται τὸ ἀπὸ τῆς ΕΘ ἀναγραφησόμενον· ὅπερ ἔδει ποιῆσαι.

equally at G, and unequally at E—the rectangle contained by BE and EF, plus the square on EG, is thus equal to the square on GF [Prop. 2.5]. And GF (is) equal to GH. Thus, the (rectangle contained) by BE and EF, plus the (square) on GE, is equal to the (square) on GH. And the (sum of the) squares on HE and EG is equal to the (square) on GH [Prop. 1.47]. Thus, the (rectangle contained) by BE and EF, plus the (square) on GE, is equal to the (sum of the squares) on HE and EG. Let the square on GE be taken from both. Thus, the remaining rectangle contained by BE and EF is equal to the square on EH. But, BD is the (rectangle contained) by BE and EF. For EF (is) equal to ED. Thus, the parallelogram BD is equal to the square on HE. And BD (is) equal to the rectilinear figure A. Thus, the rectilinear figure A is also equal to the square (which) can be described on EH.

Thus, a square—(namely), that (which) can be described on EH—has been constructed, equal to the given rectilinear figure A. (Which is) the very thing it was required to do.

ELEMENTS BOOK 3

Fundamentals of Plane Geometry Involving Circles

῞Οροι.

α΄. ῎Ισοι κύκλοι εἰσίν, ὧν αἱ διάμετροι ἴσαι εἰσίν, ἢ ὧν αἱ ἐκ τῶν κέντρων ἴσαι εἰσίν.

β΄. Εὐθεῖα κύκλου ἐφάπτεσθαι λέγεται, ἥτις ἁπτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.

γ΄. Κύκλοι ἐφάπτεσθαι ἀλλήλων λέγονται οἵτινες ἁπτόμενοι ἀλλήλων οὐ τέμνουσιν ἀλλήλους.

δ΄. ᾿Εν κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ᾿ αὐτὰς κάθετοι ἀγόμεναι ἴσαι ὦσιν.

ε΄. Μεῖζον δὲ ἀπέχειν λέγεται, ἐφ᾿ ἣν ἡ μείζων κάθετος πίπτει.

ϛ΄. Τμῆμα κύκλου ἐστὶ τὸ περιεχόμενον σχῆμα ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

ζ΄. Τμήματος δὲ γωνία ἐστὶν ἡ περιεχομένη ὑπό τε εὐθείας καὶ κύκλου περιφερείας.

η΄. ᾿Εν τμήματι δὲ γωνία ἐστίν, ὅταν ἐπὶ τῆς περιφερείας τοῦ τμήματος ληφθῇ τι σημεῖον καὶ ἀπ᾿ αὐτοῦ ἐπὶ τὰ πέρατα τῆς εὐθείας, ἥ ἐστι βάσις τοῦ τμήματος, ἐπιζευχθῶσιν εὐθεῖαι, ἡ περιεχομένη γωνία ὑπὸ τῶν ἐπιζευχθεισῶν εὐθειῶν.

θ΄. ῞Οταν δὲ αἱ περιέχουσαι τὴν γωνίαν εὐθεῖαι ἀπολαμβάνωσί τινα περιφέρειαν, ἐπ᾿ ἐκείνης λέγεται βεβηκέναι ἡ γωνία.

ι΄. Τομεὺς δὲ κύκλου ἐστίν, ὅταν πρὸς τῷ κέντρῳ τοῦ κύκλου συσταθῇ γωνία, τὸ περιεχόμενον σχῆμα ὑπό τε τῶν τὴν γωνίαν περιεχουσῶν εὐθειῶν καὶ τῆς ἀπολαμβανομένης ὑπ᾿ αὐτῶν περιφερείας.

ια΄. ῞Ομοια τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἢ ἐν οἷς αἱ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Definitions

1. Equal circles are (circles) whose diameters are equal, or whose (distances) from the centers (to the circumferences) are equal (i.e., whose radii are equal).

2. A straight-line said to touch a circle is any (straight-line) which, meeting the circle and being produced, does not cut the circle.

3. Circles said to touch one another are any (circles) which, meeting one another, do not cut one another.

4. In a circle, straight-lines are said to be equally far from the center when the perpendiculars drawn to them from the center are equal.

5. And (that straight-line) is said to be further (from the center) on which the greater perpendicular falls (from the center).

6. A segment of a circle is the figure contained by a straight-line and a circumference of a circle.

7. And the angle of a segment is that contained by a straight-line and a circumference of a circle.

8. And the angle in a segment is the angle contained by the joined straight-lines, when any point is taken on the circumference of a segment, and straight-lines are joined from it to the ends of the straight-line which is the base of the segment.

9. And when the straight-lines containing an angle cut off some circumference, the angle is said to stand upon that (circumference).

10. And a sector of a circle is the figure contained by the straight-lines surrounding an angle, and the circumference cut off by them, when the angle is constructed at the center of a circle.

11. Similar segments of circles are those accepting equal angles, or in which the angles are equal to one another.

α΄.

Τοῦ δοθέντος κύκλου τὸ κέντρον εὑρεῖν.

῎Εστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ· δεῖ δὴ τοῦ ΑΒΓ κύκλου τὸ κέντρον εὑρεῖν.

Διήχθω τις εἰς αὐτόν, ὡς ἔτυχεν, εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Δ σημεῖον, καὶ ἀπὸ τοῦ Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΔΓ καὶ διήχθω ἐπὶ τὸ Ε, καὶ τετμήσθω ἡ ΓΕ δίχα κατὰ τὸ Ζ· λέγω, ὅτι τὸ Ζ κέντρον ἐστὶ τοῦ ΑΒΓ [κύκλου].

μὴ γάρ, ἀλλ᾿ εἰ δυνατόν, ἔστω τὸ Η, καὶ ἐπεζεύχθωσαν αἱ ΗΑ, ΗΔ, ΗΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΒ, κοινὴ δὲ ἡ ΔΗ, δύο δὴ αἱ ΑΔ, ΔΗ δύο ταῖς ΗΔ, ΔΒ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΗΑ βάσει τῇ ΗΒ ἐστιν ἴση· ἐκ κέντρου γάρ· γωνία ἄρα ἡ ὑπὸ ΑΔΗ γωνίᾳ τῇ ὑπὸ ΗΔΒ ἴση ἐστίν. ὅταν δὲ εὐθεῖα ἐπ᾿ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ὀρθὴ

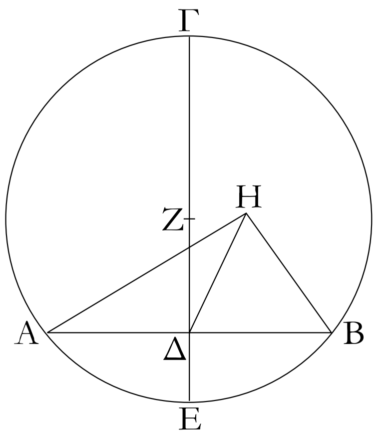
Proposition 1

To find the center of a given circle.

Let *ABC* be the given circle. So it is required to find the center of circle *ABC*.

Let some straight-line *AB* be drawn through (*ABC*), at random, and let (*AB*) be cut in half at point *D* [Prop. 1.9]. And let *DC* be drawn from *D*, at right-angles to *AB* [Prop. 1.11]. And let (*CD*) be drawn through to *E*. And let *CE* be cut in half at *F* [Prop. 1.9]. I say that (point) *F* is the center of the [circle] *ABC*.

For (if) not (then), if possible, let *G* (be the center of the circle), and let *GA*, *GD*, and *GB* be joined. And since *AD* is equal to *DB*, and *DG* (is) common, the two (straight-lines) *AD*, *DG* are equal to the two (straight-lines) *BD*, *DG*,† respectively. And the base *GA* is equal to the base *GB*. For (they are both) radii. Thus, angle *ADG* is equal to angle *GDB* *ἄρα ἐστὶν ἡ ὑπὸ ΗΔΒ. ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΔΒ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΖΔΒ τῇ ὑπὸ ΗΔΒ, ἡ μείζων τῇ ἐλάττονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Η κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾿ ἄλλο τι πλὴν τοῦ Ζ.*

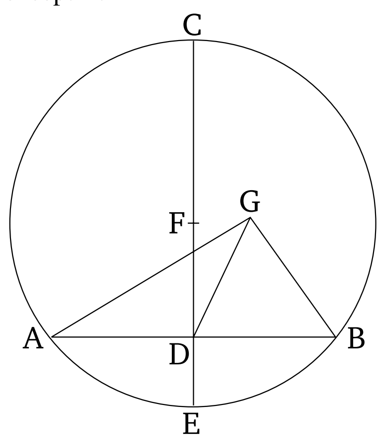


Τὸ Ζ ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ [κύκλου].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἐν κύκλῳ εὐθεῖά τις εὐθεῖάν τινα δίχα καὶ πρὸς ὀρθὰς τέμνῃ, ἐπὶ τῆς τεμνούσης ἐστὶ τὸ κέντρον τοῦ κύκλου. — ὅπερ ἔδει ποιῆσαι.

[Prop. 1.8]. And when a straight-line stood upon (another) straight-line make adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, *GDB* is a right-angle. And *FDB* is also a right-angle. Thus, *FDB* (is) equal to *GDB*, the greater to the lesser. The very thing is impossible. Thus, (point) *G* is not the center of the circle *ABC*. So, similarly, we can show that neither is any other (point) except *F*.



Thus, point *F* is the center of the [circle] *ABC*.

Corollary

So, from this, (it is) manifest that if any straight-line in a circle cuts any (other) straight-line in half, and at right-angles, (then) the center of the circle is on the former (straight-line). — (Which is) the very thing it was required to do.

† The Greek text has “*GD*, *DB*”, which is obviously a mistake.

βʹ.

Ἐὰν κύκλου ἐπὶ τῆς περιφερείας ληφθῇ δύο τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ ἐπὶ τῆς περιφερείας αὐτοῦ εἰλήφθω δύο τυχόντα σημεῖα τὰ Α, Β· λέγω, ὅτι ἡ ἀπὸ τοῦ Α ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

μὴ γάρ, ἀλλ᾿ εἰ δυνατόν, πιπτέτω ἐκτὸς ὡς ἡ ΑΕΒ, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Δ, καὶ ἐπεζεύχθωσαν αἱ ΔΑ, ΔΒ, καὶ διήχθω ἡ ΔΖΕ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΑ τῇ ΔΒ, ἴση ἄρα καὶ γωνία ἡ ὑπὸ ΔΑΕ τῇ ὑπὸ ΔΒΕ· καὶ ἐπεὶ τριγώνου τοῦ ΔΑΕ μία πλευρὰ προσεκβέβληται ἡ ΑΕΒ, μείζων ἄρα ἡ ὑπὸ ΔΕΒ γωνία τῆς ὑπὸ ΔΑΕ. ἴση δὲ ἡ ὑπὸ ΔΑΕ τῇ ὑπὸ ΔΒΕ· μείζων ἄρα ἡ ὑπὸ ΔΕΒ τῆς ὑπὸ ΔΒΕ. ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ΔΒ τῆς ΔΕ. ἴση δὲ ἡ ΔΒ τῇ ΔΖ. μείζων ἄρα ἡ ΔΖ τῆς ΔΕ ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Α ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἐκτὸς πεσεῖται τοῦ κύκλου. ὁμοίως

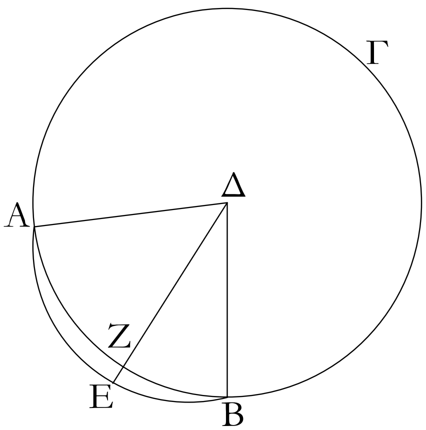
Proposition 2

If two points are taken at random on the circumference of a circle, (then) the straight-line joining the points will fall inside the circle.

Let *ABC* be a circle, and let two points *A* and *B* be taken at random on its circumference. I say that the straight-line joining *A* to *B* will fall inside the circle.

For (if) not (then), if possible, let it fall outside (the circle), like *AEB* (in the figure). And let the center of the circle *ABC* be found [Prop. 3.1], and let it be (at point) *D*. And let *DA* and *DB* be joined, and let *DFE* be drawn through.

Therefore, since *DA* is equal to *DB*, the angle *DAE* (is) thus also equal to *DBE* [Prop. 1.5]. And since in triangle *DAE* the one side, *AEB*, has been produced, angle *DEB* (is) thus greater than *DAE* [Prop. 1.16]. And *DAE* (is) equal to *DBE* [Prop. 1.5]. Thus, *DEB* (is) greater than *DBE*. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, *DB* (is) greater than *DE*. And *DB* (is) equal to *DF*. Thus, *DF* (is) greater than *DE*, the lesser than the greater. The very thing is *δὴ δείξομεν, ὅτι οὐδὲ ἐπ᾽ αὐτῆς τῆς περιφερείας· ἐντὸς ἄρα.*



Ἐὰν ἄρα κύκλου ἐπὶ τῆς περιφερείας ληφθῇ δύο τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

γ´.

Ἐὰν ἐν κύκλῳ εὐθεῖά τις διὰ τοῦ κέντρου εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐὰν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνει.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ ἐν αὐτῷ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΓΔ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΒ δίχα τεμνέτω κατὰ τὸ Ζ σημεῖον· λέγω, ὅτι καὶ πρὸς ὀρθὰς αὐτὴν τέμνει.

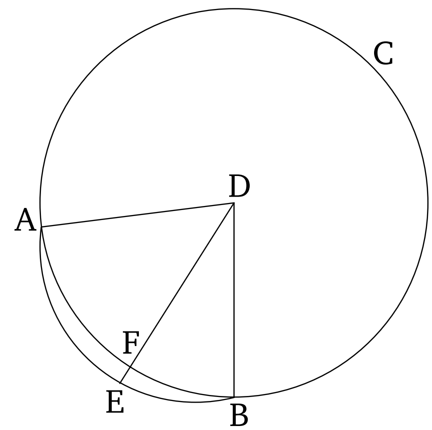
Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ε, καὶ ἐπεζεύχθωσαν αἱ ΕΑ, ΕΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, κοινὴ δὲ ἡ ΖΕ, δύο δυσὶν ἴσαι [εἰσίν]· καὶ βάσις ἡ ΕΑ βάσει τῇ ΕΒ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΖΕ γωνίᾳ τῇ ὑπὸ ΒΖΕ ἴση ἐστίν. ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστιν· ἑκατέρα ἄρα τῶν ὑπὸ ΑΖΕ, ΒΖΕ ὀρθή ἐστιν. ἡ ΓΔ ἄρα διὰ τοῦ κέντρου οὖσα τὴν ΑΒ μὴ διὰ τοῦ κέντρου οὖσαν δίχα τέμνουσα καὶ πρὸς ὀρθὰς τέμνει.

Ἀλλὰ δὴ ἡ ΓΔ τὴν ΑΒ πρὸς ὀρθὰς τεμνέτω· λέγω, ὅτι καὶ δίχα αὐτὴν τέμνει, τουτέστιν, ὅτι ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴση ἐστὶν ἡ ΕΑ τῇ ΕΒ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΕΑΖ τῇ ὑπὸ ΕΒΖ. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΖΕ ὀρθῇ τῇ ὑπὸ ΒΖΕ ἴση· δύο ἄρα τρίγωνά ἐστι ΕΑΖ, ΕΖΒ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΕΖ ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει· ἴση ἄρα ἡ ΑΖ τῇ ΖΒ.

impossible. Thus, the straight-line joining A to B will not fall outside the circle. So, similarly, we can show that neither (will it fall) on the circumference itself. Thus, (it will fall) inside (the circle).



Thus, if two points are taken at random on the circumference of a circle, (then) the straight-line joining the points will fall inside the circle. (Which is) the very thing it was required to show.

Proposition 3

In a circle, if any straight-line through the center cuts in half any straight-line not through the center, (then) it also cuts it at right-angles. And (conversely) if it cuts it at right-angles, (then) it also cuts it in half.

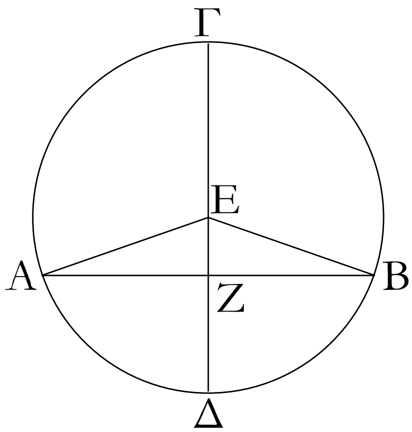
Let ABC be a circle, and, within it, let some straight-line through the center, CD, cut in half some straight-line not through the center, AB, at the point F. I say that (CD) also cuts (AB) at right-angles.

For let the center of the circle ABC be found [Prop. 3.1], and let it be (at point) E, and let EA and EB be joined.

And since AF is equal to FB, and FE (is) common, two (sides of triangle AFE) [are] equal to two (sides of triangle BFE). And the base EA (is) equal to the base EB. Thus, angle AFE is equal to angle BFE [Prop. 1.8]. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, AFE and BFE are each right-angles. Thus, the (straight-line) CD, which is through the center and cuts in half the (straight-line) AB, which is not through the center, also cuts (AB) at right-angles.

And so let CD cut AB at right-angles. I say that it also cuts (AB) in half. That is to say, that AF is equal to FB.

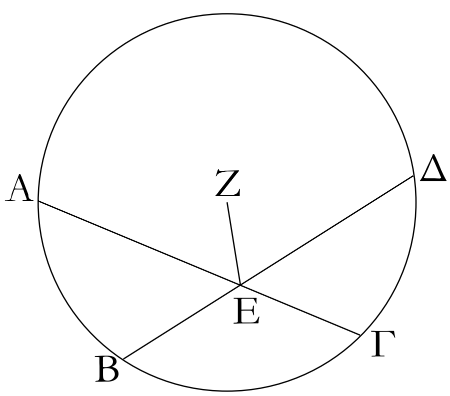
For, with the same construction, since EA is equal to EB, angle EAF is also equal to EBF [Prop. 1.5]. And the right-angle AFE is also equal to the right-angle BFE. Thus, EAF and EFB are two triangles having two angles equal to two angles, and one side equal to one side—(namely), their common



Ἐὰν ἄρα ἐν κύκλῳ εὐθεῖά τις διὰ τοῦ κέντρου εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐὰν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνει· ὅπερ ἔδει δεῖξαι.

δ'.

Ἐὰν ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ δία τοῦ κέντρου οὖσαι, οὐ τέμνουσιν ἀλλήλας δίχα.

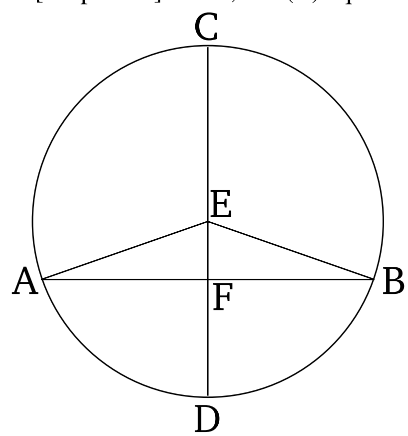


Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ δύο εὐθεῖαι αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε μὴ διὰ τοῦ κέντρου οὖσαι· λέγω, ὅτι οὐ τέμνουσιν ἀλλήλας δίχα.

Εἰ γὰρ δυνατόν, τεμνέτωσαν ἀλλήλας δίχα ὥστε ἴσην εἶναι τὴν μὲν ΑΕ τῇ ΕΓ, τὴν δὲ ΒΕ τῇ ΕΔ· καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓΔ κύκλου, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΖΕ.

Ἐπεὶ οὖν εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΖΕ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΕΑ· πάλιν, ἐπεὶ εὐθεῖά τις ἡ ΖΕ εὐθεῖάν τινα τὴν ΒΔ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἡ ὑπὸ ΖΕΒ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΕΑ

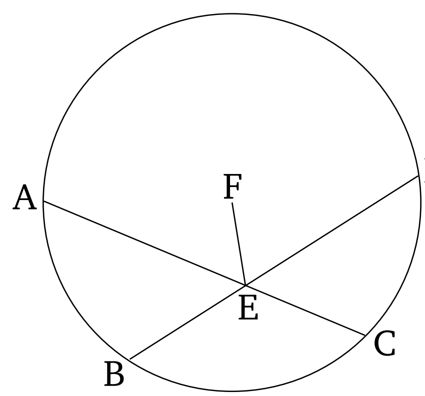
(side) EF, subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, AF (is) equal to FB.



Thus, in a circle, if any straight-line through the center cuts in half any straight-line not through the center, (then) it also cuts it at right-angles. And (conversely) if it cuts it at right-angles, (then) it also cuts it in half. (Which is) the very thing it was required to show.

Proposition 4

In a circle, if two straight-lines, which are not through the center, cut one another, (then) they do not cut one another in half.



Let $ABCD$ be a circle, and within it, let two straight-lines, AC and BD, which are not through the center, cut one another at (point) E. I say that they do not cut one another in half.

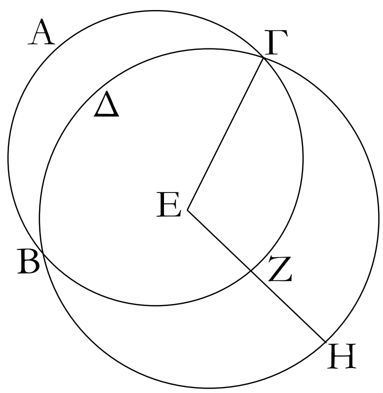
For, if possible, let them cut one another in half, such that AE is equal to EC, and BE to ED. And let the center of the circle $ABCD$ be found [Prop. 3.1], and let it be (at point) F, and let FE be joined.

Therefore, since some straight-line through the center, FE, cuts in half some straight-line not through the center, AC, it also cuts it at right-angles [Prop. 3.3]. Thus, FEA is a right-angle. Again, since some straight-line FE cuts in half some *ὀρθή· ἴση ἄρα ἡ ὑπὸ ΖΕΑ τῇ ὑπὸ ΖΕΒ ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα αἱ ΑΓ, ΒΔ τέμνουσιν ἀλλήλας δίχα.*

Ἐὰν ἄρα ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ διὰ τοῦ κέντρου οὖσαι, οὐ τέμνουσιν ἀλλήλας δίχα· ὅπερ ἔδει δεῖξαι.

εʹ.

Ἐὰν δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.



Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΓΔΗ τεμνέτωσαν ἀλλήλους κατὰ τὰ Β, Γ σημεῖα. λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Εἰ γὰρ δυνατόν, ἔστω τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΓ, καὶ διήχθω ἡ ΕΖΗ, ὡς ἔτυχεν. καὶ ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῇ ΕΖ. πάλιν, ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΓΔΗ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῇ ΕΗ· ἐδείχθη δὲ ἡ ΕΓ καὶ τῇ ΕΖ ἴση· καὶ ἡ ΕΖ ἄρα τῇ ΕΗ ἐστιν ἴση ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ε σημεῖον κέντρον ἐστὶ τῶν ΑΒΓ, ΓΔΗ κύκλων.

Ἐὰν ἄρα δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔστιν αὐτῶν τὸ αὐτὸ κέντρον· ὅπερ ἔδει δεῖξαι.

ϛʹ.

Ἐὰν δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΓΔΕ ἐφαπτέσθωσαν ἀλλήλων κατὰ τὸ Γ σημεῖον· λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Εἰ γὰρ δυνατόν, ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΖΓ, καὶ διήχθω, ὡς ἔτυχεν, ἡ ΖΕΒ.

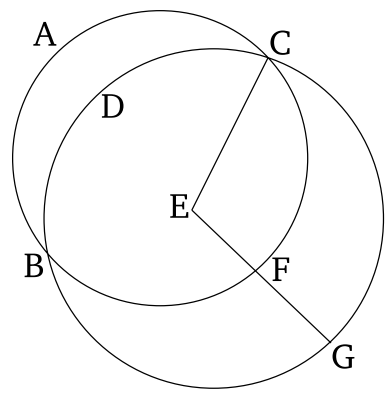
Ἐπεὶ οὖν τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῇ ΖΒ. πάλιν, ἐπεὶ τὸ Ζ σημεῖον κέντρον ἐστὶ

straight-line *BD*, it also cuts it at right-angles [Prop. 3.3]. Thus, *FEB* (is) a right-angle. But *FEA* was also shown (to be) a right-angle. Thus, *FEA* (is) equal to *FEB*, the lesser to the greater. The very thing is impossible. Thus, *AC* and *BD* do not cut one another in half.

Thus, in a circle, if two straight-lines, which are not through the center, cut one another, (then) they do not cut one another in half. (Which is) the very thing it was required to show.

Proposition 5

If two circles cut one another, (then) they will not have the same center.



For let the two circles *ABC* and *CDG* cut one another at points *B* and *C*. I say that they will not have the same center.

For, if possible, let *E* be (the common center), and let *EC* be joined, and let *EFG* be drawn through (the two circles), at random. And since point *E* is the center of the circle *ABC*, *EC* is equal to *EF*. Again, since point *E* is the center of the circle *CDG*, *EC* is equal to *EG*. But *EC* was also shown (to be) equal to *EF*. Thus, *EF* is also equal to *EG*, the lesser to the greater. The very thing is impossible. Thus, point *E* is not the (common) center of the circles *ABC* and *CDG*.

Thus, if two circles cut one another, (then) they will not have the same center. (Which is) the very thing it was required to show.

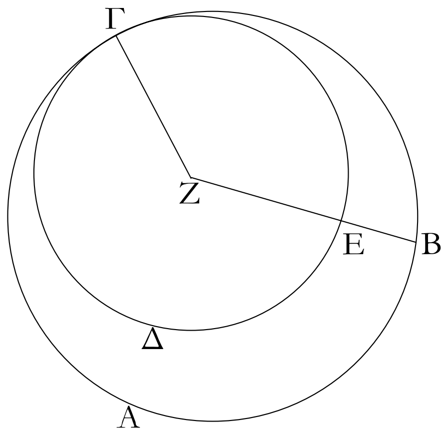
Proposition 6

If two circles touch one another, (then) they will not have the same center.

For let the two circles *ABC* and *CDE* touch one another at point *C*. I say that they will not have the same center.

For, if possible, let *F* be (the common center), and let *FC* be joined, and let *FEB* be drawn through (the two circles), at random.

Therefore, since point *F* is the center of the circle *ABC*, *FC* is equal to *FB*. Again, since point *F* is the center of the *τοῦ ΓΔΕ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῇ ΖΕ. ἐδείχθη δὲ ἡ ΖΓ τῇ ΖΒ ἴση· καὶ ἡ ΖΕ ἄρα τῇ ΖΒ ἐστιν ἴση, ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ζ σημεῖον κέντρον ἐστὶ τῶν ΑΒΓ, ΓΔΕ κύκλων.*



Ἐὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον· ὅπερ ἔδει δεῖξαι.

ζ′.

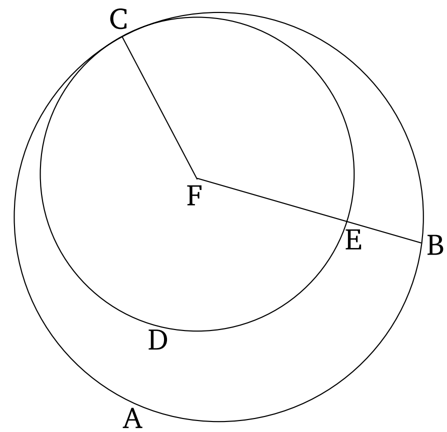
Ἐὰν κύκλου ἐπὶ τῆς διαμέτρου ληφθῇ τι σημεῖον, ὃ μή ἐστι κέντρον τοῦ κύκλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσιν εὐθεῖαί τινες, μεγίστη μὲν ἔσται, ἐφ᾿ ἧς τὸ κέντρον, ἐλαχίστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς δὶα τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾿ ἑκάτερα τῆς ἐλαχίστης.

Ἔστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΑΔ, καὶ ἐπὶ τῆς ΑΔ εἰλήφθω τι σημεῖον τὸ Ζ, ὃ μή ἐστι κέντρον τοῦ κύκλου, κέντρον δὲ τοῦ κύκλου ἔστω τὸ Ε, καὶ ἀπὸ τοῦ Ζ πρὸς τὸν ΑΒΓΔ κύκλον προσπιπτέτωσαν εὐθεῖαί τινες αἱ ΖΒ, ΖΓ, ΖΗ· λέγω, ὅτι μεγίστη μέν ἐστιν ἡ ΖΑ, ἐλαχίστη δὲ ἡ ΖΔ, τῶν δὲ ἄλλων ἡ μὲν ΖΒ τῆς ΖΓ μείζων, ἡ δὲ ΖΓ τῆς ΖΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΕ, ΗΕ. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσιν, αἱ ἄρα ΕΒ, ΕΖ τῆς ΒΖ μείζονές εἰσιν. ἴση δὲ ἡ ΑΕ τῇ ΒΕ [αἱ ἄρα ΒΕ, ΕΖ ἴσαι εἰσὶ τῇ ΑΖ]· μείζων ἄρα ἡ ΑΖ τῆς ΒΖ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΓΕ, κοινὴ δὲ ἡ ΖΕ, δύο δὴ αἱ ΒΕ, ΕΖ δυσὶ ταῖς ΓΕ, ΕΖ ἴσαι εἰσίν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΒΕΖ γωνίας τῆς ὑπὸ ΓΕΖ μείζων· βάσις ἄρα ἡ ΒΖ βάσεως τῆς ΓΖ μείζων ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΖ τῆς ΖΗ μείζων ἐστίν.

Πάλιν, ἐπεὶ αἱ ΗΖ, ΖΕ τῆς ΕΗ μείζονές εἰσιν, ἴση δὲ ἡ ΕΗ τῇ ΕΔ, αἱ ἄρα ΗΖ, ΖΕ τῆς ΕΔ μείζονές εἰσιν. κοινὴ ἀφῃρήσθω ἡ ΕΖ· λοιπὴ ἄρα ἡ ΗΖ λοιπῆς τῆς ΖΔ μείζων ἐστίν. μεγίστη μὲν ἄρα ἡ ΖΑ, ἐλαχίστη δὲ ἡ ΖΔ, μείζων δὲ ἡ μὲν ΖΒ τῆς ΖΓ, ἡ δὲ ΖΓ τῆς ΖΗ.

circle CDE, FC is equal to FE. But FC was shown (to be) equal to FB. Thus, FE is also equal to FB, the lesser to the greater. The very thing is impossible. Thus, point F is not the (common) center of the circles ABC and CDE.



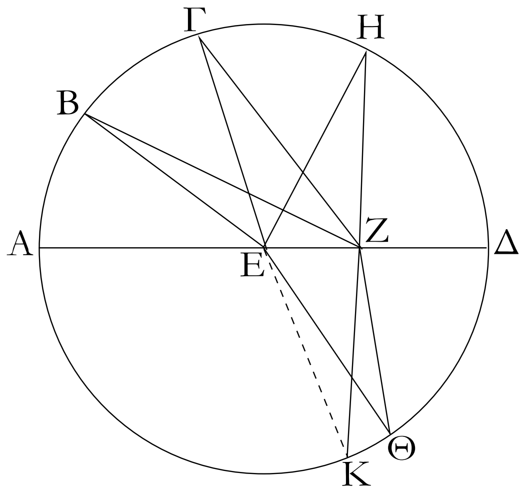
Thus, if two circles touch one another, (then) they will not have the same center. (Which is) the very thing it was required to show.

Proposition 7

If some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, (then) the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer† to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).

Let $ABCD$ be a circle, and let AD be its diameter, and let some point F, which is not the center of the circle, be taken on AD. Let E be the center of the circle. And let some straight-lines, FB, FC, and FG, radiate from F towards (the circumference of) circle $ABCD$. I say that FA is the greatest (straight-line), FD the least, and of the others, FB (is) greater than FC, and FC than FG.

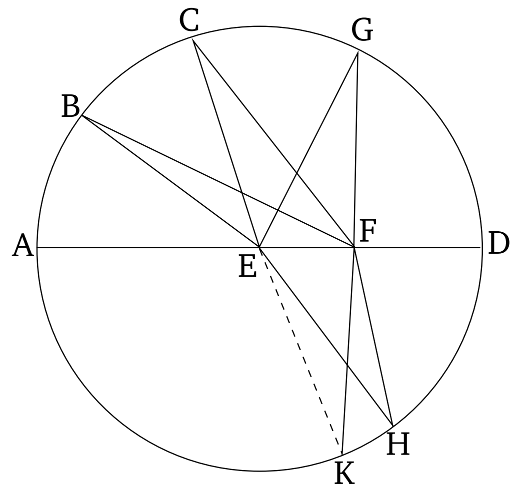
For let BE, CE, and GE be joined. And since for every triangle (any) two sides are greater than the remaining (side) [Prop. 1.20], EB and EF is thus greater than BF. And AE (is) equal to BE [thus, BE and EF is equal to AF]. Thus, AF (is) greater than BF. Again, since BE is equal to CE, and FE (is) common, the two (straight-lines) BE, EF are equal to the two (straight-lines) CE, EF (respectively). But, angle BEF (is) also greater than angle CEF.‡ Thus, the base BF is greater than the base CF. Thus, the base BF is greater than the base CF [Prop. 1.24]. So, for the same (reasons), CF is also greater than FG.



Λέγω, ὅτι καὶ ἀπὸ τοῦ Ζ σημείου δύο μόνον ἴσαι προσπεσοῦνται πρὸς τὸν ΑΒΓΔ κύκλον ἐφ᾽ ἑκάτερα τῆς ΖΔ ἐλαχίστης. συνεστάτω γὰρ πρὸς τῇ ΕΖ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ε τῇ ὑπὸ ΗΕΖ γωνίᾳ ἴση ἡ ὑπὸ ΖΕΘ, καὶ ἐπεζεύχθω ἡ ΖΘ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ΗΕ τῇ ΕΘ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἱ ΗΕ, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΗΕΖ γωνίᾳ τῇ ὑπὸ ΘΕΖ ἴση· βάσις ἄρα ἡ ΖΗ βάσει τῇ ΖΘ ἴση ἐστίν. λέγω δή, ὅτι τῇ ΖΗ ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν κύκλον ἀπὸ τοῦ Ζ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω ἡ ΖΚ. καὶ ἐπεὶ ἡ ΖΚ τῇ ΖΗ ἴση ἐστίν, ἀλλὰ ἡ ΖΘ τῇ ΖΗ [ἴση ἐστίν], καὶ ἡ ΖΚ ἄρα τῇ ΖΘ ἐστιν ἴση, ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῇ ἀπώτερον ἴση· ὅπερ ἀδύνατον. οὐκ ἄρα ἀπὸ τοῦ Ζ σημείου ἑτέρα τις προσπεσεῖται πρὸς τὸν κύκλον ἴση τῇ ΗΖ· μία ἄρα μόνη.

Ἐὰν ἄρα κύκλου ἐπὶ τῆς διαμέτρου ληφθῇ τι σημεῖον, ὃ μή ἐστι κέντρον τοῦ κύκλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσιν εὐθεῖαί τινες, μεγίστη μὲν ἔσται, ἐφ᾽ ἧς τὸ κέντρον, ἐλαχίστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς δία τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ αὐτοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ἐλαχίστης· ὅπερ ἔδει δεῖξαι.

Again, since GF and FE are greater than EG [Prop. 1.20], and EG (is) equal to ED, GF and FE are thus greater than ED. Let EF be taken from both. Thus, the remainder GF is greater than the remainder FD. Thus, FA (is) the greatest (straight-line), FD the least, and FB (is) greater than FC, and FC than FG.



I also say that from point F only two equal (straight-lines) will radiate towards (the circumference of) circle $ABCD$, (one) on each (side) of the least (straight-line) FD. For let the (angle) FEH, equal to angle GEF, be constructed on the straight-line EF, at the point E on it [Prop. 1.23], and let FH be joined. Therefore, since GE is equal to EH, and EF (is) common, the two (straight-lines) GE, EF are equal to the two (straight-lines) HE, EF (respectively). And angle GEF (is) equal to angle HEF. Thus, the base FG is equal to the base FH [Prop. 1.4]. So I say that another (straight-line) equal to FG will not radiate towards (the circumference of) the circle from point F. For, if possible, let FK (so) radiate. And since FK is equal to FG, but FH [is equal] to FG, FK is thus also equal to FH, the nearer to the (straight-line) through the center equal to the further away. The very thing (is) impossible. Thus, another (straight-line) equal to GF will not radiate from the point F towards (the circumference of) the circle. Thus, (there is) only one (such straight-line).

Thus, if some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, (then) the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the same point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

† Presumably, in an angular sense.

‡ This is not proved, except by reference to the figure.

ηʹ.

Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον διαχθῶσιν εὐθεῖαί τινες, ὧν μία μὲν διὰ τοῦ κέντρου, αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ μεταξὺ τοῦ τε σημείου καὶ τῆς διαμέτρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάττων, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ἐλαχίστης.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπ᾽ αὐτοῦ διήχθωσαν εὐθεῖαί τινες αἱ ΔΑ, ΔΕ, ΔΖ, ΔΓ, ἔστω δὲ ἡ ΔΑ διὰ τοῦ κέντρου. λέγω, ὅτι τῶν μὲν πρὸς τὴν ΑΕΖΓ κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντρου ἡ ΔΑ, μείζων δὲ ἡ μὲν ΔΕ τῆς ΔΖ ἡ δὲ ΔΖ τῆς ΔΓ, τῶν δὲ πρὸς τὴν ΘΛΚΗ κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ ΔΗ ἡ μεταξὺ τοῦ σημείου καὶ τῆς διαμέτρου τῆς ΑΗ, ἀεὶ δὲ ἡ ἔγγιον τῆς ΔΗ ἐλαχίστης ἐλάττων ἐστὶ τῆς ἀπώτερον, ἡ μὲν ΔΚ τῆς ΔΛ, ἡ δὲ ΔΛ τῆς ΔΘ.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου καὶ ἔστω τὸ Μ· καὶ ἐπεζεύχθωσαν αἱ ΜΕ, ΜΖ, ΜΓ, ΜΚ, ΜΛ, ΜΘ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΜ τῇ ΕΜ, κοινὴ προσκείσθω ἡ ΜΔ· ἡ ἄρα ΑΔ ἴση ἐστὶ ταῖς ΕΜ, ΜΔ. ἀλλ᾽ αἱ ΕΜ, ΜΔ τῆς ΕΔ μείζονές εἰσιν· καὶ ἡ ΑΔ ἄρα τῆς ΕΔ μείζων ἐστίν. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΜΕ τῇ ΜΖ, κοινὴ δὲ ἡ ΜΔ, αἱ ΕΜ, ΜΔ ἄρα ταῖς ΖΜ, ΜΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΕΜΔ γωνίας τῆς ὑπὸ ΖΜΔ μείζων ἐστίν. βάσις ἄρα ἡ ΕΔ βάσεως τῆς ΖΔ μείζων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΖΔ τῆς ΓΔ μείζων ἐστίν· μεγίστη μὲν ἄρα ἡ ΔΑ, μείζων δὲ ἡ μὲν ΔΕ τῆς ΔΖ, ἡ δὲ ΔΖ τῆς ΔΓ.

Καὶ ἐπεὶ αἱ ΜΚ, ΚΔ τῆς ΜΔ μείζονές εἰσιν, ἴση δὲ ἡ μη τῇ ΜΚ, λοιπὴ ἄρα ἡ ΚΔ λοιπῆς τῆς ΗΔ μείζων ἐστίν· ὥστε ἡ ΗΔ τῆς ΚΔ ἐλάττων ἐστίν· καὶ ἐπεὶ τριγώνου τοῦ ΜΛΔ ἐπὶ μιᾶς τῶν πλευρῶν τῆς ΜΔ δύο εὐθεῖαι ἐντὸς συνεστάθησαν αἱ ΜΚ, ΚΔ, αἱ ἄρα ΜΚ, ΚΔ τῶν ΜΛ, ΛΔ ἐλάττονές εἰσιν· ἴση δὲ ἡ ΜΚ τῇ ΜΛ· λοιπὴ ἄρα ἡ ΔΚ λοιπῆς τῆς ΔΛ ἐλάττων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΔΛ τῆς ΔΘ ἐλάττων ἐστίν· ἐλαχίστη μὲν ἄρα ἡ ΔΗ, ἐλάττων δὲ ἡ μὲν ΔΚ τῆς ΔΛ ἡ δὲ ΔΛ τῆς ΔΘ.

Proposition 8

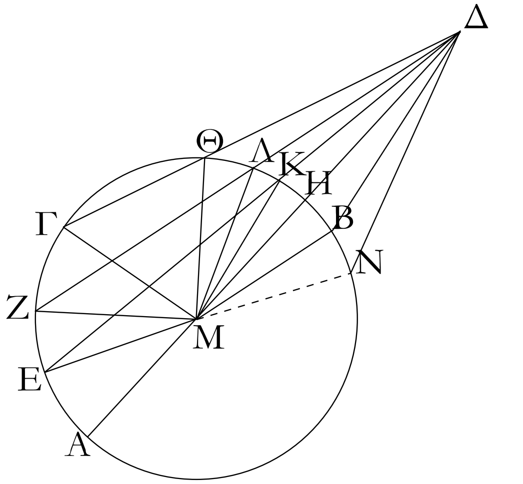
If some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, (then) for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer† to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).

Let *ABC* be a circle, and let some point *D* be taken outside *ABC*, and from it let some straight-lines, *DA*, *DE*, *DF*, and *DC*, be drawn through (the circle), and let *DA* be through the center. I say that for the straight-lines radiating towards the concave (part of the) circumference, *AEFC*, the greatest is the one (passing) through the center, (namely) *AD*, and (that) *DE* (is) greater than *DF*, and *DF* than *DC*. For the straight-lines radiating towards the convex (part of the) circumference, *HLKG*, the least is the one between the point and the diameter *AG*, (namely) *DG*, and a (straight-line) nearer to the least (straight-line) *DG* is always less than one farther away, (so that) *DK* (is less) than *DL*, and *DL* than than *DH*.

For let the center of the circle be found [Prop. 3.1], and let it be (at point) *M* [Prop. 3.1]. And let *ME*, *MF*, *MC*, *MK*, *ML*, and *mh* be joined.

And since *AM* is equal to *EM*, let *MD* be added to both. Thus, *AD* is equal to *EM* and *MD*. But, *EM* and *MD* is greater than *ED* [Prop. 1.20]. Thus, *AD* is also greater than *ED*. Again, since *ME* is equal to *MF*, and *MD* (is) common, the (straight-lines) *EM*, *MD* are thus equal to *FM*, *MD*. And angle *EMD* is greater than angle *FMD*.‡ Thus, the base *ED* is greater than the base *FD* [Prop. 1.24]. So, similarly, we can show that *FD* is also greater than *CD*. Thus, *AD* (is) the greatest (straight-line), and *DE* (is) greater than *DF*, and *DF* than *DC*.

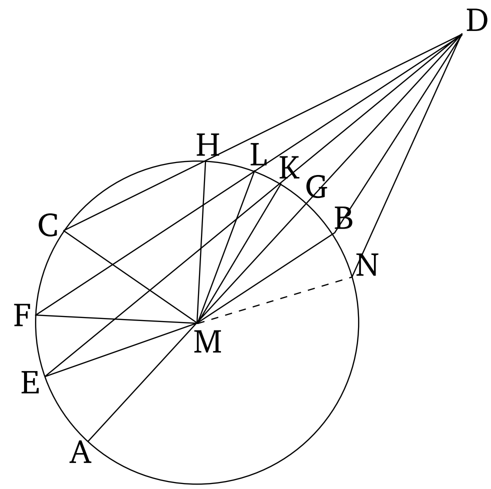
And since *MK* and *KD* is greater than *MD* [Prop. 1.20], and *MG* (is) equal to *MK*, the remainder *KD* is thus greater than the remainder *GD*. So *GD* is less than *KD*. And since in triangle *MLD*, the two internal straight-lines *MK* and *KD* were constructed on one of the sides, *MD*, (then) *MK* and *KD* are thus less than *ML* and *LD* [Prop. 1.21]. And *MK* (is) equal to *ML*. Thus, the remainder *DK* is less than the remainder *DL*.



Λέγω, ὅτι καὶ δύο μόνον ἴσαι ἀπὸ τοῦ Δ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ΔΗ ἐλαχίστης· συνεστάτω πρὸς τῇ ΜΔ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Μ τῇ ὑπὸ ΚΜΔ γωνίᾳ ἴση γωνία ἡ ὑπὸ ΔΜΒ, καὶ ἐπεζεύχθω ἡ ΔΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΜΚ τῇ ΜΒ, κοινὴ δὲ ἡ ΜΔ, δύο δὴ αἱ ΚΜ, ΜΔ δύο ταῖς ΒΜ, ΜΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΚΜΔ γωνίᾳ τῇ ὑπὸ ΒΜΔ ἴση· βάσις ἄρα ἡ ΔΚ βάσει τῇ ΔΒ ἴση ἐστίν. λέγω [δή], ὅτι τῇ ΔΚ εὐθείᾳ ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν κύκλον ἀπὸ τοῦ Δ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω καὶ ἔστω ἡ ΔΝ. ἐπεὶ οὖν ἡ ΔΚ τῇ ΔΝ ἐστιν ἴση, ἀλλ᾽ ἡ ΔΚ τῇ ΔΒ ἐστιν ἴση, καὶ ἡ ΔΒ ἄρα τῇ ΔΝ ἐστιν ἴση, ἡ ἔγγιον τῆς ΔΗ ἐλαχίστης τῇ ἀπώτερον [ἐστιν] ἴση· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα πλείους ἢ δύο ἴσαι πρὸς τὸν ΑΒΓ κύκλον ἀπὸ τοῦ Δ σημείου ἐφ᾽ ἑκάτερα τῆς ΔΗ ἐλαχίστης προσπεσοῦνται.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον διαχθῶσιν εὐθεῖαί τινες, ὧν μία μὲν διὰ τοῦ κέντρου αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μέν ἐστιν ἡ διὰ τοῦ κέντου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μέν ἐστιν ἡ μεταξὺ τοῦ τε σημείου καὶ τῆς διαμέτρου, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάττων, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ᾽ ἑκάτερα τῆς ἐλαχίστης· ὅπερ ἔδει δεῖξαι.

So, similarly, we can show that DL is also less than DH. Thus, DG (is) the least (straight-line), and DK (is) less than DL, and DL than DH.



I also say that only two equal (straight-lines) will radiate from point D towards (the circumference of) the circle, (one) on each (side) on the least (straight-line), DG. Let the angle DMB, equal to angle KMD, be constructed on the straight-line MD, at the point M on it [Prop. 1.23], and let DB be joined. And since MK is equal to MB, and MD (is) common, the two (straight-lines) KM, MD are equal to the two (straight-lines) BM, MD, respectively. And angle KMD (is) equal to angle BMD. Thus, the base DK is equal to the base DB [Prop. 1.4]. [So] I say that another (straight-line) equal to DK will not radiate towards the (circumference of the) circle from point D. For, if possible, let (such a straight-line) radiate, and let it be DN. Therefore, since DK is equal to DN, but DK is equal to DB, (then) DB is thus also equal to DN, (so that) a (straight-line) nearer to the least (straight-line) DG [is] equal to one further away. The very thing was shown (to be) impossible. Thus, not more than two equal (straight-lines) will radiate towards (the circumference of) circle ABC from point D, (one) on each side of the least (straight-line) DG.

Thus, if some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, (then) for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

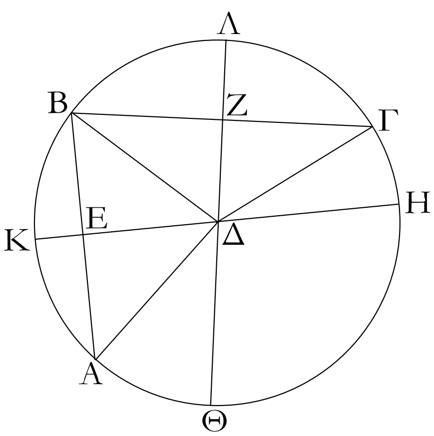
† Presumably, in an angular sense.

‡ This is not proved, except by reference to the figure.

ϑ′.

Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐντός, ἀπο δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον κέντρον ἐστὶ τοῦ κύκλου.

Ἔστω κύκλος ὁ ΑΒΓ, ἐντὸς δὲ αὐτοῦ σημεῖον τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύκλον προσπιπτέτωσαν πλείους ἢ δύο ἴσαι εὐθεῖαι αἱ ΔΑ, ΔΒ, ΔΓ· λέγω, ὅτι τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου.



Ἐπεζεύχθωσαν γὰρ αἱ ΑΒ, ΒΓ καὶ τετμήσθωσαν δίχα κατὰ τὰ Ε, Ζ σημεῖα, καὶ ἐπιζευχθεῖσαι αἱ ΕΔ, ΖΔ διήχθωσαν ἐπὶ τὰ Η, Κ, Θ, Λ σημεῖα.

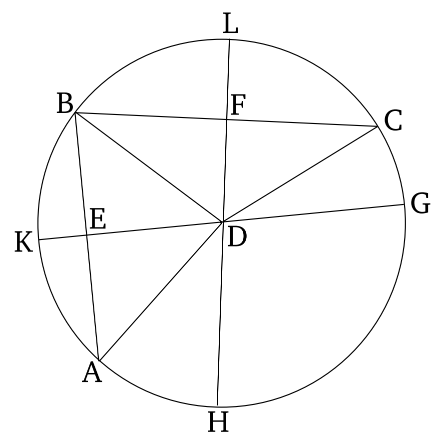
Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, κοινὴ δὲ ἡ ΕΔ, δύο δὴ αἱ ΑΕ, ΕΔ δύο ταῖς ΒΕ, ΕΔ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΑ βάσει τῇ ΔΒ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΕΔ γωνίᾳ τῇ ὑπὸ ΒΕΔ ἴση ἐστίν· ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ ΑΕΔ, ΒΕΔ γωνιῶν· ἡ ΗΚ ἄρα τὴν ΑΒ τέμνει δίχα καὶ πρὸς ὀρθάς. καὶ ἐπεί, ἐὰν ἐν κύκλῳ εὐθεῖά τις εὐθεῖάν τινα δίχα τε καὶ πρὸς ὀρθὰς τέμνῃ, ἐπὶ τῆς τεμνούσης ἐστὶ τὸ κέντρον τοῦ κύκλου, ἐπὶ τῆς ΗΚ ἄρα ἐστὶ τὸ κέντρον τοῦ κύκλου. διὰ τὰ αὐτὰ δὴ καὶ ἐπὶ τῆς ΘΛ ἐστι τὸ κέντρον τοῦ ΑΒΓ κύκλου. καὶ οὐδὲν ἕτερον κοινὸν ἔχουσιν αἱ ΗΚ, ΘΛ εὐθεῖαι ἢ τὸ Δ σημεῖον· τὸ Δ ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐντός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον κέντρον ἐστὶ τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

Proposition 9

If some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, (then) the point taken is the center of the circle.

Let *ABC* be a circle, and *D* a point inside it, and let more than two equal straight-lines, *DA*, *DB*, and *DC*, radiate from *D* towards (the circumference of) circle *ABC*. I say that point *D* is the center of circle *ABC*.



For let *AB* and *BC* be joined, and (then) be cut in half at points *E* and *F* (respectively) [Prop. 1.10]. And *ED* and *FD* being joined, let them be drawn through to points *G*, *K*, *H*, and *L*.

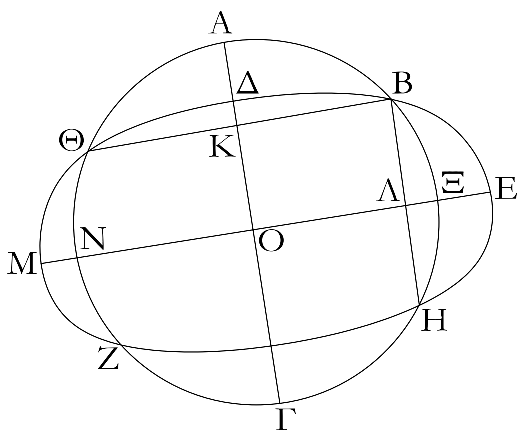
Therefore, since *AE* is equal to *EB*, and *ED* (is) common, the two (straight-lines) *AE*, *ED* are equal to the two (straight-lines) *BE*, *ED* (respectively). And the base *DA* (is) equal to the base *DB*. Thus, angle *AED* is equal to angle *BED* [Prop. 1.8]. Thus, angles *AED* and *BED* (are) each right-angles [Def. 1.10]. Thus, *GK* cuts *AB* in half, and at right-angles. And since, if some straight-line in a circle cuts some (other) straight-line in half, and at right-angles, (then) the center of the circle is on the former (straight-line) [Prop. 3.1 corr.], the center of the circle is thus on *GK*. So, for the same (reasons), the center of circle *ABC* is also on *HL*. And the straight-lines *GK* and *HL* have no common (point) other than point *D*. Thus, point *D* is the center of circle *ABC*.

Thus, if some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, (then) the point taken is the center of the circle. (Which is) the very thing it was required to show.

ι′.

Κύκλος κύκλον οὐ τέμνει κατὰ πλείονα σημεῖα ἢ δύο.

Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΒΓ κύκλον τὸν ΔΕΖ τεμνέτω κατὰ πλείονα σημεῖα ἢ δύο τὰ Β, Η, Ζ, Θ, καὶ ἐπιζευχθεῖσαι αἱ ΒΘ, ΒΗ δίχα τεμνέσθωσαν κατὰ τὰ Κ, Λ σημεῖα· καὶ ἀπὸ τῶν Κ, Λ ταῖς ΒΘ, ΒΗ πρὸς ὀρθὰς ἀχθεῖσαι αἱ ΚΓ, ΛΜ διήχθωσαν ἐπὶ τὰ Α, Ε σημεῖα.



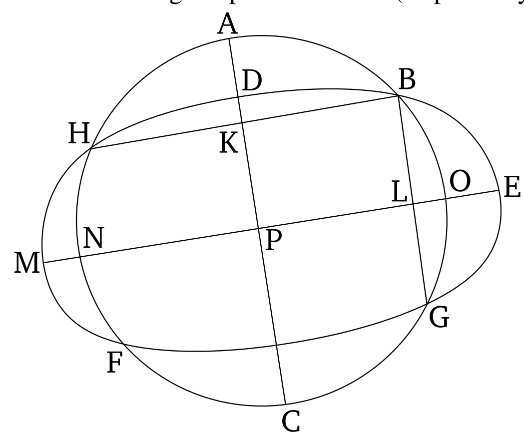
Ἐπεὶ οὖν ἐν κύκλῳ τῷ ΑΒΓ εὐθεῖά τις ἡ ΑΓ εὐθεῖάν τινα τὴν ΒΘ δίχα καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς ΑΓ ἄρα ἐστὶ τὸ κέντρον τοῦ ΑΒΓ κύκλου. πάλιν, ἐπεὶ ἐν κύκλῳ τῷ αὐτῷ τῷ ΑΒΓ εὐθεῖά τις ἡ ΝΞ εὐθεῖάν τινα τὴν ΒΗ δίχα καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς ΝΞ ἄρα ἐστὶ τὸ κέντρον τοῦ ΑΒΓ κύκλου. ἐδείχθη δὲ καὶ ἐπὶ τῆς ΑΓ, καὶ κατ᾽ οὐδὲν συμβάλλουσιν αἱ ΑΓ, ΝΞ εὐθεῖαι ἢ κατὰ τὸ Ο· τὸ Ο ἄρα σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι καὶ τοῦ ΔΕΖ κύκλου κέντρον ἐστὶ τὸ Ο· δύο ἄρα κύκλων τεμνόντων ἀλλήλους τῶν ΑΒΓ, ΔΕΖ τὸ αὐτό ἐστι κέντρον τὸ Ο· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα κύκλος κύκλον τέμνει κατὰ πλείονα σημεῖα ἢ δύο· ὅπερ ἔδει δεῖξαι.

Proposition 10

A circle does not cut a(nother) circle at more than two points.

For, if possible, let the circle *ABC* cut the circle *DEF* at more than two points, *B*, *G*, *F*, and *H*. And *BH* and *BG* being joined, let them (then) be cut in half at points *K* and *L* (respectively). And *KC* and *LM* being drawn at right-angles to *BH* and *BG* from *K* and *L* (respectively) [Prop. 1.11], let them (then) be drawn through to points *A* and *E* (respectively).



Therefore, since in circle *ABC* some straight-line *AC* cuts some (other) straight-line *BH* in half, and at right-angles, the center of circle *ABC* is thus on *AC* [Prop. 3.1 corr.]. Again, since in the same circle *ABC* some straight-line *NO* cuts some (other straight-line) *BG* in half, and at right-angles, the center of circle *ABC* is thus on *NO* [Prop. 3.1 corr.]. And it was also shown (to be) on *AC*. And the straight-lines *AC* and *NO* meet at no other (point) than *P*. Thus, point *P* is the center of circle *ABC*. So, similarly, we can show that *P* is also the center of circle *DEF*. Thus, two circles cutting one another, *ABC* and *DEF*, have the same center *P*. The very thing is impossible [Prop. 3.5].

Thus, a circle does not cut a(nother) circle at more than two points. (Which is) the very thing it was required to show.

ια′.

Ἐὰν δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐντός, καὶ ληφθῇ αὐτῶν τὰ κέντρα, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα καὶ ἐκβαλλομένη ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων.

Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΑΔΕ ἐφαπτέσθωσαν ἀλλήλων ἐντὸς κατὰ τὸ Α σημεῖον, καὶ εἰλήφθω τοῦ μὲν ΑΒΓ κύκλου κέντρον τὸ Ζ, τοῦ δὲ ΑΔΕ τὸ Η· λέγω, ὅτι ἡ ἀπὸ τοῦ Η ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖα ἐκβαλλομένη ἐπὶ τὸ Α πεσεῖται.

μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, πιπτέτω ὡς ἡ ΖΗΘ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΑΗ.

Ἐπεὶ οὖν αἱ ΑΗ, ΗΖ τῆς ΖΑ, τουτέστι τῆς ΖΘ, μείζονές

Proposition 11

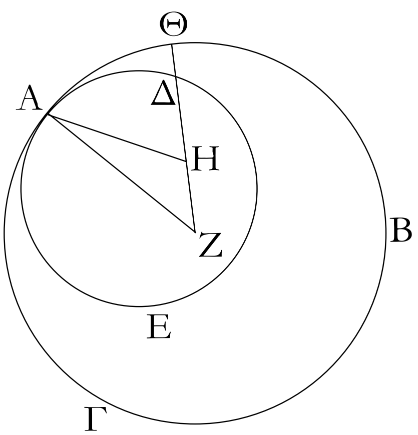
If two circles touch one another internally, and their centers are found, (then) the straight-line joining their centers, being produced, will fall upon the point of union of the circles.

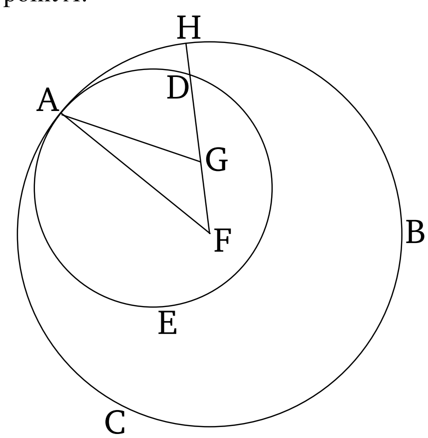
For let two circles, *ABC* and *ADE*, touch one another internally at point *A*, and let the center *F* of circle *ABC* be found [Prop. 3.1], and (the center) *G* of (circle) *ADE* [Prop. 3.1]. I say that the straight-line joining *G* to *F*, being produced, will fall on *A*.

For (if) not (then), if possible, let it fall like *FGH* (in the figure), and let *AF* and *AG* be joined.

Therefore, since *AG* and *GF* is greater than *FA*, that is to *εἰσιν, κοινὴ ἀφῃρήσθω ἡ ΖΗ· λοιπὴ ἄρα ἡ ΑΗ λοιπῆς τῆς ΗΘ μείζων ἐστίν. ἴση δὲ ἡ ΑΗ τῇ ΗΔ· καὶ ἡ ΗΔ ἄρα τῆς ΗΘ μείζων ἐστὶν ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα ἡ ἀπὸ τοῦ Ζ ἐπὶ τὸ Η ἐπιζευγνυμένη εὐθεῖα ἐκτὸς πεσεῖται· κατὰ τὸ Α ἄρα ἐπὶ τῆς συναφῆς πεσεῖται.*

say FH [Prop. 1.20], let FG be taken from both. Thus, the remainder AG is greater than the remainder GH. And AG (is) equal to GD. Thus, GD is also greater than GH, the lesser than the greater. The very thing is impossible. Thus, the straight-line joining F to G will not fall outside (one circle but inside the other). Thus, it will fall upon the point of union (of the circles) at point A.





Ἐὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐντός, [καὶ ληφθῇ αὐτῶν τὰ κέντρα], ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα [καὶ ἐκβαλλομένη] ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων· ὅπερ ἔδει δεῖξαι.

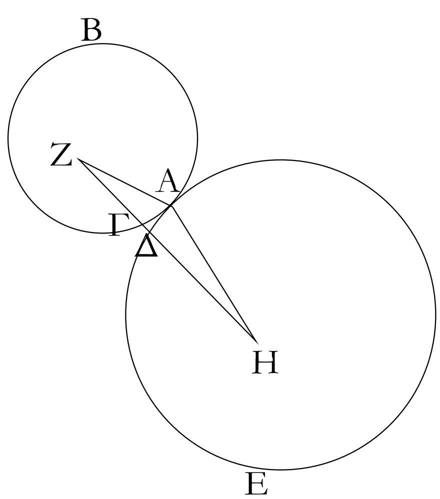
Thus, if two circles touch one another internally, [and their centers are found], (then) the straight-line joining their centers, [being produced], will fall upon the point of union of the circles. (Which is) the very thing it was required to show.

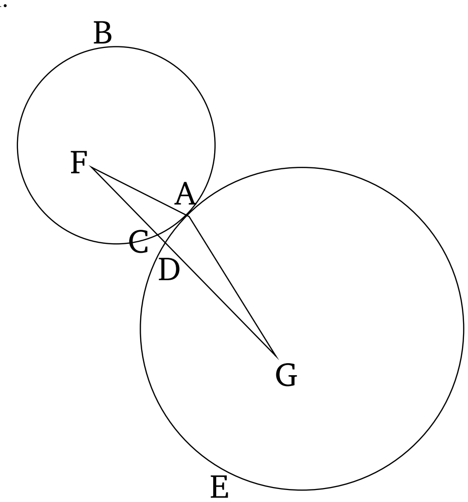
ιβ΄.

Ἐὰν δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐκτός, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη διὰ τῆς ἐπαφῆς ἐλεύσεται.

Proposition 12

If two circles touch one another externally, (then) the (straight-line) joining their centers will go through the point of union.





Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΑΔΕ ἐφαπτέσθωσαν ἀλλήλων ἐκτὸς κατὰ τὸ Α σημεῖον, καὶ εἰλήφθω τοῦ μὲν ΑΒΓ κέντρον τὸ Ζ, τοῦ δὲ ΑΔΕ τὸ Η· λέγω, ὅτι ἡ ἀπὸ τοῦ Ζ ἐπὶ τὸ Η ἐπιζευγνυμένη εὐθεῖα διὰ τῆς κατὰ τὸ Α ἐπαφῆς ἐλεύσεται.

μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, ἐρχέσθω ὡς ἡ ΖΓΔΗ, καὶ ἐπε-

For let two circles, ABC and ADE, touch one another externally at point A, and let the center F of ABC be found [Prop. 3.1], and (the center) G of ADE [Prop. 3.1]. I say that the straight-line joining F to G will go through the point of union at A.

ζεύχθωσαν αἱ ΑΖ, ΑΗ.

Ἐπεὶ οὖν τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΖΑ τῇ ΖΓ. πάλιν, ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΔΕ κύκλου, ἴση ἐστὶν ἡ ΗΑ τῇ ΗΔ. ἐδείχθη δὲ καὶ ἡ ΖΑ τῇ ΖΓ ἴση· αἱ ἄρα ΖΑ, ΑΗ ταῖς ΖΓ, ΗΔ ἴσαι εἰσίν· ὥστε ὅλη ἡ ΖΗ τῶν ΖΑ, ΑΗ μείζων ἐστίν· ἀλλὰ καὶ ἐλάττων· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Ζ ἐπὶ τὸ Η ἐπιζευγνυμένη εὐθεῖα διὰ τῆς κατὰ τὸ Α ἐπαφῆς οὐκ ἐλεύσεται· δι᾽ αὐτῆς ἄρα.

Ἐὰν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων ἐκτός, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη [εὐθεῖα] διὰ τῆς ἐπαφῆς ἐλεύσεται· ὅπερ ἔδει δεῖξαι.

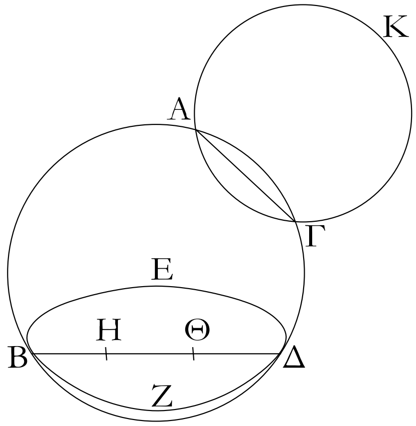
For (if) not (then), if possible, let it go like $FCDG$ (in the figure), and let AF and AG be joined.

Therefore, since point F is the center of circle ABC, FA is equal to FC. Again, since point G is the center of circle ADE, GA is equal to GD. And FA was also shown (to be) equal to FC. Thus, the (straight-lines) FA and AG are equal to the (straight-lines) FC and GD. So the whole of FG is greater than FA and AG. But, (it is) also less [Prop. 1.20]. The very thing is impossible. Thus, the straight-line joining F to G cannot not go through the point of union at A. Thus, (it will go) through it.

Thus, if two circles touch one another externally, (then) the [straight-line] joining their centers will go through the point of union. (Which is) the very thing it was required to show.

ιγ'.

Κύκλος κύκλου οὐκ ἐφάπτεται κατὰ πλείονα σημεῖα ἢ καθ᾽ ἕν, ἐάν τε ἐντὸς ἐάν τε ἐκτὸς ἐφάπτηται.



Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΒΓΔ κύκλου τοῦ ΕΒΖΔ ἐφαπτέσθω πρότερον ἐντὸς κατὰ πλείονα σημεῖα ἢ ἓν τὰ Δ, Β.

Καὶ εἰλήφθω τοῦ μὲν ΑΒΓΔ κύκλου κέντρον τὸ Η, τοῦ δὲ ΕΒΖΔ τὸ Θ.

Ἡ ἄρα ἀπὸ τοῦ Η ἐπὶ τὸ Θ ἐπιζευγνυμένη ἐπὶ τὰ Β, Δ πεσεῖται. πιπτέτω ὡς ἡ ΒΗΘΔ. καὶ ἐπεὶ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓΔ κύκλου, ἴση ἐστὶν ἡ ΒΗ τῇ ΗΔ· μείζων ἄρα ἡ ΒΗ τῆς ΘΔ· πολλῷ ἄρα μείζων ἡ ΒΘ τῆς ΘΔ. πάλιν, ἐπεὶ τὸ Θ σημεῖον κέντρον ἐστὶ τοῦ ΕΒΖΔ κύκλου, ἴση ἐστὶν ἡ ΒΘ τῇ ΘΔ· ἐδείχθη δὲ αὐτῆς καὶ πολλῷ μείζων· ὅπερ ἀδύνατον· οὐκ ἄρα κύκλος κύκλου ἐφάπτεται ἐντὸς κατὰ πλείονα σημεῖα ἢ ἕν.

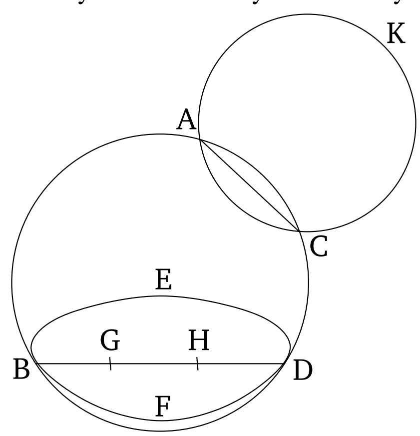
Λέγω δή, ὅτι οὐδὲ ἐκτός.

Εἰ γὰρ δυνατόν, κύκλος ὁ ΑΓΚ κύκλου τοῦ ΑΒΓΔ ἐφαπτέσθω ἐκτὸς κατὰ πλείονα σημεῖα ἢ ἓν τὰ Α, Γ, καὶ ἐπεζεύχθω ἡ ΑΓ.

Ἐπεὶ οὖν κύκλων τῶν ΑΒΓΔ, ΑΓΚ εἴληπται ἐπὶ τῆς περιφερείας ἑκατέρου δύο τυχόντα σημεῖα τὰ Α, Γ, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐντὸς ἑκατέρου πεσεῖται· ἀλλὰ

Proposition 13

A circle does not touch a(nother) circle at more than one point, whether they touch internally or externally.



For, if possible, let circle $ABDC$† touch circle $EBFD$—first of all, internally—at more than one point, D and B.

And let the center G of circle $ABDC$ be found [Prop. 3.1], and (the center) H of $EBFD$ [Prop. 3.1].

Thus, the (straight-line) joining G and H will fall on B and D [Prop. 3.11]. Let it fall like $BGHD$ (in the figure). And since point G is the center of circle $ABDC$, BG is equal to GD. Thus, BG (is) greater than HD. Thus, BH (is) much greater than HD. Again, since point H is the center of circle $EBFD$, BH is equal to HD. But it was also shown (to be) much greater than it. The very thing (is) impossible. Thus, a circle does not touch a(nother) circle internally at more than one point.

So, I say that neither (does it touch) externally (at more than one point).

For, if possible, let circle ACK touch circle $ABDC$ externally at more than one point, A and C. And let AC be joined.

Therefore, since two points, A and C, be taken at random on the circumference of each of the circles $ABDC$ and ACK, the straight-line joining the points will fall inside each (cir-

τοῦ μὲν ΑΒΓΔ ἐντὸς ἔπεσεν, τοῦ δὲ ΑΓΚ ἐκτός· ὅπερ ἄτοπον· οὐκ ἄρα κύκλος κύκλου ἐφάπτεται ἐκτὸς κατὰ πλείονα σημεῖα ἢ ἕν. ἐδείχθη δέ, ὅτι οὐδὲ ἐντός.

Κύκλος ἄρα κύκλου οὐκ ἐφάπτεται κατὰ πλείονα σημεῖα ἢ [καθ᾽] ἕν, ἐάν τε ἐντὸς ἐάν τε ἐκτὸς ἐφάπτηται· ὅπερ ἔδει δεῖξαι.

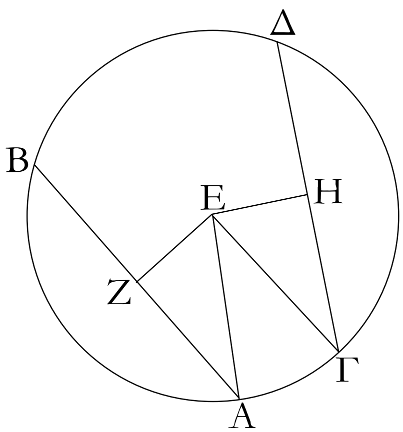
cle) [Prop. 3.2]. But, it fell inside *ABDC*, and outside *ACK* [Def. 3.3]. The very thing (is) absurd. Thus, a circle does not touch a(nother) circle externally at more than one point. And it was shown that neither (does it) internally.

Thus, a circle does not touch a(nother) circle at more than one point, whether they touch internally or externally. (Which is) the very thing it was required to show.

† The Greek text has "*ABCD*", which is obviously a mistake.

ιδʹ.

Ἐν κύκλῳ αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ αἱ ἴσον ἀπέχουσαι ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσίν.



Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ ἴσαι εὐθεῖαι ἔστωσαν αἱ ΑΒ, ΓΔ· λέγω, ὅτι αἱ ΑΒ, ΓΔ ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου.

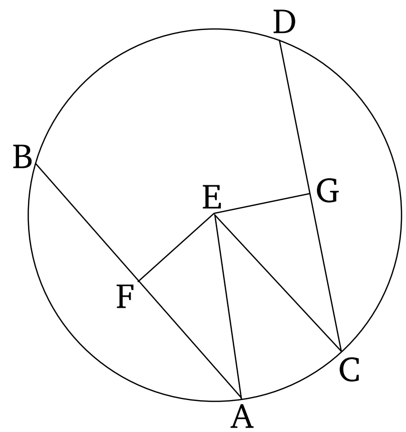
Εἰλήφθω γὰρ τὸ κέντον τοῦ ΑΒΓΔ κύκλου καὶ ἔστω τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὰς ΑΒ, ΓΔ κάθετοι ἤχθωσαν αἱ ΕΖ, ΕΗ, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΓ.

Ἐπεὶ οὖν εὐθεῖά τις δια τοῦ κέντρου ἡ ΕΖ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΒ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει. ἴση ἄρα ἡ ΑΖ τῇ ΖΒ· διπλῆ ἄρα ἡ ΑΒ τῆς ΑΖ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΔ τῆς ΓΗ ἐστι διπλῆ· καί ἐστιν ἴση ἡ ΑΒ τῇ ΓΔ· ἴση ἄρα καὶ ἡ ΑΖ τῇ ΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΓ, ἴσον καὶ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΕΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΕ ἴσα τὰ ἀπὸ τῶν ΑΖ, ΕΖ· ὀρθὴ γὰρ ἡ πρὸς τῷ Ζ γωνία· τῷ δὲ ἀπὸ τῆς ΕΓ ἴσα τὰ ἀπὸ τῶν ΕΗ, ΗΓ· ὀρθὴ γὰρ ἡ πρὸς τῷ Η γωνία· τὰ ἄρα ἀπὸ τῶν ΑΖ, ΖΕ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΓΗ, ΗΕ, ὧν τὸ ἀπὸ τῆς ΑΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΗ· ἴση γάρ ἐστιν ἡ ΑΖ τῇ ΓΗ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΖΕ τῷ ἀπὸ τῆς ΕΗ ἴσον ἐστίν· ἴση ἄρα ἡ ΕΖ τῇ ΕΗ. ἐν δὲ κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ᾽ αὐτὰς κάθετοι ἀγόμεναι ἴσαι ὦσιν· αἱ ἄρα ΑΒ, ΓΔ ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου.

Ἀλλὰ δὴ αἱ ΑΒ, ΓΔ εὐθεῖαι ἴσον ἀπεχέτωσαν ἀπὸ τοῦ κέντρου, τουτέστιν ἴση ἔστω ἡ ΕΖ τῇ ΕΗ. λέγω, ὅτι ἴση ἐστὶ

Proposition 14

In a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another.



Let *ABDC*† be a circle, and let *AB* and *CD* be equal straight-lines within it. I say that *AB* and *CD* are equally far from the center.

For let the center of circle *ABDC* be found [Prop. 3.1], and let it be (at) *E*. And let *EF* and *EG* be drawn from (point) *E*, perpendicular to *AB* and *CD* (respectively) [Prop. 1.12]. And let *AE* and *EC* be joined.

Therefore, since some straight-line, *EF*, through the center (of the circle), cuts some (other) straight-line, *AB*, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, *AF* (is) equal to *FB*. Thus, *AB* (is) double *AF*. So, for the same (reasons), *CD* is also double *CG*. And *AB* is equal to *CD*. Thus, *AF* (is) also equal to *CG*. And since *AE* is equal to *EC*, the (square) on *AE* (is) also equal to the (square) on *EC*. But, the (sum of the squares) on *AF* and *EF* (is) equal to the (square) on *AE*. For the angle at *F* (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on *EG* and *GC* (is) equal to the (square) on *EC*. For the angle at *G* (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on *AF* and *FE* is equal to the (sum of the squares) on *CG* and *GE*, of which the (square) on *AF* is equal to the (square) on *CG*. For *AF* is equal to *CG*. Thus, the remaining (square) on *FE* is equal to the (remaining square) on *EG*. Thus, *EF* (is) equal *καὶ ἡ ΑΒ τῇ ΓΔ.*

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι διπλῆ ἐστιν ἡ μὲν ΑΒ τῆς ΑΖ, ἡ δὲ ΓΔ τῆς ΓΗ· καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΓΕ, ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΕ· ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΕ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΕΖ, ΖΑ, τῷ δὲ ἀπὸ τῆς ΓΕ ἴσα τὰ ἀπὸ τῶν ΕΗ, ΗΓ. τὰ ἄρα ἀπὸ τῶν ΕΖ, ΖΑ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΕΗ, ΗΓ· ὧν τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΕΗ ἐστιν ἴσον· ἴση γὰρ ἡ ΕΖ τῇ ΕΗ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΑΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΗ· ἴση ἄρα ἡ ΑΖ τῇ ΓΗ· καὶ ἐστι τῆς μὲν ΑΖ διπλῆ ἡ ΑΒ, τῆς δὲ ΓΗ διπλῆ ἡ ΓΔ· ἴση ἄρα ἡ ΑΒ τῇ ΓΔ.

Ἐν κύκλῳ ἄρα αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ αἱ ἴσον ἀπέχουσαι ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσίν· ὅπερ ἔδει δεῖξαι.

to EG. And straight-lines in a circle are said to be equally far from the center when perpendicular (straight-lines) which are drawn to them from the center are equal [Def. 3.4]. Thus, AB and CD are equally far from the center.

So, let the straight-lines AB and CD be equally far from the center. That is to say, let EF be equal to EG. I say that AB is also equal to CD.

For, with the same construction, we can, similarly, show that AB is double AF, and CD (double) CG. And since AE is equal to CE, the (square) on AE is equal to the (square) on CE. But, the (sum of the squares) on EF and FA is equal to the (square) on AE [Prop. 1.47]. And the (sum of the squares) on EG and GC (is) equal to the (square) on CE [Prop. 1.47]. Thus, the (sum of the squares) on EF and FA is equal to the (sum of the squares) on EG and GC, of which the (square) on EF is equal to the (square) on EG. For EF (is) equal to EG. Thus, the remaining (square) on AF is equal to the (remaining square) on CG. Thus, AF (is) equal to CG. And AB is double AF, and CD double CG. Thus, AB (is) equal to CD.

Thus, in a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another. (Which is) the very thing it was required to show.

† The Greek text has "*ABCD*", which is obviously a mistake.

ιεʹ.

Ἐν κύκλῳ μεγίστη μὲν ἡ διάμετρος, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν.

Ἔστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΑΔ, κέντρον δὲ τὸ Ε, καὶ ἔγγιον μὲν τῆς ΑΔ διαμέτρου ἔστω ἡ ΒΓ, ἀπώτερον δὲ ἡ ΖΗ· λέγω, ὅτι μεγίστη μέν ἐστιν ἡ ΑΔ, μείζων δὲ ἡ ΒΓ τῆς ΖΗ.

Ἤχθωσαν γὰρ ἀπὸ τοῦ Ε κέντρου ἐπὶ τὰς ΒΓ, ΖΗ κάθετοι αἱ ΕΘ, ΕΚ. καὶ ἐπεὶ ἔγγιον μὲν τοῦ κέντρου ἐστὶν ἡ ΒΓ, ἀπώτερον δὲ ἡ ΖΗ, μείζων ἄρα ἡ ΕΚ τῆς ΕΘ. κείσθω τῇ ΕΘ ἴση ἡ ΕΛ, καὶ διὰ τοῦ Λ τῇ ΕΚ πρὸς ὀρθὰς ἀχθεῖσα ἡ ΛΜ διήχθω ἐπὶ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΜΕ, ΕΝ, ΖΕ, ΕΗ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΘ τῇ ΕΛ, ἴση ἐστὶ καὶ ἡ ΒΓ τῇ ΜΝ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΕ τῇ ΕΜ, ἡ δὲ ΕΔ τῇ ΕΝ, ἡ ἄρα ΑΔ ταῖς ΜΕ, ΕΝ ἴση ἐστίν. ἀλλ' αἱ μὲν ΜΕ, ΕΝ τῆς ΜΝ μείζονές εἰσιν [καὶ ἡ ΑΔ τῆς ΜΝ μείζων ἐστίν], ἴση δὲ ἡ ΜΝ τῇ ΒΓ· ἡ ΑΔ ἄρα τῆς ΒΓ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ΜΕ, ΕΝ δύο ταῖς ΖΕ, ΕΗ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΜΕΝ γωνίας τῆς ὑπὸ ΖΕΗ μείζων [ἐστίν], βάσις ἄρα ἡ ΜΝ βάσεως τῆς ΖΗ μείζων ἐστίν. ἀλλὰ ἡ ΜΝ τῇ ΒΓ ἐδείχθη ἴση [καὶ ἡ ΒΓ τῆς ΖΗ μείζων ἐστίν]. μεγίστη μὲν ἄρα ἡ ΑΔ διάμετρος, μείζων δὲ ἡ ΒΓ τῆς ΖΗ.

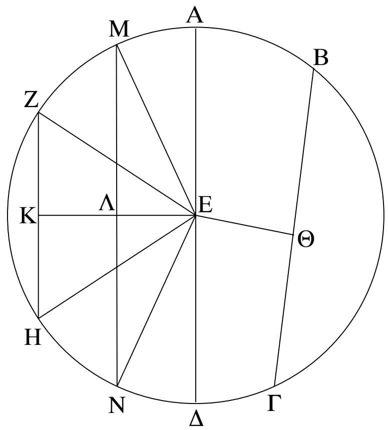
Proposition 15

In a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away.

Let $ABCD$ be a circle, and let AD be its diameter, and E (its) center. And let BC be nearer to the diameter AD,[†] and FG further away. I say that AD is the greatest (straight-line), and BC (is) greater than FG.

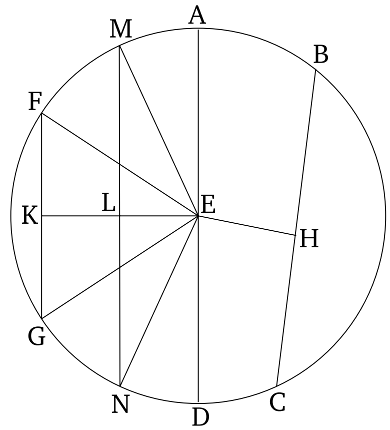
For let EH and EK be drawn from the center E, at right-angles to BC and FG (respectively) [Prop. 1.12]. And since BC is nearer to the center, and FG further away, EK (is) thus greater than EH [Def. 3.5]. Let EL be made equal to EH [Prop. 1.3]. And LM being drawn through L, at right-angles to EK [Prop. 1.11], let it be drawn through to N. And let ME, EN, FE, and EG be joined.

And since EH is equal to EL, BC is also equal to MN [Prop. 3.14]. Again, since AE is equal to EM, and ED to EN, AD is thus equal to ME and EN. But, ME and EN is greater than MN [Prop. 1.20] [also AD is greater than MN], and MN (is) equal to BC. Thus, AD is greater than BC. And since the two (straight-lines) ME, EN are equal to the two (straight-lines) FE, EG (respectively), and angle MEN [is] greater than angle FEG,[‡] the base MN is thus greater than the base FG [Prop. 1.24]. But, MN was shown (to be) equal to BC [(so)



Ἐν κύκλῳ ἄρα μεγίστη μὲν ἔστιν ἡ διάμετρος, τῶν δὲ ἄλλων ἀεὶ ἡ ἔγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

BC is also greater than *FG*]. Thus, the diameter *AD* (is) the greatest (straight-line), and *BC* (is) greater than *FG*.



Thus, in a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away. (Which is) the very thing it was required to show.

† Euclid should have said “to the center”, rather than ”to the diameter *AD*”, since *BC*, *AD* and *FG* are not necessarily parallel.

‡ This is not proved, except by reference to the figure.

ιϛ′.

Ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου, καὶ εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται, καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἁπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἐλάττων.

Ἔστω κύκλος ὁ ΑΒΓ περὶ κέντρον τὸ Δ καὶ διάμετρον τὴν ΑΒ· λέγω, ὅτι ἡ ἀπὸ τοῦ Α τῇ ΑΒ πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου.

μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, πιπτέτω ἐντὸς ὡς ἡ ΓΑ, καὶ ἐπεζεύχθω ἡ ΔΓ.

Ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΔΓ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΔΑΓ γωνίᾳ τῇ ὑπὸ ΑΓΔ. ὀρθὴ δὲ ἡ ὑπὸ ΔΑΓ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΑΓΔ· τριγώνου δὴ τοῦ ΑΓΔ αἱ δύο γωνίαι αἱ ὑπὸ ΔΑΓ, ΑΓΔ δύο ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Α σημείου τῇ ΒΑ πρὸς ὀρθὰς ἀγομένη ἐντὸς πεσεῖται τοῦ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾽ ἐπὶ τῆς περιφερείας· ἐκτὸς ἄρα.

Πιπτέτω ὡς ἡ ΑΕ· λέγω δή, ὅτι εἰς τὸν μεταξὺ τόπον τῆς τε ΑΕ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται.

Εἰ γὰρ δυνατόν, παρεμπιπτέτω ὡς ἡ ΖΑ, καὶ ἤχθω ἀπὸ τοῦ Δ σημείου ἐπὶ τὴν ΖΑ κάθετος ἡ ΔΗ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΑΗΔ, ἐλάττων δὲ ὀρθῆς ἡ ὑπὸ ΔΑΗ, μείζων ἄρα ἡ ΑΔ τῆς ΔΗ. ἴση δὲ ἡ ΔΑ τῇ ΔΘ· μείζων ἄρα ἡ ΔΘ τῆς ΔΗ, ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας

Proposition 16

A (straight-line) drawn at right-angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the (aforementioned) straight-line and the circumference. And the angle of the semi-circle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle).

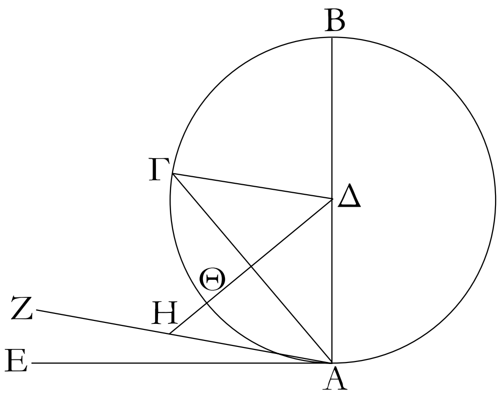
Let *ABC* be a circle around the center *D* and the diameter *AB*. I say that the (straight-line) drawn from *A*, at right-angles to *AB* [Prop 1.11], from its end, will fall outside the circle.

For (if) not (then), if possible, let it fall inside, like *CA* (in the figure), and let *DC* be joined.

Since *DA* is equal to *DC*, angle *DAC* is also equal to angle *ACD* [Prop. 1.5]. And *DAC* (is) a right-angle. Thus, *ACD* (is) also a right-angle. So, in triangle *ACD*, the two angles *DAC* and *ACD* are equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, the (straight-line) drawn from point *A*, at right-angles to *BA*, will not fall inside the circle. So, similarly, we can show that neither (will it fall) on the circumference. Thus, (it will fall) outside (the circle).

Let it fall like *AE* (in the figure). So, I say that another straight-line cannot be inserted into the space between the straight-line *AE* and the circumference *CHA*.

For, if possible, let it be inserted like *FA* (in the figure), and let *DG* be drawn from point *D*, perpendicular to *FA* [Prop. 1.12]. And since *AGD* is a right-angle, and *DAG* *ἑτέρα εὐθεῖα παρεμπεσεῖται.*



Λέγω, ὅτι καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἡ περιεχομένη ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἁπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἡ περιεχομένη ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας ἁπάσης γωνίας ὀξείας εὐθυγράμμου ἐλάττων ἐστίν.

Εἰ γὰρ ἐστί τις γωνία εὐθύγραμμος μείζων μὲν τῆς περιεχομένης ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας, ἐλάττων δὲ τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τὴς ΑΕ εὐθείας, εἰς τὸν μεταξὺ τόπον τῆς τε ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας εὐθεῖα παρεμπεσεῖται, ἥτις ποιήσει μείζονα μὲν τῆς περιεχομένης ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ὑπὸ εὐθειῶν περιεχομένην, ἐλάττονα δὲ τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας. οὐ παρεμπίπτει δέ· οὐκ ἄρα τῆς περιεχομένης γωνίας ὑπό τε τῆς ΒΑ εὐθείας καὶ τῆς ΓΘΑ περιφερείας ἔσται μείζων ὀξεῖα ὑπὸ εὐθειῶν περιεχομένη, οὐδὲ μὴν ἐλάττων τῆς περιεχομένης ὑπό τε τῆς ΓΘΑ περιφερείας καὶ τῆς ΑΕ εὐθείας.

Πόρισμα.

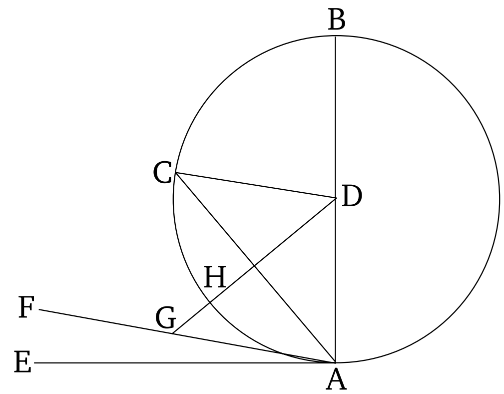
Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾿ ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου [καὶ ὅτι εὐθεῖα κύκλου καθ᾿ ἓν μόνον ἐφάπτεται σημεῖον, ἐπειδήπερ καὶ ἡ κατὰ δύο αὐτῷ συμβάλλουσα ἐντὸς αὐτοῦ πίπτουσα ἐδείχθη]· ὅπερ ἔδει δεῖξαι.

ιζ′.

Ἀπὸ τοῦ δοθέντος σημείου τοῦ δοθέντος κύκλου ἐφαπτομένην εὐθεῖαν γραμμὴν ἀγαγεῖν.

Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ὁ δὲ δοθεὶς κύκλος ὁ ΒΓΔ· δεῖ δὴ ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφα-

(is) less than a right-angle, *AD* (is) thus greater than *DG* [Prop. 1.19]. And *DA* (is) equal to *DH*. Thus, *DH* (is) greater than *DG*, the lesser than the greater. The very thing is impossible. Thus, another straight-line cannot be inserted into the space between the straight-line (*AE*) and the circumference.



And I also say that the semi-circular angle contained by the straight-line *BA* and the circumference *CHA* is greater than any acute rectilinear angle whatsoever, and the remaining (angle) contained by the circumference *CHA* and the straight-line *AE* is less than any acute rectilinear angle whatsoever.

For if any rectilinear angle is greater than the (angle) contained by the straight-line *BA* and the circumference *CHA*, or less than the (angle) contained by the circumference *CHA* and the straight-line *AE*, (then) a straight-line can be inserted into the space between the circumference *CHA* and the straight-line *AE*—anything which will make (an angle) contained by straight-lines greater than the angle contained by the straight-line *BA* and the circumference *CHA*, or less than the (angle) contained by the circumference *CHA* and the straight-line *AE*. But (such a straight-line) cannot be inserted. Thus, an acute (angle) contained by straight-lines cannot be greater than the angle contained by the straight-line *BA* and the circumference *CHA*, neither (can it be) less than the (angle) contained by the circumference *CHA* and the straight-line *AE*.

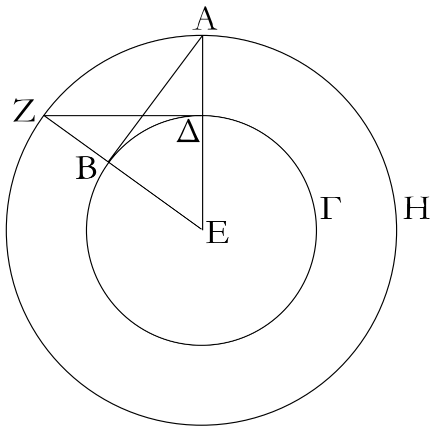
Corollary

So, from this, (it is) manifest that a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [and that the straight-line touches the circle at a single point, inasmuch as it was also shown that a (straight-line) meeting (the circle) at two (points) falls inside it [Prop. 3.2]]. (Which is) the very thing it was required to show.

Proposition 17

To draw a straight-line touching a given circle from a given point.

Let *A* be the given point, and *BCD* the given circle. So it is required to draw a straight-line touching circle *BCD* from *πτομένην εὐθεῖαν γραμμὴν ἀγαγεῖν.*

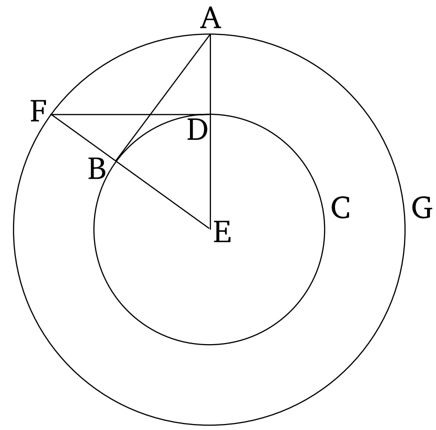


Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ε, καὶ ἐπεζεύχθω ἡ ΑΕ, καὶ κέντρῳ μὲν τῷ Ε διαστήματι δὲ τῷ ΕΑ κύκλος γεγράφθω ὁ ΑΖΗ, καὶ ἀπὸ τοῦ Δ τῇ ΕΑ πρὸς ὀρθὰς ἤχθω ἡ ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΑΒ· λέγω, ὅτι ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφαπτομένη ἦκται ἡ ΑΒ.

Ἐπεὶ γὰρ τὸ Ε κέντρον ἐστὶ τῶν ΒΓΔ, ΑΖΗ κύκλων, ἴση ἄρα ἐστὶν ἡ μὲν ΕΑ τῇ ΕΖ, ἡ δὲ ΕΔ τῇ ΕΒ· δύο δὴ αἱ ΑΕ, ΕΒ δύο ταῖς ΖΕ, ΕΔ ἴσαι εἰσίν· καὶ γωνίαν κοινὴν περιέχουσι τὴν πρὸς τῷ Ε· βάσις ἄρα ἡ ΔΖ βάσει τῇ ΑΒ ἴση ἐστίν, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΕΒΑ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΕΔΖ τῇ ὑπὸ ΕΒΑ. ὀρθὴ δὲ ἡ ὑπὸ ΕΔΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΒΑ. καί ἐστιν ἡ ΕΒ ἐκ τοῦ κέντρου· ἡ δὲ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου· ἡ ΑΒ ἄρα ἐφάπτεται τοῦ ΒΓΔ κύκλου.

Ἀπὸ τοῦ ἄρα δοθέντος σημείου τοῦ Α τοῦ δοθέντος κύκλου τοῦ ΒΓΔ ἐφαπτομένη εὐθεῖα γραμμὴ ἦκται ἡ ΑΒ· ὅπερ ἔδει ποιῆσαι.

point *A*.



For let the center *E* of the circle be found [Prop. 3.1], and let *AE* be joined. And let (the circle) *AFG* be drawn with center *E* and radius *EA*. And let *DF* be drawn from from (point) *D*, at right-angles to *EA* [Prop. 1.11]. And let *EF* and *AB* be joined. I say that the (straight-line) *AB* has been drawn from point *A* touching circle *BCD*.

For since *E* is the center of circles *BCD* and *AFG*, *EA* is thus equal to *EF*, and *ED* to *EB*. So the two (straight-lines) *AE*, *EB* are equal to the two (straight-lines) *FE*, *ED* (respectively). And they contain a common angle at *E*. Thus, the base *DF* is equal to the base *AB*, and triangle *DEF* is equal to triangle *EBA*, and the remaining angles (are equal) to the (corresponding) remaining angles [Prop. 1.4]. Thus, (angle) *EDF* (is) equal to *EBA*. And *EDF* (is) a right-angle. Thus, *EBA* (is) also a right-angle. And *EB* is a radius. And a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [Prop. 3.16 corr.]. Thus, *AB* touches circle *BCD*.

Thus, the straight-line *AB* has been drawn touching the given circle *BCD* from the given point *A*. (Which is) the very thing it was required to do.

ιη′.

Ἐὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπιζευχθῇ τις εὐθεῖα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην.

Κύκλου γὰρ τοῦ ΑΒΓ ἐφαπτέσθω τις εὐθεῖα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὸ Γ ἐπεζεύχθω ἡ ΖΓ· λέγω, ὅτι ἡ ΖΓ κάθετός ἐστιν ἐπὶ τὴν ΔΕ.

Εἰ γὰρ μή, ἤχθω ἀπὸ τοῦ Ζ ἐπὶ τὴν ΔΕ κάθετος ἡ ΖΗ.

Ἐπεὶ οὖν ἡ ὑπὸ ΖΗΓ γωνία ὀρθή ἐστιν, ὀξεῖα ἄρα ἐστὶν ἡ ὑπὸ ΖΓΗ· ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ΖΓ τῆς ΖΗ· ἴση δὲ ἡ ΖΓ τῇ ΖΒ· μείζων ἄρα καὶ ἡ ΖΒ τῆς ΖΗ ἡ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ΖΗ κάθετός ἐστιν ἐπὶ τὴν ΔΕ. ὁμοίως

Proposition 18

If some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, (then) the (straight-line) so joined will be perpendicular to the tangent.

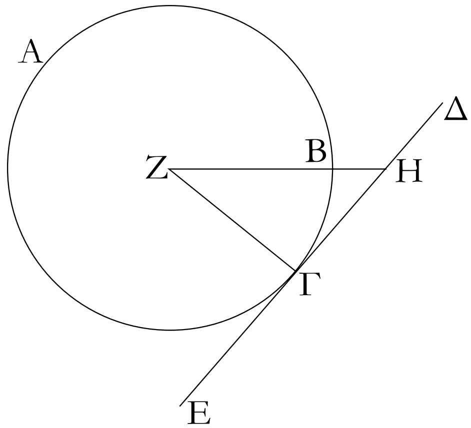
For let some straight-line *DE* touch the circle *ABC* at point *C*, and let the center *F* of circle *ABC* be found [Prop. 3.1], and let *FC* be joined from *F* to *C*. I say that *FC* is perpendicular to *DE*.

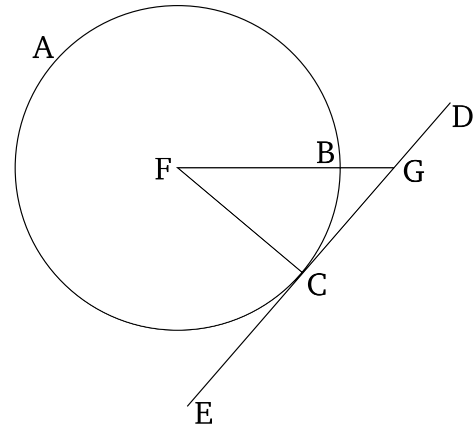
For if not, let *FG* be drawn from *F*, perpendicular to *DE* [Prop. 1.12].

Therefore, since angle *FGC* is a right-angle, (angle) *FCG* is thus acute [Prop. 1.17]. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, *FC* (is) greater than *FG*.

δὴ δεῖξομεν, ὅτι οὐδ᾽ ἄλλη τις πλὴν τῆς ΖΓ· ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΔΕ.

And FC (is) equal to FB. Thus, FB (is) also greater than FG, the lesser than the greater. The very thing is impossible. Thus, FG is not perpendicular to DE. So, similarly, we can show that neither (is) any other (straight-line) except FC. Thus, FC is perpendicular to DE.





Ἐὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπιζευχθῇ τις εὐθεῖα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην· ὅπερ ἔδει δεῖξαι.

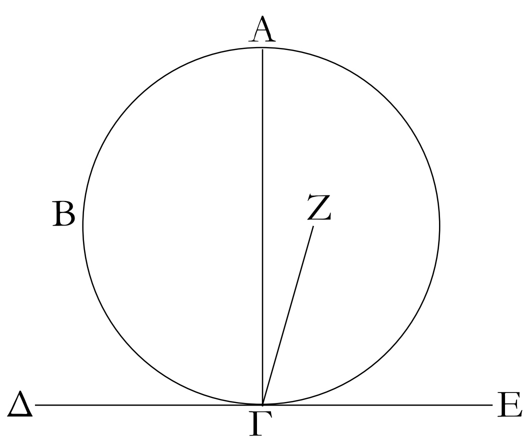
Thus, if some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, (then) the (straight-line) so joined will be perpendicular to the tangent. (Which is) the very thing it was required to show.

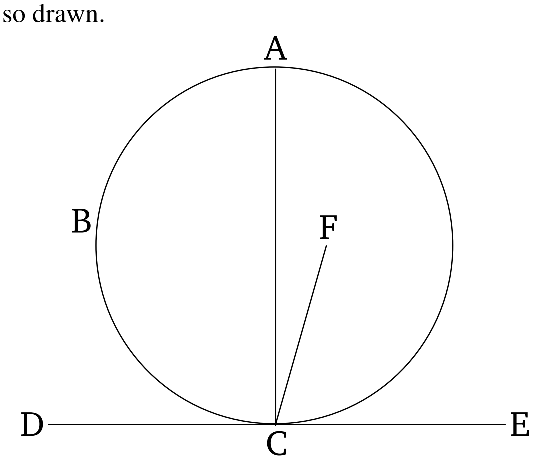
ιθ΄.

Proposition 19

Ἐὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς τῇ ἐφαπτομένῃ πρὸς ὀρθὰς [γωνίας] εὐθεῖα γραμμὴ ἀχθῇ, ἐπὶ τῆς ἀχθείσης ἔσται τὸ κέντρον τοῦ κύκλου.

If some straight-line touches a circle, and a straight-line is drawn from the point of contact, at right-[angles] to the tangent, (then) the center (of the circle) will be on the (straight-line) so drawn.





Κύκλου γὰρ τοῦ ΑΒΓ ἐφαπτέσθω τις εὐθεῖα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ ἀπὸ τοῦ Γ τῇ ΔΕ πρὸς ὀρθὰς ἤχθω ἡ ΓΑ· λέγω, ὅτι ἐπὶ τῆς ΑΓ ἐστι τὸ κέντρον τοῦ κύκλου.

μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΓΖ.

Ἐπεὶ [οὖν] κύκλου τοῦ ΑΒΓ ἐφάπτεταί τις εὐθεῖα ἡ ΔΕ, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἁφὴν ἐπέζευκται ἡ ΖΓ, ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΔΕ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΓΕ. ἐστὶ

For let some straight-line DE touch the circle ABC at point C. And let CA be drawn from C, at right-angles to DE [Prop. 1.11]. I say that the center of the circle is on AC.

For (if) not, if possible, let F be (the center of the circle), and let CF be joined.

[Therefore], since some straight-line DE touches the circle ABC, and FC has been joined from the center to the point of *δὲ καὶ ἡ ὑπὸ ΑΓΕ ὀρθή· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΖΓΕ τῇ ὑπὸ ΑΓΕ ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ζ κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾿ ἄλλο τι πλὴν ἐπὶ τῆς ΑΓ.*

Ἐὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς τῇ ἐφαπτομένῃ πρὸς ὀρθὰς εὐθεῖα γραμμὴ ἀχθῇ, ἐπὶ τῆς ἀχθείσης ἔσται τὸ κέντρον τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

contact, FC is thus perpendicular to DE [Prop. 3.18]. Thus, FCE is a right-angle. And ACE is also a right-angle. Thus, FCE is equal to ACE, the lesser to the greater. The very thing is impossible. Thus, F is not the center of circle ABC. So, similarly, we can show that neither is any (point) other (than one) on AC.

Thus, if some straight-line touches a circle, and a straight-line is drawn from the point of contact, at right-angles to the tangent, (then) the center (of the circle) will be on the (straight-line) so drawn. (Which is) the very thing it was required to show.

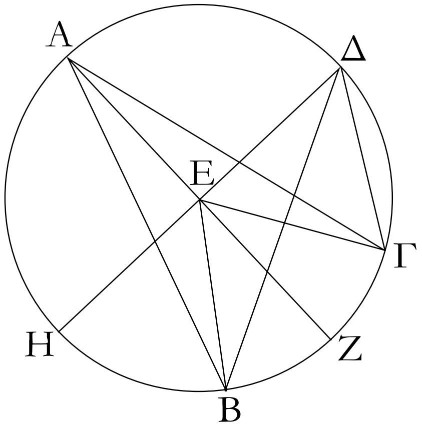
κ′.

Ἐν κύκλῳ ἡ πρὸς τῷ κέντρῳ γωνία διπλασίων ἐστὶ τῆς πρὸς τῇ περιφερείᾳ, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν αἱ γωνίαι.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ πρὸς μὲν τῷ κέντρῳ αὐτοῦ γωνία ἔστω ἡ ὑπὸ ΒΕΓ, πρὸς δὲ τῇ περιφερείᾳ ἡ ὑπὸ ΒΑΓ, ἐχέτωσαν δὲ τὴν αὐτὴν περιφέρειαν βάσιν τὴν ΒΓ· λέγω, ὅτι διπλασίων ἐστὶν ἡ ὑπὸ ΒΕΓ γωνία τῆς ὑπὸ ΒΑΓ.

Ἐπιζευχθεῖσα γὰρ ἡ ΑΕ διήχθω ἐπὶ τὸ Ζ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΕΑ τῇ ΕΒ, ἴση καὶ γωνία ἡ ὑπὸ ΕΑΒ τῇ ὑπὸ ΕΒΑ· αἱ ἄρα ὑπὸ ΕΑΒ, ΕΒΑ γωνίαι τῆς ὑπὸ ΕΑΒ διπλασίους εἰσίν. ἴση δὲ ἡ ὑπὸ ΒΕΖ ταῖς ὑπὸ ΕΑΒ, ΕΒΑ· καὶ ἡ ὑπὸ ΒΕΖ ἄρα τῆς ὑπὸ ΕΑΒ ἐστι διπλῆ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΖΕΓ τῆς ὑπὸ ΕΑΓ ἐστι διπλῆ. ὅλη ἄρα ἡ ὑπὸ ΒΕΓ ὅλης τῆς ὑπὸ ΒΑΓ ἐστι διπλῆ.



Κεκλάσθω δὴ πάλιν, καὶ ἔστω ἑτέρα γωνία ἡ ὑπὸ ΒΔΓ, καὶ ἐπιζευχθεῖσα ἡ ΔΕ ἐκβεβλήσθω ἐπὶ τὸ Η. ὁμοίως δὴ δείξομεν, ὅτι διπλῆ ἐστιν ἡ ὑπὸ ΗΕΓ γωνία τῆς ὑπὸ ΕΔΓ, ὧν ἡ ὑπὸ ΗΕΒ διπλῆ ἐστι τῆς ὑπὸ ΕΔΒ· λοιπὴ ἄρα ἡ ὑπὸ ΒΕΓ διπλῆ ἐστι τῆς ὑπὸ ΒΔΓ.

Ἐν κύκλῳ ἄρα ἡ πρὸς τῷ κέντρῳ γωνία διπλασίων ἐστὶ τῆς πρὸς τῇ περιφερείᾳ, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν [αἱ γωνίαι]· ὅπερ ἔδει δεῖξαι.

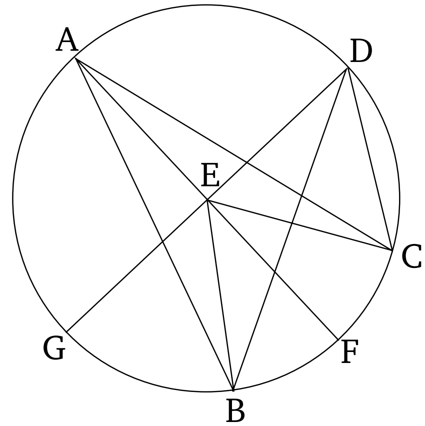
Proposition 20

In a circle, the angle at the center is double that at the circumference, when the angles have the same circumference base.

Let ABC be a circle, and let BEC be an angle at its center, and BAC (one) at (its) circumference. And let them have the same circumference base BC. I say that angle BEC is double (angle) BAC.

For being joined, let AE be drawn through to F.

Therefore, since EA is equal to EB, angle EAB (is) also equal to EBA [Prop. 1.5]. Thus, angle EAB and EBA is double (angle) EAB. And BEF (is) equal to EAB and EBA [Prop. 1.32]. Thus, BEF is also double EAB. So, for the same (reasons), FEC is also double EAC. Thus, the whole (angle) BEC is double the whole (angle) BAC.

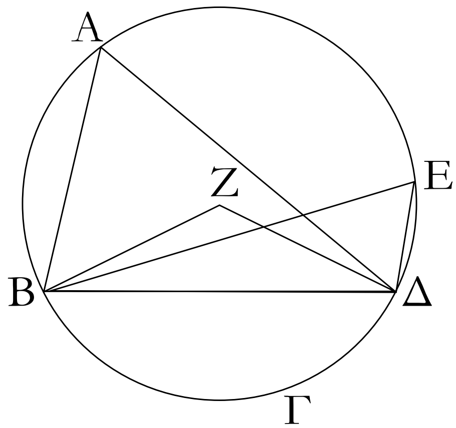


So let another (straight-line) be inflected, and let there be another angle, BDC. And DE being joined, let it be produced to G. So, similarly, we can show that angle GEC is double EDC, of which GEB is double EDB. Thus, the remaining (angle) BEC is double the (remaining angle) BDC.

Thus, in a circle, the angle at the center is double that at the circumference, when [the angles] have the same circumference base. (Which is) the very thing it was required to show.

καʹ.

Ἐν κύκλῳ αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσίν.



Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν τῷ αὐτῷ τμήματι τῷ ΒΑΕΔ γωνίαι ἔστωσαν αἱ ὑπὸ ΒΑΔ, ΒΕΔ· λέγω, ὅτι αἱ ὑπὸ ΒΑΔ, ΒΕΔ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

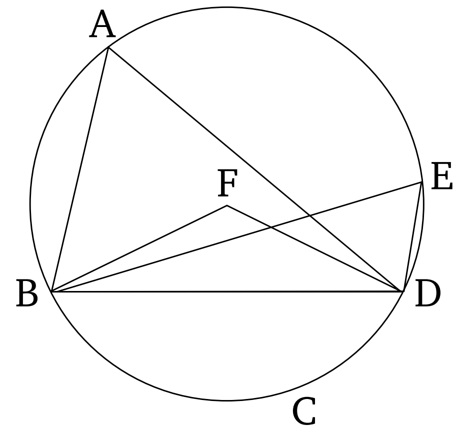
Εἰλήφθω γὰρ τοῦ ΑΒΓΔ κύκλου τὸ κέντρον, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΒΖ, ΖΔ.

Καὶ ἐπεὶ ἡ μὲν ὑπὸ ΒΖΔ γωνία πρὸς τῷ κέντρῳ ἐστίν, ἡ δὲ ὑπὸ ΒΑΔ πρὸς τῇ περιφερείᾳ, καὶ ἔχουσι τὴν αὐτὴν περιφέρειαν βάσιν τὴν ΒΓΔ, ἡ ἄρα ὑπὸ ΒΖΔ γωνία διπλασίων ἐστὶ τῆς ὑπὸ ΒΑΔ. διὰ τὰ αὐτὰ δὴ ἡ ὑπὸ ΒΖΔ καὶ τῆς ὑπὸ ΒΕΔ ἐστι διπλσίων· ἴση ἄρα ἡ ὑπὸ ΒΑΔ τῇ ὑπὸ ΒΕΔ.

Ἐν κύκλῳ ἄρα αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 21

In a circle, angles in the same segment are equal to one another.



Let $ABCD$ be a circle, and let BAD and BED be angles in the same segment $BAED$. I say that angles BAD and BED are equal to one another.

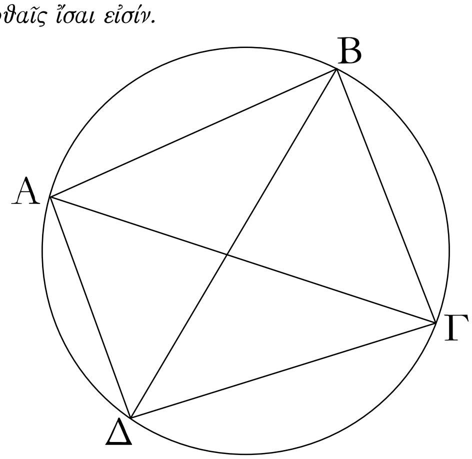
For let the center of circle $ABCD$ be found [Prop. 3.1], and let it be (at point) F. And let BF and FD be joined.

And since angle BFD is at the center, and BAD at the circumference, and they have the same circumference base BCD, angle BFD is thus double BAD [Prop. 3.20]. So, for the same (reasons), BFD is also double BED. Thus, BAD (is) equal to BED.

Thus, in a circle, angles in the same segment are equal to one another. (Which is) the very thing it was required to show.

κβʹ.

Τῶν ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

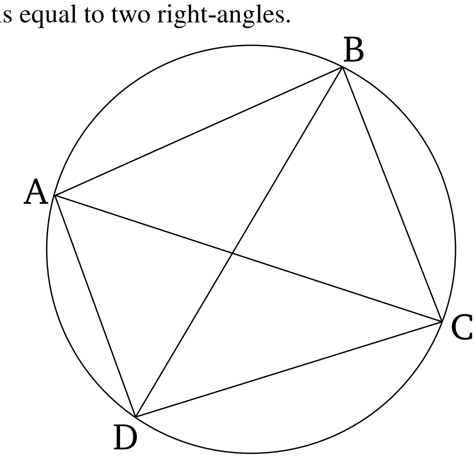


Ἔστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ τετράπλευρον ἔστω τὸ ΑΒΓΔ· λέγω, ὅτι αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἐπεζεύχθωσαν αἱ ΑΓ, ΒΔ.

Proposition 22

For quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles.



Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral within it. I say that the (sum of the) opposite angles is equal to two right-angles.

Let AC and BD be joined.

Ἐπεὶ οὖν παντὸς τριγώνου αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν, τοῦ ΑΒΓ ἄρα τριγώνου αἱ τρεῖς γωνίαι αἱ ὑπὸ ΓΑΒ, ΑΒΓ, ΒΓΑ δυσὶν ὀρθαῖς ἴσαι εἰσίν. ἴση δὲ ἡ μὲν ὑπὸ ΓΑΒ τῇ ὑπὸ ΒΔΓ· ἐν γὰρ τῷ αὐτῷ τμήματί εἰσι τῷ ΒΑΔΓ· ἡ δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΑΔΒ· ἐν γὰρ τῷ αὐτῷ τμήματί εἰσι τῷ ΑΔΓΒ· ὅλη ἄρα ἡ ὑπὸ ΑΔΓ ταῖς ὑπὸ ΒΑΓ, ΑΓΒ ἴση ἐστίν. κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· αἱ ἄρα ὑπὸ ΑΒΓ, ΒΑΓ, ΑΓΒ ταῖς ὑπὸ ΑΒΓ, ΑΔΓ ἴσαι εἰσίν. ἀλλ' αἱ ὑπὸ ΑΒΓ, ΒΑΓ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν. καὶ αἱ ὑπὸ ΑΒΓ, ΑΔΓ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ὑπὸ ΒΑΔ, ΔΓΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Τῶν ἄρα ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἔδει δεῖξαι.

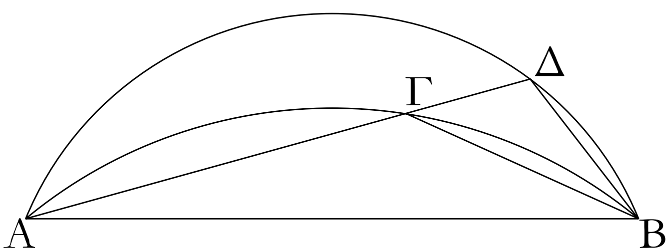
Therefore, since the three angles of any triangle are equal to two right-angles [Prop. 1.32], the three angles *CAB*, *ABC*, and *BCA* of triangle *ABC* are thus equal to two right-angles. And *CAB* (is) equal to *BDC*. For they are in the same segment *BADC* [Prop. 3.21]. And *ACB* (is equal) to *ADB*. For they are in the same segment *ADCB* [Prop. 3.21]. Thus, the whole of *ADC* is equal to *BAC* and *ACB*. Let *ABC* be added to both. Thus, *ABC*, *BAC*, and *ACB* are equal to *ABC* and *ADC*. But, *ABC*, *BAC*, and *ACB* are equal to two right-angles. Thus, *ABC* and *ADC* are also equal to two right-angles. Similarly, we can show that angles *BAD* and *DCB* are also equal to two right-angles.

Thus, for quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles. (Which is) the very thing it was required to show.

κγ'.

Ἐπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὅμοια καὶ ἄνισα οὐ συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη.

Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς ΑΒ δύο τμήματα κύκλων ὅμοια καὶ ἄνισα συνεστάτω ἐπὶ τὰ αὐτὰ μέρη τὰ ΑΓΒ, ΑΔΒ, καὶ διήχθω ἡ ΑΓΔ, καὶ ἐπεζεύχθωσαν αἱ ΓΒ, ΔΒ.



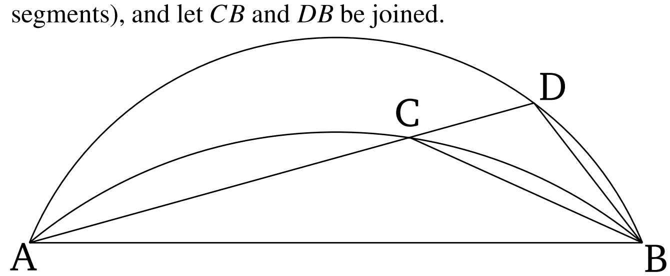
Ἐπεὶ οὖν ὅμοιόν ἐστι τὸ ΑΓΒ τμῆμα τῷ ΑΔΒ τμήματι, ὅμοια δὲ τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΑΔΒ ἡ ἐκτὸς τῇ ἐντός· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὅμοια καὶ ἄνισα συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

Proposition 23

Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

For, if possible, let the two similar and unequal segments of circles, *ACB* and *ADB*, be constructed on the same side of the same straight-line *AB*. And let *ACD* be drawn through (the segments), and let *CB* and *DB* be joined.



Therefore, since segment *ACB* is similar to segment *ADB*, and similar segments of circles are those accepting equal angles [Def. 3.11], angle *ACB* is thus equal to *ADB*, the external to the internal. The very thing is impossible [Prop. 1.16].

Thus, two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

κδ'.

Τὰ ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύλων ἴσα ἀλλήλοις ἐστίν.

Ἔστωσαν γὰρ ἐπὶ ἴσων εὐθειῶν τῶν ΑΒ, ΓΔ ὅμοια τμήματα κύκλων τὰ ΑΕΒ, ΓΖΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕΒ τμῆμα τῷ ΓΖΔ τμήματι.

Ἐφαρμοζομένου γὰρ τοῦ ΑΕΒ τμήματος ἐπὶ τὸ ΓΖΔ καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Γ τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΓΔ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Δ σημεῖον διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῇ ΓΔ· τῆς δὲ ΑΒ ἐπὶ τὴν ΓΔ ἐφαρμοσάσης ἐφαρμόσει καὶ τὸ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ. εἰ γὰρ ἡ ΑΒ εὐθεῖα ἐπὶ τὴν ΓΔ ἐφαρμόσει, τὸ δὲ ΑΕΒ τμῆμα

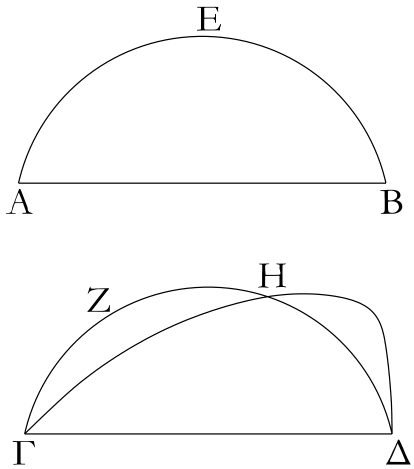
Proposition 24

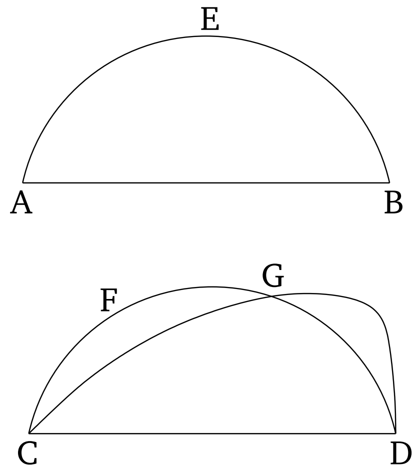
Similar segments of circles on equal straight-lines are equal to one another.

For let *AEB* and *CFD* be similar segments of circles on the equal straight-lines *AB* and *CD* (respectively). I say that segment *AEB* is equal to segment *CFD*.

For if the segment *AEB* is applied to the segment *CFD*, and point *A* is placed on (point) *C*, and the straight-line *AB* on *CD*, (then) point *B* will also coincide with point *D*, on account of *AB* being equal to *CD*. And if *AB* coincides with *CD*, (then) the segment *AEB* will also coincide with *CFD*. For if the straight-line *AB* coincides with *CD*, and the segment *AEB* ἐπὶ τὸ ΓΖΔ μὴ ἐφαρμόσει, ἤτοι ἐντὸς αὐτοῦ πεσεῖται ἢ ἐκτὸς ἢ παραλλάξει, ὡς τὸ ΓΗΔ, καὶ κύκλος κύκλον τέμνει κατὰ πλείονα σημεῖα ἢ δύο· ὅπερ ἐστίν ἀδύνατον. οὐκ ἄρα ἐφαρμοζομένης τῆς ΑΒ εὐθείας ἐπὶ τὴν ΓΔ οὐκ ἐφαρμόσει καὶ τὸ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ· ἐφαρμόσει ἄρα, καὶ ἴσον αὐτῷ ἔσται.

does not coincide with *CFD*, (then) it will surely either fall inside it, outside (it),[†] or it will miss like *CGD* (in the figure), and a circle (will) cut (another) circle at more than two points. The very thing is impossible [Prop. 3.10]. Thus, if the straight-line *AB* is applied to *CD*, the segment *AEB* cannot not also coincide with *CFD*. Thus, it will coincide, and will be equal to it [C.N. 4].





Τὰ ἄρα ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

Thus, similar segments of circles on equal straight-lines are equal to one another. (Which is) the very thing it was required to show.

[†] Both this possibility, and the previous one, are precluded by Prop. 3.23.

κεʹ.

Κύκλου τμήματος δοθέντος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα.

Ἔστω τὸ δοθὲν τμῆμα κύκλου τὸ ΑΒΓ· δεῖ δὴ τοῦ ΑΒΓ τμήματος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα.

Τετμήσθω γὰρ ἡ ΑΓ δίχα κατὰ τὸ Δ, καὶ ἤχθω ἀπὸ τοῦ Δ σημείου τῇ ΑΓ πρὸς ὀρθὰς ἡ ΔΒ, καὶ ἐπεζεύχθω ἡ ΑΒ· ἡ ὑπὸ ΑΒΔ γωνία ἄρα τῆς ὑπὸ ΒΑΔ ἤτοι μείζων ἐστὶν ἢ ἴση ἢ ἐλάττων.

Ἔστω πρότερον μείζων, καὶ συνεστάτω πρὸς τῇ ΒΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΑΒΔ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΕ, καὶ διήχθω ἡ ΔΒ ἐπὶ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΒΑΕ, ἴση ἄρα ἐστὶ καὶ ἡ ΕΒ εὐθεῖα τῇ ΕΑ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΓ, κοινὴ δὲ ἡ ΔΕ, δύο δὴ αἱ ΑΔ, ΔΕ δύο ταῖς ΓΔ, ΔΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΔΕ γωνίᾳ τῇ ὑπὸ ΓΔΕ ἐστιν ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΕ βάσει τῇ ΓΕ ἐστιν ἴση. ἀλλὰ ἡ ΑΕ τῇ ΒΕ ἐδείχθη ἴση· καὶ ἡ ΒΕ ἄρα τῇ ΓΕ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΑΕ, ΕΒ, ΕΓ ἴσαι ἀλλήλαις εἰσίν· ὁ ἄρα κέντρῳ τῷ Ε διαστήματι δὲ ἑνὶ τῶν ΑΕ, ΕΒ, ΕΓ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται προσαναγεγραμμένος. κύκλου ἄρα τμήματος δοθέντος

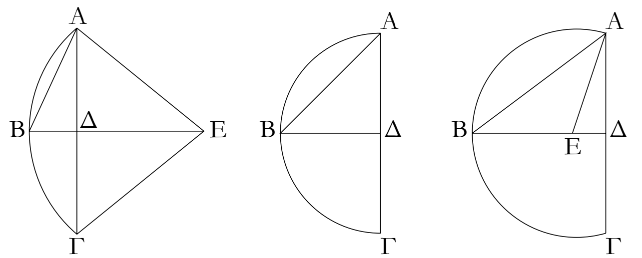
Proposition 25

For a given segment of a circle, to complete the circle, the very one of which it is a segment.

Let *ABC* be the given segment of a circle. So it is required to complete the circle for segment *ABC*, the very one of which it is a segment.

For let *AC* be cut in half at (point) *D* [Prop. 1.10], and let *DB* be drawn from point *D*, at right-angles to *AC* [Prop. 1.11]. And let *AB* be joined. Thus, angle *ABD* is surely either greater than, equal to, or less than (angle) *BAD*.

First of all, let it be greater. And let (angle) *BAE*, equal to angle *ABD*, be constructed on the straight-line *BA*, at the point *A* on it [Prop. 1.23]. And let *DB* be drawn through to *E*, and let *EC* be joined. Therefore, since angle *ABE* is equal to *BAE*, the straight-line *EB* is thus also equal to *EA* [Prop. 1.6]. And since *AD* is equal to *DC*, and *DE* (is) common, the two (straight-lines) *AD*, *DE* are equal to the two (straight-lines) *CD*, *DE*, respectively. And angle *ADE* is equal to angle *CDE*. For each (is) a right-angle. Thus, the base *AE* is equal to the base *CE* [Prop. 1.4]. But, *AE* was shown (to be) equal to *BE*. Thus, *BE* is also equal to *CE*. Thus, the three (straight-lines) *AE*, *EB*, and *EC* are equal to one another. Thus, if a circle is *προσαναγέγραπται ὁ κύκλος. καὶ δῆλον, ὡς τὸ ΑΒΓ τμῆμα ἔλαττόν ἐστιν ἡμικυκλίου διὰ τὸ τὸ Ε κέντρον ἐκτὸς αὐτοῦ τυγχάνειν.*

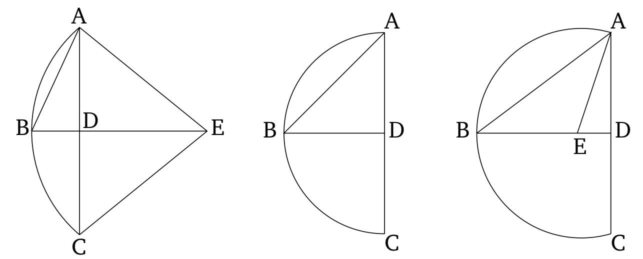


Ὁμοίως [δὲ] κἂν ᾖ ἡ ὑπὸ ΑΒΔ γωνία ἴση τῇ ὑπὸ ΒΑΔ, τῆς ΑΔ ἴσης γενομένης ἑκατέρᾳ τῶν ΒΔ, ΔΓ αἱ τρεῖς αἱ ΔΑ, ΔΒ, ΔΓ ἴσαι ἀλλήλαις ἔσονται, καὶ ἔσται τὸ Δ κέντρον τοῦ προσαναπεπληρωμένου κύκλου, καὶ δηλαδὴ ἔσται τὸ ΑΒΓ ἡμικύκλιον.

Ἐὰν δὲ ἡ ὑπὸ ΑΒΔ ἐλάττων ᾖ τῆς ὑπὸ ΒΑΔ, καὶ συστησώμεθα πρὸς τῇ ΒΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΑΒΔ γωνίᾳ ἴσην, ἐντὸς τοῦ ΑΒΓ τμήματος πεσεῖται τὸ κέντρον ἐπὶ τῆς ΔΒ, καὶ ἔσται δηλαδὴ τὸ ΑΒΓ τμῆμα μεῖζον ἡμικυκλίου.

Κύκλου ἄρα τμήματος δοθέντος προσαναγέγραπται ὁ κύκλος· ὅπερ ἔδει ποιῆσαι.

drawn with center E, and radius one of AE, EB, or EC, it will also go through the remaining points (of the segment), and the (associated circle) will be completed [Prop. 3.9]. Thus, a circle has been completed from the given segment of a circle. And (it is) clear that the segment ABC is less than a semi-circle, because the center E happens to lie outside it.



[And], similarly, even if angle ABD is equal to BAD, (since) AD becomes equal to each of BD [Prop. 1.6] and DC, the three (straight-lines) DA, DB, and DC will be equal to one another. And point D will be the center of the completed circle. And ABC will manifestly be a semi-circle.

And if ABD is less than BAD, and we construct (angle BAE), equal to angle ABD, on the straight-line BA, at the point A on it [Prop. 1.23], then the center will fall on DB, inside the segment ABC. And segment ABC will manifestly be greater than a semi-circle.

Thus, a circle has been completed from the given segment of a circle. (Which is) the very thing it was required to do.

κϛ'.

Ἐν τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι.

Ἔστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ καὶ ἐν αὐτοῖς ἴσαι γωνίαι ἔστωσαν πρὸς μὲν τοῖς κέντροις αἱ ὑπὸ ΒΗΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ ΒΑΓ, ΕΔΖ· λέγω, ὅτι ἴση ἐστὶν ἡ ΒΚΓ περιφέρεια τῇ ΕΛΖ περιφερείᾳ.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΓ, ΕΖ.

Καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΑΒΓ, ΔΕΖ κύκλοι, ἴσαι εἰσὶν αἱ ἐκ τῶν κέντρων· δύο δὴ αἱ ΒΗ, ΗΓ δύο ταῖς ΕΘ, ΘΖ ἴσαι· καὶ γωνία ἡ πρὸς τῷ Η γωνίᾳ τῇ πρὸς τῷ Θ ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ πρὸς τῷ Α γωνία τῇ πρὸς τῷ Δ, ὅμοιον ἄρα ἐστὶ τὸ ΒΑΓ τμῆμα τῷ ΕΔΖ τμήματι· καί εἰσιν ἐπὶ ἴσων εὐθειῶν [τῶν ΒΓ, ΕΖ]· τὰ δὲ ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα τὸ ΒΑΓ τμῆμα τῷ ΕΔΖ. ἔστι δὲ καὶ ὅλος ὁ ΑΒΓ κύκλος ὅλῳ τῷ ΔΕΖ κύκλῳ ἴσος· λοιπὴ ἄρα ἡ ΒΚΓ περιφέρεια τῇ ΕΛΖ περιφερείᾳ ἐστὶν ἴση.

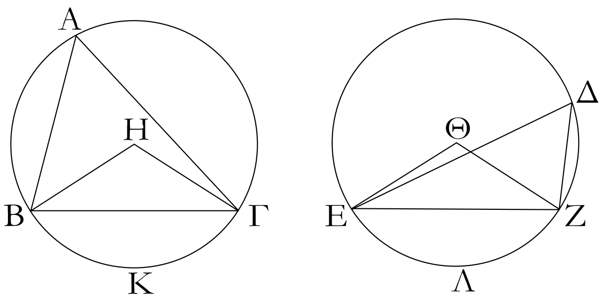
Proposition 26

In equal circles, equal angles stand upon equal circumferences whether they are standing at the center or at the circumference.

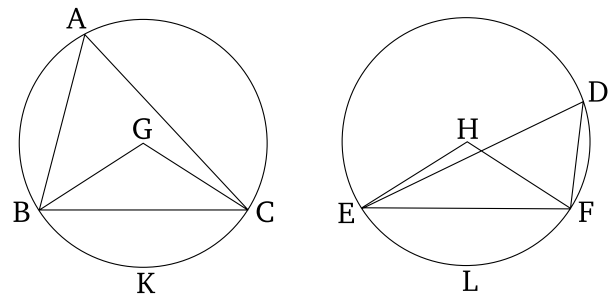
Let ABC and DEF be equal circles, and within them let BGC and EHF be equal angles at the center, and BAC and EDF (equal angles) at the circumference. I say that circumference BKC is equal to circumference ELF.

For let BC and EF be joined.

And since circles ABC and DEF are equal, their radii are equal. So the two (straight-lines) BG, GC (are) equal to the two (straight-lines) EH, HF (respectively). And the angle at G (is) equal to the angle at H. Thus, the base BC is equal to the base EF [Prop. 1.4]. And since the angle at A is equal to the (angle) at D, the segment BAC is thus similar to the segment EDF [Def. 3.11]. And they are on equal straight-lines [BC and EF]. And similar segments of circles on equal straight-lines are equal to one another [Prop. 3.24]. Thus, segment BAC is equal to (segment) EDF. And the whole circle ABC is also equal to the whole circle DEF. Thus, the remaining circumference BKC is equal to the (remaining) circumference ELF.



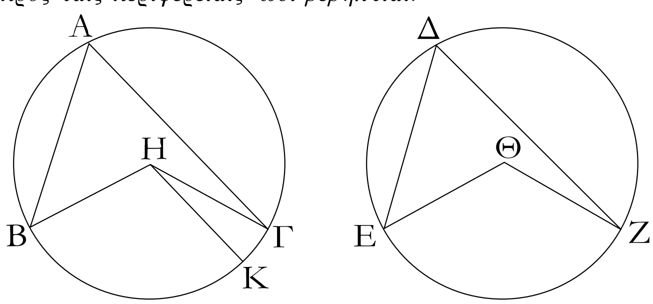
Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.



Thus, in equal circles, equal angles stand upon equal circumferences, whether they are standing at the center or at the circumference. (Which is) the very thing which it was required to show.

κζʹ.

Ἐν τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι.



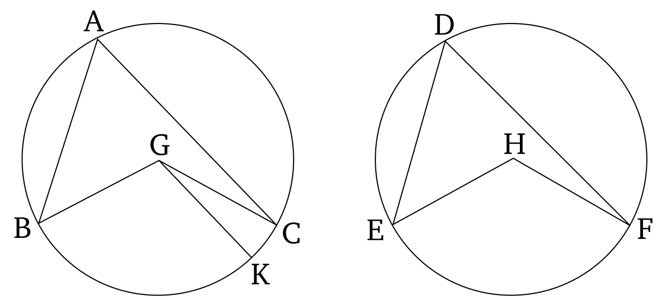
Ἐν γὰρ ἴσοις κύκλοις τοῖς ΑΒΓ, ΔΕΖ ἐπὶ ἴσων περιφερειῶν τῶν ΒΓ, ΕΖ πρὸς μὲν τοῖς Η, Θ κέντροις γωνίαι βεβηκέτωσαν αἱ ὑπὸ ΒΗΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ ΒΑΓ, ΕΔΖ· λέγω, ὅτι ἡ μὲν ὑπὸ ΒΗΓ γωνία τῇ ὑπὸ ΕΘΖ ἐστιν ἴση, ἡ δὲ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ΒΗΓ τῇ ὑπὸ ΕΘΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΒΗΓ, καὶ συνεστάτω πρὸς τῇ ΒΗ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Η τῇ ὑπὸ ΕΘΖ γωνίᾳ ἴση ἡ ὑπὸ ΒΗΚ· αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ὦσιν· ἴση ἄρα ἡ ΒΚ περιφέρεια τῇ ΕΖ περιφερείᾳ. ἀλλὰ ἡ ΕΖ τῇ ΒΓ ἐστιν ἴση· καὶ ἡ ΒΚ ἄρα τῇ ΒΓ ἐστιν ἴση ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΒΗΓ γωνία τῇ ὑπὸ ΕΘΖ· ἴση ἄρα. καί ἐστι τῆς μὲν ὑπὸ ΒΗΓ ἡμίσεια ἡ πρὸς τῷ Α, τῆς δὲ ὑπὸ ΕΘΖ ἡμίσεια ἡ πρὸς τῷ Δ· ἴση ἄρα καὶ ἡ πρὸς τῷ Α γωνία τῇ πρὸς τῷ Δ.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

Proposition 27

In equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference.



For let the angles BGC and EHF at the centers G and H, and the (angles) BAC and EDF at the circumferences, stand upon the equal circumferences BC and EF, in the equal circles ABC and DEF (respectively). I say that angle BGC is equal to (angle) EHF, and BAC is equal to EDF.

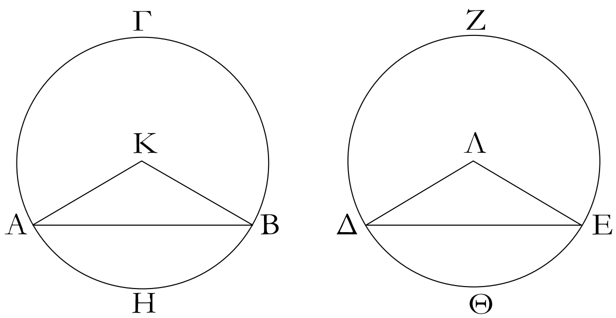
For if BGC is unequal to EHF, one of them is greater. Let BGC be greater, and let the (angle) BGK, equal to angle EHF, be constructed on the straight-line BG, at the point G on it [Prop. 1.23]. But equal angles (in equal circles) stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference BK (is) equal to circumference EF. But, EF is equal to BC. Thus, BK is also equal to BC, the lesser to the greater. The very thing is impossible. Thus, angle BGC is not unequal to EHF. Thus, (it is) equal. And the (angle) at A is half BGC, and the (angle) at D half EHF [Prop. 3.20]. Thus, the angle at A (is) also equal to the (angle) at D.

Thus, in equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference. (Which is) the very thing it was required to show.

κη'.

Ἐν τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττονα τῇ ἐλάττονι.

Ἔστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ, καὶ ἐν τοῖς κύκλοις ἴσαι εὐθεῖαι ἔστωσαν αἱ ΑΒ, ΔΕ τὰς μὲν ΑΓΒ, ΑΖΕ περιφερείας μείζονας ἀφαιροῦσαι τὰς δὲ ΑΗΒ, ΔΘΕ ἐλάττονας· λέγω, ὅτι ἡ μὲν ΑΓΒ μείζων περιφέρεια ἴση ἐστὶ τῇ ΔΖΕ μείζονι περιφερείᾳ ἡ δὲ ΑΗΒ ἐλάττων περιφέρεια τῇ ΔΘΕ.



Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων τὰ Κ, Λ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΚΒ, ΔΛ, ΛΕ.

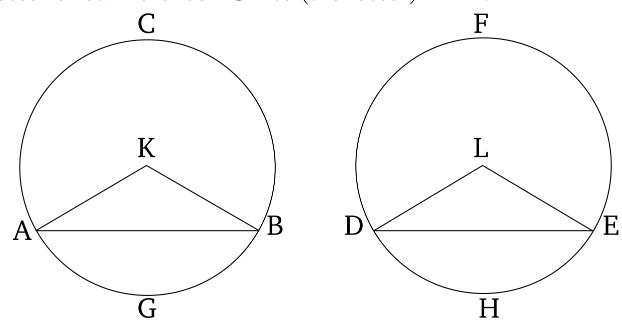
Καὶ ἐπεὶ ἴσοι κύκλοι εἰσίν, ἴσαι εἰσὶ καὶ αἱ ἐκ τῶν κέντρων· δύο δὴ αἱ ΑΚ, ΚΒ δυσὶ ταῖς ΔΛ, ΛΕ ἴσαι εἰσίν· καὶ βάσις ἡ ΑΒ βάσει τῇ ΔΕ ἴση· γωνία ἄρα ἡ ὑπὸ ΑΚΒ γωνίᾳ τῇ ὑπὸ ΔΛΕ ἴση ἐστίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ὦσιν· ἴση ἄρα ἡ ΑΗΒ περιφέρεια τῇ ΔΘΕ. ἐστὶ δὲ καὶ ὅλος ὁ ΑΒΓ κύκλος ὅλῳ τῷ ΔΕΖ κύκλῳ ἴσος· καὶ λοιπὴ ἄρα ἡ ΑΓΒ περιφέρεια λοιπῇ τῇ ΔΖΕ περιφερείᾳ ἴση ἐστίν.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττονα τῇ ἐλάττονι· ὅπερ ἔδει δεῖξαι.

Proposition 28

In equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser.

Let *ABC* and *DEF* be equal circles, and let *AB* and *DE* be equal straight-lines in these circles, cutting off the greater circumferences *ACB* and *DFE*, and the lesser (circumferences) *AGB* and *DHE* (respectively). I say that the greater circumference *ACB* is equal to the greater circumference *DFE*, and the lesser circumference *AGB* to (the lesser) *DHE*.



For let the centers of the circles, *K* and *L*, be found [Prop. 3.1], and let *AK*, *KB*, *DL*, and *LE* be joined.

And since (*ABC* and *DEF*) are equal circles, their radii are also equal [Def. 3.1]. So the two (straight-lines) *AK*, *KB* are equal to the two (straight-lines) *DL*, *LE* (respectively). And the base *AB* (is) equal to the base *DE*. Thus, angle *AKB* is equal to angle *DLE* [Prop. 1.8]. And equal angles stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference *AGB* (is) equal to *DHE*. And the whole circle *ABC* is also equal to the whole circle *DEF*. Thus, the remaining circumference *ACB* is also equal to the remaining circumference *DFE*.

Thus, in equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser. (Which is) the very thing it was required to show.

κθ'.

Ἐν τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν.

Ἔστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ, καὶ ἐν αὐτοῖς ἴσαι περιφέρειαι ἀπειλήφθωσαν αἱ ΒΗΓ, ΕΘΖ, καὶ ἐπεζεύχθωσαν αἱ ΒΓ, ΕΖ εὐθεῖαι· λέγω, ὅτι ἴση ἐστὶν ἡ ΒΓ τῇ ΕΖ.

Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων, καὶ ἔστω τὰ Κ, Λ, καὶ ἐπεζεύχθωσαν αἱ ΒΚ, ΚΓ, ΕΛ, ΛΖ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΗΓ περιφέρεια τῇ ΕΘΖ περιφερείᾳ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΚΓ τῇ ὑπὸ ΕΛΖ. καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΑΒΓ, ΔΕΖ κύκλοι, ἴσαι εἰσὶ καὶ αἱ ἐκ τῶν κέντρων· δύο δὴ αἱ ΒΚ, ΚΓ δυσὶ ταῖς ΕΛ, ΛΖ ἴσαι εἰσίν· καὶ γωνίας ἴσας περιέχουσιν· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστίν·

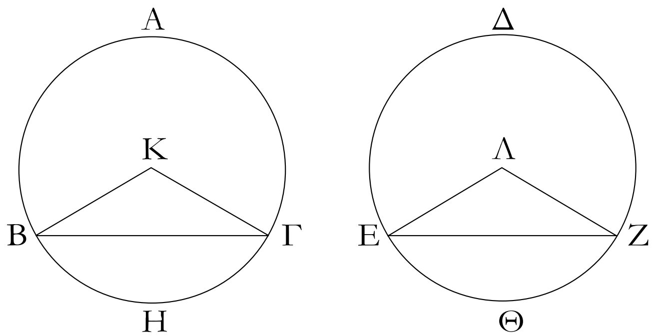
Proposition 29

In equal circles, equal straight-lines subtend equal circumferences.

Let *ABC* and *DEF* be equal circles, and within them let the equal circumferences *BGC* and *EHF* be cut off. And let the straight-lines *BC* and *EF* be joined. I say that *BC* is equal to *EF*.

For let the centers of the circles be found [Prop. 3.1], and let them be (at) *K* and *L*. And let *BK*, *KC*, *EL*, and *LF* be joined.

And since the circumference *BGC* is equal to the circumference *EHF*, the angle *BKC* is also equal to (angle) *ELF* [Prop. 3.27]. And since the circles *ABC* and *DEF* are equal,



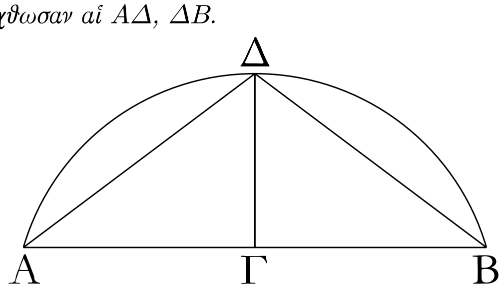
Ἐν ἄρα τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

λ΄.

Τὴν δοθεῖσαν περιφέρειαν δίχα τεμεῖν.

Ἔστω ἡ δοθεῖσα περιφέρεια ἡ ΑΔΒ· δεῖ δὴ τὴν ΑΔΒ περιφέρειαν δίχα τεμεῖν.

Ἐπεζεύχθω ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ ἀπὸ τοῦ Γ σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς ἤχθω ἡ ΓΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΒ.



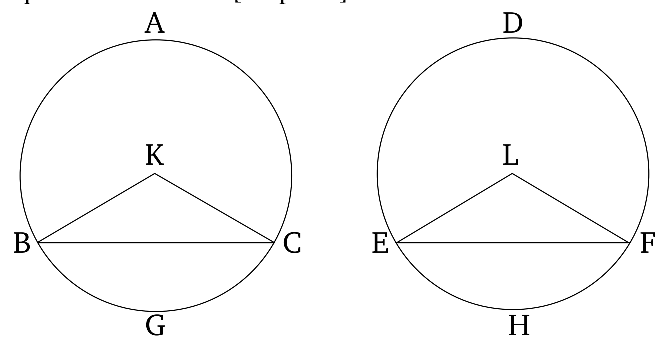
Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, κοινὴ δὲ ἡ ΓΔ, δύο δὴ αἱ ΑΓ, ΓΔ δυσὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΓΔ γωνίᾳ τῇ ὑπὸ ΒΓΔ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΔΒ ἴση ἐστίν. αἱ δὲ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττονα τῇ ἐλάττονι· κάι ἐστιν ἑκατέρα τῶν ΑΔ, ΔΒ περιφερειῶν ἐλάττων ἡμικυκλίου· ἴση ἄρα ἡ ΑΔ περιφέρεια τῇ ΔΒ περιφερείᾳ.

Ἡ ἄρα δοθεῖσα περιφέρεια δίχα τέτμηται κατὰ τὸ Δ σημεῖον· ὅπερ ἔδει ποιῆσαι.

λα΄.

Ἐν κύκλῳ ἡ μὲν ἐν τῷ ἡμικυκλίῳ γωνία ὀρθή ἐστιν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι

their radii are also equal [Def. 3.1]. So the two (straight-lines) *BK*, *KC* are equal to the two (straight-lines) *EL*, *LF* (respectively). And they contain equal angles. Thus, the base *BC* is equal to the base *EF* [Prop. 1.4].



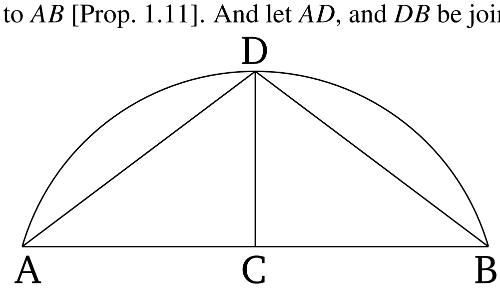
Thus, in equal circles, equal straight-lines subtend equal circumferences. (Which is) the very thing it was required to show.

Proposition 30

To cut a given circumference in half.

Let *ADB* be the given circumference. So it is required to cut circumference *ADB* in half.

Let *AB* be joined, and let it be cut in half at (point) *C* [Prop. 1.10]. And let *CD* be drawn from point *C*, at right-angles to *AB* [Prop. 1.11]. And let *AD*, and *DB* be joined.

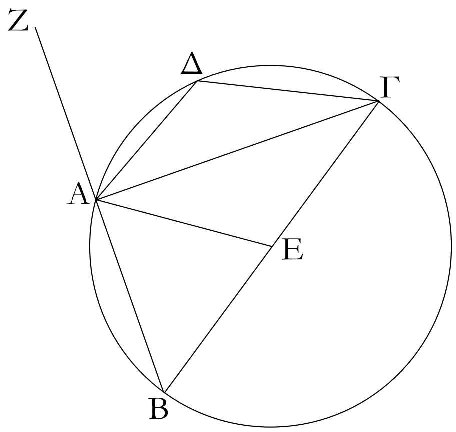


And since *AC* is equal to *CB*, and *CD* (is) common, the two (straight-lines) *AC*, *CD* are equal to the two (straight-lines) *BC*, *CD* (respectively). And angle *ACD* (is) equal to angle *BCD*. For (they are) each right-angles. Thus, the base *AD* is equal to the base *DB* [Prop. 1.4]. And equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser [Prop. 1.28]. And the circumferences *AD* and *DB* are each less than a semi-circle. Thus, circumference *AD* (is) equal to circumference *DB*.

Thus, the given circumference has been cut in half at point *D*. (Which is) the very thing it was required to do.

Proposition 31

In a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in *τμήματι μείζων ὀρθῆς· καὶ ἔτι ἡ μὲν τοῦ μείζονος τμήματος γωνία μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἐλάττων ὀρθῆς.*



Ἔστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΒΓ, κέντρον δὲ τὸ Ε, καὶ ἐπεζεύχθωσαν αἱ ΒΑ, ΑΓ, ΑΔ, ΔΓ· λέγω, ὅτι ἡ μὲν ἐν τῷ ΒΑΓ ἡμικυκλίῳ γωνία ἡ ὑπὸ ΒΑΓ ὀρθή ἐστιν, ἡ δὲ ἐν τῷ ΑΒΓ μείζονι τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ ΑΒΓ ἐλάττων ἐστὶν ὀρθῆς, ἡ δὲ ἐν τῷ ΑΔΓ ἐλάττονι τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ ΑΔΓ μείζων ἐστὶν ὀρθῆς.

Ἐπεζεύχθω ἡ ΑΕ, καὶ διήχθω ἡ ΒΑ ἐπὶ τὸ Ζ.

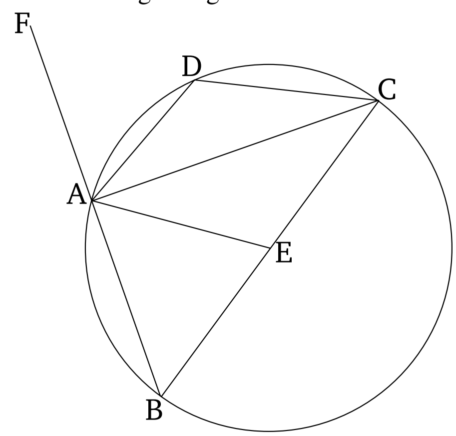
Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΑ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΒΕ τῇ ὑπὸ ΒΑΕ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΓΕ τῇ ΕΑ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΓΑΕ· ὅλη ἄρα ἡ ὑπὸ ΒΑΓ δυσὶ ταῖς ὑπὸ ΑΒΓ, ΑΓΒ ἴση ἐστίν. ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΑΓ ἐκτὸς τοῦ ΑΒΓ τριγώνου δυσὶ ταῖς ὑπὸ ΑΒΓ, ΑΓΒ γωνίαις ἴση· ἴση ἄρα καὶ ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΖΑΓ· ὀρθὴ ἄρα ἑκατέρα· ἡ ἄρα ἐν τῷ ΒΑΓ ἡμικυκλίῳ γωνία ἡ ὑπὸ ΒΑΓ ὀρθή ἐστιν.

Καὶ ἐπεὶ τοῦ ΑΒΓ τρίγωνου δύο γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΑΓ δύο ὀρθῶν ἐλάττονές εἰσιν, ὀρθὴ δὲ ἡ ὑπὸ ΒΑΓ, ἐλάττων ἄρα ὀρθῆς ἐστιν ἡ ὑπὸ ΑΒΓ γωνία· καί ἐστιν ἐν τῷ ΑΒΓ μείζονι τοῦ ἡμικυκλίου τμήματι.

Καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ΑΒΓΔ, τῶν δὲ ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν [αἱ ἄρα ὑπὸ ΑΒΓ, ΑΔΓ γωνίαι δυσὶν ὀρθαῖς ἴσας εἰσίν], καί ἐστιν ἡ ὑπὸ ΑΒΓ ἐλάττων ὀρθῆς· λοιπὴ ἄρα ἡ ὑπὸ ΑΔΓ γωνία μείζων ὀρθῆς ἐστιν· καί ἐστιν ἐν τῷ ΑΔΓ ἐλάττονι τοῦ ἡμικυκλίου τμήματι.

Λέγω, ὅτι καὶ ἡ μὲν τοῦ μείζονος τμήματος γωνία ἡ περιεχομένη ὑπό [τε] τῆς ΑΒΓ περιφερείας καὶ τῆς ΑΓ εὐθείας μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἡ περιεχομένη ὑπό [τε] τῆς ΑΔ[Γ] περιφερείας καὶ τῆς ΑΓ εὐθείας ἐλάττων ἐστὶν ὀρθῆς. καί ἐστιν αὐτόθεν φανερόν. ἐπεὶ γὰρ ἡ ὑπὸ τῶν ΒΑ, ΑΓ εὐθειῶν ὀρθή ἐστιν, ἡ ἄρα ὑπὸ τῆς ΑΒΓ περιφερείας καὶ τῆς ΑΓ εὐθείας περιεχομένη μείζων ἐστὶν ὀρθῆς. πάλιν, ἐπεὶ ἡ ὑπὸ τῶν ΑΓ, ΑΖ εὐθειῶν ὀρθή ἐστιν, ἡ ἄρα

a lesser segment (is) greater than a right-angle. And, further, the angle of a segment greater (than a semi-circle) is greater than a right-angle, and the angle of a segment less (than a semi-circle) is less than a right-angle.



Let *ABCD* be a circle, and let *BC* be its diameter, and *E* its center. And let *BA*, *AC*, *AD*, and *DC* be joined. I say that the angle *BAC* in the semi-circle *BAC* is a right-angle, and the angle *ABC* in the segment *ABC*, (which is) greater than a semi-circle, is less than a right-angle, and the angle *ADC* in the segment *ADC*, (which is) less than a semi-circle, is greater than a right-angle.

Let *AE* be joined, and let *BA* be drawn through to *F*.

And since *BE* is equal to *EA*, angle *ABE* is also equal to *BAE* [Prop. 1.5]. Again, since *CE* is equal to *EA*, *ACE* is also equal to *CAE* [Prop. 1.5]. Thus, the whole (angle) *BAC* is equal to the two (angles) *ABC* and *ACB*. And *FAC*, (which is) external to triangle *ABC*, is also equal to the two angles *ABC* and *ACB* [Prop. 1.32]. Thus, angle *BAC* (is) also equal to *FAC*. Thus, (they are) each right-angles. [Def. 1.10]. Thus, the angle *BAC* in the semi-circle *BAC* is a right-angle.

And since the two angles *ABC* and *BAC* of triangle *ABC* are less than two right-angles [Prop. 1.17], and *BAC* is a right-angle, angle *ABC* is thus less than a right-angle. And it is in segment *ABC*, (which is) greater than a semi-circle.

And since *ABCD* is a quadrilateral within a circle, and for quadrilaterals within circles the (sum of the) opposite angles is equal to two right-angles [Prop. 3.22] [angles *ABC* and *ADC* are thus equal to two right-angles], and (angle) *ABC* is less than a right-angle. The remaining angle *ADC* is thus greater than a right-angle. And it is in segment *ADC*, (which is) less than a semi-circle.

I also say that the angle of the greater segment, (namely) that contained by the circumference *ABC* and the straight-line *AC*, is greater than a right-angle. And the angle of the lesser segment, (namely) that contained by the circumference *AD*[*C*] and the straight-line *AC*, is less than a right-angle. And this is immediately apparent. For since the (angle contained by) *ὑπὸ τῆς ΓΑ εὐθείας καὶ τῆς ΑΔ[Γ] περιφερείας περιεχομένη ἐλάττων ἐστὶν ὀρθῆς.*

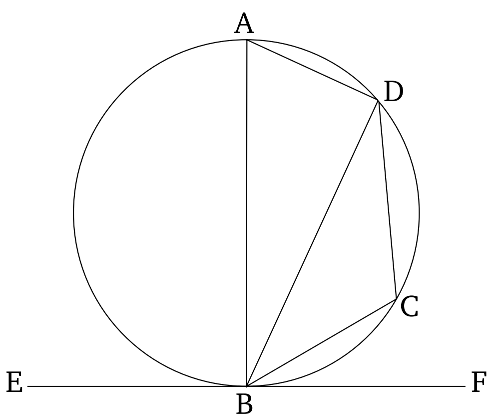
Ἐν κύκλῳ ἄρα ἡ μὲν ἐν τῷ ἡμικυκλίῳ γωνία ὀρθή ἐστιν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι [τμήματι] μείζων ὀρθῆς· καὶ ἔπι ἡ μὲν τοῦ μείζονος τμήματος [γωνία] μείζων [ἐστὶν] ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος [γωνία] ἐλάττων ὀρθῆς· ὅπερ ἔδει δεῖξαι.

the two straight-lines *BA* and *AC* is a right-angle, the (angle) contained by the circumference *ABC* and the straight-line *AC* is thus greater than a right-angle. Again, since the (angle contained by) the straight-lines *AC* and *AF* is a right-angle, the (angle) contained by the circumference *AD*[*C*] and the straight-line *CA* is thus less than a right-angle.

Thus, in a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser [segment] (is) greater than a right-angle. And, further, the [angle] of a segment greater (than a semi-circle) [is] greater than a right-angle, and the [angle] of a segment less (than a semi-circle) is less than a right-angle. (Which is) the very thing it was required to show.

λβ′.

Ἐὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς εἰς τὸν κύκλον διαχθῇ τις εὐθεῖα τέμνουσα τὸν κύκλον, ἃς ποιεῖ γωνίας πρὸς τῇ ἐφαπτομένῃ, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τοῦ κύκλου τμήμασι γωνίαις.



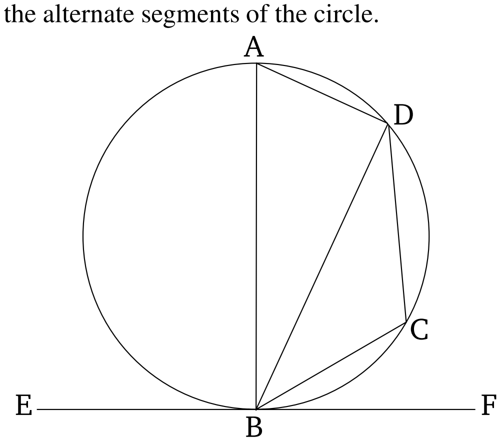
Κύκλου γὰρ τοῦ ΑΒΓΔ ἐφαπτέσθω τις εὐθεῖα ἡ ΕΖ κατὰ τὸ Β σημεῖον, καὶ ἀπὸ τοῦ Β σημείου διήχθω τις εὐθεῖα εἰς τὸν ΑΒΓΔ κύκλον τέμνουσα αὐτὸν ἡ ΒΔ. λέγω, ὅτι ἃς ποιεῖ γωνίας ἡ ΒΔ μετὰ τῆς ΕΖ ἐφαπτομένης, ἴσας ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τμήμασι τοῦ κύκλου γωνίαις, τουτέστιν, ὅτι ἡ μὲν ὑπὸ ΖΒΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ΒΑΔ τμήματι συνισταμένῃ γωνίᾳ, ἡ δὲ ὑπὸ ΕΒΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ΔΓΒ τμήματι συνισταμένῃ γωνίᾳ.

Ἤχθω γὰρ ἀπὸ τοῦ Β τῇ ΕΖ πρὸς ὀρθὰς ἡ ΒΑ, καὶ εἰλήφθω ἐπὶ τῆς ΒΔ περιφερείας τυχὸν σημεῖον τὸ Γ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, ΓΒ.

Καὶ ἐπεὶ κύκλου τοῦ ΑΒΓΔ ἐφάπτεταί τις εὐθεῖα ἡ ΕΖ κατὰ τὸ Β, καὶ ἀπὸ τῆς ἁφῆς ἦκται τῇ ἐφαπτομένῃ πρὸς ὀρθὰς ἡ ΒΑ, ἐπὶ τῆς ΒΑ ἄρα τὸ κέντρον ἐστὶ τοῦ ΑΒΓΔ κύκλου. ἡ ΒΑ ἄρα διάμετός ἐστι τοῦ ΑΒΓΔ κύκλου· ἡ ἄρα ὑπὸ ΑΔΒ γωνία ἐν ἡμικυκλίῳ οὖσα ὀρθή ἐστιν. λοιπαὶ ἄρα αἱ ὑπὸ ΒΑΔ, ΑΒΔ μιᾷ ὀρθῇ ἴσαι εἰσίν. ἐστὶ δὲ καὶ ἡ ὑπὸ

Proposition 32

If some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), (then) those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle.



For let some straight-line *EF* touch the circle *ABCD* at the point *B*, and let some (other) straight-line *BD* be drawn from point *B* into the circle *ABCD*, cutting it (in two). I say that the angles *BD* makes with the tangent *EF* will be equal to the angles in the alternate segments of the circle. That is to say, that angle *FBD* is equal to the angle constructed in segment *BAD*, and angle *EBD* is equal to the angle constructed in segment *DCB*.

For let *BA* be drawn from *B*, at right-angles to *EF* [Prop. 1.11]. And let the point *C* be taken at random on the circumference *BD*. And let *AD*, *DC*, and *CB* be joined.

And since some straight-line *EF* touches the circle *ABCD* at point *B*, and *BA* has been drawn from the point of contact, at right-angles to the tangent, the center of circle *ABCD* is thus on *BA* [Prop. 3.19]. Thus, *BA* is a diameter of circle *ABCD*. Thus, angle *ADB*, being in a semi-circle, is a right-angle [Prop. 3.31]. Thus, the remaining angles (of triangle

ΑΒΖ ὀρθή· ἡ ἄρα ὑπὸ ΑΒΖ ἴση ἐστὶ ταῖς ὑπὸ ΒΑΔ, ΑΒΔ. κοινὴ ἀφῃρήσθω ἡ ὑπὸ ΑΒΔ· λοιπὴ ἄρα ἡ ὑπὸ ΔΒΖ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τμήματι τοῦ κύκλου γωνίᾳ τῇ ὑπὸ ΒΑΔ. καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ΑΒΓΔ, αἱ ἀπεναντίον αὐτοῦ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΒΖ, ΔΒΕ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΔΒΖ, ΔΒΕ ταῖς ὑπὸ ΒΑΔ, ΒΓΔ ἴσαι εἰσίν, ὧν ἡ ὑπὸ ΒΑΔ τῇ ὑπὸ ΔΒΖ ἐδείχθη ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΔΒΕ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι τῷ ΔΓΒ τῇ ὑπὸ ΔΓΒ γωνίᾳ ἐστὶν ἴση.

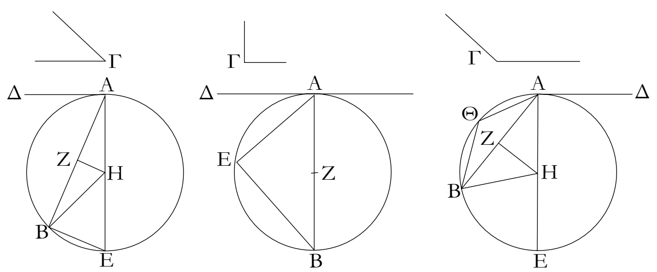
Ἐὰν ἄρα κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς εἰς τὸν κύκλον διαχθῇ τις εὐθεῖα τέμνουσα τὸν κύκλον, ἃς ποιεῖ γωνίας πρὸς τῇ ἐφαπτομένῃ, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τοῦ κύκλου τμήμασι γωνίαις· ὅπερ ἔδει δεῖξαι.

ADB) BAD and ABD are equal to one right-angle [Prop. 1.32]. And ABF is also a right-angle. Thus, ABF is equal to BAD and ABD. Let ABD be subtracted from both. Thus, the remaining angle DBF is equal to the angle BAD in the alternate segment of the circle. And since ABCD is a quadrilateral in a circle, (the sum of) its opposite angles is equal to two right-angles [Prop. 3.22]. And DBF and DBE is also equal to two right-angles [Prop. 1.13]. Thus, DBF and DBE is equal to BAD and BCD, of which BAD was shown (to be) equal to DBF. Thus, the remaining (angle) DBE is equal to the angle DCB in the alternate segment DCB of the circle.

Thus, if some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), (then) those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle. (Which is) the very thing it was required to show.

λγ'.

Ἐπὶ τῆς δοθείσης εὐθείας γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.



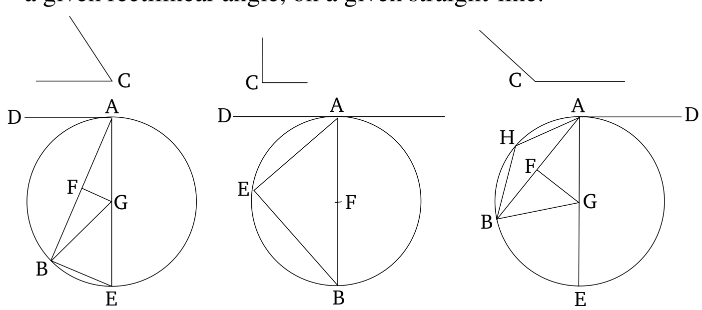
Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ πρὸς τῷ Γ· δεῖ δὴ ἐπὶ τῆς δοθείσης εὐθείας τῆς ΑΒ γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ.

Ἡ δὴ πρὸς τῷ Γ [γωνία] ἤτοι ὀξεῖά ἐστιν ἢ ὀρθὴ ἢ ἀμβλεῖα· ἔστω πρότερον ὀξεῖα, καὶ ὡς ἐπὶ τῆς πρώτης καταγραφῆς συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ Α σημείῳ τῇ πρὸς τῷ Γ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΔ· ὀξεῖα ἄρα ἐστὶ καὶ ἡ ὑπὸ ΒΑΔ. ἤχθω τῇ ΔΑ πρὸς ὀρθὰς ἡ ΑΕ, καὶ τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ ἤχθω ἀπὸ τοῦ Ζ σημείου τῇ ΑΒ πρὸς ὀρθὰς ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΗΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, κοινὴ δὲ ἡ ΖΗ, δύο δὴ αἱ ΑΖ, ΖΗ δύο ταῖς ΒΖ, ΖΗ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΖΗ [γωνίᾳ] τῇ ὑπὸ ΒΖΗ ἴση· βάσις ἄρα ἡ ΑΗ βάσει τῇ ΒΗ ἴση ἐστίν. ὁ ἄρα κέντρῳ μὲν τῷ Η διαστήματι δὲ τῷ ΗΑ κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Β. γεγράφθω καὶ ἔστω ὁ ΑΒΕ, καὶ ἐπεζεύχθω ἡ ΕΒ. ἐπεὶ οὖν ἀπ᾽ ἄκρας τῆς ΑΕ διαμέτρου ἀπὸ τοῦ Α τῇ ΑΕ πρὸς ὀρθάς ἐστιν ἡ ΑΔ, ἡ ΑΔ ἄρα ἐφάπτεται τοῦ ΑΒΕ κύκλου· ἐπεὶ οὖν κύκλου τοῦ ΑΒΕ ἐφάπτεταί τις εὐθεῖα ἡ ΑΔ, καὶ ἀπὸ τῆς κατὰ τὸ Α ἁφῆς εἰς

Proposition 33

To draw a segment of a circle, accepting an angle equal to a given rectilinear angle, on a given straight-line.



Let AB be the given straight-line, and C the given rectilinear angle. So it is required to draw a segment of a circle, accepting an angle equal to C, on the given straight-line AB.

So the [angle] C is surely either acute, a right-angle, or obtuse. First of all, let it be acute. And, as in the first diagram (from the left), let (angle) BAD, equal to angle C, be constructed on the straight-line AB, at the point A (on it) [Prop. 1.23]. Thus, BAD is also acute. Let AE be drawn, at right-angles to DA [Prop. 1.11]. And let AB be cut in half at F [Prop. 1.10]. And let FG be drawn from point F, at right-angles to AB [Prop. 1.11]. And let GB be joined.

And since AF is equal to FB, and FG (is) common, the two (straight-lines) AF, FG are equal to the two (straight-lines) BF, FG (respectively). And angle AFG (is) equal to [angle] BFG. Thus, the base AG is equal to the base BG [Prop. 1.4]. Thus, the circle drawn with center G, and radius GA, will also go through B (as well as A). Let it be drawn, and let it be (denoted) ABE. And let EB be joined. Therefore, since AD is at the extremity of diameter AE, (namely, point) A, at right-angles to AE, the (straight-line) AD thus touches the circle *τὸν ΑΒΕ κύκλον διῆκταί τις εὐθεῖα ἡ ΑΒ, ἡ ἄρα ὑπὸ ΔΑΒ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ ΑΕΒ. ἀλλ᾽ ἡ ὑπὸ ΔΑΒ τῇ πρὸς τῷ Γ ἐστιν ἴση· καὶ ἡ πρὸς τῷ Γ ἄρα γωνία ἴση ἐστὶ τῇ ὑπὸ ΑΕΒ.*

Ἐπὶ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τμῆμα κύκλου γέγραπται τὸ ΑΕΒ δεχόμενον γωνίαν τὴν ὑπὸ ΑΕΒ ἴσην τῇ δοθείσῃ τῇ πρὸς τῷ Γ.

Ἀλλὰ δὴ ὀρθὴ ἔστω ἡ πρὸς τῷ Γ· καὶ δέον πάλιν ἔστω ἐπὶ τῆς ΑΒ γράψαι τμῆμα κύκλου δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ ὀρθῇ [γωνίᾳ]. συνεστάτω [πάλιν] τῇ πρὸς τῷ Γ ὀρθῇ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΔ, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ κέντρῳ τῷ Ζ, διαστήματι δὲ ὁποτέρῳ τῶν ΖΑ, ΖΒ, κύκλος γεγράφθω ὁ ΑΕΒ.

Ἐφάπτεται ἄρα ἡ ΑΔ εὐθεῖα τοῦ ΑΒΕ κύκλου διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ Α γωνίαν. καὶ ἴση ἐστὶν ἡ ὑπὸ ΒΑΔ γωνία τῇ ἐν τῷ ΑΕΒ τμήματι· ὀρθὴ γὰρ καὶ αὐτὴ ἐν ἡμικυκλίῳ οὖσα. ἀλλὰ καὶ ἡ ὑπὸ ΒΑΔ τῇ πρὸς τῷ Γ ἴση ἐστίν. καὶ ἡ ἐν τῷ ΑΕΒ ἄρα ἴση ἐστὶ τῇ πρὸς τῷ Γ.

Γέγραπται ἄρα πάλιν ἐπὶ τῆς ΑΒ τμῆμα κύκλου τὸ ΑΕΒ δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ.

Ἀλλὰ δὴ ἡ πρὸς τῷ Γ ἀμβλεῖα ἔστω· καὶ συνεστάτω αὐτῇ ἴση πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ Α σημείῳ ἡ ὑπὸ ΒΑΔ, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καὶ τῇ ΑΔ πρὸς ὀρθὰς ἤχθω ἡ ΑΕ, καὶ τετμήσθω πάλιν ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΗΒ.

Καὶ ἐπεὶ πάλιν ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, καὶ κοινὴ ἡ ΖΗ, δύο δὴ αἱ ΑΖ, ΖΗ δύο ταῖς ΒΖ, ΖΗ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΖΗ γωνίᾳ τῇ ὑπὸ ΒΖΗ ἴση· βάσις ἄρα ἡ ΑΗ βάσει τῇ ΒΗ ἴση ἐστίν· ὁ ἄρα κέντρῳ μὲν τῷ Η διαστήματι δὲ τῷ ΗΑ κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Β. ἐρχέσθω ὡς ὁ ΑΕΒ. καὶ ἐπεὶ τῇ ΑΕ διαμέτρῳ ἀπ᾽ ἄκρας πρὸς ὀρθάς ἐστιν ἡ ΑΔ, ἡ ΑΔ ἄρα ἐφάπτεται τοῦ ΑΕΒ κύκλου. καὶ ἀπὸ τῆς κατὰ τὸ Α ἐπαφῆς διῆκται ἡ ΑΒ· ἡ ἄρα ὑπὸ ΒΑΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι τῷ ΑΘΒ συνισταμένῃ γωνίᾳ. ἀλλ᾽ ἡ ὑπὸ ΒΑΔ γωνία τῇ πρὸς τῷ Γ ἴση ἐστίν. καὶ ἡ ἐν τῷ ΑΘΒ ἄρα τμήματι γωνία ἴση ἐστὶ τῇ πρὸς τῷ Γ.

Ἐπὶ τῆς ἄρα δοθείσης εὐθείας τῆς ΑΒ γέγραπται τμῆμα κύκλου τὸ ΑΘΒ δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ· ὅπερ ἔδει ποιῆσαι.

ABE [Prop. 3.16 corr.]. Therefore, since some straight-line *AD* touches the circle *ABE*, and some (other) straight-line *AB* has been drawn across from the point of contact *A* into circle *ABE*, angle *DAB* is thus equal to the angle *AEB* in the alternate segment of the circle [Prop. 3.32]. But, *DAB* is equal to *C*. Thus, angle *C* is also equal to *AEB*.

Thus, a segment *AEB* of a circle, accepting the angle *AEB* (which is) equal to the given (angle) *C*, has been drawn on the given straight-line *AB*.

And so let *C* be a right-angle. And let it again be necessary to draw a segment of a circle on *AB*, accepting an angle equal to the right-[angle] *C*. Let the (angle) *BAD* [again] be constructed, equal to the right-angle *C* [Prop. 1.23], as in the second diagram (from the left). And let *AB* be cut in half at *F* [Prop. 1.10]. And let the circle *AEB* be drawn with center *F*, and radius either *FA* or *FB*.

Thus, the straight-line *AD* touches the circle *ABE*, on account of the angle at *A* being a right-angle [Prop. 3.16 corr.]. And angle *BAD* is equal to the angle in segment *AEB*. For (the latter angle), being in a semi-circle, is also a right-angle [Prop. 3.31]. But, *BAD* is also equal to *C*. Thus, the (angle) in (segment) *AEB* is also equal to *C*.

Thus, a segment *AEB* of a circle, accepting an angle equal to *C*, has again been drawn on *AB*.

And so let (angle) *C* be obtuse. And let (angle) *BAD*, equal to (*C*), be constructed on the straight-line *AB*, at the point *A* (on it) [Prop. 1.23], as in the third diagram (from the left). And let *AE* be drawn, at right-angles to *AD* [Prop. 1.11]. And let *AB* again be cut in half at *F* [Prop. 1.10]. And let *FG* be drawn, at right-angles to *AB* [Prop. 1.10]. And let *GB* be joined.

And again, since *AF* is equal to *FB*, and *FG* (is) common, the two (straight-lines) *AF*, *FG* are equal to the two (straight-lines) *BF*, *FG* (respectively). And angle *AFG* (is) equal to angle *BFG*. Thus, the base *AG* is equal to the base *BG* [Prop. 1.4]. Thus, a circle of center *G*, and radius *GA*, being drawn, will also go through *B* (as well as *A*). Let it go like *AEB* (in the third diagram from the left). And since *AD* is at right-angles to the diameter *AE*, at its extremity, *AD* thus touches circle *AEB* [Prop. 3.16 corr.]. And *AB* has been drawn across (the circle) from the point of contact *A*. Thus, angle *BAD* is equal to the angle constructed in the alternate segment *AHB* of the circle [Prop. 3.32]. But, angle *BAD* is equal to *C*. Thus, the angle in segment *AHB* is also equal to *C*.

Thus, a segment *AHB* of a circle, accepting an angle equal to *C*, has been drawn on the given straight-line *AB*. (Which is) the very thing it was required to do.

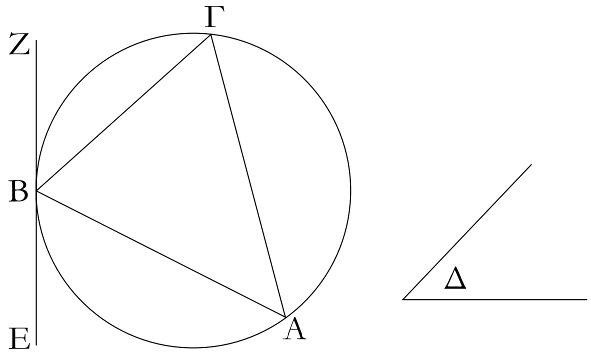
λδ'.

Ἀπὸ τοῦ δοθέντος κύκλου τμῆμα ἀφελεῖν δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Proposition 34

To cut off a segment, accepting an angle equal to a given rectilinear angle, from a given circle.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ πρὸς τῷ Δ· δεῖ δὴ ἀπὸ τοῦ ΑΒΓ κύκλου τμῆμα ἀφελεῖν δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ πρὸς τῷ Δ.

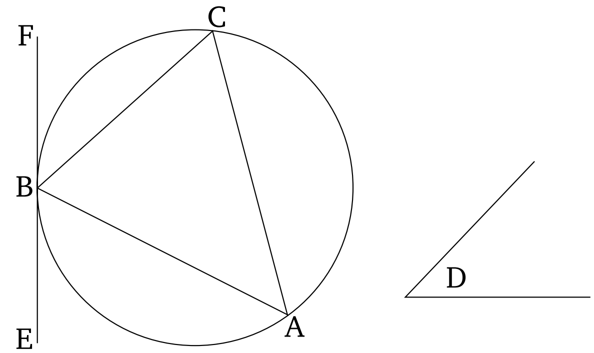


Ἤχθω τοῦ ΑΒΓ ἐφαπτομένη ἡ ΕΖ κατὰ τὸ Β σημεῖον, καὶ συνεστάτω πρὸς τῇ ΖΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β τῇ πρὸς τῷ Δ γωνίᾳ ἴση ἡ ὑπὸ ΖΒΓ.

Ἐπεὶ οὖν κύκλου τοῦ ΑΒΓ ἐφάπτεταί τις εὐθεῖα ἡ ΕΖ, καὶ ἀπὸ τῆς κατὰ τὸ Β ἐπαφῆς διῆκται ἡ ΒΓ, ἡ ὑπὸ ΖΒΓ ἄρα γωνία ἴση ἐστὶ τῇ ἐν τῷ ΒΑΓ ἐναλλὰξ τμήματι συνισταμένῃ γωνίᾳ. ἀλλ᾽ ἡ ὑπὸ ΖΒΓ τῇ πρὸς τῷ Δ ἐστιν ἴση· καὶ ἡ ἐν τῷ ΒΑΓ ἄρα τμήματι ἴση ἐστὶ τῇ πρὸς τῷ Δ [γωνίᾳ].

Ἀπὸ τοῦ δοθέντος ἄρα κύκλου τοῦ ΑΒΓ τμῆμα ἀφῄρηται τὸ ΒΑΓ δεχόμενον γωνίαν ἴσην τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ πρὸς τῷ Δ· ὅπερ ἔδει ποιῆσαι.

Let *ABC* be the given circle, and *D* the given rectilinear angle. So it is required to cut off a segment, accepting an angle equal to the given rectilinear angle *D*, from the given circle *ABC*.



Let *EF* be drawn touching *ABC* at point *B*.† And let (angle) *FBC*, equal to angle *D*, be constructed on the straight-line *FB*, at the point *B* on it [Prop. 1.23].

Therefore, since some straight-line *EF* touches the circle *ABC*, and *BC* has been drawn across (the circle) from the point of contact *B*, angle *FBC* is thus equal to the angle constructed in the alternate segment *BAC* [Prop. 1.32]. But, *FBC* is equal to *D*. Thus, the (angle) in the segment *BAC* is also equal to [angle] *D*.

Thus, the segment *BAC*, accepting an angle equal to the given rectilinear angle *D*, has been cut off from the given circle *ABC*. (Which is) the very thing it was required to do.

† Presumably, by finding the center of *ABC* [Prop. 3.1], drawing a straight-line between the center and point *B*, and then drawing *EF* through point *B*, at right-angles to the aforementioned straight-line [Prop. 1.11].

λεʹ.

Ἐὰν ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν τῆς ἑτέρας τμημάτων περιεχομένῳ ὀρθογωνίῳ.

Ἐν γὰρ κύκλῳ τῷ ΑΒΓΔ δύο εὐθεῖαι αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένῳ ὀρθογωνίῳ.

Εἰ μὲν οὖν αἱ ΑΓ, ΒΔ διὰ τοῦ κέντρου εἰσὶν ὥστε τὸ Ε κέντρον εἶναι τοῦ ΑΒΓΔ κύκλου, φανερόν, ὅτι ἴσων οὐσῶν τῶν ΑΕ, ΕΓ, ΔΕ, ΕΒ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένῳ ὀρθογωνίῳ.

μὴ ἔστωσαν δὴ αἱ ΑΓ, ΔΒ διὰ τοῦ κέντρου, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓΔ, καὶ ἔστω τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὰς ΑΓ, ΔΒ εὐθείας κάθετοι ἤχθωσαν αἱ ΖΗ, ΖΘ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΓ, ΖΕ.

Καὶ ἐπεὶ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΗΖ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν

Proposition 35

If two straight-lines in a circle cut one another, (then) the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other.

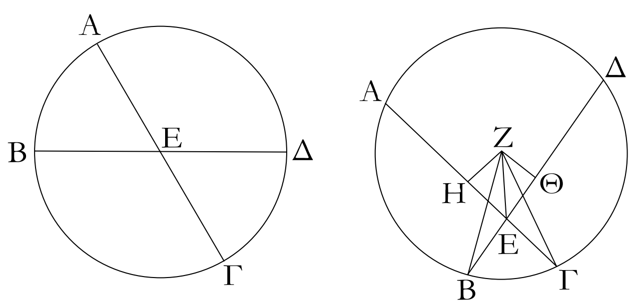
For let the two straight-lines *AC* and *BD*, in the circle *ABCD*, cut one another at point *E*. I say that the rectangle contained by *AE* and *EC* is equal to the rectangle contained by *DE* and *EB*.

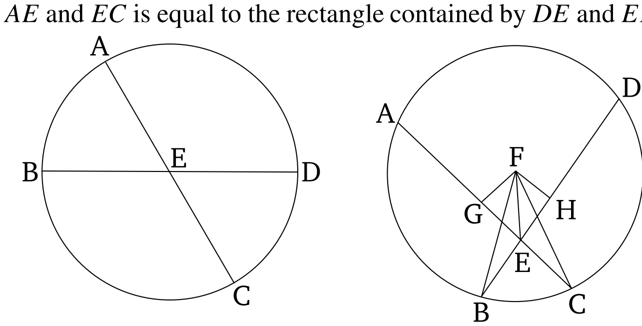
In fact, if *AC* and *BD* are through the center (as in the first diagram from the left), so that *E* is the center of circle *ABCD*, (then it is) clear that, *AE*, *EC*, *DE*, and *EB* being equal, the rectangle contained by *AE* and *EC* is also equal to the rectangle contained by *DE* and *EB*.

So let *AC* and *DB* not be though the center (as in the second diagram from the left), and let the center of *ABCD* be found [Prop. 3.1], and let it be (at) *F*. And let *FG* and *FH* be drawn from *F*, perpendicular to the straight-lines *AC* and *DB* (respectively) [Prop. 1.12]. And let *FB*, *FC*, and *FE* be joined.

τέμνει· ἴση ἄρα ἡ ΑΗ τῇ ΗΓ. ἐπεὶ οὖν εὐθεῖα ἡ ΑΓ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Η, εἰς δὲ ἄνισα κατὰ τὸ Ε, τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΗ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΓ· [κοινὸν] προσκείσθω τὸ ἀπὸ τῆς ΗΖ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τῶν ἀπὸ τῶν ΗΕ, ΗΖ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΓΗ, ΗΖ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΕΗ, ΗΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΕ, τοῖς δὲ ἀπὸ τῶν ΓΗ, ΗΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΓ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΓ. ἴση δὲ ἡ ΖΓ τῇ ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΕΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΒ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν ΔΕ, ΕΒ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΒ. ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον τῷ ἀπὸ τῆς ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΕ, ΕΓ μετὰ τοῦ ἀπὸ τῆς ΖΕ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ μετὰ τοῦ ἀπὸ τῆς ΖΕ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΖΕ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΕ, ΕΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΔΕ, ΕΒ περιεχομένῳ ὀρθογωνίῳ.

And since some straight-line, *GF*, through the center, cuts at right-angles some (other) straight-line, *AC*, not through the center, (then) it also cuts it in half [Prop. 3.3]. Thus, *AG* (is) equal to *GC*. Therefore, since the straight-line *AC* is cut equally at *G*, and unequally at *E*, the rectangle contained by *AE* and *EC* plus the square on *EG* is thus equal to the (square) on *GC* [Prop. 2.5]. Let the (square) on *GF* be added [to both]. Thus, the (rectangle contained) by *AE* and *EC* plus the (sum of the squares) on *GE* and *GF* is equal to the (sum of the squares) on *CG* and *GF*. But, the (square) on *FE* is equal to the (sum of the squares) on *EG* and *GF* [Prop. 1.47], and the (square) on *FC* is equal to the (sum of the squares) on *CG* and *GF* [Prop. 1.47]. Thus, the (rectangle contained) by *AE* and *EC* plus the (square) on *FE* is equal to the (square) on *FC*. And *FC* (is) equal to *FB*. Thus, the (rectangle contained) by *AE* and *EC* plus the (square) on *FE* is equal to the (square) on *FB*. So, for the same (reasons), the (rectangle contained) by *DE* and *EB* plus the (square) on *FE* is equal to the (square) on *FB*. And the (rectangle contained) by *AE* and *EC* plus the (square) on *FE* was also shown (to be) equal to the (square) on *FB*. Thus, the (rectangle contained) by *AE* and *EC* plus the (square) on *FE* is equal to the (rectangle contained) by *DE* and *EB* plus the (square) on *FE*. Let the (square) on *FE* be taken from both. Thus, the remaining rectangle contained by *AE* and *EC* is equal to the rectangle contained by *DE* and *EB*.





Ἐὰν ἄρα ἐν κύκλῳ εὐθεῖαι δύο τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν τῆς ἑτέρας τμημάτων περιεχομένῳ ὀρθογωνίῳ· ὅπερ ἔδει δεῖξαι.

Thus, if two straight-lines in a circle cut one another, (then) the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other. (Which is) the very thing it was required to show.

λς΄.

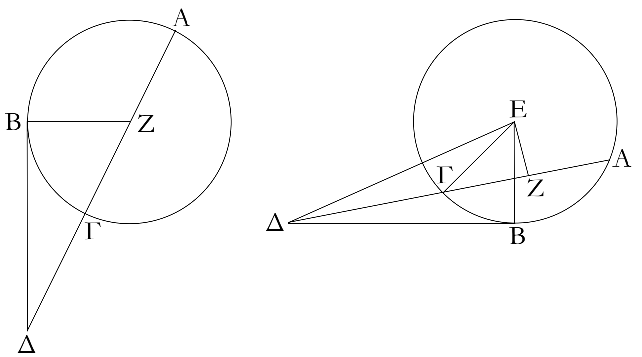
Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, καὶ ἀπ᾽ αὐτοῦ πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ ἐφάπτηται, ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ.

Κύκλου γὰρ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύκλον προσπιπτέτωσαν δύο εὐθεῖαι αἱ ΔΓ[Α], ΔΒ· καὶ ἡ μὲν ΔΓΑ τεμνέτω τὸν ΑΒΓ κύκλον, ἡ δὲ ΒΔ ἐφαπτέσθω· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ τε-

Proposition 36

If some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and the (other) touches (it), (then) the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line).

For let some point *D* be taken outside circle *ABC*, and let two straight-lines, *DC*[*A*] and *DB*, radiate from *D* towards circle *ABC*. And let *DCA* cut circle *ABC*, and let *BD* touch (it). I *τραγώνῳ.*

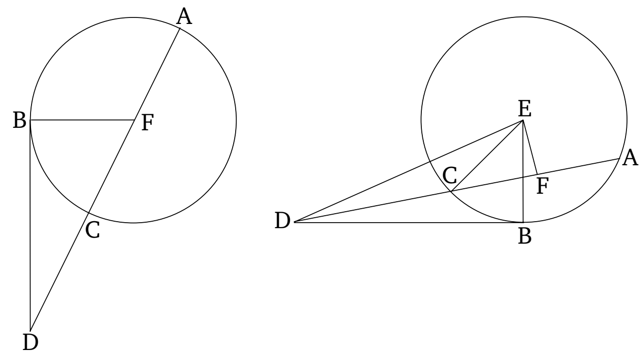


Ἡ ἄρα [Δ]ΓΑ ἤτοι διὰ τοῦ κέντρου ἐστὶν ἢ οὔ. ἔστω πρότερον διὰ τοῦ κέντρου, καὶ ἔστω τὸ Ζ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἐπεζεύχθω ἡ ΖΒ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΒΔ. καὶ ἐπεὶ εὐθεῖα ἡ ΑΓ δίχα τέτμηται κατὰ τὸ Ζ, πρόσκειται δὲ αὐτῇ ἡ ΓΔ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. ἴση δὲ ἡ ΖΓ τῇ ΖΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. τῷ δὲ ἀπὸ τῆς ΖΔ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΖΒ, ΒΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΒ, ΒΔ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΖΒ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ ἐφαπτομένης.

Ἀλλὰ δὴ ἡ ΔΓΑ μὴ ἔστω διὰ τοῦ κέντρου τοῦ ΑΒΓ κύκλου, καὶ εἰλήφθω τὸ κέντρον τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΓ κάθετος ἤχθω ἡ ΕΖ, καὶ ἐπεζεύχθωσαν αἱ ΕΒ, ΕΓ, ΕΔ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΕΒΔ. καὶ ἐπεὶ εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΕΖ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει· ἡ ΑΖ ἄρα τῇ ΖΓ ἐστιν ἴση. καὶ ἐπεὶ εὐθεῖα ἡ ΑΓ τέτμηται δίχα κατὰ τὸ Ζ σημεῖον, πρόσκειται δὲ αὐτῇ ἡ ΓΔ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΖΕ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τῶν ἀπὸ τῶν ΓΖ, ΖΕ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΔ, ΖΕ. τοῖς δὲ ἀπὸ τῶν ΓΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΓ· ὀρθὴ γὰρ [ἐστιν] ἡ ὑπὸ ΕΖΓ [γωνία]· τοῖς δὲ ἀπὸ τῶν ΔΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΔ. ἴση δὲ ἡ ΕΓ τῇ ΕΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΔ. τῷ δὲ ἀπὸ τῆς ΕΔ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΕΒ, ΒΔ· ὀρθὴ γὰρ ἡ ὑπὸ ΕΒΔ γωνία· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΕΒ, ΒΔ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΕΒ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, καὶ ἀπ' αὐτοῦ πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ ἐφάπτηται, ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

say that the rectangle contained by *AD* and *DC* is equal to the square on *DB*.



[*D*]*CA* is surely either through the center, or not. Let it first of all be through the center, and let *F* be the center of circle *ABC*, and let *FB* be joined. Thus, (angle) *FBD* is a right-angle [Prop. 3.18]. And since straight-line *AC* is cut in half at *F*, let *CD* be added to it. Thus, the (rectangle contained) by *AD* and *DC* plus the (square) on *FC* is equal to the (square) on *FD* [Prop. 2.6]. And *FC* (is) equal to *FB*. Thus, the (rectangle contained) by *AD* and *DC* plus the (square) on *FB* is equal to the (square) on *FD*. And the (square) on *FD* is equal to the (sum of the squares) on *FB* and *BD* [Prop. 1.47]. Thus, the (rectangle contained) by *AD* and *DC* plus the (square) on *FB* is equal to the (sum of the squares) on *FB* and *BD*. Let the (square) on *FB* be subtracted from both. Thus, the remaining (rectangle contained) by *AD* and *DC* is equal to the (square) on the tangent *DB*.

And so let *DCA* not be through the center of circle *ABC*, and let the center *E* be found, and let *EF* be drawn from *E*, perpendicular to *AC* [Prop. 1.12]. And let *EB*, *EC*, and *ED* be joined. (Angle) *EBD* (is) thus a right-angle [Prop. 3.18]. And since some straight-line, *EF*, through the center, cuts some (other) straight-line, *AC*, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, *AF* is equal to *FC*. And since the straight-line *AC* is cut in half at point *F*, let *CD* be added to it. Thus, the (rectangle contained) by *AD* and *DC* plus the (square) on *FC* is equal to the (square) on *FD* [Prop. 2.6]. Let the (square) on *FE* be added to both. Thus, the (rectangle contained) by *AD* and *DC* plus the (sum of the squares) on *CF* and *FE* is equal to the (sum of the squares) on *FD* and *FE*. But the (square) on *EC* is equal to the (sum of the squares) on *CF* and *FE*. For [angle] *EFC* [is] a right-angle [Prop. 1.47]. And the (square) on *ED* is equal to the (sum of the squares) on *DF* and *FE* [Prop. 1.47]. Thus, the (rectangle contained) by *AD* and *DC* plus the (square) on *EC* is equal to the (square) on *ED*. And *EC* (is) equal to *EB*. Thus, the (rectangle contained) by *AD* and *DC* plus the (square) on *EB* is equal to the (square) on *ED*. And the (sum of the squares) on *EB* and *BD* is equal to the (square) on *ED*. For *EBD* (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by

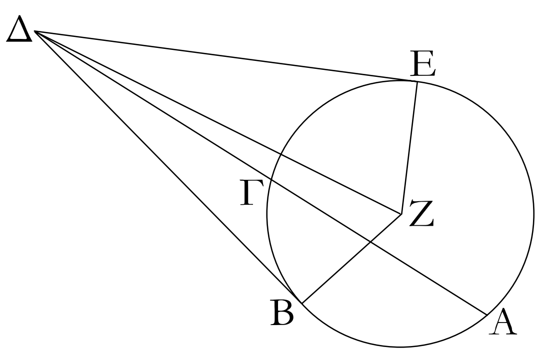
AD and DC plus the (square) on EB is equal to the (sum of the squares) on EB and BD. Let the (square) on EB be subtracted from both. Thus, the remaining (rectangle contained) by AD and DC is equal to the (square) on BD.

Thus, if some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and (the other) touches (it), (then) the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line). (Which is) the very thing it was required to show.

λζʹ.

Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ προσπίπτῃ, ᾖ δὲ τὸ ὑπὸ [τῆς] ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἡ προσπίπτουσα ἐφάψεται τοῦ κύκλου.

Κύκλου γὰρ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύκλον προσπιπτέτωσαν δύο εὐθεῖαι αἱ ΔΓΑ, ΔΒ, καὶ ἡ μὲν ΔΓΑ τεμνέτω τὸν κύκλον, ἡ δὲ ΔΒ προσπιπτέτω, ἔστω δὲ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΔΒ. λέγω, ὅτι ἡ ΔΒ ἐφάπτεται τοῦ ΑΒΓ κύκλου.

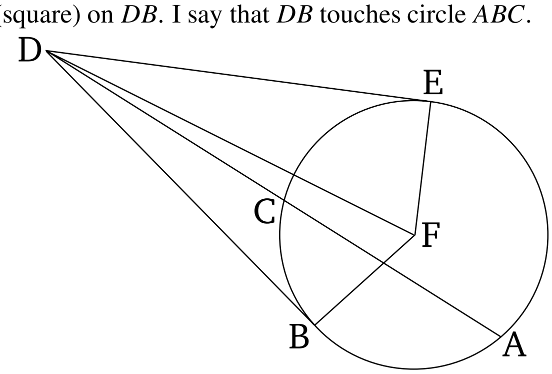


Ἤχθω γὰρ τοῦ ΑΒΓ ἐφαπτομένη ἡ ΔΕ, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΕ, ΖΒ, ΖΔ. ἡ ἄρα ὑπὸ ΖΕΔ ὀρθή ἐστιν. καὶ ἐπεὶ ἡ ΔΕ ἐφάπτεται τοῦ ΑΒΓ κύκλου, τέμνει δὲ ἡ ΔΓΑ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΕ. ἦν δὲ καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΔΒ· τὸ ἄρα ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ· ἴση ἄρα ἡ ΔΕ τῇ ΔΒ. ἐστὶ δὲ καὶ ἡ ΖΕ τῇ ΖΒ ἴση· δύο δὴ αἱ ΔΕ, ΕΖ δύο ταῖς ΔΒ, ΒΖ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΖΔ· γωνία ἄρα ἡ ὑπὸ ΔΕΖ γωνίᾳ τῇ ὑπὸ ΔΒΖ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΔΕΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΔΒΖ. καί ἐστιν ἡ ΖΒ ἐκβαλλομένη διάμετρος· ἡ δὲ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπʼ ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου· ἡ ΔΒ ἄρα ἐφάπτεται

Proposition 37

If some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), (then) the (straight-line) meeting (the circle) will touch the circle.

For let some point D be taken outside circle ABC, and let two straight-lines, DCA and DB, radiate from D towards circle ABC, and let DCA cut the circle, and let DB meet (the circle). And let the (rectangle contained) by AD and DC be equal to the (square) on DB. I say that DB touches circle ABC.



For let DE be drawn touching ABC [Prop. 3.17], and let the center of the circle ABC be found, and let it be (at) F. And let FE, FB, and FD be joined. (Angle) FED is thus a right-angle [Prop. 3.18]. And since DE touches circle ABC, and DCA cuts (it), the (rectangle contained) by AD and DC is thus equal to the (square) on DE [Prop. 3.36]. And the (rectangle contained) by AD and DC was also equal to the (square) on DB. Thus, the (square) on DE is equal to the (square) on DB. Thus, DE (is) equal to DB. And FE is also equal to FB. So the two (straight-lines) DE, EF are equal to the two (straight-lines) DB, BF (respectively). And their base, FD, is common. Thus, angle DEF is equal to angle DBF [Prop. 1.8]. And DEF (is) a right-angle. Thus, DBF (is) also a right-angle.

τοῦ ΑΒΓ κύκλου. ὁμοίως δὴ δειχθήσεται, κἂν τὸ κέντρον ἐπὶ τῆς ΑΓ τυγχάνῃ.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ προσπίπτῃ, ᾖ δὲ τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἡ προσπίπτουσα ἐφάψεται τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

And FB produced is a diameter, And a (straight-line) drawn at right-angles to a diameter of a circle, at its extremity, touches the circle [Prop. 3.16 corr.]. Thus, DB touches circle ABC. Similarly, (the same thing) can be shown, even if the center happens to be on AC.

Thus, if some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), (then) the (straight-line) meeting (the circle) will touch the circle. (Which is) the very thing it was required to show.

ELEMENTS BOOK 4

Construction of Rectilinear Figures In and Around Circles

Ὅροι.

α΄. Σχῆμα εὐθύγραμμον εἰς σχῆμα εὐθύγραμμον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη τῶν τοῦ ἐγγραφομένου σχήματος γωνιῶν ἑκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἅπτηται.

β΄. Σχῆμα δὲ ὁμοίως περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἑκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἅπτηται.

γ΄. Σχῆμα εὐθύγραμμον εἰς κύκλον ἐγγράφεσθαι λέγεται, ὅταν ἑκάστη γωνία τοῦ ἐγγραφομένου ἅπτηται τῆς τοῦ κύκλου περιφερείας.

δ΄. Σχῆμα δὲ εὐθύγραμμον περὶ κύκλον περιγράφεσθαι λέγεται, ὅταν ἑκάστη πλευρὰ τοῦ περιγραφομένου ἐφάπτηται τῆς τοῦ κύκλου περιφερείας.

ε΄. Κύκλος δὲ εἰς σχῆμα ὁμοίως ἐγγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἑκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ἅπτηται.

ϛ΄. Κύκλος δὲ περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἑκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ἅπτηται.

ζ΄. Εὐθεῖα εἰς κύκλον ἐναρμόζεσθαι λέγεται, ὅταν τὰ πέρατα αὐτῆς ἐπὶ τῆς περιφερείας ᾖ τοῦ κύκλου.

Definitions

1. A rectilinear figure is said to be inscribed in a(nother) rectilinear figure when the respective angles of the inscribed figure touch the respective sides of the (figure) in which it is inscribed.

2. And, similarly, a (rectilinear) figure is said to be circumscribed about a(nother rectilinear) figure when the respective sides of the circumscribed (figure) touch the respective angles of the (figure) about which it is circumscribed.

3. A rectilinear figure is said to be inscribed in a circle when each angle of the inscribed (figure) touches the circumference of the circle.

4. And a rectilinear figure is said to be circumscribed about a circle when each side of the circumscribed (figure) touches the circumference of the circle.

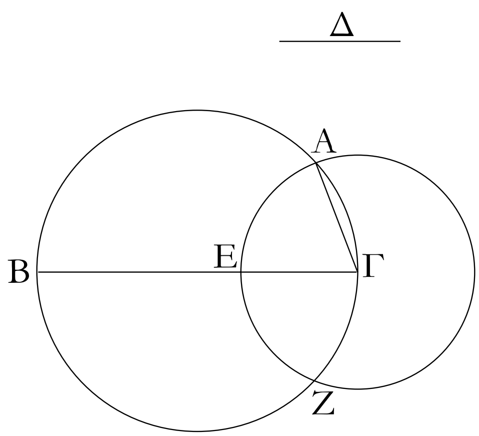
5. And, similarly, a circle is said to be inscribed in a (rectilinear) figure when the circumference of the circle touches each side of the (figure) in which it is inscribed.

6. And a circle is said to be circumscribed about a rectilinear (figure) when the circumference of the circle touches each angle of the (figure) about which it is circumscribed.

7. A straight-line is said to be inserted into a circle when its extremities are on the circumference of the circle.

α΄.

Εἰς τὸν δοθέντα κύκλον τῇ δοθείσῃ εὐθείᾳ μὴ μείζονι οὔσῃ τῆς τοῦ κύκλου διαμέτρου ἴσην εὐθεῖαν ἐναρμόσαι.

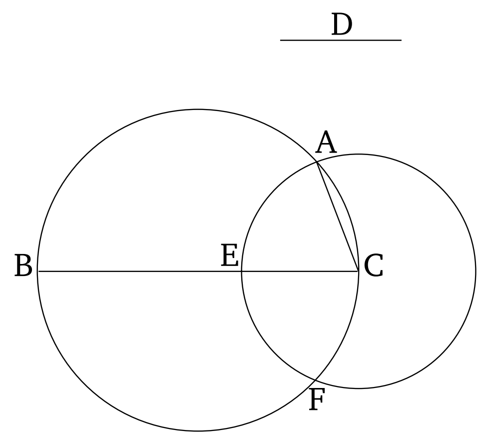


Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, ἡ δὲ δοθεῖσα εὐθεῖα μὴ μείζων τῆς τοῦ κύκλου διαμέτρου ἡ Δ. δεῖ δὴ εἰς τὸν ΑΒΓ κύκλον τῇ Δ εὐθείᾳ ἴσην εὐθεῖαν ἐναρμόσαι.

Ἤχθω τοῦ ΑΒΓ κύκλου διάμετρος ἡ ΒΓ. εἰ μὲν οὖν ἴση ἐστὶν ἡ ΒΓ τῇ Δ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· ἐνήρμοσται γὰρ εἰς τὸν ΑΒΓ κύκλον τῇ Δ εὐθείᾳ ἴση ἡ ΒΓ. εἰ δὲ μείζων ἐστὶν ἡ ΒΓ τῆς Δ, κείσθω τῇ Δ ἴση ἡ ΓΕ, καὶ κέντρῳ τῷ Γ δια-

Proposition 1

To insert a straight-line equal to a given straight-line into a circle, (the latter straight-line) not being greater than the diameter of the circle.



Let *ABC* be the given circle, and *D* the given straight-line (which is) not greater than the diameter of the circle. So it is required to insert a straight-line, equal to the straight-line *D*, into the circle *ABC*.

Let a diameter *BC* of circle *ABC* be drawn.[†] Therefore, if *BC* is equal to *D*, (then) that (which) was prescribed has taken place. For the (straight-line) *BC*, equal to the straight-line *D*, *στήματι δὲ τῷ ΓΕ κύκλος γεγράφθω ὁ ΕΑΖ, καὶ ἐπεζεύχθω ἡ ΓΑ.*

Ἐπεὶ οὖν το Γ σημεῖον κέντρον ἐστὶ τοῦ ΕΑΖ κύκλου, ἴση ἐστὶν ἡ ΓΑ τῇ ΓΕ. ἀλλὰ τῇ Δ ἡ ΓΕ ἐστιν ἴση· καὶ ἡ Δ ἄρα τῇ ΓΑ ἐστιν ἴση.

Εἰς ἄρα τὸν δοθέντα κύκλον τὸν ΑΒΓ τῇ δοθείσῃ εὐθείᾳ τῇ Δ ἴση ἐνήρμοσται ἡ ΓΑ· ὅπερ ἔδει ποιῆσαι.

has been inserted into the circle *ABC*. And if *BC* is greater than *D* then let *CE* be made equal to *D* [Prop. 1.3], and let the circle *EAF* be drawn with center *C* and radius *CE*. And let *CA* be joined.

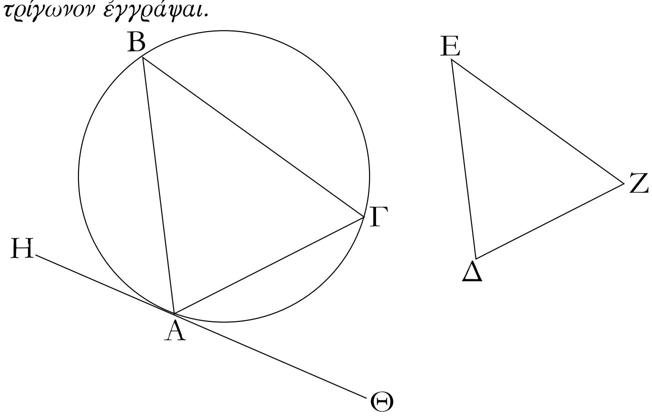
Therefore, since the point *C* is the center of circle *EAF*, *CA* is equal to *CE*. But, *CE* is equal to *D*. Thus, *D* is also equal to *CA*.

Thus, *CA*, equal to the given straight-line *D*, has been inserted into the given circle *ABC*. (Which is) the very thing it was required to do.

† Presumably, by finding the center of the circle [Prop. 3.1], and then drawing a line through it.

β′.

Εἰς τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον ἐγγράψαι.



Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, τὸ δὲ δοθὲν τριγωνον τὸ ΔΕΖ· δεῖ δὴ εἰς τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῳ ἰσογώνιον τρίγωνον ἐγγράψαι.

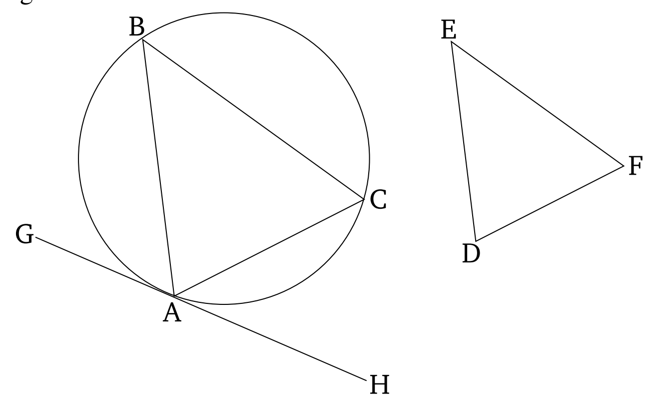
Ἤχθω τοῦ ΑΒΓ κύκλου ἐφαπτομένη ἡ ΗΘ κατὰ τὸ Α, καὶ συνεστάτω πρὸς τῇ ΑΘ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΔΕΖ γωνίᾳ ἴση ἡ ὑπὸ ΘΑΓ, πρὸς δὲ τῇ ΑΗ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΔΖΕ [γωνίᾳ] ἴση ἡ ὑπὸ ΗΑΒ, καὶ ἐπεζεύχθω ἡ ΒΓ.

Ἐπεὶ οὖν κύκλου τοῦ ΑΒΓ ἐφάπτεταί τις εὐθεῖα ἡ ΑΘ, καὶ ἀπὸ τῆς κατὰ τὸ Α ἐπαφῆς εἰς τὸν κύκλον διῆκται εὐθεῖα ἡ ΑΓ, ἡ ἄρα ὑπὸ ΘΑΓ ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ ΑΒΓ. ἀλλ᾿ ἡ ὑπὸ ΘΑΓ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση· καὶ ἡ ὑπὸ ΑΒΓ ἄρα γωνία τῇ ὑπὸ ΔΕΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΑΓ λοιπῇ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση [ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ, καὶ ἐγγέγραπται εἰς τὸν ΑΒΓ κύκλον].

Εἰς τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

Proposition 2

To inscribe a triangle, equiangular with a given triangle, in a given circle.



Let *ABC* be the given circle, and *DEF* the given triangle. So it is required to inscribe a triangle, equiangular with triangle *DEF*, in circle *ABC*.

Let *GH* be drawn touching circle *ABC* at *A*.† And let (angle) *HAC*, equal to angle *DEF*, be constructed on the straight-line *AH* at the point *A* on it, and (angle) *GAB*, equal to [angle] *DFE*, on the straight-line *AG* at the point *A* on it [Prop. 1.23]. And let *BC* be joined.

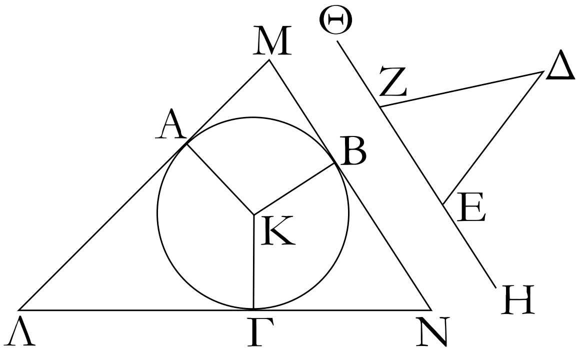
Therefore, since some straight-line *AH* touches the circle *ABC*, and the straight-line *AC* has been drawn across (the circle) from the point of contact *A*, (angle) *HAC* is thus equal to the angle *ABC* in the alternate segment of the circle [Prop. 3.32]. But, *HAC* is equal to *DEF*. Thus, angle *ABC* is also equal to *DEF*. So, for the same (reasons), *ACB* is also equal to *DFE*. Thus, the remaining (angle) *BAC* is equal to the remaining (angle) *EDF* [Prop. 1.32]. [Thus, triangle *ABC* is equiangular with triangle *DEF*, and has been inscribed in circle *ABC*].

Thus, a triangle, equiangular with the given triangle, has been inscribed in the given circle. (Which is) the very thing it was required to do.

† See the footnote to Prop. 3.34.

γ′.

Περὶ τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον περιγράψαι.



Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, τὸ δὲ δοθὲν τρίγωνον τὸ ΔΕΖ· δεῖ δὴ περὶ τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῳ ἰσογώνιον τρίγωνον περιγράψαι.

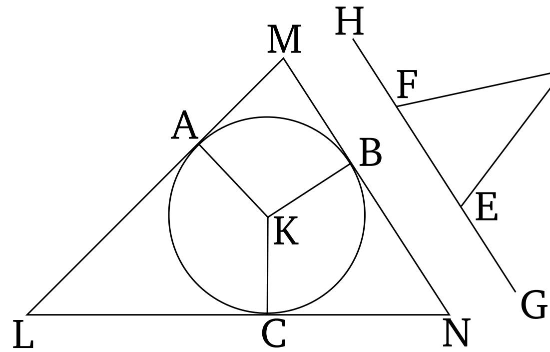
Ἐκβεβλήσθω ἡ ΕΖ ἐφ᾽ ἑκάτερα τὰ μέρη κατὰ τὰ Η, Θ σημεῖα, καὶ εἰλήφθω τοῦ ΑΒΓ κύκλου κέντρον τὸ Κ, καὶ διήχθω, ὡς ἔτυχεν, εὐθεῖα ἡ ΚΒ, καὶ συνεστάτω πρὸς τῇ ΚΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Κ τῇ μὲν ὑπὸ ΔΕΗ γωνίᾳ ἴση ἡ ὑπὸ ΒΚΑ, τῇ δὲ ὑπὸ ΔΖΘ ἴση ἡ ὑπὸ ΒΚΓ, καὶ διὰ τῶν Α, Β, Γ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ ΑΒΓ κύκλου αἱ ΛΑΜ, ΜΒΝ, ΝΓΛ.

Καὶ ἐπεὶ ἐφάπτονται τοῦ ΑΒΓ κύκλου αἱ ΛΜ, ΜΝ, ΝΛ κατὰ τὰ Α, Β, Γ σημεῖα, ἀπὸ δὲ τοῦ Κ κέντρου ἐπὶ τὰ Α, Β, Γ σημεῖα ἐπεζευγμέναι εἰσὶν αἱ ΚΑ, ΚΒ, ΚΓ, ὀρθαὶ ἄρα εἰσὶν αἱ πρὸς τοῖς Α, Β, Γ σημείοις γωνίαι. καὶ ἐπεὶ τοῦ ΑΜΒΚ τετραπλεύρου αἱ τέσσαρες γωνίαι τέτρασιν ὀρθαῖς ἴσαι εἰσίν, ἐπειδήπερ καὶ εἰς δύο τρίγωνα διαιρεῖται τὸ ΑΜΒΚ, καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΚΑΜ, ΚΒΜ γωνίαι, λοιπαὶ ἄρα αἱ ὑπὸ ΑΚΒ, ΑΜΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΕΗ, ΔΕΖ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΑΚΒ, ΑΜΒ ταῖς ὑπὸ ΔΕΗ, ΔΕΖ ἴσαι εἰσίν, ὧν ἡ ὑπὸ ΑΚΒ τῇ ὑπὸ ΔΕΗ ἐστιν ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΑΜΒ λοιπῇ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἡ ὑπὸ ΛΝΒ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΜΛΝ [λοιπῇ] τῇ ὑπὸ ΕΔΖ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΛΜΝ τρίγωνον τῷ ΔΕΖ τριγώνῳ· καὶ περιγέγραπται περὶ τὸν ΑΒΓ κύκλον.

Περὶ τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

Proposition 3

To circumscribe a triangle, equiangular with a given triangle, about a given circle.



Let ABC be the given circle, and DEF the given triangle. So it is required to circumscribe a triangle, equiangular with triangle DEF, about circle ABC.

Let EF be produced in each direction to points G and H. And let the center K of circle ABC be found [Prop. 3.1]. And let the straight-line KB be drawn, at random, across (ABC). And let (angle) BKA, equal to angle DEG, be constructed on the straight-line KB at the point K on it, and (angle) BKC, equal to DFH [Prop. 1.23]. And let the (straight-lines) LAM, MBN, and NCL be drawn through the points A, B, and C (respectively), touching the circle ABC.†

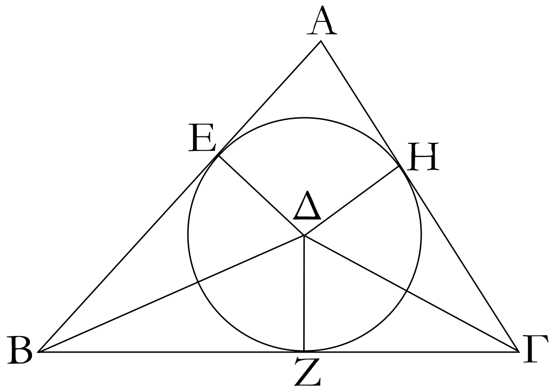
And since LM, MN, and NL touch circle ABC at points A, B, and C (respectively), and KA, KB, and KC are joined from the center K to points A, B, and C (respectively), the angles at points A, B, and C are thus right-angles [Prop. 3.18]. And since the (sum of the) four angles of quadrilateral $AMBK$ is equal to four right-angles, inasmuch as $AMBK$ (can) also (be) divided into two triangles [Prop. 1.32], and angles KAM and KBM are (both) right-angles, the (sum of the) remaining (angles), AKB and AMB, is thus equal to two right-angles. And DEG and DEF is also equal to two right-angles [Prop. 1.13]. Thus, AKB and AMB is equal to DEG and DEF, of which AKB is equal to DEG. Thus, the remainder AMB is equal to the remainder DEF. So, similarly, it can be shown that LNB is also equal to DFE. Thus, the remaining (angle) MLN is also equal to the [remaining] (angle) EDF [Prop. 1.32]. Thus, triangle LMN is equiangular with triangle DEF. And it has been drawn around circle ABC.

Thus, a triangle, equiangular with the given triangle, has been circumscribed about the given circle. (Which is) the very thing it was required to do.

† See the footnote to Prop. 3.34.

δ΄.

Εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι.



Ἔστω τὸ δοθὲν τρίγωνον τὸ ΑΒΓ· δεῖ δὴ εἰς τὸ ΑΒΓ τρίγωνον κύκλον ἐγγράψαι.

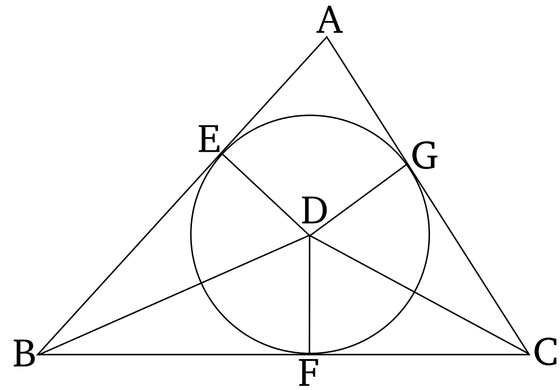
Τετμήσθωσαν αἱ ὑπὸ ΑΒΓ, ΑΓΒ γωνίαι δίχα ταῖς ΒΔ, ΓΔ εὐθείαις, καὶ συμβαλλέτωσαν ἀλλήλαις κατὰ τὸ Δ σημεῖον, καὶ ἤχθωσαν ἀπὸ τοῦ Δ ἐπὶ τὰς ΑΒ, ΒΓ, ΓΑ εὐθείας κάθετοι αἱ ΔΕ, ΔΖ, ΔΗ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΒΔ γωνία τῇ ὑπὸ ΓΒΔ, ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΒΕΔ ὀρθῇ τῇ ὑπὸ ΒΖΔ ἴση, δύο δὴ τρίγωνά ἐστι τὰ ΕΒΔ, ΖΒΔ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν κοινὴν αὐτῶν τὴν ΒΔ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν· ἴση ἄρα ἡ ΔΕ τῇ ΔΖ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΔΗ τῇ ΔΖ ἐστιν ἴση. αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΔΕ, ΔΖ, ΔΗ ἴσαι ἀλλήλαις εἰσίν· ὁ ἄρα κέντρῳ τῷ Δ καὶ διαστήματι ἑνὶ τῶν Ε, Ζ, Η κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἐφάψεται τῶν ΑΒ, ΒΓ, ΓΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Ε, Ζ, Η σημείοις γωνίας. εἰ γὰρ τεμεῖ αὐτάς, ἔσται ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένη ἐντὸς πίπτουσα τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη· οὐκ ἄρα ὁ κέντρῳ τῷ Δ διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Η γραφόμενος κύκλος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΑ εὐθείας· ἐφάψεται ἄρα αὐτῶν, καὶ ἔσται ὁ κύκλος ἐγγεγραμμένος εἰς τὸ ΑΒΓ τρίγωνον. ἐγγεγράφθω ὡς ὁ ΖΗΕ.

Εἰς ἄρα τὸ δοθὲν τρίγωνον τὸ ΑΒΓ κύκλος ἐγγέγραπται ὁ ΕΖΗ· ὅπερ ἔδει ποιῆσαι.

Proposition 4

To inscribe a circle in a given triangle.



Let *ABC* be the given triangle. So it is required to inscribe a circle in triangle *ABC*.

Let the angles *ABC* and *ACB* be cut in half by the straight-lines *BD* and *CD* (respectively) [Prop. 1.9], and let them meet one another at point *D*, and let *DE*, *DF*, and *DG* be drawn from point *D*, perpendicular to the straight-lines *AB*, *BC*, and *CA* (respectively) [Prop. 1.12].

And since angle *ABD* is equal to *CBD*, and the right-angle *BED* is also equal to the right-angle *BFD*, *EBD* and *FBD* are thus two triangles having two angles equal to two angles, and one side equal to one side—the (one) subtending one of the equal angles (which is) common to the (triangles)—(namely), *BD*. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, *DE* (is) equal to *DF*. So, for the same (reasons), *DG* is also equal to *DF*. Thus, the three straight-lines *DE*, *DF*, and *DG* are equal to one another. Thus, the circle drawn with center *D*, and radius one of *E*, *F*, or *G*,† will also go through the remaining points, and will touch the straight-lines *AB*, *BC*, and *CA*, on account of the angles at *E*, *F*, and *G* being right-angles. For if it cuts (one of) them then it will be a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, falling inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center *D*, and radius one of *E*, *F*, or *G*, does not cut the straight-lines *AB*, *BC*, and *CA*. Thus, it will touch them, and will be the circle inscribed in triangle *ABC*. Let it be (so) inscribed, like *FGE* (in the figure).

Thus, the circle *EFG* has been inscribed in the given triangle *ABC*. (Which is) the very thing it was required to do.

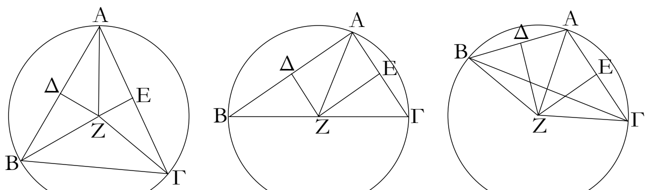
† Here, and in the following propositions, it is understood that the radius is actually one of *DE*, *DF*, or *DG*.

ε΄.

Περὶ τὸ δοθὲν τρίγωνον κύκλον περιγράψαι.

Proposition 5

To circumscribe a circle about a given triangle.



Ἔστω τὸ δοθὲν τρίγωνον τὸ ΑΒΓ· δεῖ δὲ περὶ τὸ δοθὲν τρίγωνον τὸ ΑΒΓ κύκλον περιγράψαι.

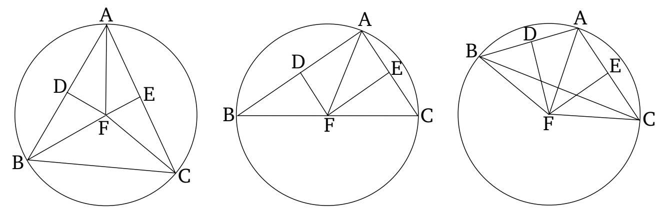
Τετμήσθωσαν αἱ ΑΒ, ΑΓ εὐθεῖαι δίχα κατὰ τὰ Δ, Ε σημεῖα, καὶ ἀπὸ τῶν Δ, Ε σημείων ταῖς ΑΒ, ΑΓ πρὸς ὀρθὰς ἤχθωσαν αἱ ΔΖ, ΕΖ· συμπεσοῦνται δὴ ἤτοι ἐντὸς τοῦ ΑΒΓ τριγώνου ἢ ἐπὶ τῆς ΒΓ εὐθείας ἢ ἐκτὸς τῆς ΒΓ.

Συμπιπτέτωσαν πρότερον ἐντὸς κατὰ τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΓ, ΖΑ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔΖ, βάσις ἄρα ἡ ΑΖ βάσει τῇ ΖΒ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΓΖ τῇ ΑΖ ἐστιν ἴση· ὥστε καὶ ἡ ΖΒ τῇ ΖΓ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΖΑ, ΖΒ, ΖΓ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν Α, Β, Γ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος ὁ κύκλος περὶ τὸ ΑΒΓ τρίγωνον. περιγεγράφθω ὡς ὁ ΑΒΓ.

Ἀλλὰ δὴ αἱ ΔΖ, ΕΖ συμπιπτέτωσαν ἐπὶ τῆς ΒΓ εὐθείας κατὰ τὸ Ζ, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ ἐπεζεύχθω ἡ ΑΖ. ὁμοίως δὴ δείξομεν, ὅτι τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ περὶ τὸ ΑΒΓ τρίγωνον περιγραφομένου κύκλου.

Ἀλλὰ δὴ αἱ ΔΖ, ΕΖ συμπιπτέτωσαν ἐκτὸς τοῦ ΑΒΓ τριγώνου κατὰ τὸ Ζ πάλιν, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΒΖ, ΓΖ. καὶ ἐπεὶ πάλιν ἴση ἐστὶν ἡ ΑΔ τῇ ΔΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔΖ, βάσις ἄρα ἡ ΑΖ βάσει τῇ ΒΖ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΓΖ τῇ ΑΖ ἐστιν ἴση· ὥστε καὶ ἡ ΒΖ τῇ ΖΓ ἐστιν ἴση· ὁ ἄρα [πάλιν] κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν ΖΑ, ΖΒ, ΖΓ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος περὶ τὸ ΑΒΓ τρίγωνον.

Περὶ τὸ δοθὲν ἄρα τρίγωνον κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.



Let *ABC* be the given triangle. So it is required to circumscribe a circle about the given triangle *ABC*.

Let the straight-lines *AB* and *AC* be cut in half at points *D* and *E* (respectively) [Prop. 1.10]. And let *DF* and *EF* be drawn from points *D* and *E*, at right-angles to *AB* and *AC* (respectively) [Prop. 1.11]. So (*DF* and *EF*) will surely either meet inside triangle *ABC*, on the straight-line *BC*, or beyond *BC*.

Let them, first of all, meet inside (triangle *ABC*) at (point) *F*, and let *FB*, *FC*, and *FA* be joined. And since *AD* is equal to *DB*, and *DF* is common and at right-angles, the base *AF* is thus equal to the base *FB* [Prop. 1.4]. So, similarly, we can show that *CF* is also equal to *AF*. So that *FB* is also equal to *FC*. Thus, the three (straight-lines) *FA*, *FB*, and *FC* are equal to one another. Thus, the circle drawn with center *F*, and radius one of *A*, *B*, or *C*, will also go through the remaining points. And the circle will be circumscribed about triangle *ABC*. Let it be (so) circumscribed, like *ABC* (in the first diagram from the left).

And so, let *DF* and *EF* meet on the straight-line *BC* at (point) *F*, like in the second diagram (from the left). And let *AF* be joined. So, similarly, we can show that point *F* is the center of the circle circumscribed about triangle *ABC*.

And so, let *DF* and *EF* meet outside triangle *ABC*, again at (point) *F*, like in the third diagram (from the left). And let *AF*, *BF*, and *CF* be joined. And, again, since *AD* is equal to *DB*, and *DF* is common and at right-angles, the base *AF* is thus equal to the base *BF* [Prop. 1.4]. So, similarly, we can show that *CF* is also equal to *AF*. So that *BF* is also equal to *FC*. Thus, [again] the circle drawn with center *F*, and radius one of *FA*, *FB*, and *FC*, will also go through the remaining points. And it will be circumscribed about triangle *ABC*.

Thus, a circle has been circumscribed about the given triangle. (Which is) the very thing it was required to do.

ϛʹ.

Εἰς τὸν δοθέντα κύκλον τετράγωνον ἐγγράψαι.

Ἔστω ἡ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον ἐγγράψαι.

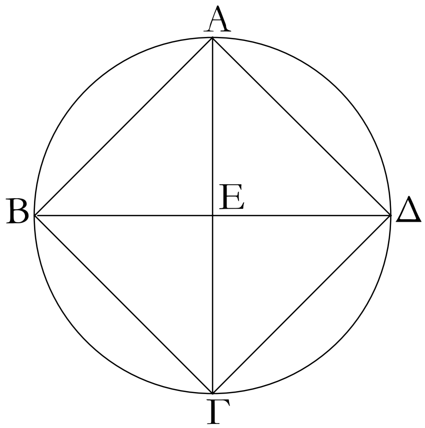
Ἤχθωσαν τοῦ ΑΒΓΔ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΑΓ, ΒΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ.

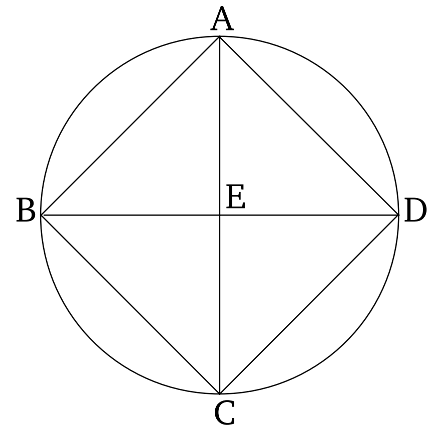
Proposition 6

To inscribe a square in a given circle.

Let *ABCD* be the given circle. So it is required to inscribe a square in circle *ABCD*.

Let two diameters of circle *ABCD*, *AC* and *BD*, be drawn at right-angles to one another.† And let *AB*, *BC*, *CD*, and *DA* be joined.





Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΔ· κέντρον γὰρ τὸ Ε· κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΑ, βάσις ἄρα ἡ ΑΒ βάσει τῇ ΑΔ ἴση ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ΒΓ, ΓΔ ἑκατέρᾳ τῶν ΑΒ, ΑΔ ἴση ἐστίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔ τετράπλευρον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ ἡ ΒΔ εὐθεῖα διάμετρός ἐστι τοῦ ΑΒΓΔ κύκλου, ἡμικύκλιον ἄρα ἐστὶ τὸ ΒΑΔ· ὀρθὴ ἄρα ἡ ὑπὸ ΒΑΔ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΑ ὀρθή ἐστιν· ὀρθογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔ τετράπλευρον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν. καὶ ἐγγέγραπται εἰς τὸν ΑΒΓΔ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον τετράγωνον ἐγγέγραπται τὸ ΑΒΓΔ· ὅπερ ἔδει ποιῆσαι.

And since *BE* is equal to *ED*, for *E* (is) the center (of the circle), and *EA* is common and at right-angles, the base *AB* is thus equal to the base *AD* [Prop. 1.4]. So, for the same (reasons), each of *BC* and *CD* is equal to each of *AB* and *AD*. Thus, the quadrilateral *ABCD* is equilateral. So I say that (it is) also right-angled. For since the straight-line *BD* is a diameter of circle *ABCD*, *BAD* is thus a semi-circle. Thus, angle *BAD* (is) a right-angle [Prop. 3.31]. So, for the same (reasons), (angles) *ABC*, *BCD*, and *CDA* are also each right-angles. Thus, the quadrilateral *ABCD* is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been inscribed in circle *ABCD*.

Thus, the square *ABCD* has been inscribed in the given circle. (Which is) the very thing it was required to do.

† Presumably, by finding the center of the circle [Prop. 3.1], drawing a line through it, and then drawing a second line through it, at right-angles to the first [Prop. 1.11].

ζʹ.

Περὶ τὸν δοθέντα κύκλον τετράγωνον περιγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ περὶ τὸν ΑΒΓΔ κύκλον τετράγωνον περιγράψαι.

Ἤχθωσαν τοῦ ΑΒΓΔ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΑΓ, ΒΔ, καὶ διὰ τῶν Α, Β, Γ, Δ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ ΑΒΓΔ κύκλου αἱ ΖΗ, ΗΘ, ΘΚ, ΚΖ.

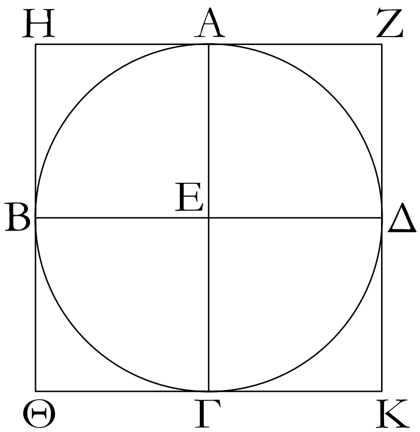
Ἐπεὶ οὖν ἐφάπτεται ἡ ΖΗ τοῦ ΑΒΓΔ κύκλου, ἀπὸ δὲ τοῦ Ε κέντρου ἐπὶ τὴν κατὰ τὸ Α ἐπαφὴν ἐπέζευκται ἡ ΕΑ, αἱ ἄρα πρὸς τῷ Α γωνίαι ὀρθαί εἰσιν. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τοῖς Β, Γ, Δ σημείοις γωνίαι ὀρθαί εἰσιν. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΑΕΒ γωνία, ἐστὶ δὲ ὀρθὴ καὶ ἡ ὑπὸ ΕΒΗ, παράλληλος ἄρα ἐστὶν ἡ ΗΘ τῇ ΑΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΑΓ τῇ ΖΚ ἐστι παράλληλος. ὥστε καὶ ἡ ΗΘ τῇ ΖΚ ἐστι παράλληλος. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΗΖ, ΘΚ τῇ ΒΕΔ ἐστι παράλληλος. παραλληλόγραμμα ἄρα ἐστὶ τὰ ΗΚ, ΗΓ, ΑΚ, ΖΒ, ΒΚ· ἴση ἄρα ἐστὶν ἡ μὲν ΗΖ τῇ ΘΚ, ἡ δὲ ΗΘ τῇ ΖΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΒΔ, ἀλλὰ καὶ ἡ μὲν ΑΓ ἑκατέρᾳ τῶν ΗΘ, ΖΚ, ἡ δὲ ΒΔ ἑκατέρᾳ τῶν ΗΖ, ΘΚ

Proposition 7

To circumscribe a square about a given circle.

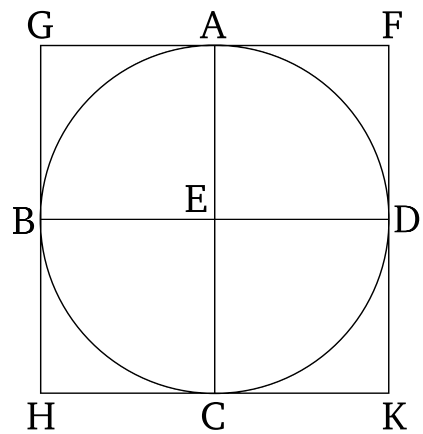
Let *ABCD* be the given circle. So it is required to circumscribe a square about circle *ABCD*. Let two diameters of circle *ABCD*, *AC* and *BD*, be drawn at right-angles to one another.† And let *FG*, *GH*, *HK*, and *KF* be drawn through points *A*, *B*, *C*, and *D* (respectively), touching circle *ABCD*.‡

Therefore, since *FG* touches circle *ABCD*, and *EA* has been joined from the center *E* to the point of contact *A*, the angles at *A* are thus right-angles [Prop. 3.18]. So, for the same (reasons), the angles at points *B*, *C*, and *D* are also right-angles. And since angle *AEB* is a right-angle, and *EBG* is also a right-angle, *GH* is thus parallel to *AC* [Prop. 1.29]. So, for the same (reasons), *AC* is also parallel to *FK*. So that *GH* is also parallel to *FK* [Prop. 1.30]. So, similarly, we can show that *GF* and *HK* are each parallel to *BED*. Thus, *GK*, *GC*, *AK*, *FB*, and *BK* are (all) parallelograms. Thus, *GF* is equal to *HK*, and *GH* to *FK* [Prop. 1.34]. And since *AC* is equal to *BD*, but *AC* (is) also (equal) to each of *GH* and *FK*, and *BD* *ἐστιν ἴση [καὶ ἑκατέρα ἄρα τῶν ΗΘ, ΖΚ ἑκατέρᾳ τῶν ΗΖ, ΘΚ ἐστιν ἴση], ἰσόπλευρον ἄρα ἐστὶ τὸ ΖΗΘΚ τετράπλευρον. λέγω δή, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παραλληλόγραμμόν ἐστι τὸ ΗΒΕΑ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΑΕΒ, ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΑΗΒ. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ πρὸς τοῖς Θ, Κ, Ζ γωνίαι ὀρθαί εἰσιν. ὀρθογώνιον ἄρα ἐστὶ τὸ ΖΗΘΚ. ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν. καὶ περιγέγραπται περὶ τὸν ΑΒΓΔ κύκλον.*



Περὶ τὸν δοθέντα ἄρα κύκλον τετράγωνον περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

is equal to each of GF and HK [Prop. 1.34] [and each of GH and FK is thus equal to each of GF and HK], the quadrilateral $FGHK$ is thus equilateral. So I say that (it is) also right-angled. For since $GBEA$ is a parallelogram, and AEB is a right-angle, AGB is thus also a right-angle [Prop. 1.34]. So, similarly, we can show that the angles at H, K, and F are also right-angles. Thus, $FGHK$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been circumscribed about circle $ABCD$.



Thus, a square has been circumscribed about the given circle. (Which is) the very thing it was required to do.

† See the footnote to the previous proposition.

‡ See the footnote to Prop. 3.34.

ηʹ.

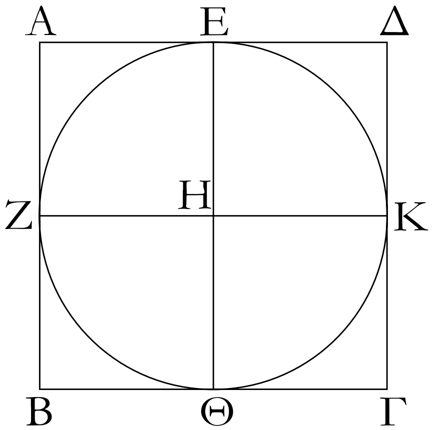
Εἰς τὸ δοθὲν τετράγωνον κύκλον ἐγγράψαι.

Ἔστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ· δεῖ δὴ εἰς τὸ ΑΒΓΔ τετράγωνον κύκλον ἐγγράψαι.

Proposition 8

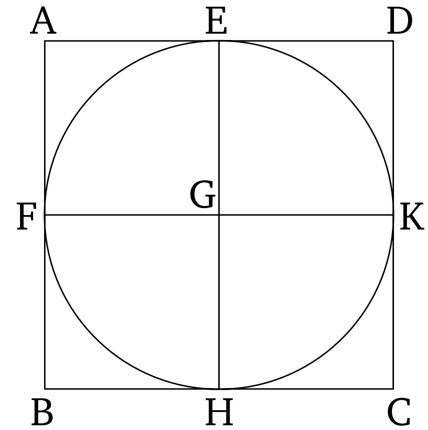
To inscribe a circle in a given square.

Let the given square be $ABCD$. So it is required to inscribe a circle in square $ABCD$.



Τετμήσθω ἑκατέρα τῶν ΑΔ, ΑΒ δίχα κατὰ τὰ Ε, Ζ σημεῖα, καὶ διὰ μὲν τοῦ Ε ὁποτέρᾳ τῶν ΑΒ, ΓΔ παράλληλος ἤχθω ὁ ΕΘ, διὰ δὲ τοῦ Ζ ὁποτέρᾳ τῶν ΑΔ, ΒΓ παράλληλος ἤχθω ἡ ΖΚ· παραλληλόγραμμον ἄρα ἐστὶν ἕκαστον τῶν ΑΚ, ΚΒ, ΑΘ, ΘΔ, ΑΗ, ΗΓ, ΒΗ, ΗΔ, καὶ αἱ ἀπεναντίον αὐτῶν πλευραὶ δηλονότι ἴσαι [εἰσίν]. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΑΒ, καί ἐστι τῆς μὲν ΑΔ ἡμίσεια ἡ ΑΕ, τῆς δὲ ΑΒ ἡμίσεια ἡ ΑΖ, ἴση ἄρα καὶ ἡ ΑΕ τῇ ΑΖ· ὥστε καὶ αἱ ἀπεναντίον· ἴση ἄρα καὶ ἡ ΖΗ τῇ ΗΕ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΗΘ, ΗΚ ἑκατέρᾳ τῶν ΖΗ, ΗΕ ἐστιν ἴση· αἱ τέσσαρες ἄρα αἱ ΗΕ, ΗΖ, ΗΘ, ΗΚ ἴσαι ἀλλήλαις [εἰσίν]. ὁ ἄρα κέντρῳ μὲν τῷ Η διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων· καὶ ἐφάψεται τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Ε, Ζ, Θ, Κ γωνίας· εἰ γὰρ τεμεῖ ὁ κύκλος τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ, ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾿ ἄκρας ἀγομένη ἐντὸς πεσεῖται τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ὁ κέντρῳ τῷ Η διαστήματι δὲ ἑνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ εὐθείας. ἐφάψεται ἄρα αὐτῶν καὶ ἔσται ἐγγεγραμμένος εἰς τὸ ΑΒΓΔ τετράγωνον.

Εἰς ἄρα τὸ δοθὲν τετράγωνον κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.



Let AD and AB each be cut in half at points E and F (respectively) [Prop. 1.10]. And let EH be drawn through E, parallel to either of AB or CD, and let FK be drawn through F, parallel to either of AD or BC [Prop. 1.31]. Thus, AK, KB, AH, HD, AG, GC, BG, and GD are each parallelograms, and their opposite sides [are] manifestly equal [Prop. 1.34]. And since AD is equal to AB, and AE is half of AD, and AF half of AB, AE (is) thus also equal to AF. So that the opposite (sides are) also (equal). Thus, FG (is) also equal to GE. So, similarly, we can also show that each of GH and GK is equal to each of FG and GE. Thus, the four (straight-lines) GE, GF, GH, and GK [are] equal to one another. Thus, the circle drawn with center G, and radius one of E, F, H, or K, will also go through the remaining points. And it will touch the straight-lines AB, BC, CD, and DA, on account of the angles at E, F, H, and K being right-angles. For if the circle cuts AB, BC, CD, or DA, (then) a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, will fall inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center G, and radius one of E, F, H, or K, does not cut the straight-lines AB, BC, CD, or DA. Thus, it will touch them, and will be inscribed in the square $ABCD$.

Thus, a circle has been inscribed in the given square. (Which is) the very thing it was required to do.

ϑʹ.

Περὶ τὸ δοθὲν τετράγωνον κύκλον περιγράψαι.

Ἔστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ· δεῖ δὴ περὶ τὸ ΑΒΓΔ τετράγωνον κύκλον περιγράψαι.

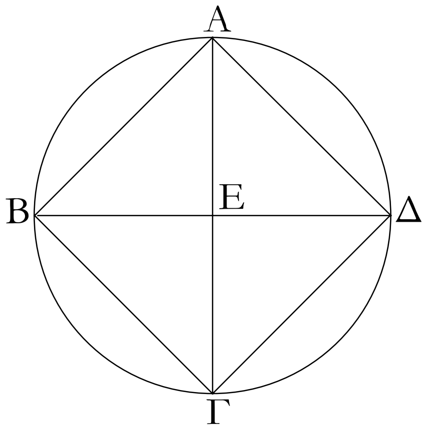
Ἐπιζευχθεῖσαι γὰρ αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε.

Proposition 9

To circumscribe a circle about a given square.

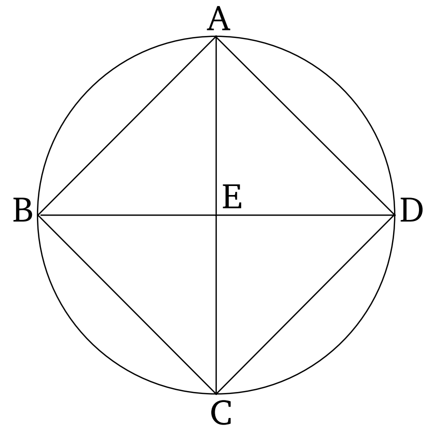
Let $ABCD$ be the given square. So it is required to circumscribe a circle about square $ABCD$.

AC and BD being joined, let them cut one another at E.



Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δὴ αἱ ΔΑ, ΑΓ δυσὶ ταῖς ΒΑ, ΑΓ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΓ βάσει τῇ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνίᾳ τῇ ὑπὸ ΒΑΓ ἴση ἐστίν· ἡ ἄρα ὑπὸ ΔΑΒ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΑ δίχα τέτμηται ὑπὸ τῶν ΑΓ, ΔΒ εὐθειῶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΑΒ γωνία τῇ ὑπὸ ΑΒΓ, καί ἐστι τῆς μὲν ὑπὸ ΔΑΒ ἡμίσεια ἡ ὑπὸ ΕΑΒ, τῆς δὲ ὑπὸ ΑΒΓ ἡμίσεια ἡ ὑπὸ ΕΒΑ, καὶ ἡ ὑπὸ ΕΑΒ ἄρα τῇ ὑπὸ ΕΒΑ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΕΑ τῇ ΕΒ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν ΕΑ, ΕΒ [εὐθειῶν] ἑκατέρᾳ τῶν ΕΓ, ΕΔ ἴση ἐστίν. αἱ τέσσαρες ἄρα αἱ ΕΑ, ΕΒ, ΕΓ, ΕΔ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ε καὶ διαστήματι ἑνὶ τῶν Α, Β, Γ, Δ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος περὶ τὸ ΑΒΓΔ τετράγωνον. περιγεγράφθω ὡς ὁ ΑΒΓΔ.

Περὶ τὸ δοθὲν ἄρα τετράγωνον κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

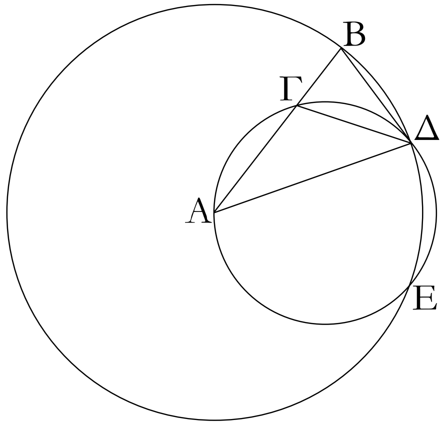


And since *DA* is equal to *AB*, and *AC* (is) common, the two (straight-lines) *DA*, *AC* are thus equal to the two (straight-lines) *BA*, *AC*. And the base *DC* (is) equal to the base *BC*. Thus, angle *DAC* is equal to angle *BAC* [Prop. 1.8]. Thus, the angle *DAB* has been cut in half by *AC*. So, similarly, we can show that *ABC*, *BCD*, and *CDA* have each been cut in half by the straight-lines *AC* and *DB*. And since angle *DAB* is equal to *ABC*, and *EAB* is half of *DAB*, and *EBA* half of *ABC*, *EAB* is thus also equal to *EBA*. So that side *EA* is also equal to *EB* [Prop. 1.6]. So, similarly, we can show that each of the [straight-lines] *EA* and *EB* are also equal to each of *EC* and *ED*. Thus, the four (straight-lines) *EA*, *EB*, *EC*, and *ED* are equal to one another. Thus, the circle drawn with center *E*, and radius one of *A*, *B*, *C*, or *D*, will also go through the remaining points, and will be circumscribed about the square *ABCD*. Let it be (so) circumscribed, like *ABCD* (in the figure).

Thus, a circle has been circumscribed about the given square. (Which is) the very thing it was required to do.

ι'.

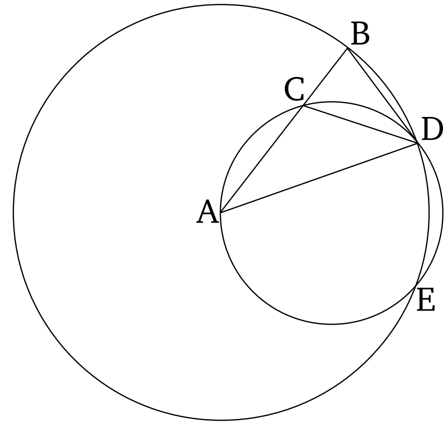
Ἰσοσκελὲς τρίγωνον συστήσασθαι ἔχον ἑκατέραν τῶν πρὸς τῇ βάσει γωνιῶν διπλασίονα τῆς λοιπῆς.



Ἐκκείσθω τις εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ

Proposition 10

To construct an isosceles triangle having each of the angles at the base double the remaining (angle).



Let some straight-line *AB* be taken, and let it be cut at point *σημεῖον, ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τῆς ΓΑ τετραγώνῳ· καὶ κέντρῳ τῷ Α καὶ διαστήματι τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΔΕ, καὶ ἐνηρμόσθω εἰς τὸν ΒΔΕ κύκλον τῇ ΑΓ εὐθείᾳ μὴ μείζονι οὔσῃ τῆς τοῦ ΒΔΕ κύκλου διαμέτρου ἴση εὐθεῖα ἡ ΒΔ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, καὶ περιγεγράφθω περὶ τὸ ΑΓΔ τρίγωνον κύκλος ὁ ΑΓΔ.*

Καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ, ἴση δὲ ἡ ΑΓ τῇ ΒΔ, τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΔ. καὶ ἐπεὶ κύκλου τοῦ ΑΓΔ εἴληπταί τι σημεῖον ἐκτὸς τὸ Β, καὶ ἀπὸ τοῦ Β πρὸς τὸν ΑΓΔ κύκλον προσπεπτώκασι δύο εὐθεῖαι αἱ ΒΑ, ΒΔ, καὶ ἡ μὲν αὐτῶν τέμνει, ἡ δὲ προσπίπτει, καί ἐστι τὸ ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τῷ ἀπὸ τῆς ΒΔ, ἡ ΒΔ ἄρα ἐφάπτεται τοῦ ΑΓΔ κύκλου. ἐπεὶ οὖν ἐφάπτεται μὲν ἡ ΒΔ, ἀπὸ δὲ τῆς κατὰ τὸ Δ ἐπαφῆς διῆκται ἡ ΔΓ, ἡ ἄρα ὑπὸ ΒΔΓ γωνιά ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ ΔΑΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΒΔΓ τῇ ὑπὸ ΔΑΓ, κοινὴ προσκείσθω ἡ ὑπὸ ΓΔΑ· ὅλη ἄρα ἡ ὑπὸ ΒΔΑ ἴση ἐστὶ δυσὶ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ. ἀλλὰ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ ἴση ἐστὶν ἡ ἐκτὸς ἡ ὑπὸ ΒΓΔ· καὶ ἡ ὑπὸ ΒΔΑ ἄρα ἴση ἐστὶ τῇ ὑπὸ ΒΓΔ. ἀλλὰ ἡ ὑπὸ ΒΔΑ τῇ ὑπὸ ΓΒΔ ἐστιν ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΑΔ τῇ ΑΒ ἐστιν ἴση· ὥστε καὶ ἡ ὑπὸ ΔΒΑ τῇ ὑπὸ ΒΓΔ ἐστιν ἴση. αἱ τρεῖς ἄρα αἱ ὑπὸ ΒΔΑ, ΔΒΑ, ΒΓΑ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῇ ὑπὸ ΒΓΔ, ἴση ἐστὶ καὶ πλευρὰ ἡ ΒΔ πλευρᾷ τῇ ΔΓ. ἀλλὰ ἡ ΒΔ τῇ ΓΑ ὑπόκειται ἴση· καὶ ἡ ΓΑ ἄρα τῇ ΓΔ ἐστιν ἴση· ὥστε καὶ γωνία ἡ ὑπὸ ΓΔΑ γωνίᾳ τῇ ὑπὸ ΔΑΓ ἐστιν ἴση· αἱ ἄρα ὑπὸ ΓΔΑ, ΔΑΓ τῆς ὑπὸ ΔΑΓ εἰσι διπλασίους. ἴση δὲ ἡ ὑπὸ ΒΓΔ ταῖς ὑπὸ ΓΔΑ, ΔΑΓ· καὶ ἡ ὑπὸ ΒΓΔ ἄρα τῆς ὑπὸ ΓΑΔ ἐστι διπλῆ. ἴση δὲ ἡ ὑπὸ ΒΓΔ ἑκατέρᾳ τῶν ὑπὸ ΒΔΑ, ΔΒΑ· καὶ ἑκατέρα ἄρα τῶν ὑπὸ ΒΔΑ, ΔΒΑ τῆς ὑπὸ ΔΑΒ ἐστι διπλῆ.

Ἰσοσκελὲς ἄρα τρίγωνον συνέσταται τὸ ΑΒΔ ἔχον ἑκατέραν τῶν πρὸς τῇ ΔΒ βάσει γωνιῶν διπλασίονα τῆς λοιπῆς· ὅπερ ἔδει ποιῆσαι.

C so that the rectangle contained by *AB* and *BC* is equal to the square on *CA* [Prop. 2.11]. And let the circle *BDE* be drawn with center *A*, and radius *AB*. And let the straight-line *BD*, equal to the straight-line *AC*, being not greater than the diameter of circle *BDE*, be inserted into circle *BDE* [Prop. 4.1]. And let *AD* and *DC* be joined. And let the circle *ACD* be circumscribed about triangle *ACD* [Prop. 4.5].

And since the (rectangle contained) by *AB* and *BC* is equal to the (square) on *AC*, and *AC* (is) equal to *BD*, the (rectangle contained) by *AB* and *BC* is thus equal to the (square) on *BD*. And since some point *B* has been taken outside of circle *ACD*, and two straight-lines *BA* and *BD* have radiated from *B* towards the circle *ACD*, and (one) of them cuts (the circle), and (the other) meets (the circle), and the (rectangle contained) by *AB* and *BC* is equal to the (square) on *BD*, *BD* thus touches circle *ACD* [Prop. 3.37]. Therefore, since *BD* touches (the circle), and *DC* has been drawn across (the circle) from the point of contact *D*, the angle *BDC* is thus equal to the angle *DAC* in the alternate segment of the circle [Prop. 3.32]. Therefore, since *BDC* is equal to *DAC*, let *CDA* be added to both. Thus, the whole of *BDA* is equal to the two (angles) *CDA* and *DAC*. But, the external (angle) *BCD* is equal to *CDA* and *DAC* [Prop. 1.32]. Thus, *BDA* is also equal to *BCD*. But, *BDA* is equal to *CBD*, since the side *AD* is also equal to *AB* [Prop. 1.5]. So that *DBA* is also equal to *BCD*. Thus, the three (angles) *BDA*, *DBA*, and *BCD* are equal to one another. And since angle *DBC* is equal to *BCD*, side *BD* is also equal to side *DC* [Prop. 1.6]. But, *BD* was assumed (to be) equal to *CA*. Thus, *CA* is also equal to *CD*. So that angle *CDA* is also equal to angle *DAC* [Prop. 1.5]. Thus, *CDA* and *DAC* is double *DAC*. But *BCD* (is) equal to *CDA* and *DAC*. Thus, *BCD* is also double *CAD*. And *BCD* (is) equal to to each of *BDA* and *DBA*. Thus, *BDA* and *DBA* are each double *DAB*.

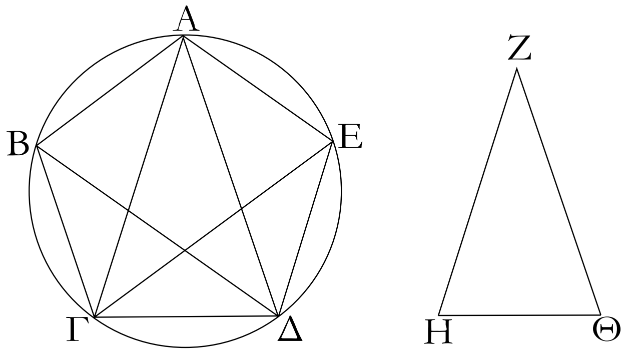
Thus, the isosceles triangle *ABD* has been constructed having each of the angles at the base *BD* double the remaining (angle). (Which is) the very thing it was required to do.

ια′.

Εἰς τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Proposition 11

To inscribe an equilateral and equiangular pentagon in a given circle.

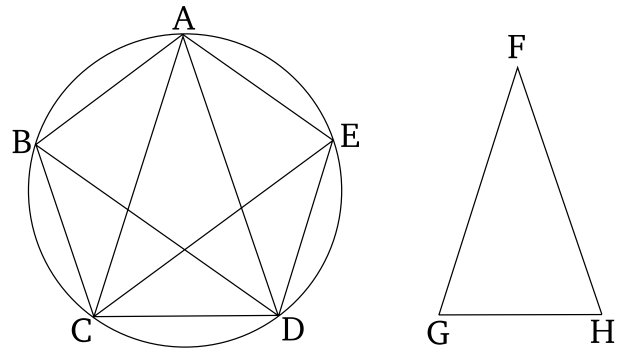


Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὴ εἰς τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἐκκείσθω τρίγωνον ἰσοσκελὲς τὸ ΖΗΘ διπλασίονα ἔχον ἑκατέραν τῶν πρὸς τοῖς Η, Θ γωνιῶν τῆς πρὸς τῷ Ζ, καὶ ἐγγεγράφθω εἰς τὸν ΑΒΓΔΕ κύκλον τῷ ΖΗΘ τριγώνῳ ἰσογώνιον τρίγωνον τὸ ΑΓΔ, ὥστε τῇ μὲν πρὸς τῷ Ζ γωνίᾳ ἴσην εἶναι τὴν ὑπὸ ΓΑΔ, ἑκατέραν δὲ τῶν πρὸς τοῖς Η, Θ ἴσην ἑκατέρᾳ τῶν ὑπὸ ΑΓΔ, ΓΔΑ· καὶ ἑκατέρα ἄρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ τῆς ὑπὸ ΓΑΔ ἐστι διπλῆ. τετμήσθω δὴ ἑκατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ δίχα ὑπὸ ἑκατέρας τῶν ΓΕ, ΔΒ εὐθειῶν, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΔΕ, ΕΑ.

Ἐπεὶ οὖν ἑκατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ γωνιῶν διπλασίων ἐστὶ τῆς ὑπὸ ΓΑΔ, καὶ τετμημέναι εἰσὶ δίχα ὑπὸ τῶν ΓΕ, ΔΒ εὐθειῶν, αἱ πέντε ἄρα γωνίαι αἱ ὑπὸ ΔΑΓ, ΑΓΕ, ΕΓΔ, ΓΔΒ, ΒΔΑ ἴσαι ἀλλήλαις εἰσίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν· αἱ πέντε ἄρα περιφέρειαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ἴσαι ἀλλήλαις εἰσίν. ὑπὸ δὲ τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν· αἱ πέντε ἄρα εὐθεῖαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἡ ΑΒ περιφέρεια τῇ ΔΕ περιφερείᾳ ἐστὶν ἴση, κοινὴ προσκείσθω ἡ ΒΓΔ· ὅλη ἄρα ἡ ΑΒΓΔ περιφέρια ὅλῃ τῇ ΕΔΓΒ περιφερείᾳ ἐστὶν ἴση. καὶ βέβηκεν ἐπὶ μὲν τῆς ΑΒΓΔ περιφερείας γωνία ἡ ὑπὸ ΑΕΔ, ἐπὶ δὲ τῆς ΕΔΓΒ περιφερείας γωνία ἡ ὑπὸ ΒΑΕ· καὶ ἡ ὑπὸ ΒΑΕ ἄρα γωνία τῇ ὑπὸ ΑΕΔ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΕ γωνιῶν ἑκατέρᾳ τῶν ὑπὸ ΒΑΕ, ΑΕΔ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον.

Εἰς ἄρα τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.



Let *ABCDE* be the given circle. So it is required to inscribed an equilateral and equiangular pentagon in circle *ABCDE*.

Let the the isosceles triangle *FGH* be set up having each of the angles at *G* and *H* double the (angle) at *F* [Prop. 4.10]. And let triangle *ACD*, equiangular to *FGH*, be inscribed in circle *ABCDE*, such that *CAD* is equal to the angle at *F*, and the (angles) at *G* and *H* (are) equal to *ACD* and *CDA*, respectively [Prop. 4.2]. Thus, *ACD* and *CDA* are each double *CAD*. So let *ACD* and *CDA* be cut in half by the straight-lines *CE* and *DB*, respectively [Prop. 1.9]. And let *AB*, *BC*, *DE* and *EA* be joined.

Therefore, since angles *ACD* and *CDA* are each double *CAD*, and are cut in half by the straight-lines *CE* and *DB*, the five angles *DAC*, *ACE*, *ECD*, *CDB*, and *BDA* are thus equal to one another. And equal angles stand upon equal circumferences [Prop. 3.26]. Thus, the five circumferences *AB*, *BC*, *CD*, *DE*, and *EA* are equal to one another [Prop. 3.29]. Thus, the pentagon *ABCDE* is equilateral. So I say that (it is) also equiangular. For since the circumference *AB* is equal to the circumference *DE*, let *BCD* be added to both. Thus, the whole circumference *ABCD* is equal to the whole circumference *EDCB*. And the angle *AED* stands upon circumference *ABCD*, and angle *BAE* upon circumference *EDCB*. Thus, angle *BAE* is also equal to *AED* [Prop. 3.27]. So, for the same (reasons), each of the angles *ABC*, *BCD*, and *CDE* is also equal to each of *BAE* and *AED*. Thus, pentagon *ABCDE* is equiangular. And it was also shown (to be) equilateral.

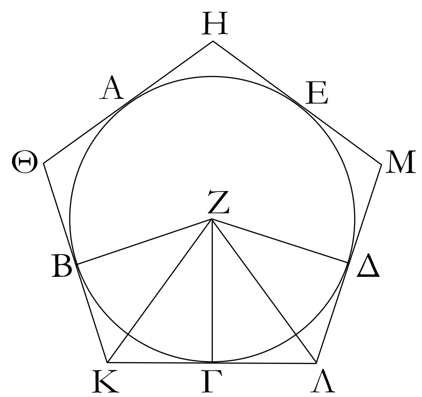
Thus, an equilateral and equiangular pentagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

ιβ'.

Περὶ τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι.

Proposition 12

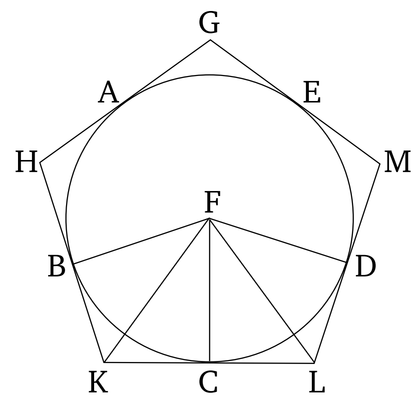
To circumscribe an equilateral and equiangular pentagon about a given circle.



Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὲ περὶ τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι.

Νενοήσθω τοῦ ἐγγεγραμμένου πενταγώνου τῶν γωνιῶν σημεῖα τὰ Α, Β, Γ, Δ, Ε, ὥστε ἴσας εἶναι τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ περιφερείας· καὶ διὰ τῶν Α, Β, Γ, Δ, Ε ἤχθωσαν τοῦ κύκλου ἐφαπτόμεναι αἱ ΗΘ, ΘΚ, ΚΛ, ΛΜ, μη, καὶ εἰλήφθω τοῦ ΑΒΓΔΕ κύκλου κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΚ, ΖΓ, ΖΛ, ΖΔ.

Καὶ ἐπεὶ ἡ μὲν ΚΛ εὐθεῖα ἐφάπτεται τοῦ ΑΒΓΔΕ κατὰ τὸ Γ, ἀπὸ δὲ τοῦ Ζ κέντρου ἐπὶ τὴν κατὰ τὸ Γ ἐπαφὴν ἐπέζευκται ἡ ΖΓ, ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΚΛ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν πρὸς τῷ Γ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τοῖς Β, Δ σημείοις γωνίαι ὀρθαί εἰσιν. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΖΓΚ γωνία, τὸ ἄρα ἀπὸ τῆς ΖΚ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΓ, ΓΚ. διὰ τὰ αὐτὰ δὴ καὶ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΚ· ὥστε τὰ ἀπὸ τῶν ΖΓ, ΓΚ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἐστιν ἴσα, ὧν τὸ ἀπὸ τῆς ΖΓ τῷ ἀπὸ τῆς ΖΒ ἐστιν ἴσον· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΓΚ τῷ ἀπὸ τῆς ΒΚ ἐστιν ἴσον. ἴση ἄρα ἡ ΒΚ τῇ ΓΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΖΒ τῇ ΖΓ, καὶ κοινὴ ἡ ΖΚ, δύο δὴ αἱ ΒΖ, ΖΚ δυσὶ ταῖς ΓΖ, ΖΚ ἴσαι εἰσίν· καὶ βάσις ἡ ΒΚ βάσει τῇ ΓΚ [ἐστιν] ἴση· γωνία ἄρα ἡ μὲν ὑπὸ ΒΖΚ [γωνίᾳ] τῇ ὑπὸ ΚΖΓ ἐστιν ἴση· ἡ δὲ ὑπὸ ΒΚΖ τῇ ὑπὸ ΖΚΓ· διπλῆ ἄρα ἡ μὲν ὑπὸ ΒΖΓ τῆς ὑπὸ ΚΖΓ, ἡ δὲ ὑπὸ ΒΚΓ τῆς ὑπὸ ΖΚΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΓΖΔ τῆς ὑπὸ ΓΖΛ ἐστι διπλῆ, ἡ δὲ ὑπὸ ΔΛΓ τῆς ὑπὸ ΖΛΓ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ περιφέρεια τῇ ΓΔ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΖΓ τῇ ὑπὸ ΓΖΔ. καί ἐστιν ἡ μὲν ὑπὸ ΒΖΓ τῆς ὑπὸ ΚΖΓ διπλῆ, ἡ δὲ ὑπὸ ΔΖΓ τῆς ὑπὸ ΛΖΓ· ἴση ἄρα καὶ ἡ ὑπὸ ΚΖΓ τῇ ὑπὸ ΛΖΓ· ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΓΚ γωνία τῇ ὑπὸ ΖΓΛ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΖΚΓ, ΖΛΓ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΖΓ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ἴση ἄρα ἡ μὲν ΚΓ εὐθεῖα τῇ ΓΛ, ἡ δὲ ὑπὸ ΖΚΓ γωνία τῇ ὑπὸ ΖΛΓ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΓ τῇ ΓΛ, διπλῆ ἄρα ἡ ΚΛ τῆς ΚΓ. διὰ τὰ αὐτα δὴ δειχθήσεται καὶ ἡ ΘΚ τῆς ΒΚ διπλῆ. καί ἐστιν ἡ ΒΚ τῇ ΚΓ ἴση· καὶ ἡ ΘΚ ἄρα τῇ ΚΛ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται καὶ ἑκάστη τῶν ΘΗ, ΗΜ, ΜΛ ἑκατέρᾳ τῶν ΘΚ,



Let $ABCDE$ be the given circle. So it is required to circumscribe an equilateral and equiangular pentagon about circle $ABCDE$.

Let A, B, C, D, and E be conceived as the angular points of a pentagon having been inscribed (in circle $ABCDE$) [Prop. 3.11], such that the circumferences AB, BC, CD, DE, and EA are equal. And let GH, HK, KL, LM, and MG be drawn through (points) A, B, C, D, and E (respectively), touching the circle.† And let the center F of the circle $ABCDE$ be found [Prop. 3.1]. And let FB, FK, FC, FL, and FD be joined.

And since the straight-line KL touches (circle) $ABCDE$ at C, and FC has been joined from the center F to the point of contact C, FC is thus perpendicular to KL [Prop. 3.18]. Thus, each of the angles at C is a right-angle. So, for the same (reasons), the angles at B and D are also right-angles. And since angle FCK is a right-angle, the (square) on FK is thus equal to the (sum of the squares) on FC and CK [Prop. 1.47]. So, for the same (reasons), the (square) on FK is also equal to the (sum of the squares) on FB and BK. So that the (sum of the squares) on FC and CK is equal to the (sum of the squares) on FB and BK, of which the (square) on FC is equal to the (square) on FB. Thus, the remaining (square) on CK is equal to the remaining (square) on BK. Thus, BK (is) equal to CK. And since FB is equal to FC, and FK (is) common, the two (straight-lines) BF, FK are equal to the two (straight-lines) CF, FK. And the base BK [is] equal to the base CK. Thus, angle BFK is equal to [angle] KFC [Prop. 1.8]. And BKF (is equal) to FKC [Prop. 1.8]. Thus, BFC (is) double KFC, and BKC (is double) FKC. So, for the same (reasons), CFD is also double CFL, and DLC (is also double) FLC. And since circumference BC is equal to CD, angle BFC is also equal to CFD [Prop. 3.27]. And BFC is double KFC, and DFC (is double) LFC. Thus, KFC is also equal to LFC. And angle FCK is also equal to FCL. So, FKC and FLC are two triangles having two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle to the remaining angle

ΚΛ ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΗΘΚΛΜ πεντάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ ΖΚΓ γωνία τῇ ὑπὸ ΖΛΓ, καὶ ἐδείχθη τῆς μὲν ὑπὸ ΖΚΓ διπλῆ ἡ ὑπὸ ΘΚΛ, τῆς δὲ ὑπὸ ΖΛΓ διπλῆ ἡ ὑπὸ ΚΛΜ, καὶ ἡ ὑπὸ ΘΚΛ ἄρα τῇ ὑπὸ ΚΛΜ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται καὶ ἑκάστη τῶν ὑπὸ ΚΘΗ, ΘΗΜ, ΗΜΛ ἑκατέρᾳ τῶν ὑπὸ ΘΚΛ, ΚΛΜ ἴση· αἱ πέντε ἄρα γωνίαι αἱ ὑπὸ ΗΘΚ, ΘΚΛ, ΚΛΜ, Λμη, μηΘ ἴσαι ἀλλήλαις εἰσίν. ἰσογώνιον ἄρα ἐστὶ τὸ ΗΘΚΛΜ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον, καὶ περιγέγραπται περὶ τὸν ΑΒΓΔΕ κύκλον.

[Περὶ τὸν δοθέντα ἄρα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγέγραπται]· ὅπερ ἔδει ποιῆσαι.

[Prop. 1.26]. Thus, the straight-line KC (is) equal to CL, and the angle FKC to FLC. And since KC is equal to CL, KL (is) thus double KC. So, for the same (reasons), it can be shown that HK (is) also double BK. And BK is equal to KC. Thus, HK is also equal to KL. So, similarly, each of HG, GM, and ML can also be shown (to be) equal to each of HK and KL. Thus, pentagon $GHKLM$ is equilateral. So I say that (it is) also equiangular. For since angle FKC is equal to FLC, and HKL was shown (to be) double FKC, and KLM double FLC, HKL is thus also equal to KLM. So, similarly, each of KHG, HGM, and GML can also be shown (to be) equal to each of HKL and KLM. Thus, the five angles GHK, HKL, KLM, LMG, and MGH are equal to one another. Thus, the pentagon $GHKLM$ is equiangular. And it was also shown (to be) equilateral, and has been circumscribed about circle $ABCDE$.

[Thus, an equilateral and equiangular pentagon has been circumscribed about the given circle]. (Which is) the very thing it was required to do.

† See the footnote to Prop. 3.34.

ιγ'.

Εἰς τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλον ἐγγράψαι.

Ἔστω τὸ δοθὲν πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ ΑΒΓΔΕ· δεῖ δὴ εἰς τὸ ΑΒΓΔΕ πεντάγωνον κύκλον ἐγγράψαι.

Τετμήσθω γὰρ ἑκατέρα τῶν ὑπὸ ΒΓΔ, ΓΔΕ γωνιῶν δίχα ὑπὸ ἑκατέρας τῶν ΓΖ, ΔΖ εὐθειῶν· καὶ ἀπὸ τοῦ Ζ σημείου, καθ᾽ ὃ συμβάλλουσιν ἀλλήλαις αἱ ΓΖ, ΔΖ εὐθεῖαι, ἐπεζεύχθωσαν αἱ ΖΒ, ΖΑ, ΖΕ εὐθεῖαι. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῇ ΓΔ, κοινὴ δὲ ἡ ΓΖ, δύο δὴ αἱ ΒΓ, ΓΖ δυσὶ ταῖς ΔΓ, ΓΖ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΒΓΖ γωνίᾳ τῇ ὑπὸ ΔΓΖ [ἐστιν] ἴση· βάσις ἄρα ἡ ΒΖ βάσει τῇ ΔΖ ἐστιν ἴση, καὶ τὸ ΒΓΖ τρίγωνον τῷ ΔΓΖ τριγώνῳ ἐστιν ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΓΒΖ γωνία τῇ ὑπὸ ΓΔΖ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ὑπὸ ΓΔΕ τῆς ὑπὸ ΓΔΖ, ἴση δὲ ἡ μὲν ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΒΓ, ἡ δὲ ὑπὸ ΓΔΖ τῇ ὑπὸ ΓΒΖ, καὶ ἡ ὑπὸ ΓΒΑ ἄρα τῆς ὑπὸ ΓΒΖ ἐστι διπλῆ· ἴση ἄρα ἡ ὑπὸ ΑΒΖ γωνία τῇ ὑπὸ ΖΒΓ· ἡ ἄρα ὑπὸ ΑΒΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΖ εὐθείας. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκατέρα τῶν ὑπὸ ΒΑΕ, ΑΕΔ δίχα τέτμηται ὑπὸ ἑκατέρας τῶν ΖΑ, ΖΕ εὐθειῶν. ἤχθωσαν δὴ ἀπὸ τοῦ Ζ σημείου ἐπὶ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας κάθετοι αἱ ΖΗ, ΖΘ, ΖΚ, ΖΛ, ΖΜ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΘΓΖ γωνία τῇ ὑπὸ ΚΓΖ, ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΖΘΓ [ὀρθῇ] τῇ ὑπὸ ΖΚΓ ἴση, δύο δὴ τρίγωνά ἐστι τὰ ΖΘΓ, ΖΚΓ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΖΓ ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει· ἴση ἄρα ἡ ΖΘ κάθετος τῇ ΖΚ καθέτῳ. ὁμοίως δὴ

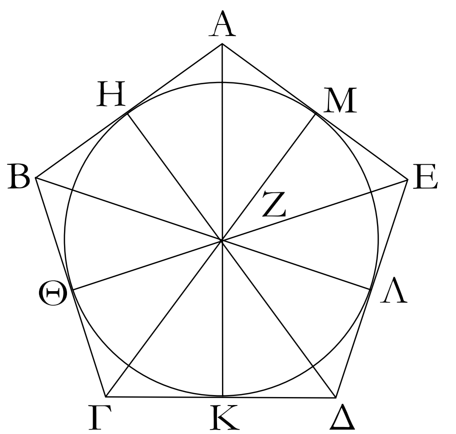
Proposition 13

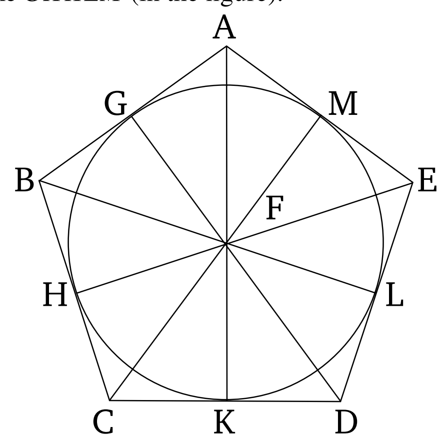
To inscribe a circle in a given pentagon, which is equilateral and equiangular.

Let $ABCDE$ be the given equilateral and equiangular pentagon. So it is required to inscribe a circle in pentagon $ABCDE$.

For let angles BCD and CDE each be cut in half by each of the straight-lines CF and DF (respectively) [Prop. 1.9]. And from the point F, at which the straight-lines CF and DF meet one another, let the straight-lines FB, FA, and FE be joined. And since BC is equal to CD, and CF (is) common, the two (straight-lines) BC, CF are equal to the two (straight-lines) DC, CF. And angle BCF [is] equal to angle DCF. Thus, the base BF is equal to the base DF, and triangle BCF is equal to triangle DCF, and the remaining angles will be equal to the (corresponding) remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle CBF (is) equal to CDF. And since CDE is double CDF, and CDE (is) equal to ABC, and CDF to CBF, CBA is thus also double CBF. Thus, angle ABF is equal to FBC. Thus, angle ABC has been cut in half by the straight-line BF. So, similarly, it can be shown that BAE and AED have been cut in half by the straight-lines FA and FE, respectively. So let FG, FH, FK, FL, and FM be drawn from point F, perpendicular to the straight-lines AB, BC, CD, DE, and EA (respectively) [Prop. 1.12]. And since angle HCF is equal to KCF, and the right-angle FHC is also equal to the [right-angle] FKC, FHC and FKC are two triangles having two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC, subtending one of the equal angles. Thus, they will also have the remaining sides equal *δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΖΛ, ΖΜ, ΖΗ ἑκατέρᾳ τῶν ΖΘ, ΖΚ ἴση ἐστίν· αἱ πέντε ἄρα εὐθεῖαι αἱ ΖΗ, ΖΘ, ΖΚ, ΖΛ, ΖΜ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν Η, Θ, Κ, Λ, Μ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἐφάψεται τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Η, Θ, Κ, Λ, Μ σημείοις γωνίας. εἰ γὰρ οὐκ ἐφάψεται αὐτῶν, ἀλλὰ τεμεῖ αὐτάς, συμβήσεται τὴν τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ᾽ ἄκρας ἀγομένην ἐντὸς πίπτειν τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ὁ κέντρῳ τῷ Ζ διαστήματι δὲ ἑνὶ τῶν Η, Θ, Κ, Λ, Μ σημείων γραφόμενος κύκλος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας· ἐφάψεται ἄρα αὐτῶν. γεγράφθω ὡς ὁ ΗΘΚΛΜ.*

to the (corresponding) remaining sides [Prop. 1.26]. Thus, the perpendicular FH (is) equal to the perpendicular FK. So, similarly, it can be shown that FL, FM, and FG are each equal to each of FH and FK. Thus, the five straight-lines FG, FH, FK, FL, and FM are equal to one another. Thus, the circle drawn with center F, and radius one of G, H, K, L, or M, will also go through the remaining points, and will touch the straight-lines AB, BC, CD, DE, and EA, on account of the angles at points G, H, K, L, and M being right-angles. For if it does not touch them, but cuts them, it follows that a (straight-line) drawn at right-angles to the diameter of the circle, from its extremity, falls inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center F, and radius one of G, H, K, L, or M, does not cut the straight-lines AB, BC, CD, DE, or EA. Thus, it will touch them. Let it be drawn, like $GHKLM$ (in the figure).





Εἰς ἄρα τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

Thus, a circle has been inscribed in the given pentagon which is equilateral and equiangular. (Which is) the very thing it was required to do.

ιδ'.

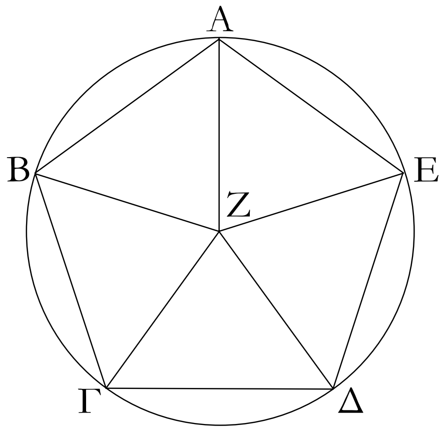
Proposition 14

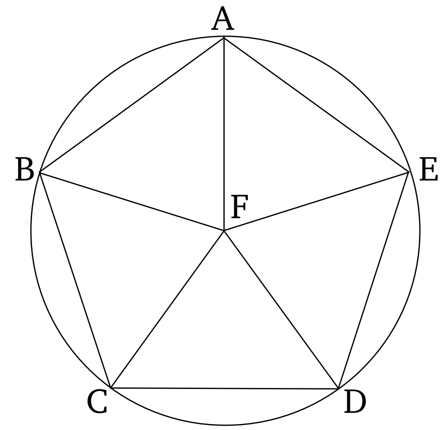
Περὶ τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλον περιγράψαι.

Ἔστω τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, τὸ ΑΒΓΔΕ· δεῖ δὴ περὶ τὸ ΑΒΓΔΕ πεντάγωνον κύκλον περιγράψαι.

To circumscribe a circle about a given pentagon which is equilateral and equiangular.

Let $ABCDE$ be the given pentagon which is equilateral and equiangular. So it is required to circumscribe a circle about the pentagon $ABCDE$.





Τετμήσθω δὴ ἑκατέρα τῶν ὑπὸ ΒΓΔ, ΓΔΕ γωνιῶν δίχα ὑπὸ ἑκατέρας τῶν ΓΖ, ΔΖ, καὶ ἀπὸ τοῦ Ζ σημείου, καθ᾿ ὃ συμβάλλουσιν αἱ εὐθεῖαι, ἐπὶ τὰ Β, Α, Ε σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΖΒ, ΖΑ, ΖΕ. ὁμοίως δὴ τῷ πρὸ τούτου δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ὑπὸ ΓΒΑ, ΒΑΕ, ΑΕΔ γωνιῶν δίχα τέτμηται ὑπὸ ἑκάστης τῶν ΖΒ, ΖΑ, ΖΕ εὐθειῶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΒΓΔ γωνία τῇ ὑπὸ ΓΔΕ, καί ἐστι τῆς μὲν ὑπὸ ΒΓΔ ἡμίσεια ἡ ὑπὸ ΖΓΔ, τῆς δὲ ὑπὸ ΓΔΕ ἡμίσεια ἡ ὑπὸ ΓΔΖ, καὶ ἡ ὑπὸ ΖΓΔ ἄρα τῇ ὑπὸ ΖΔΓ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΖΓ πλευρᾷ τῇ ΖΔ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΖΒ, ΖΑ, ΖΕ ἑκατέρᾳ τῶν ΖΓ, ΖΔ ἐστιν ἴση· αἱ πέντε ἄρα εὐθεῖαι αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ ἴσαι ἀλλήλαις εἰσίν. ὁ ἄρα κέντρῳ τῷ Ζ καὶ διαστήματι ἑνὶ τῶν ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος. περιγεγράφθω καὶ ἔστω ὁ ΑΒΓΔΕ.

Περὶ ἄρα τὸ δοθὲν πεντάγωνον, ὅ ἐστιν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

So let angles BCD and CDE have been cut in half by the (straight-lines) CF and DF, respectively [Prop. 1.9]. And let the straight-lines FB, FA, and FE be joined from point F, at which the straight-lines meet, to the points B, A, and E (respectively). So, similarly, to the (proposition) before this (one), it can be shown that angles CBA, BAE, and AED have also been cut in half by the straight-lines FB, FA, and FE, respectively. And since angle BCD is equal to CDE, and FCD is half of BCD, and CDF half of CDE, FCD is thus also equal to FDC. So that side FC is also equal to side FD [Prop. 1.6]. So, similarly, it can be shown that FB, FA, and FE are also each equal to each of FC and FD. Thus, the five straight-lines FA, FB, FC, FD, and FE are equal to one another. Thus, the circle drawn with center F, and radius one of FA, FB, FC, FD, or FE, will also go through the remaining points, and will be circumscribed. Let it be (so) circumscribed, and let it be $ABCDE$.

Thus, a circle has been circumscribed about the given pentagon, which is equilateral and equiangular. (Which is) the very thing it was required to do.

ιεʹ.

Εἰς τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕΖ· δεῖ δὴ εἰς τὸν ΑΒΓΔΕΖ κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

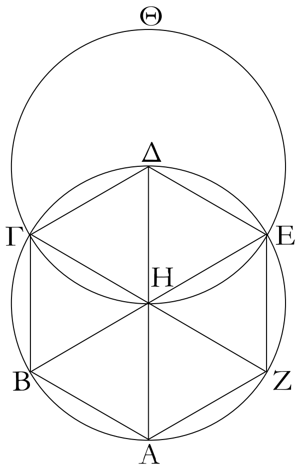
Ἤχθω τοῦ ΑΒΓΔΕΖ κύκλου διάμετρος ἡ ΑΔ, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Η, καὶ κέντρῳ μὲν τῷ Δ διαστήματι δὲ τῷ ΔΗ κύκλος γεγράφθω ὁ ΕΗΓΘ, καὶ ἐπιζευχθεῖσαι αἱ ΕΗ, ΓΗ διήχθωσαν ἐπὶ τὰ Β, Ζ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΖ, ΖΑ· λέγω, ὅτι τὸ ΑΒΓΔΕΖ ἑξάγωνον ἰσόπλευρόν τέ ἐστι καὶ ἰσογώνιον.

Proposition 15

To inscribe an equilateral and equiangular hexagon in a given circle.

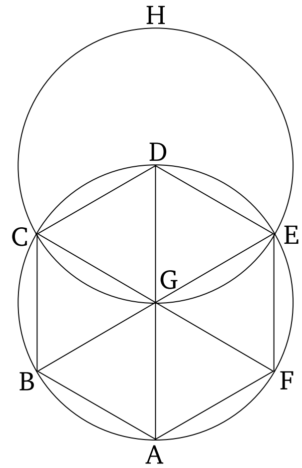
Let $ABCDEF$ be the given circle. So it is required to inscribe an equilateral and equiangular hexagon in circle $ABCDEF$.

Let the diameter AD of circle $ABCDEF$ be drawn,† and let the center G of the circle be found [Prop. 3.1]. And let the circle $EGCH$ be drawn, with center D, and radius DG. And EG and CG being joined, let them be drawn across (the circle) to points B and F (respectively). And let AB, BC, CD, DE, EF, and FA be joined. I say that the hexagon $ABCDEF$ is equilateral and equiangular.



Ἐπεὶ γὰρ τὸ Η σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓΔΕΖ κύκλου, ἴση ἐστὶν ἡ ΗΕ τῇ ΗΔ. πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΗΓΘ κύκλου, ἴση ἐστὶν ἡ ΔΕ τῇ ΔΗ. ἀλλ᾽ ἡ ΗΕ τῇ ΗΔ ἐδείχθη ἴση· καὶ ἡ ΗΕ ἄρα τῇ ΕΔ ἴση ἐστίν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΕΗΔ τρίγωνον· καὶ αἱ τρεῖς ἄρα αὐτοῦ γωνίαι αἱ ὑπὸ ΕΗΔ, ΗΔΕ, ΔΕΗ ἴσαι ἀλλήλαις εἰσίν, ἐπειδήπερ τῶν ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν· καί εἰσιν αἱ τρεῖς τοῦ τριγώνου γωνίαι δυσὶν ὀρθαῖς ἴσαι· ἡ ἄρα ὑπὸ ΕΗΔ γωνία τρίτον ἐστὶ δύο ὀρθῶν. ὁμοίως δὴ δειχθήσεται καὶ ἡ ὑπὸ ΔΗΓ τρίτον δύο ὀρθῶν. καὶ ἐπεὶ ἡ ΓΗ εὐθεῖα ἐπὶ τὴν ΕΒ σταθεῖσα τὰς ἐφεξῆς γωνίας τὰς ὑπὸ ΕΗΓ, ΓΗΒ δυσὶν ὀρθαῖς ἴσας ποιεῖ, καὶ λοιπὴ ἄρα ἡ ὑπὸ ΓΗΒ τρίτον ἐστὶ δύο ὀρθῶν· αἱ ἄρα ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ γωνίαι ἴσαι ἀλλήλαις εἰσίν· ὥστε καὶ αἱ κατὰ κορυφὴν αὐταῖς αἱ ὑπὸ ΒΗΑ, ΑΗΖ, ΖΗΕ ἴσαι εἰσὶν [ταῖς ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ]. αἱ ἓξ ἄρα γωνίαι αἱ ὑπὸ ΕΗΔ, ΔΗΓ, ΓΗΒ, ΒΗΑ, ΑΗΖ, ΖΗΕ ἴσαι ἀλλήλαις εἰσίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν· αἱ ἓξ ἄρα περιφέρειαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΖ, ΖΑ ἴσαι ἀλλήλαις εἰσίν. ὑπὸ δὲ τὰς ἴσας περιφερείας αἱ ἴσαι εὐθεῖαι ὑποτείνουσιν· αἱ ἓξ ἄρα εὐθεῖαι ἴσαι ἀλλήλαις εἰσίν· ἰσόπλευρον ἄρα ἐστὶ το ΑΒΓΔΕΖ ἑξάγωνον. λέγω δή, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΖΑ περιφέρεια τῇ ΕΔ περιφερείᾳ, κοινὴ προσκείσθω ἡ ΑΒΓΔ περιφέρεια· ὅλη ἄρα ἡ ΖΑΒΓΔ ὅλῃ τῇ ΕΔΓΒΑ ἐστιν ἴση· καὶ βέβηκεν ἐπὶ μὲν τῆς ΖΑΒΓΔ περιφερείας ἡ ὑπὸ ΖΕΔ γωνία, ἐπὶ δὲ τῆς ΕΔΓΒΑ περιφερείας ἡ ὑπὸ ΑΖΕ γωνία· ἴση ἄρα ἡ ὑπὸ ΑΖΕ γωνία τῇ ὑπὸ ΔΕΖ. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ αἱ λοιπαὶ γωνίαι τοῦ ΑΒΓΔΕΖ ἑξαγώνου κατὰ μίαν ἴσαι εἰσὶν ἑκατέρᾳ τῶν ὑπὸ ΑΖΕ, ΖΕΔ γωνιῶν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ ἑξάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· καὶ ἐγγέγραπται εἰς τὸν ΑΒΓΔΕΖ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.



For since point *G* is the center of circle *ABCDEF*, *GE* is equal to *GD*. Again, since point *D* is the center of circle *GCH*, *DE* is equal to *DG*. But, *GE* was shown (to be) equal to *GD*. Thus, *GE* is also equal to *ED*. Thus, triangle *EGD* is equilateral. Thus, its three angles *EGD*, *GDE*, and *DEG* are also equal to one another, inasmuch as the angles at the base of isosceles triangles are equal to one another [Prop. 1.5]. And the three angles of the triangle are equal to two right-angles [Prop. 1.32]. Thus, angle *EGD* is one third of two right-angles. So, similarly, *DGC* can also be shown (to be) one third of two right-angles. And since the straight-line *CG*, standing on *EB*, makes adjacent angles *EGC* and *CGB* equal to two right-angles [Prop. 1.13], the remaining angle *CGB* is thus also one third of two right-angles. Thus, angles *EGD*, *DGC*, and *CGB* are equal to one another. And hence the (angles) opposite to them *BGA*, *AGF*, and *FGE* are also equal [to *EGD*, *DGC*, and *CGB* (respectively)] [Prop. 1.15]. Thus, the six angles *EGD*, *DGC*, *CGB*, *BGA*, *AGF*, and *FGE* are equal to one another. And equal angles stand on equal circumferences [Prop. 3.26]. Thus, the six circumferences *AB*, *BC*, *CD*, *DE*, *EF*, and *FA* are equal to one another. And equal circumferences are subtended by equal straight-lines [Prop. 3.29]. Thus, the six straight-lines (*AB*, *BC*, *CD*, *DE*, *EF*, and *FA*) are equal to one another. Thus, hexagon *ABCDEF* is equilateral. So, I say that (it is) also equiangular. For since circumference *FA* is equal to circumference *ED*, let circumference *ABCD* be added to both. Thus, the whole of *FABCD* is equal to the whole of *EDCBA*. And angle *FED* stands on circumference *FABCD*, and angle *AFE* on circumference *EDCBA*. Thus, angle *AFE* is equal to *DEF* [Prop. 3.27]. Similarly, it can also be shown that the remaining angles of hexagon *ABCDEF* are individually equal to each of the angles *AFE* and *FED*. Thus, hexagon *ABCDEF* is equiangular. And it was also shown (to be) equilateral. And it has been inscribed in circle *ABCDE*.

Thus, an equilateral and equiangular hexagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τοῦ ἑξαγώνου πλευρὰ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ κύκλου.

Ὁμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐὰν διὰ τῶν κατὰ τὸν κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἀκολούθως τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις. καὶ ἔτι διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις εἰς τὸ δοθὲν ἑξάγωνον κύκλον ἐγγράψομέν τε καὶ περιγράψομεν· ὅπερ ἔδει ποιῆσαι.

Corollary

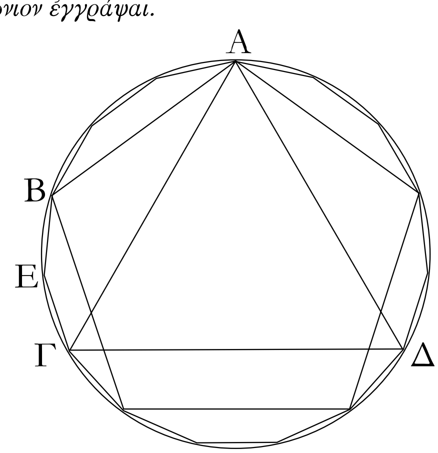
So, from this, (it is) manifest that a side of the hexagon is equal to the radius of the circle.

And similarly to a pentagon, if we draw tangents to the circle through the (sixfold) divisions of the (circumference of the) circle, an equilateral and equiangular hexagon can be circumscribed about the circle, analogously to the aforementioned pentagon. And, further, by (means) similar to the aforementioned pentagon, we can inscribe and circumscribe a circle in (and about) a given hexagon. (Which is) the very thing it was required to do.

† See the footnote to Prop. 4.6.

ιϛʹ.

Εἰς τὸν δοθέντα κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.



Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ εἰς τὸν ΑΒΓΔ κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

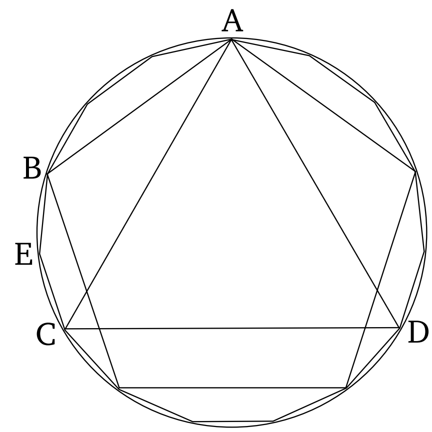
Ἐγγεγράφθω εἰς τὸν ΑΒΓΔ κύκλον τριγώνου μὲν ἰσοπλεύρου τοῦ εἰς αὐτὸν ἐγγραφομένου πλευρὰ ἡ ΑΓ, πενταγώνου δὲ ἰσοπλεύρου ἡ ΑΒ· οἵων ἄρα ἐστὶν ὁ ΑΒΓΔ κύκλος ἴσων τμήματων δεκαπέντε, τοιούτων ἡ μὲν ΑΒΓ περιφέρεια τρίτον οὖσα τοῦ κύκλου ἔσται πέντε, ἡ δὲ ΑΒ περιφέρεια πέμπτον οὖσα τοῦ κύκλου ἔσται τριῶν· λοιπὴ ἄρα ἡ ΒΓ τῶν ἴσων δύο. τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε· ἑκατέρα ἄρα τῶν ΒΕ, ΕΓ περιφερειῶν πεντεκαιδέκατόν ἐστι τοῦ ΑΒΓΔ κύκλου.

Ἐὰν ἄρα ἐπιζεύξαντες τὰς ΒΕ, ΕΓ ἴσας αὐταῖς κατὰ τὸ συνεχὲς εὐθείας ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ[Ε] κύκλον, ἔσται εἰς αὐτὸν ἐγγεγραμμένον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον· ὅπερ ἔδει ποιῆσαι.

Ὁμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐὰν διὰ τῶν κατὰ τὸν

Proposition 16

To inscribe an equilateral and equiangular fifteen-sided figure in a given circle.



Let *ABCD* be the given circle. So it is required to inscribe an equilateral and equiangular fifteen-sided figure in circle *ABCD*.

Let the side *AC* of an equilateral triangle inscribed in (the circle) [Prop. 4.2], and (the side) *AB* of an (inscribed) equilateral pentagon [Prop. 4.11], be inscribed in circle *ABCD*. Thus, just as the circle *ABCD* is (made up) of fifteen equal pieces, the circumference *ABC*, being a third of the circle, will be (made up) of five such (pieces), and the circumference *AB*, being a fifth of the circle, will be (made up) of three. Thus, the remainder *BC* (will be made up) of two equal (pieces). Let (circumference) *BC* be cut in half at *E* [Prop. 3.30]. Thus, each of the circumferences *BE* and *EC* is one fifteenth of the circle *ABCDE*.

Thus, if, joining *BE* and *EC*, we continuously insert straight-lines equal to them into circle *ABCD*[*E*] [Prop. 4.1], (then) an equilateral and equiangular fifteen-sided figure will *κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον. ἔτι δὲ διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου δείξεων καὶ εἰς τὸ δοθὲν πεντεκαιδεκάγωνον κύκλον ἐγγράψομέν τε καὶ περιγράψομεν· ὅπερ ἔδει ποιῆσαι.*

be inserted into (the circle). (Which is) the very thing it was required to do.

And similarly to the pentagon, if we draw tangents to the circle through the (fifteenfold) divisions of the (circumference of the) circle, we can circumscribe an equilateral and equiangular fifteen-sided figure about the circle. And, further, through similar proofs to the pentagon, we can also inscribe and circumscribe a circle in (and about) a given fifteen-sided figure. (Which is) the very thing it was required to do.

ELEMENTS BOOK 5

Proportion[†]

[†]The theory of proportion set out in this book is generally attributed to Eudoxus of Cnidus. The novel feature of this theory is its ability to deal with irrational magnitudes, which had hitherto been a major stumbling block for Greek mathematicians. Throughout the footnotes in this book, α, β, γ, etc., denote general (possibly irrational) magnitudes, whereas m, n, l, etc., denote positive integers.

Ὅροι.

αʹ. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

βʹ. Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος.

γʹ. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις.

δʹ. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

εʹ. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ᾽ ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ᾖ ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

ϛʹ. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω.

ζʹ. Ὅταν δὲ τῶν ἰσάκις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχῃ τοῦ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἤπερ τὸ τρίτον πρὸς τὸ τέταρτον.

ηʹ. Ἀναλογία δὲ ἐν τρισὶν ὅροις ἐλαχίστη ἐστίν.

θʹ. Ὅταν δὲ τρία μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον.

ιʹ. Ὅταν δὲ τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον, καὶ ἀεὶ ἑξῆς ὁμοίως, ὡς ἂν ἡ ἀναλογία ὑπάρχῃ.

ιαʹ. Ὁμόλογα μεγέθη λέγεται τὰ μὲν ἡγούμενα τοῖς ἡγουμένοις τὰ δὲ ἑπόμενα τοῖς ἑπομένοις.

ιβʹ. Ἐναλλὰξ λόγος ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὸ ἡγούμενον καὶ τοῦ ἑπομένου πρὸς τὸ ἑπόμενον.

ιγʹ. Ἀνάπαλιν λόγος ἐστὶ λῆψις τοῦ ἑπομένου ὡς ἡγουμένου πρὸς τὸ ἡγούμενον ὡς ἑπόμενον.

ιδʹ. Σύνθεσις λόγου ἐστὶ λῆψις τοῦ ἡγουμένου μετὰ τοῦ ἑπομένου ὡς ἑνὸς πρὸς αὐτὸ τὸ ἑπόμενον.

ιεʹ. Διαίρεσις λόγου ἐστὶ λῆψις τῆς ὑπεροχῆς, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου, πρὸς αὐτὸ τὸ ἑπόμενον.

ιϛʹ. Ἀναστροφὴ λόγου ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχήν, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου.

ιζʹ. Δι᾽ ἴσου λόγος ἐστὶ πλειόνων ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος σύνδυο λαμβανομένων καὶ ἐν τῷ αὐτῷ λόγῳ, ὅταν ᾖ ὡς ἐν τοῖς πρώτοις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον, οὕτως ἐν τοῖς δευτέροις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον· ἢ ἄλλως· λῆψις τῶν ἄκρων καθ᾽ ὑπεξαίρεσιν τῶν μέσων.

ιηʹ. Τεταραγμένη δὲ ἀναλογία ἐστίν, ὅταν τριῶν ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος γίνηται ὡς μὲν ἐν

Definitions

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.[†]

2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.

3. A ratio is a certain type of relation with respect to size of two magnitudes of the same kind.[‡]

4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.[§]

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.[¶]

6. And let magnitudes having the same ratio be called proportional.[*]

7. And when for equal multiples (as in Def. 5), the multiple of the first (magnitude) exceeds the multiple of the second, and the multiple of the third (magnitude) does not exceed the multiple of the fourth, (then) the first (magnitude) is said to have a greater ratio to the second than the third (magnitude has) to the fourth.

8. And a proportion in three terms is the smallest (possible).[$]

9. And when three magnitudes are proportional, the first is said to have to the third the squared[♭] ratio of that (it has) to the second.[††]

10. And when four magnitudes are (continuously) proportional, the first is said to have to the fourth the cubed[‡‡] ratio of that (it has) to the second.[§§] And so on, similarly, in successive order, whatever the (continuous) proportion might be.

11. These magnitudes are said to be corresponding (magnitudes): the leading to the leading (of two ratios), and the following to the following.

12. An alternate ratio is a taking of the (ratio of the) leading (magnitude) to the leading (of two equal ratios), and (setting it equal to) the (ratio of the) following (magnitude) to the following.[¶¶]

13. An inverse ratio is a taking of the (ratio of the) following (magnitude) as the leading and the leading (magnitude) as the following.[**]

14. A composition of a ratio is a taking of the (ratio of the) leading plus the following (magnitudes), as one, to the following (magnitude) by itself.[$$]

15. A separation of a ratio is a taking of the (ratio of the) excess by which the leading (magnitude) exceeds the following to the following (magnitude) by itself.[♭♭]

τοῖς πρώτοις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, οὕτως ἐν τοῖς δευτέροις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, ὡς δὲ ἐν τοῖς πρώτοις μεγέθεσιν ἑπόμενον πρὸς ἄλλο τι, οὕτως ἐν τοῖς δευτέροις ἄλλο τι πρὸς ἡγούμενον.

16. A conversion of a ratio is a taking of the (ratio of the) leading (magnitude) to the excess by which the leading (magnitude) exceeds the following.[†††]

17. There being several magnitudes, and other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, a ratio via equality (i.e., ex aequali) occurs when as the first is to the last in the first (set of) magnitudes, so the first (is) to the last in the second (set of) magnitudes. Or alternately, (it is) a taking of the (ratio of the) outer (magnitudes) by the removal of the inner (magnitudes).[‡‡‡]

18. There being three magnitudes, and other (magnitudes) of equal number to them, a perturbed proportion occurs when as the leading is to the following in the first (set of) magnitudes, so the leading (is) to the following in the second (set of) magnitudes, and as the following (is) to some other (i.e., the remaining magnitude) in the first (set of) magnitudes, so some other (is) to the leading in the second (set of) magnitudes.[§§§]

† In other words, α is said to be a part of β if $\beta = m\,\alpha$.

‡ In modern notation, the ratio of two magnitudes, α and β, is denoted $\alpha : \beta$.

§ In other words, α has a ratio with respect to β if $m\,\alpha > \beta$ and $n\,\beta > \alpha$, for some m and n.

¶ In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\,\alpha > n\,\beta$ whenever $m\,\gamma > n\,\delta$, and $m\,\alpha = n\,\beta$ whenever $m\,\gamma = n\,\delta$, and $m\,\alpha < n\,\beta$ whenever $m\,\gamma < n\,\delta$, for all m and n. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α, β, etc., are irrational.

∗ Thus if α and β have the same ratio as γ and δ then they are proportional. In modern notation, $\alpha : \beta :: \gamma : \delta$.

$ In modern notation, a proportion in three terms—α, β, and γ—is written: $\alpha : \beta :: \beta : \gamma$.

♭ Literally, "double".

†† In other words, if $\alpha : \beta :: \beta : \gamma$ then $\alpha : \gamma :: \alpha^2 : \beta^2$.

‡‡ Literally, "triple".

§§ In other words, if $\alpha : \beta :: \beta : \gamma :: \gamma : \delta$ then $\alpha : \delta :: \alpha^3 : \beta^3$.

¶¶ In other words, if $\alpha : \beta :: \gamma : \delta$ then the alternate ratio corresponds to $\alpha : \gamma :: \beta : \delta$.

∗∗ In other words, if $\alpha : \beta$ then the inverse ratio corresponds to $\beta : \alpha$.

$$ In other words, if $\alpha : \beta$ then the composed ratio corresponds to $\alpha + \beta : \beta$.

♭♭ In other words, if $\alpha : \beta$ then the separated ratio corresponds to $\alpha - \beta : \beta$.

††† In other words, if $\alpha : \beta$ then the converted ratio corresponds to $\alpha : \alpha - \beta$.

‡‡‡ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha : \beta : \gamma :: \delta : \epsilon : \zeta$, then the ratio via equality (i.e, ex aequali) corresponds to $\alpha : \gamma :: \delta : \zeta$.

§§§ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha : \beta :: \delta : \epsilon$ as well as $\beta : \gamma :: \zeta : \delta$, then the proportion is said to be perturbed.

α'.

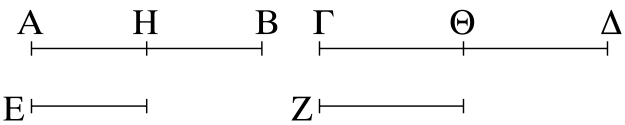
Ἐὰν ᾖ ὁποσαοῦν μεγέθη ὁποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἓν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων.

Ἔστω ὁποσαοῦν μεγέθη τὰ ΑΒ, ΓΔ ὁποσωνοῦν μεγεθῶν τῶν Ε, Ζ ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον· λέγω, ὅτι ὁσαπλάσιόν ἐστι τὸ ΑΒ τοῦ Ε, τοσαυταπλάσια ἔσται καὶ τὰ ΑΒ, ΓΔ τῶν Ε, Ζ.

Proposition 1[†]

If there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), (then) as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second).

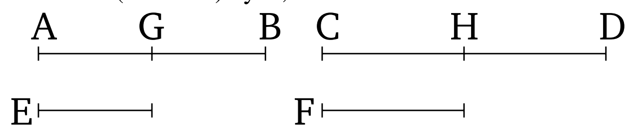
Let there be any number of magnitudes whatsoever, AB, CD, (which are) equal multiples, respectively, of some (other)



Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Ε καὶ τὸ ΓΔ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ μεγέθη ἴσα τῷ Ε, τοσαῦτα καὶ ἐν τῷ ΓΔ ἴσα τῷ Ζ. διῃρήσθω τὸ μὲν ΑΒ εἰς τὰ τῷ Ε μεγέθη ἴσα τὰ ΑΗ, ΗΒ, τὸ δὲ ΓΔ εἰς τὰ τῷ Ζ ἴσα τὰ ΓΘ, ΘΔ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΒ τῷ πλήθει τῶν ΓΘ, ΘΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν ΑΗ τῷ Ε, τὸ δὲ ΓΘ τῷ Ζ, ἴσον ἄρα τὸ ΑΗ τῷ Ε, καὶ τὰ ΑΗ, ΓΘ τοῖς Ε, Ζ. διὰ τὰ αὐτὰ δὴ ἴσον ἐστὶ τὸ ΗΒ τῷ Ε, καὶ τὰ ΗΒ, ΘΔ τοῖς Ε, Ζ· ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ ἴσα τῷ Ε, τοσαῦτα καὶ ἐν τοῖς ΑΒ, ΓΔ ἴσα τοῖς Ε, Ζ· ὁσαπλάσιον ἄρα ἐστὶ τὸ ΑΒ τοῦ Ε, τοσαυταπλάσια ἔσται καὶ τὰ ΑΒ, ΓΔ τῶν Ε, Ζ.

Ἐὰν ἄρα ᾖ ὁποσαοῦν μεγέθη ὁποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἓν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων· ὅπερ ἔδει δεῖξαι.

magnitudes, E, F, of equal number (to them). I say that as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.



For since AB, CD are equal multiples of E, F, thus as many magnitudes as (there) are in AB equal to E, so many (are there) also in CD equal to F. Let AB be divided into magnitudes AG, GB, equal to E, and CD into (magnitudes) CH, HD, equal to F. So, the number of (divisions) AG, GB will be equal to the number of (divisions) CH, HD. And since AG is equal to E, and CH to F, AG (is) thus equal to E, and AG, CH to E, F. So, for the same (reasons), GB is equal to E, and GB, HD to E, F. Thus, as many (magnitudes) as (there) are in AB equal to E, so many (are there) also in AB, CD equal to E, F. Thus, as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.

Thus, if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), (then) as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads $m\,\alpha + m\,\beta + \cdots = m\,(\alpha + \beta + \cdots)$.

β΄.

Ἐὰν πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ᾖ δὲ καὶ πέμπτον δευτέρου ἰσάκις πολλαπλάσιον καὶ ἕκτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτον δευτέρου ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τετάρτου.

Πρῶτον γὰρ τὸ ΑΒ δευτέρου τοῦ Γ ἰσάκις ἔστω πολλαπλάσιον καὶ τρίτον τὸ ΔΕ τετάρτου τοῦ Ζ, ἔστω δὲ καὶ πέμπτον τὸ ΒΗ δευτέρου τοῦ Γ ἰσάκις πολλαπλάσιον καὶ ἕκτον τὸ ΕΘ τετάρτου τοῦ Ζ· λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτον τὸ ΑΗ δευτέρου τοῦ Γ ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΔΘ τετάρτου τοῦ Ζ.

Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ ἴσα τῷ Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὅσα ἐστὶν ἐν τῷ ΒΗ ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΕΘ ἴσα τῷ Ζ· ὅσα ἄρα ἐστὶν ἐν ὅλῳ τῷ ΑΗ ἴσα τῷ Γ, τοσαῦτα καὶ ἐν ὅλῳ τῷ ΔΘ ἴσα τῷ Ζ· ὁσαπλάσιον ἄρα ἐστὶ τὸ ΑΗ τοῦ Γ, τοσαυταπλάσιον ἔσται καὶ τὸ ΔΘ τοῦ Ζ. καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ ΑΗ δευτέρου τοῦ Γ ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΔΘ τετάρτου τοῦ Ζ.

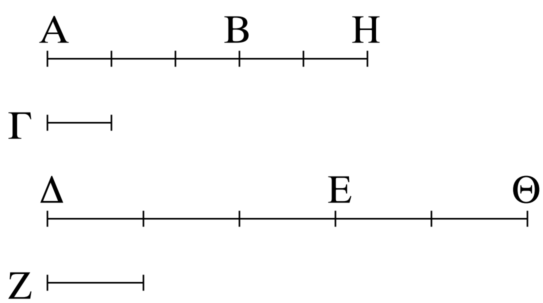
Proposition 2†

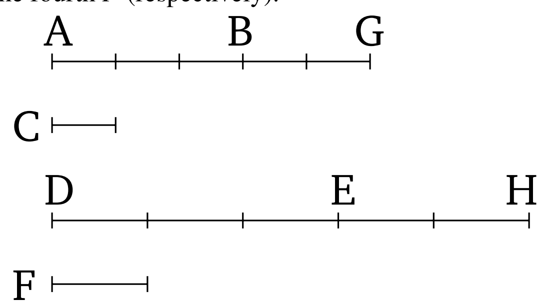
If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), (then) the first (magnitude) and the fifth, being added together, and the third and the sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively).

For let a first (magnitude) AB and a third DE be equal multiples of a second C and a fourth F (respectively). And let a fifth (magnitude) BG and a sixth EH also be (other) equal multiples of the second C and the fourth F (respectively). I say that the first (magnitude) and the fifth, being added together, (to give) AG, and the third (magnitude) and the sixth, (being added together, to give) DH, will also be equal multiples of the second (magnitude) C and the fourth F (respectively).

For since AB and DE are equal multiples of C and F (respectively), thus as many (magnitudes) as (there) are in AB equal to C, so many (are there) also in DE equal to F. And so, for the same (reasons), as many (magnitudes) as (there) are in BG equal to C, so many (are there) also in EH equal to F. Thus, as many (magnitudes) as (there) are in the whole of AG

equal to C, so many (are there) also in the whole of DH equal to F. Thus, as many times as AG is (divisible) by C, so many times will DH also be divisible by F. Thus, the first (magnitude) and the fifth, being added together, (to give) AG, and the third (magnitude) and the sixth, (being added together, to give) DH, will also be equal multiples of the second (magnitude) C and the fourth F (respectively).





Ἐὰν ἄρα πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ᾖ δὲ καὶ πέμπτον δευτέρου ἰσάκις πολλαπλάσιον καὶ ἕκτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτον δευτέρου ἰσάκις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τετάρτου· ὅπερ ἔδει δεῖξαι.

Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), (then) the first (magnitude) and the fifth, being added together, and the third and sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively). (Which is) the very thing it was required to show.

† In modern notation, this propostion reads $m\,\alpha + n\,\alpha = (m+n)\,\alpha$.

γ´.

Ἐὰν πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῇ δὲ ἰσάκις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου, καὶ δι᾽ ἴσου τῶν ληφθέντων ἑκάτερον ἑκατέρου ἰσάκις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου.

Πρῶτον γὰρ τὸ Α δευτέρου τοῦ Β ἰσάκις ἔστω πολλαπλάσιον καὶ τρίτον τὸ Γ τετάρτου τοῦ Δ, καὶ εἰλήφθω τῶν Α, Γ ἰσάκις πολλαπλάσια τὰ ΕΖ, ΗΘ· λέγω, ὅτι ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΕΖ τοῦ Β καὶ τὸ ΗΘ τοῦ Δ.

Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΕΖ τοῦ Α καὶ τὸ ΗΘ τοῦ Γ, ὅσα ἄρα ἐστὶν ἐν τῷ ΕΖ ἴσα τῷ Α, τοσαῦτα καὶ ἐν τῷ ΗΘ ἴσα τῷ Γ. διῃρήσθω τὸ μὲν ΕΖ εἰς τὰ τῷ Α μεγέθη ἴσα τὰ ΕΚ, ΚΖ, τὸ δὲ ΗΘ εἰς τὰ τῷ Γ ἴσα τὰ ΗΛ, ΛΘ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΕΚ, ΚΖ τῷ πλήθει τῶν ΗΛ, ΛΘ. καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ Α τοῦ Β καὶ τὸ

Proposition 3†

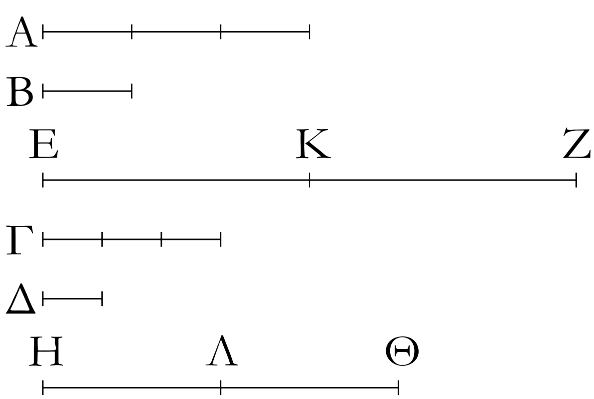
If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, (then), via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively.

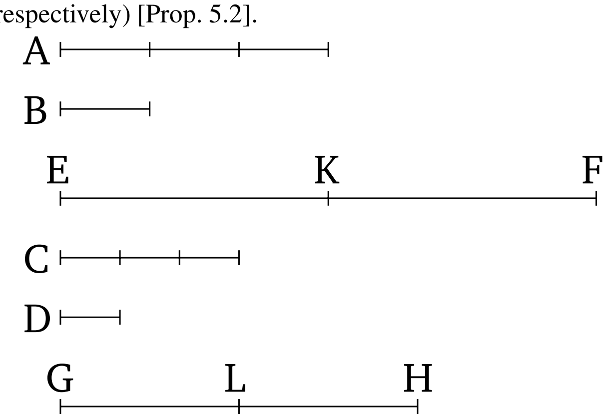
For let a first (magnitude) A and a third C be equal multiples of a second B and a fourth D (respectively), and let the equal multiples EF and GH be taken of A and C (respectively). I say that EF and GH are equal multiples of B and D (respectively).

For since EF and GH are equal multiples of A and C (respectively), thus as many (magnitudes) as (there) are in EF equal to A, so many (are there) also in GH equal to C. Let EF be divided into magnitudes EK, KF equal to A, and GH into

Γ τοῦ Δ, ἴσον δὲ τὸ μὲν ΕΚ τῷ Α, τὸ δὲ ΗΛ τῷ Γ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΕΚ τοῦ Β καὶ τὸ ΗΛ τοῦ Δ. διὰ τὰ αὐτὰ δὴ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΚΖ τοῦ Β καὶ τὸ ΛΘ τοῦ Δ. ἐπεὶ οὖν πρῶτον τὸ ΕΚ δευτέρου τοῦ Β ἴσάκις ἐστὶ πολλαπλάσιον καὶ τρίτον τὸ ΗΛ τετάρτου τοῦ Δ, ἔστι δὲ καὶ πέμπτον τὸ ΚΖ δευτέρου τοῦ Β ἰσάκις πολλαπλάσιον καὶ ἕκτον τὸ ΛΘ τετάρτου τοῦ Δ, καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ ΕΖ δευτέρου τοῦ Β ἰσάκις ἐστὶ πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΗΘ τετάρτου τοῦ Δ.

(magnitudes) *GL*, *LH* equal to *C*. So, the number of (magnitudes) *EK*, *KF* will be equal to the number of (magnitudes) *GL*, *LH*. And since *A* and *C* are equal multiples of *B* and *D* (respectively), and *EK* (is) equal to *A*, and *GL* to *C*, *EK* and *GL* are thus equal multiples of *B* and *D* (respectively). So, for the same (reasons), *KF* and *LH* are equal multiples of *B* and *D* (respectively). Therefore, since the first (magnitude) *EK* and the third *GL* are equal multiples of the second *B* and the fourth *D* (respectively), and the fifth (magnitude) *KF* and the sixth *LH* are also equal multiples of the second *B* and the fourth *D* (respectively), then the first (magnitude) and fifth, being added together, (to give) *EF*, and the third (magnitude) and sixth, (being added together, to give) *GH*, are thus also equal multiples of the second (magnitude) *B* and the fourth *D* (respectively) [Prop. 5.2].





Ἐὰν ἄρα πρῶτον δευτέρου ἰσάκις ᾖ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῇ δὲ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια, καὶ δι᾽ ἴσου τῶν ληφθέντων ἑκάτερον ἑκατέρου ἰσάκις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου· ὅπερ ἔδει δεῖξαι.

Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, (then), via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads $m\,(n\,\alpha) = (m\,n)\,\alpha$.

δ'.

Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάκις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάκις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου καθ᾽ ὁποιονοῦν πολλαπλασιασμὸν τὸν αὐτὸν ἕξει λόγον ληφθέντα κατάλληλα.

Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, καὶ εἰλήφθω τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Ε, Ζ, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Η, Θ· λέγω, ὅτι ἐστὶν ὡς τὸ Ε πρὸς τὸ Η, οὕτως τὸ Ζ πρὸς τὸ Θ.

Εἰλήφθω γὰρ τῶν μὲν Ε, Ζ ἰσάκις πολλαπλάσια τὰ Κ, Λ, τῶν δὲ Η, Θ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν.

[Καὶ] ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ μὲν Ε τοῦ Α, τὸ

Proposition 4†

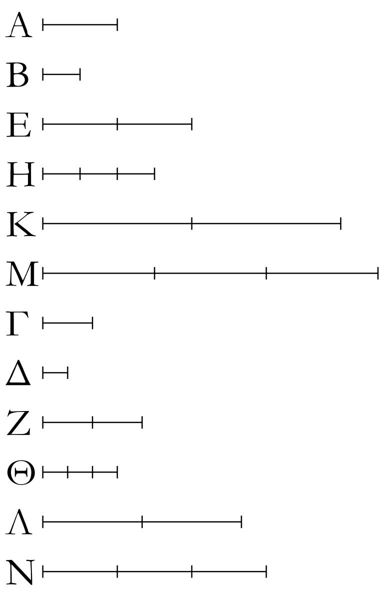
If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, (then) equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever.

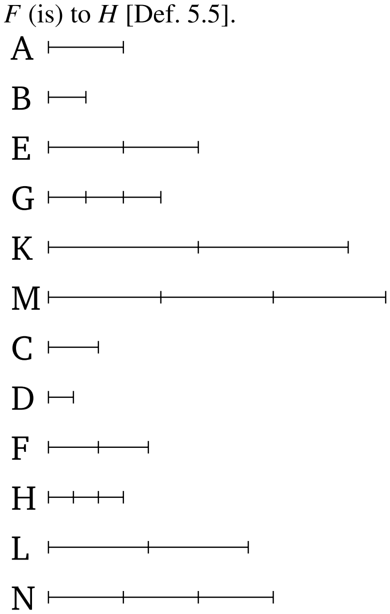
For let a first (magnitude) *A* have the same ratio to a second *B* that a third *C* (has) to a fourth *D*. And let equal multiples *E* and *F* be taken of *A* and *C* (respectively), and other random equal multiples *G* and *H* of *B* and *D* (respectively). I say that as *E* (is) to *G*, so *F* (is) to *H*.

For let equal multiples *K* and *L* be taken of *E* and *F* (respectively), and other random equal multiples *M* and *N* of *G* *δὲ Ζ τοῦ Γ, καὶ εἴληπται τῶν Ε, Ζ ἰσάκις πολλαπλάσια τὰ Κ, Λ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ Κ τοῦ Α καὶ τὸ Λ τοῦ Γ. διὰ τὰ αὐτὰ δὴ ἰσάκις ἐστὶ πολλαπλάσιον τὸ Μ τοῦ Β καὶ τὸ Ν τοῦ Δ. καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Κ, Λ, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Κ τοῦ Μ, ὑπερέχει καὶ τὸ Λ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Κ, Λ τῶν Ε, Ζ ἰσάκις πολλαπλάσια, τὰ δὲ Μ, Ν τῶν Η, Θ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Ε πρὸς τὸ Η, οὕτως τὸ Ζ πρὸς τὸ Θ.*

and *H* (respectively).

[And] since *E* and *F* are equal multiples of *A* and *C* (respectively), and the equal multiples *K* and *L* have been taken of *E* and *F* (respectively), *K* and *L* are thus equal multiples of *A* and *C* (respectively) [Prop. 5.3]. So, for the same (reasons), *M* and *N* are equal multiples of *B* and *D* (respectively). And since as *A* is to *B*, so *C* (is) to *D*, and the equal multiples *K* and *L* have been taken of *A* and *C* (respectively), and the other random equal multiples *M* and *N* of *B* and *D* (respectively), (then) if *K* exceeds *M* (then) *L* also exceeds *N*, and if (*K* is) equal (to *M* then *L* is also) equal (to *N*), and if (*K* is) less (than *M* then *L* is also) less (than *N*) [Def. 5.5]. And *K* and *L* are equal multiples of *E* and *F* (respectively), and *M* and *N* other random equal multiples of *G* and *H* (respectively). Thus, as *E* (is) to *G*, so *F* (is) to *H* [Def. 5.5].





Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάκις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάκις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου τὸν αὐτὸν ἕξει λόγον καθ᾽ ὁποιονοῦν πολλαπλασιασμὸν ληφθέντα κατάλληλα· ὅπερ ἔδει δεῖξαι.

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, (then) equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever. (Which is) the very thing it was required to show.

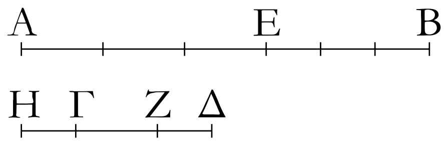
† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $m\,\alpha : n\,\beta :: m\,\gamma : n\,\delta$, for all m and n.

ε´.

Proposition 5†

Ἐὰν μέγεθος μεγέθους ἰσάκις ᾖ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, καὶ τὸ λοιπὸν τοῦ λοιποῦ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι τὸ ὅλον τοῦ ὅλου.

If a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively), (then) the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively).

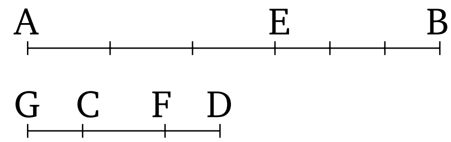


Μέγεθος γὰρ τὸ ΑΒ μεγέθους τοῦ ΓΔ ἰσάκις ἔστω πολλαπλάσιον, ὅπερ ἀφαιρεθὲν τὸ ΑΕ ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸν τὸ ΕΒ λοιποῦ τοῦ ΖΔ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ὅλον τὸ ΑΒ ὅλου τοῦ ΓΔ.

Ὁσαπλάσιον γάρ ἐστι τὸ ΑΕ τοῦ ΓΖ, τοσαυταπλάσιον γεγονέτω καὶ τὸ ΕΒ τοῦ ΓΗ.

Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΕΒ τοῦ ΗΓ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΑΒ τοῦ ΗΖ. κεῖται δὲ ἰσάκις πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΑΒ τοῦ ΓΔ. ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΒ ἑκατέρου τῶν ΗΖ, ΓΔ· ἴσον ἄρα τὸ ΗΖ τῷ ΓΔ. κοινὸν ἀφῃρήσθω τὸ ΓΖ· λοιπὸν ἄρα τὸ ΗΓ λοιπῷ τῷ ΖΔ ἴσον ἐστίν. καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΕΒ τοῦ ΗΓ, ἴσον δὲ τὸ ΗΓ τῷ ΔΖ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΕΒ τοῦ ΖΔ. ἰσάκις δὲ ὑπόκειται πολλαπλάσιον τὸ ΑΕ τοῦ ΓΖ καὶ τὸ ΑΒ τοῦ ΓΔ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΕΒ τοῦ ΖΔ καὶ τὸ ΑΒ τοῦ ΓΔ. καὶ λοιπὸν ἄρα τὸ ΕΒ λοιποῦ τοῦ ΖΔ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ὅλον τὸ ΑΒ ὅλου τοῦ ΓΔ.

Ἐὰν ἄρα μέγεθος μεγέθους ἰσάκις ᾖ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, καὶ τὸ λοιπὸν τοῦ λοιποῦ ἰσάκις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι καὶ τὸ ὅλον τοῦ ὅλου· ὅπερ ἔδει δεῖξαι.



For let the magnitude AB be the same multiple of the magnitude CD that the (part) taken away AE (is) of the (part) taken away CF (respectively). I say that the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

For as many times as AE is (divisible) by CF, so many times let EB also be made (divisible) by CG.

And since AE and EB are equal multiples of CF and GC (respectively), AE and AB are thus equal multiples of CF and GF (respectively) [Prop. 5.1]. And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, AB is an equal multiple of each of GF and CD. Thus, GF (is) equal to CD. Let CF be subtracted from both. Thus, the remainder GC is equal to the remainder FD. And since AE and EB are equal multiples of CF and GC (respectively), and GC (is) equal to DF, AE and EB are thus equal multiples of CF and FD (respectively). And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, EB and AB are equal multiples of FD and CD (respectively). Thus, the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

Thus, if a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively), (then) the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads $m\,\alpha - m\,\beta = m\,(\alpha - \beta)$.

ϛ΄.

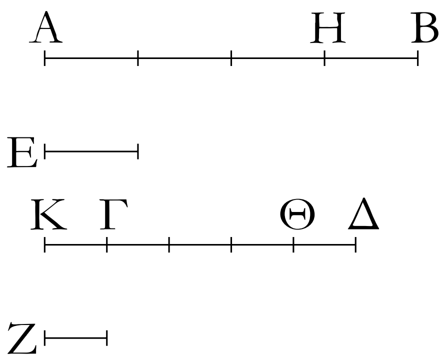
Ἐὰν δύο μεγέθη δύο μεγεθῶν ἰσάκις ᾖ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ἰσάκις ᾖ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ἤτοι ἴσα ἐστὶν ἢ ἰσάκις αὐτῶν πολλαπλάσια.

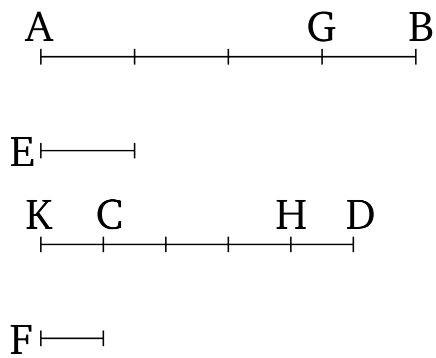
Δύο γὰρ μεγέθη τὰ ΑΒ, ΓΔ δύο μεγεθῶν τῶν Ε, Ζ ἰσάκις ἔστω πολλαπλάσια, καὶ ἀφαιρεθέντα τὰ ΑΗ, ΓΘ τῶν αὐτῶν τῶν Ε, Ζ ἰσάκις ἔστω πολλαπλάσια· λέγω, ὅτι καὶ λοιπὰ τὰ ΗΒ, ΘΔ τοῖς Ε, Ζ ἤτοι ἴσα ἐστὶν ἢ ἰσάκις αὐτῶν πολλαπλάσια.

Proposition 6†

If two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), (then) the remainders are also either equal to the latter (magnitudes), or (are) equal multiples of them (respectively).

For let two magnitudes AB and CD be equal multiples of two magnitudes E and F (respectively). And let the (parts) taken away (from the former) AG and CH be equal multiples of E and F (respectively). I say that the remainders GB and HD are also either equal to E and F (respectively), or (are) equal multiples of them.





Ἔστω γὰρ πρότερον τὸ ΗΒ τῷ Ε ἴσον· λέγω, ὅτι καὶ τὸ ΘΔ τῷ Ζ ἴσον ἐστίν.

Κείσθω γὰρ τῷ Ζ ἴσον τὸ ΓΚ. ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΗ τοῦ Ε καὶ τὸ ΓΘ τοῦ Ζ, ἴσον δὲ τὸ μὲν ΗΒ τῷ Ε, τὸ δὲ ΚΓ τῷ Ζ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Ε καὶ τὸ ΚΘ τοῦ Ζ. ἰσάκις δὲ ὑπόκειται πολλαπλάσιον τὸ ΑΒ τοῦ Ε καὶ τὸ ΓΔ τοῦ Ζ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΚΘ τοῦ Ζ καὶ τὸ ΓΔ τοῦ Ζ. ἐπεὶ οὖν ἑκάτερον τῶν ΚΘ, ΓΔ τοῦ Ζ ἰσάκις ἐστὶ πολλαπλάσιον, ἴσον ἄρα ἐστὶ τὸ ΚΘ τῷ ΓΔ. κοινὸν ἀφῃρήσθω τὸ ΓΘ· λοιπὸν ἄρα τὸ ΚΓ λοιπῷ τῷ ΘΔ ἴσον ἐστίν. ἀλλὰ τὸ Ζ τῷ ΚΓ ἐστιν ἴσον· καὶ τὸ ΘΔ ἄρα τῷ Ζ ἴσον ἐστίν. ὥστε εἰ τὸ ΗΒ τῷ Ε ἴσον ἐστίν, καὶ τὸ ΘΔ ἴσον ἔσται τῷ Ζ.

Ὁμοίως δὴ δείξομεν, ὅτι, κἂν πολλαπλάσιον ᾖ τὸ ΗΒ τοῦ Ε, τοσαυταπλάσιον ἔσται καὶ τὸ ΘΔ τοῦ Ζ.

Ἐὰν ἄρα δύο μεγέθη δύο μεγεθῶν ἰσάκις ᾖ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ἰσάκις ᾖ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ἤτοι ἴσα ἐστὶν ἢ ἰσάκις αὐτῶν πολλαπλάσια· ὅπερ ἔδει δεῖξαι.

For let GB be, first of all, equal to E. I say that HD is also equal to F.

For let CK be made equal to F. Since AG and CH are equal multiples of E and F (respectively), and GB (is) equal to E, and KC to F, AB and KH are thus equal multiples of E and F (respectively) [Prop. 5.2]. And AB and CD are assumed (to be) equal multiples of E and F (respectively). Thus, KH and CD are equal multiples of F and F (respectively). Therefore, KH and CD are each equal multiples of F. Thus, KH is equal to CD. Let CH be taken away from both. Thus, the remainder KC is equal to the remainder HD. But, F is equal to KC. Thus, HD is also equal to F. Hence, if GB is equal to E, (then) HD will also be equal to F.

So, similarly, we can show that even if GB is a multiple of E, (then) HD will also be the same multiple of F.

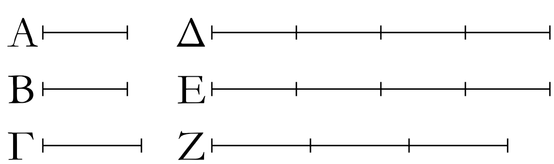
Thus, if two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), (then) the remainders are also either equal to the latter (magnitudes), or (are) equal multiples of them (respectively). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads $m\,\alpha - n\,\alpha = (m - n)\,\alpha$.

ζ'.

Τὰ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

Ἔστω ἴσα μεγέθη τὰ Α, Β, ἄλλο δέ τι, ὃ ἔτυχεν, μέγεθος τὸ Γ· λέγω, ὅτι ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν ἔχει λόγον, καὶ τὸ Γ πρὸς ἑκάτερον τῶν Α, Β.



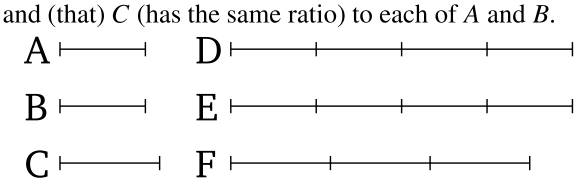
Εἰλήφθω γὰρ τῶν μὲν Α, Β ἰσάκις πολλαπλάσια τὰ Δ, Ε, τοῦ δὲ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον τὸ Ζ.

Ἐπεὶ οὖν ἰσάκις ἐστὶ πολλαπλάσιον τὸ Δ τοῦ Α καὶ τὸ Ε τοῦ Β, ἴσον δὲ τὸ Α τῷ Β, ἴσον ἄρα καὶ τὸ Δ τῷ Ε. ἄλλο δέ,

Proposition 7

Equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Let A and B be equal magnitudes, and C some other random magnitude. I say that A and B each have the same ratio to C, and (that) C (has the same ratio) to each of A and B.



For let the equal multiples D and E be taken of A and B (respectively), and the other random multiple F of C.

Therefore, since D and E are equal multiples of A and B (respectively), and A (is) equal to B, D (is) thus also equal to E.

ὃ ἔτυχεν, τὸ Ζ. εἰ ἄρα ὑπερέχει τὸ Δ τοῦ Ζ, ὑπερέχει καὶ τὸ Ε τοῦ Ζ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Δ, Ε τῶν Α, Β ἰσάκις πολλαπλάσια, τὸ δὲ Ζ τοῦ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Γ.

Λέγω [δή], ὅτι καὶ τὸ Ε πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν ἔχει λόγον.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἴσον ἐστὶ τὸ Δ τῷ Ε· ἄλλο δέ τι τὸ Ζ· εἰ ἄρα ὑπερέχει τὸ Ζ τοῦ Δ, ὑπερέχει καὶ τοῦ Ε, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὸ μὲν Ζ τοῦ Γ πολλαπλάσιον, τὰ δὲ Δ, Ε τῶν Α, Β ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως τὸ Γ πρὸς τὸ Β.

Τὰ ἴσα ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

And F (is) different, at random. Thus, if D exceeds F (then) E also exceeds F, and if (D is) equal (to F then E is also) equal (to F), and if (D is) less (than F then E is also) less (than F). And D and E are equal multiples of A and B (respectively), and F another random multiple of C. Thus, as A (is) to C, so B (is) to C [Def. 5.5].

[So] I say that $C^{\dagger}$ also has the same ratio to each of A and B.

For, similarly, we can show, by the same construction, that D is equal to E. And F (has) some other (value). Thus, if F exceeds D then it also exceeds E, and if (F is) equal (to D then it is also) equal (to E), and if (F is) less (than D then it is also) less (than E). And F is a multiple of C, and D and E other random equal multiples of A and B. Thus, as C (is) to A, so C (is) to B [Def. 5.5].

Thus, equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν μεγέθη τινὰ ἀνάλογον ᾖ, καὶ ἀνάπαλιν ἀνάλογον ἔσται. ὅπερ ἔδει δεῖξαι.

Corollary‡

So (it is) clear, from this, that if some magnitudes are proportional, (then) they will also be proportional inversely. (Which is) the very thing it was required to show.

† The Greek text has "E", which is obviously a mistake.

‡ In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\beta : \alpha :: \delta : \gamma$.

η΄.

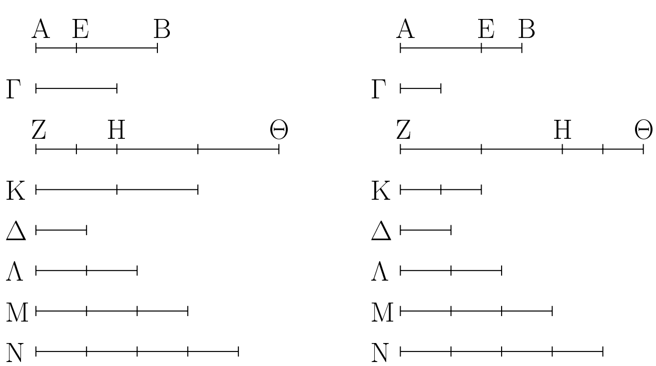
Τῶν ἀνίσων μεγεθῶν τὸ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἔλαττον. καὶ τὸ αὐτὸ πρὸς τὸ ἔλαττον μείζονα λόγον ἔχει ἤπερ πρὸς τὸ μεῖζον.

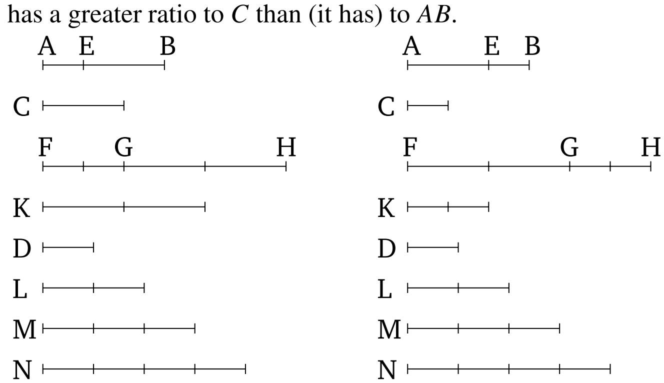
Ἔστω ἄνισα μεγέθη τὰ ΑΒ, Γ, καὶ ἔστω μεῖζον τὸ ΑΒ, ἄλλο δέ, ὃ ἔτυχεν, τὸ Δ· λέγω, ὅτι τὸ ΑΒ πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Δ, καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ πρὸς τὸ ΑΒ.

Proposition 8

For unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater.

Let AB and C be unequal magnitudes, and let AB be the greater (of the two), and D another random magnitude. I say that AB has a greater ratio to D than C (has) to D, and (that) D has a greater ratio to C than (it has) to AB.





Ἐπεὶ γὰρ μεῖζόν ἐστι τὸ ΑΒ τοῦ Γ, κείσθω τῷ Γ ἴσον τὸ ΒΕ· τὸ δὴ ἔλασσον τῶν ΑΕ, ΕΒ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μεῖζον. ἔστω πρότερον τὸ ΑΕ ἔλαττον τοῦ ΕΒ,

For since AB is greater than C, let BE be made equal to C. So, the lesser of AE and EB, being multiplied, will sometimes be greater than D [Def. 5.4]. First of all, let AE be less than *καὶ πεπολλαπλασιάσθω τὸ ΑΕ, καὶ ἔστω αὐτοῦ πολλαπλάσιον τὸ ΖΗ μεῖζον ὂν τοῦ Δ, καὶ ὁσαπλάσιόν ἐστι τὸ ΖΗ τοῦ ΑΕ, τοσαυταπλάσιον γεγονέτω καὶ τὸ μὲν ΗΘ τοῦ ΕΒ τὸ δὲ Κ τοῦ Γ· καὶ εἰλήφθω τοῦ Δ διπλάσιον μὲν τὸ Λ, τριπλάσιον δὲ τὸ Μ, καὶ ἑξῆς ἑνὶ πλεῖον, ἕως ἂν τὸ λαμβανόμενον πολλαπλάσιον μὲν γένηται τοῦ Δ, πρώτως δὲ μεῖζον τοῦ Κ. εἰλήφθω, καὶ ἔστω τὸ Ν τετραπλάσιον μὲν τοῦ Δ, πρώτως δὲ μεῖζον τοῦ Κ.*

Ἐπεὶ οὖν τὸ Κ τοῦ Ν πρώτως ἐστὶν ἔλαττον, τὸ Κ ἄρα τοῦ Μ οὐκ ἐστιν ἔλαττον. καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ ΗΘ τοῦ ΕΒ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ ΖΘ τοῦ ΑΒ. ἰσάκις δέ ἐστι πολλαπλάσιον τὸ ΖΗ τοῦ ΑΕ καὶ τὸ Κ τοῦ Γ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΖΘ τοῦ ΑΒ καὶ τὸ Κ τοῦ Γ. τὰ ΖΘ, Κ ἄρα τῶν ΑΒ, Γ ἰσάκις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΕΒ καὶ τὸ Κ τοῦ Γ, ἴσον δὲ τὸ ΕΒ τῷ Γ, ἴσον ἄρα καὶ τὸ ΗΘ τῷ Κ. τὸ δὲ Κ τοῦ Μ οὐκ ἐστιν ἔλαττον· οὐδ᾽ ἄρα τὸ ΗΘ τοῦ Μ ἔλαττόν ἐστιν. μεῖζον δὲ τὸ ΖΗ τοῦ Δ· ὅλον ἄρα τὸ ΖΘ συναμφοτέρων τῶν Δ, Μ μεῖζόν ἐστιν. ἀλλὰ συναμφότερα τὰ Δ, Μ τῷ Ν ἐστιν ἴσα, ἐπειδήπερ τὸ Μ τοῦ Δ τριπλάσιόν ἐστιν, συναμφότερα δὲ τὰ Μ, Δ τοῦ Δ ἐστι τετραπλάσια, ἔστι δὲ καὶ τὸ Ν τοῦ Δ τετραπλάσιον· συναμφότερα ἄρα τὰ Μ, Δ τῷ Ν ἴσα ἐστίν. ἀλλὰ τὸ ΖΘ τῶν Μ, Δ μεῖζόν ἐστιν· τὸ ΖΘ ἄρα τοῦ Ν ὑπερέχει· τὸ δὲ Κ τοῦ Ν οὐχ ὑπερέχει. καί ἐστι τὰ μὲν ΖΘ, Κ τῶν ΑΒ, Γ ἰσάκις πολλαπλάσια, τὸ δὲ Ν τοῦ Δ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον· τὸ ΑΒ ἄρα πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Δ.

Λέγω δή, ὅτι καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ τὸ Δ πρὸς τὸ ΑΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι τὸ μὲν Ν τοῦ Κ ὑπερέχει, τὸ δὲ Ν τοῦ ΖΘ οὐχ ὑπερέχει. καί ἐστι τὸ μὲν Ν τοῦ Δ πολλαπλάσιον, τὰ δὲ ΖΘ, Κ τῶν ΑΒ, Γ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· τὸ Δ ἄρα πρὸς τὸ Γ μείζονα λόγον ἔχει ἤπερ τὸ Δ πρὸς τὸ ΑΒ.

Ἀλλὰ δὴ τὸ ΑΕ τοῦ ΕΒ μεῖζον ἔστω. τὸ δὴ ἔλαττον τὸ ΕΒ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μεῖζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΗΘ πολλαπλάσιον μὲν τοῦ ΕΒ, μεῖζον δὲ τοῦ Δ· καὶ ὁσαπλάσιόν ἐστι τὸ ΗΘ τοῦ ΕΒ, τοσαυταπλάσιον γεγονέτω καὶ τὸ μὲν ΖΗ τοῦ ΑΕ, τὸ δὲ Κ τοῦ Γ. ὁμοίως δὴ δείξομεν, ὅτι τὰ ΖΘ, Κ τῶν ΑΒ, Γ ἰσάκις ἐστὶ πολλαπλάσια· καὶ εἰλήφθω ὁμοίως τὸ Ν πολλαπλάσιον μὲν τοῦ Δ, πρώτως δὲ μεῖζον τοῦ ΖΗ· ὥστε πάλιν τὸ ΖΗ τοῦ Μ οὐκ ἐστιν ἔλασσον. μεῖζον δὲ τὸ ΗΘ τοῦ Δ· ὅλον ἄρα τὸ ΖΘ τῶν Δ, Μ, τουτέστι τοῦ Ν, ὑπερέχει. τὸ δὲ Κ τοῦ Ν οὐχ ὑπερέχει, ἐπειδήπερ καὶ τὸ ΖΗ μεῖζον ὂν τοῦ ΗΘ, τουτέστι τοῦ Κ, τοῦ Ν οὐχ ὑπερέχει. καὶ ὡσαύτως κατακολουθοῦντες τοῖς ἐπάνω περαίνομεν τὴν ἀπόδειξιν.

Τῶν ἄρα ἀνίσων μεγεθῶν τὸ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἔλαττον· καὶ τὸ αὐτὸ πρὸς τὸ ἔλαττον μείζονα λόγον ἔχει ἤπερ πρὸς τὸ μεῖζον· ὅπερ ἔδει δεῖξαι.

EB, and let AE be multiplied, and let FG be a multiple of it which (is) greater than D. And as many times as FG is (divisible) by AE, so many times let GH also become (divisible) by EB, and K by C. And let the double multiple L of D be taken, and the triple multiple M, and several more, (each increasing) in order by one, until the (multiple) taken becomes the first multiple of D (which is) greater than K. Let it be taken, and let it also be the quadruple multiple N of D—the first (multiple) greater than K.

Therefore, since K is less than N first, K is thus not less than M. And since FG and GH are equal multiples of AE and EB (respectively), FG and FH are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. And FG and K are equal multiples of AE and C (respectively). Thus, FH and K are equal multiples of AB and C (respectively). Thus, FH, K are equal multiples of AB, C. Again, since GH and K are equal multiples of EB and C, and EB (is) equal to C, GH (is) thus also equal to K. And K is not less than M. Thus, GH not less than M either. And FG (is) greater than D. Thus, the whole of FH is greater than D and M (added) together. But, D and M (added) together is equal to N, inasmuch as M is three times D, and M and D (added) together is four times D, and N is also four times D. Thus, M and D (added) together is equal to N. But, FH is greater than M and D. Thus, FH exceeds N. And K does not exceed N. And FH, K are equal multiples of AB, C, and N another random multiple of D. Thus, AB has a greater ratio to D than C (has) to D [Def. 5.7].

So, I say that D also has a greater ratio to C than D (has) to AB.

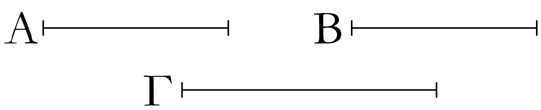
For, similarly, by the same construction, we can show that N exceeds K, and N does not exceed FH. And N is a multiple of D, and FH, K other random equal multiples of AB, C (respectively). Thus, D has a greater ratio to C than D (has) to AB [Def. 5.5].

And so let AE be greater than EB. So, the lesser, EB, being multiplied, will sometimes be greater than D. Let it be multiplied, and let GH be a multiple of EB (which is) greater than D. And as many times as GH is (divisible) by EB, so many times let FG also become (divisible) by AE, and K by C. So, similarly (to the above), we can show that FH and K are equal multiples of AB and C (respectively). And, similarly (to the above), let the multiple N of D, (which is) the first (multiple) greater than FG, be taken. So, FG is again not less than M. And GH (is) greater than D. Thus, the whole of FH exceeds D and M, that is to say N. And K does not exceed N, inasmuch as FG, which (is) greater than GH—that is to say, K—also does not exceed N. And, following the above (arguments), we (can) complete the proof in the same manner.

Thus, for unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater. (Which is) the very thing it was required to show.

θ΄.

Τὰ πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστίν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν.



Ἐχέτω γὰρ ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν λόγον· λέγω, ὅτι ἴσον ἐστὶ τὸ Α τῷ Β.

Εἰ γὰρ μή, οὐκ ἂν ἑκάτερον τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον· ἔχει δέ· ἴσον ἄρα ἐστὶ τὸ Α τῷ Β.

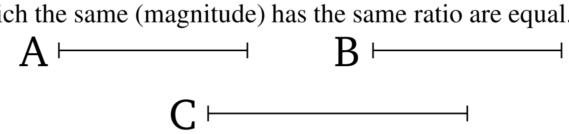
Ἐχέτω δὴ πάλιν τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν λόγον· λέγω, ὅτι ἴσον ἐστὶ τὸ Α τῷ Β.

Εἰ γὰρ μή, οὐκ ἂν τὸ Γ πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν εἶχε λόγον· ἔχει δέ· ἴσον ἄρα ἐστὶ τὸ Α τῷ Β.

Τὰ ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστίν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 9

(Magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal.



For let *A* and *B* each have the same ratio to *C*. I say that *A* is equal to *B*.

For if not, *A* and *B* would not each have the same ratio to *C* [Prop. 5.8]. But they do. Thus, *A* is equal to *B*.

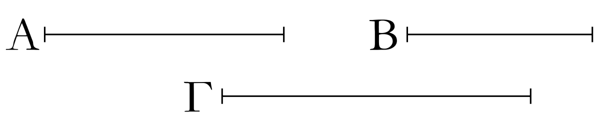
So, again, let *C* have the same ratio to each of *A* and *B*. I say that *A* is equal to *B*.

For if not, *C* would not have the same ratio to each of *A* and *B* [Prop. 5.8]. But it does. Thus, *A* is equal to *B*.

Thus, (magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal. (Which is) the very thing it was required to show.

ι΄.

Τῶν πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον ἐκεῖνο μεῖζόν ἐστιν· πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλαττόν ἐστιν.



Ἐχέτω γὰρ τὸ Α πρὸς τὸ Γ μείζονα λόγον ἤπερ τὸ Β πρὸς τὸ Γ· λέγω, ὅτι μεῖζόν ἐστι τὸ Α τοῦ Β.

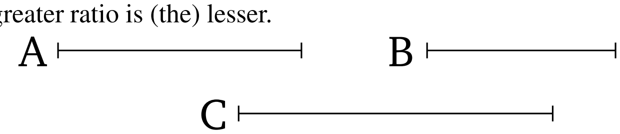
Εἰ γὰρ μή, ἤτοι ἴσον ἐστὶ τὸ Α τῷ Β ἢ ἔλασσον. ἴσον μὲν οὖν οὔκ ἐστι τὸ Α τῷ Β· ἑκάτερον γὰρ ἂν τῶν Α, Β πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον. οὐκ ἔχει δέ· οὐκ ἄρα ἴσον ἐστὶ τὸ Α τῷ Β. οὐδὲ μὴν ἔλασσόν ἐστι τὸ Α τοῦ Β· τὸ Α γὰρ ἂν πρὸς τὸ Γ ἐλάσσονα λόγον εἶχεν ἤπερ τὸ Β πρὸς τὸ Γ. οὐκ ἔχει δέ· οὐκ ἄρα ἔλασσόν ἐστι τὸ Α τοῦ Β. ἐδείχθη δὲ οὐδὲ ἴσον· μεῖζον ἄρα ἐστὶ τὸ Α τοῦ Β.

Ἐχέτω δὴ πάλιν τὸ Γ πρὸς τὸ Β μείζονα λόγον ἤπερ τὸ Γ πρὸς τὸ Α· λέγω, ὅτι ἔλασσόν ἐστι τὸ Β τοῦ Α.

Εἰ γὰρ μή, ἤτοι ἴσον ἐστὶν ἢ μεῖζον. ἴσον μὲν οὖν οὔκ ἐστι τὸ Β τῷ Α· τὸ Γ γὰρ ἂν πρὸς ἑκάτερον τῶν Α, Β τὸν αὐτὸν εἶχε λόγον. οὐκ ἔχει δέ· οὐκ ἄρα ἴσον ἐστὶ τὸ Α τῷ Β. οὐδὲ μὴν μεῖζόν ἐστι τὸ Β τοῦ Α· τὸ Γ γὰρ ἂν πρὸς τὸ Β ἐλάσσονα λόγον εἶχεν ἤπερ πρὸς τὸ Α. οὐκ ἔχει δέ· οὐκ ἄρα μεῖζόν ἐστι τὸ Β τοῦ Α. ἐδείχθη δέ, ὅτι οὐδὲ ἴσον· ἔλαττον ἄρα ἐστὶ τὸ Β τοῦ Α.

Proposition 10

For (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser.



For let *A* have a greater ratio to *C* than *B* (has) to *C*. I say that *A* is greater than *B*.

For if not, *A* is surely either equal to or less than *B*. In fact, *A* is not equal to *B*. For (then) *A* and *B* would each have the same ratio to *C* [Prop. 5.7]. But they do not. Thus, *A* is not equal to *B*. Neither, indeed, is *A* less than *B*. For (then) *A* would have a lesser ratio to *C* than *B* (has) to *C* [Prop. 5.8]. But it does not. Thus, *A* is not less than *B*. And it was shown not (to be) equal either. Thus, *A* is greater than *B*.

So, again, let *C* have a greater ratio to *B* than *C* (has) to *A*. I say that *B* is less than *A*.

For if not, (it is) surely either equal or greater. In fact, *B* is not equal to *A*. For (then) *C* would have the same ratio to each of *A* and *B* [Prop. 5.7]. But it does not. Thus, *A* is not equal to *B*. Neither, indeed, is *B* greater than *A*. For (then) *C* would have a lesser ratio to *B* than (it has) to *A* [Prop. 5.8]. But it does not. Thus, *B* is not greater than *A*. And it was shown that (it is) not equal (to *A*) either. Thus, *B* is less than *A*.

Τῶν ἄρα πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον μεῖζόν ἐστιν· καὶ πρὸς ὃ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλαττόν ἐστιν· ὅπερ ἔδει δεῖξαι.

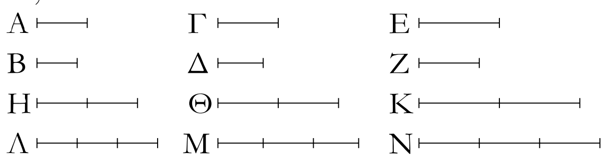
Thus, for (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser. (Which is) the very thing it was required to show.

ιαʹ.

Οἱ τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί.

Ἔστωσαν γὰρ ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, ὡς δὲ τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ.

Εἰλήφθω γὰρ τῶν Α, Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν.



Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, εἰ ἄρα ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Θ τοῦ Μ, καὶ εἰ ἴσον ἐστίν, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. πάλιν, ἐπεί ἐστιν ὡς τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ, καὶ εἴληπται τῶν Γ, Ε ἰσάκις πολλαπλάσια τὰ Θ, Κ, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Θ τοῦ Μ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλλατον, ἔλαττον. ἀλλὰ εἰ ὑπερεῖχε τὸ Θ τοῦ Μ, ὑπερεῖχε καὶ τὸ Η τοῦ Λ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον· ὥστε καὶ εἰ ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Κ τῶν Α, Ε ἰσάκις πολλαπλάσια, τὰ δὲ Λ, Ν τῶν Β, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ.

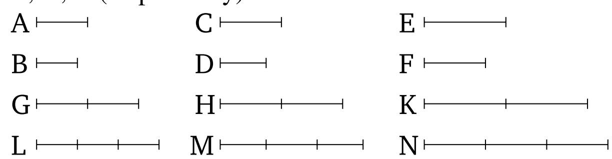
Οἱ ἄρα τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί· ὅπερ ἔδει δεῖξαι.

Proposition 11†

(Ratios which are) the same with the same ratio are also the same with one another.

For let it be that as A (is) to B, so C (is) to D, and as C (is) to D, so E (is) to F. I say that as A is to B, so E (is) to F.

For let the equal multiples G, H, K be taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively).



And since as A is to B, so C (is) to D, and the equal multiples G and H be taken of A and C (respectively), and the other random equal multiples L and M of B and D (respectively), thus if G exceeds L (then) H also exceeds M, and if (G is) equal (to L then H is also) equal (to M), and if (G is) less (than L then H is also) less (than M) [Def. 5.5]. Again, since as C is to D, so E (is) to F, and the equal multiples H and K have been taken of C and E (respectively), and the other random equal multiples M and N of D and F (respectively), thus if H exceeds M (then) K also exceeds N, and if (H is) equal (to M then K is also) equal (to N), and if (H is) less (than M then K is also) less (than N) [Def. 5.5]. But (we saw that) if H was exceeding M (then) G was also exceeding L, and if (H was) equal (to M then G was also) equal (to L), and if (H was) less (than M then G was also) less (than L). And, hence, if G exceeds L (then) K also exceeds N, and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N). And G and K are equal multiples of A and E (respectively), and L and N other random equal multiples of B and F (respectively). Thus, as A is to B, so E (is) to F [Def. 5.5].

Thus, (ratios which are) the same with the same ratio are also the same with one another. (Which is) the very thing it was required to show.

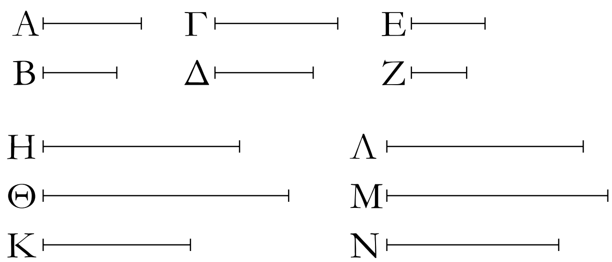
† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta :: \epsilon : \zeta$ then $\alpha : \beta :: \epsilon : \zeta$.

ιβʹ.

Ἐὰν ᾖ ὁποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα.

Proposition 12†

If there are any number of magnitudes whatsoever (which are) proportional, (then) as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes)



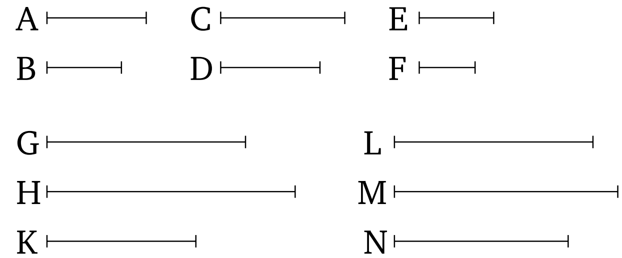
Ἔστωσαν ὁποσαοῦν μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, Ε, Ζ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ Ε πρὸς το Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὰ Α, Γ, Ε πρὸς τὰ Β, Δ, Ζ.

Εἰλήφθω γὰρ τῶν μὲν Α, Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ Ε πρὸς τὸ Ζ, καὶ εἴληπται τῶν μὲν Α, Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Θ τοῦ Μ, καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ὥστε καὶ εἰ ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὰ Η, Θ, Κ τῶν Λ, Μ, Ν, καὶ εἰ ἴσον, ἴσα, καὶ εἰ ἔλαττον, ἔλαττονα. καί ἐστι τὸ μὲν Η καὶ τὰ Η, Θ, Κ τοῦ Α καὶ τῶν Α, Γ, Ε ἰσάκις πολλαπλάσια, ἐπειδήπερ ἐὰν ᾖ ὁποσαοῦν μεγέθη ὁποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἓν τῶν μεγεθῶν ἑνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων. διὰ τὰ αὐτὰ δὴ καὶ τὸ Λ καὶ τὰ Λ, Μ, Ν τοῦ Β καὶ τῶν Β, Δ, Ζ ἰσάκις ἐστὶ πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὰ Α, Γ, Ε πρὸς τὰ Β, Δ, Ζ.

Ἐὰν ἄρα ᾖ ὁποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ὅπερ ἔδει δεῖξαι.

be to all of the following.



Let there be any number of magnitudes whatsoever, A, B, C, D, E, F, (which are) proportional, (so that) as A (is) to B, so C (is) to D, and E to F. I say that as A is to B, so A, C, E (are) to B, D, F.

For let the equal multiples G, H, K be taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively).

And since as A is to B, so C (is) to D, and E to F, and the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively), thus if G exceeds L (then) H also exceeds M, and K (exceeds) N, and if (G is) equal (to L then H is also) equal (to M, and K to N), and if (G is) less (than L then H is also) less (than M, and K than N) [Def. 5.5]. And, hence, if G exceeds L (then) G, H, K also exceed L, M, N, and if (G is) equal (to L then G, H, K are also) equal (to L, M, N) and if (G is) less (than L then G, H, K are also) less (than L, M, N). And G and G, H, K are equal multiples of A and A, C, E (respectively), inasmuch as if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), (then) as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second) [Prop. 5.1]. So, for the same (reasons), L and L, M, N are also equal multiples of B and B, D, F (respectively). Thus, as A is to B, so A, C, E (are) to B, D, F (respectively).

Thus, if there are any number of magnitudes whatsoever (which are) proportional, (then) as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following. (Which is) the very thing it was required to show.

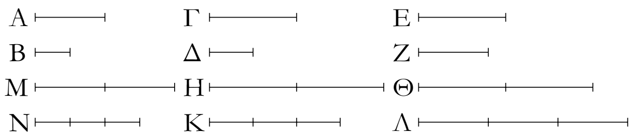
† In modern notation, this proposition reads that if $\alpha : \alpha' :: \beta : \beta' :: \gamma : \gamma'$ etc. then $\alpha : \alpha' :: (\alpha + \beta + \gamma + \cdots) : (\alpha' + \beta' + \gamma' + \cdots)$.

ιγ΄.

Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχῃ ἢ πέμπτον πρὸς ἕκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἕξει ἢ πέμπτον πρὸς ἕκτον.

Proposition 13†

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the third (magnitude) has a greater ratio to the fourth than a fifth (has) to a sixth, (then) the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth.

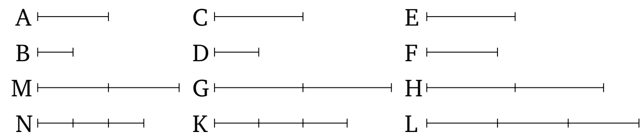


Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, τρίτον δὲ τὸ Γ πρὸς τέταρτον τὸ Δ μείζονα λόγον ἐχέτω ἢ πέμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ. λέγω, ὅτι καὶ πρῶτον τὸ Α πρὸς δεύτερον τὸ Β μείζονα λόγον ἕξει ἤπερ πέμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ.

Ἐπεὶ γὰρ ἔστι τινὰ τῶν μὲν Γ, Ε ἰσάκις πολλαπλάσια, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια, καὶ τὸ μὲν τοῦ Γ πολλαπλάσιον τοῦ τοῦ Δ πολλαπλασίου ὑπερέχει, τὸ δὲ τοῦ Ε πολλαπλάσιον τοῦ τοῦ Ζ πολλαπλασίου οὐχ ὑπερέχει, εἰλήφθω, καὶ ἔστω τῶν μὲν Γ, Ε ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Κ, Λ, ὥστε τὸ μὲν Η τοῦ Κ ὑπερέχειν, τὸ δὲ Θ τοῦ Λ μὴ ὑπερέχειν· καὶ ὁσαπλάσιον μέν ἐστι τὸ Η τοῦ Γ, τοσαυταπλάσιον ἔστω καὶ τὸ Μ τοῦ Α, ὁσαπλάσιον δὲ τὸ Κ τοῦ Δ, τοσαυταπλάσιον ἔστω καὶ τὸ Ν τοῦ Β.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκις πολλαπλάσια τὰ Μ, Η, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Ν, Κ, εἰ ἄρα ὑπερέχει τὸ Μ τοῦ Ν, ὑπερέχει καὶ τὸ Η τοῦ Κ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλλατον. ὑπερέχει δὲ τὸ Η τοῦ Κ· ὑπερέχει ἄρα καὶ τὸ Μ τοῦ Ν. τὸ δὲ Θ τοῦ Λ οὐχ ὑπερέχει· καί ἐστι τὰ μὲν Μ, Θ τῶν Α, Ε ἰσάκις πολλαπλάσια, τὰ δὲ Ν, Λ τῶν Β, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· τὸ ἄρα Α πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Ε πρὸς τὸ Ζ.

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχῃ ἢ πέμπτον πρὸς ἕκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἕξει ἢ πέμπτον πρὸς ἕκτον· ὅπερ ἔδει δεῖξαι.



For let a first (magnitude) *A* have the same ratio to a second *B* that a third *C* (has) to a fourth *D*, and let the third (magnitude) *C* have a greater ratio to the fourth *D* than a fifth *E* (has) to a sixth *F*. I say that the first (magnitude) *A* will also have a greater ratio to the second *B* than the fifth *E* (has) to the sixth *F*.

For since there are some equal multiples of *C* and *E*, and other random equal multiples of *D* and *F*, (for which) the multiple of *C* exceeds the (multiple) of *D*, and the multiple of *E* does not exceed the multiple of *F* [Def. 5.7], let them be taken. And let *G* and *H* be equal multiples of *C* and *E* (respectively), and *K* and *L* other random equal multiples of *D* and *F* (respectively), such that *G* exceeds *K*, but *H* does not exceed *L*. And as many times as *G* is (divisible) by *C*, so many times let *M* be (divisible) by *A*. And as many times as *K* (is divisible) by *D*, so many times let *N* be (divisible) by *B*.

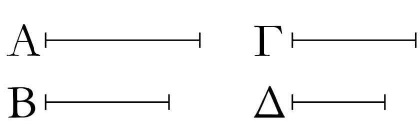
And since as *A* is to *B*, so *C* (is) to *D*, and the equal multiples *M* and *G* have been taken of *A* and *C* (respectively), and the other random equal multiples *N* and *K* of *B* and *D* (respectively), thus if *M* exceeds *N* (then) *G* exceeds *K*, and if (*M* is) equal (to *N* then *G* is also) equal (to *K*), and if (*M* is) less (than *N* then *G* is also) less (than *K*) [Def. 5.5]. And *G* exceeds *K*. Thus, *M* also exceeds *N*. And *H* does not exceed *L*. And *M* and *H* are equal multiples of *A* and *E* (respectively), and *N* and *L* other random equal multiples of *B* and *F* (respectively). Thus, *A* has a greater ratio to *B* than *E* (has) to *F* [Def. 5.7].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and a third (magnitude) has a greater ratio to a fourth than a fifth (has) to a sixth, (then) the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta > \epsilon : \zeta$ then $\alpha : \beta > \epsilon : \zeta$.

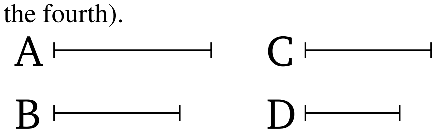
ιδ΄.

Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.



Proposition 14†

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, (then) the second will also be greater than the fourth. And if (the first magnitude is) equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth).



Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, μεῖζον δὲ ἔστω τὸ Α τοῦ Γ· λέγω, ὅτι καὶ τὸ Β τοῦ Δ μεῖζόν ἐστιν.

Ἐπεὶ γὰρ τὸ Α τοῦ Γ μεῖζόν ἐστιν, ἄλλο δέ, ὃ ἔτυχεν, [μέγεθος] τὸ Β, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. ὡς δὲ τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ τὸ Γ ἄρα πρὸς τὸ Δ μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλασσόν ἐστιν· ἔλασσον ἄρα τὸ Δ τοῦ Β· ὥστε μεῖζόν ἐστι τὸ Β τοῦ Δ.

Ὁμοίως δὴ δεῖξομεν, ὅτι κἂν ἴσον ᾖ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Β τῷ Δ, κἂν ἔλασσον ᾖ τὸ Α τοῦ Γ, ἔλασσον ἔσται καὶ τὸ Β τοῦ Δ.

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον· ὅπερ ἔδει δεῖξαι.

For let a first (magnitude) *A* have the same ratio to a second *B* that a third *C* (has) to a fourth *D*. And let *A* be greater than *C*. I say that *B* is also greater than *D*.

For since *A* is greater than *C*, and *B* (is) another random [magnitude], *A* thus has a greater ratio to *B* than *C* (has) to *B* [Prop. 5.8]. And as *A* (is) to *B*, so *C* (is) to *D*. Thus, *C* also has a greater ratio to *D* than *C* (has) to *B*. And that (magnitude) to which the same (magnitude) has a greater ratio is the lesser [Prop. 5.10]. Thus, *D* (is) less than *B*. Hence, *B* is greater than *D*.

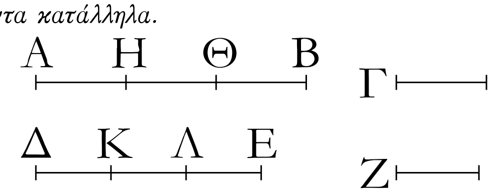
So, similarly, we can show that even if *A* is equal to *C*, (then) *B* will also be equal to *D*, and even if *A* is less than *C*, (then) *B* will also be less than *D*.

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, (then) the second will also be greater than the fourth. And if (the first magnitude is) equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha \gtreqless \gamma$ as $\beta \gtreqless \delta$.

ιε΄.

Τὰ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα.



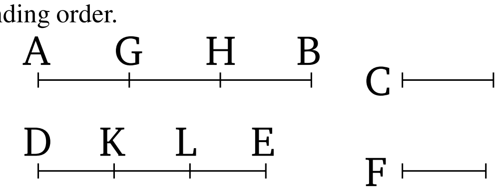
Ἔστω γὰρ ἰσάκις πολλαπλάσιον τὸ ΑΒ τοῦ Γ καὶ το ΔΕ τοῦ Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Γ πρὸς τὸ Ζ, οὕτως τὸ ΑΒ πρὸς τὸ ΔΕ.

Ἐπεὶ γὰρ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΑΒ τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ μεγέθη ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ ἴσα τῷ Ζ. διῃρήσθω τὸ μὲν ΑΒ εἰς τὰ τῷ Γ ἴσα τὰ ΑΗ, ΗΘ, ΘΒ, τὸ δὲ ΔΕ εἰς τὰ τῷ Ζ ἴσα τὰ ΔΚ, ΚΛ, ΛΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΘ, ΘΒ τῷ πλήθει τῶν ΔΚ, ΚΛ, ΛΕ. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΑΗ, ΗΘ, ΘΒ ἀλλήλοις, ἔστι δὲ καὶ τὰ ΔΚ, ΚΛ, ΛΕ ἴσα ἀλλήλοις, ἔστιν ἄρα ὡς τὸ ΑΗ πρὸς τὸ ΔΚ, οὕτως τὸ ΗΘ πρὸς τὸ ΚΛ, καὶ τὸ ΘΒ πρὸς τὸ ΛΕ. ἔσται ἄρα καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγουμένα πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα ὡς τὸ ΑΗ πρὸς τὸ ΔΚ, οὕτως τὸ ΑΒ πρὸς τὸ ΔΕ. ἴσον δὲ τὸ μὲν ΑΗ τῷ Γ, τὸ δὲ ΔΚ τῷ Ζ· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ζ οὕτως τὸ ΑΒ πρὸς τὸ ΔΕ.

Τὰ ἄρα μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα· ὅπερ ἔδει δεῖξαι.

Proposition 15†

Parts have the same ratio as similar multiples, taken in corresponding order.



For let *AB* and *DE* be equal multiples of *C* and *F* (respectively). I say that as *C* is to *F*, so *AB* (is) to *DE*.

For since *AB* and *DE* are equal multiples of *C* and *F* (respectively), thus as many magnitudes as there are in *AB* equal to *C*, so many (are there) also in *DE* equal to *F*. Let *AB* be divided into (magnitudes) *AG*, *GH*, *HB*, equal to *C*, and *DE* into (magnitudes) *DK*, *KL*, *LE*, equal to *F*. So, the number of (magnitudes) *AG*, *GH*, *HB* will equal the number of (magnitudes) *DK*, *KL*, *LE*. And since *AG*, *GH*, *HB* are equal to one another, and *DK*, *KL*, *LE* are also equal to one another, thus as *AG* is to *DK*, so *GH* (is) to *KL*, and *HB* to *LE* [Prop. 5.7]. And, thus (for proportional magnitudes), as one of the leading (magnitudes) will be to one of the following, so all of the leading (magnitudes will be) to all of the following [Prop. 5.12]. Thus, as *AG* is to *DK*, so *AB* (is) to *DE*. And *AG* is equal to *C*, and *DK* to *F*. Thus, as *C* is to *F*, so *AB* (is) to *DE*.

Thus, parts have the same ratio as similar multiples, taken in corresponding order. (Which is) the very thing it was required to show.

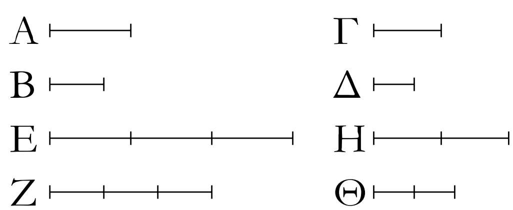
† In modern notation, this proposition reads that $\alpha : \beta :: m\,\alpha : m\beta$.

ιϛ΄.

Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, καὶ ἐναλλὰξ ἀνάλογον ἔσται.

Ἔστω τέσσαρα μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· λέγω, ὅτι καὶ ἐναλλὰξ [ἀνάλογον] ἔσται, ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Δ.

Εἰλήφθω γὰρ τῶν μὲν Α, Β ἰσάκις πολλαπλάσια τὰ Ε, Ζ, τῶν δὲ Γ, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Η, Θ.



Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ Ε τοῦ Α καὶ τὸ Ζ τοῦ Β, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ. ὡς δὲ τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ ὡς ἄρα τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ. πάλιν, ἐπεὶ τὰ Η, Θ τῶν Γ, Δ ἰσάκις ἐστὶ πολλαπλάσια, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Η πρὸς τὸ Θ. ὡς δὲ τὸ Γ πρὸς τὸ Δ, [οὕτως] τὸ Ε πρὸς τὸ Ζ· καὶ ὡς ἄρα τὸ Ε πρὸς τὸ Ζ, οὕτως τὸ Η πρὸς τὸ Θ. ἐὰν δὲ τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον. εἰ ἄρα ὑπερέχει τὸ Ε τοῦ Η, ὑπερέχει καὶ τὸ Ζ τοῦ Θ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Ε, Ζ τῶν Α, Β ἰσάκις πολλαπλάσια, τὰ δὲ Η, Θ τῶν Γ, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Δ.

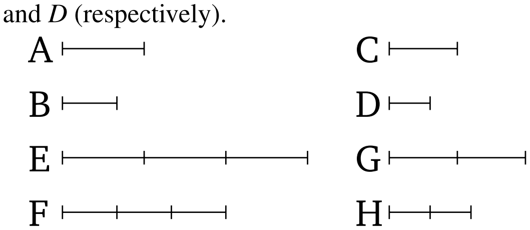
Ἐὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ᾖ, καὶ ἐναλλὰξ ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

Proposition 16†

If four magnitudes are proportional, (then) they will also be proportional alternately.

Let *A*, *B*, *C* and *D* be four proportional magnitudes, (such that) as *A* (is) to *B*, so *C* (is) to *D*. I say that they will also be [proportional] alternately, (so that) as *A* (is) to *C*, so *B* (is) to *D*.

For let the equal multiples *E* and *F* be taken of *A* and *B* (respectively), and the other random equal multiples *G* and *H* of *C* and *D* (respectively).



And since *E* and *F* are equal multiples of *A* and *B* (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as *A* is to *B*, so *E* (is) to *F*. But as *A* (is) to *B*, so *C* (is) to *D*. And, thus, as *C* (is) to *D*, so *E* (is) to *F* [Prop. 5.11]. Again, since *G* and *H* are equal multiples of *C* and *D* (respectively), thus as *C* is to *D*, so *G* (is) to *H* [Prop. 5.15]. But as *C* (is) to *D*, [so] *E* (is) to *F*. And, thus, as *E* (is) to *F*, so *G* (is) to *H* [Prop. 5.11]. And if four magnitudes are proportional, and the first is greater than the third, (then) the second will also be greater than the fourth, and if (the first is) equal (to the third then the second will also be) equal (to the fourth), and if (the first is) less (than the third then the second will also be) less (than the fourth) [Prop. 5.14]. Thus, if *E* exceeds *G* (then) *F* also exceeds *H*, and if (*E* is) equal (to *G* then *F* is also) equal (to *H*), and if (*E* is) less (than *G* then *F* is also) less (than *H*). And *E* and *F* are equal multiples of *A* and *B* (respectively), and *G* and *H* other random equal multiples of *C* and *D* (respectively). Thus, as *A* is to *C*, so *B* (is) to *D* [Def. 5.5].

Thus, if four magnitudes are proportional, (then) they will also be proportional alternately. (Which is) the very thing it was required to show.

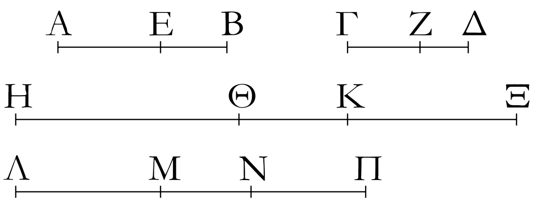
† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \gamma :: \beta : \delta$.

ιζ΄.

Ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ διαιρεθέντα ἀνάλογον ἔσται.

Proposition 17†

If composed magnitudes are proportional, (then) they will also be proportional (when) separarted.

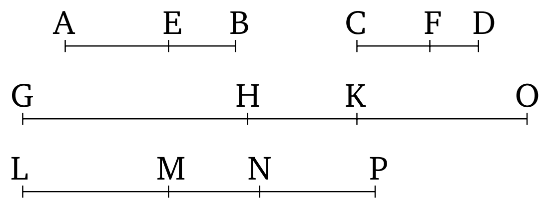


Ἔστω συγκείμενα μεγέθη ἀνάλογον τὰ ΑΒ, ΒΕ, ΓΔ, ΔΖ, ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ· λέγω, ὅτι καὶ διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΔΖ.

Εἰλήφθω γὰρ τῶν μὲν ΑΕ, ΕΒ, ΓΖ, ΖΔ ἰσάκις πολλαπλάσια τὰ ΗΘ, ΘΚ, ΛΜ, ΜΝ, τῶν δὲ ΕΒ, ΖΔ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ ΚΞ, ΝΠ.

Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΘΚ τοῦ ΕΒ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΗΚ τοῦ ΑΒ. ἰσάκις δέ ἐστι πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΛΜ τοῦ ΓΖ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ καὶ τὸ ΛΜ τοῦ ΓΖ. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΜΝ τοῦ ΖΔ, ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΛΝ τοῦ ΓΔ. ἰσάκις δὲ ἦν πολλαπλάσιον τὸ ΛΜ τοῦ ΓΖ καὶ τὸ ΗΚ τοῦ ΑΒ· ἰσάκις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ καὶ τὸ ΛΝ τοῦ ΓΔ. τὰ ΗΚ, ΛΝ ἄρα τῶν ΑΒ, ΓΔ ἰσάκις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ἰσάκις ἐστὶ πολλαπλασίον τὸ ΘΚ τοῦ ΕΒ καὶ τὸ ΜΝ τοῦ ΖΔ, ἔστι δὲ καὶ τὸ ΚΞ τοῦ ΕΒ ἰσάκις πολλαπλάσιον καὶ τὸ ΝΠ τοῦ ΖΔ, καὶ συντεθὲν τὸ ΘΞ τοῦ ΕΒ ἰσάκις ἐστὶ πολλαπλάσιον καὶ τὸ ΜΠ τοῦ ΖΔ. καὶ ἐπεί ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ, καὶ εἴληπται τῶν μὲν ΑΒ, ΓΔ ἰσάκις πολλαπλάσια τὰ ΗΚ, ΛΝ, τῶν δὲ ΕΒ, ΖΔ ἰσάκις πολλαπλάσια τὰ ΘΞ, ΜΠ, εἰ ἄρα ὑπερέχει τὸ ΗΚ τοῦ ΘΞ, ὑπερέχει καὶ τὸ ΛΝ τοῦ ΜΠ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ὑπερεχέτω δὴ τὸ ΗΚ τοῦ ΘΞ, καὶ κοινοῦ ἀφαιρεθέντος τοῦ ΘΚ ὑπερέχει ἄρα καὶ τὸ ΗΘ τοῦ ΚΞ. ἀλλα εἰ ὑπερεῖχε τὸ ΗΚ τοῦ ΘΞ ὑπερεῖχε καὶ τὸ ΛΝ τοῦ ΜΠ· ὑπερέχει ἄρα καὶ τὸ ΛΝ τοῦ ΜΠ, καὶ κοινοῦ ἀφαιρεθέντος τοῦ ΜΝ ὑπερέχει καὶ τὸ ΛΜ τοῦ ΝΠ· ὥστε εἰ ὑπερέχει τὸ ΗΘ τοῦ ΚΞ, ὑπερέχει καὶ τὸ ΛΜ τοῦ ΝΠ. ὁμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ᾖ τὸ ΗΘ τῷ ΚΞ, ἴσον ἔσται καὶ τὸ ΛΜ τῷ ΝΠ, κἂν ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν ΗΘ, ΛΜ τῶν ΑΕ, ΓΖ ἰσάκις πολλαπλάσια, τὰ δὲ ΚΞ, ΝΠ τῶν ΕΒ, ΖΔ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ.

Ἐὰν ἄρα συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ διαιρεθέντα ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.



Let AB, BE, CD, and DF be composed magnitudes (which are) proportional, (so that) as AB (is) to BE, so CD (is) to DF. I say that they will also be proportional (when) separated, (so that) as AE (is) to EB, so CF (is) to DF.

For let the equal multiples GH, HK, LM, and MN be taken of AE, EB, CF, and FD (respectively), and the other random equal multiples KO and NP of EB and FD (respectively).

And since GH and HK are equal multiples of AE and EB (respectively), GH and GK are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. But GH and LM are equal multiples of AE and CF (respectively). Thus, GK and LM are equal multiples of AB and CF (respectively). Again, since LM and MN are equal multiples of CF and FD (respectively), LM and LN are thus equal multiples of CF and CD (respectively) [Prop. 5.1]. And LM and GK were equal multiples of CF and AB (respectively). Thus, GK and LN are equal multiples of AB and CD (respectively). Thus, GK, LN are equal multiples of AB, CD. Again, since HK and MN are equal multiples of EB and FD (respectively), and KO and NP are also equal multiples of EB and FD (respectively), (then), added together, HO and MP are also equal multiples of EB and FD (respectively) [Prop. 5.2]. And since as AB (is) to BE, so CD (is) to DF, and the equal multiples GK, LN be taken of AB, CD, and the equal multiples HO, MP of EB, FD, thus if GK exceeds HO (then) LN also exceeds MP, and if (GK is) equal (to HO then LN is also) equal (to MP), and if (GK is) less (than HO then LN is also) less (than MP) [Def. 5.5]. So let GK exceed HO, and thus, HK being taken away from both, GH exceeds KO. But (we saw that) if GK was exceeding HO (then) LN was also exceeding MP. Thus, LN also exceeds MP, and, MN being taken away from both, LM also exceeds NP. Hence, if GH exceeds KO (then) LM also exceeds NP. So, similarly, we can show that even if GH is equal to KO (then) LM will also be equal to NP, and even if (GH is) less (than KO then LM will also be) less (than NP). And GH, LM are equal multiples of AE, CF, and KO, NP other random equal multiples of EB, FD. Thus, as AE is to EB, so CF (is) to FD [Def. 5.5].

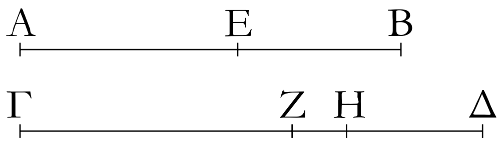
Thus, if composed magnitudes are proportional, (then) they will also be proportional (when) separarted. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha+\beta : \beta :: \gamma+\delta : \delta$ then $\alpha : \beta :: \gamma : \delta$.

ιη'.

Proposition 18[†]

Ἐὰν διῃρημένα μεγέθη ἀνάλογον ᾖ, καὶ συντεθέντα ἀνάλογον ἔσται.



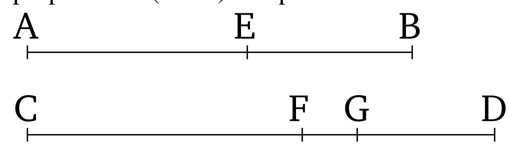
Ἔστω διῃρημένα μεγέθη ἀνάλογον τὰ ΑΕ, ΕΒ, ΓΖ, ΖΔ, ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ· λέγω, ὅτι καὶ συντεθέντα ἀνάλογον ἔσται, ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΖΔ.

Εἰ γὰρ μή ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ, ἔσται ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ ἤτοι πρὸς ἔλασσόν τι τοῦ ΔΖ ἢ πρὸς μεῖζον.

Ἔστω πρότερον πρὸς ἔλασσον τὸ ΔΗ. καὶ ἐπεί ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΗ, συγκείμενα μεγέθη ἀνάλογόν ἐστιν· ὥστε καὶ διαιρεθέντα ἀνάλογον ἔσται. ἔστιν ἄρα ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΗ πρὸς τὸ ΗΔ. ὑπόκειται δὲ καὶ ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. καὶ ὡς ἄρα τὸ ΓΗ πρὸς τὸ ΗΔ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. μεῖζον δὲ τὸ πρῶτον τὸ ΓΗ τοῦ τρίτου τοῦ ΓΖ· μεῖζον ἄρα καὶ τὸ δεύτερον τὸ ΗΔ τοῦ τετάρτου τοῦ ΖΔ. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα ἐστὶν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς ἔλασσον τοῦ ΖΔ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ πρὸς μεῖζον· πρὸς αὐτὸ ἄρα.

Ἐὰν ἄρα διῃρημένα μεγέθη ἀνάλογον ᾖ, καὶ συντεθέντα ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

If separated magnitudes are proportional, (then) they will also be proportional (when) composed.



Let AE, EB, CF, and FD be separated magnitudes (which are) proportional, (so that) as AE (is) to EB, so CF (is) to FD. I say that they will also be proportional (when) composed, (so that) as AB (is) to BE, so CD (is) to FD.

For if (it is) not (the case that) as AB is to BE, so CD (is) to FD, (then) it will surely be (the case that) as AB (is) to BE, so CD is either to some (magnitude) less than DF, or (some magnitude) greater (than DF).‡

Let it, first of all, be to (some magnitude) less (than DF), (namely) DG. And since composed magnitudes are proportional, (so that) as AB is to BE, so CD (is) to DG, they will thus also be proportional (when) separated [Prop. 5.17]. Thus, as AE is to EB, so CG (is) to GD. But it was also assumed that as AE (is) to EB, so CF (is) to FD. Thus, (it is) also (the case that) as CG (is) to GD, so CF (is) to FD [Prop. 5.11]. And the first (magnitude) CG (is) greater than the third CF. Thus, the second (magnitude) GD (is) also greater than the fourth FD [Prop. 5.14]. But (it is) also less. The very thing is impossible. Thus, (it is) not (the case that) as AB is to BE, so CD (is) to less than FD. Similarly, we can show that neither (is it the case) to greater (than FD). Thus, (it is the case) to the same (as FD).

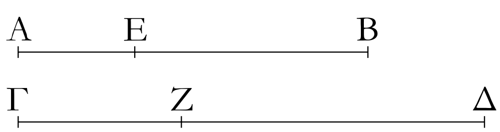
Thus, if separated magnitudes are proportional, (then) they will also be proportional (when) composed. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha + \beta : \beta :: \gamma + \delta : \delta$.

‡ Here, Euclid assumes, without proof, that a fourth magnitude proportional to three given magnitudes can always be found.

ιθ΄.

Ἐὰν ᾖ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς ἀφαιρεθέν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον.

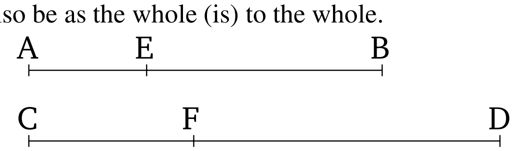


Ἔστω γὰρ ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ ΑΕ πρὸς ἀφαιρεθὲν τὸ ΓΖ· λέγω, ὅτι καὶ λοιπὸν τὸ ΕΒ πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ.

Ἐπεὶ γάρ ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΑΕ πρὸς τὸ ΓΖ, καὶ ἐναλλὰξ ὡς τὸ ΒΑ πρὸς τὸ ΑΕ, οὕτως τὸ ΔΓ πρὸς τὸ ΓΖ. καὶ ἐπεὶ συγκείμενα μεγέθη ἀνάλογόν ἐστιν, καὶ

Proposition 19†

If as the whole is to the whole so the (part) taken away is to the (part) taken away, (then) the remainder to the remainder will also be as the whole (is) to the whole.



For let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF. I say that the remainder EB to the remainder FD will also be as the whole AB (is) to the whole CD.

For since as AB is to CD, so AE (is) to CF, (it is) also (the case), alternately, (that) as BA (is) to AE, so DC (is) to CF [Prop. 5.16]. And since composed magnitudes are propor-

διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ ΒΕ πρὸς τὸ ΕΑ, οὕτως τὸ ΔΖ πρὸς τὸ ΓΖ· καὶ ἐναλλάξ, ὡς τὸ ΒΕ πρὸς τὸ ΔΖ, οὕτως τὸ ΕΑ πρὸς τὸ ΖΓ. ὡς δὲ τὸ ΑΕ πρὸς τὸ ΓΖ, οὕτως ὑπόκειται ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ. καὶ λοιπὸν ἄρα τὸ ΕΒ πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ.

Ἐὰν ἄρα ᾖ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς ἀφαιρεθέν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον [ὅπερ ἔδει δεῖξαι].

[Καὶ ἐπεὶ ἐδείχθη ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΕΒ πρὸς τὸ ΖΔ, καὶ ἐναλλὰξ ὡς τὸ ΑΒ πρὸς τὸ ΒΕ οὕτως τὸ ΓΔ πρὸς τὸ ΖΔ, συγκείμενα ἄρα μεγέθη ἀνάλογόν ἐστιν· ἐδείχθη δὲ ὡς τὸ ΒΑ πρὸς τὸ ΑΕ, οὕτως τὸ ΔΓ πρὸς τὸ ΓΖ· καί ἐστιν ἀναστρέψαντι].

tional, (then) they will also be proportional (when) separated, (so that) as *BE* (is) to *EA*, so *DF* (is) to *CF* [Prop. 5.17]. Also, alternately, as *BE* (is) to *DF*, so *EA* (is) to *FC* [Prop. 5.16]. And it was assumed that as *AE* (is) to *CF*, so the whole *AB* (is) to the whole *CD*. And, thus, as the remainder *EB* (is) to the remainder *FD*, so the whole *AB* will be to the whole *CD*.

Thus, if as the whole is to the whole so the (part) taken away is to the (part) taken away, (then) the remainder to the remainder will also be as the whole (is) to the whole. [(Which is) the very thing it was required to show.]

[And since it was shown (that) as *AB* (is) to *CD*, so *EB* (is) to *FD*, (it is) also (the case), alternately, (that) as *AB* (is) to *BE*, so *CD* (is) to *FD*. Thus, composed magnitudes are proportional. And it was shown (that) as *BA* (is) to *AE*, so *DC* (is) to *CF*. And (the latter) is converted (from the former).]

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ ἀναστρέψαντι ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

Corollary‡

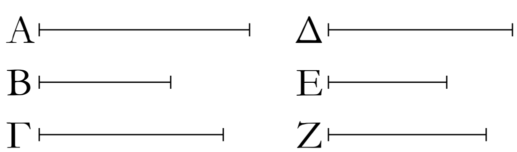
So (it is) clear, from this, that if composed magnitudes are proportional, (then) they will also be proportional (when) converted. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \beta :: \alpha - \gamma : \beta - \delta$.

‡ In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \alpha - \beta :: \gamma : \gamma - \delta$.

κ′.

Ἐὰν ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι᾿ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

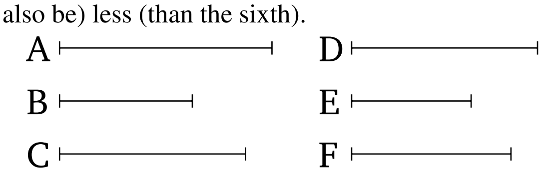


Ἔστω τρία μεγέθη τὰ Α, Β, Γ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, Ε, Ζ, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Δ πρὸς τὸ Ε, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Ε πρὸς τὸ Ζ, δι᾿ ἴσου δὲ μεῖζον ἔστω τὸ Α τοῦ Γ· λέγω, ὅτι καὶ τὸ Δ τοῦ Ζ μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

Ἐπεὶ γὰρ μεῖζόν ἐστι τὸ Α τοῦ Γ, ἄλλο δέ τι τὸ Β, τὸ δὲ μεῖζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἤπερ τὸ ἔλαττον, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. ἀλλ᾿ ὡς μὲν τὸ Α πρὸς τὸ Β [οὕτως] τὸ Δ πρὸς τὸ Ε, ὡς δὲ τὸ Γ πρὸς τὸ Β, ἀνάπαλιν οὕτως τὸ Ζ πρὸς τὸ Ε· καὶ τὸ Δ ἄρα πρὸς τὸ Ε μείζονα λόγον ἔχει ἤπερ τὸ Ζ πρὸς τὸ Ε. τῶν δὲ πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον μεῖζόν

Proposition 20†

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third (then) the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



Let *A*, *B*, and *C* be three magnitudes, and *D*, *E*, *F* other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, (so that) as *A* (is) to *B*, so *D* (is) to *E*, and as *B* (is) to *C*, so *E* (is) to *F*. And let *A* be greater than *C*, via equality. I say that *D* will also be greater than *F*. And if (*A* is) equal (to *C* then *D* will also be) equal (to *F*). And if (*A* is) less (than *C* then *D* will also be) less (than *F*).

For since *A* is greater than *C*, and *B* some other (magnitude), and the greater (magnitude) has a greater ratio than the lesser to the same (magnitude) [Prop. 5.8], *A* thus has a greater ratio to *B* than *C* (has) to *B*. But as *A* (is) to *B*, [so] *D* (is) to *E*. And, inversely, as *C* (is) to *B*, so *F* (is) to *E* [Prop. 5.7 corr.]. Thus, *D* also has a greater ratio to *E* than *F* *ἐστιν. μεῖζον ἄρα τὸ Δ τοῦ Ζ. ὁμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ᾖ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Δ τῷ Ζ, κἂν ἔλαττον, ἔλαττον.*

Ἐὰν ἄρα ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον· ὅπερ ἔδει δεῖξαι.

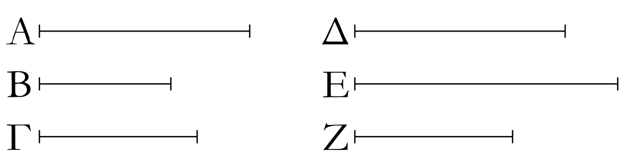
(has) to E [Prop. 5.13]. And for (magnitudes) having a ratio to the same (magnitude), that having the greater ratio is greater [Prop. 5.10]. Thus, D (is) greater than F. Similarly, we can show that even if A is equal to C (then) D will also be equal to F, and even if (A is) less (than C then D will also be) less (than F).

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third, (then) the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And (if the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \delta : \epsilon$ and $\beta : \gamma :: \epsilon : \zeta$ then $\alpha \gtreqless \gamma$ as $\delta \gtreqless \zeta$.

κα΄.

Ἐὰν ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.



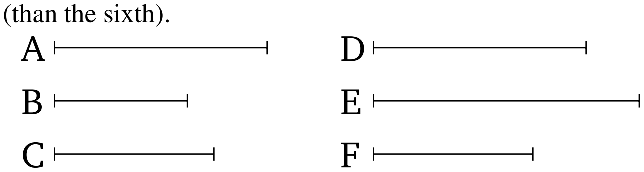
Ἔστω τρία μεγέθη τὰ Α, Β, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, Ε, Ζ, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ἔστω δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ε, δι᾽ ἴσου δὲ τὸ Α τοῦ Γ μεῖζον ἔστω· λέγω, ὅτι καὶ τὸ Δ τοῦ Ζ μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν ἔλαττον, ἔλαττον.

Ἐπεὶ γὰρ μεῖζόν ἐστι τὸ Α τοῦ Γ, ἄλλο δέ τι τὸ Β, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἤπερ τὸ Γ πρὸς τὸ Β. ἀλλ᾽ ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ, ὡς δὲ τὸ Γ πρὸς τὸ Β, ἀνάπαλιν οὕτως τὸ Ε πρὸς τὸ Δ. καὶ τὸ Ε ἄρα πρὸς τὸ Ζ μείζονα λόγον ἔχει ἤπερ τὸ Ε πρὸς τὸ Δ. πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλασσόν ἐστιν· ἔλασσον ἄρα ἐστὶ τὸ Ζ τοῦ Δ· μεῖζον ἄρα ἐστὶ τὸ Δ τοῦ Ζ. ὁμοίως δὴ δείξομεν, ὅτι κἂν ἴσον ᾖ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Δ τῷ Ζ, κἂν ἔλαττον, ἔλαττον.

Ἐὰν ἄρα ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι᾽ ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, κἂν ἴσον, ἴσον, κἂν

Proposition 21†

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third (then) the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



Let A, B, and C be three magnitudes, and D, E, F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B, so E (is) to F, and as B (is) to C, so D (is) to E. And let A be greater than C, via equality. I say that D will also be greater than F. And if (A is) equal (to C then D will also be) equal (to F). And if (A is) less (than C then D will also be) less (than F).

For since A is greater than C, and B some other (magnitude), A thus has a greater ratio to B than C (has) to B [Prop. 5.8]. But as A (is) to B, so E (is) to F. And, inversely, as C (is) to B, so E (is) to D [Prop. 5.7 corr.]. Thus, E also has a greater ratio to F than E (has) to D [Prop. 5.13]. And that (magnitude) to which the same (magnitude) has a greater ratio is (the) lesser (magnitude) [Prop. 5.10]. Thus, F is less than D. Thus, D is greater than F. Similarly, we can show that even if A is equal to C (then) D will also be equal to F, and even if (A is) less (than C then D will also be) less (than F).

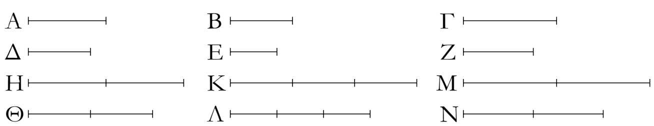
Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by *ἔλαττον, ἔλαττον· ὅπερ ἔδει δεῖξαι.*

two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third (then) the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha \gtreqless \gamma$ as $\delta \gtreqless \zeta$.

κβ′.

Ἐὰν ᾖ ὁποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.



Ἔστω ὁποσαοῦν μεγέθη τὰ Α, Β, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, Ε, Ζ, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Δ πρὸς τὸ Ε, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Ε πρὸς τὸ Ζ· λέγω, ὅτι καὶ δι᾽ ἴσου ἐν τῷ αὐτῳ λόγῳ ἔσται.

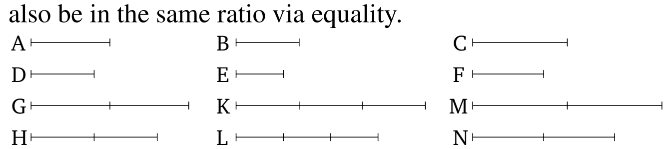
Εἰλήφθω γὰρ τῶν μὲν Α, Δ ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Ε ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Κ, Λ, καὶ ἔτι τῶν Γ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Μ, Ν.

Καὶ ἐπεί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Δ πρὸς τὸ Ε, καὶ εἴληπται τῶν μὲν Α, Δ ἰσάκις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Β, Ε ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Κ, Λ, ἔστιν ἄρα ὡς τὸ Η πρὸς τὸ Κ, οὕτως τὸ Θ πρὸς τὸ Λ. δὶα τὰ αὐτὰ δὴ καὶ ὡς τὸ Κ πρὸς τὸ Μ, οὕτως τὸ Λ πρὸς τὸ Ν. ἐπεὶ οὖν τρία μεγέθη ἐστὶ τὰ Η, Κ, Μ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Θ, Λ, Ν, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι᾽ ἴσου ἄρα, εἰ ὑπερέχει τὸ Η τοῦ Μ, ὑπερέχει καὶ τὸ Θ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Θ τῶν Α, Δ ἰσάκις πολλαπλάσια, τὰ δὲ Μ, Ν τῶν Γ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια. ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ζ.

Ἐὰν ἄρα ᾖ ὁποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι.

Proposition 22†

If there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, (then) they will also be in the same ratio via equality.



Let there be any number of magnitudes whatsoever, *A*, *B*, *C*, and (some) other (magnitudes), *D*, *E*, *F*, of equal number to them, (which are) in the same ratio taken two by two, (so that) as *A* (is) to *B*, so *D* (is) to *E*, and as *B* (is) to *C*, so *E* (is) to *F*. I say that they will also be in the same ratio via equality. (That is, as *A* is to *C*, so *D* is to *F*.)

For let the equal multiples *G* and *H* be taken of *A* and *D* (respectively), and the other random equal multiples *K* and *L* of *B* and *E* (respectively), and the yet other random equal multiples *M* and *N* of *C* and *F* (respectively).

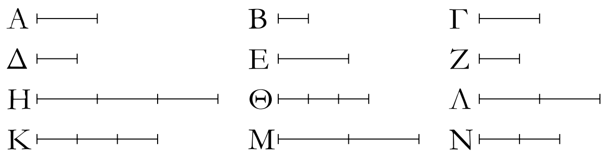
And since as *A* is to *B*, so *D* (is) to *E*, and the equal multiples *G* and *H* have been taken of *A* and *D* (respectively), and the other random equal multiples *K* and *L* of *B* and *E* (respectively), thus as *G* is to *K*, so *H* (is) to *L* [Prop. 5.4]. And, so, for the same (reasons), as *K* (is) to *M*, so *L* (is) to *N*. Therefore, since *G*, *K*, and *M* are three magnitudes, and *H*, *L*, and *N* other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, thus, via equality, if *G* exceeds *M* (then) *H* also exceeds *N*, and if (*G* is) equal (to *M* then *H* is also) equal (to *N*), and if (*G* is) less (than *M* then *H* is also) less (than *N*) [Prop. 5.20]. And *G* and *H* are equal multiples of *A* and *D* (respectively), and *M* and *N* other random equal multiples of *C* and *F* (respectively). Thus, as *A* is to *C*, so *D* (is) to *F* [Def. 5.5].

Thus, if there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, (then) they will also be in the same ratio via equality. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \zeta : \eta$ and $\gamma : \delta :: \eta : \theta$ then $\alpha : \delta :: \epsilon : \theta$.

κγ΄.

Ἐὰν ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, καὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.



Ἔστω τρία μεγέθη τὰ Α, Β, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ τὰ Δ, Ε, Ζ, ἔστω δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, ὡς μὲν τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ, ὡς δὲ τὸ Β πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ε· λέγω, ὅτι ἐστὶν ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ζ.

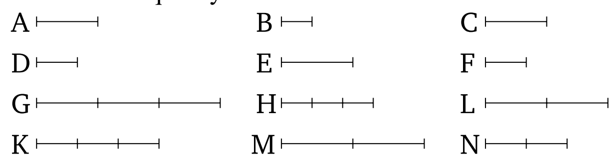
Εἰλήφθω τῶν μὲν Α, Β, Δ ἰσάκις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Γ, Ε, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεὶ ἰσάκις ἐστὶ πολλαπλάσια τὰ Η, Θ τῶν Α, Β, τὰ δὲ μέρη τοὶς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Η πρὸς τὸ Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς τὸ Ε πρὸς τὸ Ζ, οὕτως τὸ Μ πρὸς τὸ Ν· καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ· καὶ ὡς ἄρα τὸ Η πρὸς τὸ Θ, οὕτως τὸ Μ πρὸς τὸ Ν. καὶ ἐπεί ἐστιν ὡς τὸ Β πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ε, καὶ ἐναλλὰξ ὡς τὸ Β πρὸς τὸ Δ, οὕτως τὸ Γ πρὸς τὸ Ε. καὶ ἐπεὶ τὰ Θ, Κ τῶν Β, Δ ἰσάκις ἐστὶ πολλαπλάσια, τὰ δὲ μέρη τοῖς ἰσάκις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ Β πρὸς τὸ Δ, οὕτως τὸ Θ πρὸς τὸ Κ. ἀλλ᾽ ὡς τὸ Β πρὸς τὸ Δ, οὕτως τὸ Γ πρὸς τὸ Ε· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ Κ, οὕτως τὸ Γ πρὸς τὸ Ε. πάλιν, ἐπεὶ τὰ Λ, Μ τῶν Γ, Ε ἰσάκις ἐστι πολλαπλάσια, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ε, οὕτως τὸ Λ πρὸς τὸ Μ. ἀλλ᾽ ὡς τὸ Γ πρὸς τὸ Ε, οὕτως τὸ Θ πρὸς τὸ Κ· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ Κ, οὕτως τὸ Λ πρὸς τὸ Μ, καὶ ἐναλλὰξ ὡς τὸ Θ πρὸς τὸ Λ, τὸ Κ πρὸς τὸ Μ. ἐδείχθη δὲ καὶ ὡς τὸ Η πρὸς τὸ Θ, οὕτως τὸ Μ πρὸς τὸ Ν. ἐπεὶ οὖν τρία μεγέθη ἐστὶ τὰ Η, Θ, Λ, καὶ ἄλλα αὐτοις ἴσα τὸ πλῆθος τὰ Κ, Μ, Ν σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καί ἐστιν αὐτῶν τεταραγμένη ἡ ἀναλογία, δι᾽ ἴσου ἄρα, εἰ ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καί ἐστι τὰ μὲν Η, Κ τῶν Α, Δ ἰσάκις πολλαπλάσια, τὰ δὲ Λ, Ν τῶν Γ, Ζ. ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Ζ.

Ἐὰν ἄρα ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ᾖ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, καὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι.

Proposition 23†

If there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, (then) they will also be in the same ratio via equality.



Let A, B, and C be three magnitudes, and D, E and F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B, so E (is) to F, and as B (is) to C, so D (is) to E. I say that as A is to C, so D (is) to F.

Let the equal multiples G, H, and K be taken of A, B, and D (respectively), and the other random equal multiples L, M, and N of C, E, and F (respectively).

And since G and H are equal multiples of A and B (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as A (is) to B, so G (is) to H. And, so, for the same (reasons), as E (is) to F, so M (is) to N. And as A is to B, so E (is) to F. And, thus, as G (is) to H, so M (is) to N [Prop. 5.11]. And since as B is to C, so D (is) to E, also, alternately, as B (is) to D, so C (is) to E [Prop. 5.16]. And since H and K are equal multiples of B and D (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as B is to D, so H (is) to K. But, as B (is) to D, so C (is) to E. And, thus, as H (is) to K, so C (is) to E [Prop. 5.11]. Again, since L and M are equal multiples of C and E (respectively), thus as C is to E, so L (is) to M [Prop. 5.15]. But, as C (is) to E, so H (is) to K. And, thus, as H (is) to K, so L (is) to M [Prop. 5.11]. Also, alternately, as H (is) to L, so K (is) to M [Prop. 5.16]. And it was also shown (that) as G (is) to H, so M (is) to N. Therefore, since G, H, and L are three magnitudes, and K, M, and N other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, and their proportion is perturbed, thus, via equality, if G exceeds L (then) K also exceeds N, and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N) [Prop. 5.21]. And G and K are equal multiples of A and D (respectively), and L and N of C and F (respectively). Thus, as A (is) to C, so D (is) to F [Def. 5.5].

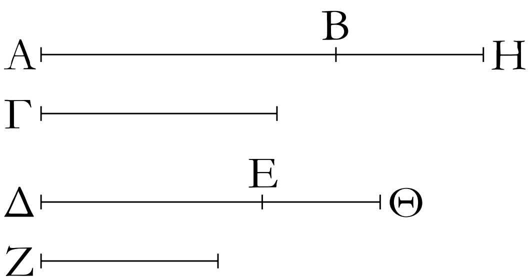
Thus, if there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, (then) they will also be in the same ratio via equality. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha : \gamma :: \delta : \zeta$.

κδ΄.

Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἕκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἕξει λόγον καὶ τρίτον καὶ ἕκτον πρὸς τέταρτον.

Πρῶτον γὰρ τὸ AB *πρὸς δεύρερον τὸ* Γ *τὸν αὐτὸν ἐχέτω λόγον καὶ τρίτον τὸ* ΔE *πρὸς τέταρτον τὸ* Z*, ἐχέτω δὲ καὶ πέμπτον τὸ* BH *πρὸς δεύτερον τὸ* Γ *τὸν αὐτὸν λόγον καὶ ἕκτον τὸ* $E\Theta$ *πρὸς τέταρτον τὸ* Z*· λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτον τὸ* AH *πρὸς δεύτερον τὸ* Γ *τὸν αὐτὸν ἕξει λόγον, καὶ τρίτον καὶ ἕκτον τὸ* $\Delta\Theta$ *πρὸς τέταρτον τὸ* Z*.*



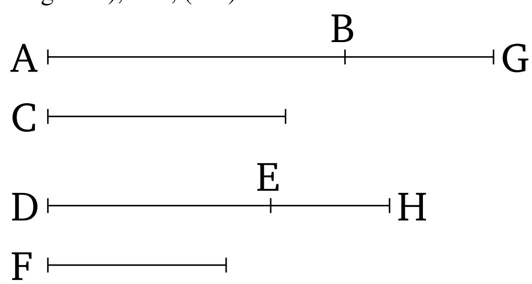
Ἐπεὶ γάρ ἐστιν ὡς τὸ BH *πρὸς τὸ* Γ*, οὕτως τὸ* $E\Theta$ *πρὸς τὸ* Z*, ἀνάπαλιν ἄρα ὡς τὸ* Γ *πρὸς τὸ* BH*, οὕτως τὸ* Z *πρὸς τὸ* $E\Theta$*. ἐπεὶ οὖν ἐστιν ὡς τὸ* AB *πρὸς τὸ* Γ*, οὕτως τὸ* ΔE *πρὸς τὸ* Z*, ὡς δὲ τὸ* Γ *πρὸς τὸ* BH*, οὕτως τὸ* Z *πρὸς τὸ* $E\Theta$*, δι᾽ ἴσου ἄρα ἐστὶν ὡς τὸ* AB *πρὸς τὸ* BH*, οὕτως τὸ* ΔE *πρὸς τὸ* $E\Theta$*. καὶ ἐπεὶ διῃρημένα μεγέθη ἀνάλογόν ἐστιν, καὶ συντεθέντα ἀνάλογον ἔσται· ἔστιν ἄρα ὡς τὸ* AH *πρὸς τὸ* HB*, οὕτως τὸ* $\Delta\Theta$ *πρὸς τὸ* ΘE*. ἔστι δὲ καὶ ὡς τὸ* BH *πρὸς τὸ* Γ*, οὕτως τὸ* $E\Theta$ *πρὸς τὸ* Z*· δι᾽ ἴσου ἄρα ἐστὶν ὡς τὸ* AH *πρὸς τὸ* Γ*, οὕτως τὸ* $\Delta\Theta$ *πρὸς τὸ* Z*.*

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἕκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἕξει λόγον καὶ τρίτον καὶ ἕκτον πρὸς τέταρτον· ὅπερ ἔδει δεῖξαι.

Proposition 24†

If a first (magnitude) has to a second the same ratio that third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, (then) the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and sixth (added together, have) to the fourth.

For let a first (magnitude) AB have the same ratio to a second C that a third DE (has) to a fourth F. And let a fifth (magnitude) BG also have the same ratio to the second C that a sixth EH (has) to the fourth F. I say that the first (magnitude) and the fifth, added together, AG, will also have the same ratio to the second C that the third (magnitude) and the sixth, (added together), DH, (has) to the fourth F.



For since as BG is to C, so EH (is) to F, thus, inversely, as C (is) to BG, so F (is) to EH [Prop. 5.7 corr.]. Therefore, since as AB is to C, so DE (is) to F, and as C (is) to BG, so F (is) to EH, thus, via equality, as AB is to BG, so DE (is) to EH [Prop. 5.22]. And since separated magnitudes are proportional, (then) they will also be proportional (when) composed [Prop. 5.18]. Thus, as AG is to GB, so DH (is) to HE. And, also, as BG is to C, so EH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F [Prop. 5.22].

Thus, if a first (magnitude) has to a second the same ratio that a third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, (then) the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and the sixth (added together, have) to the fourth. (Which is) the very thing it was required to show.

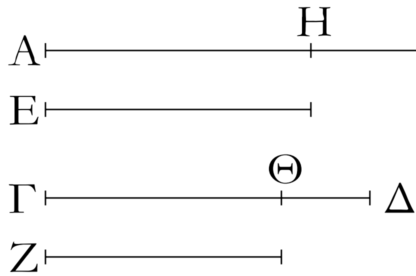
† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\epsilon : \beta :: \zeta : \delta$ then $\alpha + \epsilon : \beta :: \gamma + \zeta : \delta$.

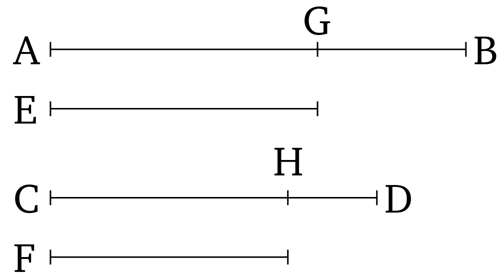
κε΄.

Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ μέγιστον [αὐτῶν] καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν.

Proposition 25†

If four magnitudes are proportional, (then) the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes).





Ἔστω τέσσαρα μεγέθη ἀνάλογον τὰ ΑΒ, ΓΔ, Ε, Ζ, ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ Ε πρὸς τὸ Ζ, ἔστω δὲ μέγιστον μὲν αὐτῶν τὸ ΑΒ, ἐλάχιστον δὲ τὸ Ζ· λέγω, ὅτι τὰ ΑΒ, Ζ τῶν ΓΔ, Ε μείζονά ἐστιν.

Κείσθω γὰρ τῷ μὲν Ε ἴσον τὸ ΑΗ, τῷ δὲ Ζ ἴσον τὸ ΓΘ.

Ἐπεὶ [οὖν] ἐστιν ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ Ε πρὸς τὸ Ζ, ἴσον δὲ τὸ μὲν Ε τῷ ΑΗ, τὸ δὲ Ζ τῷ ΓΘ, ἔστιν ἄρα ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΑΗ πρὸς τὸ ΓΘ. καὶ ἐπεί ἐστιν ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ ΑΗ πρὸς ἀφαιρεθὲν τὸ ΓΘ, καὶ λοιπὸν ἄρα τὸ ΗΒ πρὸς λοιπὸν τὸ ΘΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ. μεῖζον δὲ τὸ ΑΒ τοῦ ΓΔ· μεῖζον ἄρα καὶ τὸ ΗΒ τοῦ ΘΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν ΑΗ τῷ Ε, τὸ δὲ ΓΘ τῷ Ζ, τὰ ἄρα ΑΗ, Ζ ἴσα ἐστὶ τοῖς ΓΘ, Ε. καὶ [ἐπεὶ] ἐὰν [ἀνίσοις ἴσα προστεθῇ, τὰ ὅλα ἄνισά ἐστιν, ἐὰν ἄρα] τῶν ΗΒ, ΘΔ ἀνίσων ὄντων καὶ μείζονος τοῦ ΗΒ τῷ μὲν ΗΒ προστεθῇ τὰ ΑΗ, Ζ, τῷ δὲ ΘΔ προστεθῇ τὰ ΓΘ, Ε, συνάγεται τὰ ΑΒ, Ζ μείζονα τῶν ΓΔ, Ε.

Ἐὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ μέγιστον αὐτῶν καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν. ὅπερ ἔδει δεῖξαι.

Let AB, CD, E, and F be four proportional magnitudes, (such that) as AB (is) to CD, so E (is) to F. And let AB be the greatest of them, and F the least. I say that AB and F is greater than CD and E.

For let AG be made equal to E, and CH equal to F.

[In fact,] since as AB is to CD, so E (is) to F, and E (is) equal to AG, and F to CH, thus as AB is to CD, so AG (is) to CH. And since the whole AB is to the whole CD as the (part) taken away AG (is) to the (part) taken away CH, thus the remainder GB will also be to the remainder HD as the whole AB (is) to the whole CD [Prop. 5.19]. And AB (is) greater than CD. Thus, GB (is) also greater than HD. And since AG is equal to E, and CH to F, thus AG and F is equal to CH and E. And [since] if [equal (magnitudes) are added to unequal (magnitudes then) the wholes are unequal, thus if] AG and F are added to GB, and CH and E to HD—GB and HD being unequal, and GB greater—it is inferred that AB and F (is) greater than CD and E.

Thus, if four magnitudes are proportional, (then) the (sum of the) largest and the smallest of them is greater than the (sum of the) remaining two (magnitudes). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$, and α is the greatest and δ the least, then $\alpha + \delta > \beta + \gamma$.

ELEMENTS BOOK 6

Similar Figures

Ὅροι.

αʹ. Ὅμοια σχήματα εὐθύγραμμά ἐστιν, ὅσα τάς τε γωνίας ἴσας ἔχει κατὰ μίαν καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον.

βʹ. Ἄκρον καὶ μέσον λόγον εὐθεῖα τετμῆσθαι λέγεται, ὅταν ᾖ ὡς ἡ ὅλη πρὸς τὸ μεῖζον τμῆμα, οὕτως τὸ μεῖζον πρὸς τὸ ἔλαττὸν.

γʹ. Ὕψος ἐστὶ πάντος σχήματος ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἀγομένη.

Definitions

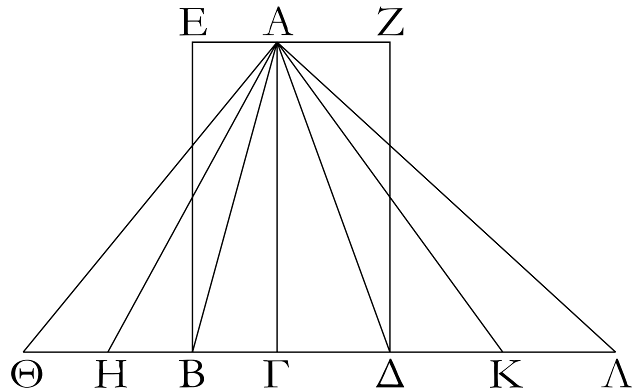
1. Similar rectilinear figures are those (which) have (their) angles separately equal, and the (corresponding) sides about the equal angles proportional.

2. A straight-line is said to be cut in extreme and mean ratio when as the whole is to the greater segment so the greater (segment is) to the lesser.

3. The height of any figure is the (straight-line) drawn from the vertex perpendicular to the base.

αʹ.

Τὰ τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.



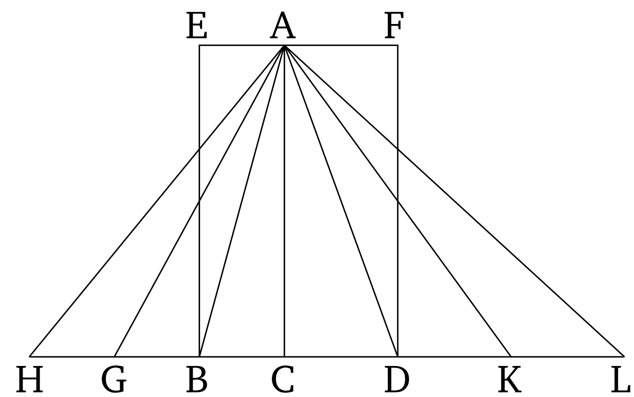
Ἔστω τρίγωνα μὲν τὰ ΑΒΓ, ΑΓΔ, παραλληλόγραμμα δὲ τὰ ΕΓ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος τὸ ΑΓ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, καὶ τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον.

Ἐκβεβλήσθω γὰρ ἡ ΒΔ ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ τὰ Θ, Λ σημεῖα, καὶ κείσθωσαν τῇ μὲν ΒΓ βάσει ἴσαι [ὁσαιδηποτοῦν] αἱ ΒΗ, ΗΘ, τῇ δὲ ΓΔ βάσει ἴσαι ὁσαιδηποτοῦν αἱ ΔΚ, ΚΛ, καὶ ἐπεζεύχθωσαν αἱ ΑΗ, ΑΘ, ΑΚ, ΑΛ.

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΓΒ, ΒΗ, ΗΘ ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ ΑΘΗ, ΑΗΒ, ΑΒΓ τρίγωνα ἀλλήλοις. ὁσαπλασίων ἄρα ἐστὶν ἡ ΘΓ βάσις τῆς ΒΓ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΒΓ τριγώνου. διὰ τὰ αὐτὰ δὴ ὁσαπλασίων ἐστὶν ἡ ΛΓ βάσις τῆς ΓΔ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΑΛΓ τρίγωνον τοῦ ΑΓΔ τριγώνου· καὶ εἰ ἴση ἐστὶν ἡ ΘΓ βάσις τῇ ΓΛ βάσει, ἴσον ἐστὶ καὶ τὸ ΑΘΓ τρίγωνον τῳ ΑΓΛ τριγώνῳ, καὶ εἰ ὑπερέχει ἡ ΘΓ βάσις τῆς ΓΛ βάσεως, ὑπερέχει καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΓΛ τριγώνου, καὶ εἰ ἐλάσσων, ἔλασσον. τεσσάρων δὴ ὄντων μεγεθῶν δύο μὲν βάσεων τῶν ΒΓ, ΓΔ, δύο δὲ τριγώνων τῶν ΑΒΓ, ΑΓΔ εἴληπται ἰσάκις πολλαπλάσια τῆς μὲν ΒΓ βάσεως καὶ τοῦ ΑΒΓ τριγώνου ἥ τε ΘΓ βάσις καὶ τὸ ΑΘΓ τρίγωνον, τῆς δὲ ΓΔ βάσεως καὶ τοῦ ΑΔΓ τριγώνου ἄλλα, ἃ ἔτυχεν,

Proposition 1†

Triangles and parallelograms which are of the same height are to one another as their bases.



Let *ABC* and *ACD* be triangles, and *EC* and *CF* parallelograms, of the same height *AC*. I say that as base *BC* is to base *CD*, so triangle *ABC* (is) to triangle *ACD*, and parallelogram *EC* to parallelogram *CF*.

For let the (straight-line) *BD* be produced in each direction to points *H* and *L*, and let [any number] (of straight-lines) *BG* and *GH* be made equal to base *BC*, and any number (of straight-lines) *DK* and *KL* equal to base *CD*. And let *AG*, *AH*, *AK*, and *AL* be joined.

And since *CB*, *BG*, and *GH* are equal to one another, triangles *AHG*, *AGB*, and *ABC* are also equal to one another [Prop. 1.38]. Thus, as many times as base *HC* is (divisible by) base *BC*, so many times is triangle *AHC* also (divisible) by triangle *ABC*. So, for the same (reasons), as many times as base *LC* is (divisible) by base *CD*, so many times is triangle *ALC* also (divisible) by triangle *ACD*. And if base *HC* is equal to base *CL*, (then) triangle *AHC* is also equal to triangle *ACL* [Prop. 1.38]. And if base *HC* exceeds base *CL*, (then) triangle *AHC* also exceeds triangle *ACL*.‡ And if (*HC* is) less (than *CL* then *AHC* is also) less (than *ACL*). So, their being four magnitudes, two bases, *BC* and *CD*, and two triangles, *ABC* and *ACD*, equal multiples have been taken of base *BC* and triangle *ABC*—(namely), base *HC* and triangle *AHC*—and other ran-

ἰσάκις πολλαπλάσια ἥ τε ΛΓ βάσις καὶ τὸ ΑΛΓ τρίγωνον· καὶ δέδεικται, ὅτι, εἰ ὑπερέχει ἡ ΘΓ βάσις τῆς ΓΛ βάσεως, ὑπερέχει καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΛΓ τριγώνου, καί εἰ ἴση, ἴσον, καὶ εἰ ἔλασσων, ἔλασσον· ἔστιν ἄρα ὡς ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον.

Καὶ ἐπεὶ τοῦ μὲν ΑΒΓ τριγώνου διπλάσιόν ἐστι τὸ ΕΓ παραλληλόγραμμον, τοῦ δὲ ΑΓΔ τριγώνου διπλάσιόν ἐστι τὸ ΖΓ παραλληλόγραμμον, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΖΓ παραλληλόγραμμον. ἐπεὶ οὖν ἐδείχθη, ὡς μὲν ἡ ΒΓ βάσις πρὸς τὴν ΓΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, ὡς δὲ τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον, καὶ ὡς ἄρα ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΖΓ παραλληλόγραμμον.

Τὰ ἄρα τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

dom equal multiples of base *CD* and triangle *ADC*—(namely), base *LC* and triangle *ALC*. And it has been shown that if base *HC* exceeds base *CL*, (then) triangle *AHC* also exceeds triangle *ALC*, and if (*HC* is) equal (to *CL* then *AHC* is also) equal (to *ALC*), and if (*HC* is) less (than *CL* then *AHC* is also) less (than *ALC*). Thus, as base *BC* is to base *CD*, so triangle *ABC* (is) to triangle *ACD* [Def. 5.5]. And since parallelogram *EC* is double triangle *ABC*, and parallelogram *FC* is double triangle *ACD* [Prop. 1.34], and parts have the same ratio as similar multiples [Prop. 5.15], thus as triangle *ABC* is to triangle *ACD*, so parallelogram *EC* (is) to parallelogram *FC*. In fact, since it was shown that as base *BC* (is) to *CD*, so triangle *ABC* (is) to triangle *ACD*, and as triangle *ABC* (is) to triangle *ACD*, so parallelogram *EC* (is) to parallelogram *CF*, thus, also, as base *BC* (is) to base *CD*, so parallelogram *EC* (is) to parallelogram *FC* [Prop. 5.11].

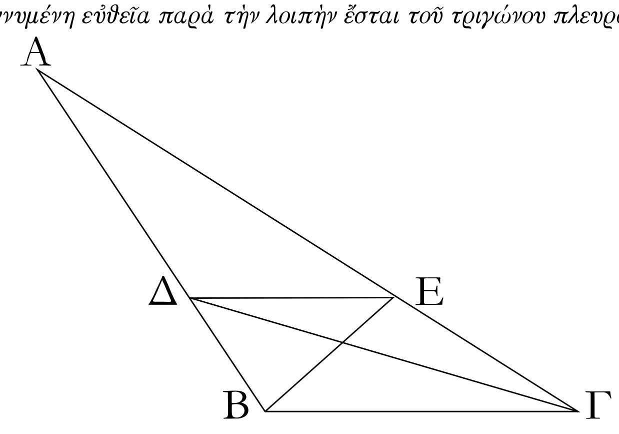
Thus, triangles and parallelograms which are of the same height are to one another as their bases. (Which is) the very thing it was required to show.

† As is easily demonstrated, this proposition holds even when the triangles, or parallelograms, do not share a common side, and/or are not right-angled.

‡ This is a straight-forward generalization of Prop. 1.38.

βʹ.

Ἐὰν τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῇ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς· καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν.



Proposition 2

If some straight-line is drawn parallel to one of the sides of a triangle, (then) it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally, (then) the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle.

Τριγώνου γὰρ τοῦ ΑΒΓ παράλληλος μιᾷ τῶν πλευρῶν τῇ ΒΓ ἤχθω ἡ ΔΕ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ.

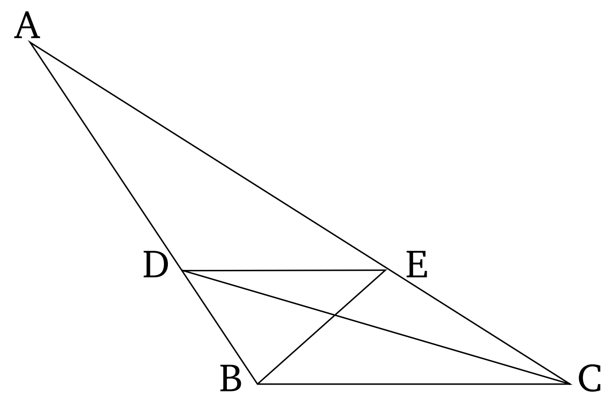
Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΔ.

Ἴσον ἄρα ἐστὶ τὸ ΒΔΕ τρίγωνον τῷ ΓΔΕ τριγώνῳ· ἐπὶ γὰρ τῆς αὐτῆς βάσεώς ἐστι τῆς ΔΕ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΔΕ, ΒΓ· ἄλλο δέ τι τὸ ΑΔΕ τρίγωνον. τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον· ἔστιν ἄρα ὡς τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ [τρίγωνον], οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον. ἀλλ᾽ ὡς μὲν τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ· ὑπὸ γὰρ τὸ αὐτὸ ὕψος ὄντα τὴν ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΒ κάθετον ἀγομένην πρὸς ἄλληλά εἰσιν ὡς αἱ βάσεις. διὰ τὰ αὐτὰ δὴ ὡς τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ· καὶ ὡς ἄρα ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ.

Ἀλλὰ δὴ αἱ τοῦ ΑΒΓ τριγώνου πλευραὶ αἱ ΑΒ, ΑΓ ἀνάλογον τετμήσθωσαν, ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ, καὶ ἐπεζεύχθω ἡ ΔΕ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΔΕ τῇ ΒΓ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΑ, ἀλλ᾽ ὡς μὲν ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, ὡς δὲ ἡ ΓΕ πρὸς τὴν ΕΑ, οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, καὶ ὡς ἄρα τὸ ΒΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον, οὕτως τὸ ΓΔΕ τρίγωνον πρὸς τὸ ΑΔΕ τρίγωνον. ἑκάτερον ἄρα τῶν ΒΔΕ, ΓΔΕ τριγώνων πρὸς τὸ ΑΔΕ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΒΔΕ τρίγωνον τῷ ΓΔΕ τριγώνῳ· καί εἰσιν ἐπὶ τῆς αὐτῆς βάσεως τῆς ΔΕ. τὰ δὲ ἴσα τρίγωνα καὶ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν. παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΓ.

Ἐὰν ἄρα τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῇ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς· καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν· ὅπερ ἔδει δεῖξαι.



For let *DE* be drawn parallel to one of the sides *BC* of triangle *ABC*. I say that as *BD* is to *DA*, so *CE* (is) to *EA*.

For let *BE* and *CD* be joined.

Thus, triangle *BDE* is equal to triangle *CDE*. For they are on the same base *DE* and between the same parallels *DE* and *BC* [Prop. 1.38]. And *ADE* is some other triangle. And equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7]. Thus, as triangle *BDE* is to [triangle] *ADE*, so triangle *CDE* (is) to triangle *ADE*. But, as triangle *BDE* (is) to triangle *ADE*, so (is) *BD* to *DA*. For, having the same height—(namely), the (straight-line) drawn from *E* perpendicular to *AB*—they are to one another as their bases [Prop. 6.1]. So, for the same (reasons), as triangle *CDE* (is) to *ADE*, so *CE* (is) to *EA*. And, thus, as *BD* (is) to *DA*, so *CE* (is) to *EA* [Prop. 5.11].

And so, let the sides *AB* and *AC* of triangle *ABC* be cut proportionally (such that) as *BD* (is) to *DA*, so *CE* (is) to *EA*. And let *DE* be joined. I say that *DE* is parallel to *BC*.

For, by the same construction, since as *BD* is to *DA*, so *CE* (is) to *EA*, but as *BD* (is) to *DA*, so triangle *BDE* (is) to triangle *ADE*, and as *CE* (is) to *EA*, so triangle *CDE* (is) to triangle *ADE* [Prop. 6.1], thus, also, as triangle *BDE* (is) to triangle *ADE*, so triangle *CDE* (is) to triangle *ADE* [Prop. 5.11]. Thus, triangles *BDE* and *CDE* each have the same ratio to *ADE*. Thus, triangle *BDE* is equal to triangle *CDE* [Prop. 5.9]. And they are on the same base *DE*. And equal triangles, which are also on the same base, are also between the same parallels [Prop. 1.39]. Thus, *DE* is parallel to *BC*.

Thus, if some straight-line is drawn parallel to one of the sides of a triangle, (then) it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally, (then) the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle. (Which is) the very thing it was required to show.

γ′.

Ἐὰν τριγώνου ἡ γωνία δίχα τμηθῇ, ἡ δὲ τέμνουσα τὴν

Proposition 3

If an angle of a triangle is cut in half, and the straight-line *γωνίαν εὐθεῖα τέμνῃ καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἕξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχῃ λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένη εὐθεῖα δίχα τεμεῖ τὴν τοῦ τριγώνου γωνίαν.*

Ἔστω τρίγωνον τὸ ΑΒΓ, καὶ τετμήσθω ἡ ὑπὸ ΒΑΓ γωνία δίχα ὑπὸ τῆς ΑΔ εὐθείας· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΓΔ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ.

Ἤχθω γὰρ διὰ τοῦ Γ τῇ ΔΑ παράλληλος ἡ ΓΕ, καὶ διαχθεῖσα ἡ ΒΑ συμπιπτέτω αὐτῇ κατὰ τὸ Ε.

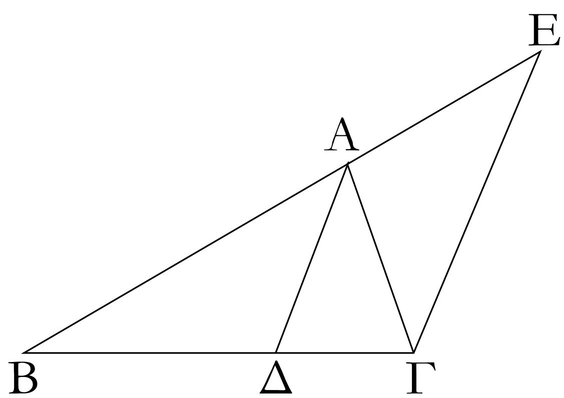
Καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΔ, ΕΓ εὐθεῖα ἐνέπεσεν ἡ ΑΓ, ἡ ἄρα ὑπὸ ΑΓΕ γωνία ἴση ἐστὶ τῇ ὑπὸ ΓΑΔ. ἀλλ᾽ ἡ ὑπὸ ΓΑΔ τῇ ὑπὸ ΒΑΔ ὑπόκειται ἴση· καὶ ἡ ὑπὸ ΒΑΔ ἄρα τῇ ὑπὸ ΑΓΕ ἐστιν ἴση. πάλιν, ἐπεὶ εἰς παραλλήλους τὰς ΑΔ, ΕΓ εὐθεῖα ἐνέπεσεν ἡ ΒΑΕ, ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΑΔ ἴση ἐστὶ τῇ ἐντὸς τῇ ὑπὸ ΑΕΓ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΕ τῇ ὑπὸ ΒΑΔ ἴση· καὶ ἡ ὑπὸ ΑΓΕ ἄρα γωνία τῇ ὑπὸ ΑΕΓ ἐστιν ἴση· ὥστε καὶ πλευρὰ ἡ ΑΕ πλευρᾷ τῇ ΑΓ ἐστιν ἴση. καὶ ἐπεὶ τριγώνου τοῦ ΒΓΕ παρὰ μίαν τῶν πλευρῶν τὴν ΕΓ ἦκται ἡ ΑΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΕ. ἴση δὲ ἡ ΑΕ τῇ ΑΓ· ὡς ἄρα ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ.

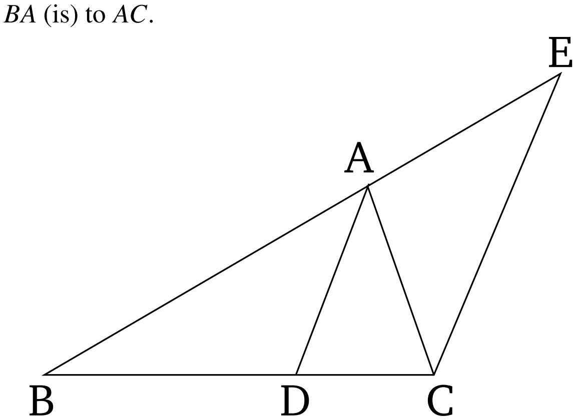
cutting the angle also cuts the base, (then) the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, (then) the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half.

Let ABC be a triangle. And let the angle BAC be cut in half by the straight-line AD. I say that as BD is to CD, so BA (is) to AC.

For let CE be drawn through (point) C parallel to DA. And, BA being drawn through, let it meet (CE) at (point) E.†

And since the straight-line AC falls across the parallel (straight-lines) AD and EC, angle ACE is thus equal to CAD [Prop. 1.29]. But, (angle) CAD is assumed (to be) equal to BAD. Thus, (angle) BAD is also equal to ACE. Again, since the straight-line BAE falls across the parallel (straight-lines) AD and EC, the external angle BAD is equal to the internal (angle) AEC [Prop. 1.29]. And (angle) ACE was also shown (to be) equal to BAD. Thus, angle ACE is also equal to AEC. And, hence, side AE is equal to side AC [Prop. 1.6]. And since AD has been drawn parallel to one of the sides EC of triangle BCE, thus, proportionally, as BD is to DC, so BA (is) to AE [Prop. 6.2]. And AE (is) equal to AC. Thus, as BD (is) to DC, so BA (is) to AC.





Ἀλλὰ δὴ ἔστω ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ, καὶ ἐπεζεύχθω ἡ ΑΔ· λέγω, ὅτι δίχα τέτμηται ἡ ὑπὸ ΒΑΓ γωνία ὑπὸ τῆς ΑΔ εὐθείας.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΓ, ἀλλὰ καὶ ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἐστὶν ἡ ΒΑ πρὸς τὴν ΑΕ· τριγώνου γὰρ τοῦ ΒΓΕ παρὰ μίαν τὴν ΕΓ ἦκται ἡ ΑΔ· καὶ ὡς ἄρα ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΕ. ἴση ἄρα ἡ ΑΓ τῇ ΑΕ· ὥστε καὶ γωνία ἡ ὑπὸ ΑΕΓ τῇ ὑπὸ ΑΓΕ ἐστιν ἴση. ἀλλ᾽ ἡ μὲν ὑπὸ ΑΕΓ τῇ ἐκτὸς τῇ ὑπὸ ΒΑΔ [ἐστιν] ἴση, ἡ δὲ ὑπὸ ΑΓΕ τῇ ἐναλλὰξ τῇ ὑπὸ ΓΑΔ ἐστιν ἴση· καὶ ἡ ὑπὸ ΒΑΔ ἄρα τῇ ὑπὸ ΓΑΔ ἐστιν ἴση. ἡ ἄρα ὑπὸ ΒΑΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΔ εὐθείας.

Ἐὰν ἄρα τριγώνου ἡ γωνία δίχα τμηθῇ, ἡ δὲ τέμνουσα τὴν

And so, let BD be to DC, as BA (is) to AC. And let AD be joined. I say that angle BAC has been cut in half by the straight-line AD.

For, by the same construction, since as BD is to DC, so BA (is) to AC, (then) also as BD (is) to DC, so BA is to AE. For AD has been drawn parallel to one (of the sides) EC of triangle BCE [Prop. 6.2]. Thus, also, as BA (is) to AC, so BA (is) to AE [Prop. 5.11]. Thus, AC (is) equal to AE [Prop. 5.9]. And, hence, angle AEC is equal to ACE [Prop. 1.5]. But, AEC [is] equal to the external (angle) BAD, and ACE is equal to the alternate (angle) CAD [Prop. 1.29]. Thus, (angle) BAD is also equal to CAD. Thus, angle BAC has been cut in half by the straight-line AD.

Thus, if an angle of a triangle is cut in half, and the straight-γωνίαν εὐθεῖα τέμνῃ καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἕξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχῃ λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένη εὐθεῖα δίχα τέμνει τὴν τοῦ τριγώνου γωνίαν· ὅπερ ἔδει δεῖξαι.

line cutting the angle also cuts the base, (then) the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, (then) the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half. (Which is) the very thing it was required to show.

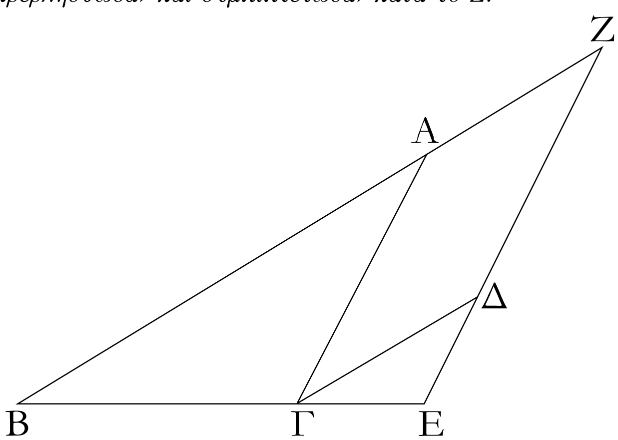
† The fact that the two straight-lines meet follows because the sum of ACE and CAE is less than two right-angles, as can easily be demonstrated. See Post. 5.

δʹ.

Τῶν ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.

Ἔστω ἰσογώνια τρίγωνα τὰ ΑΒΓ, ΔΓΕ ἴσην ἔχοντα τὴν μὲν ὑπὸ ΑΒΓ γωνίαν τῇ ὑπὸ ΔΓΕ, τὴν δὲ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΕ καὶ ἔτι τὴν ὑπὸ ΑΓΒ τῇ ὑπὸ ΓΕΔ· λέγω, ὅτι τῶν ΑΒΓ, ΔΓΕ τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.

Κείσθω γὰρ ἐπ᾽ εὐθείας ἡ ΒΓ τῇ ΓΕ. καὶ ἐπεὶ αἱ ὑπὸ ΑΒΓ, ΑΓΒ γωνίαι δύο ὀρθῶν ἐλάττονές εἰσιν, ἴση δὲ ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΕΓ, αἱ ἄρα ὑπὸ ΑΒΓ, ΔΕΓ δύο ὀρθῶν ἐλάττονές εἰσιν· αἱ ΒΑ, ΕΔ ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Ζ.



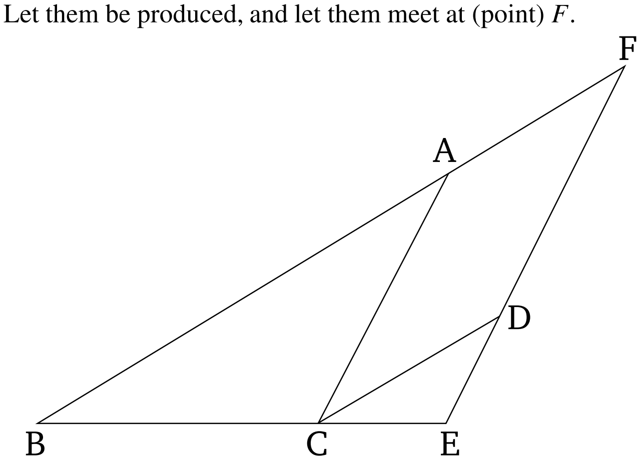
Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΓΕ γωνία τῇ ὑπὸ ΑΒΓ, παράλληλός ἐστιν ἡ ΒΖ τῇ ΓΔ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΕΓ, παράλληλός ἐστιν ἡ ΑΓ τῇ ΖΕ. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΖΑΓΔ· ἴση ἄρα ἡ μὲν ΖΑ τῇ ΔΓ, ἡ δὲ ΑΓ τῇ ΖΔ. καὶ ἐπεὶ τριγώνου τοῦ ΖΒΕ παρὰ μίαν τὴν ΖΕ ἦκται ἡ ΑΓ, ἔστιν ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΑΖ, οὕτως ἡ ΒΓ πρὸς τὴν ΓΕ. ἴση δὲ ἡ ΑΖ τῇ ΓΔ· ὡς ἄρα ἡ ΒΑ πρὸς τὴν ΓΔ, οὕτως ἡ ΒΓ πρὸς τὴν ΓΕ, καὶ ἐναλλὰξ ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΕ. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ ΓΔ τῇ ΒΖ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΕ, οὕτως ἡ ΖΔ πρὸς τὴν ΔΕ. ἴση δὲ ἡ ΖΔ τῇ ΑΓ· ὡς ἄρα ἡ ΒΓ πρὸς τὴν ΓΕ, οὕτως ἡ ΑΓ πρὸς τὴν ΔΕ, καὶ ἐναλλὰξ ὡς ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΔ. ἐπεὶ οὖν ἐδείχθη ὡς

Proposition 4

In equiangular triangles the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.

Let ABC and DCE be equiangular triangles, having angle ABC equal to DCE, and (angle) BAC to CDE, and, further, (angle) ACB to CED. I say that in triangles ABC and DCE the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.

Let BC be placed straight-on to CE. And since angles ABC and ACB are less than two right-angles [Prop 1.17], and ACB (is) equal to DEC, thus ABC and DEC are less than two right-angles. Thus, BA and ED, being produced, will meet [C.N. 5]. Let them be produced, and let them meet at (point) F.



And since angle DCE is equal to ABC, BF is parallel to CD [Prop. 1.28]. Again, since (angle) ACB is equal to DEC, AC is parallel to FE [Prop. 1.28]. Thus, $FACD$ is a parallelogram. Thus, FA is equal to DC, and AC to FD [Prop. 1.34]. And since AC has been drawn parallel to one (of the sides) FE of triangle FBE, thus as BA is to AF, so BC (is) to CE [Prop. 6.2]. And AF (is) equal to CD. Thus, as BA (is) to CD, so BC (is) to CE, and, alternately, as AB (is) to BC, so DC (is) to CE [Prop. 5.16]. Again, since CD is parallel to BF, thus as BC (is) to CE, so FD (is) to DE [Prop. 6.2]. And FD (is) equal to AC. Thus, as BC is to CE, so AC (is) to DE, and, alternately, as BC (is) to CA, so CE (is) to ED [Prop. 6.2]. Therefore, since it was shown that as AB (is) to BC, so DC

μὲν ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΕ, ὡς δὲ ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΓΕ πρὸς τὴν ΕΔ, δι᾽ ἴσου ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΕ.

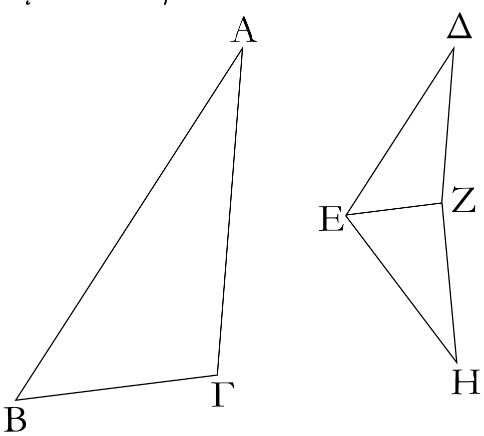
Τῶν ἄρα ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι· ὅπερ ἔδει δεῖξαι.

ε΄.

Ἐὰν δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ τὰς πλευρὰς ἀνάλογον ἔχοντα, ὡς μὲν τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, ὡς δὲ τὴν ΒΓ πρὸς τὴν ΓΑ, οὕτως τὴν ΕΖ πρὸς τὴν ΖΔ, καὶ ἔτι ὡς τὴν ΒΑ πρὸς τὴν ΑΓ, οὕτως τὴν ΕΔ πρὸς τὴν ΔΖ. λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ καὶ ἴσας ἕξουσι τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν, τὴν μὲν ὑπὸ ΑΒΓ τῇ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ ΒΓΑ τῇ ὑπὸ ΕΖΔ καὶ ἔτι τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ.

Συνεστάτω γὰρ πρὸς τῇ ΕΖ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Ε, Ζ τῇ μὲν ὑπὸ ΑΒΓ γωνίᾳ ἴση ἡ ὑπὸ ΖΕΗ, τῇ δὲ ὑπὸ ΑΓΒ ἴση ἡ ὑπὸ ΕΖΗ· λοιπὴ ἄρα ἡ πρὸς τῷ Α λοιπῇ τῇ πρὸς τῷ Η ἐστιν ἴση.



Ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΕΗΖ [τριγώνῳ]. τῶν ἄρα ΑΒΓ, ΕΗΖ τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι· ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, [οὕτως] ἡ ΗΕ πρὸς τὴν ΕΖ. ἀλλ᾽ ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ὑπόκειται ἡ ΔΕ πρὸς τὴν ΕΖ· ὡς ἄρα ἡ ΔΕ πρὸς τὴν ΕΖ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΖ. ἑκατέρα ἄρα τῶν ΔΕ, ΗΕ πρὸς τὴν ΕΖ τὸν αὐτὸν ἔχει λόγον· ἴση ἄρα ἐστὶν ἡ ΔΕ τῇ ΗΕ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΔΖ τῇ ΗΖ ἐστιν ἴση. ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔΕ τῇ ΕΗ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἱ ΔΕ, ΕΖ δυσὶ ταῖς ΗΕ, ΕΖ ἴσαι εἰσίν· καὶ βάσις ἡ ΔΖ βάσει τῇ ΖΗ [ἐστιν] ἴση· γωνία ἄρα ἡ ὑπὸ ΔΕΖ γωνίᾳ τῇ ὑπὸ ΗΕΖ ἐστιν ἴση, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΗΕΖ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶ καὶ

(is) to *CE*, and as *BC* (is) to *CA*, so *CE* (is) to *ED*, thus, via equality, as *BA* (is) to *AC*, so *CD* (is) to *DE* [Prop. 5.22].

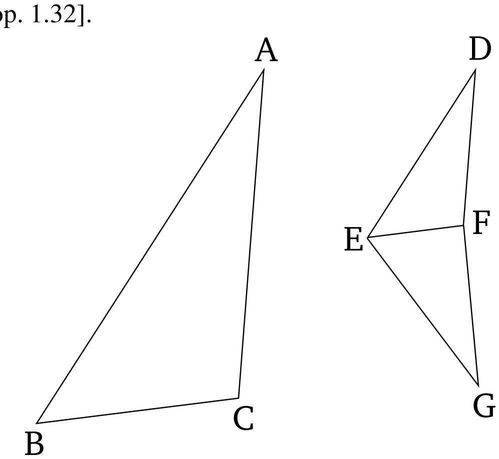
Thus, in equiangular triangles the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond. (Which is) the very thing it was required to show.

Proposition 5

If two triangles have proportional sides, (then) the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.

Let *ABC* and *DEF* be two triangles having proportional sides, (so that) as *AB* (is) to *BC*, so *DE* (is) to *EF*, and as *BC* (is) to *CA*, so *EF* (is) to *FD*, and, further, as *BA* (is) to *AC*, so *ED* (is) to *DF*. I say that triangle *ABC* is equiangular to triangle *DEF*, and (that the triangles) will have the angles which corresponding sides subtend equal. (That is), (angle) *ABC* (equal) to *DEF*, *BCA* to *EFD*, and, further, *BAC* to *EDF*.

For let (angle) *FEG*, equal to angle *ABC*, and (angle) *EFG*, equal to *ACB*, be constructed on the straight-line *EF* at the points *E* and *F* on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at *A* is equal to the remaining (angle) at *G* [Prop. 1.32].



Thus, triangle *ABC* is equiangular to [triangle] *EGF*. Thus, for triangles *ABC* and *EGF*, the sides about the equal angles are proportional, and (those) sides subtending equal angles correspond [Prop. 6.4]. Thus, as *AB* is to *BC*, [so] *GE* (is) to *EF*. But, as *AB* (is) to *BC*, so, it was assumed, (is) *DE* to *EF*. Thus, as *DE* (is) to *EF*, so *GE* (is) to *EF* [Prop. 5.11]. Thus, *DE* and *GE* each have the same ratio to *EF*. Thus, *DE* is equal to *GE* [Prop. 5.9]. So, for the same (reasons), *DF* is also equal to *GF*. Therefore, since *DE* is equal to *EG*, and *EF* (is) common, the two (sides) *DE*, *EF* are equal to the two (sides) *GE*, *EF* (respectively). And base *DF* [is] equal to base *FG*. Thus, angle *DEF* is equal to angle *GEF* [Prop. 1.8], and triangle *DEF* (is) equal to triangle *GEF*, and the remaining angles (are) equal to the remaining angles which the equal *ἡ μὲν ὑπὸ ΔΖΕ γωνία τῇ ὑπὸ ΗΖΕ, ἡ δὲ ὑπὸ ΕΔΖ τῇ ὑπὸ ΕΗΖ. καὶ ἐπεὶ ἡ μὲν ὑπὸ ΖΕΔ τῇ ὑπὸ ΗΕΖ ἐστιν ἴση, ἀλλ᾽ ἡ ὑπὸ ΗΕΖ τῇ ὑπὸ ΑΒΓ, καὶ ἡ ὑπὸ ΑΒΓ ἄρα γωνία τῇ ὑπὸ ΔΕΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση, καὶ ἔτι ἡ πρὸς τῷ Α τῇ πρὸς τῷ Δ· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.*

Ἐὰν ἄρα δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

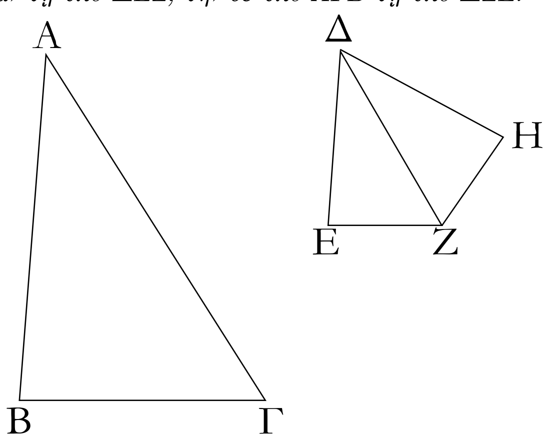
sides subtend [Prop. 1.4]. Thus, angle DFE is also equal to GFE, and (angle) EDF to EGF. And since (angle) FED is equal to GEF, and (angle) GEF to ABC, angle ABC is thus also equal to DEF. So, for the same (reasons), (angle) ACB is also equal to DFE, and, further, the (angle) at A to the (angle) at D. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have proportional sides, (then) the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

ϛʹ.

Ἐὰν δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ᾽ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν.

Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ μίαν γωνίαν τὴν ὑπὸ ΒΑΓ μιᾷ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὡς τὴν ΒΑ πρὸς τὴν ΑΓ, οὕτως τὴν ΕΔ πρὸς τὴν ΔΖ· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ καὶ ἴσην ἕξει τὴν ὑπὸ ΑΒΓ γωνίαν τῇ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ.



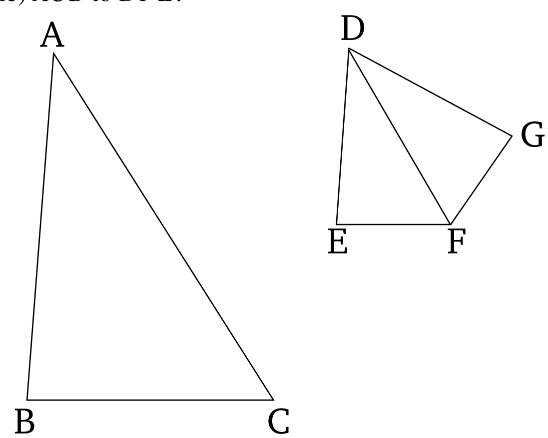
Συνεστάτω γὰρ πρὸς τῇ ΔΖ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Δ, Ζ ὁποτέρᾳ μὲν τῶν ὑπὸ ΒΑΓ, ΕΔΖ ἴση ἡ ὑπὸ ΖΔΗ, τῇ δὲ ὑπὸ ΑΓΒ ἴση ἡ ὑπὸ ΔΖΗ· λοιπὴ ἄρα ἡ πρὸς τῷ Β γωνία λοιπῇ τῇ πρὸς τῷ Η ἴση ἐστίν.

Ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΗΖ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ. ὑπόκειται δὲ καὶ ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΕΔ πρὸς τὴν ΔΖ· καὶ ὡς ἄρα ἡ ΕΔ πρὸς τὴν ΔΖ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ. ἴση ἄρα ἡ ΕΔ τῇ ΔΗ· καὶ κοινὴ ἡ ΔΖ· δύο δὴ αἱ ΕΔ, ΔΖ δυσὶ ταῖς ΗΔ, ΔΖ ἴσας εἰσίν· καὶ γωνία ἡ ὑπὸ ΕΔΖ γωνίᾳ τῇ ὑπὸ ΗΔΖ [ἐστιν] ἴση· βάσις ἄρα ἡ ΕΖ βάσει τῇ ΗΖ ἐστιν ἴση, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΗΔΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ᾽ ἃς αἱ ἴσας πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΔΖΗ τῇ ὑπὸ ΔΖΕ, ἡ δὲ ὑπὸ ΔΗΖ τῇ ὑπὸ ΔΕΖ.

Proposition 6

If two triangles have one angle equal to one angle, and the sides about the equal angles proportional, (then) the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.

Let ABC and DEF be two triangles having one angle, BAC, equal to one angle, EDF (respectively), and the sides about the equal angles proportional, (so that) as BA (is) to AC, so ED (is) to DF. I say that triangle ABC is equiangular to triangle DEF, and will have angle ABC equal to DEF, and (angle) ACB to DFE.



For let (angle) FDG, equal to each of BAC and EDF, and (angle) DFG, equal to ACB, be constructed on the straight-line AF at the points D and F on it (respectively) [Prop. 1.23]. Thus, the remaining angle at B is equal to the remaining angle at G [Prop. 1.32].

Thus, triangle ABC is equiangular to triangle DGF. Thus, proportionally, as BA (is) to AC, so GD (is) to DF [Prop. 6.4]. And it was also assumed that as BA (is) to AC, so ED (is) to DF. And, thus, as ED (is) to DF, so GD (is) to DF [Prop. 5.11]. Thus, ED (is) equal to DG [Prop. 5.9]. And DF (is) common. So, the two (sides) ED, DF are equal to the two (sides) GD, DF (respectively). And angle EDF [is] equal to angle GDF. Thus, base EF is equal to base GF, and triangle DEF is equal to triangle GDF, and the remaining angles will be equal to the remaining angles which the equal sides subtend *ἀλλ᾿ ἡ ὑπὸ ΔΖΗ τῇ ὑπὸ ΑΓΒ ἐστιν ἴση· καὶ ἡ ὑπὸ ΑΓΒ ἄρα τῇ ὑπὸ ΔΖΕ ἐστιν ἴση. ὑπόκειται δὲ καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ ἴση· καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Β λοιπῇ τῇ πρὸς τῷ Ε ἴση ἐστίν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.*

Ἐὰν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, ὑφ᾿ ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

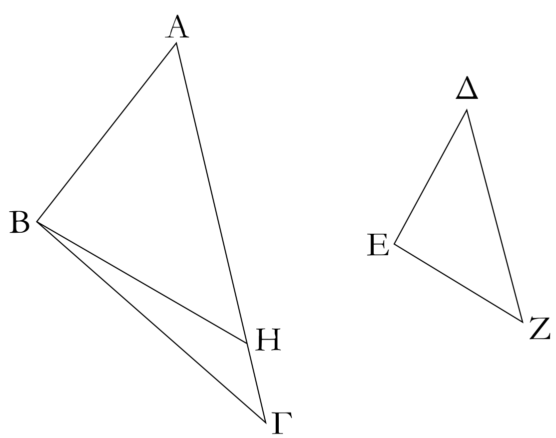
[Prop. 1.4]. Thus, (angle) DFG is equal to DFE, and (angle) DGF to DEF. But, (angle) DFG is equal to ACB. Thus, (angle) ACB is also equal to DFE. And (angle) BAC was also assumed (to be) equal to EDF. Thus, the remaining (angle) at B is equal to the remaining (angle) at E [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have one angle equal to one angle, and the sides about the equal angles proportional, (then) the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

ζʹ.

Ἐὰν δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἅμα ἤτοι ἐλάσσονα ἢ μὴ ἐλάσσονα ὀρθῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, περὶ ἃς ἀνάλογόν εἰσιν αἱ πλευραί.

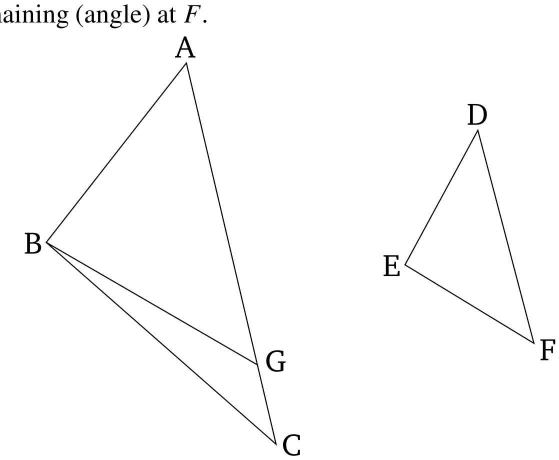
Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ, περὶ δὲ ἄλλας γωνίας τὰς ὑπὸ ΑΒΓ, ΔΕΖ τὰς πλευρὰς ἀνάλογον, ὡς τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, τῶν δὲ λοιπῶν τῶν πρὸς τοῖς Γ, Ζ πρότερον ἑκατέραν ἅμα ἐλάσσονα ὀρθῆς· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ, καὶ ἴση ἔσται ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ, καὶ λοιπὴ δηλονότι ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Ζ ἴση.



Proposition 7

If two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles either both less than, or both not less than, right-angles, (then) the triangles will be equiangular, and will have the angles about which the sides are proportional equal.

Let ABC and DEF be two triangles having one angle, BAC, equal to one angle, EDF (respectively), and the sides about (some) other angles, ABC and DEF (respectively), proportional, (so that) as AB (is) to BC, so DE (is) to EF, and the remaining (angles) at C and F, first of all, both less than right-angles. I say that triangle ABC is equiangular to triangle DEF, and (that) angle ABC will be equal to DEF, and (that) the remaining (angle) at C (will be) manifestly equal to the remaining (angle) at F.



Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΑΒΓ. καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β τῇ ὑπὸ ΔΕΖ γωνίᾳ ἴση ἡ ὑπὸ ΑΒΗ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν Α γωνία τῇ Δ, ἡ δὲ ὑπὸ ΑΒΗ τῇ ὑπὸ ΔΕΖ, λοιπὴ ἄρα ἡ ὑπὸ ΑΗΒ λοιπῇ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ τριγώνῳ. ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΗ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΖ. ὡς δὲ ἡ ΔΕ πρὸς τὴν ΕΖ, [οὕτως] ὑπόκειται ἡ ΑΒ πρὸς τὴν ΒΓ· ἡ ΑΒ ἄρα πρὸς ἑκατέραν τῶν ΒΓ, ΒΗ τὸν αὐτὸν ἔχει

For if angle ABC is not equal to (angle) DEF, (then) one of them is greater. Let ABC be greater. And let (angle) ABG, equal to (angle) DEF, be constructed on the straight-line AB at the point B on it [Prop. 1.23].

And since angle A is equal to (angle) D, and (angle) ABG to DEF, the remaining (angle) AGB is thus equal to the remaining (angle) DFE [Prop. 1.32]. Thus, triangle ABG is equiangular to triangle DEF. Thus, as AB is to BG, so DE (is) to EF [Prop. 6.4]. And as DE (is) to EF, [so] it was assumed (is) AB to BC. Thus, AB has the same ratio to each of BC and *λόγον· ἴση ἄρα ἡ ΒΓ τῇ ΒΗ. ὥστε καὶ γωνία ἡ πρὸς τῷ Γ γωνίᾳ τῇ ὑπὸ ΒΗΓ ἐστιν ἴση. ἐλάττων δὲ ὀρθῆς ὑπόκειται ἡ πρὸς τῷ Γ· ἐλάττων ἄρα ἐστὶν ὀρθῆς καὶ ὑπὸ ΒΗΓ· ὥστε ἡ ἐφεξῆς αὐτῇ γωνία ἡ ὑπὸ ΑΗΒ μείζων ἐστὶν ὀρθῆς. καὶ ἐδείχθη ἴση οὖσα τῇ πρὸς τῷ Ζ· καὶ ἡ πρὸς τῷ Ζ ἄρα μείζων ἐστὶν ὀρθῆς. ὑπόκειται δὲ ἐλάσσων ὀρθῆς· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ· ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ Α ἴση τῇ πρὸς τῷ Δ· καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Ζ ἴση ἐστίν. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.*

Ἀλλὰ δὴ πάλιν ὑποκείσθω ἑκατέρα τῶν πρὸς τοῖς Γ, Ζ μὴ ἐλάσσων ὀρθῆς· λέγω πάλιν, ὅτι καὶ οὕτως ἐστὶν ἰσογώνιον τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἴση ἐστὶν ἡ ΒΓ τῇ ΒΗ· ὥστε καὶ γωνία ἡ πρὸς τῷ Γ τῇ ὑπὸ ΒΗΓ ἴση ἐστίν. οὐκ ἐλάττων δὲ ὀρθῆς ἡ πρὸς τῷ Γ· οὐκ ἐλάττων ἄρα ὀρθῆς οὐδὲ ἡ ὑπὸ ΒΗΓ. τριγώνου δὴ τοῦ ΒΗΓ αἱ δύο γωνίαι δύο ὀρθῶν οὔκ εἰσιν ἐλάττονες· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα πάλιν ἄνισός ἐστιν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ· ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ Α τῇ πρὸς τῷ Δ ἴση· λοιπὴ ἄρα ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Ζ ἴση ἐστίν. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἐὰν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἅμα ἐλάττονα ἢ μὴ ἐλάττονα ὀρθῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἕξει τὰς γωνίας, περὶ ἃς ἀνάλογόν εἰσιν αἱ πλευραί· ὅπερ ἔδει δεῖξαι.

BG [Prop. 5.11]. Thus, *BC* (is) equal to *BG* [Prop. 5.9]. And, hence, the angle at *C* is equal to angle *BGC* [Prop. 1.5]. And the angle at *C* was assumed (to be) less than a right-angle. Thus, (angle) *BGC* is also less than a right-angle. Hence, the adjacent angle to it, *AGB*, is greater than a right-angle [Prop. 1.13]. And (*AGB*) was shown to be equal to the (angle) at *F*. Thus, the (angle) at *F* is also greater than a right-angle. But it was assumed (to be) less than a right-angle. The very thing is absurd. Thus, angle *ABC* is not unequal to (angle) *DEF*. Thus, (it is) equal. And the (angle) at *A* is also equal to the (angle) at *D*. And thus the remaining (angle) at *C* is equal to the remaining (angle) at *F* [Prop. 1.32]. Thus, triangle *ABC* is equiangular to triangle *DEF*.

But, again, let each of the (angles) at *C* and *F* be assumed (to be) not less than a right-angle. I say, again, that triangle *ABC* is equiangular to triangle *DEF* in this case also.

For, with the same construction, we can similarly show that *BC* is equal to *BG*. Hence, also, the angle at *C* is equal to (angle) *BGC*. And the (angle) at *C* (is) not less than a right-angle. Thus, *BGC* (is) not less than a right-angle either. So, in triangle *BGC* the (sum of) two angles is not less than two right-angles. The very thing is impossible [Prop. 1.17]. Thus, again, angle *ABC* is not unequal to *DEF*. Thus, (it is) equal. And the (angle) at *A* is also equal to the (angle) at *D*. Thus, the remaining (angle) at *C* is equal to the remaining (angle) at *F* [Prop. 1.32]. Thus, triangle *ABC* is equiangular to triangle *DEF*.

Thus, if two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles both less than, or both not less than, right-angles, (then) the triangles will be equiangular, and will have the angles about which the sides (are) proportional equal. (Which is) the very thing it was required to show.

ηʹ.

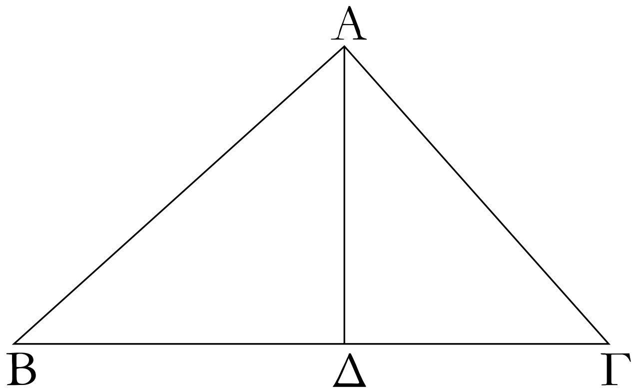
Ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, τὰ πρὸς τῇ καθέτῳ τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις.

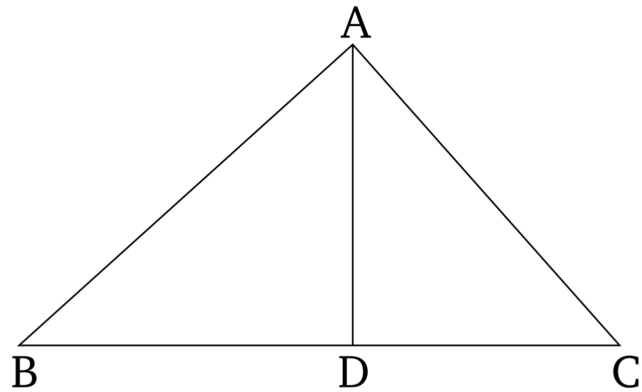
Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν, καὶ ἤχθω ἀπὸ τοῦ Α ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ· λέγω, ὅτι ὅμοιόν ἐστιν ἑκάτερον τῶν ΑΒΔ, ΑΔΓ τριγώνων ὅλῳ τῷ ΑΒΓ καὶ ἔτι ἀλλήλοις.

Proposition 8

If, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base, (then) the triangles around the perpendicular are similar to the whole (triangle), and to one another.

Let *ABC* be a right-angled triangle having the angle *BAC* a right-angle, and let *AD* be drawn from *A*, perpendicular to *BC* [Prop. 1.12]. I say that triangles *ABD* and *ADC* are each similar to the whole (triangle) *ABC* and, further, to one another.





Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΑΔΒ· ὀρθὴ γὰρ ἑκατέρα· καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΒΓ καὶ τοῦ ΑΒΔ ἡ πρὸς τῷ Β, λοιπὴ ἄρα ἡ ὑπὸ ΑΓΒ λοιπῇ τῇ ὑπὸ ΒΑΔ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνῳ. ἔστιν ἄρα ὡς ἡ ΒΓ ὑποτείνουσα τὴν ὀρθὴν τοῦ ΑΒΓ τριγώνου πρὸς τὴν ΒΑ ὑποτείνουσαν τὴν ὀρθὴν τοῦ ΑΒΔ τριγώνου, οὕτως αὐτὴ ἡ ΑΒ ὑποτείνουσα τὴν πρὸς τῷ Γ γωνίαν τοῦ ΑΒΓ τριγώνου πρὸς τὴν ΒΔ ὑποτείνουσαν τὴν ἴσην τὴν ὑπὸ ΒΑΔ τοῦ ΑΒΔ τριγώνου, καὶ ἔτι ἡ ΑΓ πρὸς τὴν ΑΔ ὑποτείνουσαν τὴν πρὸς τῷ Β γωνίαν κοινὴν τῶν δύο τριγώνων. τὸ ΑΒΓ ἄρα τρίγωνον τῷ ΑΒΔ τριγώνῳ ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. ὅμοιον ἄμα [ἐστὶ] τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνῳ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τῷ ΑΔΓ τριγώνῳ ὅμοιόν ἐστι τὸ ΑΒΓ τρίγωνον· ἑκάτερον ἄρα τῶν ΑΒΔ, ΑΔΓ [τριγώνων] ὅμοιόν ἐστιν ὅλῳ τῷ ΑΒΓ.

Λέγω δή, ὅτι καὶ ἀλλήλοις ἐστὶν ὅμοια τὰ ΑΒΔ, ΑΔΓ τρίγωνα.

Ἐπεὶ γὰρ ὀρθὴ ἡ ὑπὸ ΒΔΑ ὀρθῇ τῇ ὑπὸ ΑΔΓ ἐστιν ἴση, ἀλλὰ μὴν καὶ ἡ ὑπὸ ΒΑΔ τῇ πρὸς τῷ Γ ἐδείχθη ἴση, καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Β λοιπῇ τῇ ὑπὸ ΔΑΓ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΑΔΓ τριγώνῳ. ἔστιν ἄρα ὡς ἡ ΒΔ τοῦ ΑΒΔ τριγώνου ὑποτείνουσα τὴν ὑπὸ ΒΑΔ πρὸς τὴν ΔΑ τοῦ ΑΔΓ τριγώνου ὑποτείνουσαν τὴν πρὸς τῷ Γ ἴσην τῇ ὑπὸ ΒΑΔ, οὕτως αὐτὴ ἡ ΑΔ τοῦ ΑΒΔ τριγώνου ὑποτείνουσα τὴν πρὸς τῷ Β γωνίαν πρὸς τὴν ΔΓ ὑποτείνουσαν τὴν ὑπὸ ΔΑΓ τοῦ ΑΔΓ τριγώνου ἴσην τῇ πρὸς τῷ Β, καὶ ἔτι ἡ ΒΑ πρὸς τὴν ΑΓ ὑποτείνουσαι τὰς ὀρθάς· ὅμοιον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΑΔΓ τριγώνῳ.

Ἐὰν ἄρα ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, τὰ πρὸς τῇ καθέτῳ τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις [ὅπερ ἔδει δεῖξαι].

For since (angle) *BAC* is equal to *ADB*—for each (are) right-angles—and the (angle) at *B* (is) common to the two triangles *ABC* and *ABD*, the remaining (angle) *ACB* is thus equal to the remaining (angle) *BAD* [Prop. 1.32]. Thus, triangle *ABC* is equiangular to triangle *ABD*. Thus, as *BC*, subtending the right-angle in triangle *ABC*, is to *BA*, subtending the right-angle in triangle *ABD*, so the same *AB*, subtending the angle at *C* in triangle *ABC*, (is) to *BD*, subtending the equal (angle) *BAD* in triangle *ABD*, and, further, (so is) *AC* to *AD*, (both) subtending the angle at *B* common to the two triangles [Prop. 6.4]. Thus, triangle *ABC* is equiangular to triangle *ABD*, and has the sides about the equal angles proportional. Thus, triangle *ABC* [is] similar to triangle *ABD* [Def. 6.1]. So, similarly, we can show that triangle *ABC* is also similar to triangle *ADC*. Thus, [triangles] *ABD* and *ADC* are each similar to the whole (triangle) *ABC*.

So I say that triangles *ABD* and *ADC* are also similar to one another.

For since the right-angle *BDA* is equal to the right-angle *ADC*, and, indeed, (angle) *BAD* was also shown (to be) equal to the (angle) at *C*, thus the remaining (angle) at *B* is also equal to the remaining (angle) *DAC* [Prop. 1.32]. Thus, triangle *ABD* is equiangular to triangle *ADC*. Thus, as *BD*, subtending (angle) *BAD* in triangle *ABD*, is to *DA*, subtending the (angle) at *C* in triangle *ADC*, (which is) equal to (angle) *BAD*, so (is) the same *AD*, subtending the angle at *B* in triangle *ABD*, to *DC*, subtending (angle) *DAC* in triangle *ADC*, (which is) equal to the (angle) at *B*, and, further, (so is) *BA* to *AC*, (each) subtending right-angles [Prop. 6.4]. Thus, triangle *ABD* is similar to triangle *ADC* [Def. 6.1].

Thus, if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base, (then) the triangles around the perpendicular are similar to the whole (triangle), and to one another. [(Which is) the very thing it was required to show.]

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ

Corollary

So (it is) clear, from this, that if, in a right-angled triangle, *τῆς ὀρθῆς γωνάις ἐπὶ τὴν βάσις κάθετος ἀχθῇ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν· ὅπερ ἔδει δεῖξαι.*

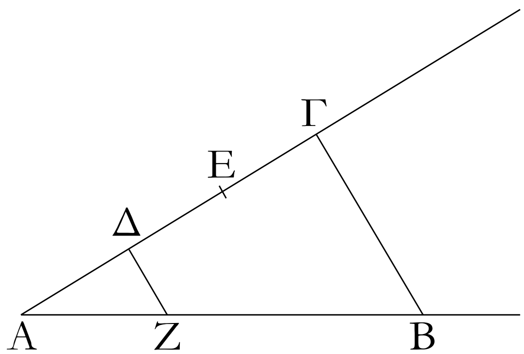
a (straight-line) is drawn from the right-angle perpendicular to the base, (then) the (straight-line so) drawn is in mean proportion to the pieces of the base.† (Which is) the very thing it was required to show.

† In other words, the perpendicular is the geometric mean of the pieces.

ϑ'.

Τῆς δοθείσης εὐθείας τὸ προσταχθὲν μέρος ἀφελεῖν.

Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ· δεῖ δὴ τῆς ΑΒ τὸ προσταχθὲν μέρος ἀφελεῖν.



Ἐπιτετάχθω δὴ τὸ τρίτον. [καὶ] διήχθω τις ἀπὸ τοῦ Α εὐθεῖα ἡ ΑΓ γωνίαν περιέχουσα μετὰ τῆς ΑΒ τυχοῦσαν· καὶ εἰλήφθω τυχὸν σημεῖον ἐπὶ τῆς ΑΓ τὸ Δ, καὶ κείσθωσαν τῇ ΑΔ ἴσαι αἱ ΔΕ, ΕΓ. καὶ ἐπεζεύχθω ἡ ΒΓ, καὶ διὰ τοῦ Δ παράλληλος αὐτῇ ἤχθω ἡ ΔΖ.

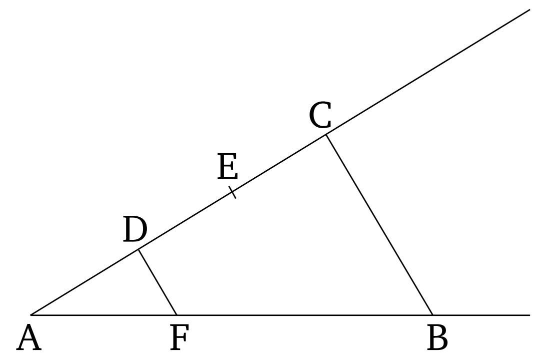
Ἐπεὶ οὖν τριγώνου τοῦ ΑΒΓ παρὰ μίαν τῶν πλευρῶν τὴν ΒΓ ἦκται ἡ ΖΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΒΖ πρὸς τὴν ΖΑ. διπλῆ δὲ ἡ ΓΔ τῆς ΔΑ· διπλῆ ἄρα καὶ ἡ ΒΖ τῆς ΖΑ· τριπλῆ ἄρα ἡ ΒΑ τῆς ΑΖ.

Τῆς ἄρα δοθείσης εὐθείας τῆς ΑΒ τὸ ἐπιταχθὲν τρίτον μέρος ἀφῄρηται τὸ ΑΖ· ὅπερ ἔδει ποιῆσαι.

Proposition 9

To cut off a prescribed part from a given straight-line.

Let *AB* be the given straight-line. So it is required to cut off a prescribed part from *AB*.



So let a third (part) be prescribed. [And] let some straight-line *AC* be drawn from (point) *A*, encompassing a random angle with *AB*. And let a random point *D* be taken on *AC*. And let *DE* and *EC* be made equal to *AD* [Prop. 1.3]. And let *BC* be joined. And let *DF* be drawn through *D* parallel to it [Prop. 1.31].

Therefore, since *FD* has been drawn parallel to one of the sides, *BC*, of triangle *ABC*, then, proportionally, as *CD* is to *DA*, so *BF* (is) to *FA* [Prop. 6.2]. And *CD* (is) double *DA*. Thus, *BF* (is) also double *FA*. Thus, *BA* (is) triple *AF*.

Thus, the prescribed third part, *AF*, has been cut off from the given straight-line, *AB*. (Which is) the very thing it was required to do.

ι'.

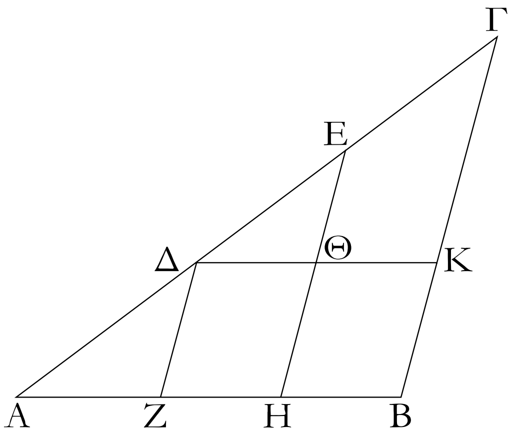
Τὴν δοθεῖσαν εὐθεῖαν ἄτμητον τῇ δοθείσῃ τετμημένῃ ὁμοίως τεμεῖν.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἄτμητος ἡ ΑΒ, ἡ δὲ τετμημένη ἡ ΑΓ κατὰ τὰ Δ, Ε σημεῖα, καὶ κείσθωσαν ὥστε γωνίαν τυχοῦσαν περιέχειν, καὶ ἐπεζεύχθω ἡ ΓΒ, καὶ διὰ τῶν Δ, Ε τῇ ΒΓ παράλληλοι ἤχθωσαν αἱ ΔΖ, ΕΗ, διὰ δὲ τοῦ Δ τῇ ΑΒ παράλληλος ἤχθω ἡ ΔΘΚ.

Proposition 10

To cut a given uncut straight-line similarly to a given cut (straight-line).

Let *AB* be the given uncut straight-line, and *AC* a (straight-line) cut at points *D* and *E*, and let (*AC*) be laid down so as to encompass a random angle (with *AB*). And let *CB* be joined. And let *DF* and *EG* be drawn through (points) *D* and *E* (respectively), parallel to *BC*, and let *DHK* be drawn through (point) *D*, parallel to *AB* [Prop. 1.31].



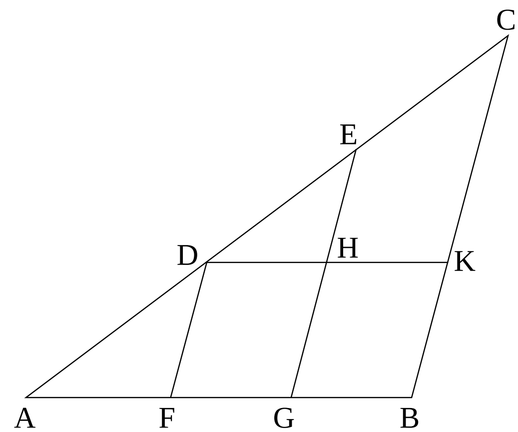
Παραλληλόγραμμον ἄρα ἐστὶν ἑκάτερον τῶν ΖΘ, ΘΒ· ἴση ἄρα ἡ μὲν ΔΘ τῇ ΖΗ, ἡ δὲ ΘΚ τῇ ΗΒ. καὶ ἐπεὶ τριγώνου τοῦ ΔΚΓ παρὰ μίαν τῶν πλευρῶν τὴν ΚΓ εὐθεῖα ἦκται ἡ ΘΕ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΚΘ πρὸς τὴν ΘΔ. ἴση δὲ ἡ μὲν ΚΘ τῇ ΒΗ, ἡ δὲ ΘΔ τῇ ΗΖ. ἔστιν ἄρα ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ. πάλιν, ἐπεὶ τριγώνου τοῦ ΑΗΕ παρὰ μίαν τῶν πλευρῶν τὴν ΗΕ ἦκται ἡ ΖΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΕΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ. ἐδείχθη δὲ καὶ ὡς ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ· ἔστιν ἄρα ὡς μὲν ἡ ΓΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΒΗ πρὸς τὴν ΗΖ, ὡς δὲ ἡ ΕΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ.

Ἡ ἄρα δοθεῖσα εὐθεῖα ἄτμητος ἡ ΑΒ τῇ δοθείσῃ εὐθείᾳ τετμημένῃ τῇ ΑΓ ὁμοίως τέτμηται· ὅπερ ἔδει ποιῆσαι·

ια′.

Δύο δοθεισῶν εὐθειῶν τρίτην ἀνάλογον προσευρεῖν.

Ἔστωσαν αἱ δοθεῖσαι [δύο εὐθεῖαι] αἱ ΒΑ, ΑΓ καὶ κείσθωσαν γωνίαν περιέχουσαι τυχοῦσαν. δεῖ δὴ τῶν ΒΑ, ΑΓ τρίτην ἀνάλογον προσευρεῖν. ἐκβεβλήσθωσαν γὰρ ἐπὶ τὰ Δ, Ε σημεῖα, καὶ κείσθω τῇ ΑΓ ἴση ἡ ΒΔ, καὶ ἐπεζεύχθω ἡ ΒΓ, καὶ διὰ τοῦ Δ παράλληλος αὐτῇ ἤχθω ἡ ΔΕ.



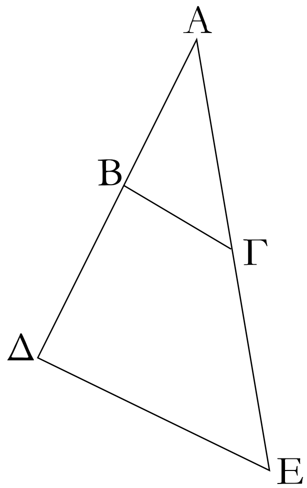
Thus, *FH* and *HB* are each parallelograms. Thus, *DH* (is) equal to *FG*, and *HK* to *GB* [Prop. 1.34]. And since the straight-line *HE* has been drawn parallel to one of the sides, *KC*, of triangle *DKC*, thus, proportionally, as *CE* is to *ED*, so *KH* (is) to *HD* [Prop. 6.2]. And *KH* (is) equal to *BG*, and *HD* to *GF*. Thus, as *CE* is to *ED*, so *BG* (is) to *GF*. Again, since *FD* has been drawn parallel to one of the sides, *GE*, of triangle *AGE*, thus, proportionally, as *ED* is to *DA*, so *GF* (is) to *FA* [Prop. 6.2]. And it was also shown that as *CE* (is) to *ED*, so *BG* (is) to *GF*. Thus, as *CE* is to *ED*, so *BG* (is) to *GF*, and as *ED* (is) to *DA*, so *GF* (is) to *FA*.

Thus, the given uncut straight-line, *AB*, has been cut similarly to the given cut straight-line, *AC*. (Which is) the very thing it was required to do.

Proposition 11

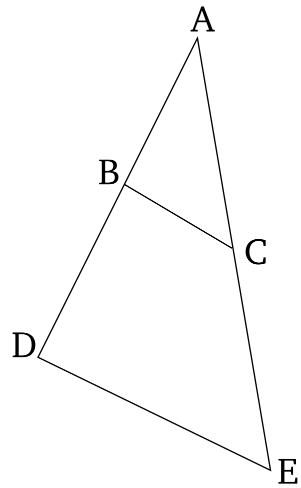
To find a third (straight-line) proportional to two given straight-lines.

Let *BA* and *AC* be the [two] given [straight-lines], and let them be laid down encompassing a random angle. So it is required to find a third (straight-line) proportional to *BA* and *AC*. For let (*BA* and *AC*) be produced to points *D* and *E* (respectively), and let *BD* be made equal to *AC* [Prop. 1.3]. And let *BC* be joined. And let *DE* be drawn through (point) *D* parallel to it [Prop. 1.31].



Ἐπεὶ οὖν τριγώνου τοῦ ΑΔΕ παρὰ μίαν τῶν πλευρῶν τὴν ΔΕ ἦκται ἡ ΒΓ, ἀνάλογόν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως ἡ ΑΓ πρὸς τὴν ΓΕ. ἴση δὲ ἡ ΒΔ τῇ ΑΓ. ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΑΓ, οὕτως ἡ ΑΓ πρὸς τὴν ΓΕ.

Δύο ἄρα δοθεισῶν εὐθειῶν τῶν ΑΒ, ΑΓ τρίτη ἀνάλογον αὐταῖς προσεύρηται ἡ ΓΕ· ὅπερ ἔδει ποιῆσαι.

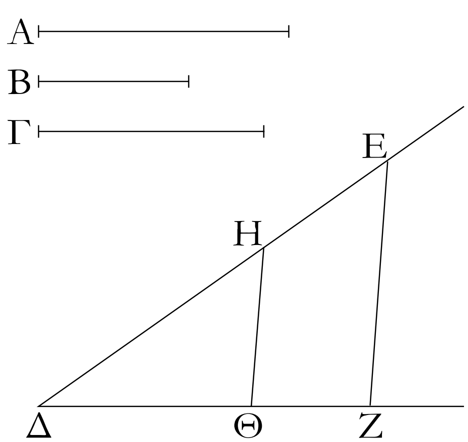


Therefore, since *BC* has been drawn parallel to one of the sides *DE* of triangle *ADE*, proportionally, as *AB* is to *BD*, so *AC* (is) to *CE* [Prop. 6.2]. And *BD* (is) equal to *AC*. Thus, as *AB* is to *AC*, so *AC* (is) to *CE*.

Thus, a third (straight-line), *CE*, has been found (which is) proportional to the two given straight-lines, *AB* and *AC*. (Which is) the very thing it was required to do.

ιβ΄.

Τριῶν δοθεισῶν εὐθειῶν τετάρτην ἀνάλογον προσευρεῖν.



Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τετράτην ἀνάλογον προσευρεῖν.

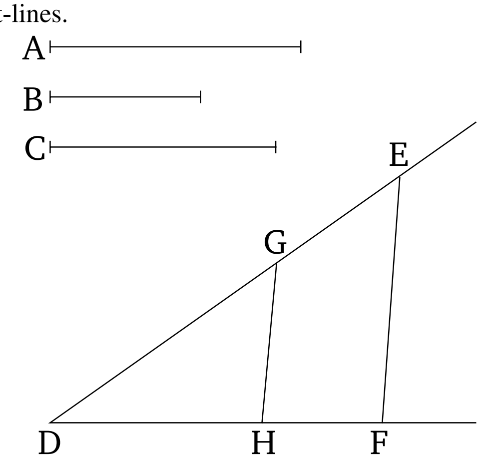
Ἐκκείσθωσαν δύο εὐθεῖαι αἱ ΔΕ, ΔΖ γωνίαν περιέχουσαι [τυχοῦσαν] τὴν ὑπὸ ΕΔΖ· καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΗ, τῇ δὲ Β ἴση ἡ ΗΕ, καὶ ἔτι τῇ Γ ἴση ἡ ΔΘ· καὶ ἐπιζευχθείσης τῆς ΗΘ παράλληλος αὐτῇ ἤχθω διὰ τοῦ Ε ἡ ΕΖ.

Ἐπεὶ οὖν τριγώνου τοῦ ΔΕΖ παρὰ μίαν τὴν ΕΖ ἦκται ἡ ΗΘ, ἔστιν ἄρα ὡς ἡ ΔΗ πρὸς τὴν ΗΕ, οὕτως ἡ ΔΘ πρὸς τὴν ΘΖ. ἴση δὲ ἡ μὲν ΔΗ τῇ Α, ἡ δὲ ΗΕ τῇ Β, ἡ δὲ ΔΘ τῇ Γ· ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν ΘΖ.

Τριῶν ἄρα δοθεισῶν εὐθειῶν τῶν Α, Β, Γ τετάρτη

Proposition 12

To find a fourth (straight-line) proportional to three given straight-lines.



Let *A*, *B*, and *C* be the three given straight-lines. So it is required to find a fourth (straight-line) proportional to *A*, *B*, and *C*.

Let the two straight-lines *DE* and *DF* be set out encompassing the [random] angle *EDF*. And let *DG* be made equal to *A*, and *GE* to *B*, and, further, *DH* to *C* [Prop. 1.3]. And *GH* being joined, let *EF* be drawn through (point) *E* parallel to it [Prop. 1.31].

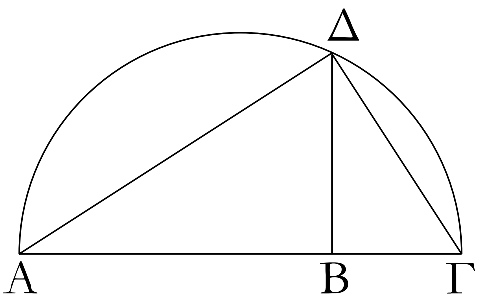
Therefore, since *GH* has been drawn parallel to one of the sides *EF* of triangle *DEF*, thus as *DG* is to *GE*, so *DH* (is) to *HF* [Prop. 6.2]. And *DG* (is) equal to *A*, and *GE* to *B*, and *DH* *ἀνάλογον προσεύρηται ἡ ΘΖ· ὅπερ ἔδει ποιῆσαι.*

to C. Thus, as A is to B, so C (is) to HF.

Thus, a fourth (straight-line), HF, has been found (which is) proportional to the three given straight-lines, A, B, and C. (Which is) the very thing it was required to do.

ιγʹ.

Δύο δοθεισῶν εὐθειῶν μέσην ἀνάλογον προσευρεῖν.



Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΒ, ΒΓ· δεῖ δὴ τῶν ΑΒ, ΒΓ μέσην ἀνάλογον προσευρεῖν.

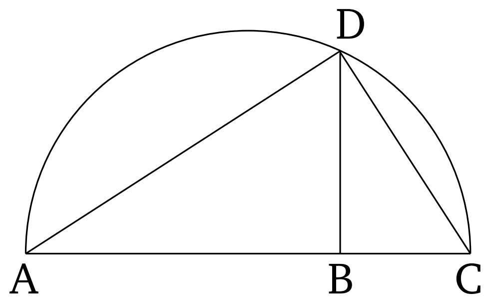
Κείσθωσαν ἐπ᾽ εὐθείας, καὶ γεγράφθω ἐπὶ τῆς ΑΓ ἡμικύκλιον τὸ ΑΔΓ, καὶ ἤχθω ἀπὸ τοῦ Β σημείου τῇ ΑΓ εὐθείᾳ πρὸς ὀρθὰς ἡ ΒΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ.

Ἐπεὶ ἐν ἡμικυκλίῳ γωνία ἐστὶν ἡ ὑπὸ ΑΔΓ, ὀρθή ἐστιν. καὶ ἐπεὶ ἐν ὀρθογωνίῳ τριγώνῳ τῷ ΑΔΓ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΔΒ, ἡ ΔΒ ἄρα τῶν τῆς βάσεως τμημάτων τῶν ΑΒ, ΒΓ μέση ἀνάλογόν ἐστιν.

Δύο ἄρα δοθεισῶν εὐθειῶν τῶν ΑΒ, ΒΓ μέση ἀνάλογον προσεύρηται ἡ ΔΒ· ὅπερ ἔδει ποιῆσαι.

Proposition 13

To find the (straight-line) in mean proportion to two given straight-lines.†



Let AB and BC be the two given straight-lines. So it is required to find the (straight-line) in mean proportion to AB and BC.

Let (AB and BC) be laid down straight-on (with respect to one another), and let the semi-circle ADC be drawn on AC [Prop. 1.10]. And let BD be drawn from (point) B, at right-angles to AC [Prop. 1.11]. And let AD and DC be joined.

And since ADC is an angle in a semi-circle, it is a right-angle [Prop. 3.31]. And since, in the right-angled triangle ADC, the (straight-line) DB has been drawn from the right-angle perpendicular to the base, DB is thus the mean proportional to the pieces of the base, AB and BC [Prop. 6.8 corr.].

Thus, DB has been found (which is) in mean proportion to the two given straight-lines, AB and BC. (Which is) the very thing it was required to do.

† In other words, to find the geometric mean of two given straight-lines.

ιδʹ.

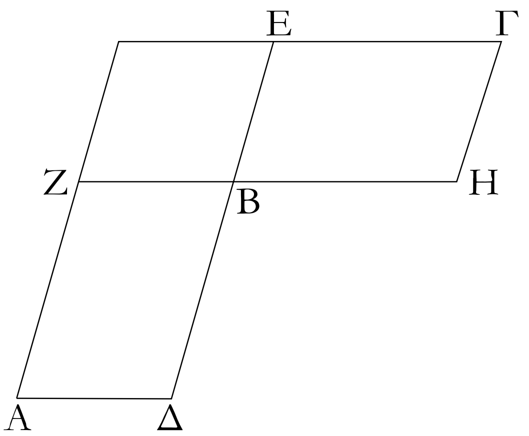
Τῶν ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα.

Ἔστω ἴσα τε καὶ ἰσογώνια παραλληλόγραμμα τὰ ΑΒ, ΒΓ ἴσας ἔχοντα τὰς πρὸς τῷ Β γωνίας, καὶ κείσθωσαν ἐπ᾽ εὐθείας αἱ ΔΒ, ΒΕ· ἐπ᾽ εὐθείας ἄρα εἰσὶ καὶ αἱ ΖΒ, ΒΗ. λέγω, ὅτι τῶν ΑΒ, ΒΓ ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ.

Proposition 14

In equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let AB and BC be equal and equiangular parallelograms having the angles at B equal. And let DB and BE be laid down straight-on (with respect to one another). Thus, FB and BG are also straight-on (with respect to one another) [Prop. 1.14]. I say that the sides of AB and BC about the equal angles are reciprocally proportional, that is to say, that as DB is to BE, so GB (is) to BF.



Συμπεπληρώσθω γὰρ τὸ ΖΕ παραλληλόγραμμον. ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ, ἄλλο δέ τι τὸ ΖΕ, ἔστιν ἄρα ὡς τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως τὸ ΒΓ πρὸς τὸ ΖΕ. ἀλλ᾿ ὡς μὲν τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΕ, ὡς δὲ τὸ ΒΓ πρὸς τὸ ΖΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ· καὶ ὡς ἄρα ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ. τῶν ἄρα ΑΒ, ΒΓ παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Ἀλλὰ δὴ ἔστω ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ.

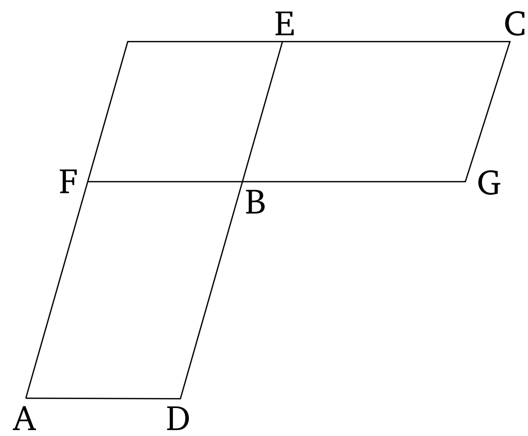
Ἐπεὶ γάρ ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΗΒ πρὸς τὴν ΒΖ, ἀλλ᾿ ὡς μὲν ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως τὸ ΑΒ παραλληλόγραμμον πρὸς τὸ ΖΕ παραλληλόγραμμον, ὡς δὲ ἡ ΗΒ πρὸς τὴν ΒΖ, οὕτως τὸ ΒΓ παραλληλόγραμμον πρὸς τὸ ΖΕ παραλληλόγραμμον, καὶ ὡς ἄρα τὸ ΑΒ πρὸς τὸ ΖΕ, οὕτως τὸ ΒΓ πρὸς τὸ ΖΕ· ἴσον ἄρα ἐστὶ τὸ ΑΒ παραλληλόγραμμον τῷ ΒΓ παραλληλογράμμῳ.

Τῶν ἄρα ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα· ὅπερ ἔδει δεῖξαι.

ιε΄.

Τῶν ἴσων καὶ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα.

Ἔστω ἴσα τρίγωνα τὰ ΑΒΓ, ΑΔΕ μίαν μιᾷ ἴσην ἔχοντα γωνίαν τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΔΑΕ· λέγω, ὅτι τῶν ΑΒΓ, ΑΔΕ τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ.



For let the parallelogram *FE* be completed. Therefore, since parallelogram *AB* is equal to parallelogram *BC*, and *FE* (is) some other (parallelogram), thus as (parallelogram) *AB* is to *FE*, so (parallelogram) *BC* (is) to *FE* [Prop. 5.7]. But, as (parallelogram) *AB* (is) to *FE*, so *DB* (is) to *BE*, and as (parallelogram) *BC* (is) to *FE*, so *GB* (is) to *BF* [Prop. 6.1]. Thus, also, as *DB* (is) to *BE*, so *GB* (is) to *BF*. Thus, in parallelograms *AB* and *BC* the sides about the equal angles are reciprocally proportional.

And so, let *DB* be to *BE*, as *GB* (is) to *BF*. I say that parallelogram *AB* is equal to parallelogram *BC*.

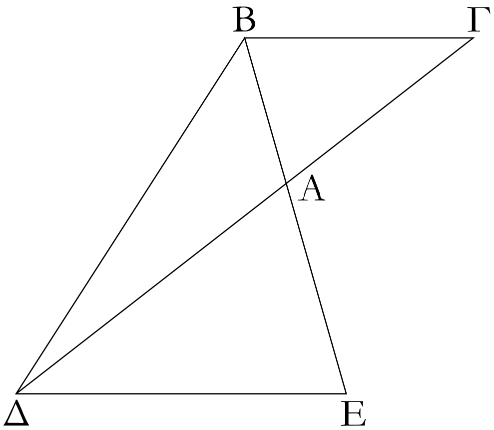
For since as *DB* is to *BE*, so *GB* (is) to *BF*, but as *DB* (is) to *BE*, so parallelogram *AB* (is) to parallelogram *FE*, and as *GB* (is) to *BF*, so parallelogram *BC* (is) to parallelogram *FE* [Prop. 6.1], thus, also, as (parallelogram) *AB* (is) to *FE*, so (parallelogram) *BC* (is) to *FE* [Prop. 5.11]. Thus, parallelogram *AB* is equal to parallelogram *BC* [Prop. 5.9].

Thus, in equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal. (Which is) the very thing it was required to show.

Proposition 15

In equal triangles also having one angle equal to one (angle), the sides about the equal angles are reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal.

Let *ABC* and *ADE* be equal triangles having one angle equal to one (angle), (namely) *BAC* (equal) to *DAE*. I say that, in triangles *ABC* and *ADE*, the sides about the equal angles are reciprocally proportional, that is to say, that as *CA* is to *AD*, so *EA* (is) to *AB*.



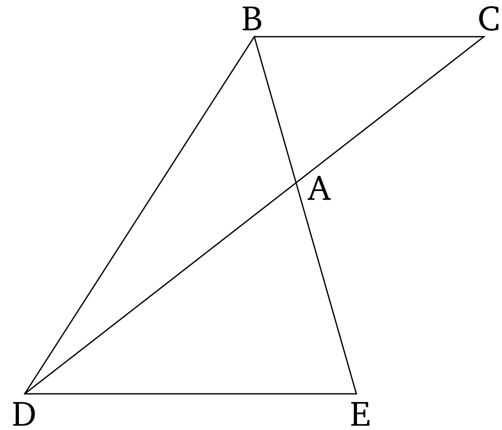
Κείσθω γὰρ ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΓΑ τῇ ΑΔ· ἐπ᾽ εὐθείας ἄρα ἐστὶ καὶ ἡ ΕΑ τῇ ΑΒ. καὶ ἐπεζεύχθω ἡ ΒΔ.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΔΕ τριγώνῳ, ἄλλο δέ τι τὸ ΒΑΔ, ἔστιν ἄρα ὡς τὸ ΓΑΒ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον. ἀλλ᾽ ὡς μὲν τὸ ΓΑΒ πρὸς τὸ ΒΑΔ, οὕτως ἡ ΓΑ πρὸς τὴν ΑΔ, ὡς δὲ τὸ ΕΑΔ πρὸς τὸ ΒΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ. καὶ ὡς ἄρα ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ. τῶν ΑΒΓ, ΑΔΕ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Ἀλλὰ δὴ ἀντιπεπονθέτωσαν αἱ πλευραὶ τῶν ΑΒΓ, ΑΔΕ τριγώνων, καὶ ἔστω ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΔΕ τριγώνῳ.

Ἐπιζευχθείσης γὰρ πάλιν τῆς ΒΔ, ἐπεί ἐστιν ὡς ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΒ, ἀλλ᾽ ὡς μὲν ἡ ΓΑ πρὸς τὴν ΑΔ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, ὡς δὲ ἡ ΕΑ πρὸς τὴν ΑΒ, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον, οὕτως τὸ ΕΑΔ τρίγωνον πρὸς τὸ ΒΑΔ τρίγωνον. ἑκάτερον ἄρα τῶν ΑΒΓ, ΕΑΔ πρὸς τὸ ΒΑΔ τὸν αὐτὸν ἔχει λόγον. ἴσων ἄρα ἐστὶ τὸ ΑΒΓ [τρίγωνον] τῷ ΕΑΔ τριγώνῳ.

Τῶν ἄρα ἴσων καὶ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧς μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἐκεῖνα ἴσα ἐστίν· ὅπερ ἔδει δεῖξαι.



For let *CA* be laid down so as to be straight-on (with respect) to *AD*. Thus, *EA* is also straight-on (with respect) to *AB* [Prop. 1.14]. And let *BD* be joined.

Therefore, since triangle *ABC* is equal to triangle *ADE*, and *BAD* (is) some other (triangle), thus as triangle *CAB* is to triangle *BAD*, so triangle *EAD* (is) to triangle *BAD* [Prop. 5.7]. But, as (triangle) *CAB* (is) to *BAD*, so *CA* (is) to *AD*, and as (triangle) *EAD* (is) to *BAD*, so *EA* (is) to *AB* [Prop. 6.1]. And thus, as *CA* (is) to *AD*, so *EA* (is) to *AB*. Thus, in triangles *ABC* and *ADE* the sides about the equal angles (are) reciprocally proportional.

And so, let the sides of triangles *ABC* and *ADE* be reciprocally proportional, and (thus) let *CA* be to *AD*, as *EA* (is) to *AB*. I say that triangle *ABC* is equal to triangle *ADE*.

For, *BD* again being joined, since as *CA* is to *AD*, so *EA* (is) to *AB*, but as *CA* (is) to *AD*, so triangle *ABC* (is) to triangle *BAD*, and as *EA* (is) to *AB*, so triangle *EAD* (is) to triangle *BAD* [Prop. 6.1], thus as triangle *ABC* (is) to triangle *BAD*, so triangle *EAD* (is) to triangle *BAD*. Thus, (triangles) *ABC* and *EAD* each have the same ratio to *BAD*. Thus, [triangle] *ABC* is equal to triangle *EAD* [Prop. 5.9].

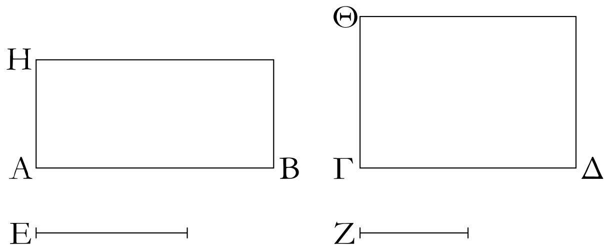
Thus, in equal triangles also having one angle equal to one (angle), the sides about the equal angles (are) reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal. (Which is) the very thing it was required to show.

ιϛʹ.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται.

Proposition 16

If four straight-lines are proportional, (then) the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two), (then) the four straight-lines will be proportional.



Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ ΑΒ, ΓΔ, Ε, Ζ, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΒ, Ζ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένῳ ὀρθογωνίῳ.

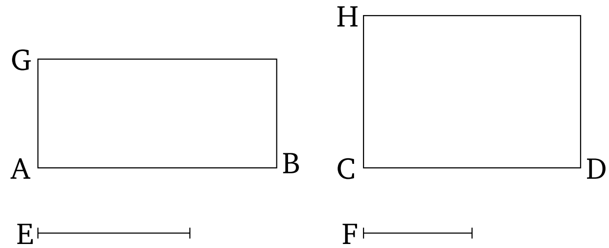
Ἤχθωσαν [γὰρ] ἀπὸ τῶν Α, Γ σημείων ταῖς ΑΒ, ΓΔ εὐθείαις πρὸς ὀρθὰς αἱ ΑΗ, ΓΘ, καὶ κείσθω τῇ μὲν Ζ ἴση ἡ ΑΗ, τῇ δὲ Ε ἴση ἡ ΓΘ. καὶ συμπεπληρώσθω τὰ ΒΗ, ΔΘ παραλληλόγραμμα.

Καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ, ἴση δὲ ἡ μὲν Ε τῇ ΓΘ, ἡ δὲ Ζ τῇ ΑΗ, ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΘ πρὸς τὴν ΑΗ. τῶν ΒΗ, ΔΘ ἄρα παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ὧν δὲ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραί αἱ περὶ τὰς ἴσας γωνάις, ἴσα ἐστὶν ἐκεῖνα· ἴσον ἄρα ἐστὶ τὸ ΒΗ παραλληλόγραμμον τῷ ΔΘ παραλληλογράμμῳ. καί ἐστι τὸ μὲν ΒΗ τὸ ὑπὸ τῶν ΑΒ, Ζ· ἴση γὰρ ἡ ΑΗ τῇ Ζ· τὸ δὲ ΔΘ τὸ ὑπὸ τῶν ΓΔ, Ε· ἴση γὰρ ἡ Ε τῇ ΓΘ· τὸ ἄρα ὑπὸ τῶν ΑΒ, Ζ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένῳ ὀρθογώνιῳ.

Ἀλλὰ δὴ τὸ ὑπὸ τῶν ΑΒ, Ζ περιεχόμενον ὀρθογώνιον ἴσον ἔστω τῷ ὑπὸ τῶν ΓΔ, Ε περιεχομένῳ ὀρθογωνίῳ. λέγω, ὅτι αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν ΑΒ, Ζ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, Ε, καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒ, Ζ τὸ ΒΗ· ἴση γάρ ἐστιν ἡ ΑΗ τῇ Ζ· τὸ δὲ ὑπὸ τῶν ΓΔ, Ε τὸ ΔΘ· ἴση γὰρ ἡ ΓΘ τῇ Ε· τὸ ἄρα ΒΗ ἴσον ἐστὶ τῷ ΔΘ. καί ἐστιν ἰσογώνια. τῶν δὲ ἴσων καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΘ πρὸς τὴν ΑΗ. ἴση δὲ ἡ μὲν ΓΘ τῇ Ε, ἡ δὲ ΑΗ τῇ Ζ· ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ Ε πρὸς τὴν Ζ.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι.



Let *AB*, *CD*, *E*, and *F* be four proportional straight-lines, (such that) as *AB* (is) to *CD*, so *E* (is) to *F*. I say that the rectangle contained by *AB* and *F* is equal to the rectangle contained by *CD* and *E*.

[For] let *AG* and *CH* be drawn from points *A* and *C* at right-angles to the straight-lines *AB* and *CD* (respectively) [Prop. 1.11]. And let *AG* be made equal to *F*, and *CH* to *E* [Prop. 1.3]. And let the parallelograms *BG* and *DH* be completed.

And since as *AB* is to *CD*, so *E* (is) to *F*, and *E* (is) equal *CH*, and *F* to *AG*, thus as *AB* is to *CD*, so *CH* (is) to *AG*. Thus, in the parallelograms *BG* and *DH*, the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.14]. Thus, parallelogram *BG* is equal to parallelogram *DH*. And *BG* is the (rectangle contained) by *AB* and *F*. For *AG* (is) equal to *F*. And *DH* (is) the (rectangle contained) by *CD* and *E*. For *E* (is) equal to *CH*. Thus, the rectangle contained by *AB* and *F* is equal to the rectangle contained by *CD* and *E*.

And so, let the rectangle contained by *AB* and *F* be equal to the rectangle contained by *CD* and *E*. I say that the four straight-lines will be proportional, (so that) as *AB* (is) to *CD*, so *E* (is) to *F*.

For, with the same construction, since the (rectangle contained) by *AB* and *F* is equal to the (rectangle contained) by *CD* and *E*. And *BG* is the (rectangle contained) by *AB* and *F*. For *AG* is equal to *F*. And *DH* (is) the (rectangle contained) by *CD* and *E*. For *CH* (is) equal to *E*. *BG* is thus equal to *DH*. And they are equiangular. And in equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as *AB* is to *CD*, so *CH* (is) to *AG*. And *CH* (is) equal to *E*, and *AG* to *F*. Thus, as *AB* is to *CD*, so *E* (is) to *F*.

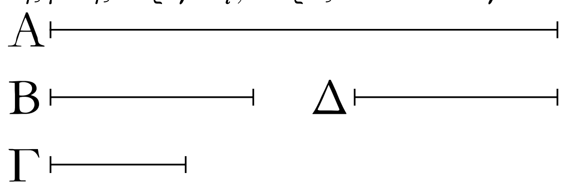
Thus, if four straight-lines are proportional, (then) the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two) then the four straight-lines will be proportional. (Which is) the very thing it was required to show.

ιζʹ.

Ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περι-

Proposition 17

If three straight-lines are proportional, (then) the rectangle εχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ἀπὸ τῆς μέσης τετραγώνῳ, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσονται.



Ἔστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ· λέγω, ὅτι τὸ ὑπὸ τῶν Α, Γ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β τετραγώνῳ.

Κείσθω τῇ Β ἴση ἡ Δ.

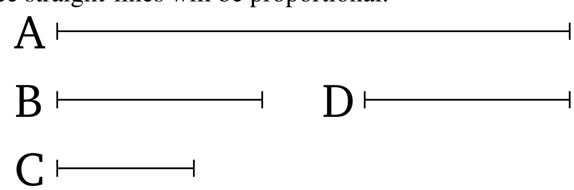
Καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ, ἴση δὲ ἡ Β τῇ Δ, ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Β, ἡ Δ πρὸς τὴν Γ. ἐὰν δὲ τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον [ὀρθογώνιον] ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογωνίῳ. τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Β, Δ. ἀλλὰ τὸ ὑπὸ τῶν Β, Δ τὸ ἀπὸ τῆς Β ἐστιν· ἴση γὰρ ἡ Β τῇ Δ· τὸ ἄρα ὑπὸ τῶν Α, Γ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β τετραγώνῳ.

Ἀλλὰ δὴ τὸ ὑπὸ τῶν Α, Γ ἴσον ἔστω τῷ ἀπὸ τῆς Β· λέγω, ὅτι ἐστὶν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β, ἀλλὰ τὸ ἀπὸ τῆς Β τὸ ὑπὸ τῶν Β, Δ ἐστιν· ἴση γὰρ ἡ Β τῇ Δ· τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Β, Δ. ἐὰν δὲ τὸ ὑπὸ τῶν ἄκρων ἴσον ᾖ τῷ ὑπὸ τῶν μέσων, αἱ τέσσαρες εὐθεῖαι ἀνάλογόν εἰσιν. ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Δ πρὸς τὴν Γ. ἴση δὲ ἡ Β τῇ Δ· ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ.

Ἐὰν ἄρα τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνῳ· κἂν τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ᾖ τῷ ἀπὸ τῆς μέσης τετραγώνῳ, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι.

contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one), (then) the three straight-lines will be proportional.



Let *A*, *B* and *C* be three proportional straight-lines, (such that) as *A* (is) to *B*, so *B* (is) to *C*. I say that the rectangle contained by *A* and *C* is equal to the square on *B*.

Let *D* be made equal to *B* [Prop. 1.3].

And since as *A* is to *B*, so *B* (is) to *C*, and *B* (is) equal to *D*, thus as *A* is to *B*, (so) *D* (is) to *C*. And if four straight-lines are proportional, (then) the [rectangle] contained by the (two) outermost is equal to the rectangle contained by the middle (two) [Prop. 6.16]. Thus, the (rectangle contained) by *A* and *C* is equal to the (rectangle contained) by *B* and *D*. But, the (rectangle contained) by *B* and *D* is the (square) on *B*. For *B* (is) equal to *D*. Thus, the rectangle contained by *A* and *C* is equal to the square on *B*.

And so, let the (rectangle contained) by *A* and *C* be equal to the (square) on *B*. I say that as *A* is to *B*, so *B* (is) to *C*.

For, with the same construction, since the (rectangle contained) by *A* and *C* is equal to the (square) on *B*. But, the (square) on *B* is the (rectangle contained) by *B* and *D*. For *B* (is) equal to *D*. The (rectangle contained) by *A* and *C* is thus equal to the (rectangle contained) by *B* and *D*. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two), (then) the four straight-lines are proportional [Prop. 6.16]. Thus, as *A* is to *B*, so *D* (is) to *C*. And *B* (is) equal to *D*. Thus, as *A* (is) to *B*, so *B* (is) to *C*.

Thus, if three straight-lines are proportional, (then) the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one), (then) the three straight-lines will be proportional. (Which is) the very thing it was required to show.

ιηʹ.

Ἀπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι εὐθυγράμμῳ ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγράψαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμ-

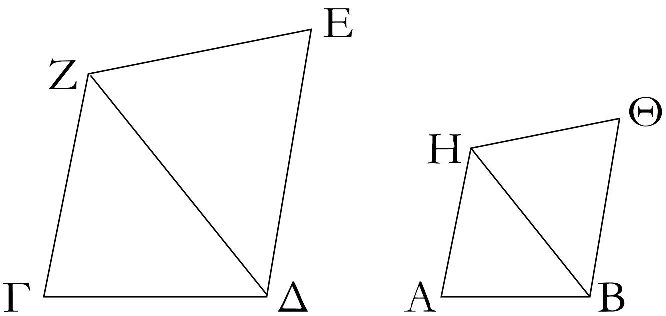
Proposition 18

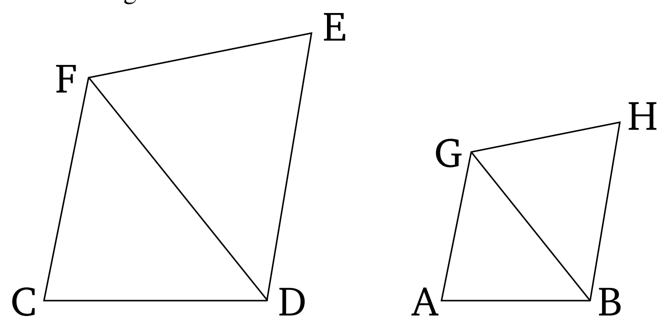
To describe a rectilinear figure similar, and similarly laid down, to a given rectilinear figure on a given straight-line.

Let *AB* be the given straight-line, and *CE* the given recti-

ον τὸ ΓΕ· δεῖ δὴ ἀπὸ τῆς ΑΒ εὐθείας τῷ ΓΕ εὐθυγράμμῳ ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγράψαι.

linear figure. So it is required to describe a rectilinear figure similar, and similarly laid down, to the rectilinear figure *CE* on the straight-line *AB*.





Ἐπεζεύχθω ἡ ΔΖ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Α, Β τῇ μὲν πρὸς τῷ Γ γωνίᾳ ἴση ἡ ὑπὸ ΗΑΒ, τῇ δὲ ὑπὸ ΓΔΖ ἴση ἡ ὑπὸ ΑΒΗ. λοιπὴ ἄρα ἡ ὑπὸ ΓΖΔ τῇ ὑπὸ ΑΗΒ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΖΓΔ τρίγωνον τῷ ΗΑΒ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΓ πρὸς τὴν ΗΑ, καὶ ἡ ΓΔ πρὸς τὴν ΑΒ. πάλιν συνεστάτω πρὸς τῇ ΒΗ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Β, Η τῇ μὲν ὑπὸ ΔΖΕ γωνίᾳ ἴση ἡ ὑπὸ ΒΗΘ, τῇ δὲ ὑπὸ ΖΔΕ ἴση ἡ ὑπὸ ΗΒΘ. λοιπὴ ἄρα ἡ πρὸς τῷ Ε λοιπῇ τῇ πρὸς τῷ Θ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΖΔΕ τρίγωνον τῷ ΗΘΒ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΕ πρὸς τὴν ΗΘ καὶ ἡ ΕΔ πρὸς τὴν ΘΒ. ἐδείχθη δὲ καὶ ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΓ πρὸς τὴν ΗΑ καὶ ἡ ΓΔ πρὸς τὴν ΑΒ· καὶ ὡς ἄρα ἡ ΖΓ πρὸς τὴν ΑΗ, οὕτως ἥ τε ΓΔ πρὸς τὴν ΑΒ καὶ ἡ ΖΕ πρὸς τὴν ΗΘ καὶ ἔτι ἡ ΕΔ πρὸς τὴν ΘΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ὑπὸ ΓΖΔ γωνία τῇ ὑπὸ ΑΗΒ, ἡ δὲ ὑπὸ ΔΖΕ τῇ ὑπὸ ΒΗΘ, ὅλη ἄρα ἡ ὑπὸ ΓΖΕ ὅλη τῇ ὑπὸ ΑΗΘ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΒΘ ἐστιν ἴση. ἔστι δὲ καὶ ἡ μὲν πρὸς τῷ Γ τῇ πρὸς τῷ Α ἴση, ἡ δὲ πρὸς τῷ Ε τῇ πρὸς τῷ Θ. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΘ τῷ ΓΕ· καὶ τὰς περὶ τὰς ἴσας γωνίας αὐτῶν πλευρὰς ἀνάλογον ἔχει· ὅμοιον ἄρα ἐστὶ τὸ ΑΘ εὐθύγραμμον τῷ ΓΕ εὐθυγράμμῳ.

Ἀπὸ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ ΓΕ ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον ἀναγέγραπται τὸ ΑΘ· ὅπερ ἔδει ποιῆσαι.

Let *DF* be joined, and let *GAB*, equal to the angle at *C*, and *ABG*, equal to (angle) *CDF*, be constructed on the straight-line *AB* at the points *A* and *B* on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) *CFD* is equal to *AGB* [Prop. 1.32]. Thus, triangle *FCD* is equiangular to triangle *GAB*. Thus, proportionally, as *FD* is to *GB*, so *FC* (is) to *GA*, and *CD* to *AB* [Prop. 6.4]. Again, let *BGH*, equal to angle *DFE*, and *GBH* equal to (angle) *FDE*, be constructed on the straight-line *BG* at the points *G* and *B* on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at *E* is equal to the remaining (angle) at *H* [Prop. 1.32]. Thus, triangle *FDE* is equiangular to triangle *GHB*. Thus, proportionally, as *FD* is to *GB*, so *FE* (is) to *GH*, and *ED* to *HB* [Prop. 6.4]. And it was also shown (that) as *FD* (is) to *GB*, so *FC* (is) to *GA*, and *CD* to *AB*. Thus, also, as *FC* (is) to *AG*, so *CD* (is) to *AB*, and *FE* to *GH*, and, further, *ED* to *HB*. And since angle *CFD* is equal to *AGB*, and *DFE* to *BGH*, thus the whole (angle) *CFE* is equal to the whole (angle) *AGH*. So, for the same (reasons), (angle) *CDE* is also equal to *ABH*. And the (angle) at *C* is also equal to the (angle) at *A*, and the (angle) at *E* to the (angle) at *H*. Thus, (figure) *AH* is equiangular to *CE*. And (the two figures) have the sides about their equal angles proportional. Thus, the rectilinear figure *AH* is similar to the rectilinear figure *CE* [Def. 6.1].

Thus, the rectilinear figure *AH*, similar, and similarly laid down, to the given rectilinear figure *CE* has been constructed on the given straight-line *AB*. (Which is) the very thing it was required to do.

ιθʹ.

Τὰ ὅμοια τρίγωνα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν.

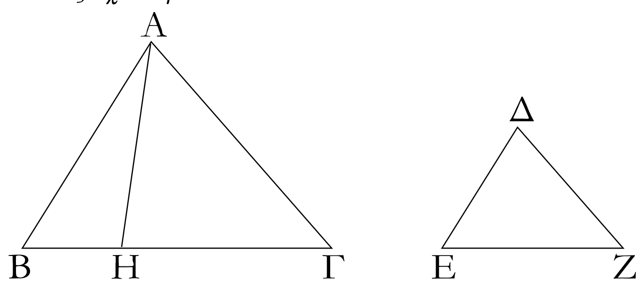
Ἔστω ὅμοια τρίγωνα τὰ ΑΒΓ, ΔΕΖ ἴσην ἔχοντα τὴν πρὸς τῷ Β γωνίαν τῇ πρὸς τῷ Ε, ὡς δὲ τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, ὥστε ὁμόλογον εἶναι τὴν ΒΓ τῇ ΕΖ· λέγω, ὅτι τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Proposition 19

Similar triangles are to one another in the squared† ratio of (their) corresponding sides.

Let *ABC* and *DEF* be similar triangles having the angle at *B* equal to the (angle) at *E*, and *AB* to *BC*, as *DE* (is) to *EF*, such that *BC* corresponds to *EF*. I say that triangle *ABC* has a squared ratio to triangle *DEF* with respect to (that side) *BC* (has) to *EF*.

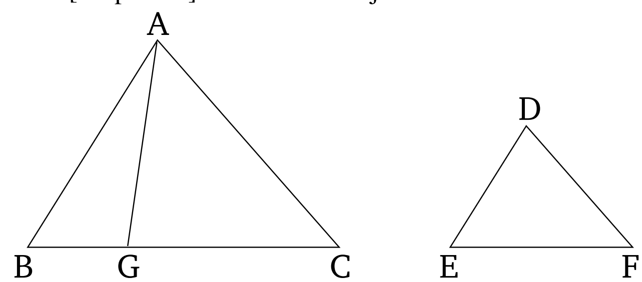
Εἰλήφθω γὰρ τῶν ΒΓ, ΕΖ τρίτη ἀνάλογον ἡ ΒΗ, ὥστε εἶναι ὡς τὴν ΒΓ πρὸς τὴν ΕΖ, οὕτως τὴν ΕΖ πρὸς τὴν ΒΗ· καὶ ἐπεζεύχθω ἡ ΑΗ.



Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΖ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρὸς τὴν ΕΖ. ἀλλ᾽ ὡς ἡ ΒΓ πρὸς ΕΖ, οὕτως ἐστὶν ἡ ΕΖ πρὸς ΒΗ. καὶ ὡς ἄρα ἡ ΑΒ πρὸς ΔΕ, οὕτως ἡ ΕΖ πρὸς ΒΗ· τῶν ΑΒΗ, ΔΕΖ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνάις. ὧν δὲ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνάις, ἴσα ἐστὶν ἐκεῖνα. ἴσον ἄρα ἐστὶ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ τριγώνῳ. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΕΖ, οὕτως ἡ ΕΖ πρὸς τὴν ΒΗ, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἡ πρώτη πρὸς τὴν τρίτην διπλασίονα λόγον ἔχει ἤπερ πρὸς τὴν δευτέραν, ἡ ΒΓ ἄρα πρὸς τὴν ΒΗ διπλασίονα λόγον ἔχει ἤπερ ἡ ΓΒ πρὸς τὴν ΕΖ. ὡς δὲ ἡ ΓΒ πρὸς τὴν ΒΗ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΒΗ τρίγωνον· καὶ τὸ ΑΒΓ ἄρα τρίγωνον πρὸς τὸ ΑΒΗ διπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἴσον δὲ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ τριγώνῳ· καὶ τὸ ΑΒΓ ἄρα τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Τὰ ἄρα ὅμοια τρίγωνα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. [ὅπερ ἔδει δεῖξαι.]

For let a third (straight-line), *BG*, be taken (which is) proportional to *BC* and *EF*, so that as *BC* (is) to *EF*, so *EF* (is) to *BG* [Prop. 6.11]. And let *AG* be joined.



Therefore, since as *AB* is to *BC*, so *DE* (is) to *EF*, thus, alternately, as *AB* is to *DE*, so *BC* (is) to *EF* [Prop. 5.16]. But, as *BC* (is) to *EF*, so *EF* is to *BG*. And, thus, as *AB* (is) to *DE*, so *EF* (is) to *BG*. Thus, for triangles *ABG* and *DEF*, the sides about the equal angles are reciprocally proportional. And those triangles having one (angle) equal to one (angle) for which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.15]. Thus, triangle *ABG* is equal to triangle *DEF*. And since as *BC* (is) to *EF*, so *EF* (is) to *BG*, and if three straight-lines are proportional, (then) the first has a squared ratio to the third with respect to the second [Def. 5.9], *BC* thus has a squared ratio to *BG* with respect to (that) *CB* (has) to *EF*. And as *CB* (is) to *BG*, so triangle *ABC* (is) to triangle *ABG* [Prop. 6.1]. Thus, triangle *ABC* also has a squared ratio to (triangle) *ABG* with respect to (that side) *BC* (has) to *EF*. And triangle *ABG* (is) equal to triangle *DEF*. Thus, triangle *ABC* also has a squared ratio to triangle *DEF* with respect to (that side) *BC* (has) to *EF*.

Thus, similar triangles are to one another in the squared ratio of (their) corresponding sides. [(Which is) the very thing it was required to show].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ὅπερ ἔδει δεῖξαι.

Corollary

So it is clear, from this, that if three straight-lines are proportional, (then) as the first is to the third, so the figure (described) on the first (is) to the similar, and similarly described, (figure) on the second. (Which is) the very thing it was required to show.

† Literally, "double".

κ΄.

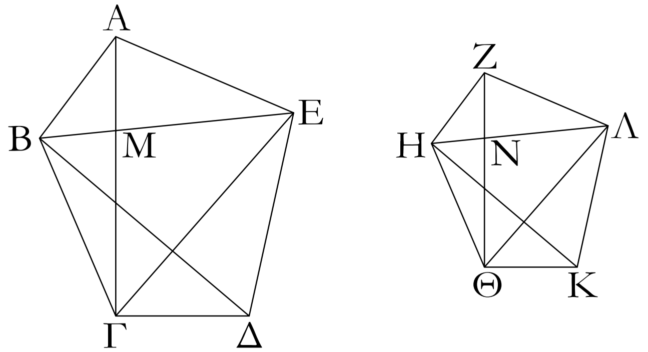
Τὰ ὅμοια πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.

Ἔστω ὅμοια πολύγωνα τὰ ΑΒΓΔΕ, ΖΗΘΚΛ, ὁμόλογος δὲ ἔστω ἡ ΑΒ τῇ ΖΗ· λέγω, ὅτι τὰ ΑΒΓΔΕ, ΖΗΘΚΛ

Proposition 20

Similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side.

Let *ABCDE* and *FGHKL* be similar polygons, and let *AB* *πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ πρὸς τὴν ΖΗ.*



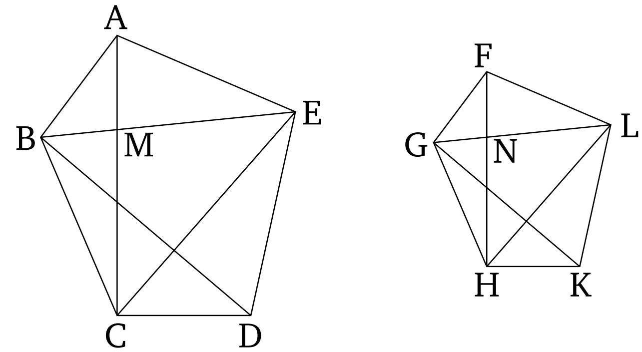
Ἐπεζεύχθωσαν αἱ ΒΕ, ΕΓ, ΗΛ, ΛΘ.

Καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνῳ, ἴση ἐστὶν ἡ ὑπὸ ΒΑΕ γωνία τῇ ὑπὸ ΗΖΛ. καί ἐστιν ὡς ἡ ΒΑ πρὸς ΑΕ, οὕτως ἡ ΗΖ πρὸς ΖΛ. ἐπεὶ οὖν δύο τρίγωνά ἐστι τὰ ΑΒΕ, ΖΗΛ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΗΛ τριγώνῳ· ὥστε καὶ ὅμοιον· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΖΗΛ. ἔστι δὲ καὶ ὅλη ἡ ὑπὸ ΑΒΓ ὅλῃ τῇ ὑπὸ ΖΗΘ ἴση διὰ τὴν ὁμοιότητα τῶν πολυγώνων· λοιπὴ ἄρα ἡ ὑπὸ ΕΒΓ γωνία τῇ ὑπὸ ΛΗΘ ἐστιν ἴση. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΑΒΕ, ΖΗΛ τριγώνων ἐστὶν ὡς ἡ ΕΒ πρὸς ΒΑ, οὕτως ἡ ΛΗ πρὸς ΗΖ, ἀλλὰ μὴν καὶ διὰ τὴν ὁμοιότητα τῶν πολυγώνων ἐστὶν ὡς ἡ ΑΒ πρὸς ΒΓ, οὕτως ἡ ΖΗ πρὸς ΗΘ, δι᾽ ἴσου ἄρα ἐστὶν ὡς ἡ ΕΒ πρὸς ΒΓ, οὕτως ἡ ΛΗ πρὸς ΗΘ, καὶ περὶ τὰς ἴσας γωνάις τὰς ὑπὸ ΕΒΓ, ΛΗΘ αἱ πλευραὶ ἀνάλογόν εἰσιν· ἰσογώνιον ἄρα ἐστὶ τὸ ΕΒΓ τρίγωνον τῷ ΛΗΘ τριγώνῳ· ὥστε καὶ ὅμοιόν ἐστι τὸ ΕΒΓ τρίγωνον τῷ ΛΗΘ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΕΓΔ τρίγωνον ὅμοιόν ἐστι τῷ ΛΘΚ τριγώνῳ. τὰ ἄρα ὅμοια πολύγωνα τὰ ΑΒΓΔΕ, ΖΗΘΚΛ εἴς τε ὅμοια τρίγωνα διῄρηται καὶ εἰς ἴσα τὸ πλῆθος.

Λέγω, ὅτι καὶ ὁμόλογα τοῖς ὅλοις, τουτέστιν ὥστε ἀνάλογον εἶναι τὰ τρίγωνα, καὶ ἡγούμενα μὲν εἶναι τὰ ΑΒΕ, ΕΒΓ, ΕΓΔ, ἑπόμενα δὲ αὐτῶν τὰ ΖΗΛ, ΛΗΘ, ΛΘΚ, καὶ ὅτι τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἡ ΑΒ πρὸς τὴν ΖΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΖΘ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν πολυγώνων ἴση ἐστὶν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΖΗΘ, καί ἐστιν ὡς ἡ ΑΒ πρὸς ΒΓ, οὕτως ἡ ΖΗ πρὸς ΗΘ, ἰσογώνιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΖΗΘ τριγώνῳ· ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΗΖΘ, ἡ δὲ ὑπὸ ΒΓΑ τῇ ὑπὸ ΗΘΖ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΒΑΜ γωνία τῇ ὑπὸ ΗΖΝ, ἔστι δὲ καὶ ἡ ὑπὸ ΑΒΜ τῇ ὑπὸ ΖΗΝ ἴση, καὶ λοιπὴ ἄρα ἡ ὑπὸ ΑΜΒ λοιπῇ τῇ ὑπὸ ΖΝΗ ἴση ἐστίν· ἰσογώνιον ἄρα ἐστὶ

correspond to *FG*. I say that polygons *ABCDE* and *FGHKL* can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and (that) polygon *ABCDE* has a squared ratio to polygon *FGHKL* with respect to that *AB* (has) to *FG*.



Let *BE*, *EC*, *GL*, and *LH* be joined.

And since polygon *ABCDE* is similar to polygon *FGHKL*, angle *BAE* is equal to angle *GFL*, and as *BA* is to *AE*, so *GF* (is) to *FL* [Def. 6.1]. Therefore, since *ABE* and *FGL* are two triangles having one angle equal to one angle and the sides about the equal angles proportional, triangle *ABE* is thus equiangular to triangle *FGL* [Prop. 6.6]. Hence, (they are) also similar [Prop. 6.4, Def. 6.1]. Thus, angle *ABE* is equal to (angle) *FGL*. And the whole (angle) *ABC* is equal to the whole (angle) *FGH*, on account of the similarity of the polygons. Thus, the remaining angle *EBC* is equal to *LGH*. And since, on account of the similarity of triangles *ABE* and *FGL*, as *EB* is to *BA*, so *LG* (is) to *GF*, but also, on account of the similarity of the polygons, as *AB* is to *BC*, so *FG* (is) to *GH*, thus, via equality, as *EB* is to *BC*, so *LG* (is) to *GH* [Prop. 5.22], and the sides about the equal angles, *EBC* and *LGH*, are proportional. Thus, triangle *EBC* is equiangular to triangle *LGH* [Prop. 6.6]. Hence, triangle *EBC* is also similar to triangle *LGH* [Prop. 6.4, Def. 6.1]. So, for the same (reasons), triangle *ECD* is also similar to triangle *LHK*. Thus, the similar polygons *ABCDE* and *FGHKL* have been divided into equal numbers of similar triangles.

I also say that (the triangles) correspond (in proportion) to the wholes. That is to say, the triangles are proportional: *ABE*, *EBC*, and *ECD* are the leading (magnitudes), and their (associated) following (magnitudes are) *FGL*, *LGH*, and *LHK* (respectively). (I) also (say) that polygon *ABCDE* has a squared ratio to polygon *FGHKL* with respect to (that) a corresponding side (has) to a corresponding side—that is to say, (side) *AB* to *FG*.

For let *AC* and *FH* be joined. And since angle *ABC* is equal to *FGH*, and as *AB* is to *BC*, so *FG* (is) to *GH*, on account of the similarity of the polygons, triangle *ABC* is equiangular to triangle *FGH* [Prop. 6.6]. Thus, angle *BAC* is equal to *GFH*, and (angle) *BCA* to *GHF*. And since angle *BAM* is *τὸ ΑΒΜ τρίγωνον τῷ ΖΗΝ τριγώνῳ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ ΒΜΓ τρίγωνον ἰσογώνιόν ἐστι τῷ ΗΝΘ τριγώνῳ. ἀνάλογον ἄρα ἐστίν, ὡς μὲν ἡ ΑΜ πρὸς ΜΒ, οὕτως ἡ ΖΝ πρὸς ΝΗ, ὡς δὲ ἡ ΒΜ πρὸς ΜΓ, οὕτως ἡ ΗΝ πρὸς ΝΘ· ὥστε καὶ δι᾿ ἴσου, ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως ἡ ΖΝ πρὸς ΝΘ. ἀλλ᾿ ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως τὸ ΑΒΜ [τρίγωνον] πρὸς τὸ ΜΒΓ, καὶ τὸ ΑΜΕ πρὸς τὸ ΕΜΓ· πρὸς ἄλληλα γάρ εἰσιν ὡς αἱ βάσεις. καὶ ὡς ἄρα ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπόμενων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ὡς ἄρα τὸ ΑΜΒ τρίγωνον πρὸς τὸ ΒΜΓ, οὕτως τὸ ΑΒΕ πρὸς τὸ ΓΒΕ. αλλ᾿ ὡς τὸ ΑΜΒ πρὸς τὸ ΒΜΓ, οὕτως ἡ ΑΜ πρὸς ΜΓ· καὶ ὡς ἄρα ἡ ΑΜ πρὸς ΜΓ, οὕτως τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΕΒΓ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ ΖΝ πρὸς ΝΘ, οὕτως τὸ ΖΗΛ τρίγωνον πρὸς τὸ ΗΛΘ τρίγωνον. καί ἐστιν ὡς ἡ ΑΜ πρὸς ΜΓ, οὕτως ἡ ΖΝ πρὸς ΝΘ· καὶ ὡς ἄρα τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΒΕΓ τρίγωνον, οὕτως τὸ ΖΗΛ τρίγωνον πρὸς τὸ ΗΛΘ τρίγωνον, καὶ ἐναλλὰξ ὡς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΒΕΓ τρίγωνον πρὸς τὸ ΗΛΘ τρίγωνον. ὁμοίως δὴ δείξομεν ἐπιζευχθεισῶν τῶν ΒΔ, ΗΚ, ὅτι καὶ ὡς τὸ ΒΕΓ τρίγωνον πρὸς τὸ ΛΗΘ τρίγωνον, οὕτως τὸ ΕΓΔ τρίγωνον πρὸς τὸ ΛΘΚ τρίγωνον. καὶ ἐπεί ἐστιν ὡς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΕΒΓ πρὸς τὸ ΛΗΘ, καὶ ἔτι τὸ ΕΓΔ πρὸς τὸ ΛΘΚ, καὶ ὡς ἄρα ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα ὡς τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον. ἀλλὰ τὸ ΑΒΕ τρίγωνον πρὸς τὸ ΖΗΛ τρίγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ ὁμόλογος πλευρὰ πρὸς τὴν ΖΗ ὁμόλογον πλευράν· τὰ γὰρ ὅμοια τρίγωνα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. καὶ τὸ ΑΒΓΔΕ ἄρα πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ ὁμόλογος πλευρὰ πρὸς τὴν ΖΗ ὁμόλογον πλευράν.*

Τὰ ἄρα ὅμοια πολύγωνα εἴς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν [ὅπερ ἔδει δεῖξαι].

equal to GFN, and (angle) ABM is also equal to FGN (see earlier), the remaining (angle) AMB is thus also equal to the remaining (angle) FNG [Prop. 1.32]. Thus, triangle ABM is equiangular to triangle FGN. So, similarly, we can show that triangle BMC is also equiangular to triangle GNH. Thus, proportionally, as AM is to MB, so FN (is) to NG, and as BM (is) to MC, so GN (is) to NH [Prop. 6.4]. Hence, also, via equality, as AM (is) to MC, so FN (is) to NH [Prop. 5.22]. But, as AM (is) to MC, so [triangle] ABM is to MBC, and AME to EMC. For they are to one another as their bases [Prop. 6.1]. And as one of the leading (magnitudes) is to one of the following (magnitudes), so (the sum of) all the leading (magnitudes) is to (the sum of) all the following (magnitudes) [Prop. 5.12]. Thus, as triangle AMB (is) to BMC, so (triangle) ABE (is) to CBE. But, as (triangle) AMB (is) to BMC, so AM (is) to MC. Thus, also, as AM (is) to MC, so triangle ABE (is) to triangle EBC. And so, for the same (reasons), as FN (is) to NH, so triangle FGL (is) to triangle GLH. And as AM is to MC, so FN (is) to NH. Thus, also, as triangle ABE (is) to triangle BEC, so triangle FGL (is) to triangle GLH, and, alternately, as triangle ABE (is) to triangle FGL, so triangle BEC (is) to triangle GLH [Prop. 5.16]. So, similarly, we can also show, by joining BD and GK, (that) as triangle BEC (is) to triangle LGH, so triangle ECD (is) to triangle LHK. And since as triangle ABE is to triangle FGL, so (triangle) EBC (is) to LGH, and, further, (triangle) ECD to LHK, and also as one of the leading (magnitudes is) to one of the following, so (the sum of) all the leading (magnitudes is) to (the sum of) all the following [Prop. 5.12], thus as triangle ABE is to triangle FGL, so polygon $ABCDE$ (is) to polygon $FGHKL$. But, triangle ABE has a squared ratio to triangle FGL with respect to (that) the corresponding side AB (has) to the corresponding side FG. For, similar triangles are in the squared ratio of corresponding sides [Prop. 6.14]. Thus, polygon $ABCDE$ also has a squared ratio to polygon $FGHKL$ with respect to (that) the corresponding side AB (has) to the corresponding side FG.

Thus, similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side. [(Which is) the very thing it was required to show].

Πόρισμα.

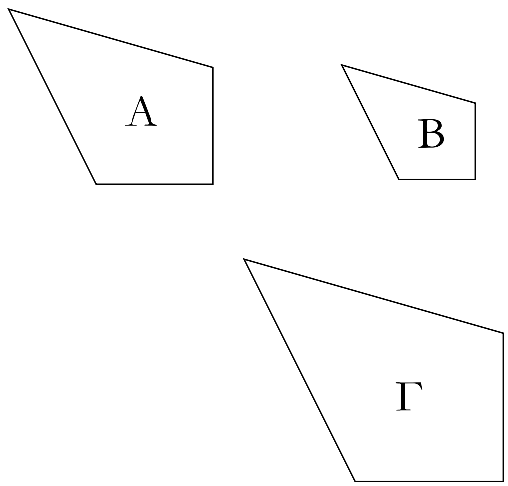
Ὡσαύτως δὲ καὶ ἐπὶ τῶν [ὁμοίων] τετραπλεύρων δειχθήσεται, ὅτι ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ἐδείχθη δὲ καὶ ἐπὶ τῶν τριγώνων· ὥστε καὶ καθόλου τὰ ὅμοια εὐθύγραμμα σχήματα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ὅπερ ἔδει δεῖξαι.

Corollary

And, in the same manner, it can also be shown for [similar] quadrilaterals, (that) they are in the squared ratio of (their) corresponding sides. And it was also shown for triangles. Hence, in general, similar rectilinear figures are also to one another in the squared ratio of (their) corresponding sides. (Which is) the very thing it was required to show.

κα΄.

Τὰ τῷ αὐτῷ εὐθυγράμμῳ ὅμοια καὶ ἀλλήλοις ἐστὶν ὅμοια.

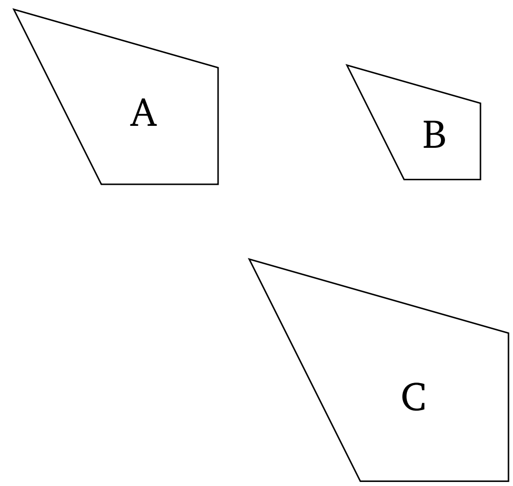


Ἔστω γὰρ ἑκάτερον τῶν Α, Β εὐθυγράμμων τῷ Γ ὅμοιον· λέγω, ὅτι καὶ τὸ Α τῷ Β ἐστιν ὅμοιον.

Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ Α τῷ Γ, ἰσογώνιόν τέ ἐστιν αὐτῷ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. πάλιν, ἐπεὶ ὅμοιόν ἐστι τὸ Β τῷ Γ, ἰσογώνιόν τέ ἐστιν αὐτῷ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. ἑκάτερον ἄρα τῶν Α, Β τῷ Γ ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει [ὥστε καὶ τὸ Α τῷ Β ἰσογώνιόν τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει]. ὅμοιον ἄρα ἐστὶ τὸ Α τῷ Β· ὅπερ ἔδει δεῖξαι.

Proposition 21

(Rectilinear figures) similar to the same rectilinear figure are also similar to one another.



Let each of the rectilinear figures *A* and *B* be similar to (the rectilinear figure) *C*. I say that *A* is also similar to *B*.

For since *A* is similar to *C*, (*A*) is equiangular to (*C*), and has the sides about the equal angles proportional [Def. 6.1]. Again, since *B* is similar to *C*, (*B*) is equiangular to (*C*), and has the sides about the equal angles proportional [Def. 6.1]. Thus, *A* and *B* are each equiangular to *C*, and have the sides about the equal angles proportional [hence, *A* is also equiangular to *B*, and has the sides about the equal angles proportional]. Thus, *A* is similar to *B* [Def. 6.1]. (Which is) the very thing it was required to show.

κβ΄.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ᾽ αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· κἂν τὰ ἀπ᾽ αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ᾖ, καὶ αὐτὰι αἱ εὐθεῖαι ἀνάλογον ἔσονται.

Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ ΑΒ, ΓΔ, ΕΖ, ΗΘ, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, καὶ ἀναγεγράφθωσαν ἀπὸ μὲν τῶν ΑΒ, ΓΔ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ ΚΑΒ, ΛΓΔ, ἀπὸ δὲ τῶν ΕΖ, ΗΘ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ ΜΖ, ΝΘ· λέγω, ὅτι ἐστὶν ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ.

Εἰλήφθω γὰρ τῶν μὲν ΑΒ, ΓΔ τρίτη ἀνάλογον ἡ Ξ, τῶν δὲ ΕΖ, ΗΘ τρίτη ἀνάλογον ἡ Ο. καὶ ἐπεί ἐστιν ὡς μὲν ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, ὡς δὲ ἡ ΓΔ πρὸς τὴν Ξ, οὕτως ἡ ΗΘ πρὸς τὴν Ο, δι᾽ ἴσου ἄρα ἐστὶν ὡς ἡ ΑΒ

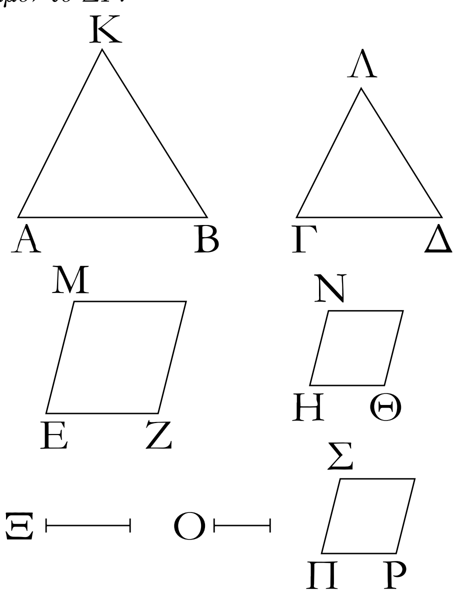
Proposition 22

If four straight-lines are proportional, (then) similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional, (then) the straight-lines themselves will also be proportional.

Let *AB*, *CD*, *EF*, and *GH* be four proportional straight-lines, (such that) as *AB* (is) to *CD*, so *EF* (is) to *GH*. And let the similar, and similarly laid out, rectilinear figures *KAB* and *LCD* be described on *AB* and *CD* (respectively), and the similar, and similarly laid out, rectilinear figures *MF* and *NH* on *EF* and *GH* (respectively). I say that as *KAB* is to *LCD*, so *MF* (is) to *NH*.

For let a third (straight-line) *O* be taken (which is) proportional to *AB* and *CD*, and a third (straight-line) *P* proportional to *EF* and *GH* [Prop. 6.11]. And since as *AB* is to *CD*, so *EF* (is) to *GH*, and as *CD* (is) to *O*, so *GH* (is) to *P*, thus, via *πρὸς τὴν Ξ, οὕτως ἡ ΕΖ πρὸς τὴν Ο. ἀλλ᾽ ὡς μὲν ἡ ΑΒ πρὸς τὴν Ξ, οὕτως [καὶ] τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, ὡς δὲ ἡ ΕΖ πρὸς τὴν Ο, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ· καὶ ὡς ἄρα τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ.*

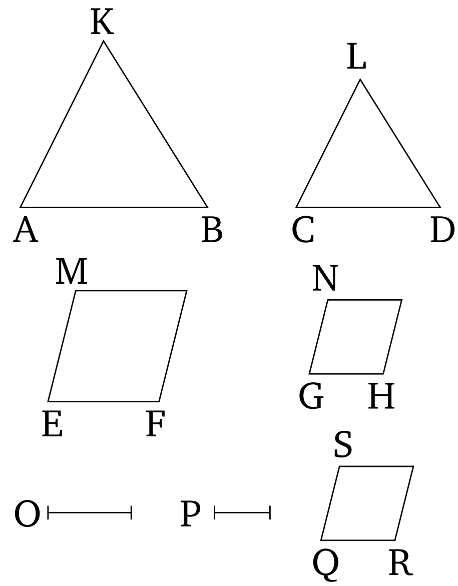
Ἀλλὰ δὴ ἔστω ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ· λέγω, ὅτι ἐστὶ καὶ ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ. εἰ γὰρ μή ἐστιν, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, ἔστω ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΠΡ ὁποτέρῳ τῶν ΜΖ, ΝΘ ὅμοιόν τε καὶ ὁμοίως κείμενον εὐθύγραμμον τὸ ΣΡ.



Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, καὶ ἀναγέγραπται ἀπὸ μὲν τῶν ΑΒ, ΓΔ ὅμοιά τε καὶ ὁμοίως κείμενα τὰ ΚΑΒ, ΛΓΔ, ἀπὸ δὲ τῶν ΕΖ, ΠΡ ὅμοιά τε καὶ ὁμοίως κείμενα τὰ ΜΖ, ΣΡ, ἔστιν ἄρα ὡς τὸ ΚΑΒ

equality, as AB is to O, so EF (is) to P [Prop. 5.22]. But, as AB (is) to O, so [also] KAB (is) to LCD, and as EF (is) to P, so MF (is) to NH [Prop. 5.19 corr.]. And, thus, as KAB (is) to LCD, so MF (is) to NH.

And so let KAB be to LCD, as MF (is) to NH. I say also that as AB is to CD, so EF (is) to GH. For if as AB is to CD, so EF (is) not to GH, let AB be to CD, as EF (is) to QR [Prop. 6.12]. And let the rectilinear figure SR, similar, and similarly laid down, to either of MF or NH, be described on QR [Props. 6.18, 6.21].



Therefore, since as AB is to CD, so EF (is) to QR, and the similar, and similarly laid out, (rectilinear figures) KAB and LCD have been described on AB and CD (respectively), and the similar, and similarly laid out, (rectilinear figures) MF and SR on EF and QR (respectively), thus as KAB is to LCD, *πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΣΡ. ὑπόκειται δὲ καὶ ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ· καὶ ὡς ἄρα τὸ ΜΖ πρὸς τὸ ΣΡ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ. τὸ ΜΖ ἄρα πρὸς ἑκάτερον τῶν ΝΘ, ΣΡ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΝΘ τῷ ΣΡ. ἔστι δὲ αὐτῷ καὶ ὅμοιον καὶ ὁμοίως κείμενον· ἴση ἄρα ἡ ΗΘ τῇ ΠΡ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, ἴση δὲ ἡ ΠΡ τῇ ΗΘ, ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.*

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ᾽ αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· κἂν τὰ ἀπ᾽ αὐτῶν εὐθύγραμμα ὅμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ᾖ, καὶ αὐτὰι αἱ εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι.

so *MF* (is) to *SR* (see above). And it was also assumed that as *KAB* (is) to *LCD*, so *MF* (is) to *NH*. Thus, also, as *MF* (is) to *SR*, so *MF* (is) to *NH* [Prop. 5.11]. Thus, *MF* has the same ratio to each of *NH* and *SR*. Thus, *NH* is equal to *SR* [Prop. 5.9]. And it is also similar, and similarly laid out, to it. Thus, *GH* (is) equal to *QR*.[†] And since *AB* is to *CD*, as *EF* (is) to *QR*, and *QR* (is) equal to *GH*, thus as *AB* is to *CD*, so *EF* (is) to *GH*.

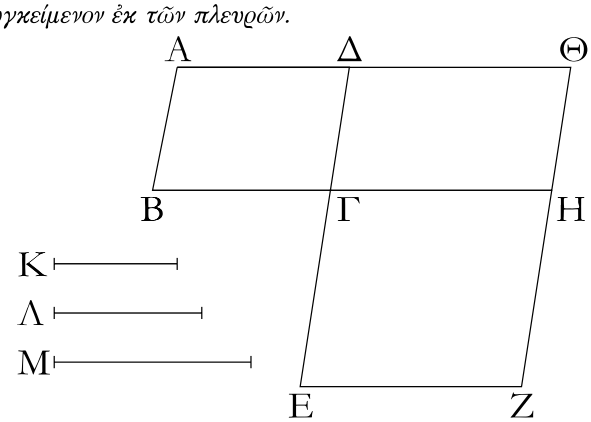
Thus, if four straight-lines are proportional, (then) similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional, (then) the straight-lines themselves will also be proportional. (Which is) the very thing it was required to show.

[†] Here, Euclid assumes, without proof, that if two similar figures are equal then any pair of corresponding sides is also equal.

κγ΄.

Τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Ἔστω ἰσογώνια παραλληλόγραμμα τὰ ΑΓ, ΓΖ ἴσην ἔχοντα τὴν ὑπὸ ΒΓΔ γωνίαν τῇ ὑπὸ ΕΓΗ· λέγω, ὅτι τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.



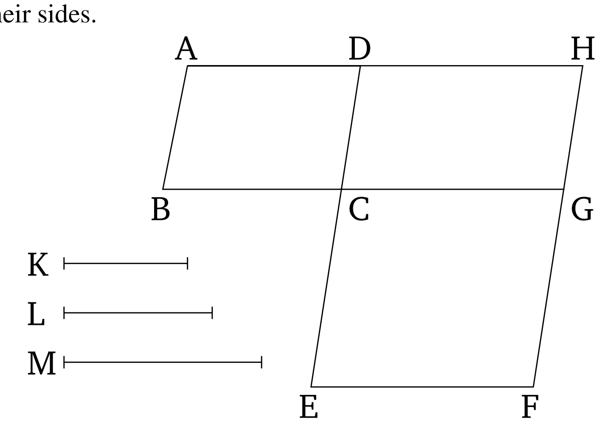
Κείσθω γὰρ ὥστε ἐπ᾽ εὐθείας εἶναι τὴν ΒΓ τῇ ΓΗ· ἐπ᾽ εὐθείας ἄρα ἐστὶ καὶ ἡ ΔΓ τῇ ΓΕ. καὶ συμπεπληρώσθω τὸ ΔΗ παραλληλόγραμμον, καὶ ἐκκείσθω τις εὐθεῖα ἡ Κ, καὶ γεγονέτω ὡς μὲν ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, ὡς δὲ ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως ἡ Λ πρὸς τὴν Μ.

Οἱ ἄρα λόγοι τῆς τε Κ πρὸς τὴν Λ καὶ τῆς Λ πρὸς τὴν Μ οἱ αὐτοί εἰσι τοῖς λόγοις τῶν πλευρῶν, τῆς τε ΒΓ πρὸς τὴν ΓΗ καὶ τῆς ΔΓ πρὸς τὴν ΓΕ. ἀλλ᾽ ὁ τῆς Κ πρὸς Μ λόγος σύγκειται ἔκ τε τοῦ τῆς Κ πρὸς Λ λόγου καὶ τοῦ τῆς Λ πρὸς Μ· ὥστε καὶ ἡ Κ πρὸς τὴν Μ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΘ, ἀλλ᾽ ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, καὶ ὡς ἄρα ἡ Κ πρὸς τὴν Λ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΘ. πάλιν, ἐπεί ἐστιν ὡς ἡ ΔΓ

Proposition 23

Equiangular parallelograms have to one another the ratio compounded[†] out of (the ratios of) their sides.

Let *AC* and *CF* be equiangular parallelograms having angle *BCD* equal to *ECG*. I say that parallelogram *AC* has to parallelogram *CF* the ratio compounded out of (the ratios of) their sides.



For let *BC* be laid down so as to be straight-on to *CG*. Thus, *DC* is also straight-on to *CE* [Prop. 1.14]. And let the parallelogram *DG* be completed. And let some straight-line *K* be laid down. And let it be contrived that as *BC* (is) to *CG*, so *K* (is) to *L*, and as *DC* (is) to *CE*, so *L* (is) to *M* [Prop. 6.12].

Thus, the ratios of *K* to *L* and of *L* to *M* are the same as the ratios of the sides, (namely), *BC* to *CG* and *DC* to *CE* (respectively). But, the ratio of *K* to *M* is compounded out of the ratio of *K* to *L* and (the ratio) of *L* to *M*. Hence, *K* also has to *M* the ratio compounded out of (the ratios of) the sides (of the parallelograms). And since as *BC* is to *CG*, so parallelogram *AC* (is) to *CH* [Prop. 6.1], but as *BC* (is) to *CG*, so *K* (is) to *L*, thus, also, as *K* (is) to *L*, so (parallelogram) *AC* (is) to *CH*. Again, since as *DC* (is) to *CE*, so parallelogram *CH* *πρὸς τὴν ΓΕ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ, ἀλλ᾿ ὡς ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως ἡ Λ πρὸς τὴν Μ, καὶ ὡς ἄρα ἡ Λ πρὸς τὴν Μ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον. ἐπεὶ οὖν ἐδείχθη, ὡς μὲν ἡ Κ πρὸς τὴν Λ, οὕτως τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΘ παραλληλόγραμμον, ὡς δὲ ἡ Λ πρὸς τὴν Μ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον, δι᾿ ἴσου ἄρα ἐστὶν ὡς ἡ Κ πρὸς τὴν Μ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΖ παραλληλόγραμμον. ἡ δὲ Κ πρὸς τὴν Μ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· καὶ τὸ ΑΓ ἄρα πρὸς τὸ ΓΖ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.*

Τὰ ἄρα ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

(is) to CF [Prop. 6.1], but as DC (is) to CE, so L (is) to M, thus, also, as L (is) to M, so parallelogram CH (is) to parallelogram CF. Therefore, since it was shown that as K (is) to L, so parallelogram AC (is) to parallelogram CH, and as L (is) to M, so parallelogram CH (is) to parallelogram CF, thus, via equality, as K is to M, so (parallelogram) AC (is) to parallelogram CF [Prop. 5.22]. And K has to M the ratio compounded out of (the ratios of) the sides (of the parallelograms). Thus, (parallelogram) AC also has to (parallelogram) CF the ratio compounded out of (the ratio of) their sides.

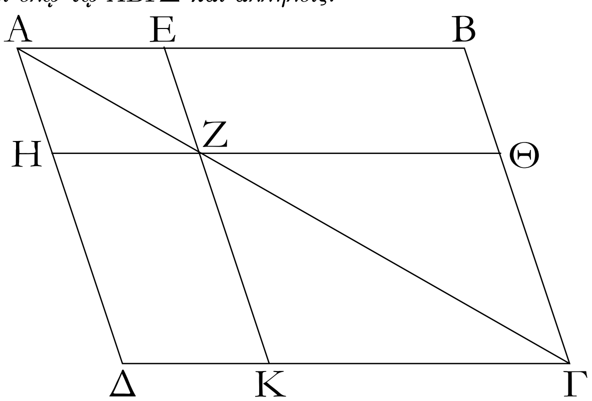
Thus, equiangular parallelograms have to one another the ratio compounded out of (the ratio of) their sides. (Which is) the very thing it was required to show.

† In modern terminology, if two ratios are "compounded" then they are multiplied together.

κδʹ.

Παντὸς παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις.

Ἔστω παραλληλόγραμμον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, περὶ δὲ τὴν ΑΓ παραλληλόγραμμα ἔστω τὰ ΕΗ, ΘΚ· λέγω, ὅτι ἑκάτερον τῶν ΕΗ, ΘΚ παραλληλογράμμων ὅμοιόν ἐστι ὅλῳ τῷ ΑΒΓΔ καὶ ἀλλήλοις.

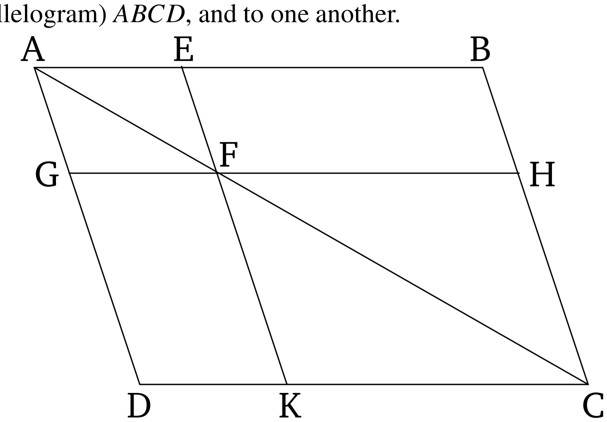


Ἐπεὶ γὰρ τριγώνου τοῦ ΑΒΓ παρὰ μίαν τῶν πλευρῶν τὴν ΒΓ ἦκται ἡ ΕΖ, ἀνάλογόν ἐστιν ὡς ἡ ΒΕ πρὸς τὴν ΕΑ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΑ. πάλιν, ἐπεὶ τριγώνου τοῦ ΑΓΔ παρὰ μίαν τὴν ΓΔ ἦκται ἡ ΖΗ, ἀνάλογόν ἐστιν ὡς ἡ ΓΖ πρὸς τὴν ΖΑ, οὕτως ἡ ΔΗ πρὸς τὴν ΗΑ. ἀλλ᾿ ὡς ἡ ΓΖ πρὸς τὴν ΖΑ, οὕτως ἐδείχθη καὶ ἡ ΒΕ πρὸς τὴν ΕΑ· καὶ ὡς ἄρα ἡ ΒΕ πρὸς τὴν ΕΑ, οὕτως ἡ ΔΗ πρὸς τὴν ΗΑ, καὶ συνθέντι ἄρα ὡς ἡ ΒΑ πρὸς ΑΕ, οὕτως ἡ ΔΑ πρὸς ΑΗ, καὶ ἐναλλὰξ ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΕΑ πρὸς τὴν ΑΗ. τῶν ἄρα ΑΒΓΔ, ΕΗ παραλληλογράμμων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὴν κοινὴν γωνίαν τὴν ὑπὸ ΒΑΔ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΗΖ τῇ ΔΓ, ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΖΗ γωνία τῇ ὑπὸ ΔΓΑ· καὶ κοινὴ τῶν δύο τριγώνων τῶν ΑΔΓ, ΑΗΖ ἡ ὑπὸ ΔΑΓ γωνία· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΔΓ τρίγωνον τῷ ΑΗΖ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΓΒ τρίγωνον ἰσογώνιόν ἐστι

Proposition 24

In any parallelogram, the parallelograms about the diagonal are similar to the whole, and to one another.

Let $ABCD$ be a parallelogram, and AC its diagonal. And let EG and HK be parallelograms about AC. I say that the parallelograms EG and HK are each similar to the whole (parallelogram) $ABCD$, and to one another.



For since EF has been drawn parallel to one of the sides BC of triangle ABC, proportionally, as BE is to EA, so CF (is) to FA [Prop. 6.2]. Again, since FG has been drawn parallel to one (of the sides) CD of triangle ACD, proportionally, as CF is to FA, so DG (is) to GA [Prop. 6.2]. But, as CF (is) to FA, so it was also shown (is) BE to EA. And thus as BE (is) to EA, so DG (is) to GA. And, thus, compounding, as BA (is) to AE, so DA (is) to AG [Prop. 5.18]. And, alternately, as BA (is) to AD, so EA (is) to AG [Prop. 5.16]. Thus, in parallelograms $ABCD$ and EG, the sides about the common angle BAD are proportional. And since GF is parallel to DC, angle AFG is equal to DCA [Prop. 1.29]. And angle DAC (is) common to the two triangles ADC and AGF. Thus, triangle ADC is equiangular to triangle AGF [Prop. 1.32]. So, for the same (reasons), triangle ACB is equiangular to triangle AFE, *τῷ ΑΖΕ τριγώνῳ, καὶ ὅλον τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΗ παραλληλογράμμῳ ἰσογώνιόν ἐστιν. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΑΗ πρὸς τὴν ΗΖ, ὡς δὲ ἡ ΔΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΑΖ πρὸς τὴν ΖΕ, καὶ ἔτι ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΖΕ πρὸς τὴν ΕΑ. καὶ ἐπεὶ ἐδείχθη ὡς μὲν ἡ ΔΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΑ, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΑΖ πρὸς τὴν ΖΕ, δι᾽ ἴσου ἄρα ἐστὶν ὡς ἡ ΔΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΕ. τῶν ἄρα ΑΒΓΔ, ΕΗ παραλληλογράμμων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ὅμοιον ἄρα ἐστὶ τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΕΗ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ τὸ ΑΒΓΔ παραλληλόγραμμον καὶ τῷ ΚΘ παραλληλογράμμῳ ὅμοιόν ἐστιν· ἑκάτερον ἄρα τῶν ΕΗ, ΘΚ παραλληλογράμμων τῷ ΑΒΓΔ [παραλληλογράμμῳ] ὅμοιόν ἐστιν. τὰ δὲ τῷ αὐτῷ εὐθυγράμμῳ ὅμοια καὶ ἀλλήλοις ἐστὶν ὅμοια· καὶ τὸ ΕΗ ἄρα παραλληλόγραμμον τῷ ΘΚ παραλληλογράμμῳ ὅμοιόν ἐστιν.*

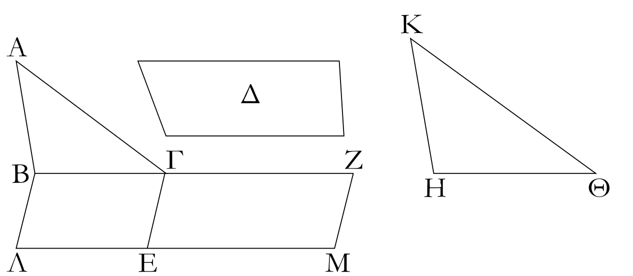
Παντὸς ἄρα παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὅμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις· ὅπερ ἔδει δεῖξαι.

and the whole parallelogram *ABCD* is equiangular to parallelogram *EG*. Thus, proportionally, as *AD* (is) to *DC*, so *AG* (is) to *GF*, and as *DC* (is) to *CA*, so *GF* (is) to *FA*, and as *AC* (is) to *CB*, so *AF* (is) to *FE*, and, further, as *CB* (is) to *BA*, so *FE* (is) to *EA* [Prop. 6.4]. And since it was shown that as *DC* is to *CA*, so *GF* (is) to *FA*, and as *AC* (is) to *CB*, so *AF* (is) to *FE*, thus, via equality, as *DC* is to *CB*, so *GF* (is) to *FE* [Prop. 5.22]. Thus, in parallelograms *ABCD* and *EG*, the sides about the equal angles are proportional. Thus, parallelogram *ABCD* is similar to parallelogram *EG* [Def. 6.1]. So, for the same (reasons), parallelogram *ABCD* is also similar to parallelogram *KH*. Thus, parallelograms *EG* and *HK* are each similar to [parallelogram] *ABCD*. And (rectilinear figures) similar to the same rectilinear figure are also similar to one another [Prop. 6.21]. Thus, parallelogram *EG* is also similar to parallelogram *HK*.

Thus, in any parallelogram, the parallelograms about the diagonal are similar to the whole, and to one another. (Which is) the very thing it was required to show.

κεʹ.

Τῷ δοθέντι εὐθυγράμμῳ ὅμοιον καὶ ἄλλῳ τῷ δοθέντι ἴσον τὸ αὐτὸ συστήσασθαι.



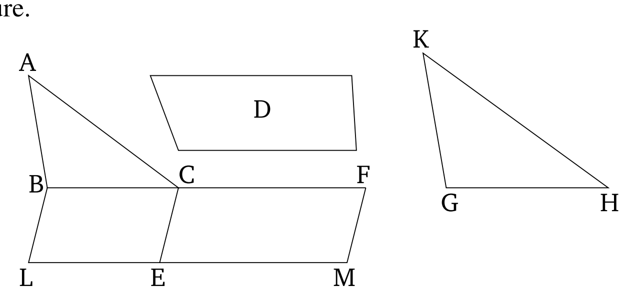
Ἔστω τὸ μὲν δοθὲν εὐθύγραμμον, ᾧ δεῖ ὅμοιον συστήσασθαι, τὸ ΑΒΓ, ᾧ δὲ δεῖ ἴσον, τὸ Δ· δεῖ δὴ τῷ μὲν ΑΒΓ ὅμοιον, τῷ δὲ Δ ἴσον τὸ αὐτὸ συστήσασθαι.

Παραβεβλήσθω γὰρ παρὰ μὲν τὴν ΒΓ τῷ ΑΒΓ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΒΕ, παρὰ δὲ τὴν ΓΕ τῷ Δ ἴσον παραλληλόγραμμον τὸ ΓΜ ἐν γωνίᾳ τῇ ὑπὸ ΖΓΕ, ἥ ἐστιν ἴση τῇ ὑπὸ ΓΒΛ. ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ μὲν ΒΓ τῇ ΓΖ, ἡ δὲ ΛΕ τῇ ΕΜ. καὶ εἰλήφθω τῶν ΒΓ, ΓΖ μέση ἀνάλογον ἡ ΗΘ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΗΘ τῷ ΑΒΓ ὅμοιόν τε καὶ ὁμοίως κείμενον τὸ ΚΗΘ.

Καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΗΘ, οὕτως ἡ ΗΘ πρὸς τὴν ΓΖ, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΖ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΚΗΘ τρίγωνον. ἀλλὰ καὶ ὡς ἡ ΒΓ πρὸς τὴν ΓΖ, οὕτως τὸ ΒΕ παραλληλόγραμμον πρὸς τὸ ΕΖ παραλληλόγραμμον. καὶ ὡς

Proposition 25

To construct a single (rectilinear figure) similar to a given rectilinear figure, and equal to a different given rectilinear figure.



Let *ABC* be the given rectilinear figure to which it is required to construct a similar (rectilinear figure), and *D* the (rectilinear figure) to which (the constructed figure) is required (to be) equal. So it is required to construct a single (rectilinear figure) similar to *ABC*, and equal to *D*.

For let the parallelogram *BE*, equal to triangle *ABC*, be applied to (the straight-line) *BC* [Prop. 1.44], and the parallelogram *CM*, equal to *D*, (be applied) to (the straight-line) *CE*, in the angle *FCE*, which is equal to *CBL* [Prop. 1.45]. Thus, *BC* is straight-on to *CF*, and *LE* to *EM* [Prop. 1.14]. And let the mean proportion *GH* be taken of *BC* and *CF* [Prop. 6.13]. And let *KGH*, similar, and similarly laid out, to *ABC*, be described on *GH* [Prop. 6.18].

And since as *BC* is to *GH*, so *GH* (is) to *CF*, and if three straight-lines are proportional, (then) as the first is to the third, so the figure (described) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.], ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΚΗΘ τρίγωνον, οὕτως τὸ ΒΕ παραλληλόγραμμον πρὸς τὸ ΕΖ παραλληλόγραμμον· ἐναλλὰξ ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΒΕ παραλληλόγραμμον, οὕτως τὸ ΚΗΘ τρίγωνον πρὸς τὸ ΕΖ παραλληλόγραμμον. ἴσον δὲ τὸ ΑΒΓ τρίγωνον τῷ ΒΕ παραλληλογράμμῳ· ἴσον ἄρα καὶ τὸ ΚΗΘ τρίγωνον τῷ ΕΖ παραλληλογράμμῳ. ἀλλὰ τὸ ΕΖ παραλληλόγραμμον τῷ Δ ἐστιν ἴσον· καὶ τὸ ΚΗΘ ἄρα τῷ Δ ἐστιν ἴσον. ἔστι δὲ τὸ ΚΗΘ καὶ τῷ ΑΒΓ ὅμοιον.

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ ΑΒΓ ὅμοιον καὶ ἄλλῳ τῷ δοθέντι τῷ Δ ἴσον τὸ αὐτὸ συνέσταται τὸ ΚΗΘ· ὅπερ ἔδει ποιῆσαι.

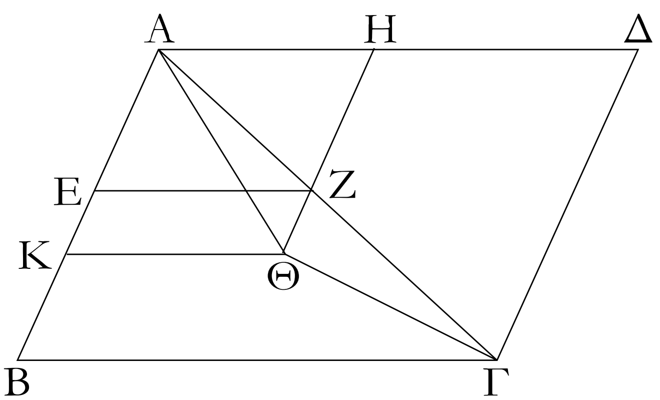
thus as *BC* is to *CF*, so triangle *ABC* (is) to triangle *KGH*. But, also, as *BC* (is) to *CF*, so parallelogram *BE* (is) to parallelogram *EF* [Prop. 6.1]. And, thus, as triangle *ABC* (is) to triangle *KGH*, so parallelogram *BE* (is) to parallelogram *EF*. Thus, alternately, as triangle *ABC* (is) to parallelogram *BE*, so triangle *KGH* (is) to parallelogram *EF* [Prop. 5.16]. And triangle *ABC* (is) equal to parallelogram *BE*. Thus, triangle *KGH* (is) also equal to parallelogram *EF*. But, parallelogram *EF* is equal to *D*. Thus, *KGH* is also equal to *D*. And *KGH* is also similar to *ABC*.

Thus, a single (rectilinear figure) *KGH* has been constructed (which is) similar to the given rectilinear figure *ABC*, and equal to a different given (rectilinear figure) *D*. (Which is) the very thing it was required to do.

κϛʹ.

Ἐὰν ἀπὸ παραλληλογράμμου παραλληλόγραμμον ἀφαιρεθῇ ὅμοιόν τε τῷ ὅλῳ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ, περὶ τὴν αὐτὴν διάμετρόν ἐστι τῷ ὅλῳ.

Ἀπὸ γὰρ παραλληλογράμμου τοῦ ΑΒΓΔ παραλληλόγραμμον ἀφῃρήσθω τὸ ΑΖ ὅμοιον τῷ ΑΒΓΔ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ τὴν ὑπὸ ΔΑΒ· λέγω, ὅτι περὶ τὴν αὐτὴν διάμετρόν ἐστι τὸ ΑΒΓΔ τῷ ΑΖ.



μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, ἔστω [αὐτῶν] διάμετρος ἡ ΑΘΓ, καὶ ἐκβληθεῖσα ἡ ΗΖ διήχθω ἐπὶ τὸ Θ, καὶ ἤχθω διὰ τοῦ Θ ὁποτέρᾳ τῶν ΑΔ, ΒΓ παράλληλος ἡ ΘΚ.

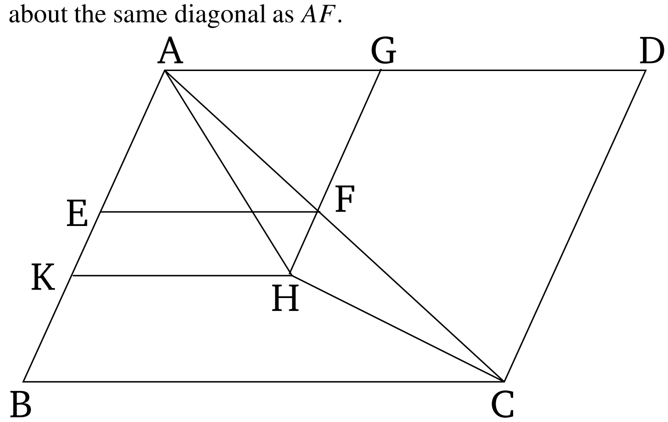
Ἐπεὶ οὖν περὶ τὴν αὐτὴν διάμετρόν ἐστι τὸ ΑΒΓΔ τῷ ΚΗ, ἔστιν ἄρα ὡς ἡ ΔΑ πρὸς τὴν ΑΒ, οὕτως ἡ ΗΑ πρὸς τὴν ΑΚ. ἔστι δὲ καὶ διὰ τὴν ὁμοιότητα τῶν ΑΒΓΔ, ΕΗ καὶ ὡς ἡ ΔΑ πρὸς τὴν ΑΒ, οὕτως ἡ ΗΑ πρὸς τὴν ΑΕ· καὶ ὡς ἄρα ἡ ΗΑ πρὸς τὴν ΑΚ, οὕτως ἡ ΗΑ πρὸς τὴν ΑΕ. ἡ ΗΑ ἄρα πρὸς ἑκατέραν τῶν ΑΚ, ΑΕ τὸν αὐτὸν ἔχει λόγον. ἴση ἄρα ἐστὶν ἡ ΑΕ τῇ ΑΚ ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὔκ ἐστι περὶ τὴν αὐτὴν διάμετρον τὸ ΑΒΓΔ τῷ ΑΖ· περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὸ ΑΒΓΔ παραλληλόγραμμον τῷ ΑΖ παραλληλογράμμῳ.

Ἐὰν ἄρα ἀπὸ παραλληλογράμμου παραλληλόγραμμον

Proposition 26

If from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, (then the subtracted parallelogram) is about the same diagonal as the whole.

For, from parallelogram *ABCD*, let (parallelogram) *AF* be subtracted (which is) similar, and similarly laid out, to *ABCD*, having the common angle *DAB* with it. I say that *ABCD* is about the same diagonal as *AF*.



For (if) not, (then), if possible, let *AHC* be [*ABCD*'s] diagonal. And producing *GF*, let it be drawn through to (point) *H*. And let *HK* be drawn through (point) *H*, parallel to either of *AD* or *BC* [Prop. 1.31].

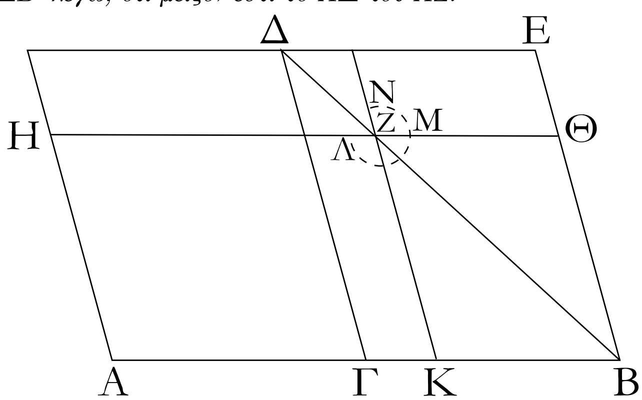
Therefore, since *ABCD* is about the same diagonal as *KG*, thus as *DA* is to *AB*, so *GA* (is) to *AK* [Prop. 6.24]. And, on account of the similarity of *ABCD* and *EG*, also, as *DA* (is) to *AB*, so *GA* (is) to *AE*. Thus, also, as *GA* (is) to *AK*, so *GA* (is) to *AE*. Thus, *GA* has the same ratio to each of *AK* and *AE*. Thus, *AE* is equal to *AK* [Prop. 5.9], the lesser to the greater. The very thing is impossible. Thus, *ABCD* is not not about the same diagonal as *AF*. Thus, parallelogram *ABCD* is about the same diagonal as parallelogram *AF*.

Thus, if from a parallelogram a(nother) parallelogram *ἀφαιρεθῇ ὅμοιόν τε τῷ ὅλῳ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ, περὶ τὴν αὐτὴν διάμετρόν ἐστι τῷ ὅλῳ· ὅπερ ἔδει δεῖξαι.*

κζ΄.

Πάντων τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβαλλόμενον [παραλληλόγραμμον] ὅμοιον ὂν τῷ ἐλλείμματι.

Ἔστω εὐθεῖα ἡ ΑΒ καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ παραβεβλήσθω παρὰ τὴν ΑΒ εὐθεῖαν τὸ ΑΔ παραλληλόγραμμον ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΔΒ ἀναγραφέντι ἀπὸ τῆς ἡμισείας τῆς ΑΒ, τουτέστι τῆς ΓΒ· λέγω, ὅτι πάντων τῶν παρὰ τὴν ΑΒ παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι [παραλληλογράμμοις] ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ΔΒ μέγιστόν ἐστι τὸ ΑΔ. παραβεβλήσθω γὰρ παρὰ τὴν ΑΒ εὐθεῖαν τὸ ΑΖ παραλληλόγραμμον ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΖΒ ὁμοίῳ τε καὶ ὁμοίως κειμένῳ τῷ ΔΒ· λέγω, ὅτι μεῖζόν ἐστι τὸ ΑΔ τοῦ ΑΖ.



Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ ΔΒ παραλληλόγραμμον τῷ ΖΒ παραλληλογράμμῳ, περὶ τὴν αὐτήν εἰσι διάμετρον. ἤχθω αὐτῶν διάμετρος ἡ ΔΒ, καὶ καταγεγράφθω τὸ σχῆμα.

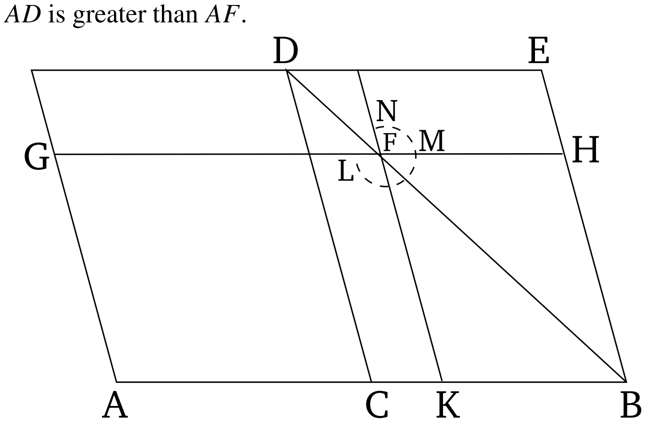
Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΓΖ τῷ ΖΕ, κοινὸν δὲ τὸ ΖΒ, ὅλον ἄρα τὸ ΓΘ ὅλῳ τῷ ΚΕ ἐστιν ἴσον. ἀλλὰ τὸ ΓΘ τῷ ΓΗ ἐστιν ἴσον, ἐπεὶ καὶ ἡ ΑΓ τῇ ΓΒ. καὶ τὸ ΗΓ ἄρα τῷ ΕΚ ἐστιν ἴσον.

is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, (then the subtracted parallelogram) is about the same diagonal as the whole. (Which is) the very thing it was required to show.

Proposition 27

Of all the parallelograms applied to the same straight-line, and falling short by parallelogrammic figures similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line) which (is) similar to (that parallelogram) by which it falls short.

Let AB be a straight-line, and let it be cut in half at (point) C [Prop. 1.10]. And let the parallelogram AD be applied to the straight-line AB, falling short by the parallelogramic figure DB (which is) applied to half of AB—that is to say, CB. I say that of all the parallelograms applied to AB, and falling short by [parallelogramic] figures similar, and similarly laid out, to DB, the greatest is AD. For let the parallelogram AF be applied to the straight-line AB, falling short by the parallelogramic figure FB (which is) similar, and similarly laid out, to DB. I say that AD is greater than AF.



For since parallelogram DB is similar to parallelogram FB, they are about the same diagonal [Prop. 6.26]. Let their (common) diagonal DB be drawn, and let the (rest of the) figure be described.

Therefore, since (complement) CF is equal to (comple-

κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ τῷ ΛΜΝ γνώμονί ἐστιν ἴσον· ὥστε τὸ ΔΒ παραλληλόγραμμον, τουτέστι τὸ ΑΔ, τοῦ ΑΖ παραλληλογράμμου μεῖζόν ἐστιν.

Πάντων ἄρα τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβληθέν· ὅπερ ἔδει δεῖξαι.

ment) FE [Prop. 1.43], and (parallelogram) FB is common, the whole (parallelogram) CH is thus equal to the whole (parallelogram) KE. But, (parallelogram) CH is equal to CG, since AC (is) also (equal) to CB [Prop. 6.1]. Thus, (parallelogram) GC is also equal to EK. Let (parallelogram) CF be added to both. Thus, the whole (parallelogram) AF is equal to the gnomon LMN. Hence, parallelogram DB—that is to say, AD—is greater than parallelogram AF.

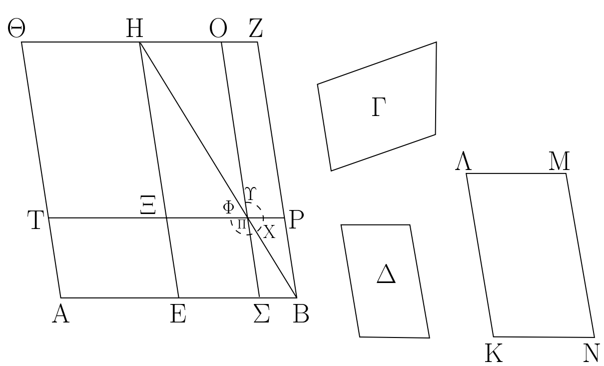
Thus, for all parallelograms applied to the same straight-line, and falling short by a parallelogramic figure similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line). (Which is) the very thing it was required to show.

κη'.

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι· δεῖ δὲ τὸ διδόμενον εὐθύγραμμον [ᾧ δεῖ ἴσον παραβαλεῖν] μὴ μεῖζον εἶναι τοῦ ἀπὸ τῆς ἡμισείας ἀναγραφομένου ὁμοίου τῷ ἐλλείμματι [τοῦ τε ἀπὸ τῆς ἡμισείας καὶ ᾧ δεῖ ὅμοιον ἐλλείπειν].

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμον, ᾧ δεῖ ἴσον παρὰ τὴν ΑΒ παραβαλεῖν, τὸ Γ μὴ μεῖζον [ὂν] τοῦ ἀπὸ τῆς ἡμισείας τῆς ΑΒ ἀναγραφομένου ὁμοίου τῷ ἐλλείμματι, ᾧ δὲ δεῖ ὅμοιον ἐλλείπειν, τὸ Δ· δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῳ ὁμοίῳ ὄντι τῷ Δ.

Τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ε σημεῖον, καὶ ἀναγεγράφθω ἀπὸ τῆς ΕΒ τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ ΕΒΖΗ, καὶ συμπεπληρώσθω τὸ ΑΗ παραλληλόγραμμον.



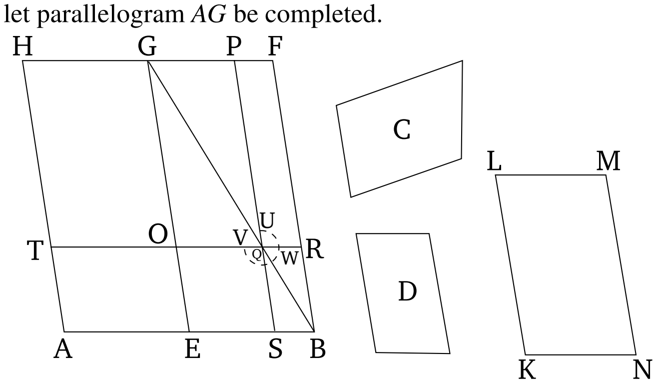
Εἰ μὲν οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ Γ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· παραβέβληται γὰρ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον τὸ ΑΗ ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΗΒ ὁμοίῳ ὄντι τῷ

Proposition 28†

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) falling short by a parallelogrammic figure similar to a given (parallelogram). It is necessary for the given rectilinear figure [to which it is required to apply an equal (parallelogram)] not to be greater than the (parallelogram) described on half (of the straight-line) and similar to the deficit.

Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to AB is required (to be) equal, [being] not greater than the (parallelogram) described on half of AB and similar to the deficit, and D the (parallelogram) to which the deficit is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C, to the straight-line AB, falling short by a parallelogramic figure which is similar to D.

Let AB be cut in half at point E [Prop. 1.10], and let (parallelogram) $EBFG$, (which is) similar, and similarly laid out, to (parallelogram) D, be described on EB [Prop. 6.18]. And let parallelogram AG be completed.



Therefore, if AG is equal to C, (then) the thing prescribed has happened. For a parallelogram AG, equal to the given rectilinear figure C, has been applied to the given straight-line AB, falling short by a parallelogramic figure GB which is similar

Δ. εἰ δὲ οὔ, μεῖζόν ἔστω τὸ ΘΕ τοῦ Γ. ἴσον δὲ τὸ ΘΕ τῷ ΗΒ· μεῖζον ἄρα καὶ τὸ ΗΒ τοῦ Γ. ᾧ δὴ μεῖζόν ἐστι τὸ ΗΒ τοῦ Γ, ταύτῃ τῇ ὑπεροχῇ ἴσον, τῷ δὲ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ ΚΛΜΝ. ἀλλὰ τὸ Δ τῷ ΗΒ [ἐστιν] ὅμοιον· καὶ τὸ ΚΜ ἄρα τῷ ΗΒ ἐστιν ὅμοιον. ἔστω οὖν ὁμόλογος ἡ μὲν ΚΛ τῇ ΗΕ, ἡ δὲ ΛΜ τῇ ΗΖ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΗΒ τοῖς Γ, ΚΜ, μεῖζον ἄρα ἐστὶ τὸ ΗΒ τοῦ ΚΜ· μείζων ἄρα ἐστὶ καὶ ἡ μὲν ΗΕ τῆς ΚΛ, ἡ δὲ ΗΖ τῆς ΛΜ. κείσθω τῇ μὲν ΚΛ ἴση ἡ ΗΞ, τῇ δὲ ΛΜ ἴση ἡ ΗΟ, καὶ συμπεπληρώσθω τὸ ΞΗΟΠ παραλληλόγραμμον· ἴσον ἄρα καὶ ὅμοιον ἐστι [τὸ ΗΠ] τῷ ΚΜ [ἀλλὰ τὸ ΚΜ τῷ ΗΒ ὅμοιόν ἐστιν]. καὶ τὸ ΗΠ ἄρα τῷ ΗΒ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΗΠ τῷ ΗΒ. ἔστω αὐτῶν διάμετρος ἡ ΗΠΒ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΒΗ τοῖς Γ, ΚΜ, ὧν τὸ ΗΠ τῷ ΚΜ ἐστιν ἴσον, λοιπὸς ἄρα ὁ ΥΧΦ γνόμων λοιπῷ τῷ Γ ἴσος ἐστίν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΟΡ τῷ ΞΣ, κοινὸν προσκείσθω τὸ ΠΒ· ὅλον ἄρα τὸ ΟΒ ὅλῳ τῷ ΞΒ ἴσον ἐστίν. ἀλλὰ τὸ ΞΒ τῷ ΤΕ ἐστιν ἴσον, ἐπεὶ καὶ πλευρὰ ἡ ΑΕ πλευρᾷ τῇ ΕΒ ἐστιν ἴση· καὶ τὸ ΤΕ ἄρα τῷ ΟΒ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΞΣ· ὅλον ἄρα τὸ ΤΣ ὅλῳ τῷ ΦΧΥ γνώμονί ἐστιν ἴσον. ἀλλ᾽ ὁ ΦΧΥ γνώμων τῷ Γ ἐδείχθη ἴσος· καὶ τὸ ΤΣ ἄρα τῷ Γ ἐστιν ἴσον.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΣΤ ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΠΒ ὁμοίῳ ὄντι τῷ Δ [ἐπειδήπερ τὸ ΠΒ τῷ ΗΠ ὅμοιόν ἐστιν]· ὅπερ ἔδει ποιῆσαι.

to D. And if not, let HE be greater than C. And HE (is) equal to GB [Prop. 6.1]. Thus, GB (is) also greater than C. So, let (parallelogram) $KLMN$ be constructed (so as to be) both similar, and similarly laid out, to D, and equal to the excess by which GB is greater than C [Prop. 6.25]. But, GB [is] similar to D. Thus, KM is also similar to GB [Prop. 6.21]. Therefore, let KL correspond to GE, and LM to GF. And since (parallelogram) GB is equal to (figure) C and (parallelogram) KM, GB is thus greater than KM. Thus, GE is also greater than KL, and GF than LM. Let GO be made equal to KL, and GP to LM [Prop. 1.3]. And let the parallelogram $OGPQ$ be completed. Thus, [GQ] is equal and similar to KM [but, KM is similar to GB]. Thus, GQ is also similar to GB [Prop. 6.21]. Thus, GQ and GB are about the same diagonal [Prop. 6.26]. Let GQB be their (common) diagonal, and let the (remainder of the) figure be described.

Therefore, since BG is equal to C and KM, of which GQ is equal to KM, the remaining gnomon UWV is thus equal to the remainder C. And since (the complement) PR is equal to (the complement) OS [Prop. 1.43], let (parallelogram) QB be added to both. Thus, the whole (parallelogram) PB is equal to the whole (parallelogram) OB. But, OB is equal to TE, since side AE is equal to side EB [Prop. 6.1]. Thus, TE is also equal to PB. Let (parallelogram) OS be added to both. Thus, the whole (parallelogram) TS is equal to the gnomon VWU. But, gnomon VWU was shown (to be) equal to C. Therefore, (parallelogram) TS is also equal to (figure) C.

Thus, the parallelogram ST, equal to the given rectilinear figure C, has been applied to the given straight-line AB, falling short by the parallelogramic figure QB, which is similar to D [inasmuch as QB is similar to GQ [Prop. 6.24]]. (Which is) the very thing it was required to do.

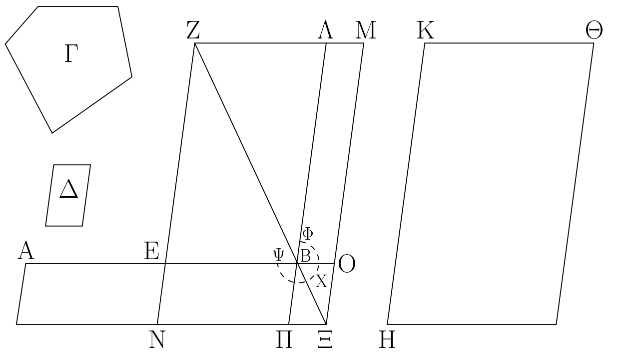
† This proposition is a geometric solution of the quadratic equation $x^2 - \alpha\, x + \beta = 0$. Here, x is the ratio of a side of the deficit to the corresponding side of figure D, α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the deficit running along AB, and β is the ratio of the areas of figures C and D. The constraint corresponds to the condition $\beta < \alpha^2/4$ for the equation to have real roots. Only the smaller root of the equation is found. The larger root can be found by a similar method.

κθʹ.

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι.

Proposition 29†

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) overshooting by a parallelogramic figure similar to a given (parallelogram).

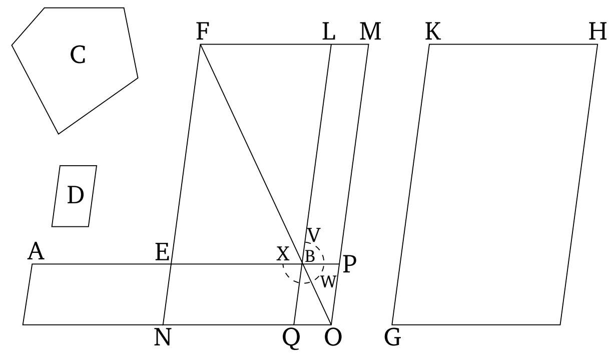


Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμον, ᾧ δεῖ ἴσον παρὰ τὴν ΑΒ παραβαλεῖν, τὸ Γ, ᾧ δὲ δεῖ ὅμοιον ὑπερβάλλειν, τὸ Δ· δεῖ δὴ παρὰ τὴν ΑΒ εὐθεῖαν τῷ Γ εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ Δ.

Τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ε, καὶ ἀναγεγράφθω ἀπὸ τῆς ΕΒ τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον παραλληλόγραμμον τὸ ΒΖ, καὶ συναμφοτέροις μὲν τοῖς ΒΖ, Γ ἴσον, τῷ δὲ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ ΗΘ. ὁμόλογος δὲ ἔστω ἡ μὲν ΚΘ τῇ ΖΛ, ἡ δὲ ΚΗ τῇ ΖΕ. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΘ τοῦ ΖΒ, μείζων ἄρα ἐστὶ καὶ ἡ μὲν ΚΘ τῆς ΖΛ, ἡ δὲ ΚΗ τῇ ΖΕ. ἐκβεβλήσθωσαν αἱ ΖΛ, ΖΕ, καὶ τῇ μὲν ΚΘ ἴση ἔστω ἡ ΖΛΜ, τῇ δὲ ΚΗ ἴση ἡ ΖΕΝ, καὶ συμπεπληρώσθω τὸ ΜΝ· τὸ ΜΝ ἄρα τῷ ΗΘ ἴσον τέ ἐστι καὶ ὅμοιον. ἀλλὰ τὸ ΗΘ τῷ ΕΛ ἐστιν ὅμοιον· καὶ τὸ ΜΝ ἄρα τῷ ΕΛ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΕΛ τῷ ΜΝ. ἤχθω αὐτῶν διάμετρος ἡ ΖΞ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ ἴσον ἐστὶ τὸ ΗΘ τοῖς ΕΛ, Γ, ἀλλὰ τὸ ΗΘ τῷ ΜΝ ἴσον ἐστίν, καὶ τὸ ΜΝ ἄρα τοῖς ΕΛ, Γ ἴσον ἐστίν. κοινὸν ἀφῃρήσθω τὸ ΕΛ· λοιπὸς ἄρα ὁ ΨΧΦ γνώμων τῷ Γ ἐστιν ἴσος. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, ἴσον ἐστὶ καὶ τὸ ΑΝ τῷ ΝΒ, τουτέστι τῷ ΛΟ. κοινὸν προσκείσθω τὸ ΕΞ· ὅλον ἄρα τὸ ΑΞ ἴσον ἐστὶ τῷ ΦΧΨ γνώμονι. ἀλλὰ ὁ ΦΧΨ γνώμων τῷ Γ ἴσος ἐστίν· καὶ τὸ ΑΞ ἄρα τῷ Γ ἴσον ἐστίν.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΑΞ ὑπερβάλλον εἴδει παραλληλογράμμῳ τῷ ΠΟ ὁμοίῳ ὄντι τῷ Δ, ἐπεὶ καὶ τῷ ΕΛ ἐστιν ὅμοιον τὸ ΟΠ· ὅπερ ἔδει ποιῆσαι.



Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to AB is required (to be) equal, and D the (parallelogram) to which the excess is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C, to the given straight-line AB, overshooting by a parallelogrammic figure similar to D.

Let AB be cut in half at (point) E [Prop. 1.10], and let the parallelogram BF, (which is) similar, and similarly laid out, to D, be described on EB [Prop. 6.18]. And let (parallelogram) GH be constructed (so as to be) both similar, and similarly laid out, to D, and equal to the sum of BF and C [Prop. 6.25]. And let KH correspond to FL, and KG to FE. And since (parallelogram) GH is greater than (parallelogram) FB, KH is thus also greater than FL, and KG than FE. Let FL and FE be produced, and let FLM be (made) equal to KH, and FEN to KG [Prop. 1.3]. And let (parallelogram) MN be completed. Thus, MN is equal and similar to GH. But, GH is similar to EL. Thus, MN is also similar to EL [Prop. 6.21]. EL is thus about the same diagonal as MN [Prop. 6.26]. Let their (common) diagonal FO be drawn, and let the (remainder of the) figure be described.

And since (parallelogram) GH is equal to (parallelogram) EL and (figure) C, but GH is equal to (parallelogram) MN, MN is thus also equal to EL and C. Let EL be subtracted from both. Thus, the remaining gnomon XWV is equal to (figure) C. And since AE is equal to EB, (parallelogram) AN is also equal to (parallelogram) NB [Prop. 6.1], that is to say, (parallelogram) LP [Prop. 1.43]. Let (parallelogram) EO be added to both. Thus, the whole (parallelogram) AO is equal to the gnomon VWX. But, the gnomon VWX is equal to (figure) C. Thus, (parallelogram) AO is also equal to (figure) C.

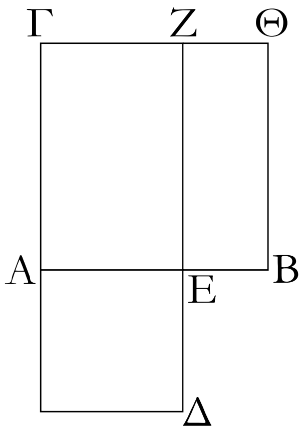
Thus, the parallelogram AO, equal to the given rectilinear figure C, has been applied to the given straight-line AB, overshooting by the parallelogramic figure QP which is similar to D, since PQ is also similar to EL [Prop. 6.24]. (Which is) the very thing it was required to do.

† This proposition is a geometric solution of the quadratic equation $x^2 + \alpha\, x - \beta = 0$. Here, x is the ratio of a side of the excess to the corresponding side of figure D, α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the excess running along AB, and β is the ratio

of the areas of figures C and D. Only the positive root of the equation is found.

λ'.

Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην ἄκρον καὶ μέσον λόγον τεμεῖν.



Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ εὐθεῖαν ἄκρον καὶ μέσον λόγον τεμεῖν.

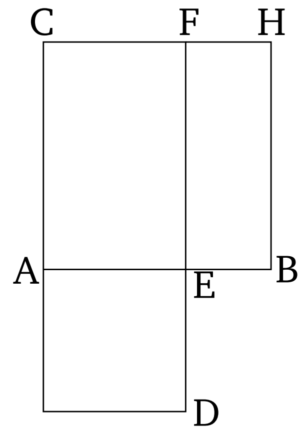
Ἀναγεγράφθω ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΒΓ, καὶ παραβεβλήσθω παρὰ τὴν ΑΓ τῷ ΒΓ ἴσον παραλληλόγραμμον τὸ ΓΔ ὑπερβάλλον εἴδει τῷ ΑΔ ὁμοίῳ τῷ ΒΓ.

Τετράγωνον δέ ἐστι τὸ ΒΓ· τετράγωνον ἄρα ἐστὶ καὶ τὸ ΑΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΒΓ τῷ ΓΔ, κοινὸν ἀφῃρήσθω τὸ ΓΕ· λοιπὸν ἄρα τὸ ΒΖ λοιπῷ τῷ ΑΔ ἐστιν ἴσον. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον· τῶν ΒΖ, ΑΔ ἄρα ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ἔστιν ἄρα ὡς ἡ ΖΕ πρὸς τὴν ΕΔ, οὕτως ἡ ΑΕ πρὸς τὴν ΕΒ. ἴση δὲ ἡ μὲν ΖΕ τῇ ΑΒ, ἡ δὲ ΕΔ τῇ ΑΕ. ἔστιν ἄρα ὡς ἡ ΒΑ πρὸς τὴν ΑΕ, οὕτως ἡ ΑΕ πρὸς τὴν ΕΒ. μείζων δὲ ἡ ΑΒ τῆς ΑΕ· μείζων ἄρα καὶ ἡ ΑΕ τῆς ΕΒ.

Ἡ ἄρα ΑΒ εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ε, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστι τὸ ΑΕ· ὅπερ ἔδει ποιῆσαι.

Proposition 30†

To cut a given finite straight-line in extreme and mean ratio.



Let AB be the given finite straight-line. So it is required to cut the straight-line AB in extreme and mean ratio.

Let the square BC be described on AB [Prop. 1.46], and let the parallelogram CD, equal to BC, be applied to AC, overshooting by the figure AD (which is) similar to BC [Prop. 6.29].

And BC is a square. Thus, AD is also a square. And since BC is equal to CD, let (rectangle) CE be subtracted from both. Thus, the remaining (rectangle) BF is equal to the remaining (square) AD. And it is also equiangular to it. Thus, the sides of BF and AD about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as FE is to ED, so AE (is) to EB. And FE (is) equal to AB, and ED to AE. Thus, as BA is to AE, so AE (is) to EB. And AB (is) greater than AE. Thus, AE (is) also greater than EB [Prop. 5.14].

Thus, the straight-line AB has been cut in extreme and mean ratio at E, and AE is its greater piece. (Which is) the very thing it was required to do.

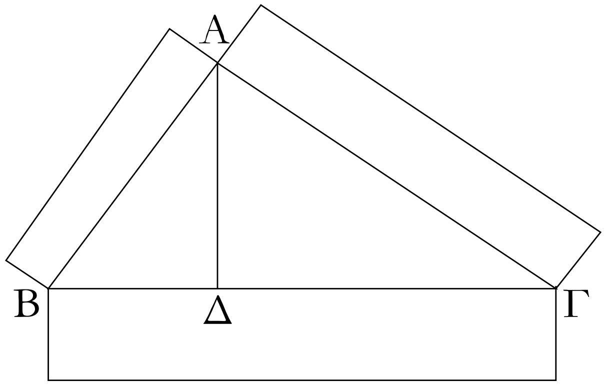
† This method of cutting a straight-line is sometimes called the "Golden Section"—see Prop. 2.11.

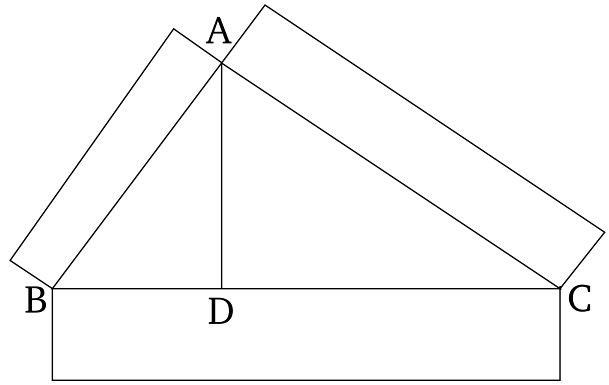
λα'.

Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.

Proposition 31

In right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle.





Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς ΒΓ εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.

Ἤχθω κάθετος ἡ ΑΔ.

Ἐπεὶ οὖν ἐν ὀρθογωνίῳ τριγώνῳ τῷ ΑΒΓ ἀπὸ τῆς πρὸς τῷ Α ὀρθῆς γωνίας ἐπὶ τὴν ΒΓ βάσιν κάθετος ἦκται ἡ ΑΔ, τὰ ΑΒΔ, ΑΔΓ πρὸς τῇ καθέτῳ τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ τῷ ΑΒΓ καὶ ἀλλήλοις. καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓ τῷ ΑΒΔ, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΑΒ πρὸς τὴν ΒΔ. καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ὡς ἄρα ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΓΒ εἶδος πρὸς τὸ ἀπὸ τῆς ΒΑ τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ ΒΓ πρὸς τὴν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὸ ἀπὸ τῆς ΓΑ. ὥστε καὶ ὡς ἡ ΒΓ πρὸς τὰς ΒΔ, ΔΓ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὰ ἀπὸ τῶν ΒΑ, ΑΓ τὰ ὅμοια καὶ ὁμοίως ἀναγραφόμενα. ἴση δὲ ἡ ΒΓ ταῖς ΒΔ, ΔΓ· ἴσον ἄρα καὶ τὸ ἀπὸ τῆς ΒΓ εἶδος τοῖς ἀπὸ τῶν ΒΑ, ΑΓ εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις.

Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν εἴδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφομένοις· ὅπερ ἔδει δεῖξαι.

Let *ABC* be a right-angled triangle having the angle *BAC* a right-angle. I say that the figure (drawn) on *BC* is equal to the (sum of the) similar, and similarly described, figures on *BA* and *AC*.

Let the perpendicular *AD* be drawn [Prop. 1.12].

Therefore, since, in the right-angled triangle *ABC*, the (straight-line) *AD* has been drawn from the right-angle at *A* perpendicular to the base *BC*, the triangles *ABD* and *ADC* about the perpendicular are similar to the whole (triangle) *ABC*, and to one another [Prop. 6.8]. And since *ABC* is similar to *ABD*, thus as *CB* is to *BA*, so *AB* (is) to *BD* [Def. 6.1]. And since three straight-lines are proportional, as the first is to the third, so the figure (drawn) on the first is to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. Thus, as *CB* (is) to *BD*, so the figure (drawn) on *CB* (is) to the similar, and similarly described, (figure) on *BA*. And so, for the same (reasons), as *BC* (is) to *CD*, so the figure (drawn) on *BC* (is) to the (figure) on *CA*. Hence, also, as *BC* (is) to *BD* and *DC*, so the figure (drawn) on *BC* (is) to the (sum of the) similar, and similarly described, (figures) on *BA* and *AC* [Prop. 5.24]. And *BC* is equal to *BD* and *DC*. Thus, the figure (drawn) on *BC* (is) also equal to the (sum of the) similar, and similarly described, figures on *BA* and *AC* [Prop. 5.9].

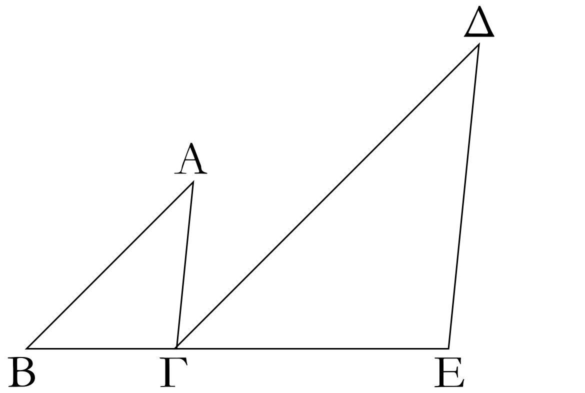
Thus, in right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle. (Which is) the very thing it was required to show.

λβʹ.

Ἐὰν δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ᾽ εὐθείας ἔσονται.

Proposition 32

If two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, (then) the remaining sides of the triangles will be straight-on (with respect to one another).



Ἔστω δύο τρίγωνα τὰ ΑΒΓ, ΔΓΕ τὰς δύο πλευρὰς τὰς ΒΑ, ΑΓ ταῖς δυσὶ πλευραῖς ταῖς ΔΓ, ΔΕ ἀνάλογον ἔχοντα, ὡς μὲν τὴν ΑΒ πρὸς τὴν ΑΓ, οὕτως τὴν ΔΓ πρὸς τὴν ΔΕ, παράλληλον δὲ τὴν μὲν ΑΒ τῇ ΔΓ, τὴν δὲ ΑΓ τῇ ΔΕ· λέγω, ὅτι ἐπ᾽ εὐθείας ἐστὶν ἡ ΒΓ τῇ ΓΕ.

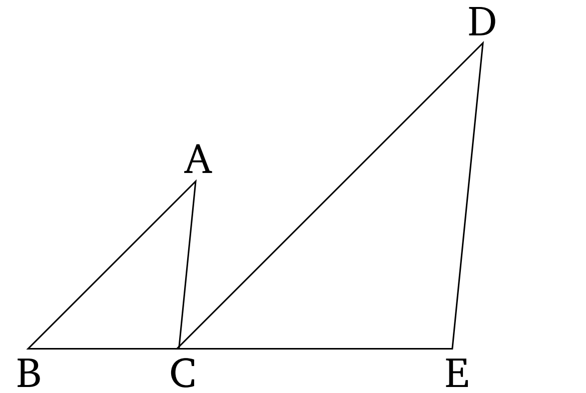
Ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΑΒ τῇ ΔΓ, καὶ εἰς αὐτὰς ἐμπέπτωκεν εὐθεῖα ἡ ΑΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΔ ἴσαι ἀλλήλαις εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΓΔ ἴση ἐστίν. ὥστε καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΕ ἐστιν ἴση. καὶ ἐπεὶ δύο τρίγωνά ἐστι τὰ ΑΒΓ, ΔΓΕ μίαν γωνίαν τὴν πρὸς τῷ Α μιᾷ γωνίᾳ τῇ πρὸς τῷ Δ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὡς τὴν ΒΑ πρὸς τὴν ΑΓ, οὕτως τὴν ΓΔ πρὸς τὴν ΔΕ, ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΓΕ τριγώνῳ· ἴση ἄρα ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΓΕ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΑΓΔ τῇ ὑπὸ ΒΑΓ ἴση· ὅλη ἄρα ἡ ὑπὸ ΑΓΕ δυσὶ ταῖς ὑπὸ ΑΒΓ, ΒΑΓ ἴση ἐστίν. κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΕ, ΑΓΒ ταῖς ὑπὸ ΒΑΓ, ΑΓΒ, ΓΒΑ ἴσαι εἰσίν. ἀλλ᾽ αἱ ὑπὸ ΒΑΓ, ΑΒΓ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ ΑΓΕ, ΑΓΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν. πρὸς δή τινι εὐθείᾳ τῇ ΑΓ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Γ δύο εὐθεῖαι αἱ ΒΓ, ΓΕ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνάις τὰς ὑπὸ ΑΓΕ, ΑΓΒ δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΒΓ τῇ ΓΕ.

Ἐὰν ἄρα δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ᾽ εὐθείας ἔσονται· ὅπερ ἔδει δεῖξαι.

λγʹ.

Ἐν τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ᾽ ὧν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι.

Ἔστωσαν ἴσοι κύκλοι οἱ ΑΒΓ, ΔΕΖ, καὶ πρὸς μὲν τοῖς κέντροις αὐτῶν τοῖς Η, Θ γωνίαι ἔστωσαν αἱ ὑπὸ ΒΗΓ, ΕΘΖ, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ ΒΑΓ, ΕΔΖ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΓ περιφέρεια πρὸς τὴν ΕΖ περιφέρειαν, οὕτως ἥ τε



Let *ABC* and *DCE* be two triangles having the two sides *BA* and *AC* proportional to the two sides *DC* and *DE*—so that as *AB* (is) to *AC*, so *DC* (is) to *DE*—and (having side) *AB* parallel to *DC*, and *AC* to *DE*. I say that (side) *BC* is straight-on to *CE*.

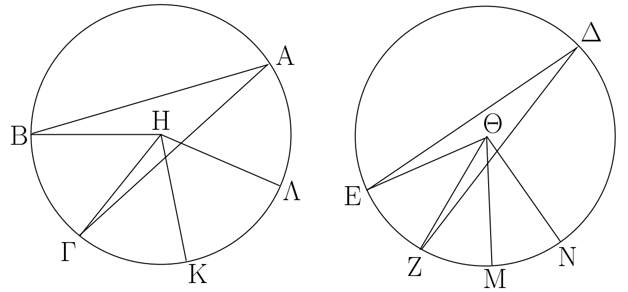
For since *AB* is parallel to *DC*, and the straight-line *AC* has fallen across them, the alternate angles *BAC* and *ACD* are equal to one another [Prop. 1.29]. So, for the same (reasons), *CDE* is also equal to *ACD*. And, hence, *BAC* is equal to *CDE*. And since *ABC* and *DCE* are two triangles having the one angle at *A* equal to the one angle at *D*, and the sides about the equal angles proportional, (so that) as *BA* (is) to *AC*, so *CD* (is) to *DE*, triangle *ABC* is thus equiangular to triangle *DCE* [Prop. 6.6]. Thus, angle *ABC* is equal to *DCE*. And (angle) *ACD* was also shown (to be) equal to *BAC*. Thus, the whole (angle) *ACE* is equal to the two (angles) *ABC* and *BAC*. Let *ACB* be added to both. Thus, *ACE* and *ACB* are equal to *BAC*, *ACB*, and *CBA*. But, *BAC*, *ABC*, and *ACB* are equal to two right-angles [Prop. 1.32]. Thus, *ACE* and *ACB* are also equal to two right-angles. Thus, the two straight-lines *BC* and *CE*, not lying on the same side, make adjacent angles *ACE* and *ACB* (whose sum is) equal to two right-angles with some straight-line *AC*, at the point *C* on it. Thus, *BC* is straight-on to *CE* [Prop. 1.14].

Thus, if two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, (then) the remaining sides of the triangles will be straight-on (with respect to one another). (Which is) the very thing it was required to show.

Proposition 33

In equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences.

Let *ABC* and *DEF* be equal circles, and let *BGC* and *EHF* be angles at their centers, *G* and *H* (respectively), and *BAC* and *EDF* (angles) at their circumferences. I say that as circumference *BC* is to circumference *EF*, so angle *BGC* (is) *ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ καὶ ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ.*

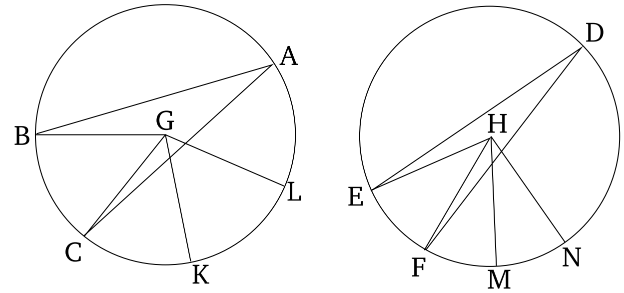


Κείσθωσαν γὰρ τῇ μὲν ΒΓ περιφερείᾳ ἴσαι κατὰ τὸ ἑξῆς ὁσαιδηποτοῦν αἱ ΓΚ, ΚΛ, τῇ δὲ ΕΖ περιφερείᾳ ἴσαι ὁσαιδηποτοῦν αἱ ΖΜ, ΜΝ, καὶ ἐπεζεύχθωσαν αἱ ΗΚ, ΗΛ, ΘΜ, ΘΝ.

Ἐπεὶ οὖν ἴσαι εἰσὶν αἱ ΒΓ, ΓΚ, ΚΛ περιφέρειαι ἀλλήλαις, ἴσαι εἰσὶ καὶ αἱ ὑπὸ ΒΗΓ, ΓΗΚ, ΚΗΛ γωνίαι ἀλλήλαις· ὁσαπλασίων ἄρα ἐστὶν ἡ ΒΛ περιφέρεια τῆς ΒΓ, τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπὸ ΒΗΓ. διὰ τὰ αὐτὰ δὴ καὶ ὁσαπλασίων ἐστὶν ἡ ΝΕ περιφέρεια τῆς ΕΖ, τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ ΝΘΕ γωνία τῆς ὑπὸ ΕΘΖ. εἰ ἄρα ἴση ἐστὶν ἡ ΒΛ περιφέρεια τῇ ΕΝ περιφερείᾳ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΗΛ τῇ ὑπὸ ΕΘΝ, καὶ εἰ μείζων ἐστὶν ἡ ΒΛ περιφέρεια τῆς ΕΝ περιφερείας, μείζων ἐστὶ καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπὸ ΕΘΝ, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὴ ὄντων μεγεθῶν, δύο μὲν περιφερειῶν τῶν ΒΓ, ΕΖ, δύο δὲ γωνιῶν τῶν ὑπὸ ΒΗΓ, ΕΘΖ, εἴληπται τῆς μὲν ΒΓ περιφερείας καὶ τῆς ὑπὸ ΒΗΓ γωνίας ἰσάκις πολλαπλασίων ἥ τε ΒΛ περιφέρεια καὶ ἡ ὑπὸ ΒΗΛ γωνία, τῆς δὲ ΕΖ περιφερείας καὶ τῆς ὑπὸ ΕΘΖ γωνίας ἥ τε ΕΝ περιφέρια καὶ ἡ ὑπὸ ΕΘΝ γωνία. καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ ΒΛ περιφέρεια τῆς ΕΝ περιφερείας, ὑπερέχει καὶ ἡ ὑπὸ ΒΗΛ γωνία τῆς ὑπο ΕΘΝ γωνίας, καὶ εἰ ἴση, ἴση, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα, ὡς ἡ ΒΓ περιφέρεια πρὸς τὴν ΕΖ, οὕτως ἡ ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ. ἀλλ᾽ ὡς ἡ ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ, οὕτως ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ. διπλασία γὰρ ἑκατέρα ἑκατέρας. καὶ ὡς ἄρα ἡ ΒΓ περιφέρεια πρὸς τὴν ΕΖ περιφέρειαν, οὕτως ἥ τε ὑπὸ ΒΗΓ γωνία πρὸς τὴν ὑπὸ ΕΘΖ καὶ ἡ ὑπὸ ΒΑΓ πρὸς τὴν ὑπὸ ΕΔΖ.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ᾽ ὧν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὦσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

to EHF, and (angle) BAC to EDF.



For let any number whatsoever of consecutive (circumferences), CK and KL, be made equal to circumference BC, and any number whatsoever, FM and MN, to circumference EF. And let GK, GL, HM, and HN be joined.

Therefore, since circumferences BC, CK, and KL are equal to one another, angles BGC, CGK, and KGL are also equal to one another [Prop. 3.27]. Thus, as many times as circumference BL is (divisible) by BC, so many times is angle BGL also (divisible) by BGC. And so, for the same (reasons), as many times as circumference NE is (divisible) by EF, so many times is angle NHE also (divisible) by EHF. Thus, if circumference BL is equal to circumference EN (then) angle BGL is also equal to EHN [Prop. 3.27], and if circumference BL is greater than circumference EN (then) angle BGL is also greater than EHN,† and if (BL is) less (than EN then BGL is also) less (than EHN). So there are four magnitudes, two circumferences BC and EF, and two angles BGC and EHF. And equal multiples have been taken of circumference BC and angle BGC, (namely) circumference BL and angle BGL, and of circumference EF and angle EHF, (namely) circumference EN and angle EHN. And it has been shown that if circumference BL exceeds circumference EN (then) angle BGL also exceeds angle EHN, and if (BL is) equal (to EN then BGL is also) equal (to EHN), and if (BL is) less (than EN then BGL is also) less (than EHN). Thus, as circumference BC (is) to EF, so angle BGC (is) to EHF [Def. 5.5]. But as angle BGC (is) to EHF, so (angle) BAC (is) to EDF [Prop. 5.15]. For the former (are) double the latter (respectively) [Prop. 3.20]. Thus, also, as circumference BC (is) to circumference EF, so angle BGC (is) to EHF, and BAC to EDF.

Thus, in equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences. (Which is) the very thing it was required to show.

† This is a straight-forward generalization of Prop. 3.27

ELEMENTS BOOK 7

Elementary Number Theory[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

Ὅροι.

αʹ. Μονάς ἐστιν, καθʼ ἣν ἕκαστον τῶν ὄντων ἓν λέγεται.

βʹ. Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.

γʹ. Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα.

δʹ. Μέρη δέ, ὅταν μὴ καταμετρῇ.

εʹ. Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

ϛʹ. Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.

ζʹ. Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.

ηʹ. Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.

θʹ. Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

ιʹ. Περισσάκις δὲ περισσὸς ἀριθμός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

ιαʹ. Πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος.

ιβʹ. Πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ.

ιγʹ. Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος.

ιδʹ. Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.

ιεʹ. Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις.

ιϛʹ. Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιζʹ. Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιηʹ. Τετράγωνος ἀριθμός ἐστιν ὁ ἰσάκις ἴσος ἢ [ὁ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.

ιθʹ. Κύβος δὲ ὁ ἰσάκις ἴσος ἰσάκις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.

κʹ. Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.

καʹ. Ὅμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς.

κβʹ. Τέλειος ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν.

Definitions

1. A unit is (that) according to which each existing (thing) is said (to be) one.

2. And a number (is) a multitude composed of units.[†]

3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.[‡]

4. But (the lesser is) parts (of the greater) when it does not measure it.[§]

5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.

6. An even number is one (which can be) divided in half.

7. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.

8. An even-times-even number is one (which is) measured by an even number according to an even number.[¶]

9. And an even-times-odd number is one (which is) measured by an even number according to an odd number.[∗]

10. And an odd-times-odd number is one (which is) measured by an odd number according to an odd number.[$]

11. A prime[♭] number is one (which is) measured by a unit alone.

12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.

13. A composite number is one (which is) measured by some number.

14. And numbers composite to one another are those (which are) measured by some number as a common measure.

15. A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.

16. And when two numbers multiplying one another make some (other number then) the (number so) created is called plane, and its sides (are) the numbers which multiply one another.

17. And when three numbers multiplying one another make some (other number then) the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.

18. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.

19. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.

20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.

21. Similar plane and solid numbers are those having proportional sides.

22. A perfect number is that which is equal to its own parts.[††]

[†] In other words, a “number” is a positive integer greater than unity.

[‡] In other words, a number a is part of another number b if there exists some number n such that $n\,a = b$.

[§] In other words, a number a is parts of another number b (where $a < b$) if there exist distinct numbers, m and n, such that $n\,a = m\,b$.

[¶] In other words. an even-times-even number is the product of two even numbers.

[∗] In other words, an even-times-odd number is the product of an even and an odd number.

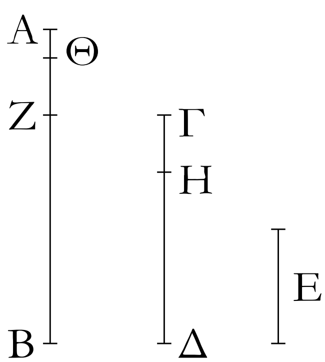
[$] In other words, an odd-times-odd number is the product of two odd numbers.

[♭] Literally, “first”.

[††] In other words, a perfect number is equal to the sum of its own factors.

α′.

Δύο ἀριθμῶν ἀνίσων ἐκκειμένων, ἀνθυφαιρουμένου δὲ ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, ἐὰν ὁ λειπόμενος μηδέποτε καταμετρῇ τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς, οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.



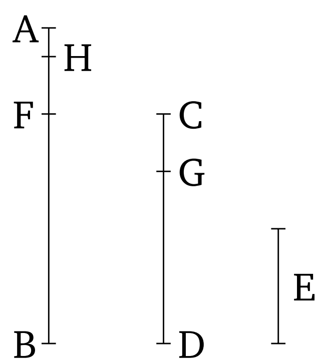
Δύο γὰρ [ἀνίσων] ἀριθμῶν τῶν ΑΒ, ΓΔ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος ὁ λειπόμενος μηδέποτε καταμετρείτω τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς· λέγω, ὅτι οἱ ΑΒ, ΓΔ πρῶτοι πρὸς ἀλλήλους εἰσίν, τουτέστιν ὅτι τοὺς ΑΒ, ΓΔ μονὰς μόνη μετρεῖ.

Εἰ γὰρ μή εἰσιν οἱ ΑΒ, ΓΔ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Ε· καὶ ὁ μὲν ΓΔ τὸν ΒΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΖΑ, ὁ δὲ ΑΖ τὸν ΔΗ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΗΓ, ὁ δὲ ΗΓ τὸν ΖΘ μετρῶν λειπέτω μονάδα τὴν ΘΑ.

Ἐπεὶ οὖν ὁ Ε τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν ΒΖ μετρεῖ, καὶ ὁ Ε ἄρα τὸν ΒΖ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΒΑ· καὶ λοιπὸν ἄρα τὸν ΑΖ μετρήσει. ὁ δὲ ΑΖ τὸν ΔΗ μετρεῖ· καὶ ὁ Ε ἄρα τὸν ΔΗ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓΗ μετρήσει. ὁ δὲ ΓΗ τὸν ΖΘ μετρεῖ· καὶ ὁ Ε ἄρα τὸν ΖΘ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΖΑ· καὶ λοιπὴν ἄρα τὴν ΑΘ μονάδα μετρήσει ἀριθμὸς ὤν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΑΒ, ΓΔ ἀριθμοὺς μετρήσει τις ἀριθμός· οἱ ΑΒ, ΓΔ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 1

Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, (then) the original numbers will be prime to one another.



For two [unequal] numbers, AB and CD, the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains. I say that AB and CD are prime to one another—that is to say, that a unit alone measures (both) AB and CD.

For if AB and CD are not prime to one another, (then) some number will measure them. Let (some number) measure them, and let it be E. And let CD measuring BF leave FA less than itself, and let AF measuring DG leave GC less than itself, and let GC measuring FH leave a unit, HA.

In fact, since E measures CD, and CD measures BF, E thus also measures BF.[†] And (E) also measures the whole of BA. Thus, (E) will also measure the remainder AF.[‡] And AF measures DG. Thus, E also measures DG. And (E) also measures the whole of DC. Thus, (E) will also measure the remainder CG. And CG measures FH. Thus, E also measures FH. And (E) also measures the whole of FA. Thus, (E) will also measure the remaining unit AH, (despite) being a number. The very thing is impossible. Thus, some number does not measure (both) the numbers AB and CD. Thus, AB and

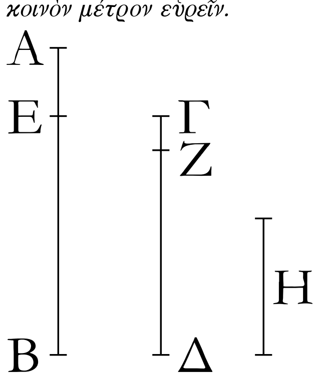
CD are prime to one another. (Which is) the very thing it was required to show.

† Here, use is made of the unstated common notion that if a measures b, and b measures c, then a also measures c, where all symbols denote numbers.

‡ Here, use is made of the unstated common notion that if a measures b, and a measures part of b, then a also measures the remainder of b, where all symbols denote numbers.

β'.

Δύο ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.



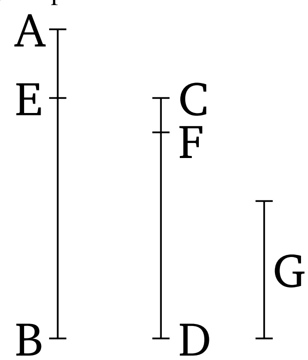
Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ ΑΒ, ΓΔ. δεῖ δὴ τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰ μὲν οὖν ὁ ΓΔ τὸν ΑΒ μετρεῖ, μετρεῖ δὲ καὶ ἑαυτόν, ὁ ΓΔ ἄρα τῶν ΓΔ, ΑΒ κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· οὐδεὶς γὰρ μείζων τοῦ ΓΔ τὸν ΓΔ μετρήσει.

Εἰ δὲ οὐ μετρεῖ ὁ ΓΔ τὸν ΑΒ, τῶν ΑΒ, ΓΔ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος λειφθήσεταί τις ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. μονὰς μὲν γὰρ οὐ λειφθήσεται· εἰ δὲ μή, ἔσονται οἱ ΑΒ, ΓΔ πρῶτοι πρὸς ἀλλήλους· ὅπερ οὐχ ὑπόκειται. λειφθήσεταί τις ἄρα ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. καὶ ὁ μὲν ΓΔ τὸν ΒΕ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΕΑ, ὁ δὲ ΕΑ τὸν ΔΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΖΓ, ὁ δὲ ΓΖ τὸν ΑΕ μετρείτω. ἐπεὶ οὖν ὁ ΓΖ τὸν ΑΕ μετρεῖ, ὁ δὲ ΑΕ τὸν ΔΖ μετρεῖ, καὶ ὁ ΓΖ ἄρα τὸν ΔΖ μετρήσει. μετρεῖ δὲ καὶ ἑαυτόν· καὶ ὅλον ἄρα τὸν ΓΔ μετρήσει. ὁ δὲ ΓΔ τὸν ΒΕ μετρεῖ· καὶ ὁ ΓΖ ἄρα τὸν ΒΕ μετρεῖ· μετρεῖ δὲ καὶ τὸν ΕΑ· καὶ ὅλον ἄρα τὸν ΒΑ μετρήσει· μετρεῖ δὲ καὶ τὸν ΓΔ· ὁ ΓΖ ἄρα τοὺς ΑΒ, ΓΔ μετρεῖ. ὁ ΓΖ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ ΓΖ τῶν ΑΒ, ΓΔ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ ΓΖ. μετρείτω, καὶ ἔστω ὁ Η. καὶ ἐπεὶ ὁ Η τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν ΒΕ μετρεῖ, καὶ ὁ Η ἄρα τὸν ΒΕ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΒΑ· καὶ λοιπὸν ἄρα τὸν ΑΕ μετρήσει. ὁ δὲ ΑΕ τὸν ΔΖ μετρεῖ· καὶ ὁ Η ἄρα τὸν ΔΖ μετρήσει· μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓΖ μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ὢν

Proposition 2

To find the greatest common measure of two given numbers (which are) not prime to one another.



Let AB and CD be the two given numbers (which are) not prime to one another. So it is required to find the greatest common measure of AB and CD.

In fact, if CD measures AB, CD is thus a common measure of CD and AB, (since CD) also measures itself. And (it is) manifest that (it is) also the greatest (common measure). For nothing greater than CD can measure CD.

But if CD does not measure AB, (then) some number will remain from AB and CD, the lesser being continually subtracted, in turn, from the greater, which will measure the (number) preceding it. For a unit will not be left. But if not, AB and CD will be prime to one another [Prop. 7.1]. The very opposite thing was assumed. Thus, some number will remain which will measure the (number) preceding it. And let CD measuring BE leave EA less than itself, and let EA measuring DF leave FC less than itself, and let CF measure AE. Therefore, since CF measures AE, and AE measures DF, CF will thus also measure DF. And it also measures itself. Thus, it will also measure the whole of CD. And CD measures BE. Thus, CF also measures BE. And it also measures EA. Thus, it will also measure the whole of BA. And it also measures CD. Thus, CF measures (both) AB and CD. Thus, CF is a common measure of AB and CD. So I say that (it is) also the greatest (common measure). For if CF is not the greatest common measure of AB and CD, (then) some number which is greater than CF will measure the numbers AB and CD. Let it (so) measure (AB and CD), and let it be G. And since G measures CD, and CD measures BE, G thus also measures BE. And it also measures the whole of BA. Thus, it will also *τοῦ ΓΖ· ὁ ΓΖ ἄρα τῶν ΑΒ, ΓΔ μέγιστόν ἐστι κοινὸν μέτρον [ὅπερ ἔδει δεῖξαι].*

measure the remainder AE. And AE measures DF. Thus, G will also measure DF. And it also measures the whole of DC. Thus, it will also measure the remainder CF, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than CF cannot measure the numbers AB and CD. Thus, CF is the greatest common measure of AB and CD. [(Which is) the very thing it was required to show].

Πόρισμα.

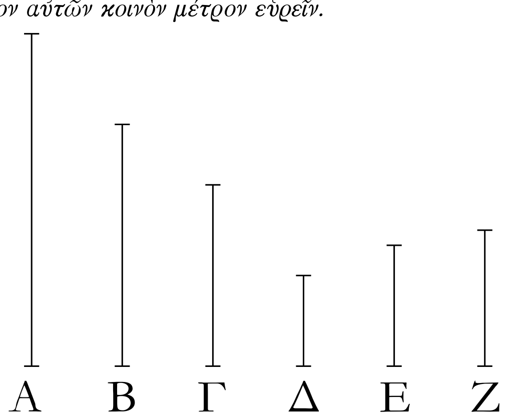
Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἀριθμὸς δύο ἀριθμοὺς μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει· ὅπερ ἔδει δεῖξαι.

Corollary

So it is manifest, from this, that if a number measures two numbers, (then) it will also measure their greatest common measure. (Which is) the very thing it was required to show.

γ′.

Τριῶν ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.



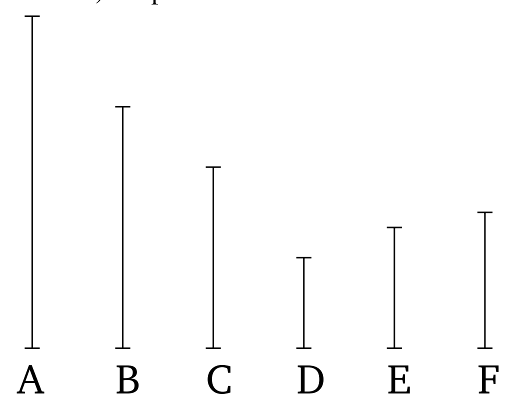
Ἔστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰλήφθω γὰρ δύο τῶν Α, Β τὸ μέγιστον κοινὸν μέτρον ὁ Δ· ὁ δὴ Δ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον· μετρεῖ δέ καὶ τοὺς Α, Β· ὁ Δ ἄρα τοὺς Α, Β, Γ μετρεῖ· ὁ Δ ἄρα τῶν Α, Β, Γ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ Δ τῶν Α, Β, Γ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ Δ. μετρείτω, καὶ ἔστω ὁ Ε. ἐπεὶ οὖν ὁ Ε τοὺς Α, Β, Γ μετρεῖ, καὶ τοὺς Α, Β ἄρα μετρήσει· καὶ τὸ τῶν Α, Β ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Α, Β μέγιστον κοινὸν μέτρον ἐστὶν ὁ Δ· ὁ Ε ἄρα τὸν Δ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ὢν τοῦ Δ· ὁ Δ ἄρα τῶν Α, Β, Γ μέγιστόν ἐστι κοινὸν μέτρον.

μὴ μετρείτω δὴ ὁ Δ τὸν Γ· λέγω πρῶτον, ὅτι οἱ Γ, Δ οὔκ εἰσι πρῶτοι πρὸς ἀλλήλους. ἐπεὶ γὰρ οἱ Α, Β, Γ οὔκ εἰσι πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. ὁ δὴ τοὺς Α, Β, Γ μετρῶν καὶ τοὺς Α, Β μετρήσει, καὶ τὸ τῶν Α, Β μέγιστον κοινὸν μέτρον τὸν Δ μετρήσει· μετρεῖ δὲ καὶ τὸν Γ· τοὺς Δ, Γ ἄρα ἀριθμοὺς ἀριθμός τις μετρήσει· οἱ Δ, Γ

Proposition 3

To find the greatest common measure of three given numbers (which are) not prime to one another.



Let A, B, and C be the three given numbers (which are) not prime to one another. So it is required to find the greatest common measure of A, B, and C.

For let the greatest common measure, D, of the two (numbers) A and B be taken [Prop. 7.2]. So D either measures, or does not measure, C. First of all, let it measure (C). And it also measures A and B. Thus, D measures A, B, and C. Thus, D is a common measure of A, B, and C. So I say that (it is) also the greatest (common measure). For if D is not the greatest common measure of A, B, and C, (then) some number greater than D will measure the numbers A, B, and C. Let it (so) measure (A, B, and C), and let it be E. Therefore, since E measures A, B, and C, it will thus also measure A and B. Thus, it will also measure the greatest common measure of A and B [Prop. 7.2 corr.]. And D is the greatest common measure of A and B. Thus, E measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than D cannot measure the numbers A, B, and C. Thus, D is the greatest common measure of A, B, and C.

So let D not measure C. I say, first of all, that C and D are not prime to one another. For since A, B, C are not prime to one another, some number will measure them. So the (num-

ἄρα οὐκ εἰσι πρῶτοι πρὸς ἀλλήλους. εἰλήφθω οὖν αὐτῶν τὸ μέγιστον κοινὸν μέτρον ὁ Ε. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, ὁ δὲ Δ τοὺς Α, Β μετρεῖ, καὶ ὁ Ε ἄρα τοὺς Α, Β μετρεῖ· μετρεῖ δὲ καὶ τὸν Γ· ὁ Ε ἄρα τοὺς Α, Β, Γ μετρεῖ. ὁ Ε ἄρα τῶν Α, Β, Γ κοινόν ἐστι μέτρον. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή ἐστιν ὁ Ε τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ Ε. μετρείτω, καὶ ἔστω ὁ Ζ. καὶ ἐπεὶ ὁ Ζ τοὺς Α, Β, Γ μετρεῖ, καὶ τοὺς Α, Β μετρεῖ· καὶ τὸ τῶν Α, Β ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Α, Β μέγιστον κοινὸν μέτρον ἐστὶν ὁ Δ· ὁ Ζ ἄρα τὸν Δ μετρεῖ· μετρεῖ δὲ καὶ τὸν Γ· ὁ Ζ ἄρα τοὺς Δ, Γ μετρεῖ· καὶ τὸ τῶν Δ, Γ ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Δ, Γ μέγιστον κοινὸν μέτρον ἐστὶν ὁ Ε· ὁ Ζ ἄρα τὸν Ε μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β, Γ ἀριθμοὺς ἀριθμός τις μετρήσει μείζων ὢν τοῦ Ε· ὁ Ε ἄρα τῶν Α, Β, Γ μέγιστόν ἐστι κοινὸν μέτρον· ὅπερ ἔδει δεῖξαι.

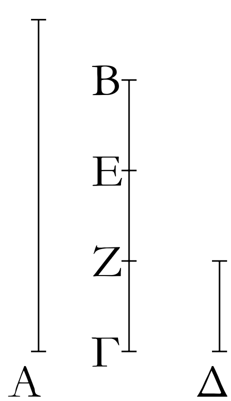
ber) measuring *A*, *B*, and *C* will also measure *A* and *B*, and it will also measure the greatest common measure, *D*, of *A* and *B* [Prop. 7.2 corr.]. And it also measures *C*. Thus, some number will measure the numbers *D* and *C*. Thus, *D* and *C* are not prime to one another. Therefore, let their greatest common measure, *E*, be taken [Prop. 7.2]. And since *E* measures *D*, and *D* measures *A* and *B*, *E* thus also measures *A* and *B*. And it also measures *C*. Thus, *E* measures *A*, *B*, and *C*. Thus, *E* is a common measure of *A*, *B*, and *C*. So I say that (it is) also the greatest (common measure). For if *E* is not the greatest common measure of *A*, *B*, and *C*, (then) some number greater than *E* will measure the numbers *A*, *B*, and *C*. Let it (so) measure (*A*, *B*, and *C*), and let it be *F*. And since *F* measures *A*, *B*, and *C*, it also measures *A* and *B*. Thus, it will also measure the greatest common measure of *A* and *B* [Prop. 7.2 corr.]. And *D* is the greatest common measure of *A* and *B*. Thus, *F* measures *D*. And it also measures *C*. Thus, *F* measures *D* and *C*. Thus, it will also measure the greatest common measure of *D* and *C* [Prop. 7.2 corr.]. And *E* is the greatest common measure of *D* and *C*. Thus, *F* measures *E*, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than *E* does not measure the numbers *A*, *B*, and *C*. Thus, *E* is the greatest common measure of *A*, *B*, and *C*. (Which is) the very thing it was required to show.

δ′.

Ἅπας ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη.

Ἔστωσαν δύο ἀριθμοὶ οἱ Α, ΒΓ, καὶ ἔστω ἐλάσσων ὁ ΒΓ· λέγω, ὅτι ὁ ΒΓ τοῦ Α ἤτοι μέρος ἐστὶν ἢ μέρη.

Οἱ Α, ΒΓ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. ἔστωσαν πρότερον οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους. διαιρεθέντος δὴ τοῦ ΒΓ εἰς τὰς ἐν αὐτῷ μονάδας ἔσται ἑκάστη μονὰς τῶν ἐν τῷ ΒΓ μέρος τι τοῦ Α· ὥστε μέρη ἐστὶν ὁ ΒΓ τοῦ Α.



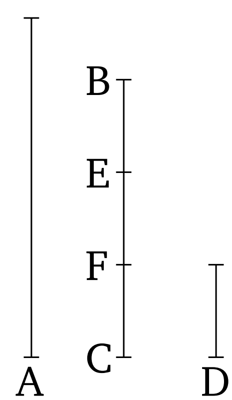
μὴ ἔστωσαν δὴ οἱ Α, ΒΓ πρῶτοι πρὸς ἀλλήλους· ὁ δὴ ΒΓ τὸν Α ἤτοι μετρεῖ ἢ οὐ μετρεῖ. εἰ μὲν οὖν ὁ ΒΓ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ ΒΓ τοῦ Α. εἰ δὲ οὔ, εἰλήφθω τῶν Α, ΒΓ

Proposition 4

Any number is either part or parts of any (other) number, the lesser of the greater.

Let *A* and *BC* be two numbers, and let *BC* be the lesser. I say that *BC* is either part or parts of *A*.

For *A* and *BC* are either prime to one another, or not. Let *A* and *BC*, first of all, be prime to one another. So separating *BC* into its constituent units, each of the units in *BC* will be some part of *A*. Hence, *BC* is parts of *A*.



So let *A* and *BC* be not prime to one another. So *BC* either measures, or does not measure, *A*. Therefore, if *BC* measures *A*, (then) *BC* is part of *A*. And if not, let the greatest common *μέγιστον κοινὸν μέτρον ὁ Δ, καὶ διῃρήσθω ὁ ΒΓ εἰς τοὺς τῷ Δ ἴσους τοὺς ΒΕ, ΕΖ, ΖΓ. καὶ ἐπεὶ ὁ Δ τὸν Α μετρεῖ, μέρος ἐστὶν ὁ Δ τοῦ Α· ἴσος δὲ ὁ Δ ἑκάστῳ τῶν ΒΕ, ΕΖ, ΖΓ· καὶ ἕκαστος ἄρα τῶν ΒΕ, ΕΖ, ΖΓ τοῦ Α μέρος ἐστίν· ὥστε μέρη ἐστὶν ὁ ΒΓ τοῦ Α.*

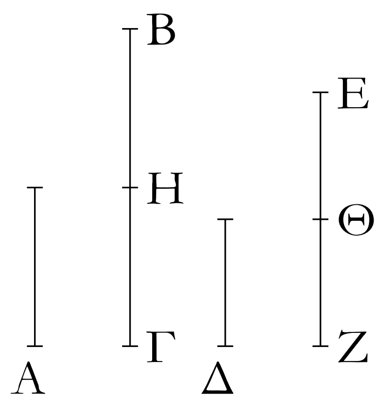
Ἅπας ἄρα ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη· ὅπερ ἔδει δεῖξαι.

measure, D, of A and BC be taken [Prop. 7.2], and let BC be divided into BE, EF, and FC, equal to D. And since D measures A, D is a part of A. And D is equal to each of BE, EF, and FC. Thus, BE, EF, and FC are also each part of A. Hence, BC is parts of A.

Thus, any number is either part or parts of any (other) number, the lesser of the greater. (Which is) the very thing it was required to show.

εʹ.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾖ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾖ, καὶ συναμφότερος συναμφοτέρου τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ εἷς τοῦ ἑνός.

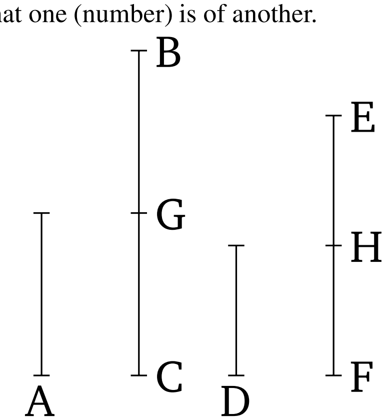


Ἀριθμὸς γὰρ ὁ Α [ἀριθμοῦ] τοῦ ΒΓ μέρος ἔστω, καὶ ἕτερος ὁ Δ ἑτέρου τοῦ ΕΖ τὸ αὐτὸ μέρος, ὅπερ ὁ Α τοῦ ΒΓ· λέγω, ὅτι καὶ συναμφότερος ὁ Α, Δ συναμφοτέρου τοῦ ΒΓ, ΕΖ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ Α τοῦ ΒΓ.

Ἐπεὶ γάρ, ὃ μέρος ἐστὶν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Δ τοῦ ΕΖ, ὅσοι ἄρα εἰσὶν ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. διῃρήσθω ὁ μὲν ΒΓ εἰς τοὺς τῷ Α ἴσους τοὺς ΒΗ, ΗΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΓ τῷ πλήθει τῶν ΕΘ, ΘΖ. καὶ ἐπεὶ ἴσος ἐστὶν ὁ μὲν ΒΗ τῷ Α, ὁ δὲ ΕΘ τῷ Δ, καὶ οἱ ΒΗ, ΕΘ ἄρα τοῖς Α, Δ ἴσοι. διὰ τὰ αὐτὰ δὴ καὶ οἱ ΗΓ, ΘΖ τοῖς Α, Δ. ὅσοι ἄρα [εἰσὶν] ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τοῖς ΒΓ, ΕΖ ἴσοι τοῖς Α, Δ. ὁσαπλασίων ἄρα ἐστὶν ὁ ΒΓ τοῦ Α, τοσαυταπλασίων ἐστὶ καὶ συναμφότερος ὁ ΒΓ, ΕΖ συναμφοτέρου τοῦ Α, Δ. ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ Α, Δ συναμφοτέρου τοῦ ΒΓ, ΕΖ· ὅπερ ἔδει δεῖξαι.

Proposition 5†

If a number is part of a number, and another (number) is the same part of another, (then) the sum (of the leading numbers) will also be the same part of the sum (of the following numbers) that one (number) is of another.



For let a number A be part of a [number] BC, and another (number) D (be) the same part of another (number) EF that A (is) of BC. I say that the sum A, D is also the same part of the sum BC, EF that A (is) of BC.

For since which(ever) part A is of BC, D is the same part of EF, thus as many numbers as are in BC equal to A, so many numbers are also in EF equal to D. Let BC be divided into BG and GC, equal to A, and EF into EH and HF, equal to D. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF. And since BG is equal to A, and EH to D, thus BG, EH (is) also equal to A, D. So, for the same (reasons), GC, HF (is) also (equal) to A, D. Thus, as many numbers as [are] in BC equal to A, so many are also in BC, EF equal to A, D. Thus, as many times as BC is (divisible) by A, so many times is the sum BC, EF also (divisible) by the sum A, D. Thus, which(ever) part A is of BC, the sum A, D is also the same part of the sum BC, EF. (Which is) the very thing it was required to show.

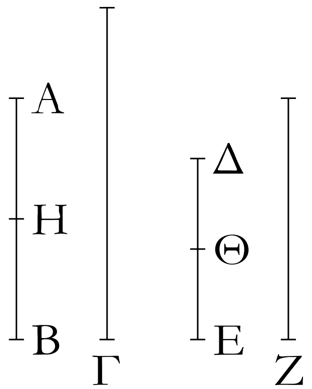
† In modern notation, this proposition states that if $a = (1/n)\, b$ and $c = (1/n)\, d$ then $(a + c) = (1/n)\,(b + d)$, where all symbols denote numbers.

ϛʹ.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾖ, καὶ ἕτερος ἑτέρου τὰ αὐτὰ

Proposition 6†

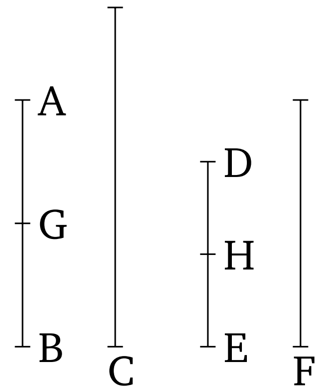
If a number is parts of a number, and another (number) is *μέρη ᾖ, καὶ συναμφότερος συναμφοτέρου τὰ αὐτὰ μέρη ἔσται, ὅπερ ὁ εἷς τοῦ ἑνός.*



Ἀριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ Γ μέρη ἔστω, καὶ ἕτερος ὁ ΔE ἑτέρου τοῦ Z τὰ αὐτὰ μέρη, ἅπερ ὁ AB τοῦ Γ· λέγω, ὅτι καὶ συναμφότερος ὁ AB, ΔE συναμφοτέρου τοῦ Γ, Z τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὁ AB τοῦ Γ.

Ἐπεὶ γάρ, ἃ μέρη ἐστὶν ὁ AB τοῦ Γ, τὰ αὐτὰ μέρη καὶ ὁ ΔE τοῦ Z, ὅσα ἄρα ἐστὶν ἐν τῷ AB μέρη τοῦ Γ, τοσαῦτά ἐστι καὶ ἐν τῷ ΔE μέρη τοῦ Z. διῃρήσθω ὁ μὲν AB εἰς τὰ τοῦ Γ μέρη τὰ AH, HB, ὁ δὲ ΔE εἰς τὰ τοῦ Z μέρη τὰ ΔΘ, ΘE· ἔσται δὴ ἴσον τὸ πλῆθος τῶν AH, HB τῷ πλήθει τῶν ΔΘ, ΘE. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ AH τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΔΘ τοῦ Z, ὃ ἄρα μέρος ἐστὶν ὁ AH τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ AH, ΔΘ συναμφοτέρου τοῦ Γ, Z. διὰ τὰ αὐτὰ δὴ καὶ ὃ μέρος ἐστὶν ὁ HB τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ HB, ΘE συναμφοτέρου τοῦ Γ, Z. ἃ ἄρα μέρη ἐστὶν ὁ AB τοῦ Γ, τὰ αὐτὰ μέρη ἐστὶ καὶ συναμφότερος ὁ AB, ΔE συναμφοτέρου τοῦ Γ, Z· ὅπερ ἔδει δεῖξαι.

the same parts of another, (then) the sum (of the leading numbers) will also be the same parts of the sum (of the following numbers) that one (number) is of another.



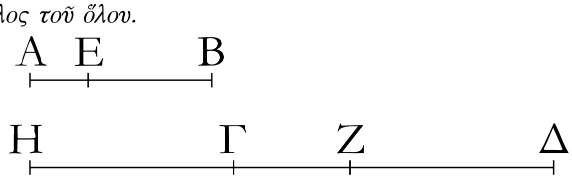
For let a number *AB* be parts of a number *C*, and another (number) *DE* (be) the same parts of another (number) *F* that *AB* (is) of *C*. I say that the sum *AB*, *DE* is also the same parts of the sum *C*, *F* that *AB* (is) of *C*.

For since which(ever) parts *AB* is of *C*, *DE* (is) also the same parts of *F*, thus as many parts of *C* as are in *AB*, so many parts of *F* are also in *DE*. Let *AB* be divided into the parts of *C*, *AG* and *GB*, and *DE* into the parts of *F*, *DH* and *HE*. So the multitude of (divisions) *AG*, *GB* will be equal to the multitude of (divisions) *DH*, *HE*. And since which(ever) part *AG* is of *C*, *DH* is also the same part of *F*, thus which(ever) part *AG* is of *C*, the sum *AG*, *DH* is also the same part of the sum *C*, *F* [Prop. 7.5]. And so, for the same (reasons), which(ever) part *GB* is of *C*, the sum *GB*, *HE* is also the same part of the sum *C*, *F*. Thus, which(ever) parts *AB* is of *C*, the sum *AB*, *DE* is also the same parts of the sum *C*, *F*. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (m/n)\, b$ and $c = (m/n)\, d$ then $(a + c) = (m/n)\,(b + d)$, where all symbols denote numbers.

ζʹ.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾖ, ὅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ ὅλος τοῦ ὅλου.

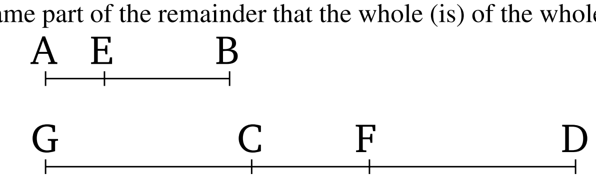


Ἀριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ ΓΔ μέρος ἔστω, ὅπερ ἀφαιρεθεὶς ὁ AE ἀφαιρεθέντος τοῦ ΓZ· λέγω, ὅτι καὶ λοιπὸς ὁ EB λοιποῦ τοῦ ZΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ AB ὅλου τοῦ ΓΔ.

Ὃ γὰρ μέρος ἐστὶν ὁ AE τοῦ ΓZ, τὸ αὐτὸ μέρος ἔστω καὶ ὁ EB τοῦ ΓH. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ AE τοῦ ΓZ, τὸ

Proposition 7†

If a number is that part of a number that a (part) taken away (is) of a (part) taken away, (then) the remainder will also be the same part of the remainder that the whole (is) of the whole.



For let a number *AB* be that part of a number *CD* that a (part) taken away *AE* (is) of a part taken away *CF*. I say that the remainder *EB* is also the same part of the remainder *FD* that the whole *AB* (is) of the whole *CD*.

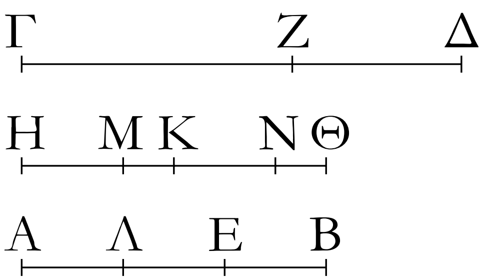
For which(ever) part *AE* is of *CF*, let *EB* also be the same part of *CG*. And since which(ever) part *AE* is of *CF*, *EB* is αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΒ τοῦ ΓΗ, ὃ ἄρα μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΗΖ. ὃ δὲ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ὑπόκειται καὶ ὁ ΑΒ τοῦ ΓΔ· ὃ ἄρα μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΗΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ τοῦ ΓΔ· ἴσος ἄρα ἐστὶν ὁ ΗΖ τῷ ΓΔ. κοινὸς ἀφῃρήσθω ὁ ΓΖ· λοιπὸς ἄρα ὁ ΗΓ λοιπῷ τῷ ΖΔ ἐστιν ἴσος. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος [ἐστὶ] καὶ ὁ ΕΒ τοῦ ΗΓ, ἴσος δὲ ὁ ΗΓ τῷ ΖΔ, ὃ ἄρα μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΒ τοῦ ΖΔ. ἀλλὰ ὃ μέρος ἐστὶν ὁ ΑΕ τοῦ ΓΖ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΓΔ· καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ· ὅπερ ἔδει δεῖξαι.

also the same part of *CG*, thus which(ever) part *AE* is of *CF*, *AB* is also the same part of *GF* [Prop. 7.5]. And which(ever) part *AE* is of *CF*, *AB* is also assumed (to be) the same part of *CD*. Thus, also, which(ever) part *AB* is of *GF*, (*AB*) is also the same part of *CD*. Thus, *GF* is equal to *CD*. Let *CF* be subtracted from both. Thus, the remainder *GC* is equal to the remainder *FD*. And since which(ever) part *AE* is of *CF*, *EB* [is] also the same part of *GC*, and *GC* (is) equal to *FD*, thus which(ever) part *AE* is of *CF*, *EB* is also the same part of *FD*. But, which(ever) part *AE* is of *CF*, *AB* is also the same part of *CD*. Thus, the remainder *EB* is also the same part of the remainder *FD* that the whole *AB* (is) of the whole *CD*. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (1/n)\,b$ and $c = (1/n)\,d$ then $(a - c) = (1/n)\,(b - d)$, where all symbols denote numbers.

η΄.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾖ, ἅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὰ αὐτὰ μέρη ἔσται, ἅπερ ὁ ὅλος τοῦ ὅλου.

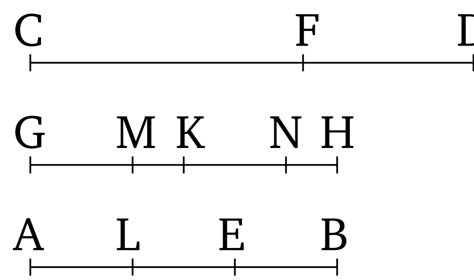


Ἀριθμὸς γὰρ ὁ ΑΒ ἀριθμοῦ τοῦ ΓΔ μέρη ἔστω, ἅπερ ἀφαιρεθεὶς ὁ ΑΕ ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ.

Κείσθω γὰρ τῷ ΑΒ ἴσος ὁ ΗΘ, ἃ ἄρα μέρη ἐστὶν ὁ ΗΘ τοῦ ΓΔ, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ ΑΕ τοῦ ΓΖ. διῃρήσθω ὁ μὲν ΗΘ εἰς τὰ τοῦ ΓΔ μέρη τὰ ΗΚ, ΚΘ, ὁ δὲ ΑΕ εἰς τὰ τοῦ ΓΖ μέρη τὰ ΑΛ, ΛΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΗΚ, ΚΘ τῷ πλήθει τῶν ΑΛ, ΛΕ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΗΚ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΛ τοῦ ΓΖ, μείζων δὲ ὁ ΓΔ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΗΚ τοῦ ΑΛ. κείσθω τῷ ΑΛ ἴσος ὁ ΗΜ. ὃ ἄρα μέρος ἐστὶν ὁ ΗΚ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΜ τοῦ ΓΖ· καὶ λοιπὸς ἄρα ὁ ΜΚ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΗΚ ὅλου τοῦ ΓΔ. πάλιν ἐπεί, ὃ μέρος ἐστὶν ὁ ΚΘ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΛ τοῦ ΓΖ, μείζων δὲ ὁ ΓΔ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΘΚ τοῦ ΕΛ. κείσθω τῷ ΕΛ ἴσος ὁ ΚΝ. ὃ ἄρα μέρος ἐστὶν ὁ ΚΘ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΚΝ τοῦ ΓΖ· καὶ λοιπὸς ἄρα ὁ ΝΘ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΚΘ ὅλου τοῦ ΓΔ. ἐδείχθη δὲ καὶ λοιπὸς ὁ ΜΚ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ὤν, ὅπερ ὅλος ὁ ΗΚ ὅλου τοῦ ΓΔ· καὶ συναμφότερος ἄρα ὁ

Proposition 8†

If a number is those parts of a number that a (part) taken away (is) of a (part) taken away, (then) the remainder will also be the same parts of the remainder that the whole (is) of the whole.



For let a number *AB* be those parts of a number *CD* that a (part) taken away *AE* (is) of a (part) taken away *CF*. I say that the remainder *EB* is also the same parts of the remainder *FD* that the whole *AB* (is) of the whole *CD*.

For let *GH* be laid down equal to *AB*. Thus, which(ever) parts *GH* is of *CD*, *AE* is also the same parts of *CF*. Let *GH* be divided into the parts of *CD*, *GK* and *KH*, and *AE* into the parts of *CF*, *AL* and *LE*. So the multitude of (divisions) *GK*, *KH* will be equal to the multitude of (divisions) *AL*, *LE*. And since which(ever) part *GK* is of *CD*, *AL* is also the same part of *CF*, and *CD* (is) greater than *CF*, *GK* (is) thus also greater than *AL*. Let *GM* be made equal to *AL*. Thus, which(ever) part *GK* is of *CD*, *GM* is also the same part of *CF*. Thus, the remainder *MK* is also the same part of the remainder *FD* that the whole *GK* (is) of the whole *CD* [Prop. 7.5]. Again, since which(ever) part *KH* is of *CD*, *EL* is also the same part of *CF*, and *CD* (is) greater than *CF*, *HK* (is) thus also greater than *EL*. Let *KN* be made equal to *EL*. Thus, which(ever) part *KH* (is) of *CD*, *KN* is also the same part of *CF*. Thus, the remainder *NH* is also the same part of the remainder *FD* that the whole *KH* (is) of the whole *CD* [Prop. 7.5]. And the

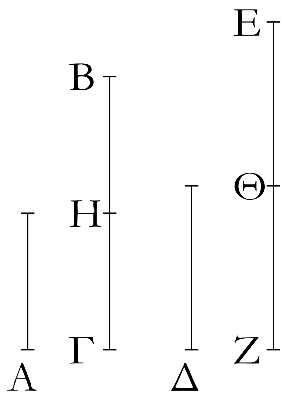
ΜΚ, ΝΘ τοῦ ΔΖ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΘΗ ὅλου τοῦ ΓΔ. ἴσος δὲ συναμφότερος μὲν ὁ ΜΚ, ΝΘ τῷ ΕΒ, ὁ δὲ ΘΗ τῷ ΒΑ· καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ· ὅπερ ἔδει δεῖξαι.

remainder MK was also shown to be the same part of the remainder FD that the whole GK (is) of the whole CD. Thus, the sum MK, NH is the same parts of DF that the whole HG (is) of the whole CD. And the sum MK, NH (is) equal to EB, and HG to BA. Thus, the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (m/n)\, b$ and $c = (m/n)\, d$ then $(a - c) = (m/n)\,(b - d)$, where all symbols denote numbers.

θ'.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾖ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾖ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ἢ μέρη ὁ πρῶτος τοῦ τρίτου, τὸ αὐτὸ μέρος ἔσται ἢ τὰ αὐτὰ μέρη καὶ ὁ δεύτερος τοῦ τετάρτου.



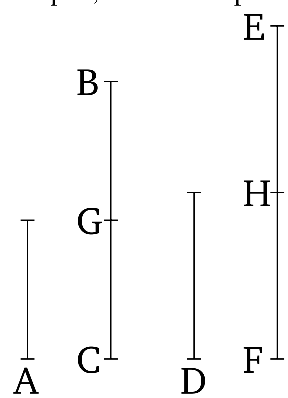
Ἀριθμὸς γὰρ ὁ Α ἀριθμοῦ τοῦ ΒΓ μέρος ἔστω, καὶ ἕτερος ὁ Δ ἑτέρου τοῦ ΕΖ τὸ αὐτὸ μέρος, ὅπερ ὁ Α τοῦ ΒΓ· λέγω, ὅτι καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ὁ Α τοῦ Δ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΒΓ τοῦ ΕΖ ἢ μέρη.

Ἐπεὶ γὰρ ὃ μέρος ἐστὶν ὁ Α τοῦ ΒΓ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Δ τοῦ ΕΖ, ὅσοι ἄρα εἰσὶν ἐν τῷ ΒΓ ἀριθμοὶ ἴσοι τῷ Α, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἴσοι τῷ Δ. διῃρήσθω ὁ μὲν ΒΓ εἰς τοὺς τῷ Α ἴσους τοὺς ΒΗ, ΗΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΓ τῷ πλήθει τῶν ΕΘ, ΘΖ.

Καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΒΗ, ΗΓ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΕΘ, ΘΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΓ τῷ πλήθει τῶν ΕΘ, ΘΖ, ὃ ἄρα μέρος ἐστὶν ὁ ΒΗ τοῦ ΕΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΓ τοῦ ΘΖ ἢ τὰ αὐτὰ μέρη· ὥστε καὶ ὃ μέρος ἐστὶν ὁ ΒΗ τοῦ ΕΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφότερος ὁ ΒΓ συναμφοτέρου τοῦ ΕΖ ἢ τὰ αὐτὰ μέρη. ἴσος δὲ ὁ μὲν ΒΗ τῷ Α, ὁ δὲ ΕΘ τῷ Δ· ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ Δ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΒΓ τοῦ ΕΖ ἢ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

Proposition 9†

If a number is part of a number, and another (number) is the same part of another, also, alternately, which(ever) part, or parts, the first (number) is of the third, the second (number) will also be the same part, or the same parts, of the fourth.



For let a number A be part of a number BC, and another (number) D (be) the same part of another EF that A (is) of BC. I say that, also, alternately, which(ever) part, or parts, A is of D, BC is also the same part, or parts, of EF.

For since which(ever) part A is of BC, D is also the same part of EF, thus as many numbers as are in BC equal to A, so many are also in EF equal to D. Let BC be divided into BG and GC, equal to A, and EF into EH and HF, equal to D. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF.

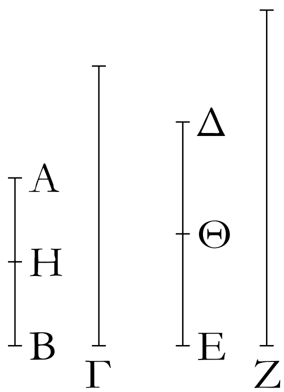
And since the numbers BG and GC are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) BG, GC is equal to the multitude of (divisions) EH, HC, thus which(ever) part, or parts, BG is of EH, GC is also the same part, or the same parts, of HF. And hence, which(ever) part, or parts, BG is of EH, the sum BC is also the same part, or the same parts, of the sum EF [Props. 7.5, 7.6]. And BG (is) equal to A, and EH to D. Thus, which(ever) part, or parts, A is of D, BC is also the same part, or the same parts, of EF. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (1/n)\,b$ and $c = (1/n)\,d$ then if $a = (k/l)\,c$ then $b = (k/l)\,d$, where all symbols denote numbers.

ι'.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾖ, καὶ ἕτερος ἑτέρου τὰ αὐτὰ μέρη ᾖ, καὶ ἐναλλάξ, ἃ μέρη ἐστὶν ὁ πρῶτος τοῦ τρίτου ἢ μέρος, τὰ αὐτὰ μέρη ἔσται καὶ ὁ δεύτερος τοῦ τετάρτου ἢ τὸ αὐτὸ μέρος.

Ἀριθμὸς γὰρ ὁ ΑΒ ἀριθμοῦ τοῦ Γ μέρη ἔστω, καὶ ἕτερος ὁ ΔΕ ἑτέρου τοῦ Ζ τὰ αὐτὰ μέρη· λέγω, ὅτι καὶ ἐναλλάξ, ἃ μέρη ἐστὶν ὁ ΑΒ τοῦ ΔΕ ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὸ αὐτὸ μέρος.

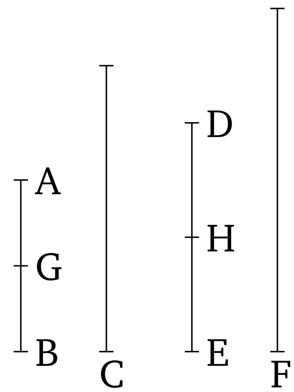


Ἐπεὶ γάρ, ἃ μέρη ἐστὶν ὁ ΑΒ τοῦ Γ, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ ΑΒ μέρη τοῦ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ μέρη τοῦ Ζ. διῃρήσθω ὁ μὲν ΑΒ εἰς τὰ τοῦ Γ μέρη τὰ ΑΗ, ΗΒ, ὁ δὲ ΔΕ εἰς τὰ τοῦ Ζ μέρη τὰ ΔΘ, ΘΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΒ τῷ πλήθει τῶν ΔΘ, ΘΕ. καὶ ἐπεί, ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ Γ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΔΘ τοῦ Ζ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὰ αὐτὰ μέρη. διὰ τὰ αὐτὰ δὴ καί, ὃ μέρος ἐστὶν ὁ ΗΒ τοῦ ΘΕ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὰ αὐτὰ μέρη· ὥστε καί [ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΒ τοῦ ΘΕ ἢ τὰ αὐτὰ μέρη· καὶ ὃ ἄρα μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΒ τοῦ ΔΕ ἢ τὰ αὐτὰ μέρη· ἀλλ' ὃ μέρος ἐστὶν ὁ ΑΗ τοῦ ΔΘ ἢ μέρη, τὸ αὐτὸ μέρος ἐδείχθη καὶ ὁ Γ τοῦ Ζ ἢ τὰ αὐτὰ μέρη, καὶ] ἃ [ἄρα] μέρη ἐστὶν ὁ ΑΒ τοῦ ΔΕ ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ Γ τοῦ Ζ ἢ τὸ αὐτὸ μέρος· ὅπερ ἔδει δεῖξαι.

Proposition 10†

If a number is parts of a number, and another (number) is the same parts of another, also, alternately, which(ever) parts, or part, the first (number) is of the third, the second will also be the same parts, or the same part, of the fourth.

For let a number AB be parts of a number C, and another (number) DE (be) the same parts of another F. I say that, also, alternately, which(ever) parts, or part, AB is of DE, C is also the same parts, or the same part, of F.



For since which(ever) parts AB is of C, DE is also the same parts of F, thus as many parts of C as are in AB, so many parts of F (are) also in DE. Let AB be divided into the parts of C, AG and GB, and DE into the parts of F, DH and HE. So the multitude of (divisions) AG, GB will be equal to the multitude of (divisions) DH, HE. And since which(ever) part AG is of C, DH is also the same part of F, also, alternately, which(ever) part, or parts, AG is of DH, C is also the same part, or the same parts, of F [Prop. 7.9]. And so, for the same (reasons), which(ever) part, or parts, GB is of HE, C is also the same part, or the same parts, of F [Prop. 7.9]. And so [which(ever) part, or parts, AG is of DH, GB is also the same part, or the same parts, of HE. And thus, which(ever) part, or parts, AG is of DH, AB is also the same part, or the same parts, of DE [Props. 7.5, 7.6]. But, which(ever) part, or parts, AG is of DH, C was also shown (to be) the same part, or the same parts, of F. And, thus] which(ever) parts, or part, AB is of DE, C is also the same parts, or the same part, of F. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (m/n)\,b$ and $c = (m/n)\,d$ then if $a = (k/l)\,c$ then $b = (k/l)\,d$, where all symbols denote numbers.

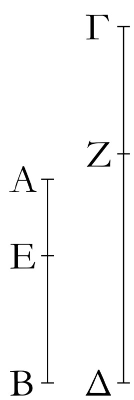
ια'.

Ἐαν ᾖ ὡς ὅλος πρὸς ὅλον, οὕτως ἀφαιρεθεὶς πρὸς ἀφαιρεθέντα, καὶ ὁ λοιπὸς πρὸς τὸν λοιπὸν ἔσται, ὡς ὅλος πρὸς ὅλον.

Proposition 11

If as the whole (of a number) is to the whole (of another), so a (part) taken away (is) to a (part) taken away, (then) the remainder will also be to the remainder as the whole (is) to the

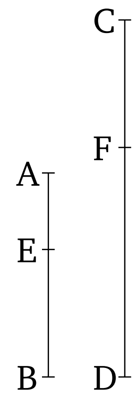
Ἔστω ὡς ὅλος ὁ ΑΒ πρὸς ὅλον τὸν ΓΔ, οὕτως ἀφαιρεθεὶς ὁ ΑΕ πρὸς ἀφαιρεθέντα τὸν ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ ΕΒ πρὸς λοιπὸν τὸν ΖΔ ἐστιν, ὡς ὅλος ὁ ΑΒ πρὸς ὅλον τὸν ΓΔ.



Ἐπεί ἐστιν ὡς ὁ ΑΒ πρὸς τὸν ΓΔ, οὕτως ὁ ΑΕ πρὸς τὸν ΓΖ, ὃ ἄρα μέρος ἐστὶν ὁ ΑΒ τοῦ ΓΔ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΕ τοῦ ΓΖ ἢ τὰ αὐτὰ μέρη. καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστὶν ἢ μέρη, ἅπερ ὁ ΑΒ τοῦ ΓΔ. ἔστιν ἄρα ὡς ὁ ΕΒ πρὸς τὸν ΖΔ, οὕτως ὁ ΑΒ πρὸς τὸν ΓΔ· ὅπερ ἔδει δεῖξαι.

whole.

Let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF. I say that the remainder EB is to the remainder FD as the whole AB (is) to the whole CD.



(For) since as AB is to CD, so AE (is) to CF, thus which(ever) part, or parts, AB is of CD, AE is also the same part, or the same parts, of CF [Def. 7.20]. Thus, the remainder EB is also the same part, or parts, of the remainder FD that AB (is) of CD [Props. 7.7, 7.8]. Thus, as EB is to FD, so AB (is) to CD [Def. 7.20]. (Which is) the very thing it was required to show.

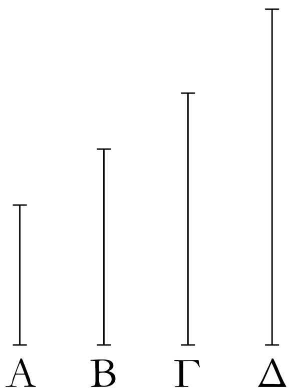
† In modern notation, this proposition states that if $a : b :: c : d$ then $a : b :: a - c : b - d$, where all symbols denote numbers.

ιβ'.

Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἀνάλογον, ἔσται ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους.

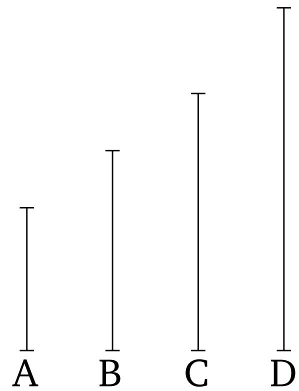
Proposition 12†

If any multitude whatsoever of numbers are proportional, (then) as one of the leading (numbers is) to one of the following so (the sum of) all of the leading (numbers) will be to (the sum of) all of the following.



Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ· λέγω, ὅτι ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως οἱ Α, Γ πρὸς τοὺς Β, Δ.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ, ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ Β ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ μέρη. καὶ συναμφότερος ἄρα ὁ Α, Γ συναμφοτέρου τοῦ Β, Δ τὸ αὐτὸ μέρος ἐστὶν ἢ τὰ αὐτὰ μέρη, ἅπερ ὁ Α τοῦ Β. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως οἱ Α, Γ πρὸς τοὺς Β, Δ· ὅπερ ἔδει δεῖξαι.



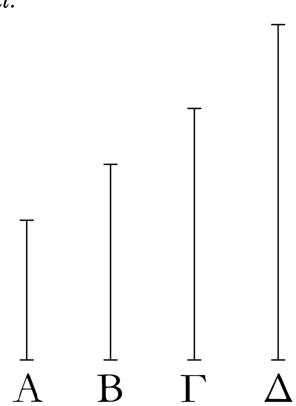
Let any multitude whatsoever of numbers, *A*, *B*, *C*, *D*, be proportional, (such that) as *A* (is) to *B*, so *C* (is) to *D*. I say that as *A* is to *B*, so *A*, *C* (is) to *B*, *D*.

For since as *A* is to *B*, so *C* (is) to *D*, thus which(ever) part, or parts, *A* is of *B*, *C* is also the same part, or parts, of *D* [Def. 7.20]. Thus, the sum *A*, *C* is also the same part, or the same parts, of the sum *B*, *D* that *A* (is) of *B* [Props. 7.5, 7.6]. Thus, as *A* is to *B*, so *A*, *C* (is) to *B*, *D* [Def. 7.20]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a : b :: c : d$ then $a : b :: a + c : b + d$, where all symbols denote numbers.

ιγ'.

Ἐὰν τέσσαρες ἀριθμοὶ ἀνάλογον ὦσιν, καὶ ἐναλλὰξ ἀνάλογον ἔσονται.

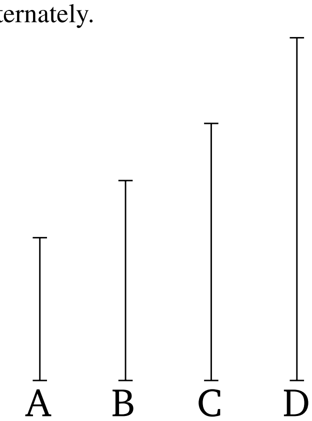


Ἔστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ· λέγω, ὅτι καὶ ἐναλλὰξ ἀνάλογον ἔσονται, ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Β πρὸς τὸν Δ.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ, ὃ ἄρα μέρος ἐστὶν ὁ Α τοῦ Β ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἐναλλὰξ ἄρα, ὃ μέρος ἐστὶν ὁ Α τοῦ Γ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Β τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Β πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

Proposition 13†

If four numbers are proportional, (then) they will also be proportional alternately.



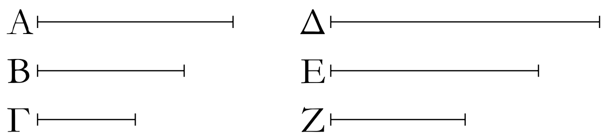
Let the four numbers *A*, *B*, *C*, and *D* be proportional, (such that) as *A* (is) to *B*, so *C* (is) to *D*. I say that they will also be proportional alternately, (such that) as *A* (is) to *C*, so *B* (is) to *D*.

For since as *A* is to *B*, so *C* (is) to *D*, thus which(ever) part, or parts, *A* is of *B*, *C* is also the same part, or the same parts, of *D* [Def. 7.20]. Thus, alternately, which(ever) part, or parts, *A* is of *C*, *B* is also the same part, or the same parts, of *D* [Props. 7.9, 7.10]. Thus, as *A* is to *C*, so *B* (is) to *D* [Def. 7.20]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a : b :: c : d$ then $a : c :: b : d$, where all symbols denote numbers.

ιδ′.

Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι᾽ ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσονται.

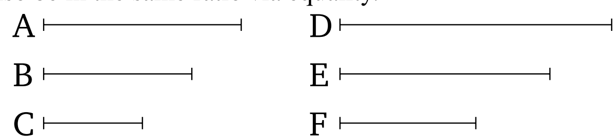


Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ οἱ Α, Β, Γ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι ἐν τῷ αὐτῷ λόγῳ οἱ Δ, Ε, Ζ, ὡς μὲν ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε, ὡς δὲ ὁ Β πρὸς τὸν Γ, οὕτως ὁ Ε πρὸς τὸν Ζ· λέγω, ὅτι καὶ δι᾽ ἴσου ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ζ.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, οὕτως ὁ Β πρὸς τὸν Ε. πάλιν, ἐπεί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ Ε πρὸς τὸν Ζ, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Β πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Ζ. ὡς δὲ ὁ Β πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Δ, οὕτως ὁ Γ πρὸς τὸν Ζ· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ζ· ὅπερ ἔδει δεῖξαι.

Proposition 14†

If there are any multitude of numbers whatsoever, and (some) other (numbers) of equal multitude to them, (which are) also in the same ratio taken two by two, (then) they will also be in the same ratio via equality.



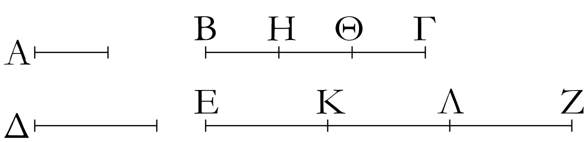
Let there be any multitude of numbers whatsoever, A, B, C, and (some) other (numbers), D, E, F, of equal multitude to them, (which are) in the same ratio taken two by two, (such that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that also, via equality, as A is to C, so D (is) to F.

For since as A is to B, so D (is) to E, thus, alternately, as A is to D, so B (is) to E [Prop. 7.13]. Again, since as B is to C, so E (is) to F, thus, alternately, as B is to E, so C (is) to F [Prop. 7.13]. And as B (is) to E, so A (is) to D. Thus, also, as A (is) to D, so C (is) to F. Thus, alternately, as A is to C, so D (is) to F [Prop. 7.13]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a : b :: d : e$ and $b : c :: e : f$ then $a : c :: d : f$, where all symbols denote numbers.

ιε′.

Ἐὰν μονὰς ἀριθμόν τινα μετρῇ, ἰσάκις δὲ ἕτερος ἀριθμὸς ἄλλον τινὰ ἀριθμὸν μετρῇ, καὶ ἐναλλὰξ ἰσάκις ἡ μονὰς τὸν τρίτον ἀριθμὸν μετρήσει καὶ ὁ δεύτερος τὸν τέταρτον.

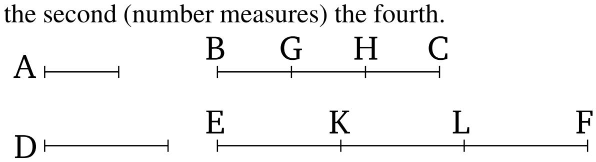


Μονὰς γὰρ ἡ Α ἀριθμόν τινα τὸν ΒΓ μετρείτω, ἰσάκις δὲ ἕτερος ἀριθμὸς ὁ Δ ἄλλον τινὰ ἀριθμὸν τὸν ΕΖ μετρείτω· λέγω, ὅτι καὶ ἐναλλὰξ ἰσάκις ἡ Α μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ ΒΓ τὸν ΕΖ.

Ἐπεὶ γὰρ ἰσάκις ἡ Α μονὰς τὸν ΒΓ ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν ΕΖ, ὅσαι ἄρα εἰσὶν ἐν τῷ ΒΓ μονάδες, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. διῃρήσθω ὁ μὲν ΒΓ εἰς τὰς ἐν ἑαυτῷ μονάδας τὰς ΒΗ, ΗΘ, ΘΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΚ, ΚΛ, ΛΖ. ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ τῷ πλήθει τῶν ΕΚ, ΚΛ, ΛΖ. καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΒΗ, ΗΘ, ΘΓ μονάδες ἀλλήλαις, εἰσὶ δὲ καὶ οἱ ΕΚ, ΚΛ, ΛΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ μονάδων τῷ πλήθει τῶν ΕΚ, ΚΛ, ΛΖ ἀριθμῶν, ἔσται ἄρα

Proposition 15

If a unit measures some number, and another number measures some other number as many times, (then), also, alternately, the unit will measure the third number as many times as the second (number measures) the fourth.



For let a unit A measure some number BC, and let another number D measure some other number EF as many times. I say that, also, alternately, the unit A also measures the number D as many times as BC (measures) EF.

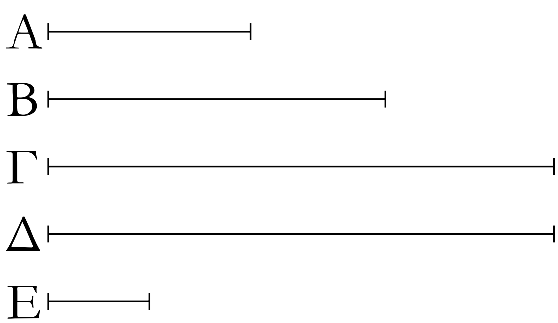
For since the unit A measures the number BC as many times as D (measures) EF, thus as many units as are in BC, so many numbers are also in EF equal to D. Let BC be divided into its constituent units, BG, GH, and HC, and EF into the (divisions) EK, KL, and LF, equal to D. So the multitude of (units) BG, GH, HC will be equal to the multitude of (divisions) EK, KL, LF. And since the units BG, GH, and HC are equal to one another, and the numbers EK, KL, and LF are also equal to one another, and the multitude of the (units) BG, ὡς ἡ ΒΗ μονὰς πρὸς τὸν ΕΚ ἀριθμόν, οὕτως ἡ ΗΘ μονὰς πρὸς τὸν ΚΛ ἀριθμὸν καὶ ἡ ΘΓ μονὰς πρὸς τὸν ΛΖ ἀριθμόν. ἔσται ἄρα καὶ ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους· ἔστιν ἄρα ὡς ἡ ΒΗ μονὰς πρὸς τὸν ΕΚ ἀριθμόν, οὕτως ὁ ΒΓ πρὸς τὸν ΕΖ. ἴση δὲ ἡ ΒΗ μονὰς τῇ Α μονάδι, ὁ δὲ ΕΚ ἀριθμὸς τῷ Δ ἀριθμῷ. ἔστιν ἄρα ὡς ἡ Α μονὰς πρὸς τὸν Δ ἀριθμόν, οὕτως ὁ ΒΓ πρὸς τὸν ΕΖ. ἰσάκις ἄρα ἡ Α μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ ΒΓ τὸν ΕΖ· ὅπερ ἔδει δεῖξαι.

GH, *HC* is equal to the multitude of the numbers *EK*, *KL*, *LF*, thus as the unit *BG* (is) to the number *EK*, so the unit *GH* will be to the number *KL*, and the unit *HC* to the number *LF*. And thus, as one of the leading (numbers is) to one of the following, so (the sum of) all of the leading will be to (the sum of) all of the following [Prop. 7.12]. Thus, as the unit *BG* (is) to the number *EK*, so *BC* (is) to *EF*. And the unit *BG* (is) equal to the unit *A*, and the number *EK* to the number *D*. Thus, as the unit *A* is to the number *D*, so *BC* (is) to *EF*. Thus, the unit *A* measures the number *D* as many times as *BC* (measures) *EF* [Def. 7.20]. (Which is) the very thing it was required to show.

† This proposition is a special case of Prop. 7.9.

ιϛʹ.

Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινας, οἱ γενόμενοι ἐξ αὐτῶν ἴσοι ἀλλήλοις ἔσονται.

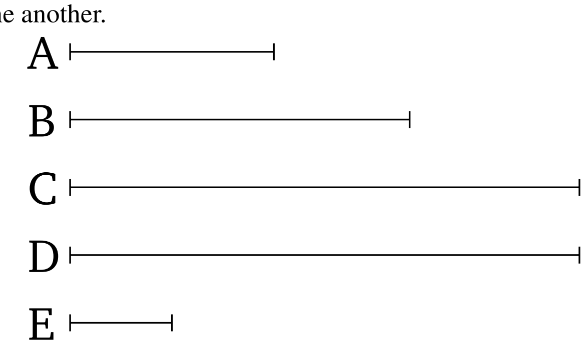


Ἔστωσαν δύο ἀριθμοὶ οἱ Α, Β, καὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω, ὁ δὲ Β τὸν Α πολλαπλασιάσας τὸν Δ ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ Γ τῷ Δ.

Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Α ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Γ. ἐναλλὰξ ἄρα ἰσάκις ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Γ. πάλιν, ἐπεὶ ὁ Β τὸν Α πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Α ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Β μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Β κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Δ. ἰσάκις δὲ ἡ Ε μονὰς τὸν Β ἀριθμὸν ἐμέτρει καὶ ὁ Α τὸν Γ· ἰσάκις ἄρα ὁ Α ἑκάτερον τῶν Γ, Δ μετρεῖ. ἴσος ἄρα ἐστὶν ὁ Γ τῷ Δ· ὅπερ ἔδει δεῖξαι.

Proposition 16†

If two numbers multiplying one another make some (numbers, then) the (numbers) generated from them will be equal to one another.



Let *A* and *B* be two numbers. And let *A* make *C* (by) multiplying *B*, and let *B* make *D* (by) multiplying *A*. I say that *C* is equal to *D*.

For since *A* has made *C* (by) multiplying *B*, *B* thus measures *C* according to the units in *A* [Def. 7.15]. And the unit *E* also measures the number *A* according to the units in it. Thus, the unit *E* measures the number *A* as many times as *B* (measures) *C*. Thus, alternately, the unit *E* measures the number *B* as many times as *A* (measures) *C* [Prop. 7.15]. Again, since *B* has made *D* (by) multiplying *A*, *A* thus measures *D* according to the units in *B* [Def. 7.15]. And the unit *E* also measures *B* according to the units in it. Thus, the unit *E* measures the number *B* as many times as *A* (measures) *D*. And the unit *E* was measuring the number *B* as many times as *A* (measures) *C*. Thus, *A* measures each of *C* and *D* an equal number of times. Thus, *C* is equal to *D*. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that $a\,b = b\,a$, where all symbols denote numbers.

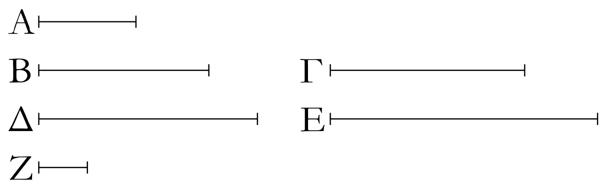
ιζʹ.

Ἐὰν ἀριθμὸς δύο ἀριθμοὺς πολλαπλασιάσας ποιῇ τινας,

Proposition 17†

If a number multiplying two numbers makes some (num-

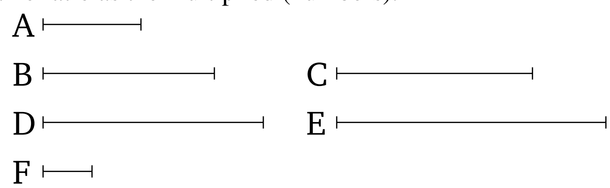
οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἕξουσι λόγον τοῖς πολλαπλασιασθεῖσιν.



Ἀριθμὸς γὰρ ὁ Α δύο ἀριθμοὺς τοὺς Β, Γ πολλαπλασιάσας τοὺς Δ, Ε ποιείτω· λέγω, ὅτι ἐστὶν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ε.

Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Β ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ δὲ καὶ ἡ Ζ μονὰς τὸν Α ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ζ μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Δ. ἔστιν ἄρα ὡς ἡ Ζ μονὰς πρὸς τὸν Α ἀριθμόν, οὕτως ὁ Β πρὸς τὸν Δ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ Ζ μονὰς πρὸς τὸν Α ἀριθμόν, οὕτως ὁ Γ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Β πρὸς τὸν Δ, οὕτως ὁ Γ πρὸς τὸν Ε. ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Β πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ε· ὅπερ ἔδει δεῖξαι.

bers, then) the (numbers) generated from them will have the same ratio as the multiplied (numbers).



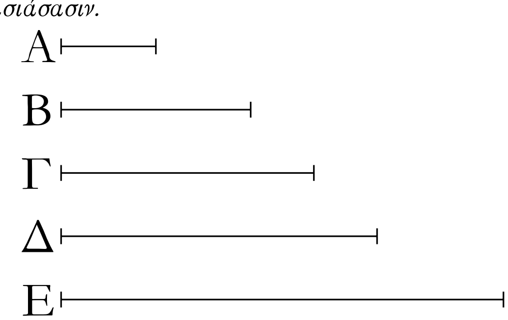
For let the number A make (the numbers) D and E (by) multiplying the two numbers B and C (respectively). I say that as B is to C, so D (is) to E.

For since A has made D (by) multiplying B, B thus measures D according to the units in A [Def. 7.15]. And the unit F also measures the number A according to the units in it. Thus, the unit F measures the number A as many times as B (measures) D. Thus, as the unit F is to the number A, so B (is) to D [Def. 7.20]. And so, for the same (reasons), as the unit F (is) to the number A, so C (is) to E. And thus, as B (is) to D, so C (is) to E. Thus, alternately, as B is to C, so D (is) to E [Prop. 7.13]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $d = a\,b$ and $e = a\,c$ then $d : e :: b : c$, where all symbols denote numbers.

ιη'.

Ἐὰν δύο ἀριθμοὶ ἀριθμόν τινα πολλαπλασιάσαντες ποιῶσί τινας, οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἕξουσι λόγον τοῖς πολλαπλασιάσασιν.

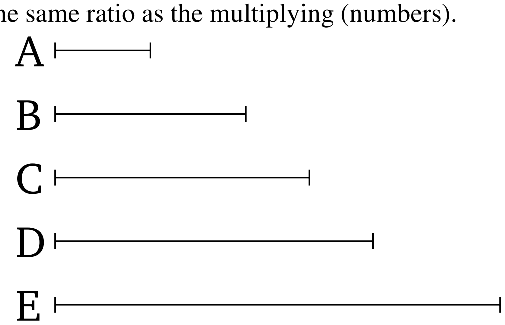


Δύο γὰρ ἀριθμοὶ οἱ Α, Β ἀριθμόν τινα τὸν Γ πολλαπλασιάσαντες τοὺς Δ, Ε ποιείτωσαν· λέγω, ὅτι ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε.

Ἐπεὶ γὰρ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν, καὶ ὁ Γ ἄρα τὸν Α πολλαπλασιάσας τὸν Δ πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Β πολλαπλασιάσας τὸν Ε πεποίηκεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς Α, Β πολλαπλασιάσας τοὺς Δ, Ε πεποίηκεν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· ὅπερ ἔδει δεῖξαι.

Proposition 18†

If two numbers multiplying some number make some (other numbers, then) the (numbers) generated from them will have the same ratio as the multiplying (numbers).



For let the two numbers A and B make (the numbers) D and E (respectively, by) multiplying some number C. I say that as A is to B, so D (is) to E.

For since A has made D (by) multiplying C, C has thus also made D (by) multiplying A [Prop. 7.16]. So, for the same (reasons), C has also made E (by) multiplying B. So the number C has made D and E (by) multiplying the two numbers A and B (respectively). Thus, as A is to B, so D (is) to E [Prop. 7.17]. (Which is) the very thing it was required to show.

† In modern notation, this propositions states that if $a\,c = d$ and $b\,c = e$ then $a : b :: d : e$, where all symbols denote numbers.

ιθ'.

Ἐὰν τέσσαρες ἀριθμοὶ ἀνάλογον ὦσιν, ὁ ἐκ πρώτου καὶ

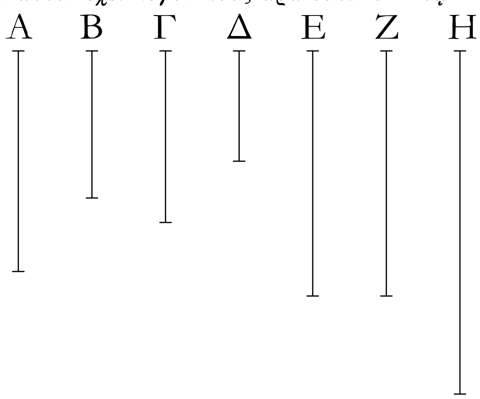
Proposition 19†

If four number are proportional, (then) the number cre-

τετάρτου γενόμενος ἀριθμὸς ἴσος ἔσται τῷ ἐκ δευτέρου καὶ τρίτου γενομένῳ ἀριθμῷ· καὶ ἐὰν ὁ ἐκ πρώτου καὶ τετάρτου γενόμενος ἀριθμὸς ἴσος ᾖ τῷ ἐκ δευτέρου καὶ τρίτου, οἱ τέσσαρες ἀριθμοὶ ἀνάλογον ἔσονται.

Ἔστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ, καὶ ὁ μὲν Α τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ Β τὸν Γ πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ Ε τῷ Ζ.

Ὁ γὰρ Α τὸν Γ πολλαπλασιάσας τὸν Η ποιείτω. ἐπεὶ οὖν ὁ Α τὸν Γ πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Δ πολλαπλασιάσας τὸν Ε πεποίηκεν, ἀριθμὸς δὴ ὁ Α δύο ἀριθμοὺς τοὺς Γ, Δ πολλαπλασιάσας τοὺς Η, Ε πεποίηκεν. ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Η πρὸς τὸν Ε. ἀλλʼ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ε. πάλιν, ἐπεὶ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Η πεποίηκεν, ἀλλὰ μὴν καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Ζ πεποίηκεν, δύο δὴ ἀριθμοὶ οἱ Α, Β ἀριθμόν τινα τὸν Γ πολλαπλασιάσαντες τοὺς Η, Ζ πεποιήκασιν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ζ. ἀλλὰ μὴν καὶ ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Ζ. ὁ Η ἄρα πρὸς ἑκάτερον τῶν Ε, Ζ τὸν αὐτὸν ἔχει λόγον· ἴσος ἄρα ἐστὶν ὁ Ε τῷ Ζ.



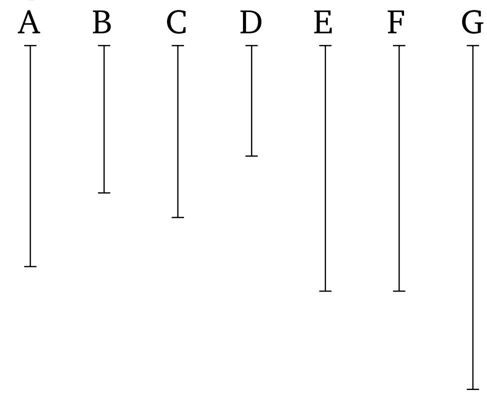
Ἔστω δὴ πάλιν ἴσος ὁ Ε τῷ Ζ· λέγω, ὅτι ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴσος ἐστὶν ὁ Ε τῷ Ζ, ἔστιν ἄρα ὡς ὁ Η πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Ζ. ἀλλʼ ὡς μὲν ὁ Η πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ, ὡς δὲ ὁ Η πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Γ πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

ated from (multiplying) the first and fourth will be equal to the number created from (multiplying) the second and third. And if the number created from (multiplying) the first and fourth is equal to the (number created) from (multiplying) the second and third, (then) the four numbers will be proportional.

Let A, B, C, and D be four proportional numbers, (such that) as A (is) to B, so C (is) to D. And let A make E (by) multiplying D, and let B make F (by) multiplying C. I say that E is equal to F.

For let A make G (by) multiplying C. Therefore, since A has made G (by) multiplying C, and has made E (by) multiplying D, the number A has made G and E by multiplying the two numbers C and D (respectively). Thus, as C is to D, so G (is) to E [Prop. 7.17]. But, as C (is) to D, so A (is) to B. Thus, also, as A (is) to B, so G (is) to E. Again, since A has made G (by) multiplying C, but, in fact, B has also made F (by) multiplying C, the two numbers A and B have made G and F (respectively, by) multiplying some number C. Thus, as A is to B, so G (is) to F [Prop. 7.18]. But, also, as A (is) to B, so G (is) to E. And thus, as G (is) to E, so G (is) to F. Thus, G has the same ratio to each of E and F. Thus, E is equal to F [Prop. 5.9].



So, again, let E be equal to F. I say that as A is to B, so C (is) to D.

For, with the same construction, since E is equal to F, thus as G is to E, so G (is) to F [Prop. 5.7]. But, as G (is) to E, so C (is) to D [Prop. 7.17]. And as G (is) to F, so A (is) to B [Prop. 7.18]. And, thus, as A (is) to B, so C (is) to D. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $a : b :: c : d$ then $a\,d = b\,c$, and vice versa, where all symbols denote numbers.

κʹ.

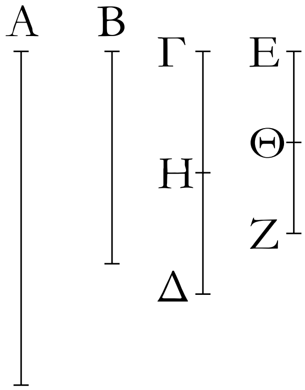
Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα.

Ἔστωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β οἱ ΓΔ, ΕΖ· λέγω, ὅτι ἰσάκις ὁ ΓΔ τὸν Α

Proposition 20

The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

For let CD and EF be the least numbers having the same μετρεῖ καὶ ὁ ΕΖ τὸν Β.



Ὁ ΓΔ γὰρ τοῦ Α οὔκ ἐστι μέρη. εἰ γὰρ δυνατόν, ἔστω· καὶ ὁ ΕΖ ἄρα τοῦ Β τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὁ ΓΔ τοῦ Α. ὅσα ἄρα ἐστὶν ἐν τῷ ΓΔ μέρη τοῦ Α, τοσαῦτά ἐστι καὶ ἐν τῷ ΕΖ μέρη τοῦ Β. διῃρήσθω ὁ μὲν ΓΔ εἰς τὰ τοῦ Α μέρη τὰ ΓΗ, ΗΔ, ὁ δὲ ΕΖ εἰς τὰ τοῦ Β μέρη τὰ ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΓΗ, ΗΔ τῷ πλήθει τῶν ΕΘ, ΘΖ. καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΓΗ, ΗΔ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΕΘ, ΘΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῶν ΓΗ, ΗΔ τῷ πλήθει τῶν ΕΘ, ΘΖ, ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν ΕΘ, οὕτως ὁ ΗΔ πρὸς τὸν ΘΖ. ἔσται ἄρα καὶ ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους. ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν ΕΘ, οὕτως ὁ ΓΔ πρὸς τὸν ΕΖ· οἱ ΓΗ, ΕΘ ἄρα τοῖς ΓΔ, ΕΖ ἐν τῷ αὐτῷ λόγῳ εἰσὶν ἐλάσσονες ὄντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον· ὑπόκεινται γὰρ οἱ ΓΔ, ΕΖ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οὐκ ἄρα μέρη ἐστὶν ὁ ΓΔ τοῦ Α· μέρος ἄρα. καὶ ὁ ΕΖ τοῦ Β τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ ΓΔ τοῦ Α· ἰσάκις ἄρα ὁ ΓΔ τὸν Α μετρεῖ καὶ ὁ ΕΖ τὸν Β· ὅπερ ἔδει δεῖξαι.

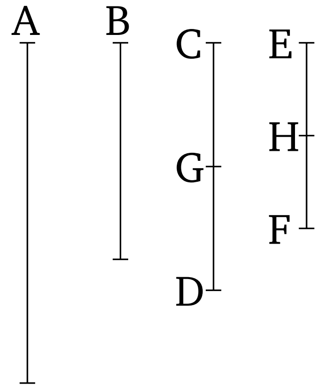
καʹ.

Οἱ πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Ἔστωσαν πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι οἱ Α, Β ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔσονταί τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β. ἔστωσαν οἱ Γ, Δ.

ratio as *A* and *B* (respectively). I say that *CD* measures *A* the same number of times as *EF* (measures) *B*.



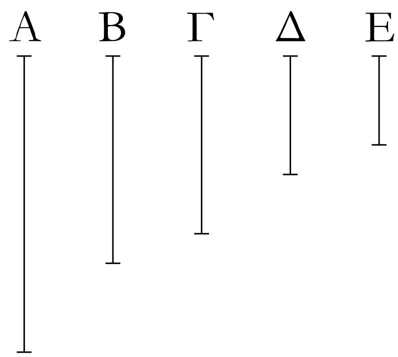
For *CD* is not parts of *A*. For, if possible, let it be (parts of *A*). Thus, *EF* is also the same parts of *B* that *CD* (is) of *A* [Def. 7.20, Prop. 7.13]. Thus, as many parts of *A* as are in *CD*, so many parts of *B* are also in *EF*. Let *CD* be divided into the parts of *A*, *CG* and *GD*, and *EF* into the parts of *B*, *EH* and *HF*. So the multitude of (divisions) *CG*, *GD* will be equal to the multitude of (divisions) *EH*, *HF*. And since the numbers *CG* and *GD* are equal to one another, and the numbers *EH* and *HF* are also equal to one another, and the multitude of (divisions) *CG*, *GD* is equal to the multitude of (divisions) *EH*, *HF*, thus as *CG* is to *EH*, so *GD* (is) to *HF*. Thus, as one of the leading (numbers is) to one of the following, so will (the sum of) all of the leading (numbers) be to (the sum of) all of the following [Prop. 7.12]. Thus, as *CG* is to *EH*, so *CD* (is) to *EF*. Thus, *CG* and *EH* are in the same ratio as *CD* and *EF*, being less than them. The very thing is impossible. For *CD* and *EF* were assumed (to be) the least of those (numbers) having the same ratio as them. Thus, *CD* is not parts of *A*. Thus, (it is) a part (of *A*) [Prop. 7.4]. And *EF* is the same part of *B* that *CD* (is) of *A* [Def. 7.20, Prop 7.13]. Thus, *CD* measures *A* the same number of times that *EF* (measures) *B*. (Which is) the very thing it was required to show.

Proposition 21

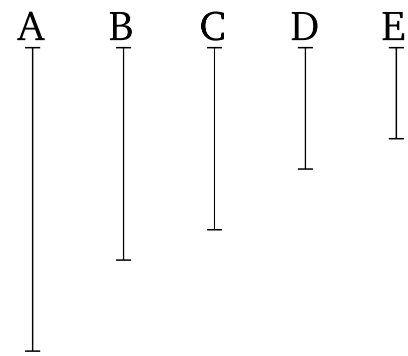
Numbers prime to one another are the least of those (numbers) having the same ratio as them.

Let *A* and *B* be numbers prime to one another. I say that *A* and *B* are the least of those (numbers) having the same ratio as them.

For if not then there will be some numbers less than *A* and *B* which are in the same ratio as *A* and *B*. Let them be *C* and *D*.



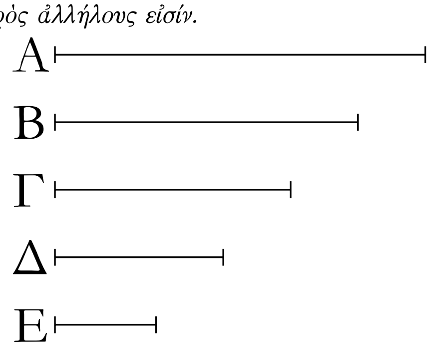
Ἐπεὶ οὖν οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάττων τὸν ἐλάττονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ἰσάκις ἄρα ὁ Γ τὸν Α μετρεῖ καὶ ὁ Δ τὸν Β. ὁσάκις δὴ ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. καὶ ὁ Δ ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας. καὶ ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, καί ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας. διὰ τὰ αὐτὰ δὴ ὁ Ε καὶ τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας. ὁ Ε ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσονταί τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β. οἱ Α, Β ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.



Therefore, since the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following—*C* thus measures *A* the same number of times that *D* (measures) *B* [Prop. 7.20]. So as many times as *C* measures *A*, so many units let there be in *E*. Thus, *D* also measures *B* according to the units in *E*. And since *C* measures *A* according to the units in *E*, *E* thus also measures *A* according to the units in *C* [Prop. 7.16]. So, for the same (reasons), *E* also measures *B* according to the units in *D* [Prop. 7.16]. Thus, *E* measures *A* and *B*, which are prime to one another. The very thing is impossible. Thus, there cannot be any numbers less than *A* and *B* which are in the same ratio as *A* and *B*. Thus, *A* and *B* are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

κβʹ.

Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς πρῶτοι πρὸς ἀλλήλους εἰσίν.



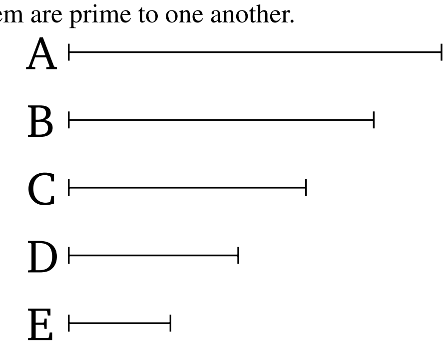
Ἔστωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β· λέγω, ὅτι οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσι πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Γ. καὶ ὁσάκις μὲν ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ, ὁσάκις δὲ ὁ Γ τὸν Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.

Ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, ὁ Γ ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ

Proposition 22

The least numbers of those (numbers) having the same ratio as them are prime to one another.



Let *A* and *B* be the least numbers of those (numbers) having the same ratio as them. I say that *A* and *B* are prime to one another.

For if they are not prime to one another, (then) some number will measure them. Let it (so measure them), and let it be *C*. And as many times as *C* measures *A*, so many units let there be in *D*. And as many times as *C* measures *B*, so many units let there be in *E*.

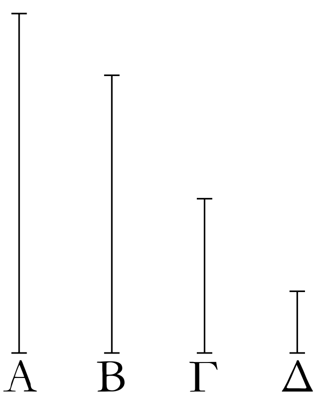
δὴ καὶ ὁ Γ τὸν Ε πολλαπλασιάσας τὸν Β πεποίηκεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς Δ, Ε πολλαπλασιάσας τοὺς Α, Β πεποίηκεν· ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Β· οἱ Δ, Ε ἄρα τοῖς Α, Β ἐν τῷ αὐτῷ λόγῳ εἰσὶν ἐλάσσονες ὄντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Since C measures A according to the units in D, C has thus made A (by) multiplying D [Def. 7.15]. So, for the same (reasons), C has also made B (by) multiplying E. So the number C has made A and B (by) multiplying the two numbers D and E (respectively). Thus, as D is to E, so A (is) to B [Prop. 7.17]. Thus, D and E are in the same ratio as A and B, being less than them. The very thing is impossible. Thus, some number does not measure the numbers A and B. Thus, A and B are prime to one another. (Which is) the very thing it was required to show.

κγʹ.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ τὸν ἕνα αὐτῶν μετρῶν ἀριθμὸς πρὸς τὸν λοιπὸν πρῶτος ἔσται.

Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, τὸν δὲ Α μετρείτω τις ἀριθμὸς ὁ Γ· λέγω, ὅτι καὶ οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους εἰσίν.

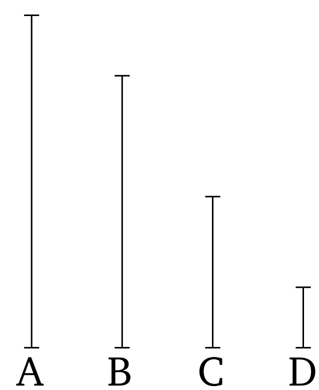


Εἰ γὰρ μή εἰσιν οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Β ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. ἐπεὶ ὁ Δ τὸν Γ μετρεῖ, ὁ δὲ Γ τὸν Α μετρεῖ, καὶ ὁ Δ ἄρα τὸν Α μετρεῖ. μετρεῖ δὲ καὶ τὸν Β· ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Γ, Β ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 23

If two numbers are prime to one another, (then) a number measuring one of them will be prime to the remaining (one).

Let A and B be two numbers (which are) prime to one another, and let some number C measure A. I say that C and B are also prime to one another.



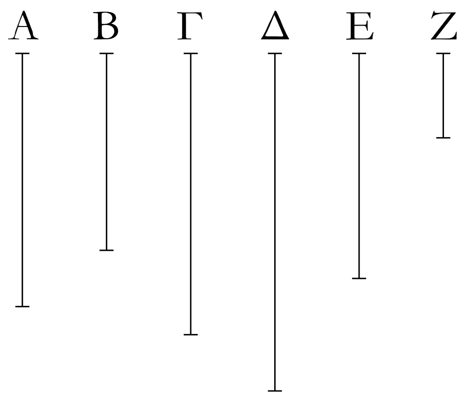
For if C and B are not prime to one another, (then) [some] number will measure C and B. Let it (so) measure (them), and let it be D. Since D measures C, and C measures A, D thus also measures A. And (D) also measures B. Thus, D measures A and B, which are prime to one another. The very thing is impossible. Thus, some number does not measure the numbers C and B. Thus, C and B are prime to one another. (Which is) the very thing it was required to show.

κδʹ.

Ἐὰν δύο ἀριθμοὶ πρός τινα ἀριθμὸν πρῶτοι ὦσιν, καὶ ὁ ἐξ αὐτῶν γενόμενος πρὸς τὸν αὐτὸν πρῶτος ἔσται.

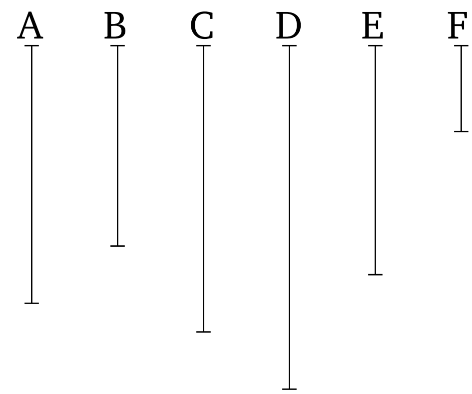
Proposition 24

If two numbers are prime to some number, (then) the number created from (multiplying) the former (two numbers) will also be prime to the latter (number).



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρός τινα ἀριθμὸν τὸν Γ πρῶτοι ἔστωσαν, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ ποιείτω· λέγω, ὅτι οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Δ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Ε. καὶ ἐπεὶ οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν, τὸν δὲ Γ μετρεῖ τις ἀριθμὸς ὁ Ε, οἱ Α, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ὁσάκις δὴ ὁ Ε τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ζ· καὶ ὁ Ζ ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας. ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Ε, Ζ τῷ ἐκ τῶν Α, Β. ἐὰν δὲ ὁ ὑπὸ τῶν ἄκρων ἴσος ᾖ τῷ ὑπὸ τῶν μέσων, οἱ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν· ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Ζ. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· ὁ Ε ἄρα τὸν Β μετρεῖ. μετρεῖ δὲ καὶ τὸν Γ· ὁ Ε ἄρα τοὺς Β, Γ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Γ, Δ ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ Γ, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.



For let *A* and *B* be two numbers (which are both) prime to some number *C*. And let *A* make *D* (by) multiplying *B*. I say that *C* and *D* are prime to one another.

For if *C* and *D* are not prime to one another, (then) [some] number will measure *C* and *D*. Let it (so) measure them, and let it be *E*. And since *C* and *A* are prime to one another, and some number *E* measures *C*, *A* and *E* are thus prime to one another [Prop. 7.23]. So as many times as *E* measures *D*, so many units let there be in *F*. Thus, *F* also measures *D* according to the units in *E* [Prop. 7.16]. Thus, *E* has made *D* (by) multiplying *F* [Def. 7.15]. But, in fact, *A* has also made *D* (by) multiplying *B*. Thus, the (number created) from (multiplying) *E* and *F* is equal to the (number created) from (multiplying) *A* and *B*. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two then) the four numbers are proportional [Prop. 6.15]. Thus, as *E* is to *A*, so *B* (is) to *F*. And *A* and *E* (are) prime (to one another). And (numbers) prime (to one another) are also the least (of those numbers having the same ratio) [Prop. 7.21]. And the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, *E* measures *B*. And it also measures *C*. Thus, *E* measures *B* and *C*, which are prime to one another. The very thing is impossible. Thus, some number cannot measure the numbers *C* and *D*. Thus, *C* and *D* are prime to one another. (Which is) the very thing it was required to show.

κεʹ.

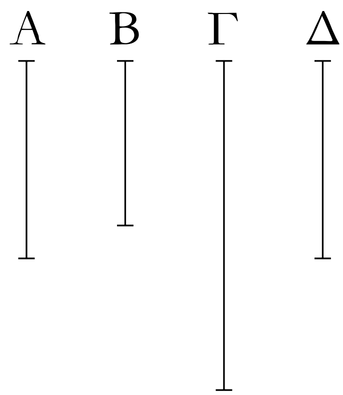
Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτος ἔσται.

Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι οἱ Β, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν.

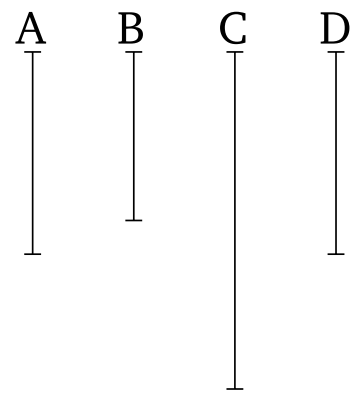
Proposition 25

If two numbers are prime to one another, (then) the number created from (squaring) one of them will be prime to the remaining (number).

Let *A* and *B* be two numbers (which are) prime to one another. And let *A* make *C* (by) multiplying itself. I say that *B* and *C* are prime to one another.



Κείσθω γὰρ τῷ Α ἴσος ὁ Δ. ἐπεὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, ἴσος δὲ ὁ Α τῷ Δ, καί οἱ Δ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἑκάτερος ἄρα τῶν Δ, Α πρὸς τὸν Β πρῶτός ἐστιν· καὶ ὁ ἐκ τῶν Δ, Α ἄρα γενόμενος πρὸς τὸν Β πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Δ, Α γενόμενος ἀριθμός ἐστιν ὁ Γ. οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

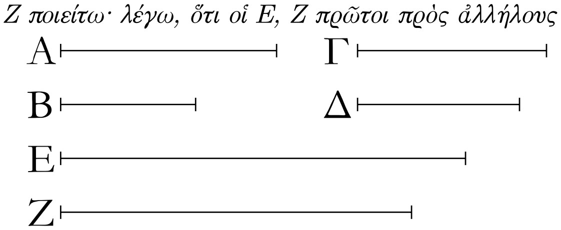


For let *D* be made equal to *A*. Since *A* and *B* are prime to one another, and *A* (is) equal to *D*, *D* and *B* are thus also prime to one another. Thus, *D* and *A* are each prime to *B*. Thus, the (number) created from (multilying) *D* and *A* will also be prime to *B* [Prop. 7.24]. And *C* is the number created from (multiplying) *D* and *A*. Thus, *C* and *B* are prime to one another. (Which is) the very thing it was required to show.

κςʹ.

Ἐὰν δύο ἀριθμοὶ πρὸς δύο ἀριθμοὺς ἀμφότεροι πρὸς ἑκάτερον πρῶτοι ὦσιν, καὶ οἱ ἐξ αὐτῶν γενόμενοι πρῶτοι πρὸς ἀλλήλους ἔσονται.

Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς δύο ἀριθμοὺς τοὺς Γ, Δ ἀμφότεροι πρὸς ἑκάτερον πρῶτοι ἔστωσαν, καὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ Γ τὸν Δ πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι οἱ Ε, Ζ πρῶτοι πρὸς ἀλλήλους εἰσίν.

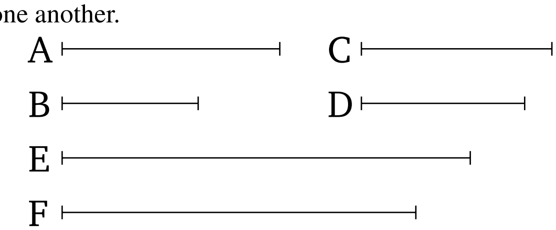


Ἐπεὶ γὰρ ἑκάτερος τῶν Α, Β πρὸς τὸν Γ πρῶτός ἐστιν, καὶ ὁ ἐκ τῶν Α, Β ἄρα γενόμενος πρὸς τὸν Γ πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Α, Β γενόμενός ἐστιν ὁ Ε· οἱ Ε, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ οἱ Ε, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν. ἑκάτερος ἄρα τῶν Γ, Δ πρὸς τὸν Ε πρῶτός ἐστιν. καὶ ὁ ἐκ τῶν Γ, Δ ἄρα γενόμενος πρὸς τὸν Ε πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Γ, Δ γενόμενός ἐστιν ὁ Ζ. οἱ Ε, Ζ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 26

If two numbers are both prime to each of two numbers, (then) the (numbers) created from (multiplying) them will also be prime to one another.

For let two numbers, *A* and *B*, both be prime to each of two numbers, *C* and *D*. And let *A* make *E* (by) multiplying *B*, and let *C* make *F* (by) multiplying *D*. I say that *E* and *F* are prime to one another.



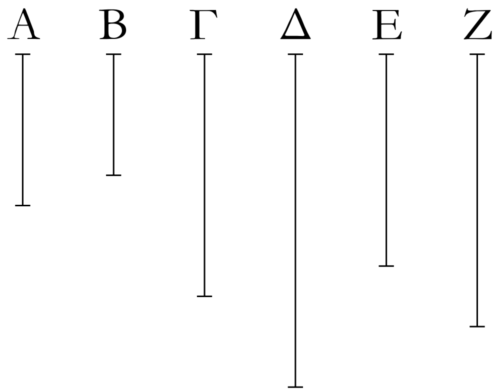
For since *A* and *B* are each prime to *C*, the (number) created from (multiplying) *A* and *B* will thus also be prime to *C* [Prop. 7.24]. And *E* is the (number) created from (multiplying) *A* and *B*. Thus, *E* and *C* are prime to one another. So, for the same (reasons), *E* and *D* are also prime to one another. Thus, *C* and *D* are each prime to *E*. Thus, the (number) created from (multiplying) *C* and *D* will also be prime to *E* [Prop. 7.24]. And *F* is the (number) created from (multiplying) *C* and *D*. Thus, *E* and *F* are prime to one another. (Which is) the very thing it was required to show.

κζʹ.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ πολλαπλασιάσας ἑκάτερος ἑαυτὸν ποιῇ τινα, οἱ γενόμενοι ἐξ αὐτῶν πρῶτοι πρὸς ἀλλήλους ἔσονται, κἂν οἱ ἐξ ἀρχῆς τοὺς γε-

Proposition 27†

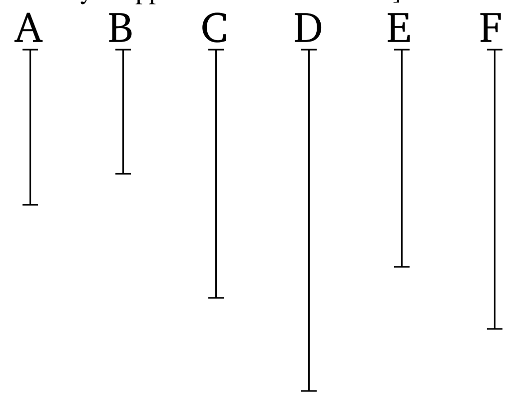
If two numbers are prime to one another and each makes some (number by) multiplying itself, (then) the numbers created from them will be prime to one another, and if the original *νομένους πολλαπλασιάσαντες ποιῶσί τινας, κἀκεῖνοι πρῶτοι πρὸς ἀλλήλους ἔσονται [καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].*



Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ ποιείτω, τὸν δὲ Γ πολλαπλασιάσας τὸν Δ ποιείτω, ὁ δὲ Β ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιείτω, τὸν δὲ Ε πολλαπλασιάσας τὸν Ζ ποιείτω· λέγω, ὅτι οἵ τε Γ, Ε καὶ οἱ Δ, Ζ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Ἐπεὶ γὰρ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν, οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐπεὶ οὖν οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Γ, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. πάλιν, ἐπεὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, οἱ Α, Ε ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐπεὶ οὖν δύο ἀριθμοὶ οἱ Α, Γ πρὸς δύο ἀριθμοὺς τοὺς Β, Ε ἀμφότεροι πρὸς ἑκάτερον πρῶτοί εἰσιν, καὶ ὁ ἐκ τῶν Α, Γ ἄρα γενόμενος πρὸς τὸν ἐκ τῶν Β, Ε πρῶτός ἐστιν. καί ἐστιν ὁ μὲν ἐκ τῶν Α, Γ ὁ Δ, ὁ δὲ ἐκ τῶν Β, Ε ὁ Ζ. οἱ Δ, Ζ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

(numbers) make some (more numbers by) multiplying the created (numbers, then) these will also be prime to one another [and this always happens with the extremes].



Let *A* and *B* be two numbers prime to one another, and let *A* make *C* (by) multiplying itself, and let it make *D* (by) multiplying *C*. And let *B* make *E* (by) multiplying itself, and let it make *F* by multiplying *E*. I say that *C* and *E*, and *D* and *F*, are prime to one another.

For since *A* and *B* are prime to one another, and *A* has made *C* (by) multiplying itself, *C* and *B* are thus prime to one another [Prop. 7.25]. Therefore, since *C* and *B* are prime to one another, and *B* has made *E* (by) multiplying itself, *C* and *E* are thus prime to one another [Prop. 7.25]. Again, since *A* and *B* are prime to one another, and *B* has made *E* (by) multiplying itself, *A* and *E* are thus prime to one another [Prop. 7.25]. Therefore, since the two numbers *A* and *C* are both prime to each of the two numbers *B* and *E*, the (number) created from (multiplying) *A* and *C* is thus prime to the (number created) from (multiplying) *B* and *E* [Prop. 7.26]. And *D* is the (number created) from (multiplying) *A* and *C*, and *F* the (number created) from (multiplying) *B* and *E*. Thus, *D* and *F* are prime to one another. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if a is prime to b, then a^2 is also prime to b^2, as well as a^3 to b^3, etc., where all symbols denote numbers.

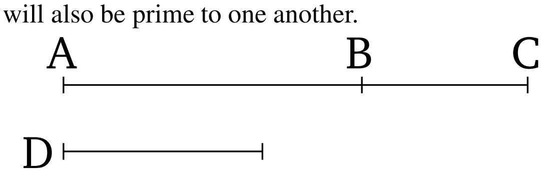
κη΄.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ συναμφότερος πρὸς ἑκάτερον αὐτῶν πρῶτος ἔσται· καὶ ἐὰν συναμφότερος πρὸς ἕνα τινὰ αὐτῶν πρῶτος ᾖ, καὶ οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.

Συγκείσθωσαν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ ΑΒ, ΒΓ· λέγω, ὅτι καὶ συναμφότερος ὁ ΑΓ πρὸς ἑκάτερον τῶν ΑΒ, ΒΓ πρῶτός ἐστιν.

Proposition 28

If two numbers are prime to one another, (then) their sum will also be prime to each of them. And if the sum (of two numbers) is prime to any one of them, (then) the original numbers will also be prime to one another.



For let the two numbers, *AB* and *BC*, (which are) prime to one another, be laid down together. I say that their sum *AC* is also prime to each of *AB* and *BC*.

Εἰ γὰρ μή εἰσιν οἱ ΓΑ, ΑΒ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς ΓΑ, ΑΒ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. ἐπεὶ οὖν ὁ Δ τοὺς ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸν ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸν ΒΑ· ὁ Δ ἄρα τοὺς ΑΒ, ΒΓ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΓΑ, ΑΒ ἀριθμοὺς ἀριθμός τις μετρήσει· οἱ ΓΑ, ΑΒ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ οἱ ΑΓ, ΓΒ πρῶτοι πρὸς ἀλλήλους εἰσίν. ὁ ΓΑ ἄρα πρὸς ἑκάτερον τῶν ΑΒ, ΒΓ πρῶτός ἐστιν.

Ἔστωσαν δὴ πάλιν οἱ ΓΑ, ΑΒ πρῶτοι πρὸς ἀλλήλους· λέγω, ὅτι καὶ οἱ ΑΒ, ΒΓ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μή εἰσιν οἱ ΑΒ, ΒΓ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς ΑΒ, ΒΓ ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. καὶ ἐπεὶ ὁ Δ ἑκάτερον τῶν ΑΒ, ΒΓ μετρεῖ, καὶ ὅλον ἄρα τὸν ΓΑ μετρήσει. μετρεῖ δὲ καὶ τὸν ΑΒ· ὁ Δ ἄρα τοὺς ΓΑ, ΑΒ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς ΑΒ, ΒΓ ἀριθμοὺς ἀριθμός τις μετρήσει. οἱ ΑΒ, ΒΓ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

For if CA and AB are not prime to one another, (then) some number will measure CA and AB. Let it (so) measure (them), and let it be D. Therefore, since D measures CA and AB, it will thus also measure the remainder BC. And it also measures BA. Thus, D measures AB and BC, which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers CA and AB. Thus, CA and AB are prime to one another. So, for the same (reasons), AC and CB are also prime to one another. Thus, CA is prime to each of AB and BC.

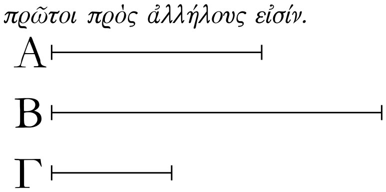
So, again, let CA and AB be prime to one another. I say that AB and BC are also prime to one another.

For if AB and BC are not prime to one another, (then) some number will measure AB and BC. Let it (so) measure (them), and let it be D. And since D measures each of AB and BC, it will thus also measure the whole of CA. And it also measures AB. Thus, D measures CA and AB, which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers AB and BC. Thus, AB and BC are prime to one another. (Which is) the very thing it was required to show.

κθʹ.

Ἅπας πρῶτος ἀριθμὸς πρὸς ἅπαντα ἀριθμόν, ὃν μὴ μετρεῖ, πρῶτός ἐστιν.

Ἔστω πρῶτος ἀριθμὸς ὁ Α καὶ τὸν Β μὴ μετρείτω· λέγω, ὅτι οἱ Β, Α πρῶτοι πρὸς ἀλλήλους εἰσίν.

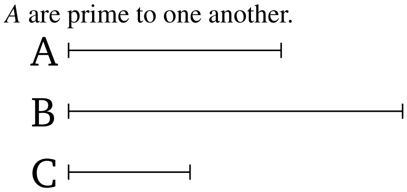


Εἰ γὰρ μή εἰσιν οἱ Β, Α πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω ὁ Γ. ἐπεὶ ὁ Γ τὸν Β μετρεῖ, ὁ δὲ Α τὸν Β οὐ μετρεῖ, ὁ Γ ἄρα τῷ Α οὔκ ἐστιν ὁ αὐτός. καὶ ἐπεὶ ὁ Γ τοὺς Β, Α μετρεῖ, καὶ τὸν Α ἄρα μετρεῖ πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Β, Α μετρήσει τις ἀριθμός. οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 29

Every prime number is prime to every number which it does not measure.

Let A be a prime number, and let it not measure B. I say that B and A are prime to one another.



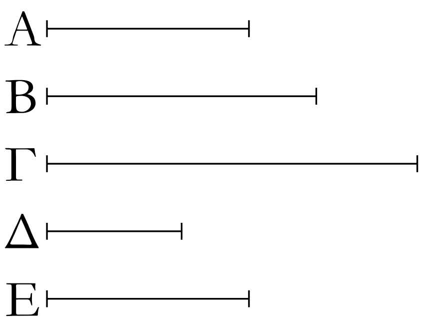
For if B and A are not prime to one another, (then) some number will measure them. Let C measure (them). Since C measures B, and A does not measure B, C is thus not the same as A. And since C measures B and A, it thus also measures A, which is prime, (despite) not being the same as it. The very thing is impossible. Thus, some number cannot measure (both) B and A. Thus, A and B are prime to one another. (Which is) the very thing it was required to show.

λʹ.

Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γενόμενον ἐξ αὐτῶν μετρῇ τις πρῶτος ἀριθμός, καὶ ἕνα τῶν ἐξ ἀρχῆς μετρήσει.

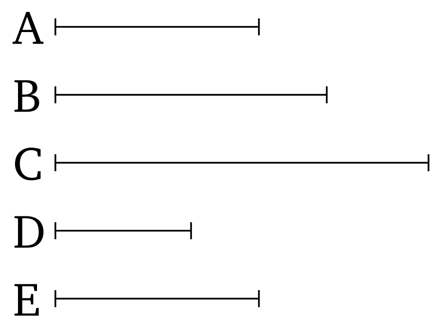
Proposition 30

If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, (then) it will also measure one of the original (numbers).



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πολλαπλασιάσαντες ἀλλήλους τὸν Γ ποιείτωσαν, τὸν δὲ Γ μετρείτω τις πρῶτος ἀριθμὸς ὁ Δ· λέγω, ὅτι ὁ Δ ἕνα τῶν Α, Β μετρεῖ.

Τὸν γὰρ Α μὴ μετρείτω· καί ἐστι πρῶτος ὁ Δ· οἱ Α, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ὁσάκις ὁ Δ τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. ἐπεὶ οὖν ὁ Δ τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, ὁ Δ ἄρα τὸν Ε πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Δ, Ε τῷ ἐκ τῶν Α, Β. ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Ε. οἱ δὲ Δ, Α πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· ὁ Δ ἄρα τὸν Β μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἐὰν τὸν Β μὴ μετρῇ, τὸν Α μετρήσει. ὁ Δ ἄρα ἕνα τῶν Α, Β μετρεῖ· ὅπερ ἔδει δεῖξαι.



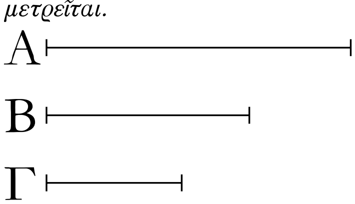
For let two numbers A and B make C (by) multiplying one another, and let some prime number D measure C. I say that D measures one of A and B.

For let it not measure A. And since D is prime, A and D are thus prime to one another [Prop. 7.29]. And as many times as D measures C, so many units let there be in E. Therefore, since D measures C according to the units E, D has thus made C (by) multiplying E [Def. 7.15]. But, in fact, A has also made C (by) multiplying B. Thus, the (number created) from (multiplying) D and E is equal to the (number created) from (multiplying) A and B. Thus, as D is to A, so B (is) to E [Prop. 7.19]. And D and A (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, D measures B. So, similarly, we can also show that if (D) does not measure B, (then) it will measure A. Thus, D measures one of A and B. (Which is) the very thing it was required to show.

λα΄.

Ἅπας σύνθεντος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

Ἔστω σύνθεντος ἀριθμὸς ὁ Α· λέγω, ὅτι ὁ Α ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

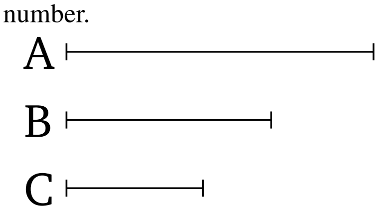


Ἐπεὶ γὰρ σύνθετός ἐστιν ὁ Α, μετρήσει τις αὐτὸν ἀριθμός. μετρείτω, καὶ ἔστω ὁ Β. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Β, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. μετρείτω, καὶ ἔστω ὁ Γ. καὶ ἐπεὶ ὁ Γ τὸν Β μετρεῖ, ὁ δὲ Β τὸν Α μετρεῖ, καὶ ὁ Γ ἄρα τὸν Α μετρεῖ. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Γ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. τοιαύτης δὴ γινομένης ἐπισκέψεως ληφθήσεταί τις πρῶτος ἀριθμός, ὃς μετρήσει. εἰ

Proposition 31

Every composite number is measured by some prime number.

Let A be a composite number. I say that A is measured by some prime number.



For since A is composite, some number will measure it. Let it (so) measure (A), and let it be B. And if B is prime then that which was prescribed has happened. And if (B is) composite, (then) some number will measure it. Let it (so) measure (B), and let it be C. And since C measures B, and B measures A, C thus also measures A. And if C is prime then that which was prescribed has happened. And if (C is) composite, (then) some number will measure it. So, in this manner of continued *γὰρ οὐ ληφθήσεται, μετρήσουσι τὸν Α ἀριθμὸν ἄπειροι ἀριθμοί, ὧν ἕτερος ἑτέρου ἐλάσσων ἐστίν· ὅπερ ἐστὶν ἀδύνατον ἐν ἀριθμοῖς. ληφθήσεταί τις ἄρα πρῶτος ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ, ὃς καὶ τὸν Α μετρήσει.*

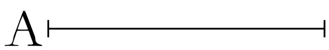
Ἅπας ἄρα σύνθεντος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

investigation, some prime number will be found which will measure (the number preceding it, which will also measure *A*). And if (such a number) cannot be found, (then) an infinite (series of) numbers, each of which is less than the preceding, will measure the number *A*. The very thing is impossible for numbers. Thus, some prime number will (eventually) be found which will measure the (number) preceding it, which will also measure *A*.

Thus, every composite number is measured by some prime number. (Which is) the very thing it was required to show.

λβ′.

Ἅπας ἀριθμὸς ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.



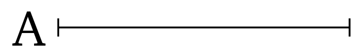
Ἔστω ἀριθμὸς ὁ Α· λέγω, ὅτι ὁ Α ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται.

Εἰ μὲν οὖν πρῶτός ἐστιν ὁ Α, γεγονὸς ἂν εἴη τό ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν πρῶτος ἀριθμός.

Ἅπας ἄρα ἀριθμὸς ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

Proposition 32

Every number is either prime, or is measured by some prime number.



Let *A* be a number. I say that *A* is either prime, or is measured by some prime number.

In fact, if *A* is prime, (then) that which was prescribed has happened. And if (it is) composite, (then) some prime number will measure it [Prop. 7.31].

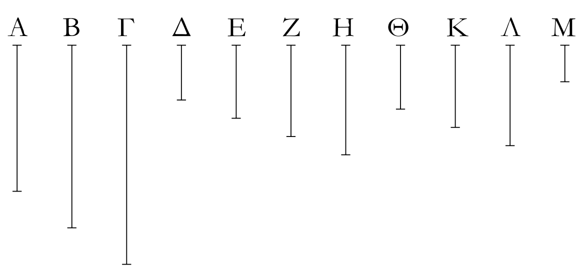
Thus, every number is either prime, or is measured by some prime number. (Which is) the very thing it was required to show.

λγ′.

Ἀριθμῶν δοθέντων ὁποσωνοῦν εὑρεῖν τοὺς ἐλαχίστους τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Ἔστωσαν οἱ δοθέντες ὁποσοιοῦν ἀριθμοὶ οἱ Α, Β, Γ· δεῖ δὴ εὑρεῖν τοὺς ἐλαχίστους τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ.

Οἱ Α, Β, Γ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. εἰ μὲν οὖν οἱ Α, Β, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.



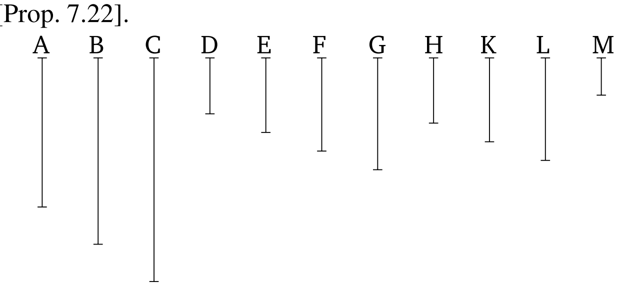
Εἰ δὲ οὔ, εἰλήφθω τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον ὁ Δ, καὶ ὁσάκις ὁ Δ ἕκαστον τῶν Α, Β, Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν ἑκάστῳ τῶν Ε, Ζ, Η. καὶ ἕκαστος ἄρα τῶν Ε, Ζ, Η ἕκαστον τῶν Α, Β, Γ μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας. οἱ Ε, Ζ, Η ἄρα τοὺς Α, Β, Γ ἰσάκις μετροῦσιν· οἱ Ε, Ζ, Η ἄρα τοῖς Α, Β, Γ ἐν τῷ αὐτῷ λόγῳ εἰσίν. λέγω δή, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μή εἰσιν οἱ Ε, Ζ, Η ἐλάχιστοι τῶν

Proposition 33

To find the least of those (numbers) having the same ratio as any given multitude of numbers.

Let *A*, *B*, and *C* be any given multitude of numbers. So it is required to find the least of those (numbers) having the same ratio as *A*, *B*, and *C*.

For *A*, *B*, and *C* are either prime to one another, or not. In fact, if *A*, *B*, and *C* are prime to one another, (then) they are the least of those (numbers) having the same ratio as them [Prop. 7.22].



And if not, let the greatest common measure, *D*, of *A*, *B*, and *C* have be taken [Prop. 7.3]. And as many times as *D* measures *A*, *B*, *C*, so many units let there be in *E*, *F*, *G*, respectively. And thus *E*, *F*, *G* measure *A*, *B*, *C*, respectively, according to the units in *D* [Prop. 7.15]. Thus, *E*, *F*, *G* measure *A*, *B*, *C* (respectively) an equal number of times. Thus, *E*, *F*, *G* are in the same ratio as *A*, *B*, *C* (respectively) [Def. 7.20].

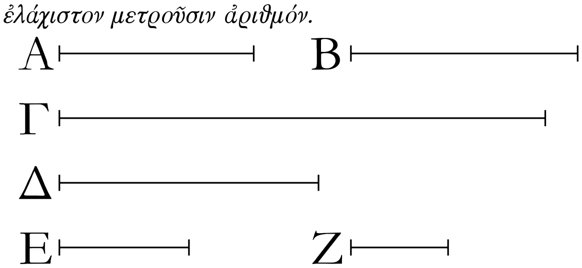
τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ, ἔσονται [τινες] τῶν Ε, Ζ, Η ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β, Γ. ἔστωσαν οἱ Θ, Κ, Λ· ἰσάκις ἄρα ὁ Θ τὸν Α μετρεῖ καὶ ἑκάτερος τῶν Κ, Λ ἑκάτερον τῶν Β, Γ. ὁσάκις δὲ ὁ Θ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Μ· καὶ ἑκάτερος ἄρα τῶν Κ, Λ ἑκάτερον τῶν Β, Γ μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας. καὶ ἐπεὶ ὁ Θ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας, καὶ ὁ Μ ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Θ μονάδας. διὰ τὰ αὐτὰ δὴ ὁ Μ καὶ ἑκάτερον τῶν Β, Γ μετρεῖ κατὰ τὰς ἐν ἑκατέρῳ τῶν Κ, Λ μονάδας· ὁ Μ ἄρα τοὺς Α, Β, Γ μετρεῖ. καὶ ἐπεὶ ὁ Θ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Μ μονάδας, ὁ Θ ἄρα τὸν Μ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Ε, Δ τῷ ἐκ τῶν Θ, Μ. ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Μ πρὸς τὸν Δ. μείζων δὲ ὁ Ε τοῦ Θ· μείζων ἄρα καὶ ὁ Μ τοῦ Δ. καὶ μετρεῖ τοὺς Α, Β, Γ· ὅπερ ἐστὶν ἀδύνατον· ὑπόκειται γὰρ ὁ Δ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον. οὐκ ἄρα ἔσονταί τινες τῶν Ε, Ζ, Η ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β, Γ. οἱ Ε, Ζ, Η ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ· ὅπερ ἔδει δεῖξαι.

So I say that (they are) also the least (of those numbers having the same ratio as *A*, *B*, *C*). For if *E*, *F*, *G* are not the least of those (numbers) having the same ratio as *A*, *B*, *C* (respectively, then) there will be [some] numbers less than *E*, *F*, *G* which are in the same ratio as *A*, *B*, *C* (respectively). Let them be *H*, *K*, *L*. Thus, *H* measures *A* the same number of times that *K*, *L* also measure *B*, *C*, respectively. And as many times as *H* measures *A*, so many units let there be in *M*. Thus, *K*, *L* measure *B*, *C*, respectively, according to the units in *M*. And since *H* measures *A* according to the units in *M*, *M* thus also measures *A* according to the units in *H* [Prop. 7.15]. So, for the same (reasons), *M* also measures *B*, *C* according to the units in *K*, *L*, respectively. Thus, *M* measures *A*, *B*, and *C*. And since *H* measures *A* according to the units in *M*, *H* has thus made *A* (by) multiplying *M*. So, for the same (reasons), *E* has also made *A* (by) multiplying *D*. Thus, the (number created) from (multiplying) *E* and *D* is equal to the (number created) from (multiplying) *H* and *M*. Thus, as *E* (is) to *H*, so *M* (is) to *D* [Prop. 7.19]. And *E* (is) greater than *H*. Thus, *M* (is) also greater than *D* [Prop. 5.13]. And (*M*) measures *A*, *B*, and *C*. The very thing is impossible. For *D* was assumed (to be) the greatest common measure of *A*, *B*, and *C*. Thus, there cannot be any numbers less than *E*, *F*, *G* which are in the same ratio as *A*, *B*, *C* (respectively). Thus, *E*, *F*, *G* are the least of (those numbers) having the same ratio as *A*, *B*, *C* (respectively). (Which is) the very thing it was required to show.

λδ΄.

Δύο ἀριθμῶν δοθέντων εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β· δεῖ δὴ εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

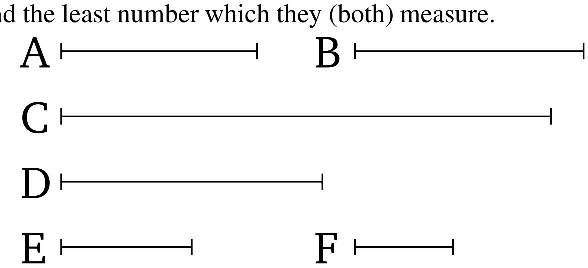


Οἱ Α, Β γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. ἔστωσαν πρότερον οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· καὶ ὁ Β ἄρα τὸν Α πολλαπλασιάσας τὸν Γ πεποίηκεν. οἱ Α, Β ἄρα τὸν Γ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα ἀριθμὸν οἱ Α, Β ἐλάσσονα ὄντα τοῦ Γ. μετρείτωσαν τὸν Δ. καὶ ὁσάκις ὁ Α τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε, ὁσάκις δὲ ὁ Β τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ζ. ὁ μὲν Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ Β τὸν

Proposition 34

To find the least number which two given numbers (both) measure.

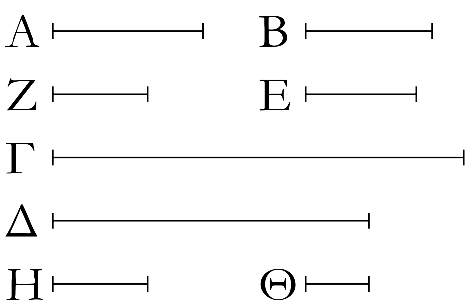
Let *A* and *B* be the two given numbers. So it is required to find the least number which they (both) measure.

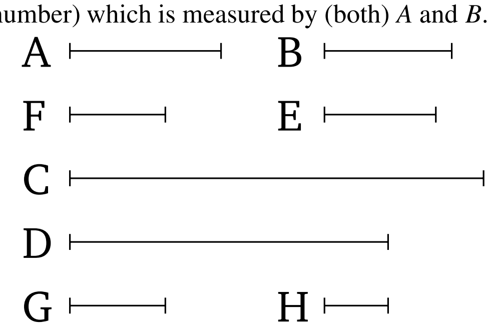


For *A* and *B* are either prime to one another, or not. Let them, first of all, be prime to one another. And let *A* make *C* (by) multiplying *B*. Thus, *B* has also made *C* (by) multiplying *A* [Prop. 7.16]. Thus, *A* and *B* (both) measure *C*. So I say that (*C*) is also the least (number which they both measure). For if not, *A* and *B* will (both) measure some (other) number which is less than *C*. Let them (both) measure *D* (which is less than *C*). And as many times as *A* measures *D*, so many units let there be in *E*. And as many times as *B* measures *D*, so many

Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Α, Ε τῷ ἐκ τῶν Β, Ζ. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Ε. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα· ὁ Β ἄρα τὸν Ε μετρεῖ, ὡς ἑπόμενος ἑπόμενον. καὶ ἐπεὶ ὁ Α τοὺς Β, Ε πολλαπλασιάσας τοὺς Γ, Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Β πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ. μετρεῖ δὲ ὁ Β τὸν Ε· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β μετροῦσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάχιστος ὢν ὑπὸ τῶν Α, Β μετρεῖται.

units let there be in *F*. Thus, *A* has made *D* (by) multiplying *E*, and *B* has made *D* (by) multiplying *F*. Thus, the (number created) from (multiplying) *A* and *E* is equal to the (number created) from (multiplying) *B* and *F*. Thus, as *A* (is) to *B*, so *F* (is) to *E* [Prop. 7.19]. And *A* and *B* are prime (to one another), and prime (numbers) are the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, *B* measures *E*, as the following (number measuring) the following. And since *A* has made *C* and *D* (by) multiplying *B* and *E* (respectively), thus as *B* is to *E*, so *C* (is) to *D* [Prop. 7.17]. And *B* measures *E*. Thus, *C* also measures *D*, the greater (measuring) the lesser. The very thing is impossible. Thus, *A* and *B* do not (both) measure some number which is less than *C*. Thus, *C* is the least (number) which is measured by (both) *A* and *B*.





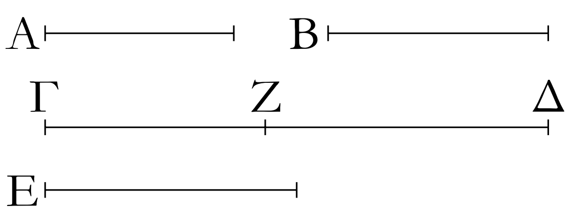
μὴ ἔστωσαν δὴ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β οἱ Ζ, Ε· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Α, Ε τῷ ἐκ τῶν Β, Ζ. καὶ ὁ Α τὸν Ε πολλαπλασιάσας τὸν Γ ποιείτω· καὶ ὁ Β ἄρα τὸν Ζ πολλαπλασιάσας τὸν Γ πεποίηκεν· οἱ Α, Β ἄρα τὸν Γ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα ἀριθμὸν οἱ Α, Β ἐλάσσονα ὄντα τοῦ Γ. μετρείτωσαν τὸν Δ. καὶ ὁσάκις μὲν ὁ Α τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Η, ὁσάκις δὲ ὁ Β τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Θ. ὁ μὲν Α ἄρα τὸν Η πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ Β τὸν Θ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Α, Η τῷ ἐκ τῶν Β, Θ· ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Θ πρὸς τὸν Η. ὡς δὲ ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Ζ πρὸς τὸν Ε, οὕτως ὁ Θ πρὸς τὸν Η. οἱ δὲ Ζ, Ε ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα· ὁ Ε ἄρα τὸν Η μετρεῖ. καὶ ἐπεὶ

So let *A* and *B* be not prime to one another. And let the least numbers, *F* and *E*, be taken having the same ratio as *A* and *B* (respectively) [Prop. 7.33]. Thus, the (number created) from (multiplying) *A* and *E* is equal to the (number created) from (multiplying) *B* and *F* [Prop. 7.19]. And let *A* make *C* (by) multiplying *E*. Thus, *B* has also made *C* (by) multiplying *F*. Thus, *A* and *B* (both) measure *C*. So I say that (*C*) is also the least (number which they both measure). For if not, *A* and *B* will (both) measure some number which is less than *C*. Let them (both) measure *D* (which is less than *C*). And as many times as *A* measures *D*, so many units let there be in *G*. And as many times as *B* measures *D*, so many units let there be in *H*. Thus, *A* has made *D* (by) multiplying *G*, and *B* has made *D* (by) multiplying *H*. Thus, the (number created) from (multiplying) *A* and *G* is equal to the (number created) from (multiplying) *B* and *H*. Thus, as *A* is to *B*, so *H* (is) to *G* *ὁ Α τοὺς Ε, Η πολλαπλασιάσας τοὺς Γ, Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ δὲ Ε τὸν Η μετρεῖ· καὶ ὁ Γ ἄρα τὸν Δ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάχιστος ὢν ὑπὸ τῶν Α, Β μετρεῖται· ὅπερ ἔπει δεῖξαι.*

[Prop. 7.19]. And as *A* (is) to *B*, so *F* (is) to *E*. Thus, also, as *F* (is) to *E*, so *H* (is) to *G*. And *F* and *E* are the least (numbers having the same ratio as *A* and *B*), and the least (numbers) measure those (numbers) having the same ratio an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, *E* measures *G*. And since *A* has made *C* and *D* (by) multiplying *E* and *G* (respectively), thus as *E* is to *G*, so *C* (is) to *D* [Prop. 7.17]. And *E* measures *G*. Thus, *C* also measures *D*, the greater (measuring) the lesser. The very thing is impossible. Thus, *A* and *B* do not (both) measure some (number) which is less than *C*. Thus, *C* (is) the least (number) which is measured by (both) *A* and *B*. (Which is) the very thing it was required to show.

λε'.

Ἐὰν δύο ἀριθμοὶ ἀριθμόν τινα μετρῶσιν, καὶ ὁ ἐλάχιστος ὑπ᾽ αὐτῶν μετρούμενος τὸν αὐτὸν μετρήσει.

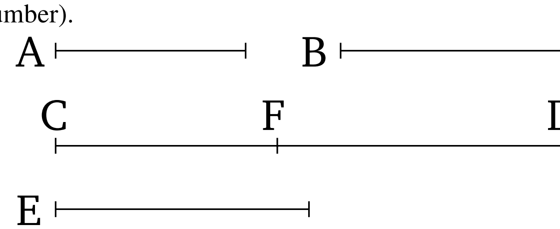


Δύο γὰρ ἀριθμοὶ οἱ Α, Β ἀριθμόν τινα τὸν ΓΔ μετρείτωσαν, ἐλάχιστον δὲ τὸν Ε· λέγω, ὅτι καὶ ὁ Ε τὸν ΓΔ μετρεῖ.

Εἰ γὰρ οὐ μετρεῖ ὁ Ε τὸν ΓΔ, ὁ Ε τὸν ΔΖ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΓΖ. καὶ ἐπεὶ οἱ Α, Β τὸν Ε μετροῦσιν, ὁ δὲ Ε τὸν ΔΖ μετρεῖ, καὶ οἱ Α, Β ἄρα τὸν ΔΖ μετρήσουσιν. μετροῦσι δὲ καὶ ὅλον τὸν ΓΔ· καὶ λοιπὸν ἄρα τὸν ΓΖ μετρήσουσιν ἐλάσσονα ὄντα τοῦ Ε· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὐ μετρεῖ ὁ Ε τὸν ΓΔ· μετρεῖ ἄρα· ὅπερ ἔδει δεῖξαι.

Proposition 35

If two numbers (both) measure some number, (then) the least (number) measured by them will also measure the same (number).



For let two numbers, *A* and *B*, (both) measure some number *CD*, and (let) *E* (be the) least (number measured by both *A* and *B*). I say that *E* also measures *CD*.

For if *E* does not measure *CD*, (then) let *E* leave *CF* less than itself (in) measuring *DF*. And since *A* and *B* (both) measure *E*, and *E* measures *DF*, *A* and *B* will thus also measure *DF*. And (*A* and *B*) also measure the whole of *CD*. Thus, they will also measure the remainder *CF*, which is less than *E*. The very thing is impossible. Thus, *E* cannot not measure *CD*. Thus, (*E*) measures (*CD*). (Which is) the very thing it was required to show.

λϛ'.

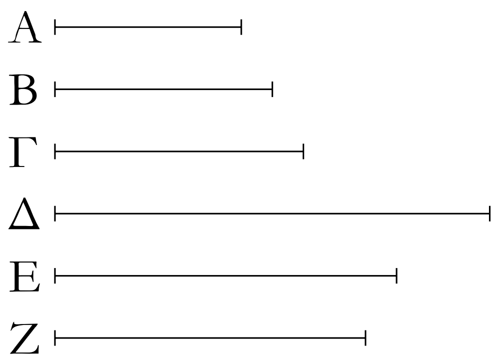
Τριῶν ἀριθμῶν δοθέντων εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Ἔστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ Α, Β, Γ· δεῖ δὴ εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Proposition 36

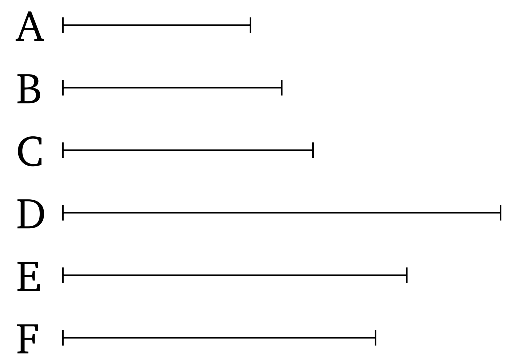
To find the least number which three given numbers (all) measure.

Let *A*, *B*, and *C* be the three given numbers. So it is required to find the least number which they (all) measure.



Εἰλήφθω γὰρ ὑπὸ δύο τῶν A, B ἐλάχιστος μετρούμενος ὁ Δ. ὁ δὴ Γ τὸν Δ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον. μετροῦσι δὲ καὶ οἱ A, B τὸν Δ· οἱ A, B, Γ ἄρα τὸν Δ μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσιν [τινα] ἀριθμὸν οἱ A, B, Γ ἐλάσσονα ὄντα τοῦ Δ. μετρείτωσαν τὸν E. ἐπεὶ οἱ A, B, Γ τὸν E μετροῦσιν, καὶ οἱ A, B ἄρα τὸν E μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν A, B μετρούμενος [τὸν E] μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν A, B μετρούμενός ἐστιν ὁ Δ· ὁ Δ ἄρα τὸν E μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ A, B, Γ μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ Δ· οἱ A, B, Γ ἄρα ἐλάχιστον τὸν Δ μετροῦσιν.

μὴ μετρείτω δὴ πάλιν ὁ Γ τὸν Δ, καὶ εἰλήφθω ὑπὸ τῶν Γ, Δ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ E. ἐπεὶ οἱ A, B τὸν Δ μετροῦσιν, ὁ δὲ Δ τὸν E μετρεῖ, καὶ οἱ A, B ἄρα τὸν E μετροῦσιν. μετρεῖ δὲ καὶ ὁ Γ [τὸν E· καὶ] οἱ A, B, Γ ἄρα τὸν E μετροῦσιν. λέγω δή, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσί τινα οἱ A, B, Γ ἐλάσσονα ὄντα τοῦ E. μετρείτωσαν τὸν Z. ἐπεὶ οἱ A, B, Γ τὸν Z μετροῦσιν, καὶ οἱ A, B ἄρα τὸν Z μετροῦσιν· καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν A, B μετρούμενος τὸν Z μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν A, B μετρούμενός ἐστιν ὁ Δ· ὁ Δ ἄρα τὸν Z μετρεῖ. μετρεῖ δὲ καὶ ὁ Γ τὸν Z· οἱ Δ, Γ ἄρα τὸν Z μετροῦσιν· ὥστε καὶ ὁ ἐλάχιστος ὑπὸ τῶν Δ, Γ μετρούμενος τὸν Z μετρήσει. ὁ δὲ ἐλάχιστος ὑπὸ τῶν Γ, Δ μετρούμενός ἐστιν ὁ E· ὁ E ἄρα τὸν Z μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ A, B, Γ μετρήσουσί τινα ἀριθμὸν ἐλάσσονα ὄντα τοῦ E. ὁ E ἄρα ἐλάχιστος ὢν ὑπὸ τῶν A, B, Γ μετρεῖται· ὅπερ ἔδει δεῖξαι.



For let the least (number), *D*, measured by the two (numbers) *A* and *B* be taken [Prop. 7.34]. So *C* either measures, or does not measure, *D*. Let it, first of all, measure (*D*). And *A* and *B* also measure *D*. Thus, *A*, *B*, and *C* (all) measure *D*. So I say that (*D* is) also the least (number measured by *A*, *B*, and *C*). For if not, *A*, *B*, and *C* will (all) measure [some] number which is less than *D*. Let them measure *E* (which is less than *D*). Since *A*, *B*, and *C* (all) measure *E*, (then) *A* and *B* thus also measure *E*. Thus, the least (number) measured by *A* and *B* will also measure [*E*] [Prop. 7.35]. And *D* is the least (number) measured by *A* and *B*. Thus, *D* will measure *E*, the greater (measuring) the lesser. The very thing is impossible. Thus, *A*, *B*, and *C* cannot (all) measure some number which is less than *D*. Thus, *A*, *B*, and *C* (all) measure the least (number) *D*.

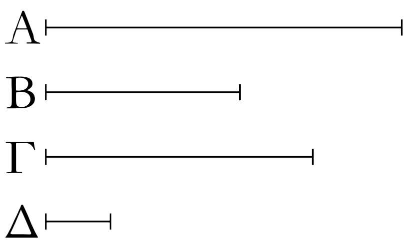
So, again, let *C* not measure *D*. And let the least number, *E*, measured by *C* and *D* be taken [Prop. 7.34]. Since *A* and *B* measure *D*, and *D* measures *E*, *A* and *B* thus also measure *E*. And *C* also measures [*E*]. Thus, *A*, *B*, and *C* [also] measure *E*. So I say that (*E* is) also the least (number measured by *A*, *B*, and *C*). For if not, *A*, *B*, and *C* will (all) measure some (number) which is less than *E*. Let them measure *F* (which is less than *E*). Since *A*, *B*, and *C* (all) measure *F*, *A* and *B* thus also measure *F*. Thus, the least (number) measured by *A* and *B* will also measure *F* [Prop. 7.35]. And *D* is the least (number) measured by *A* and *B*. Thus, *D* measures *F*. And *C* also measures *F*. Thus, *D* and *C* (both) measure *F*. Hence, the least (number) measured by *D* and *C* will also measure *F* [Prop. 7.35]. And *E* is the least (number) measured by *C* and *D*. Thus, *E* measures *F*, the greater (measuring) the lesser. The very thing is impossible. Thus, *A*, *B*, and *C* cannot measure some number which is less than *E*. Thus, *E* (is) the least (number) which is measured by *A*, *B*, and *C*. (Which is) the very thing it was required to show.

λζ΄.

Ἐὰν ἀριθμὸς ὑπό τινος ἀριθμοῦ μετρῆται, ὁ μετρούμενος ὁμώνυμον μέρος ἕξει τῷ μετροῦντι.

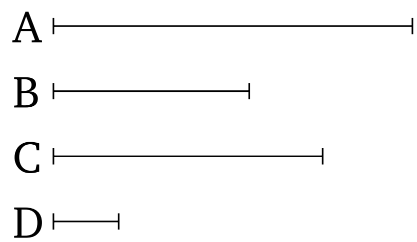
Proposition 37

If a number is measured by some number, (then) the (number) measured will have a part called the same as the measuring (number).



Ἀριθμὸς γὰρ ὁ Α ὑπό τινος ἀριθμοῦ τοῦ Β μετρείσθω· λέγω, ὅτι ὁ Α ὁμώνυμον μέρος ἔχει τῷ Β.

Ὁσάκις γὰρ ὁ Β τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Γ. ἐπεὶ ὁ Β τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, μετρεῖ δὲ καὶ ἡ Δ μονὰς τὸν Γ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Α. ἐναλλὰξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν Α· ὃ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ Β ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Α. ἡ δὲ Δ μονὰς τοῦ Β ἀριθμοῦ μέρος ἐστὶν ὁμώνυμον αὐτῷ· καὶ ὁ Γ ἄρα τοῦ Α μέρος ἐστὶν ὁμώνυμον τῷ Β. ὥστε ὁ Α μέρος ἔχει τὸν Γ ὁμώνυμον ὄντα τῷ Β· ὅπερ ἔδει δεῖξαι.

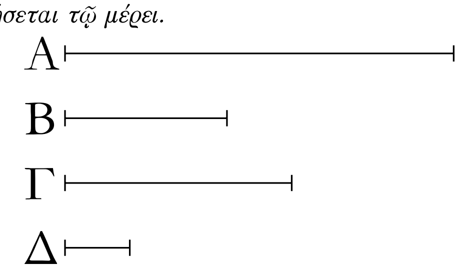


For let the number *A* be measured by some number *B*. I say that *A* has a part called the same as *B*.

For as many times as *B* measures *A*, so many units let there be in *C*. Since *B* measures *A* according to the units in *C*, and the unit *D* also measures *C* according to the units in it, the unit *D* thus measures the number *C* as many times as *B* (measures) *A*. Thus, alternately, the unit *D* measures the number *B* as many times as *C* (measures) *A* [Prop. 7.15]. Thus, which(ever) part the unit *D* is of the number *B*, *C* is also the same part of *A*. And the unit *D* is a part of the number *B* called the same as it (i.e., a *B*th part). Thus, *C* is also a part of *A* called the same as *B* (i.e., *C* is the *B*th part of *A*). Hence, *A* has a part *C* which is called the same as *B* (i.e., *A* has a *B*th part). (Which is) the very thing it was required to show.

λη΄.

Ἐὰν ἀριθμος μέρος ἔχῃ ὁτιοῦν, ὑπὸ ὁμωνύμου ἀριθμοῦ μετρηθήσεται τῷ μέρει.

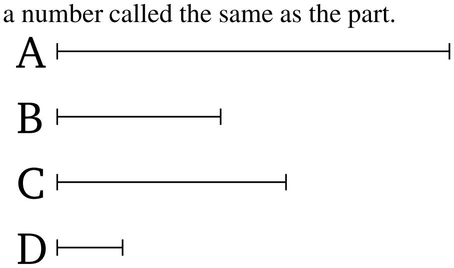


Ἀριθμὸς γὰρ ὁ Α μέρος ἐχέτω ὁτιοῦν τὸν Β, καὶ τῷ Β μέρει ὁμώνυμος ἔστω [ἀριθμὸς] ὁ Γ· λέγω, ὅτι ὁ Γ τὸν Α μετρεῖ.

Ἐπεὶ γὰρ ὁ Β τοῦ Α μέρος ἐστὶν ὁμώνυμον τῷ Γ, ἔστι δὲ καὶ ἡ Δ μονὰς τοῦ Γ μέρος ὁμώνυμον αὐτῷ, ὃ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ Γ ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Β τοῦ Α· ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Α. ἐναλλὰξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν Α. ὁ Γ ἄρα τὸν Α μετρεῖ· ὅπερ ἔδει δεῖξαι.

Proposition 38

If a number has any part whatever, (then) it will be measured by a number called the same as the part.



For let the number *A* have any part whatever, *B*. And let the [number] *C* be called the same as the part *B* (i.e., *B* is the *C*th part of *A*). I say that *C* measures *A*.

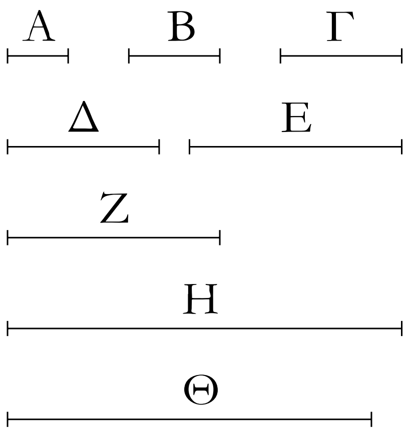
For since *B* is a part of *A* called the same as *C*, and the unit *D* is also a part of *C* called the same as it (i.e., *D* is the *C*th part of *C*), thus which(ever) part the unit *D* is of the number *C*, *B* is also the same part of *A*. Thus, the unit *D* measures the number *C* as many times as *B* (measures) *A*. Thus, alternately, the unit *D* measures the number *B* as many times as *C* (measures) *A* [Prop. 7.15]. Thus, *C* measures *A*. (Which is) the very thing it was required to show.

λθ΄.

Ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ δοθέντα μέρη.

Proposition 39

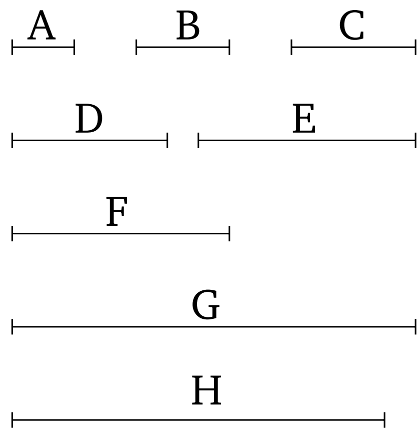
To find the least number that will have given parts.



Ἔστω τὰ δοθέντα μέρη τὰ Α, Β, Γ· δεῖ δὴ ἀριθμὸν εὑρεῖν, ὃς ἐλάχιστος ὢν ἕξει τὰ Α, Β, Γ μέρη.

Ἔστωσαν γὰρ τοῖς Α, Β, Γ μέρεσιν ὁμώνυμοι ἀριθμοὶ οἱ Δ, Ε, Ζ, καὶ εἰλήφθω ὑπὸ τῶν Δ, Ε, Ζ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Η.

Ὁ Η ἄρα ὁμώνυμα μέρη ἔχει τοῖς Δ, Ε, Ζ. τοῖς δὲ Δ, Ε, Ζ ὁμώνυμα μέρη ἐστὶ τὰ Α, Β, Γ· ὁ Η ἄρα ἔχει τὰ Α, Β, Γ μέρη. λέγω δή, ὅτι καὶ ἐλάχιστος ὤν, εἰ γὰρ μή, ἔσται τις τοῦ Η ἐλάσσων ἀριθμός, ὃς ἕξει τὰ Α, Β, Γ μέρη. ἔστω ὁ Θ. ἐπεὶ ὁ Θ ἔχει τὰ Α, Β, Γ μέρη, ὁ Θ ἄρα ὑπὸ ὁμωνύμων ἀριθμῶν μετρηθήσεται τοῖς Α, Β, Γ μέρεσιν. τοῖς δὲ Α, Β, Γ μέρεσιν ὁμώνυμοι ἀριθμοί εἰσιν οἱ Δ, Ε, Ζ· ὁ Θ ἄρα ὑπὸ τῶν Δ, Ε, Ζ μετρεῖται. καί ἐστιν ἐλάσσων τοῦ Η· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσται τις τοῦ Η ἐλάσσων ἀριθμός, ὃς ἕξει τὰ Α, Β, Γ μέρη· ὅπερ ἔδει δεῖξαι.



Let *A*, *B*, and *C* be the given parts. So it is required to find the least number which will have the parts *A*, *B*, and *C* (i.e., an *A*th part, a *B*th part, and a *C*th part).

For let *D*, *E*, and *F* be numbers having the same names as the parts *A*, *B*, and *C* (respectively). And let the least number, *G*, measured by *D*, *E*, and *F*, be taken [Prop. 7.36].

Thus, *G* has parts called the same as *D*, *E*, and *F* [Prop. 7.37]. And *A*, *B*, and *C* are parts called the same as *D*, *E*, and *F* (respectively). Thus, *G* has the parts *A*, *B*, and *C*. So I say that (*G*) is also the least (number having the parts *A*, *B*, and *C*). For if not, there will be some number less than *G* which will have the parts *A*, *B*, and *C*. Let it be *H*. Since *H* has the parts *A*, *B*, and *C*, *H* will thus be measured by numbers called the same as the parts *A*, *B*, and *C* [Prop. 7.38]. And *D*, *E*, and *F* are numbers called the same as the parts *A*, *B*, and *C* (respectively). Thus, *H* is measured by *D*, *E*, and *F*. And (*H*) is less than *G*. The very thing is impossible. Thus, there cannot be some number less than *G* which will have the parts *A*, *B*, and *C*. (Which is) the very thing it was required to show.

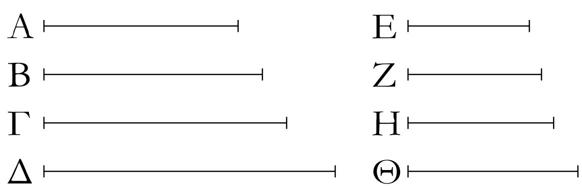
ELEMENTS BOOK 8

Continued Proportion[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

α'.

Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

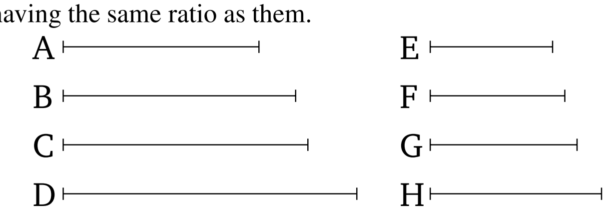


Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, οἱ δὲ ἄκροι αὐτῶν οἱ Α, Δ, πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οἱ Α, Β, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔστωσαν ἐλάττονες τῶν Α, Β, Γ, Δ οἱ Ε, Ζ, Η, Θ ἐν τῷ αὐτῷ λόγῳ ὄντες αὐτοῖς. καὶ ἐπεὶ οἱ Α, Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς Ε, Ζ, Η, Θ, καί ἐστιν ἴσον τὸ πλῆθος [τῶν Α, Β, Γ, Δ] τῷ πλήθει [τῶν Ε, Ζ, Η, Θ], δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Θ. οἱ δὲ Α, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Ε ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Ε, Ζ, Η, Θ ἐλάσσονες ὄντες τῶν Α, Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶν αὐτοῖς. οἱ Α, Β, Γ, Δ ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.

Proposition 1

If there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them.



Let *A*, *B*, *C*, *D* be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, *A* and *D*, be prime to one another. I say that *A*, *B*, *C*, *D* are the least of those (numbers) having the same ratio as them.

For if not, let *E*, *F*, *G*, *H* be less than *A*, *B*, *C*, *D* (respectively), being in the same ratio as them. And since *A*, *B*, *C*, *D* are in the same ratio as *E*, *F*, *G*, *H*, and the multitude [of *A*, *B*, *C*, *D*] is equal to the multitude [of *E*, *F*, *G*, *H*], thus, via equality, as *A* is to *D*, (so) *E* (is) to *H* [Prop. 7.14]. And *A* and *D* (are) prime (to one another). And prime (numbers are) also the least of those (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, *A* measures *E*, the greater (measuring) the lesser. The very thing is impossible. Thus, *E*, *F*, *G*, *H*, being less than *A*, *B*, *C*, *D*, are not in the same ratio as them. Thus, *A*, *B*, *C*, *D* are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

β'.

Ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους, ὅσους ἂν ἐπιτάξῃ τις, ἐν τῷ δοθέντι λόγῳ.

Ἔστω ὁ δοθεὶς λόγος ἐν ἐλαχίστοις ἀριθμοῖς ὁ τοῦ Α πρὸς τὸν Β· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους, ὅσους ἄν τις ἐπιτάξῃ, ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ.

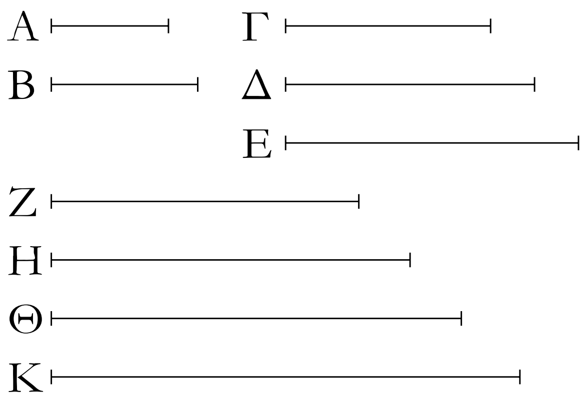
Ἐπιτετάχθωσαν δὴ τέσσαρες, καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω, τὸν δὲ Β πολλαπλασιάσας τὸν Δ ποιείτω, καὶ ἔτι ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε ποιείτω, καὶ ἔτι ὁ Α τοὺς Γ, Δ, Ε πολλαπλασιάσας τοὺς Ζ, Η, Θ ποιείτω, ὁ δὲ Β τὸν Ε πολλαπλασιάσας τὸν Κ ποιείτω.

Proposition 2

To find the least numbers, as many as may be prescribed, (which are) continuously proportional in a given ratio.

Let the given ratio, (expressed) in the least numbers, be that of *A* to *B*. So it is required to find the least numbers, as many as may be prescribed, (which are) in the ratio of *A* to *B*.

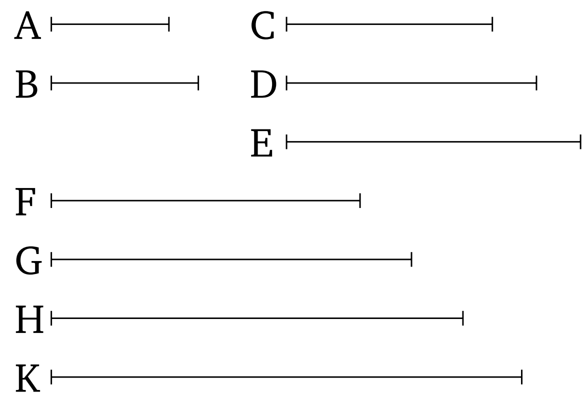
Let four (numbers) be prescribed. And let *A* make *C* (by) multiplying itself, and let it make *D* (by) multiplying *B*. And, further, let *B* make *E* (by) multiplying itself. And, further, let *A* make *F*, *G*, *H* (by) multiplying *C*, *D*, *E*. And let *B* make *K* (by) multiplying *E*.



Καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, [οὕτως] ὁ Γ πρὸς τὸν Δ. πάλιν, ἐπεὶ ὁ μὲν Α τὸν Β πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ Β ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν, ἑκάτερος ἄρα τῶν Α, Β τὸν Β πολλαπλασιάσας ἑκάτερον τῶν Δ, Ε πεποίηκεν. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε. ἀλλ᾽ ὡς ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, ὁ Δ πρὸς τὸν Ε. καὶ ἐπεὶ ὁ Α τοὺς Γ, Δ πολλαπλασιάσας τοὺς Ζ, Η πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, [οὕτως] ὁ Ζ πρὸς τὸν Η. ὡς δὲ ὁ Γ πρὸς τὸν Δ, οὕτως ἦν ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, ὁ Ζ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Α τοὺς Δ, Ε πολλαπλασιάσας τοὺς Η, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, ὁ Η πρὸς τὸν Θ. ἀλλ᾽ ὡς ὁ Δ πρὸς τὸν Ε, ὁ Α πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Θ. καὶ ἐπεὶ οἱ Α, Β τὸν Ε πολλαπλασιάσαντες τοὺς Θ, Κ πεποιήκασιν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Θ πρὸς τὸν Κ. ἀλλ᾽ ὡς ὁ Α πρὸς τὸν Β, οὕτως ὅ τε Ζ πρὸς τὸν Η καὶ ὁ Η πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Ζ πρὸς τὸν Η, οὕτως ὅ τε Η πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Κ· οἱ Γ, Δ, Ε ἄρα καὶ οἱ Ζ, Η, Θ, Κ ἀνάλογόν εἰσιν ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ. λέγω δή, ὅτι καὶ ἐλάχιστοι. ἐπεὶ γὰρ οἱ Α, Β ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ δὲ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων πρῶτοι πρὸς ἀλλήλους εἰσίν, οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἑκάτερος μὲν τῶν Α, Β ἑαυτὸν πολλαπλασιάσας ἑκάτερον τῶν Γ, Ε πεποίηκεν, ἑκάτερον δὲ τῶν Γ, Ε πολλαπλασιάσας ἑκάτερον τῶν Ζ, Κ πεποίηκεν· οἱ Γ, Ε ἄρα καὶ οἱ Ζ, Κ πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ ὦσιν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οἱ Γ, Δ, Ε ἄρα καὶ οἱ Ζ, Η, Θ, Κ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β· ὅπερ ἔδει δεῖξαι.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι ὦσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκρον αὐτῶν τετράγωνοί εἰσιν, ἐὰν δὲ τέσσαρες, κύβοι.



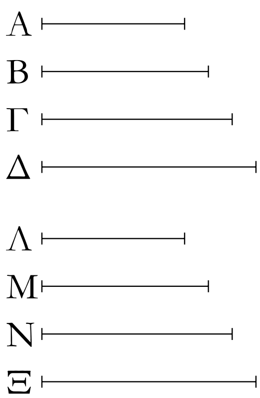
And since *A* has made *C* (by) multiplying itself, and has made *D* (by) multiplying *B*, thus as *A* is to *B*, [so] *C* (is) to *D* [Prop. 7.17]. Again, since *A* has made *D* (by) multiplying *B*, and *B* has made *E* (by) multiplying itself, *A*, *B* have thus made *D*, *E*, respectively, (by) multiplying *B*. Thus, as *A* is to *B*, so *D* (is) to *E* [Prop. 7.18]. But, as *A* (is) to *B*, (so) *C* (is) to *D*. And thus as *C* (is) to *D*, (so) *D* (is) to *E*. And since *A* has made *F*, *G* (by) multiplying *C*, *D*, thus as *C* is to *D*, [so] *F* (is) to *G* [Prop. 7.17]. And as *C* (is) to *D*, so *A* was to *B*. And thus as *A* (is) to *B*, (so) *F* (is) to *G*. Again, since *A* has made *G*, *H* (by) multiplying *D*, *E*, thus as *D* is to *E*, (so) *G* (is) to *H* [Prop. 7.17]. But, as *D* (is) to *E*, (so) *A* (is) to *B*. And thus as *A* (is) to *B*, so *G* (is) to *H*. And since *A*, *B* have made *H*, *K* (by) multiplying *E*, thus as *A* is to *B*, so *H* (is) to *K*. But, as *A* (is) to *B*, so *F* (is) to *G*, and *G* to *H*. And thus as *F* (is) to *G*, so *G* (is) to *H*, and *H* to *K*. Thus, *C*, *D*, *E* and *F*, *G*, *H*, *K* are (both continuously) proportional in the ratio of *A* to *B*. So I say that (they are) also the least (sets of numbers continuously proportional in that ratio). For since *A* and *B* are the least of those (numbers) having the same ratio as them, and the least of those (numbers) having the same ratio are prime to one another [Prop. 7.22], *A* and *B* are thus prime to one another. And *A*, *B* have made *C*, *E*, respectively, (by) multiplying themselves, and have made *F*, *K* by multiplying *C*, *E*, respectively. Thus, *C*, *E* and *F*, *K* are prime to one another [Prop. 7.27]. And if there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them [Prop. 8.1]. Thus, *C*, *D*, *E* and *F*, *G*, *H*, *K* are the least of those (continuously proportional sets of numbers) having the same ratio as *A* and *B*. (Which is) the very thing it was required to show.

Corollary

So it is clear, from this, that if three continuously proportional numbers are the least of those (numbers) having the same ratio as them then the outermost of them are square, and, if four (numbers), cube.

γ'.

Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.



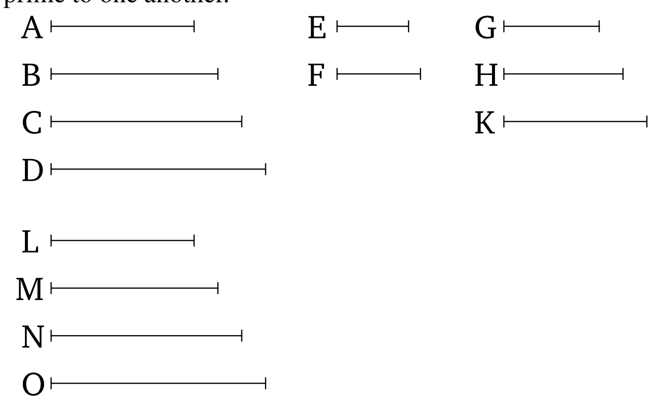
Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β, Γ, Δ· λέγω, ὅτι οἱ ἄκροι αὐτῶν οἱ Α, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν Α, Β, Γ, Δ λόγῳ οἱ Ε, Ζ, τρεῖς δὲ οἱ Η, Θ, Κ, καὶ ἑξῆς ἑνὶ πλείους, ἕως τὸ λαμβανόμενον πλῆθος ἴσον γένηται τῷ πλήθει τῶν Α, Β, Γ, Δ. εἰλήφθωσαν καὶ ἔστωσαν οἱ Λ, Μ, Ν, Ξ.

Καὶ ἐπεὶ οἱ Ε, Ζ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ἑκάτερος τῶν Ε, Ζ ἑαυτὸν μὲν πολλαπλασιάσας ἑκάτερον τῶν Η, Κ πεποίηκεν, ἑκάτερον δὲ τῶν Η, Κ πολλαπλασιάσας ἑκάτερον τῶν Λ, Ξ πεποίηκεν, καὶ οἱ Η, Κ ἄρα καὶ οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ Α, Β, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, εἰσὶ δὲ καὶ οἱ Λ, Μ, Ν, Ξ ἐλάχιστοι ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β, Γ, Δ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Α, Β, Γ, Δ τῷ πλήθει τῶν Λ, Μ, Ν, Ξ, ἕκαστος ἄρα τῶν Α, Β, Γ, Δ ἑκάστῳ τῶν Λ, Μ, Ν, Ξ ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν Α τῷ Λ, ὁ δὲ Δ τῷ Ξ. καί εἰσιν οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους. καὶ οἱ Α, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 3

If there are any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them then the outermost of them are prime to one another.



Let *A*, *B*, *C*, *D* be any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that the outermost of them, *A* and *D*, are prime to one another.

For let the two least (numbers) *E*, *F* (which are) in the same ratio as *A*, *B*, *C*, *D* be taken [Prop. 7.33]. And the three (least numbers) *G*, *H*, *K* [Prop. 8.2]. And (so on), successively increasing by one, until the multitude of (numbers) taken is made equal to the multitude of *A*, *B*, *C*, *D*. Let them be taken, and let them be *L*, *M*, *N*, *O*.

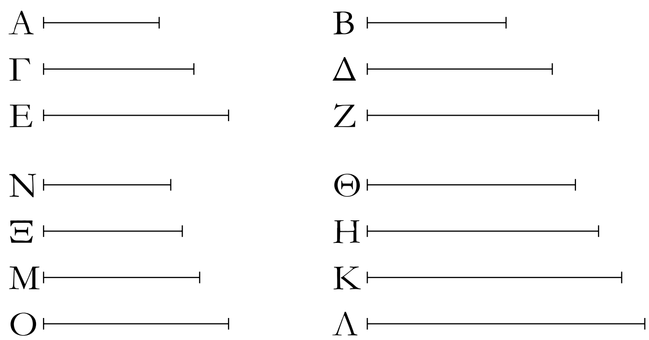
And since *E* and *F* are the least of those (numbers) having the same ratio as them they are prime to one another [Prop. 7.22]. And since *E*, *F* have made *G*, *K*, respectively, (by) multiplying themselves [Prop. 8.2 corr.], and have made *L*, *O* (by) multiplying *G*, *K*, respectively, *G*, *K* and *L*, *O* are thus also prime to one another [Prop. 7.27]. And since *A*, *B*, *C*, *D* are the least of those (numbers) having the same ratio as them, and *L*, *M*, *N*, *O* are also the least (of those numbers having the same ratio as them), being in the same ratio as *A*, *B*, *C*, *D*, and the multitude of *A*, *B*, *C*, *D* is equal to the multitude of *L*, *M*, *N*, *O*, thus *A*, *B*, *C*, *D* are equal to *L*, *M*, *N*, *O*, respectively. Thus, *A* is equal to *L*, and *D* to *O*. And *L* and *O* are prime to one another. Thus, *A* and *D* are also prime to one another. (Which is) the very thing it was required to show.

δ'.

Λόγων δοθέντων ὁποσωνοῦν ἐν ἐλαχίστοις ἀριθμοῖς ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους ἐν τοῖς δοθεῖσι λόγοις.

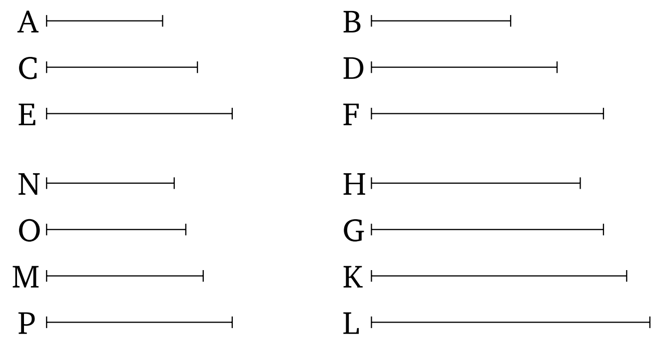
Proposition 4

For any multitude whatsoever of given ratios, (expressed) in the least numbers, to find the least numbers continuously proportional in these given ratios.



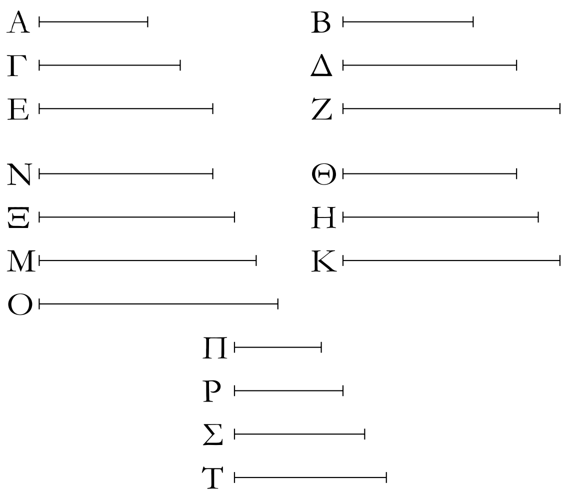
Ἔστωσαν οἱ δοθέντες λόγοι ἐν ἐλαχίστοις ἀριθμοῖς ὅ τε τοῦ Α πρὸς τὸν Β καὶ ὁ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ὁ τοῦ Ε πρὸς τὸν Ζ· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἑξῆς ἀνάλογον ἐλαχίστους ἔν τε τῷ τοῦ Α πρὸς τὸν Β λόγῳ καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Ζ.

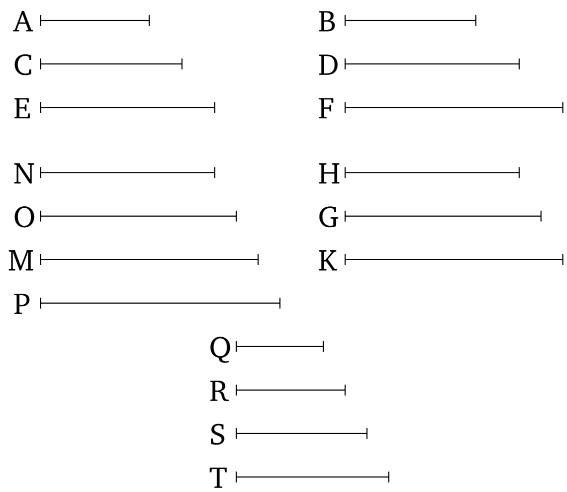
Εἰλήφθω γὰρ ὁ ὑπὸ τῶν Β, Γ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Η. καὶ ὁσάκις μὲν ὁ Β τὸν Η μετρεῖ, τοσαυτάκις καὶ ὁ Α τὸν Θ μετρείτω, ὁσάκις δὲ ὁ Γ τὸν Η μετρεῖ, τοσαυτάκις καὶ ὁ Δ τὸν Κ μετρείτω. ὁ δὲ Ε τὸν Κ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον. καὶ ὁσάκις ὁ Ε τὸν Κ μετρεῖ, τοσαυτάκις καὶ ὁ Ζ τὸν Λ μετρείτω. καὶ ἐπεὶ ἰσάκις ὁ Α τὸν Θ μετρεῖ καὶ ὁ Β τὸν Η, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Θ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Η πρὸς τὸν Κ, καὶ ἔτι ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Λ· οἱ Θ, Η, Κ, Λ ἄρα ἑξῆς ἀνάλογόν εἰσιν ἔν τε τῷ τοῦ Α πρὸς τὸν Β καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ἐν τῷ τοῦ Ε πρὸς τὸν Ζ λόγῳ. λέγω δή, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μή εἰσιν οἱ Θ, Η, Κ, Λ ἑξῆς ἀνάλογον ἐλάχιστοι ἔν τε τοῖς τοῦ Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἐν τῷ τοῦ Ε πρὸς τὸν Ζ λόγοις, ἔστωσαν οἱ Ν, Ξ, Μ, Ο. καὶ ἐπεί ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ὁ Β ἄρα τὸν Ξ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ξ μετρεῖ· οἱ Β, Γ ἄρα τὸν Ξ μετροῦσιν· καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Β, Γ μετρούμενος τὸν Ξ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρεῖται ὁ Η· ὁ Η ἄρα τὸν Ξ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσονταί τινες τῶν Θ, Η, Κ, Λ ἐλάσσονες ἀριθμοὶ ἑξῆς ἔν τε τῷ τοῦ Α πρὸς τὸν Β καὶ τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Ζ λόγῳ.



Let the given ratios, (expressed) in the least numbers, be the (ratios) of A to B, and of C to D, and, further, of E to F. So it is required to find the least numbers continuously proportional in the ratio of A to B, and of C to B, and, further, of E to F.

For let the least number, G, measured by (both) B and C have be taken [Prop. 7.34]. And as many times as B measures G, so many times let A also measure H. And as many times as C measures G, so many times let D also measure K. And E either measures, or does not measure, K. Let it, first of all, measure (K). And as many times as E measures K, so many times let F also measure L. And since A measures H the same number of times that B also (measures) G, thus as A is to B, so H (is) to G [Def. 7.20, Prop. 7.13]. And so, for the same (reasons), as C (is) to D, so G (is) to K, and, further, as E (is) to F, so K (is) to L. Thus, H, G, K, L are continuously proportional in the ratio of A to B, and of C to D, and, further, of E to F. So I say that (they are) also the least (numbers continuously proportional in these ratios). For if H, G, K, L are not the least numbers continuously proportional in the ratios of A to B, and of C to D, and of E to F, let N, O, M, P be (the least such numbers). And since as A is to B, so N (is) to O, and A and B are the least (numbers which have the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures O. So, for the same (reasons), C also measures O. Thus, B and C (both) measure O. Thus, the least number measured by (both) B and C will also measure O [Prop. 7.35]. And G (is) the least number measured by (both) B and C. Thus, G measures O, the greater (measuring) the lesser. The very thing is impossible. Thus, there cannot be any numbers less than H, G, K, L (which are) continuously (proportional) in the ratio of A to B, and of C to D, and, further, of E to F.





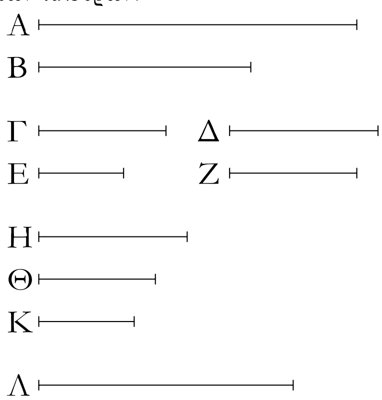
μὴ μετρείτω δὴ ὁ Ε τὸν Κ, καὶ εἰλήφθω ὑπὸ τῶν Ε, Κ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Μ. καὶ ὁσάκις μὲν ὁ Κ τὸν Μ μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Η ἑκάτερον τῶν Ν, Ξ μετρείτω, ὁσάακις δὲ ὁ Ε τὸν Μ μετρεῖ, τοσαυτάκις καὶ ὁ Ζ τὸν Ο μετρείτω. ἐπεὶ ἰσάκις ὁ Θ τὸν Ν μετρεῖ καὶ ὁ Η τὸν Ξ, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Ν πρὸς τὸν Ξ. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ξ πρὸς τὸν Μ. πάλιν, ἐπεὶ ἰσάκις ὁ Ε τὸν Μ μετρεῖ καὶ ὁ Ζ τὸν Ο, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Μ πρὸς τὸν Ο· οἱ Ν, Ξ, Μ, Ο ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τοῖς τοῦ τε Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγοις. λέγω δή, ὅτι καὶ ἐλάχιστοι ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις. εἰ γὰρ μή, ἔσονταί τινες τῶν Ν, Ξ, Μ, Ο ἐλάσσονες ἀριθμοὶ ἑξῆς ἀνάλογον ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις. ἔστωσαν οἱ Π, Ρ, Σ, Τ. καὶ ἐπεί ἐστιν ὡς ὁ Π πρὸς τὸν Ρ, οὕτως ὁ Α πρὸς τὸν Β, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον, ὁ Β ἄρα τὸν Ρ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ρ μετρεῖ· οἱ Β, Γ ἄρα τὸν Ρ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Β, Γ μετούμενος τὸν Ρ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρούμενος ἐστιν ὁ Η· ὁ Η ἄρα τὸν Ρ μετρεῖ. καί ἐστιν ὡς ὁ Η πρὸς τὸν Ρ, οὕτως ὁ Κ πρὸς τὸν Σ· καὶ ὁ Κ ἄρα τὸν Σ μετρεῖ. μετρεῖ δὲ καὶ ὁ Ε τὸν Σ· οἱ Ε, Κ ἄρα τὸν Σ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Ε, Κ μετρούμενος τὸν Σ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Ε, Κ μετρούμενός ἐστιν ὁ Μ· ὁ Μ ἄρα τὸν Σ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσονταί τινες τῶν Ν, Ξ, Μ, Ο ἐλάσσονες ἀριθμοὶ ἑξῆς ἀνάλογον ἔν τε τοῖς τοῦ Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγοις· οἱ Ν, Ξ, Μ, Ο ἄρα ἑξῆς ἀνάλογον ἐλάχιστοί εἰσιν ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις· ὅπερ ἔδει δεῖξαι.

So let E not measure K. And let the least number, M, measured by (both) E and K be taken [Prop. 7.34]. And as many times as K measures M, so many times let H, G also measure N, O, respectively. And as many times as E measures M, so many times let F also measure P. Since H measures N the same number of times as G (measures) O, thus as H is to G, so N (is) to O [Def. 7.20, Prop. 7.13]. And as H (is) to G, so A (is) to B. And thus as A (is) to B, so N (is) to O. And so, for the same (reasons), as C (is) to D, so O (is) to M. Again, since E measures M the same number of times as F (measures) P, thus as E is to F, so M (is) to P [Def. 7.20, Prop. 7.13]. Thus, N, O, M, P are continuously proportional in the ratios of A to B, and of C to D, and, further, of E to F. So I say that (they are) also the least (numbers) in the ratios of A B, C D, E F. For if not, then there will be some numbers less than N, O, M, P (which are) continuously proportional in the ratios of A B, C D, E F. Let them be Q, R, S, T. And since as Q is to R, so A (is) to B, and A and B (are) the least (numbers having the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures R. So, for the same (reasons), C also measures R. Thus, B and C (both) measure R. Thus, the least (number) measured by (both) B and C will also measure R [Prop. 7.35]. And G is the least number measured by (both) B and C. Thus, G measures R. And as G is to R, so K (is) to S. Thus, K also measures S [Def. 7.20]. And E also measures S [Prop. 7.20]. Thus, E and K (both) measure S. Thus, the least (number) measured by (both) E and K will also measure S [Prop. 7.35]. And M is the least (number) measured by (both) E and K. Thus, M measures S, the greater (measuring) the lesser. The very thing is impossible. Thus there cannot be any numbers less than N, O, M, P (which are) continuously proportional in the ratios of A to B, and of C to D, and, further, of E to F. Thus, N, O, M, P are the least (numbers) continuously proportional in the ratios of A B, C D,

$E F$. (Which is) the very thing it was required to show.

ε΄.

Οἱ ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσι τὸν συγκείμενον ἐκ τῶν πλευρῶν.



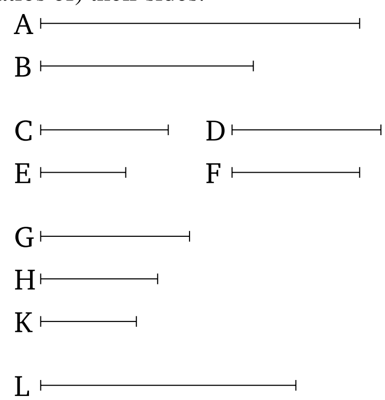
Ἔστωσαν ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ Β οἱ Ε, Ζ· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Λόγων γὰρ δοθέντων τοῦ τε ὃν ἔχει ὁ Γ πρὸς τὸν Ε καὶ ὁ Δ πρὸς τὸν Ζ εἰλήφθωσαν ἀριθμοὶ ἑξῆς ἐλάχιστοι ἐν τοῖς Γ Ε, Δ Ζ λόγοις, οἱ Η, Θ, Κ, ὥστε εἶναι ὡς μὲν τὸν Γ πρὸς τὸν Ε, οὕτως τὸν Η πρὸς τὸν Θ, ὡς δὲ τὸν Δ πρὸς τὸν Ζ, οὕτως τὸν Θ πρὸς τὸν Κ. καὶ ὁ Δ τὸν Ε πολλαπλασιάσας τὸν Λ ποιείτω.

Καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Λ. ὡς δὲ ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Θ· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Λ. πάλιν, ἐπεὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν Λ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Ζ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Λ πρὸς τὸν Β. ἀλλ᾽ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Θ πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Κ, οὕτως ὁ Λ πρὸς τὸν Β. ἐδείχθη δὲ καὶ ὡς ὁ Η πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Λ· δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Η πρὸς τὸν Κ, [οὕτως] ὁ Α πρὸς τὸν Β. ὁ δὲ Η πρὸς τὸν Κ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· καὶ ὁ Α ἄρα πρὸς τὸν Β λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

Proposition 5

Plane numbers have to one another the ratio compounded† out of (the ratios of) their sides.



Let A and B be plane numbers, and let the numbers C, D be the sides of A, and (the numbers) E, F (the sides) of B. I say that A has to B the ratio compounded out of (the ratios of) their sides.

For given the ratios which C has to E, and D (has) to F, let the least numbers, G, H, K, continuously proportional in the ratios $C E$, $D F$ be taken [Prop. 8.4], so that as C is to E, so G (is) to H, and as D (is) to F, so H (is) to K. And let D make L (by) multiplying E.

And since D has made A (by) multiplying C, and has made L (by) multiplying E, thus as C is to E, so A (is) to L [Prop. 7.17]. And as C (is) to E, so G (is) to H. And thus as G (is) to H, so A (is) to L. Again, since E has made L (by) multiplying D [Prop. 7.16], but, in fact, has also made B (by) multiplying F, thus as D is to F, so L (is) to B [Prop. 7.17]. But, as D (is) to F, so H (is) to K. And thus as H (is) to K, so L (is) to B. And it was also shown that as G (is) to H, so A (is) to L. Thus, via equality, as G is to K, [so] A (is) to B [Prop. 7.14]. And G has to K the ratio compounded out of (the ratios of) the sides (of A and B). Thus, A also has to B the ratio compounded out of (the ratios of) the sides (of A and B). (Which is) the very thing it was required to show.

† i.e., multiplied.

ϛ΄.

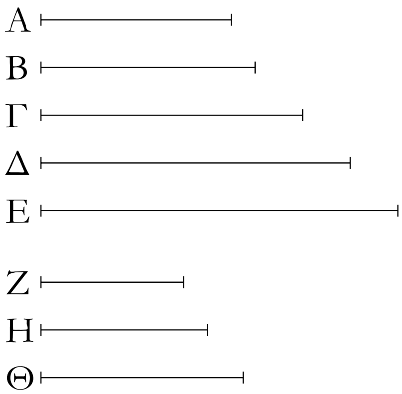
Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, ὁ δὲ πρῶτος τὸν δεύτερον μὴ μετρῇ, οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, Ε, ὁ δὲ Α τὸν Β μὴ μετρείτω· λέγω, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.

Proposition 6

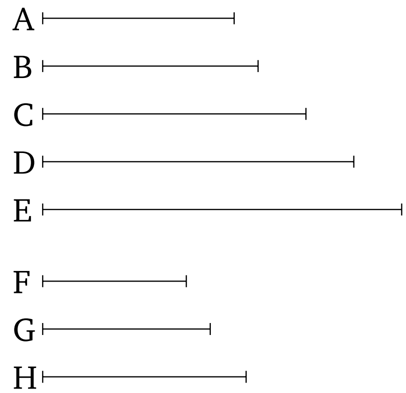
If there are any multitude whatsoever of continuously proportional numbers, and the first does not measure the second, then no other (number) will measure any other (number) either.

Let A, B, C, D, E be any multitude whatsoever of contin-



Ὅτι μὲν οὖν οἱ Α, Β, Γ, Δ, Ε ἑξῆς ἀλλήλους οὐ μετροῦσιν, φανερόν· οὐδὲ γὰρ ὁ Α τὸν Β μετρεῖ. λέγω δή, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει. εἰ γὰρ δυνατόν, μετρείτω ὁ Α τὸν Γ. καὶ ὅσοι εἰσὶν οἱ Α, Β, Γ, τοσοῦτοι εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ οἱ Ζ, Η, Θ. καὶ ἐπεὶ οἱ Ζ, Η, Θ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς Α, Β, Γ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Α, Β, Γ τῷ πλήθει τῶν Ζ, Η, Θ, δι᾿ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Ζ πρὸς τὸν Θ. καὶ ἐπεί ἐστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν Η, οὐ μετρεῖ δὲ ὁ Α τὸν Β, οὐ μετρεῖ ἄρα οὐδὲ ὁ Ζ τὸν Η· οὐκ ἄρα μονάς ἐστιν ὁ Ζ· ἡ γὰρ μονὰς πάντα ἀριθμὸν μετρεῖ. καί εἰσιν οἱ Ζ, Θ πρῶτοι πρὸς ἀλλήλους [οὐδὲ ὁ Ζ ἄρα τὸν Θ μετρεῖ]. καί ἐστιν ὡς ὁ Ζ πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Γ· οὐδὲ ὁ Α ἄρα τὸν Γ μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· ὅπερ ἔδει δεῖξαι.

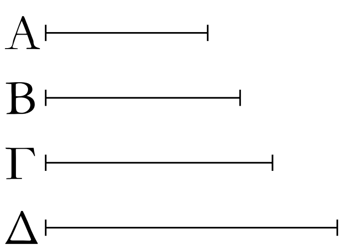
uously proportional numbers, and let A not measure B. I say that no other (number) will measure any other (number) either.



Now, (it is) clear that A, B, C, D, E do not successively measure one another. For A does not even measure B. So I say that no other (number) will measure any other (number) either. For, if possible, let A measure C. And as many (numbers) as are A, B, C, let so many of the least numbers, F, G, H, be taken of those (numbers) having the same ratio as A, B, C [Prop. 7.33]. And since F, G, H are in the same ratio as A, B, C, and the multitude of A, B, C is equal to the multitude of F, G, H, thus, via equality, as A is to C, so F (is) to H [Prop. 7.14]. And since as A is to B, so F (is) to G, and A does not measure B, F does not measure G either [Def. 7.20]. Thus, F is not a unit. For a unit measures all numbers. And F and H are prime to one another [Prop. 8.3] [and thus F does not measure H]. And as F is to H, so A (is) to C. And thus A does not measure C either [Def. 7.20]. So, similarly, we can show that no other (number) can measure any other (number) either. (Which is) the very thing it was required to show.

ζ΄.

Ἐὰν ὦσιν ὁποσοιοῦν ἀριθμοὶ [ἑξῆς] ἀνάλογον, ὁ δὲ πρῶτος τὸν ἔσχατον μετρῇ, καὶ τὸν δεύτερον μετρήσει.

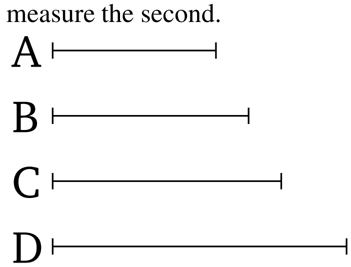


Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, ὁ δὲ Α τὸν Δ μετρείτω· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεῖ.

Εἰ γὰρ οὐ μετρεῖ ὁ Α τὸν Β, οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· μετρεῖ δὲ ὁ Α τὸν Δ. μετρεῖ ἄρα καὶ ὁ Α τὸν Β· ὅπερ ἔδει δεῖξαι.

Proposition 7

If there are any multitude whatsoever of [continuously] proportional numbers, and the first measures the last, then (the first) will also measure the second.

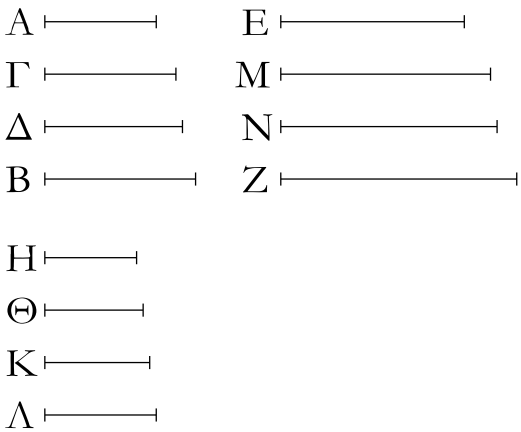


Let A, B, C, D be any number whatsoever of continuously proportional numbers. And let A measure D. I say that A also measures B.

For if A does not measure B then no other (number) will measure any other (number) either [Prop. 8.6]. But A measures D. Thus, A also measures B. (Which is) the very thing it was required to show.

η´.

Ἐὰν δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνόλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας [αὐτοῖς] μεταξὺ κατὰ τὸ συνέχες ἀνάλογον ἐμπεσοῦνται

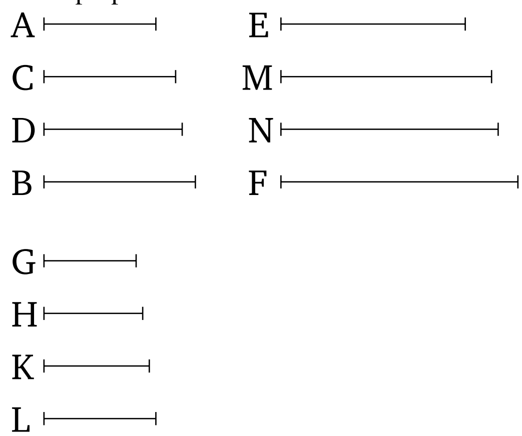


Δύο γὰρ ἀριθμῶν τῶν Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἱ Γ, Δ, καὶ πεποιήσθω ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ε πρὸς τὸν Ζ· λέγω, ὅτι ὅσοι εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Ε, Ζ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Ὅσοι γάρ εἰσι τῷ πλήθει οἱ Α, Β, Γ, Δ, τοσοῦτοι εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ, Β οἱ Η, Θ, Κ, Λ· οἱ ἄρα ἄκροι αὐτῶν οἱ Η, Λ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ Α, Γ, Δ, Β τοῖς Η, Θ, Κ, Λ ἐν τῷ αὐτῷ λόγῳ εἰσίν, καί ἐστιν ἴσον τὸ πλῆθος τῶν Α, Γ, Δ, Β τῷ πλήθει τῶν Η, Θ, Κ, Λ, δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Η πρὸς τὸν Λ. ὡς δὲ ὁ Α πρὸς τὸν Β, οὕτως ὁ Ε πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Λ, οὕτως ὁ Ε πρὸς τὸν Ζ. οἱ δὲ Η, Λ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. ἰσάκις ἄρα ὁ Η τὸν Ε μετρεῖ καὶ ὁ Λ τὸν Ζ. ὁσάκις δὴ ὁ Η τὸν Ε μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Κ ἑκάτερον τῶν Μ, Ν μετρείτω· οἱ Η, Θ, Κ, Λ ἄρα τοὺς Ε, Μ, Ν, Ζ ἰσάκις μετροῦσιν. οἱ Η, Θ, Κ, Λ ἄρα τοῖς Ε, Μ, Ν, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσίν. ἀλλὰ οἱ Η, Θ, Κ, Λ τοῖς Α, Γ, Δ, Β ἐν τῷ αὐτῷ λόγῳ εἰσίν· καὶ οἱ Α, Γ, Δ, Β ἄρα τοῖς Ε, Μ, Ν, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσίν. οἱ δὲ Α, Γ, Δ, Β ἑξῆς ἀνάλογόν εἰσιν· καὶ οἱ Ε, Μ, Ν, Ζ ἄρα ἑξῆς ἀνάλογόν εἰσιν. ὅσοι ἄρα εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Ε, Ζ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

Proposition 8

If between two numbers there fall (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall in between (any two numbers) having the same ratio [as them] in continued proportion.



For let the numbers, *C* and *D*, fall between two numbers, *A* and *B*, in continued proportion, and let it be contrived (that) as *A* (is) to *B*, so *E* (is) to *F*. I say that, as many numbers as have fallen in between *A* and *B* in continued proportion, so many (numbers) will also fall in between *E* and *F* in continued proportion.

For as many as *A*, *B*, *C*, *D* are in multitude, let so many of the least numbers, *G*, *H*, *K*, *L*, having the same ratio as *A*, *B*, *C*, *D*, be taken [Prop. 7.33]. Thus, the outermost of them, *G* and *L*, are prime to one another [Prop. 8.3]. And since *A*, *B*, *C*, *D* are in the same ratio as *G*, *H*, *K*, *L*, and the multitude of *A*, *B*, *C*, *D* is equal to the multitude of *G*, *H*, *K*, *L*, thus, via equality, as *A* is to *B*, so *G* (is) to *L* [Prop. 7.14]. And as *A* (is) to *B*, so *E* (is) to *F*. And thus as *G* (is) to *L*, so *E* (is) to *F*. And *G* and *L* (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, *G* measures *E* the same number of times as *L* (measures) *F*. So as many times as *G* measures *E*, so many times let *H*, *K* also measure *M*, *N*, respectively. Thus, *G*, *H*, *K*, *L* measure *E*, *M*, *N*, *F* (respectively) an equal number of times. Thus, *G*, *H*, *K*, *L* are in the same ratio as *E*, *M*, *N*, *F* [Def. 7.20]. But, *G*, *H*, *K*, *L* are in the same ratio as *A*, *C*, *D*, *B*. Thus, *A*, *C*, *D*, *B* are also in the same ratio as *E*, *M*, *N*, *F*. And *A*, *C*, *D*, *B* are continuously proportional. Thus, *E*, *M*, *N*, *F* are also continuously proportional. Thus, as many numbers as have fallen in between A and B in continued proportion, so many numbers have also fallen in between E and F in continued proportion. (Which is) the very thing it was required to show.

ϑ′.

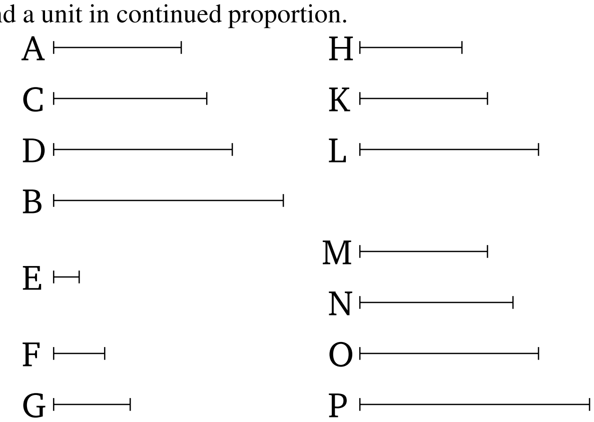
Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Ἔστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν οἱ Γ, Δ, καὶ ἐκκείσθω ἡ Ε μονάς· λέγω, ὅτι ὅσοι εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν Α, Β καὶ τῆς μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν Α, Γ, Δ, Β λόγῳ ὄντες οἱ Ζ, Η, τρεῖς δὲ οἱ Θ, Κ, Λ, καὶ ἀεὶ ἑξῆς ἑνὶ πλείους, ἕως ἂν ἴσον γένηται τὸ πλῆθος αὐτῶν τῷ πλήθει τῶν Α, Γ, Δ, Β. εἰλήφθωσαν, καὶ ἔστωσαν οἱ Μ, Ν, Ξ, Ο. φανερὸν δή, ὅτι ὁ μὲν Ζ ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, τὸν δὲ Θ πολλαπλασιάσας τὸν Μ πεποίηκεν, καὶ ὁ Η ἑαυτὸν μὲν πολλαπλασιάσας τὸν Λ πεποίηκεν, τὸν δὲ Λ πολλαπλασιάσας τὸν Ο πεποίηκεν. καὶ ἐπεὶ οἱ Μ, Ν, Ξ, Ο ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Ζ, Η, εἰσὶ δὲ καὶ οἱ Α, Γ, Δ, Β ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Ζ, Η, καί ἐστιν ἴσον τὸ πλῆθος τῶν Μ, Ν, Ξ, Ο τῷ πλήθει τῶν Α, Γ, Δ, Β, ἕκαστος ἄρα τῶν Μ, Ν, Ξ, Ο ἑκάστῳ τῶν Α, Γ, Δ, Β ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν Μ τῷ Α, ὁ δὲ Ο τῷ Β. καὶ ἐπεὶ ὁ Ζ ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, ὁ Ζ ἄρα τὸν Θ μετρεῖ κατὰ τὰς ἐν τῷ Ζ μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Ζ κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα ἡ Ε μονὰς τὸν Ζ ἀριθμὸν μετρεῖ καὶ ὁ Ζ τὸν Θ. ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ. πάλιν, ἐπεὶ ὁ Ζ τὸν Θ πολλαπλασιάσας τὸν Μ πεποίηκεν, ὁ Θ ἄρα τὸν Μ μετρεῖ κατὰ τὰς ἐν τῷ Ζ μονάδας. μετρεῖ δὲ καὶ ἡ Ε μονὰς τὸν Ζ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκις ἄρα

Proposition 9

If two numbers are prime to one another and there fall in between them (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall between each of them and a unit in continued proportion.



Let A and B be two numbers (which are) prime to one another, and let the (numbers) C and D fall in between them in continued proportion. And let the unit E be set out. I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall between each of A and B and the unit in continued proportion.

For let the least two numbers, F and G, which are in the ratio of A, C, D, B, be taken [Prop. 7.33]. And the (least) three (numbers), H, K, L. And so on, successively increasing by one, until the multitude of the (least numbers taken) is made equal to the multitude of A, C, D, B [Prop. 8.2]. Let them be taken, and let them be M, N, O, P. So (it is) clear that F has made H (by) multiplying itself, and has made M (by) multiplying H. And G has made L (by) multiplying itself, and has made P (by) multiplying L [Prop. 8.2 corr.]. And since M, N, O, P are the least of those (numbers) having the same ratio as F, G, and A, C, D, B are also the least of those (numbers) having the same ratio as F, G [Prop. 8.2], and the multitude of M, N, O, P is equal to the multitude of A, C, D, B, thus M, N, O, P are equal to A, C, D, B, respectively. Thus, M is equal to A, and P to B. And since F has made H (by) multiplying itself, F thus measures H according to the units in F [Def. 7.15]. And the unit E also measures F according to the units in it. Thus, the unit E measures the number F as many times as F (measures) H. Thus, as the unit E is to the number F, so F (is) to H [Def. 7.20]. Again, since F has made M (by) multiplying H, H thus measures M according to the units in F [Def. 7.15].

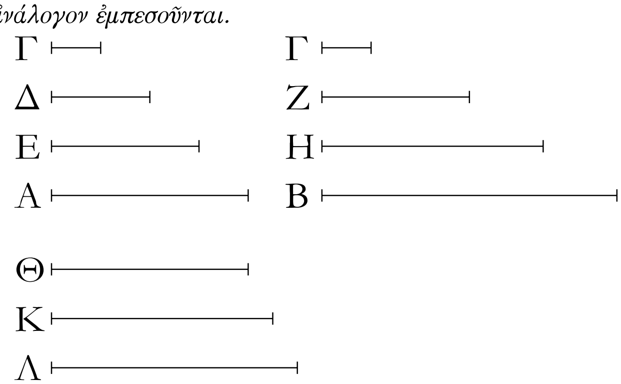
ἡ Ε μονὰς τὸν Ζ ἀριθμὸν μετρεῖ καὶ ὁ Θ τὸν Μ. ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Θ πρὸς τὸν Μ. ἐδείχθη δὲ καὶ ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ· καὶ ὡς ἄρα ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Μ. ἴσος δὲ ὁ Μ τῷ Α· ἔστιν ἄρα ὡς ἡ Ε μονὰς πρὸς τὸν Ζ ἀριθμόν, οὕτως ὁ Ζ πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Α. διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ Ε μονὰς πρὸς τὸν Η ἀριθμόν, οὕτως ὁ Η πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν Β. ὅσοι ἄρα εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν Α, Β καὶ μονάδος τῆς Ε μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

And the unit *E* also measures the number *F* according to the units in it. Thus, the unit *E* measures the number *F* as many times as *H* (measures) *M*. Thus, as the unit *E* is to the number *F*, so *H* (is) to *M* [Prop. 7.20]. And it was shown that as the unit *E* (is) to the number *F*, so *F* (is) to *H*. And thus as the unit *E* (is) to the number *F*, so *F* (is) to *H*, and *H* (is) to *M*. And *M* (is) equal to *A*. Thus, as the unit *E* is to the number *F*, so *F* (is) to *H*, and *H* to *A*. And so, for the same (reasons), as the unit *E* (is) to the number *G*, so *G* (is) to *L*, and *L* to *B*. Thus, as many (numbers) as have fallen in between *A* and *B* in continued proportion, so many numbers have also fallen between each of *A* and *B* and the unit *E* in continued proportion. (Which is) the very thing it was required to show.

ι'.

Ἐάν δύο ἀριθμῶν ἑκατέρου καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Δύο γὰρ ἀριθμῶν τῶν Α, Β καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἵ τε Δ, Ε καὶ οἱ Ζ, Η· λέγω, ὅτι ὅσοι ἑκατέρου τῶν Α, Β καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.



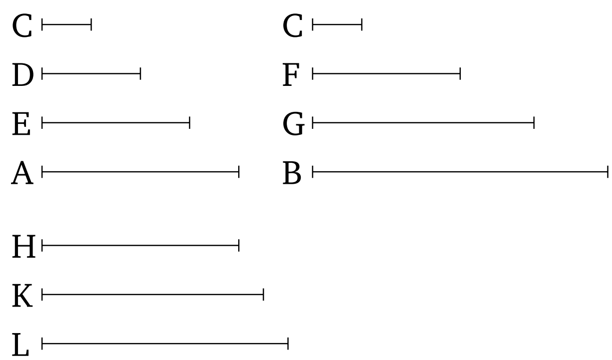
Ὁ Δ γὰρ τὸν Ζ πολλαπλασιάσας τὸν Θ ποιείτω, ἑκάτερος δὲ τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ ποιείτω.

Καὶ ἐπεί ἐστιν ὡς ἡ Γ μονὰς πρὸς τὸν Δ ἀριθμόν, οὕτως ὁ Δ πρὸς τὸν Ε, ἰσάκις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν Ε. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Δ ἄρα ἀριθμὸς τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν. πάλιν, ἐπεί ἐστιν ὡς ἡ Γ [μονὰς] πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ Ε πρὸς τὸν Α, ἰσάκις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Ε τὸν Α. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα τὸν Ε

Proposition 10

If (some) numbers fall between each of two numbers and a unit in continued proportion then, as many (numbers) as fall between each of the (two numbers) and the unit in continued proportion, so many (numbers) will also fall in between the (two numbers) themselves in continued proportion.

For let the numbers *D*, *E* and *F*, *G* fall between the numbers *A* and *B* (respectively) and the unit *C* in continued proportion. I say that, as many numbers as have fallen between each of *A* and *B* and the unit *C* in continued proportion, so many will also fall in between *A* and *B* in continued proportion.



For let *D* make *H* (by) multiplying *F*. And let *D*, *F* make *K*, *L*, respectively, by multiplying *H*.

As since as the unit *C* is to the number *D*, so *D* (is) to *E*, the unit *C* thus measures the number *D* as many times as *D* (measures) *E* [Def. 7.20]. And the unit *C* measures the number *D* according to the units in *D*. Thus, the number *D* also measures *E* according to the units in *D*. Thus, *D* has made *E* (by) multiplying itself. Again, since as the [unit] *C* is to the number *D*, so *E* (is) to *A*, the unit *C* thus measures the number *D* as many times as *E* (measures) *A* [Def. 7.20]. And the unit *C* measures the number *D* according to the units in *D*. Thus, *E* also measures *A* according to the units in *D*. Thus, *D* has made *A* (by) multiplying *E*. And so, for the same (reasons), *F* *πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ μὲν Ζ ἑαυτὸν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Ε πρὸς τὸν Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Θ πρὸς τὸν Η. καὶ ὡς ἄρα ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ε, Θ πολλαπλασιάσας ἑκάτερον τῶν Α, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Κ. ἀλλ᾽ ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ. πάλιν, ἐπεὶ ἑκάτερος τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Λ. ἀλλ᾽ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ. ἔτι ἐπεὶ ὁ Ζ ἑκάτερον τῶν Θ, Η πολλαπλασιάσας ἑκάτερον τῶν Λ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Λ πρὸς τὸν Β. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Λ πρὸς τὸν Β. ἐδείχθη δὲ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὅ τε Α πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Λ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν Β. οἱ Α, Κ, Λ, Β ἄρα κατὰ τὸ συνεχὲς ἑξῆς εἰσιν ἀνάλογον. ὅσοι ἄρα ἑκατέρου τῶν Α, Β καὶ τῆς Γ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἐμπεσοῦνται· ὅπερ ἔδει δεῖξαι.*

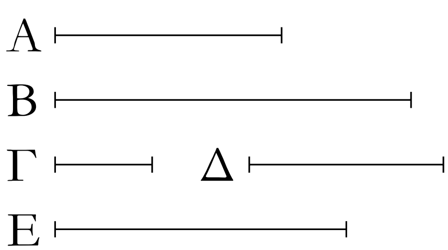
has made *G* (by) multiplying itself, and has made *B* (by) multiplying *G*. And since *D* has made *E* (by) multiplying itself, and has made *H* (by) multiplying *F*, thus as *D* is to *F*, so *E* (is) to *H* [Prop 7.17]. And so, for the same reasons, as *D* (is) to *F*, so *H* (is) to *G* [Prop. 7.18]. And thus as *E* (is) to *H*, so *H* (is) to *G*. Again, since *D* has made *A*, *K* (by) multiplying *E*, *H*, respectively, thus as *E* is to *H*, so *A* (is) to *K* [Prop 7.17]. But, as *E* (is) to *H*, so *D* (is) to *F*. And thus as *D* (is) to *F*, so *A* (is) to *K*. Again, since *D*, *F* have made *K*, *L*, respectively, (by) multiplying *H*, thus as *D* is to *F*, so *K* (is) to *L* [Prop. 7.18]. But, as *D* (is) to *F*, so *A* (is) to *K*. And thus as *A* (is) to *K*, so *K* (is) to *L*. Further, since *F* has made *L*, *B* (by) multiplying *H*, *G*, respectively, thus as *H* is to *G*, so *L* (is) to *B* [Prop 7.17]. And as *H* (is) to *G*, so *D* (is) to *F*. And thus as *D* (is) to *F*, so *L* (is) to *B*. And it was also shown that as *D* (is) to *F*, so *A* (is) to *K*, and *K* to *L*. And thus as *A* (is) to *K*, so *K* (is) to *L*, and *L* to *B*. Thus, *A*, *K*, *L*, *B* are successively in continued proportion. Thus, as many numbers as fall between each of *A* and *B* and the unit *C* in continued proportion, so many will also fall in between *A* and *B* in continued proportion. (Which is) the very thing it was required to show.

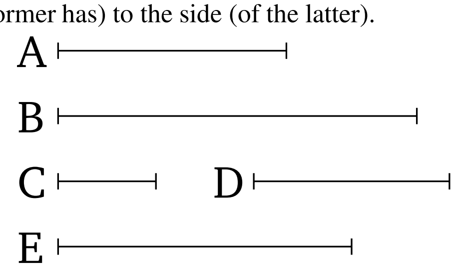
ια΄.

Proposition 11

Δύο τετραγώνων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

There exists one number in mean proportion to two (given) square numbers.† And (one) square (number) has to the (other) square (number) a squared‡ ratio with respect to (that) the side (of the former has) to the side (of the latter).





Ἔστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι τῶν Α, Β εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ.

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω. καὶ ἐπεὶ τετράγωνός ἐστιν ὁ Α, πλευρὰ δὲ αὐτοῦ ἐστιν ὁ Γ, ὁ Γ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν. ἐπεὶ οὖν ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν Α, Ε πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς

Let *A* and *B* be square numbers, and let *C* be the side of *A*, and *D* (the side) of *B*. I say that there exists one number in mean proportion to *A* and *B*, and that *A* has to *B* a squared ratio with respect to (that) *C* (has) to *D*.

For let *C* make *E* (by) multiplying *D*. And since *A* is square, and *C* is its side, *C* has thus made *A* (by) multiplying itself. And so, for the same (reasons), *D* has made *B* (by) multiplying itself. Therefore, since *C* has made *A*, *E* (by) multiplying *C*, *D*, respectively, thus as *C* is to *D*, so *A* (is) to *E* [Prop. 7.17]. And so, for the same (reasons), as *C* (is) to *D*, so τὸν Ε. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Β. καὶ ὡς ἄρα ὁ Α πρὸς τὸν Ε, οὕτως ὁ Ε πρὸς τὸν Β. τῶν Α, Β ἄρα εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός.

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τρεῖς ἀριθμοὶ ἀνάλογόν εἰσιν οἱ Α, Ε, Β, ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Α πρὸς τὸν Ε. ὡς δὲ ὁ Α πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ἡ Γ πλευρὰ πρὸς τὴν Δ· ὅπερ ἔδει δεῖξαι.

E (is) to *B* [Prop. 7.18]. And thus as *A* (is) to *E*, so *E* (is) to *B*. Thus, one number (namely, *E*) is in mean proportion to *A* and *B*.

So I say that *A* also has to *B* a squared ratio with respect to (that) *C* (has) to *D*. For since *A*, *E*, *B* are three (continuously) proportional numbers, *A* thus has to *B* a squared ratio with respect to (that) *A* (has) to *E* [Def. 5.9]. And as *A* (is) to *E*, so *C* (is) to *D*. Thus, *A* has to *B* a squared ratio with respect to (that) side *C* (has) to (side) *D*. (Which is) the very thing it was required to show.

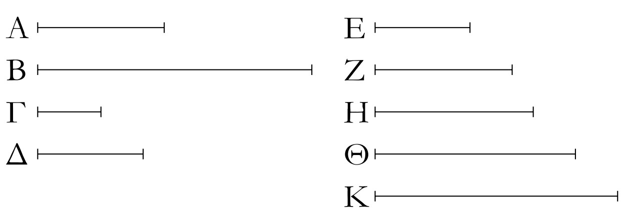
† In other words, between two given square numbers there exists a number in continued proportion.

‡ Literally, "double".

ιβʹ.

Δύο κύβων ἀριθμῶν δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ κύβος πρὸς τὸν κύβον τριπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

Ἔστωσαν κύβοι ἀριθμοὶ οἱ Α, Β καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι τῶν Α, Β δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ.



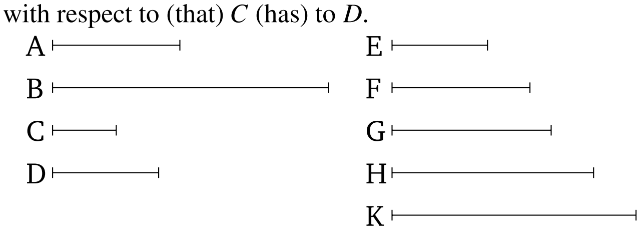
Ὁ γὰρ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιείτω, τὸν δὲ Δ πολλαπλασιάσας τὸν Ζ ποιείτω, ὁ δὲ Δ ἑαυτὸν πολλαπλασιάσας τὸν Η ποιείτω, ἑκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ ποιείτω.

Καὶ ἐπεὶ κύβος ἐστὶν ὁ Α, πλευρὰ δὲ αὐτοῦ ὁ Γ, καὶ ὁ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, ὁ Γ ἄρα ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν Ε, Ζ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Γ ἑκάτερον τῶν Ε, Ζ πολλαπλασιάσας ἑκάτερον τῶν Α, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ. πάλιν, ἐπεὶ ἑκάτερος τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ζ, Η πολλαπλασιάσας ἑκάτερον τῶν Κ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρὸς τὸν Η,

Proposition 12

There exist two numbers in mean proportion to two (given) cube numbers.† And (one) cube (number) has to the (other) cube (number) a cubed‡ ratio with respect to (that) the side (of the former has) to the side (of the latter).

Let *A* and *B* be cube numbers, and let *C* be the side of *A*, and *D* (the side) of *B*. I say that there exist two numbers in mean proportion to *A* and *B*, and that *A* has to *B* a cubed ratio with respect to (that) *C* (has) to *D*.



For let *C* make *E* (by) multiplying itself, and let it make *F* (by) multiplying *D*. And let *D* make *G* (by) multiplying itself, and let *C*, *D* make *H*, *K*, respectively, (by) multiplying *F*.

And since *A* is cube, and *C* (is) its side, and *C* has made *E* (by) multiplying itself, *C* has thus made *E* (by) multiplying itself, and has made *A* (by) multiplying *E*. And so, for the same (reasons), *D* has made *G* (by) multiplying itself, and has made *B* (by) multiplying *G*. And since *C* has made *E*, *F* (by) multiplying *C*, *D*, respectively, thus as *C* is to *D*, so *E* (is) to *F* [Prop. 7.17]. And so, for the same (reasons), as *C* (is) to *D*, so *F* (is) to *G* [Prop. 7.18]. Again, since *C* has made *A*, *H* (by) multiplying *E*, *F*, respectively, thus as *E* is to *F*, so *A* (is) to *H* [Prop. 7.17]. And as *E* (is) to *F*, so *C* (is) to *D*. And thus as *C* (is) to *D*, so *A* (is) to *H*. Again, since *C*, *D* have made *H*, *K*, respectively, (by) multiplying *F*, thus as *C* is to *D*, so *H* (is) to *K* [Prop. 7.18]. Again, since *D* has made *K*, *B* (by) multiplying *F*, *G*, respectively, thus as *F* is to *G*, so *K* (is) to *B* [Prop. 7.17]. And as *F* (is) to *G*, so *C* (is) to *D*. And thus as *C* (is) to *D*, so *A* (is) to *H*, and *H* to *K*, and *K* to *B*. Thus, *H* and *K* are two (numbers) in mean proportion to *A* and *B*.

So I say that *A* also has to *B* a cubed ratio with respect to *οὕτως ὁ Κ πρὸς τὸν Β. ὡς δὲ ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὅ τε Α πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογόν εἰσιν οἱ Θ, Κ.*

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν οἱ Α, Θ, Κ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὁ Α [ἄρα] πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

(that) *C* (has) to *D*. For since *A*, *H*, *K*, *B* are four (continuously) proportional numbers, *A* thus has to *B* a cubed ratio with respect to (that) *A* (has) to *H* [Def. 5.10]. And as *A* (is) to *H*, so *C* (is) to *D*. And [thus] *A* has to *B* a cubed ratio with respect to (that) *C* (has) to *D*. (Which is) the very thing it was required to show.

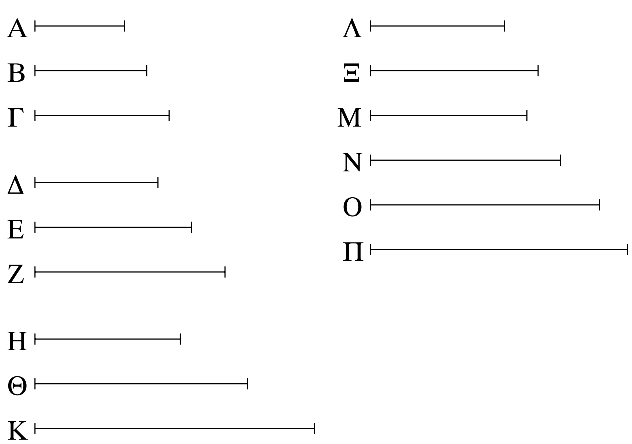
† In other words, between two given cube numbers there exist two numbers in continued proportion.

‡ Literally, "triple".

ιγ'.

Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, καὶ πολλαπλασιάσας ἕκαστος ἑαυτὸν ποιῇ τινα, οἱ γενόμενοι ἐξ αὐτῶν ἀνάλογον ἔσονται· καὶ ἐὰν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσί τινας, καὶ αὐτοὶ ἀνάλογον ἔσονται [καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].

Ἔστωσαν ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ Α, Β, Γ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ, καὶ οἱ Α, Β, Γ ἑαυτοὺς μὲν πολλαπλασιάσαντες τοὺς Δ, Ε, Ζ ποιείτωσαν, τοὺς δὲ Δ, Ε, Ζ πολλαπλασιάσαντες τοὺς Η, Θ, Κ ποιείτωσαν· λέγω, ὅτι οἵ τε Δ, Ε, Ζ καὶ οἱ Η, Θ, Κ ἑξῆς ἀνάλογον εἰσιν.



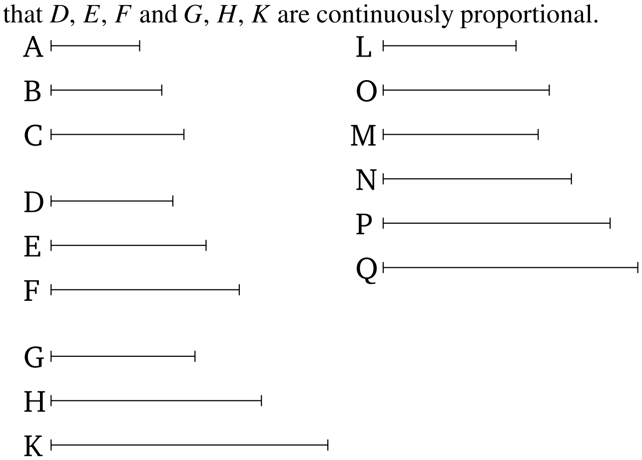
Ὁ μὲν γὰρ Α τὸν Β πολλαπλασιάσας τὸν Λ ποιείτω, ἑκάτερος δὲ τῶν Α, Β τὸν Λ πολλαπλασιάσας ἑκάτερον τῶν Μ, Ν ποιείτω. καὶ πάλιν ὁ μὲν Β τὸν Γ πολλαπλασιάσας τὸν Ξ ποιείτω, ἑκάτερος δὲ τῶν Β, Γ τὸν Ξ πολλαπλασιάσας ἑκάτερον τῶν Ο, Π ποιείτω.

Ὁμοίως δὴ τοῖς ἐπάνω δείξομεν, ὅτι οἱ Δ, Λ, Ε καὶ οἱ Η, Μ, Ν, Θ ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ, καὶ ἔτι οἱ Ε, Ξ, Ζ καὶ οἱ Θ, Ο, Π, Κ ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Β πρὸς τὸν Γ λόγῳ. καὶ ἔστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· καὶ οἱ Δ, Λ, Ε ἄρα τοῖς Ε, Ξ, Ζ ἐν

Proposition 13

If there are any multitude whatsoever of continuously proportional numbers, and each makes some (number by) multiplying itself, then the (numbers) created from them will (also) be (continuously) proportional. And if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be (continuously) proportional [and this always happens with the extremes].

Let *A*, *B*, *C* be any multitude whatsoever of continuously proportional numbers, (such that) as *A* (is) to *B*, so *B* (is) to *C*. And let *A*, *B*, *C* make *D*, *E*, *F* (by) multiplying themselves, and let them make *G*, *H*, *K* (by) multiplying *D*, *E*, *F*. I say that *D*, *E*, *F* and *G*, *H*, *K* are continuously proportional.



For let *A* make *L* (by) multiplying *B*. And let *A*, *B* make *M*, *N*, respectively, (by) multiplying *L*. And, again, let *B* make *O* (by) multiplying *C*. And let *B*, *C* make *P*, *Q*, respectively, (by) multplying *O*.

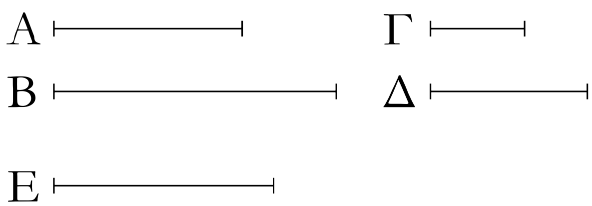
So, similarly to the above, we can show that *D*, *L*, *E* and *G*, *M*, *N*, *H* are continuously proportional in the ratio of *A* to *B*, and, further, (that) *E*, *O*, *F* and *H*, *P*, *Q*, *K* are continuously proportional in the ratio of *B* to *C*. And as *A* is to *B*, so *B* (is) to *C*. And thus *D*, *L*, *E* are in the same ratio as *E*, *O*, *F*, and, further, *G*, *M*, *N*, *H* (are in the same ratio) as *H*, *P*, *Q*, *τῷ αὐτῷ λόγῳ εἰσὶ καὶ ἔτι οἱ Η, Μ, Ν, Θ τοῖς Θ, Ο, Π, Κ. καί ἐστιν ἴσον τὸ μὲν τῶν Δ, Λ, Ε πλῆθος τῷ τῶν Ε, Ξ, Ζ πλήθει, τὸ δὲ τῶν Η, Μ, Ν, Θ τῷ τῶν Θ, Ο, Π, Κ· δι᾽ ἴσου ἄρα ἐστὶν ὡς μὲν ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Ε πρὸς τὸν Ζ, ὡς δὲ ὁ Η πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Κ· ὅπερ ἔδει δεῖξαι.*

K. And the multitude of *D*, *L*, *E* is equal to the multitude of *E*, *O*, *F*, and that of *G*, *M*, *N*, *H* to that of *H*, *P*, *Q*, *K*. Thus, via equality, as *D* is to *E*, so *E* (is) to *F*, and as *G* (is) to *H*, so *H* (is) to *K* [Prop. 7.14]. (Which is) the very thing it was required to show.

ιδ΄.

Ἐὰν τετράγωνος τετράγωνον μετρῇ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῇ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.

Ἔστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, ὁ δὲ Α τὸν Β μετρείτω· λέγω, ὅτι καὶ ὁ Γ τὸν Δ μετρεῖ.



Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιείτω· οἱ Α, Ε, Β ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Ε, Β ἑξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καί ἐστιν ὡς ὁ Α πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Πάλιν δὴ ὁ Γ τὸν Δ μετρείτω· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεῖ.

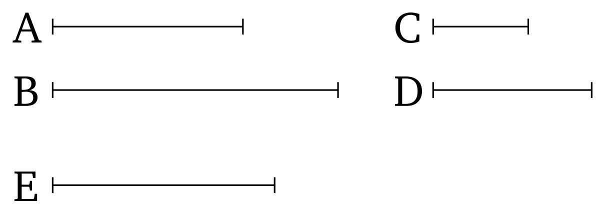
Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι οἱ Α, Ε, Β ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Ε, μετρεῖ δὲ ὁ Γ τὸν Δ, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καί εἰσιν οἱ Α, Ε, Β ἑξῆς ἀνάλογον· μετρεῖ ἄρα καὶ ὁ Α τὸν Β.

Ἐὰν ἄρα τετράγωνος τετράγωνον μετρῇ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῇ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσει· ὅπερ ἔδει δεῖξαι.

Proposition 14

If a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number).

Let *A* and *B* be square numbers, and let *C* and *D* be their sides (respectively). And let *A* measure *B*. I say that *C* also measures *D*.



For let *C* make *E* (by) multiplying *D*. Thus, *A*, *E*, *B* are continuously proportional in the ratio of *C* to *D* [Prop. 8.11]. And since *A*, *E*, *B* are continuously proportional, and *A* measures *B*, *A* thus also measures *E* [Prop. 8.7]. And as *A* is to *E*, so *C* (is) to *D*. Thus, *C* also measures *D* [Def. 7.20].

So, again, let *C* measure *D*. I say that *A* also measures *B*.

For similarly, with the same construction, we can show that *A*, *E*, *B* are continuously proportional in the ratio of *C* to *D*. And since as *C* is to *D*, so *A* (is) to *E*, and *C* measures *D*, *A* thus also measures *E* [Def. 7.20]. And *A*, *E*, *B* are continuously proportional. Thus, *A* also measures *B*.

Thus, if a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number). (Which is) the very thing it was required to show.

ιε΄.

Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μετρῇ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῇ, καὶ ὁ κύβος τὸν κύβον μετρήσει.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον τὸν Β μετρείτω, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ μετρεῖ.

Ὁ Γ γὰρ ἑαυτὸν πολλαπλασιάσας τὸν Ε ποιείτω, ὁ δὲ Δ ἑαυτὸν πολλαπλασιάσας τὸν Η ποιείτω, καὶ ἔτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Ζ [ποιείτω], ἑκάτερος δὲ τῶν Γ, Δ τὸν Ζ

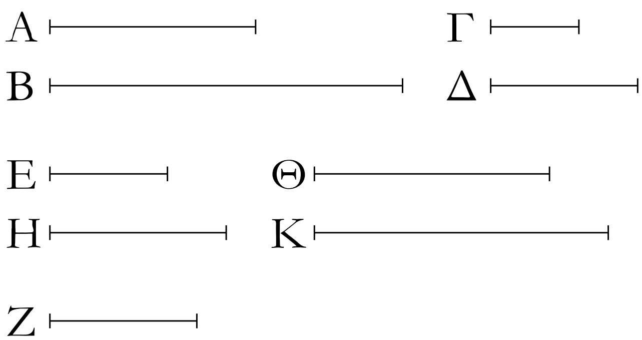
Proposition 15

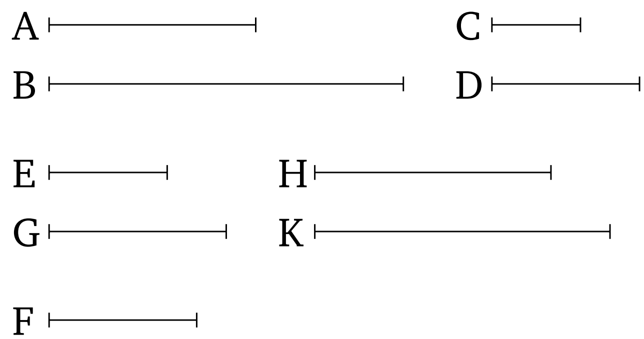
If a cube number measures a(nother) cube number then the side (of the former) will also measure the side (of the latter). And if the side (of a cube number) measures the side (of another cube number) then the (former) cube (number) will also measure the (latter) cube (number).

For let the cube number *A* measure the cube (number) *B*, and let *C* be the side of *A*, and *D* (the side) of *B*. I say that *C* measures *D*.

For let *C* make *E* (by) multiplying itself. And let *D* make *πολλαπλασιάσας ἑκάτερον τῶν Θ, Κ ποιείτω. φανερὸν δή, ὅτι οἱ Ε, Ζ, Η καὶ οἱ Α, Θ, Κ, Β ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Θ, Κ, Β ἑξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ τὸν Θ. καί ἐστιν ὡς ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.*

G (by) multiplying itself. And, further, [let] *C* [make] *F* (by) multiplying *D*, and let *C*, *D* make *H*, *K*, respectively, (by) multiplying *F*. So it is clear that *E*, *F*, *G* and *A*, *H*, *K*, *B* are continuously proportional in the ratio of *C* to *D* [Prop. 8.12]. And since *A*, *H*, *K*, *B* are continuously proportional, and *A* measures *B*, (*A*) thus also measures *H* [Prop. 8.7]. And as *A* is to *H*, so *C* (is) to *D*. Thus, *C* also measures *D* [Def. 7.20].





Ἀλλὰ δὴ μετρείτω ὁ Γ τὸν Δ· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρήσει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δὴ δείξομεν, ὅτι οἱ Α, Θ, Κ, Β ἑξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ ὁ Γ τὸν Δ μετρεῖ, καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ, καὶ ὁ Α ἄρα τὸν Θ μετρεῖ· ὥστε καὶ τὸν Β μετρεῖ ὁ Α· ὅπερ ἔδει δεῖξαι.

And so let *C* measure *D*. I say that *A* will also measure *B*.

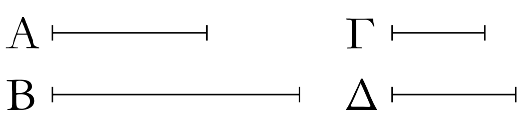
For similarly, with the same construction, we can show that *A*, *H*, *K*, *B* are continuously proportional in the ratio of *C* to *D*. And since *C* measures *D*, and as *C* is to *D*, so *A* (is) to *H*, *A* thus also measures *H* [Def. 7.20]. Hence, *A* also measures *B*. (Which is) the very thing it was required to show.

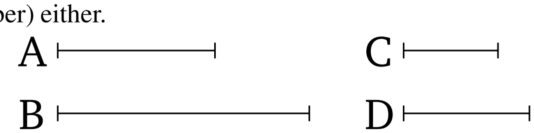
ιϛ'.

Proposition 16

Ἐὰν τετράγωνος ἀριθμὸς τετράγωνον ἀριθμὸν μὴ μετρῇ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· κἂν ἡ πλευρὰ τὴν πλευρὰν μὴ μετρῇ, οὐδὲ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.

If a square number does not measure a(nother) square number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a square number) does not measure the side (of another square number) then the (former) square (number) will not measure the (latter) square (number) either.





Ἔστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, καὶ μὴ μετρείτω ὁ Α τὸν Β· λέγω, ὅτι οὐδὲ ὁ Γ τὸν Δ μετρεῖ.

Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ, μετρήσει καὶ ὁ Α τὸν Β. οὐ μετρεῖ δὲ ὁ Α τὸν Β· οὐδὲ ἄρα ὁ Γ τὸν Δ μετρήσει.

μὴ μετρείτω [δὴ] πάλιν ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Εἰ γὰρ μετρεῖ ὁ Α τὸν Β, μετρήσει καὶ ὁ Γ τὸν Δ. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ᾽ ἄρα ὁ Α τὸν Β μετρήσει· ὅπερ ἔδει δεῖξαι.

Let *A* and *B* be square numbers, and let *C* and *D* be their sides (respectively). And let *A* not measure *B*. I say that *C* does not measure *D* either.

For if *C* measures *D* then *A* will also measure *B* [Prop. 8.14]. And *A* does not measure *B*. Thus, *C* will not measure *D* either.

[So], again, let *C* not measure *D*. I say that *A* will not measure *B* either.

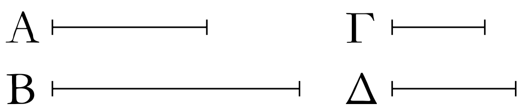
For if *A* measures *B* then *C* will also measure *D* [Prop. 8.14]. And *C* does not measure *D*. Thus, *A* will not measure *B* either. (Which is) the very thing it was required to show.

ιζ'.

Proposition 17

Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μὴ μετρῇ, οὐδὲ ἡ

If a cube number does not measure a(nother) cube number *πλευρὰ τὴν πλευρὰν μετρήσει· κἂν ἡ πλευρὰ τὴν πλευρὰν μὴ μετρῇ, οὐδὲ ὁ κύβος τὸν κύβον μετρήσει.*



Κύβος γὰρ ἀριθμὸς ὁ Α κύβον ἀριθμὸν τὸν Β μὴ μετρείτω, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ οὐ μετρήσει.

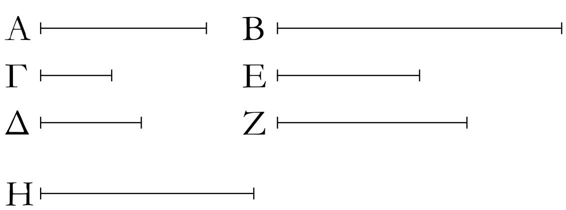
Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ, καὶ ὁ Α τὸν Β μετρήσει. οὐ μετρεῖ δὲ ὁ Α τὸν Β· οὐδ᾿ ἄρα ὁ Γ τὸν Δ μετρεῖ.

Ἀλλὰ δὴ μὴ μετρείτω ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Εἰ γὰρ ὁ Α τὸν Β μετρεῖ, καὶ ὁ Γ τὸν Δ μετρήσει. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ᾿ ἄρα ὁ Α τὸν Β μετρήσει· ὅπερ ἔδει δεῖξαι.

ιη΄.

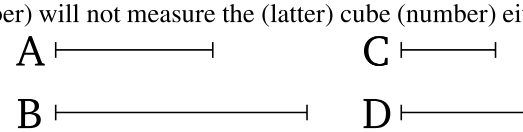
Δύο ὁμοίων ἐπιπέδων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός· καὶ ὁ ἐπίπεδος πρὸς τὸν ἐπίπεδον διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.



Ἔστωσαν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ Β οἱ Ε, Ζ. καὶ ἐπεὶ ὅμοιοι ἐπίπεδοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. λέγω οὖν, ὅτι τῶν Α, Β εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ, τουτέστιν ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον [πλευράν].

Καὶ ἐπεί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ε, ὁ Δ πρὸς τὸν Ζ. καὶ ἐπεὶ ἐπίπεδός ἐστιν ὁ Α, πλευραὶ δὲ αὐτοῦ οἱ Γ, Δ, ὁ Δ ἄρα τὸν Γ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε τὸν Ζ πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ Δ δὴ τὸν Ε πολλαπλασιάσας τὸν Η ποιείτω. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Η πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Η. ἀλλ᾿ ὡς ὁ Γ πρὸς τὸν Ε, [οὕτως] ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Ε τὸν μὲν Δ πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς

then the side (of the former) will not measure the side (of the latter) either. And if the side (of a cube number) does not measure the side (of another cube number) then the (former) cube (number) will not measure the (latter) cube (number) either.



For let the cube number *A* not measure the cube number *B*. And let *C* be the side of *A*, and *D* (the side) of *B*. I say that *C* will not measure *D*.

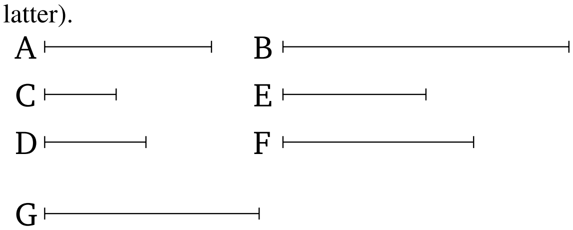
For if *C* measures *D* then *A* will also measure *B* [Prop. 8.15]. And *A* does not measure *B*. Thus, *C* does not measure *D* either.

And so let *C* not measure *D*. I say that *A* will not measure *B* either.

For if *A* measures *B* then *C* will also measure *D* [Prop. 8.15]. And *C* does not measure *D*. Thus, *A* will not measure *B* either. (Which is) the very thing it was required to show.

Proposition 18

There exists one number in mean proportion to two similar plane numbers. And (one) plane (number) has to the (other) plane (number) a squared† ratio with respect to (that) a corresponding side (of the former has) to a corresponding side (of the latter).



Let *A* and *B* be two similar plane numbers. And let the numbers *C*, *D* be the sides of *A*, and *E*, *F* (the sides) of *B*. And since similar numbers are those having proportional sides [Def. 7.21], thus as *C* is to *D*, so *E* (is) to *F*. Therefore, I say that there exists one number in mean proportion to *A* and *B*, and that *A* has to *B* a squared ratio with respect to that *C* (has) to *E*, or *D* to *F*—that is to say, with respect to (that) a corresponding side (has) to a corresponding [side].

For since as *C* is to *D*, so *E* (is) to *F*, thus, alternately, as *C* is to *E*, so *D* (is) to *F* [Prop. 7.13]. And since *A* is plane, and *C*, *D* its sides, *D* has thus made *A* (by) multiplying *C*. And so, for the same (reasons), *E* has made *B* (by) multiplying *F*. So let *D* make *G* (by) multiplying *E*. And since *D* has made *A* (by) multiplying *C*, and has made *G* (by) multiplying *E*, thus as *C* is to *E*, so *A* (is) to *G* [Prop. 7.17]. But as *C* (is) to *E*, [so] *D* (is) to *F*. And thus as *D* (is) to *F*, so *A* (is) to *G*. Again, since *E* has made *G* (by) multiplying *D*, and has made *B* (by) multiplying *F*, thus as *D* is to *F*, so *G* (is) to *B* [Prop. 7.17]. And it was also shown that as *D* (is) to *F*, so *A* (is) to *G*. And thus as *A* (is) to *G*, so *G* (is) to *B*. Thus, *A*, *G*, *τὸν Ζ, οὕτως ὁ Η πρὸς τὸν Β. ἐδείχθη δὲ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Η· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Η, οὕτως ὁ Η πρὸς τὸν Β. οἱ Α, Η, Β ἄρα ἑξῆς ἀνάλογόν εἰσιν. τῶν Α, Β ἄρα εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός.*

Λέγω δή, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἤπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ. ἐπεὶ γὰρ οἱ Α, Η, Β ἑξῆς ἀνάλογόν εἰσιν, ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ πρὸς τὸν Η. καί ἐστιν ὡς ὁ Α πρὸς τὸν Η, οὕτως ὅ τε Γ πρὸς τὸν Ε καὶ ὁ Δ πρὸς τὸν Ζ. καὶ ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ· ὅπερ ἔδει δεῖξαι.

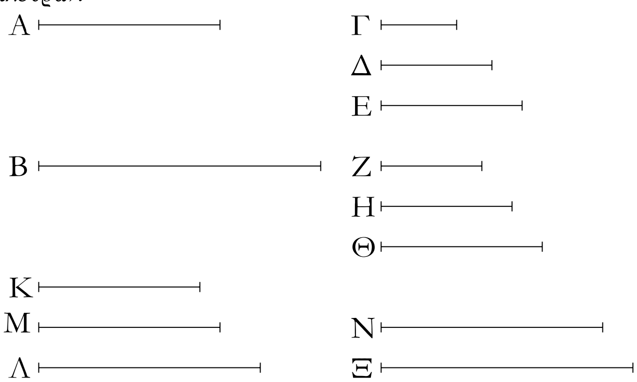
B are continously proportional. Thus, there exists one number (namely, *G*) in mean proportion to *A* and *B*.

So I say that *A* also has to *B* a squared ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) *C* (has) to *E*, or *D* to *F*. For since *A*, *G*, *B* are continuously proportional, *A* has to *B* a squared ratio with respect to (that *A* has) to *G* [Prop. 5.9]. And as *A* is to *G*, so *C* (is) to *E*, and *D* to *F*. And thus *A* has to *B* a squared ratio with respect to (that) *C* (has) to *E*, or *D* to *F*. (Which is) the very thing it was required to show.

† Literally, "double".

ιθ'.

Δύο ὁμοίων στερεῶν ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· καὶ ὁ στερεὸς πρὸς τὸν ὅμοιον στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.

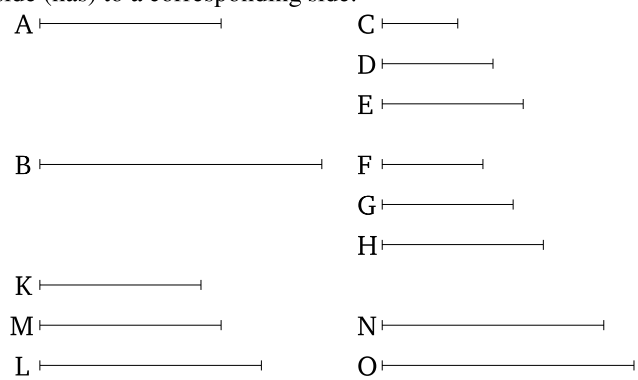


Ἔστωσαν δύο ὅμοιοι στερεοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ, Ε, τοῦ δὲ Β οἱ Ζ, Η, Θ. καὶ ἐπεὶ ὅμοιοι στερεοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς μὲν ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ὡς δὲ ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Θ. λέγω, ὅτι τῶν Α, Β δύο μέσοι ἀνάλογόν ἐμπίπτουσιν ἀριθμοί, καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ.

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Κ ποιείτω, ὁ δὲ Ζ τὸν Η πολλαπλασιάσας τὸν Λ ποιείτω. καὶ ἐπεὶ οἱ Γ, Δ τοῖς Ζ, Η ἐν τῷ αὐτῷ λόγῳ εἰσίν, καὶ ἐκ μὲν τῶν Γ, Δ ἐστιν ὁ Κ, ἐκ δὲ τῶν Ζ, Η ὁ Λ, οἱ Κ, Λ [ἄρα] ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί· τῶν Κ, Λ ἄρα εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός. ἔστω ὁ Μ. ὁ Μ ἄρα ἐστὶν ὁ ἐκ τῶν Δ, Ζ, ὡς ἐν τῷ πρὸ τούτου θεωρήματι ἐδείχθη. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Κ πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Μ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Μ. ἀλλ' ὡς ὁ Κ πρὸς τὸν Μ, ὁ Μ πρὸς τὸν Λ. οἱ Κ, Μ, Λ ἄρα ἑξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ. καὶ ἐπεί ἐστιν

Proposition 19

Two numbers fall (between) two similar solid numbers in mean proportion. And a solid (number) has to a similar solid (number) a cubed† ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let *A* and *B* be two similar solid numbers, and let *C*, *D*, *E* be the sides of *A*, and *F*, *G*, *H* (the sides) of *B*. And since similar solid (numbers) are those having proportional sides [Def. 7.21], thus as *C* is to *D*, so *F* (is) to *G*, and as *D* (is) to *E*, so *G* (is) to *H*. I say that two numbers fall (between) *A* and *B* in mean proportion, and (that) *A* has to *B* a cubed ratio with respect to (that) *C* (has) to *F*, and *D* to *G*, and, further, *E* to *H*.

For let *C* make *K* (by) multiplying *D*, and let *F* make *L* (by) multiplying *G*. And since *C*, *D* are in the same ratio as *F*, *G*, and *K* is the (number created) from (multiplying) *C*, *D*, and *L* the (number created) from (multiplying) *F*, *G*, [thus] *K* and *L* are similar plane numbers [Def. 7.21]. Thus, there exits one number in mean proportion to *K* and *L* [Prop. 8.18]. Let it be *M*. Thus, *M* is the (number created) from (multiplying) *D*, *F*, as shown in the theorem before this (one). And since *D* has made *K* (by) multiplying *C*, and has made *M* (by) multiplying *F*, thus as *C* is to *F*, so *K* (is) to *M* [Prop. 7.17]. But, as *K* (is) to *M*, (so) *M* (is) to *L*. Thus, *K*, *M*, *L* are continuously *ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Δ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Η, οὕτως ὁ Ε πρὸς τὸν Θ. οἱ Κ, Μ, Λ ἄρα ἑξῆς εἰσιν ἀνάλογον ἔν τε τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ καὶ τῷ τοῦ Δ πρὸς τὸν Η καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Θ. ἑκατερος δὴ τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ ποιείτω. καὶ ἐπεὶ στερεός ἐστιν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Γ, Δ, Ε, ὁ Ε ἄρα τὸν ἐκ τῶν Γ, Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ ἐκ τῶν Γ, Δ ἐστιν ὁ Κ· ὁ Ε ἄρα τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Θ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Ε τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Μ πολλαπλασιάσας τὸν Ν πεποίηκεν, ἔστιν ἄρα ὡς ὁ Κ πρὸς τὸν Μ, οὕτως ὁ Α πρὸς τὸν Ν. ὡς δὲ ὁ Κ πρὸς τὸν Μ, οὕτως ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Ν. πάλιν, ἐπεὶ ἑκάτερος τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Ν πρὸς τὸν Ξ. ἀλλ᾽ ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὅ τε Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. πάλιν, ἐπεὶ ὁ Θ τὸν Μ πολλαπλασιάσας τὸν Ξ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὁ Ξ πρὸς τὸν Β. ἀλλ᾽ ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως οὐ μόνον ὁ Ξ πρὸς τὸν Β, ἀλλὰ καὶ ὁ Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. οἱ Α, Ν, Ξ, Β ἄρα ἑξῆς εἰσιν ἀνάλογον ἐν τοῖς εἰρημένοις τῶν πλευρῶν λόγοις.*

Λέγω, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἤπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ ἢ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογόν εἰσιν οἱ Α, Ν, Ξ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ὁ Α πρὸς τὸν Ν. ἀλλ᾽ ὡς ὁ Α πρὸς τὸν Ν, οὕτως ἐδείχθη ὅ τε Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. καὶ ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἤπερ ἡ ομόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἤπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· ὅπερ ἔδει δεῖξαι.

proportional in the ratio of C to F. And since as C is to D, so F (is) to G, thus, alternately, as C is to F, so D (is) to G [Prop. 7.13]. And so, for the same (reasons), as D (is) to G, so E (is) to H. Thus, K, M, L are continuously proportional in the ratio of C to F, and of D to G, and, further, of E to H. So let E, H make N, O, respectively, (by) multiplying M. And since A is solid, and C, D, E are its sides, E has thus made A (by) multiplying the (number created) from (multiplying) C, D. And K is the (number created) from (multiplying) C, D. Thus, E has made A (by) multiplying K. And so, for the same (reasons), H has made B (by) multiplying L. And since E has made A (by) multiplying K, but has, in fact, also made N (by) multiplying M, thus as K is to M, so A (is) to N [Prop. 7.17]. And as K (is) to M, so C (is) to F, and D to G, and, further, E to H. And thus as C (is) to F, and D to G, and E to H, so A (is) to N. Again, since E, H have made N, O, respectively, (by) multiplying M, thus as E is to H, so N (is) to O [Prop. 7.18]. But, as E (is) to H, so C (is) to F, and D to G. And thus as C (is) to F, and D to G, and E to H, so (is) A to N, and N to O. Again, since H has made O (by) multiplying M, but has, in fact, also made B (by) multiplying L, thus as M (is) to L, so O (is) to B [Prop. 7.17]. But, as M (is) to L, so C (is) to F, and D to G, and E to H. And thus as C (is) to F, and D to G, and E to H, so not only (is) O to B, but also A to N, and N to O. Thus, A, N, O, B are continuously proportional in the aforementioned ratios of the sides.

So I say that A also has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C (has) to F, or D to G, and, further, E to H. For since A, N, O, B are four continuously proportional numbers, A thus has to B a cubed ratio with respect to (that) A (has) to N [Def. 5.10]. But, as A (is) to N, so it was shown (is) C to F, and D to G, and, further, E to H. And thus A has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C (has) to F, and D to G, and, further, E to H. (Which is) the very thing it was required to show.

† Literally, "triple".

κ'.

Ἐὰν δύο ἀριθμῶν εἷς μέσος ἀνάλογον ἐμπίπτῃ ἀριθμός, ὅμοιοι ἐπίπεδοι ἔσονται οἱ ἀριθμοί.

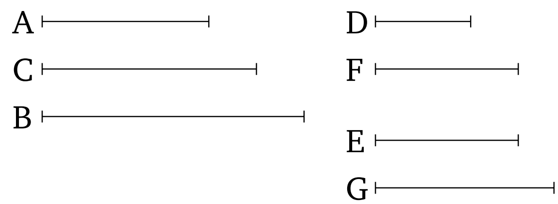
Δύο γὰρ ἀριθμῶν τῶν Α, Β εἷς μέσος ἀνάλογον ἐμπιπτέτω ἀριθμὸς ὁ Γ· λέγω, ὅτι οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν

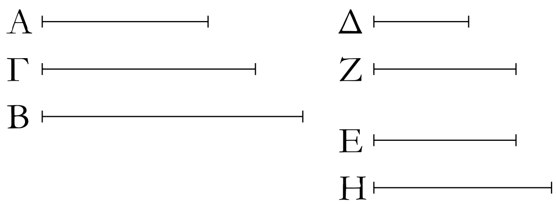
Proposition 20

If one number falls between two numbers in mean proportion then the numbers will be similar plane (numbers).

For let one number C fall between the two numbers A and B in mean proportion. I say that A and B are similar plane *ἀριθμοί.*

numbers.





Εἰλήφθωσαν [γὰρ] ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ οἱ Δ, Ε· ἰσάκις ἄρα ὁ Δ τὸν Α μετρεῖ καὶ ὁ Ε τὸν Γ. ὁσάκις δὴ ὁ Δ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ζ· ὁ Ζ ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ὥστε ὁ Α ἐπίπεδός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ Δ, Ζ. πάλιν, ἐπεὶ οἱ Δ, Ε ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, Β, ἰσάκις ἄρα ὁ Δ τὸν Γ μετρεῖ καὶ ὁ Ε τὸν Β. ὁσάκις δὴ ὁ Ε τὸν Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Η. ὁ Ε ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Η μονάδας· ὁ Η ἄρα τὸν Ε πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ Β ἄρα ἐπίπεδος ἐστι, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Ε, Η. οἱ Α, Β ἄρα ἐπίπεδοί εἰσιν ἀριθμοί. λέγω δή, ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ ὁ Ζ τὸν μὲν Δ πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Γ πρὸς τὸν Β. πάλιν, ἐπεὶ ὁ Ε ἑκάτερον τῶν Ζ, Η πολλαπλασιάσας τοὺς Γ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Β. ὡς δὲ ὁ Γ πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Ζ πρὸς τὸν Η· καὶ ἐναλλὰξ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Ε πρὸς τὸν Η. οἱ Α, Β ἄρα ὅμοιοι ἐπίπεδοι ἀριθμοί εἰσιν· αἱ γὰρ πλευραὶ αὐτῶν ἀνάλογόν εἰσιν· ὅπερ ἔδει δεῖξαι.

[For] let the least numbers, D and E, having the same ratio as A and C be taken [Prop. 7.33]. Thus, D measures A as many times as E (measures) C [Prop. 7.20]. So as many times as D measures A, so many units let there be in F. Thus, F has made A (by) multiplying D [Def. 7.15]. Hence, A is plane, and D, F (are) its sides. Again, since D and E are the least of those (numbers) having the same ratio as C and B, D thus measures C as many times as E (measures) B [Prop. 7.20]. So as many times as E measures B, so many units let there be in G. Thus, E measures B according to the units in G. Thus, G has made B (by) multiplying E [Def. 7.15]. Thus, B is plane, and E, G are its sides. Thus, A and B are (both) plane numbers. So I say that (they are) also similar. For since F has made A (by) multiplying D, and has made C (by) multiplying E, thus as D is to E, so A (is) to C—that is to say, C to B [Prop. 7.17].† Again, since E has made C, B (by) multiplying F, G, respectively, thus as F is to G, so C (is) to B [Prop. 7.17]. And as C (is) to B, so D (is) to E. And thus as D (is) to E, so F (is) to G. And, alternately, as D (is) to F, so E (is) to G [Prop. 7.13]. Thus, A and B are similar plane numbers. For their sides are proportional [Def. 7.21]. (Which is) the very thing it was required to show.

† This part of the proof is defective, since it is not demonstrated that $F \times E = C$. Furthermore, it is not necessary to show that $D : E :: A : C$, because this is true by hypothesis.

κα'.

Ἐὰν δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅμοιοι στερεοί εἰσιν οἱ ἀριθμοί.

Δύο γὰρ ἀριθμῶν τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπιπτέτωσαν ἀριθμοὶ οἱ Γ, Δ· λέγω, ὅτι οἱ Α, Β ὅμοιοι στερεοί εἰσιν.

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ τρεῖς οἱ Ε, Ζ, Η· οἱ ἄρα ἄκροι αὐτῶν οἱ Ε, Η πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ τῶν Ε, Η εἷς μέσος ἀνάλογον ἐμπέπτωκεν ἀριθμὸς ὁ Ζ, οἱ Ε, Η ἄρα ἀριθμοὶ ὅμοιοι ἐπίπεδοί εἰσιν. ἔστωσαν οὖν τοῦ μὲν Ε πλευραὶ

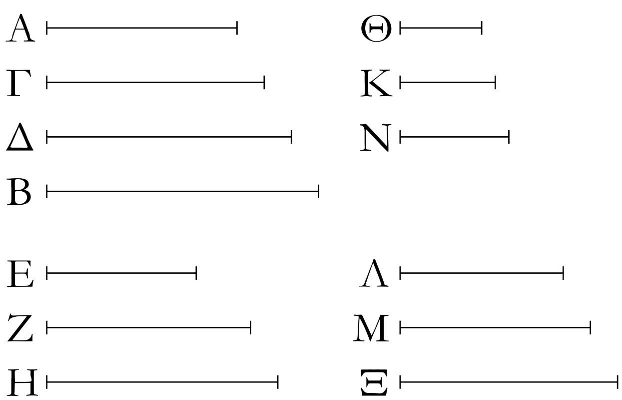
Proposition 21

If two numbers fall between two numbers in mean proportion then the (latter) are similar solid (numbers).

For let the two numbers C and D fall between the two numbers A and B in mean proportion. I say that A and B are similar solid (numbers).

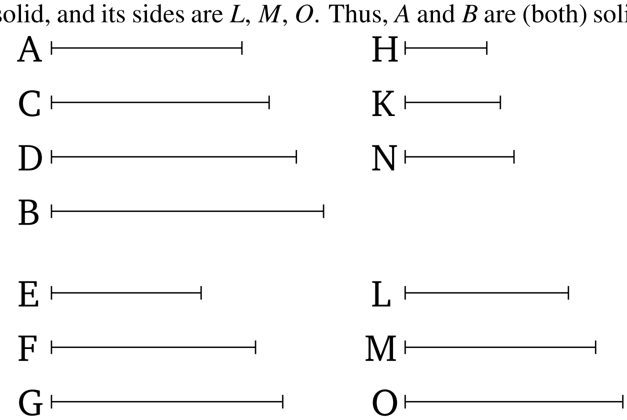
For let the three least numbers E, F, G having the same ratio as A, C, D be taken [Prop. 8.2]. Thus, the outermost of them, E and G, are prime to one another [Prop. 8.3]. And since one number, F, has fallen (between) E and G in mean proportion, E and G are thus similar plane numbers [Prop. 8.20].

οἱ Θ, Κ, τοῦ δὲ Η οἱ Λ, Μ. φανερὸν ἄρα ἐστὶν ἐκ τοῦ πρὸ τούτου, ὅτι οἱ Ε, Ζ, Η ἑξῆς εἰσιν ἀνάλογον ἔν τε τῷ τοῦ Θ πρὸς τὸν Λ λόγῳ καὶ τῷ τοῦ Κ πρὸς τὸν Μ. καὶ ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ, καί ἐστιν ἴσον τὸ πλῆθος τῶν Ε, Ζ, Η τῷ πλήθει τῶν Α, Γ, Δ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Δ. οἱ δὲ Ε, Η πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· ἰσάκις ἄρα ὁ Ε τὸν Α μετρεῖ καὶ ὁ Η τὸν Δ. ὁσάκις δὴ ὁ Ε τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ν. ὁ Ν ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ Ε ἐστιν ὁ ἐκ τῶν Θ, Κ· ὁ Ν ἄρα τὸν ἐκ τῶν Θ, Κ πολλαπλασιάσας τὸν Α πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Θ, Κ, Ν. πάλιν, ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, Δ, Β, ἰσάκις ἄρα ὁ Ε τὸν Γ μετρεῖ καὶ ὁ Η τὸν Β. ὁσάκις δὴ ὁ Ε τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ξ. ὁ Η ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Ξ μονάδας· ὁ Ξ ἄρα τὸν Η πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ δὲ Η ἐστιν ὁ ἐκ τῶν Λ, Μ· ὁ Ξ ἄρα τὸν ἐκ τῶν Λ, Μ πολλαπλασιάσας τὸν Β πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Β, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Λ, Μ, Ξ· οἱ Α, Β ἄρα στερεοί εἰσιν.



Λέγω [δή], ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ οἱ Ν, Ξ τὸν Ε πολλαπλασιάσαντες τοὺς Α, Γ πεποιήκασιν, ἔστιν ἄρα ὡς ὁ Ν πρὸς τὸν Ξ, ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Ε πρὸς τὸν Ζ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Θ πρὸς τὸν Λ καὶ ὁ Κ πρὸς τὸν Μ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Λ, οὕτως ὁ Κ πρὸς τὸν Μ καὶ ὁ Ν πρὸς τὸν Ξ. καί εἰσιν οἱ μὲν Θ, Κ, Ν πλευραὶ τοῦ Α, οἱ δὲ Ξ, Λ, Μ πλευραὶ τοῦ Β. οἱ Α, Β ἄρα ἀριθμοὶ ὅμοιοι στερεοί εἰσιν· ὅπερ ἔδει δεῖξαι.

Therefore, let *H*, *K* be the sides of *E*, and *L*, *M* (the sides) of *G*. Thus, it is clear from the (proposition) before this (one) that *E*, *F*, *G* are continuously proportional in the ratio of *H* to *L*, and of *K* to *M*. And since *E*, *F*, *G* are the least (numbers) having the same ratio as *A*, *C*, *D*, and the multitude of *E*, *F*, *G* is equal to the multitude of *A*, *C*, *D*, thus, via equality, as *E* is to *G*, so *A* (is) to *D* [Prop. 7.14]. And *E* and *G* (are) prime (to one another), and prime (numbers) are also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, *E* measures *A* the same number of times as *G* (measures) *D*. So as many times as *E* measures *A*, so many units let there be in *N*. Thus, *N* has made *A* (by) multiplying *E* [Def. 7.15]. And *E* is the (number created) from (multiplying) *H* and *K*. Thus, *N* has made *A* (by) multiplying the (number created) from (multiplying) *H* and *K*. Thus, *A* is solid, and its sides are *H*, *K*, *N*. Again, since *E*, *F*, *G* are the least (numbers) having the same ratio as *C*, *D*, *B*, thus *E* measures *C* the same number of times as *G* (measures) *B* [Prop. 7.20]. So as many times as *E* measures *C*, so many units let there be in *O*. Thus, *G* measures *B* according to the units in *O*. Thus, *O* has made *B* (by) multiplying *G*. And *G* is the (number created) from (multiplying) *L* and *M*. Thus, *O* has made *B* (by) multiplying the (number created) from (multiplying) *L* and *M*. Thus, *B* is solid, and its sides are *L*, *M*, *O*. Thus, *A* and *B* are (both) solid.



[So] I say that (they are) also similar. For since *N*, *O* have made *A*, *C* (by) multiplying *E*, thus as *N* is to *O*, so *A* (is) to *C*—that is to say, *E* to *F* [Prop. 7.18]. But, as *E* (is) to *F*, so *H* (is) to *L*, and *K* to *M*. And thus as *H* (is) to *L*, so *K* (is) to *M*, and *N* to *O*. And *H*, *K*, *N* are the sides of *A*, and *L*, *M*, *O* the sides of *B*. Thus, *A* and *B* are similar solid numbers [Def. 7.21]. (Which is) the very thing it was required to show.

† The Greek text has "*O*, *L*, *M*", which is obviously a mistake.

κβ'.

Ἐὰν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ δὲ πρῶτος τετράγωνος ᾖ, καὶ ὁ τρίτος τετράγωνος ἔσται.

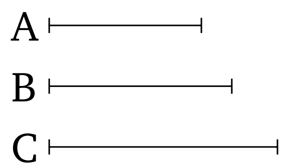
Ἔστωσαν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, ὁ δὲ πρῶτος ὁ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ τρίτος ὁ Γ τετράγωνός ἐστιν.

Ἐπεὶ γὰρ τῶν Α, Γ εἷς μέσος ἀνάλογόν ἐστιν ἀριθμὸς ὁ Β, οἱ Α, Γ ἄρα ὅμοιοι ἐπίπεδοί εἰσιν. τετράγωνος δὲ ὁ Α· τετράγωνος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.

Proposition 22

If three numbers are continuously proportional, and the first is square, then the third will also be square.

Let *A*, *B*, *C* be three continuously proportional numbers, and let the first *A* be square. I say that the third *C* is also square.



For since one number, *B*, is in mean proportion to *A* and *C*, *A* and *C* are thus similar plane (numbers) [Prop. 8.20]. And *A* is square. Thus, *C* is also square [Def. 7.21]. (Which is) the very thing it was required to show.

κγ'.

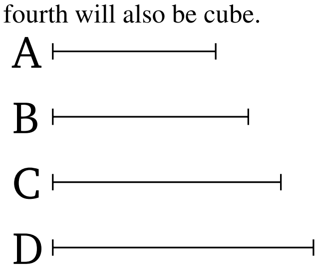
Ἐὰν τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ δὲ πρῶτος κύβος ᾖ, καὶ ὁ τέταρτος κύβος ἔσται.

Ἔστωσαν τέσσαρες ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, ὁ δὲ Α κύβος ἔστω· λέγω, ὅτι καὶ ὁ Δ κύβος ἐστίν.

Ἐπεὶ γὰρ τῶν Α, Δ δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοὶ οἱ Β, Γ, οἱ Α, Δ ἄρα ὅμοιοί εἰσι στερεοὶ ἀριθμοί. κύβος δὲ ὁ Α· κύβος ἄρα καὶ ὁ Δ· ὅπερ ἔδει δεῖξαι.

Proposition 23

If four numbers are continuously proportional, and the first is cube, then the fourth will also be cube.

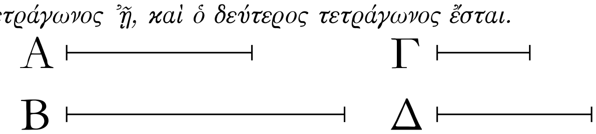


Let *A*, *B*, *C*, *D* be four continuously proportional numbers, and let *A* be cube. I say that *D* is also cube.

For since two numbers, *B* and *C*, are in mean proportion to *A* and *D*, *A* and *D* are thus similar solid numbers [Prop. 8.21]. And *A* (is) cube. Thus, *D* (is) also cube [Def. 7.21]. (Which is) the very thing it was required to show.

κδ'.

Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὁ δὲ πρῶτος τετράγωνος ᾖ, καὶ ὁ δεύτερος τετράγωνος ἔσται.

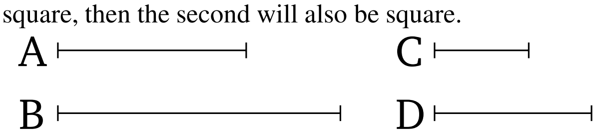


Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς ἀλλήλους λόγον ἐχέτωσαν, ὃν τετράγωνος ἀριθμὸς ὁ Γ πρὸς τετράγωνον ἀριθμὸν τὸν Δ, ὁ δὲ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ Β τετράγωνός ἐστιν.

Ἐπεὶ γὰρ οἱ Γ, Δ τετράγωνοί εἰσιν, οἱ Γ, Δ ἄρα ὅμοιοι ἐπίπεδοί εἰσιν. τῶν Γ, Δ ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί ἐστιν ὁ Α τετράγωνος· καὶ ὁ Β ἄρα τετράγωνός ἐστιν· ὅπερ ἔδει

Proposition 24

If two numbers have to one another the ratio which a square number (has) to a(nother) square number, and the first is square, then the second will also be square.



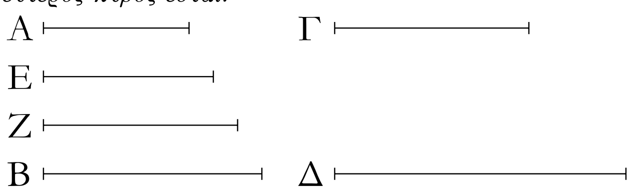
For let two numbers, *A* and *B*, have to one another the ratio which the square number *C* (has) to the square number *D*. And let *A* be square. I say that *B* is also square.

For since *C* and *D* are square, *C* and *D* are thus similar plane (numbers). Thus, one number falls (between) *C* and *D* in mean proportion [Prop. 8.18]. And as *C* is to *D*, (so) *A* (is) to *B*. Thus, one number also falls (between) *A* and *B* in mean proportion [Prop. 8.8]. And *A* is square. Thus, *B* is also square *δεῖξαι.*

[Prop. 8.22]. (Which is) the very thing it was required to show.

κε'.

Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν, ὁ δὲ πρῶτος κύβος ᾖ, καὶ ὁ δεύτερος κύβος ἔσται.

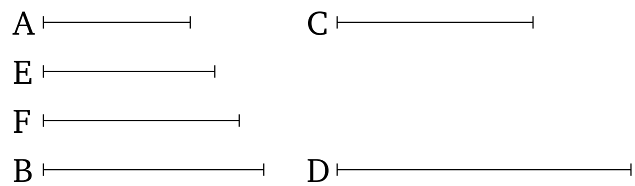


Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς ἀλλήλους λόγον ἐχέτωσαν, ὃν κύβος ἀριθμὸς ὁ Γ πρὸς κύβον ἀριθμὸν τὸν Δ, κύβος δὲ ἔστω ὁ Α· λέγω [δή], ὅτι καὶ ὁ Β κύβος ἐστίν.

Ἐπεὶ γὰρ οἱ Γ, Δ κύβοι εἰσίν, οἱ Γ, Δ ὅμοιοι στερεοί εἰσιν· τῶν Γ, Δ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ὅσοι δὲ εἰς τοὺς Γ, Δ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς· ὥστε καὶ τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Ε, Ζ. ἐπεὶ οὖν τέσσαρες ἀριθμοὶ οἱ Α, Ε, Ζ, Β ἑξῆς ἀνάλογόν εἰσιν, καί ἐστι κύβος ὁ Α, κύβος ἄρα καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

Proposition 25

If two numbers have to one another the ratio which a cube number (has) to a(nother) cube number, and the first is cube, then the second will also be cube.

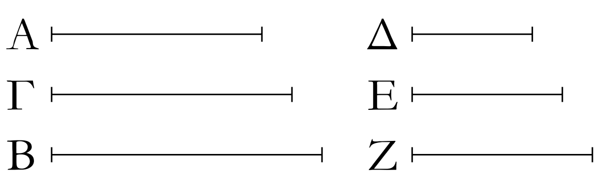


For let two numbers, *A* and *B*, have to one another the ratio which the cube number *C* (has) to the cube number *D*. And let *A* be cube. [So] I say that *B* is also cube.

For since *C* and *D* are cube (numbers), *C* and *D* are (thus) similar solid (numbers). Thus, two numbers fall (between) *C* and *D* in mean proportion [Prop. 8.19]. And as many (numbers) as fall in between *C* and *D* in continued proportion, so many also (fall) in (between) those (numbers) having the same ratio as them (in continued proportion) [Prop. 8.8]. And hence two numbers fall (between) *A* and *B* in mean proportion. Let *E* and *F* (so) fall. Therefore, since the four numbers *A*, *E*, *F*, *B* are continuously proportional, and *A* is cube, *B* (is) thus also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

κϛ'.

Οἱ ὅμοιοι ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

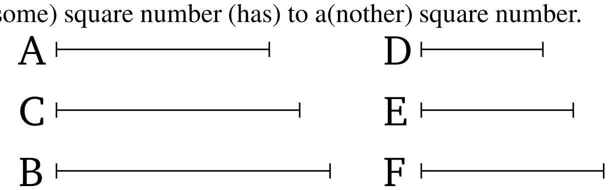


Ἔστωσαν ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Ἐπεὶ γὰρ οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν, τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐμπιπτέτω καὶ ἔστω ὁ Γ, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Β οἱ Δ, Ε, Ζ· οἱ ἄρα ἄκροι αὐτῶν οἱ Δ, Ζ τετράγωνοί εἰσιν. καὶ ἐπεὶ ἐστιν ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Β, καί εἰσιν οἱ Δ, Ζ τετράγωνοι, ὁ Α ἄρα πρὸς

Proposition 26

Similar plane numbers have to one another the ratio which (some) square number (has) to a(nother) square number.

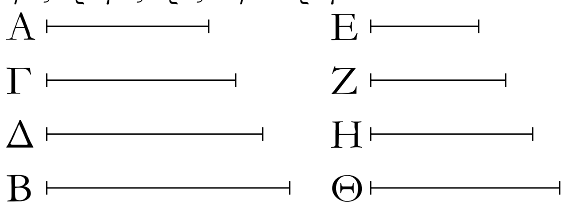


Let *A* and *B* be similar plane numbers. I say that *A* has to *B* the ratio which (some) square number (has) to a(nother) square number.

For since *A* and *B* are similar plane numbers, one number thus falls (between) *A* and *B* in mean proportion [Prop. 8.18]. Let it (so) fall, and let it be *C*. And let the least numbers, *D*, *E*, *F*, having the same ratio as *A*, *C*, *B* be taken [Prop. 8.2]. The outermost of them, *D* and *F*, are thus square [Prop. 8.2 corr.]. And since as *D* is to *F*, so *A* (is) to *B*, and *D* and *F* are square, *A* thus has to *B* the ratio which (some) square number (has) *τὸν Β λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὅπερ ἔδει δεῖξαι.*

κζ΄.

Οἱ ὅμοιοι στερεοὶ ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.



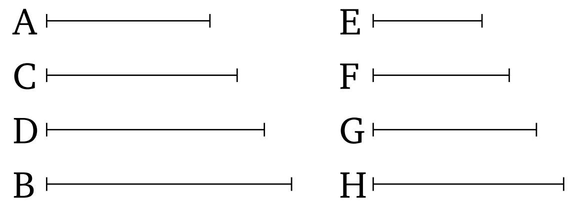
Ἔστωσαν ὅμοιοι στερεοὶ ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.

Ἐπεὶ γὰρ οἱ Α, Β ὅμοιοι στερεοί εἰσιν, τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Γ, Δ, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ, Β ἴσοι αὐτοῖς τὸ πλῆθος οἱ Ε, Ζ, Η, Θ· οἱ ἄρα ἄκροι αὐτῶν οἱ Ε, Θ κύβοι εἰσίν. καί ἐστιν ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὁ Α ἄρα πρὸς τὸν Β λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

to a(nother) square number. (Which is) the very thing it was required to show.

Proposition 27

Similar solid numbers have to one another the ratio which (some) cube number (has) to a(nother) cube number.



Let A and B be similar solid numbers. I say that A has to B the ratio which (some) cube number (has) to a(nother) cube number.

For since A and B are similar solid (numbers), two numbers thus fall (between) A and B in mean proportion [Prop. 8.19]. Let C and D have (so) fallen. And let the least numbers, E, F, G, H, having the same ratio as A, C, D, B, (and) equal in multitude to them, be taken [Prop. 8.2]. Thus, the outermost of them, E and H, are cube [Prop. 8.2 corr.]. And as E is to H, so A (is) to B. And thus A has to B the ratio which (some) cube number (has) to a(nother) cube number. (Which is) the very thing it was required to show.

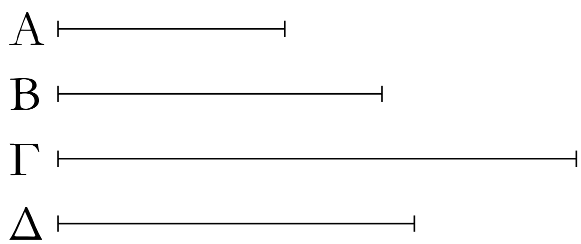
ELEMENTS BOOK 9

Applications of Number Theory[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

α'.

Ἐὰν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος τετράγωνος ἔσται.

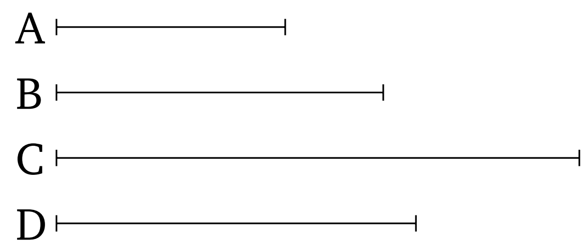


Ἔστωσαν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β, καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ τετράγωνός ἐστιν.

Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω. ὁ Δ ἄρα τετράγωνός ἐστιν. ἐπεὶ οὖν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί, τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐὰν δὲ δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς ἐμπίπτουσι, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας· ὥστε καὶ τῶν Δ, Γ εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καί ἐστι τετράγωνος ὁ Δ· τετράγωνος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.

Proposition 1

If two similar plane numbers make some (number by) multiplying one another then the created (number) will be square.

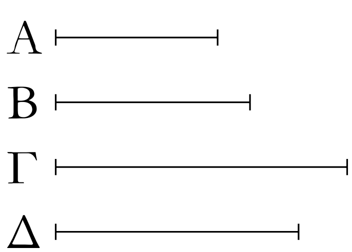


Let *A* and *B* be two similar plane numbers, and let *A* make *C* (by) multiplying *B*. I say that *C* is square.

For let *A* make *D* (by) multiplying itself. *D* is thus square. Therefore, since *A* has made *D* (by) multiplying itself, and has made *C* (by) multiplying *B*, thus as *A* is to *B*, so *D* (is) to *C* [Prop. 7.17]. And since *A* and *B* are similar plane numbers, one number thus falls (between) *A* and *B* in mean proportion [Prop. 8.18]. And if (some) numbers fall between two numbers in continued proportion then, as many (numbers) as fall in (between) them (in continued proportion), so many also (fall) in (between numbers) having the same ratio (as them in continued proportion) [Prop. 8.8]. And hence one number falls (between) *D* and *C* in mean proportion. And *D* is square. Thus, *C* (is) also square [Prop. 8.22]. (Which is) the very thing it was required to show.

β'.

Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσι τετράγωνον, ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.

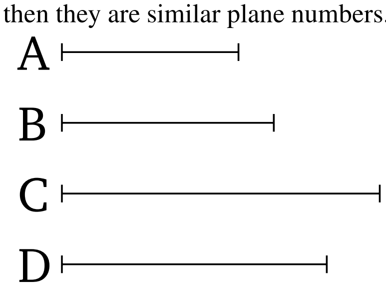


Ἔστωσαν δύο ἀριθμοὶ οἱ Α, Β, καὶ ὁ Α τὸν Β πολλαπλασιάσας τετράγωνον τὸν Γ ποιείτω· λέγω, ὅτι οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν ἀριθμοί.

Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Δ ἄρα τετράγωνός ἐστιν. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ ὁ Δ τετράγωνός ἐστιν, ἀλλὰ καὶ ὁ Γ, οἱ Δ, Γ ἄρα ὅμοιοι ἐπίπεδοί εἰσιν. τῶν Δ, Γ ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει. καί ἐστιν ὡς ὁ Δ πρὸς τὸν Γ, οὕτως ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα

Proposition 2

If two numbers make a square (number by) multiplying one another then they are similar plane numbers.



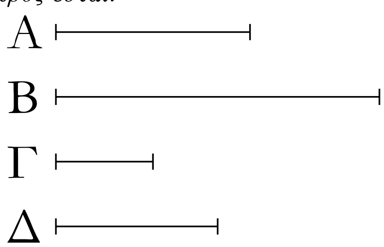
Let *A* and *B* be two numbers, and let *A* make the square (number) *C* (by) multiplying *B*. I say that *A* and *B* are similar plane numbers.

For let *A* make *D* (by) multiplying itself. Thus, *D* is square. And since *A* has made *D* (by) multiplying itself, and has made *C* (by) multiplying *B*, thus as *A* is to *B*, so *D* (is) to *C* [Prop. 7.17]. And since *D* is square, and *C* (is) also, *D* and *C* are thus similar plane numbers. Thus, one (number) falls (between) *D* and *C* in mean proportion [Prop. 8.18]. And as *D* is to *C*, so *A* (is) to *B*. Thus, one (number) also falls *εἷς μέσος ἀνάλογον ἐμπίπτει. ἐὰν δὲ δύο ἀριθμῶν εἷς μέσος ἀνάλογον ἐμπίπτῃ, ὅμοιοι ἐπίπεδοί εἰσιν [οἱ] ἀριθμοί· οἱ ἄρα Α, Β ὅμοιοί εἰσιν ἐπίπεδοι· ὅπερ ἔδει δεῖξαι.*

(between) *A* and *B* in mean proportion [Prop. 8.8]. And if one (number) falls (between) two numbers in mean proportion then [the] numbers are similar plane (numbers) [Prop. 8.20]. Thus, *A* and *B* are similar plane (numbers). (Which is) the very thing it was required to show.

γ'.

Ἐὰν κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἔσται.

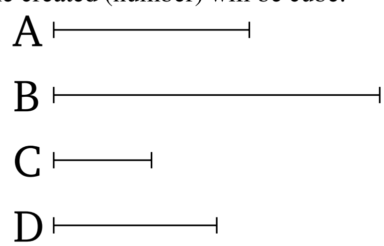


Κύβος γὰρ ἀριθμὸς ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β ποιείτω· λέγω, ὅτι ὁ Β κύβος ἐστίν.

Εἰλήφθω γὰρ τοῦ Α πλευρὰ ὁ Γ, καὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω. φανερὸν δή ἐστιν, ὅτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Γ ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας. ἀλλὰ μὴν καὶ ἡ μονὰς τὸν Γ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ. πάλιν, ἐπεὶ ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν, ὁ Δ ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Γ κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Α. ἀλλ' ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ἡ μονὰς πρὸς τὸν Γ, οὕτως ὁ Γ πρὸς τὸν Δ καὶ ὁ Δ πρὸς τὸν Α. τῆς ἄρα μονάδος καὶ τοῦ Α ἀριθμοῦ δύο μέσοι ἀνάλογον κατὰ τὸ συνεχὲς ἐμπεπτώκασιν ἀριθμοὶ οἱ Γ, Δ. πάλιν, ἐπεὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν, ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, ὁ Α πρὸς τὸν Β. τῆς δὲ μονάδος καὶ τοῦ Α δύο μέσοι ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. ἐὰν δὲ δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν, ὁ δὲ πρῶτος κύβος ᾖ, καὶ ὁ δεύτερος κύβος ἔσται. καί ἐστιν ὁ Α κύβος· καὶ ὁ Β ἄρα κύβος ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 3

If a cube number makes some (number by) multiplying itself then the created (number) will be cube.



For let the cube number *A* make *B* (by) multiplying itself. I say that *B* is cube.

For let the side *C* of *A* be taken. And let *C* make *D* by multiplying itself. So it is clear that *C* has made *A* (by) multiplying *D*. And since *C* has made *D* (by) multiplying itself, *C* thus measures *D* according to the units in it [Def. 7.15]. But, in fact, a unit also measures *C* according to the units in it [Def. 7.20]. Thus, as a unit is to *C*, so *C* (is) to *D*. Again, since *C* has made *A* (by) multiplying *D*, *D* thus measures *A* according to the units in *C*. And a unit also measures *C* according to the units in it. Thus, as a unit is to *C*, so *D* (is) to *A*. But, as a unit (is) to *C*, so *C* (is) to *D*. And thus as a unit (is) to *C*, so *C* (is) to *D*, and *D* to *A*. Thus, two numbers, *C* and *D*, have fallen (between) a unit and the number *A* in continued mean proportion. Again, since *A* has made *B* (by) multiplying itself, *A* thus measures *B* according to the units in it. And a unit also measures *A* according to the units in it. Thus, as a unit is to *A*, so *A* (is) to *B*. And two numbers have fallen (between) a unit and *A* in mean proportion. Thus two numbers will also fall (between) *A* and *B* in mean proportion [Prop. 8.8]. And if two (numbers) fall (between) two numbers in mean proportion, and the first (number) is cube, then the second will also be cube [Prop. 8.23]. And *A* is cube. Thus, *B* is also cube. (Which is) the very thing it was required to show.

δ'.

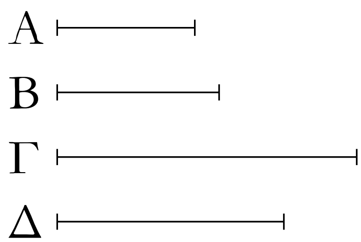
Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἔσται.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον ἀριθμὸν τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ κύβος ἐστίν.

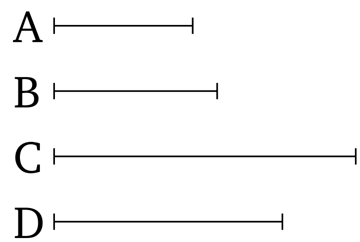
Proposition 4

If a cube number makes some (number by) multiplying a(nother) cube number then the created (number) will be cube.

For let the cube number *A* make *C* (by) multiplying the cube number *B*. I say that *C* is cube.



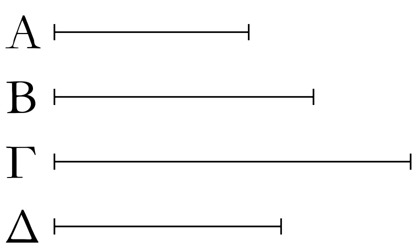
Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Δ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Α, Β κύβοι εἰσίν, ὅμοιοι στερεοί εἰσιν οἱ Α, Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· ὥστε καὶ τῶν Δ, Γ δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. καί ἐστι κύβος ὁ Δ· κύβος ἄρα καὶ ὁ Γ· ὅπερ ἔδει δεῖξαι.



For let *A* make *D* (by) multiplying itself. Thus, *D* is cube [Prop. 9.3]. And since *A* has made *D* (by) multiplying itself, and has made *C* (by) multiplying *B*, thus as *A* is to *B*, so *D* (is) to *C* [Prop. 7.17]. And since *A* and *B* are cube, *A* and *B* are similar solid (numbers). Thus, two numbers fall (between) *A* and *B* in mean proportion [Prop. 8.19]. Hence, two numbers will also fall (between) *D* and *C* in mean proportion [Prop. 8.8]. And *D* is cube. Thus, *C* (is) also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

ε'.

Ἐὰν κύβος ἀριθμὸς ἀριθμόν τινα πολλαπλασιάσας κύβον ποιῇ, καὶ ὁ πολλαπλασιασθεὶς κύβος ἔσται.

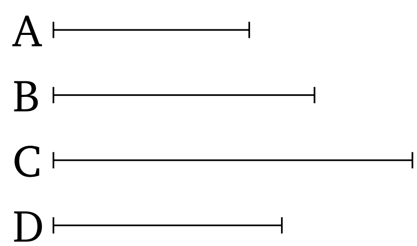


Κύβος γὰρ ἀριθμὸς ὁ Α ἀριθμόν τινα τὸν Β πολλαπλασιάσας κύβον τὸν Γ ποιείτω· λέγω, ὅτι ὁ Β κύβος ἐστίν.

Ὁ γὰρ Α ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· κύβος ἄρα ἐστὶν ὁ Δ. καὶ ἐπεὶ ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Δ πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Δ, Γ κύβοι εἰσίν, ὅμοιοι στερεοί εἰσιν. τῶν Δ, Γ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καί ἐστιν ὡς ὁ Δ πρὸς τὸν Γ, οὕτως ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καί ἐστι κύβος ὁ Α· κύβος ἄρα ἐστὶ καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

Proposition 5

If a cube number makes a(nother) cube number (by) multiplying some (number) then the (number) multiplied will also be cube.



For let the cube number *A* make the cube (number) *C* (by) multiplying some number *B*. I say that *B* is cube.

For let *A* make *D* (by) multiplying itself. *D* is thus cube [Prop. 9.3]. And since *A* has made *D* (by) multiplying itself, and has made *C* (by) multiplying *B*, thus as *A* is to *B*, so *D* (is) to *C* [Prop. 7.17]. And since *D* and *C* are (both) cube, they are similar solid (numbers). Thus, two numbers fall (between) *D* and *C* in mean proportion [Prop. 8.19]. And as *D* is to *C*, so *A* (is) to *B*. Thus, two numbers also fall (between) *A* and *B* in mean proportion [Prop. 8.8]. And *A* is cube. Thus, *B* is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

ϛ'.

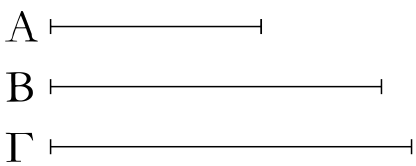
Ἐὰν ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῇ, καὶ αὐτὸς κύβος ἔσται.

Ἀριθμὸς γὰρ ὁ Α ἑαυτὸν πολλαπλασιάσας κύβον τὸν Β ποιείτω· λέγω, ὅτι καὶ ὁ Α κύβος ἐστίν.

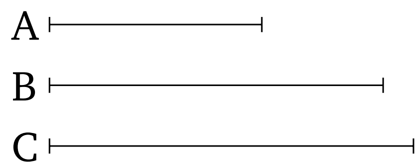
Proposition 6

If a number makes a cube (number by) multiplying itself then it itself will also be cube.

For let the number *A* make the cube (number) *B* (by) multiplying itself. I say that *A* is also cube.



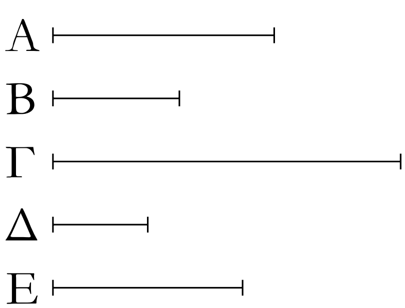
Ὁ γὰρ Α τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω. ἐπεὶ οὖν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Β πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν, ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β. καὶ ἐπεὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Γ. ἀλλ᾽ ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. καὶ ἐπεὶ οἱ Β, Γ κύβοι εἰσίν, ὅμοιοι στερεοί εἰσιν. τῶν Β, Γ ἄρα δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί. καί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, ὁ Α πρὸς τὸν Β. καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί. καί ἐστιν κύβος ὁ Β· κύβος ἄρα ἐστὶ καὶ ὁ Α· ὅπερ ἔδει δεῖξαι.



For let A make C (by) multiplying B. Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B, C is thus cube. And since A has made B (by) multiplying itself, A thus measures B according to the units in (A). And a unit also measures A according to the units in it. Thus, as a unit is to A, so A (is) to B. And since A has made C (by) multiplying B, B thus measures C according to the units in A. And a unit also measures A according to the units in it. Thus, as a unit is to A, so B (is) to C. But, as a unit (is) to A, so A (is) to B. And thus as A (is) to B, (so) B (is) to C. And since B and C are cube, they are similar solid (numbers). Thus, there exist two numbers in mean proportion (between) B and C [Prop. 8.19]. And as B is to C, (so) A (is) to B. Thus, there also exist two numbers in mean proportion (between) A and B [Prop. 8.8]. And B is cube. Thus, A is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

ζʹ.

Ἐὰν σύνθετος ἀριθμὸς ἀριθμόν τινα πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος στερεὸς ἔσται.

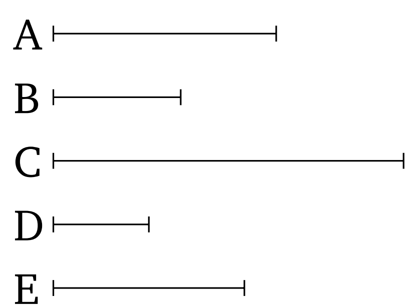


Σύνθετος γὰρ ἀριθμὸς ὁ Α ἀριθμόν τινα τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ στερεός ἐστιν.

Ἐπεὶ γὰρ ὁ Α σύνθετός ἐστιν, ὑπὸ ἀριθμοῦ τινος μετρηθήσεται. μετρείσθω ὑπὸ τοῦ Δ, καὶ ὁσάκις ὁ Δ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. ἐπεὶ οὖν ὁ Δ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, ὁ Ε ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ ἐπεὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ δὲ Α ἐστιν ὁ ἐκ τῶν Δ, Ε, ὁ ἄρα ἐκ τῶν Δ, Ε τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ Γ ἄρα στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Δ, Ε, Β· ὅπερ ἔδει δεῖξαι.

Proposition 7

If a composite number makes some (number by) multiplying some (other) number then the created (number) will be solid.



For let the composite number A make C (by) multiplying some number B. I say that C is solid.

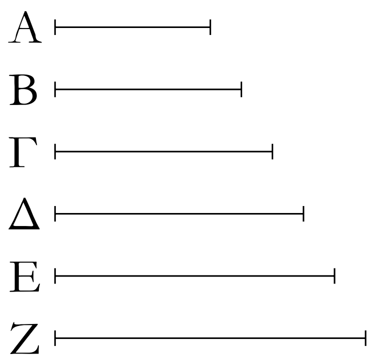
For since A is a composite (number), it will be measured by some number. Let it be measured by D. And, as many times as D measures A, so many units let there be in E. Therefore, since D measures A according to the units in E, E has thus made A (by) multiplying D [Def. 7.15]. And since A has made C (by) multiplying B, and A is the (number created) from (multiplying) D, E, the (number created) from (multiplying) D, E has thus made C (by) multiplying B. Thus, C is solid, and its sides are D, E, B. (Which is) the very thing it was required to show.

ηʹ.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν,

Proposition 8

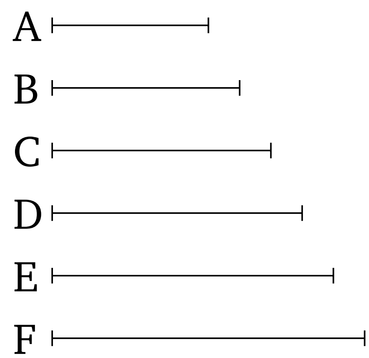
If any multitude whatsoever of numbers is continuously *ὁ μὲν τρίτος ἀπὸ τῆς μονάδος τετράγωνος ἔσται καὶ οἱ ἕνα διαλείποντες, ὁ δὲ τέταρτος κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἕβδομος κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες.*



Ἔστωσαν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, Ε, Ζ· λέγω, ὅτι ὁ μὲν τρίτος ἀπὸ τῆς μονάδος ὁ Β τετράγωνός ἐστι καὶ οἱ ἕνα διαλείποντες πάντες, ὁ δὲ τέταρτος ὁ Γ κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἕβδομος ὁ Ζ κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες πάντες.

Ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β, ἰσάκις ἄρα ἡ μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Α τὸν Β. ἡ δὲ μονὰς τὸν Α ἀριθμὸν μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· καὶ ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· τετράγωνος ἄρα ἐστὶν ὁ Β. καὶ ἐπεὶ οἱ Β, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, ὁ δὲ Β τετράγωνός ἐστιν, καὶ ὁ Δ ἄρα τετράγωνός ἐστινμήιά τὰ αὐτὰ δὴ καὶ ὁ Ζ τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ ἕνα διαλείποντες πάντες τετράγωνοί εἰσιν. λέγω δή, ὅτι καὶ ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες. ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Β πρὸς τὸν Γ, ἰσάκις ἄρα ἡ μονὰς τὸν Α ἀριθμὸν μετρεῖ καὶ ὁ Β τὸν Γ. ἡ δὲ μονὰς τὸν Α ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας· καὶ ὁ Β ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας· ὁ Α ἄρα τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ἐπεὶ οὖν ὁ Α ἑαυτὸν μὲν πολλαπλασιάσας τὸν Β πεποίηκεν, τὸν δὲ Β πολλαπλασιάσας τὸν Γ πεποίηκεν, κύβος ἄρα ἐστὶν ὁ Γ. καὶ ἐπεὶ οἱ Γ, Δ, Ε, Ζ ἑξῆς ἀνάλογόν εἰσιν, ὁ δὲ Γ κύβος ἐστίν, καὶ ὁ Ζ ἄρα κύβος ἐστίν. ἐδείχθη δὲ καὶ τετράγωνος· ὁ ἄρα ἕβδομος ἀπὸ τῆς μονάδος κύβος τέ ἐστι καὶ τετράγωνος. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ πέντε διαλείποντες πάντες κύβοι τέ εἰσι καὶ τετράγωνοι· ὅπερ ἔδει δεῖξαι.

proportional, (starting) from a unit, then the third from the unit will be square, and (all) those (numbers after that) which leave an interval of one (number), and the fourth (will be) cube, and all those (numbers after that) which leave an interval of two (numbers), and the seventh (will be) both cube and square, and (all) those (numbers after that) which leave an interval of five (numbers).

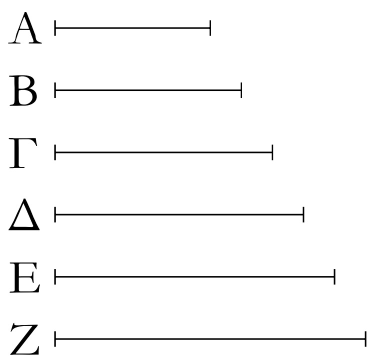


Let any multitude whatsoever of numbers, *A*, *B*, *C*, *D*, *E*, *F*, be continuously proportional, (starting) from a unit. I say that the third from the unit, *B*, is square, and all those (numbers after that) which leave an interval of one (number). And the fourth (from the unit), *C*, (is) cube, and all those (numbers after that) which leave an interval of two (numbers). And the seventh (from the unit), *F*, (is) both cube and square, and all those (numbers after that) which leave an interval of five (numbers).

For since as the unit is to *A*, so *A* (is) to *B*, the unit thus measures the number *A* the same number of times as *A* (measures) *B* [Def. 7.20]. And the unit measures the number *A* according to the units in it. Thus, *A* also measures *B* according to the units in *A*. *A* has thus made *B* (by) multiplying itself [Def. 7.15]. Thus, *B* is square. And since *B*, *C*, *D* are continuously proportional, and *B* is square, *D* is thus also square [Prop. 8.22]. So, for the same (reasons), *F* is also square. So, similarly, we can also show that all those (numbers after that) which leave an interval of one (number) are square. So I also say that the fourth (number) from the unit, *C*, is cube, and all those (numbers after that) which leave an interval of two (numbers). For since as the unit is to *A*, so *B* (is) to *C*, the unit thus measures the number *A* the same number of times that *B* (measures) *C*. And the unit measures the number *A* according to the units in *A*. And thus *B* measures *C* according to the units in *A*. *A* has thus made *C* (by) multiplying *B*. Therefore, since *A* has made *B* (by) multiplying itself, and has made *C* (by) multiplying *B*, *C* is thus cube. And since *C*, *D*, *E*, *F* are continuously proportional, and *C* is cube, *F* is thus also cube [Prop. 8.23]. And it was also shown (to be) square. Thus, the seventh (number) from the unit is (both) cube and square. So, similarly, we can show that all those (numbers after that) which leave an interval of five (numbers) are (both) cube and

θʹ.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἑξῆς κατὰ τὸ συνεχὲς ἀριθμοὶ ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα τετράγωνος ᾖ, καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος ᾖ, καὶ οἱ λοιποὶ πάντες κύβοι ἔσονται.



Ἔστωσαν ἀπὸ μονάδος ἑξῆς ἀνάλογον ὁσοιδηποτοῦν ἀριθμοὶ οἱ Α, Β, Γ, Δ, Ε, Ζ, ὁ δὲ μετὰ τὴν μονάδα ὁ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται.

Ὅτι μὲν οὖν ὁ τρίτος ἀπὸ τῆς μονάδος ὁ Β τετράγωνός ἐστι καὶ οἱ ἕνα διαπλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν. ἐπεὶ γὰρ οἱ Α, Β, Γ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Α τετράγωνος, καὶ ὁ Γ [ἄρα] τετράγωνος ἐστιν. πάλιν, ἐπεὶ [καὶ] οἱ Β, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Β τετράγωνος, καὶ ὁ Δ [ἄρα] τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν.

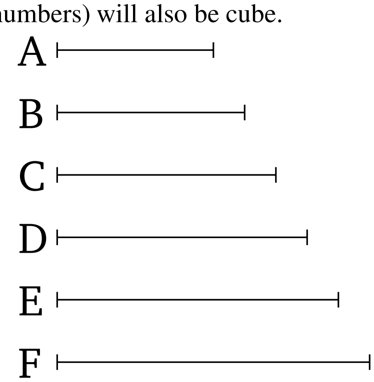
Ἀλλὰ δὴ ἔστω ὁ Α κύβος· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσίν.

Ὅτι μὲν οὖν ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσίν. ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, οὕτως ὁ Α πρὸς τὸν Β, ἰσάκις ἄρα ἡ μονὰς τὸν Α μετρεῖ καὶ ὁ Α τὸν Β. ἡ δὲ μονὰς τὸν Α μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· καὶ ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν. καί ἐστιν ὁ Α κύβος. ἐὰν δὲ κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἐστίν· καὶ ὁ Β ἄρα κύβος ἐστίν. καὶ ἐπεὶ τέσσαρες ἀριθμοὶ οἱ Α, Β, Γ, Δ ἑξῆς ἀνάλογόν εἰσιν, καί ἐστιν ὁ Α κύβος, καὶ ὁ Δ ἄρα κύβος ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Ε κύβος ἐστίν, καὶ ὁμοίως οἱ λοιποὶ πάντες κύβοι εἰσίν· ὅπερ ἔδει δεῖξαι.

square. (Which is) the very thing it was required to show.

Proposition 9

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is square, then all the remaining (numbers) will also be square. And if the (number) after the unit is cube, then all the remaining (numbers) will also be cube.



Let any multitude whatsoever of numbers, *A*, *B*, *C*, *D*, *E*, *F*, be continuously proportional, (starting) from a unit. And let the (number) after the unit, *A*, be square. I say that all the remaining (numbers) will also be square.

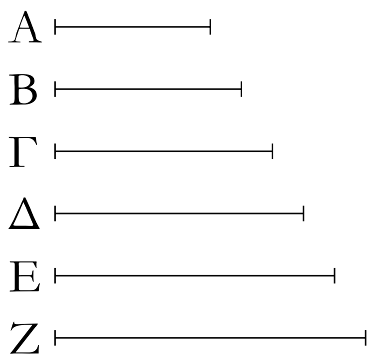
In fact, it has (already) been shown that the third (number) from the unit, *B*, is square, and all those (numbers after that) which leave an interval of one (number) [Prop. 9.8]. [So] I say that all the remaining (numbers) are also square. For since *A*, *B*, *C* are continuously proportional, and *A* (is) square, *C* is [thus] also square [Prop. 8.22]. Again, since *B*, *C*, *D* are [also] continuously proportional, and *B* is square, *D* is [thus] also square [Prop. 8.22]. So, similarly, we can show that all the remaining (numbers) are also square.

And so let *A* be cube. I say that all the remaining (numbers) are also cube.

In fact, it has (already) been shown that the fourth (number) from the unit, *C*, is cube, and all those (numbers after that) which leave an interval of two (numbers) [Prop. 9.8]. [So] I say that all the remaining (numbers) are also cube. For since as the unit is to *A*, so *A* (is) to *B*, the unit thus measures *A* the same number of times as *A* (measures) *B*. And the unit measures *A* according to the units in it. Thus, *A* also measures *B* according to the units in (*A*). *A* has thus made *B* (by) multiplying itself. And *A* is cube. And if a cube number makes some (number by) multiplying itself then the created (number) is cube [Prop. 9.3]. Thus, *B* is also cube. And since the four numbers *A*, *B*, *C*, *D* are continuously proportional, and *A* is cube, *D* is thus also cube [Prop. 8.23]. So, for the same (reasons), *E* is also cube, and, similarly, all the remaining (numbers) are cube. (Which is) the very thing it was required to

ιʹ.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ [ἑξῆς] ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα μὴ ᾖ τετράγωνος, οὐδ᾿ ἄλλος οὐδεὶς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἕνα διαλειπόντων πάντων. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος μὴ ᾖ, οὐδὲ ἄλλος οὐδεὶς κύβος ἔσται χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων πάντων.



Ἔστωσαν ἀπὸ μονάδος ἑξῆς ἀνάλογον ὁσοιδηποτοῦν ἀριθμοὶ οἱ Α, Β, Γ, Δ, Ε, Ζ, ὁ μετὰ τὴν μονάδα ὁ Α μὴ ἔστω τετράγωνος· λέγω, ὅτι οὐδὲ ἄλλος οὐδεὶς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος [καὶ τῶν ἕνα διαλειπόντων].

Εἰ γὰρ δυνατόν, ἔστω ὁ Γ τετράγωνος. ἔστι δὲ καὶ ὁ Β τετράγωνος· οἱ Β, Γ ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, ὁ Α πρὸς τὸν Β· οἱ Α, Β ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὥστε οἱ Α, Β ὅμοιοι ἐπίπεδοί εἰσιν. καί ἐστι τετράγωνος ὁ Β· τετράγωνος ἄρα ἐστὶ καὶ ὁ Α· ὅπερ οὐχ ὑπέκειτο. οὐκ ἄρα ὁ Γ τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾿ ἄλλος οὐδεὶς τετράγωνός ἐστι χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἕνα διαλειπόντων.

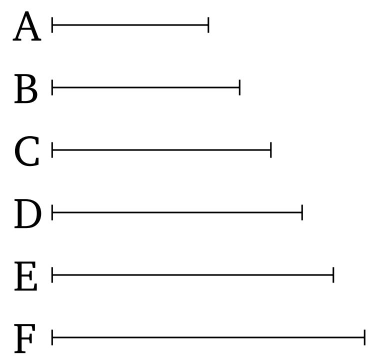
Ἀλλὰ δὴ μὴ ἔστω ὁ Α κύβος. λέγω, ὅτι οὐδ᾿ ἄλλος οὐδεὶς κύβος ἔσται χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων.

Εἰ γὰρ δυνατόν, ἔστω ὁ Δ κύβος. ἔστι δὲ καὶ ὁ Γ κύβος· τέταρτος γάρ ἐστιν ἀπὸ τῆς μονάδος. καί ἐστιν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Β πρὸς τὸν Γ· καὶ ὁ Β ἄρα πρὸς τὸν Γ λόγον ἔχει, ὃν κύβος πρὸς κύβον. καί ἐστιν ὁ Γ κύβος· καὶ ὁ Β ἄρα κύβος ἐστίν. καὶ ἐπεί ἐστιν ὡς ἡ μονὰς πρὸς τὸν Α, ὁ Α πρὸς τὸν Β, ἡ δὲ μονὰς τὸν Α μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας, καὶ ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ὁ Α ἄρα ἑαυτὸν πολλαπλασιάσας κύβον τὸν Β πεποίηκεν. ἐὰν δὲ ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῇ, καὶ αὐτὸς κύβος ἔσται. κύβος ἄρα καὶ ὁ Α· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ὁ

show.

Proposition 10

If any multitude whatsoever of numbers is [continuously] proportional, (starting) from a unit, and the (number) after the unit is not square, then no other (number) will be square either, apart from the third from the unit, and all those (numbers after that) which leave an interval of one (number). And if the (number) after the unit is not cube, then no other (number) will be cube either, apart from the fourth from the unit, and all those (numbers after that) which leave an interval of two (numbers).



Let any multitude whatsoever of numbers, *A*, *B*, *C*, *D*, *E*, *F*, be continuously proportional, (starting) from a unit. And let the (number) after the unit, *A*, not be square. I say that no other (number) will be square either, apart from the third from the unit [and (all) those (numbers after that) which leave an interval of one (number)].

For, if possible, let *C* be square. And *B* is also square [Prop. 9.8]. Thus, *B* and *C* have to one another (the) ratio which (some) square number (has) to (some other) square number. And as *B* is to *C*, (so) *A* (is) to *B*. Thus, *A* and *B* have to one another (the) ratio which (some) square number has to (some other) square number. Hence, *A* and *B* are similar plane (numbers) [Prop. 8.26]. And *B* is square. Thus, *A* is also square. The very opposite thing was assumed. *C* is thus not square. So, similarly, we can show that no other (number is) square either, apart from the third from the unit, and (all) those (numbers after that) which leave an interval of one (number).

And so let *A* not be cube. I say that no other (number) will be cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers).

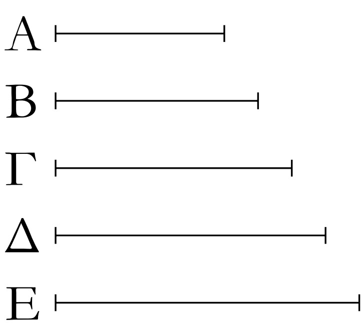
For, if possible, let *D* be cube. And *C* is also cube [Prop. 9.8]. For it is the fourth (number) from the unit. And as *C* is to *D*, (so) *B* (is) to *C*. And *B* thus has to *C* the ratio which (some) cube (number has) to (some other) cube (number). And *C* is cube. Thus, *B* is also cube [Props. 7.13, 8.25]. And since

Δ κύβος ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ᾿ ἄλλος οὐδεὶς κύβος ἐστὶ χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων· ὅπερ ἔδει δεῖξαι.

as the unit is to *A*, (so) *A* (is) to *B*, and the unit measures *A* according to the units in it, *A* thus also measures *B* according to the units in (*A*). Thus, *A* has made the cube (number) *B* (by) multiplying itself. And if a number makes a cube (number by) multiplying itself then it itself will be cube [Prop. 9.6]. Thus, *A* (is) also cube. The very opposite thing was assumed. Thus, *D* is not cube. So, similarly, we can show that no other (number) is cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers). (Which is) the very thing it was required to show.

ιαʹ.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ ἐλάττων τὸν μείζονα μετρεῖ κατά τινα τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

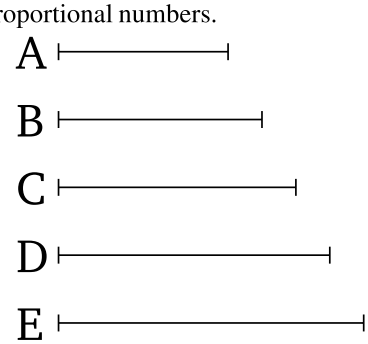


Ἔστωσαν ἀπὸ μονάδος τῆς Α ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Β, Γ, Δ, Ε· λέγω, ὅτι τῶν Β, Γ, Δ, Ε ὁ ἐλάχιστος ὁ Β τὸν Ε μετρεῖ κατά τινα τῶν Γ, Δ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ Α μονὰς πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε, ἰσάκις ἄρα ἡ Α μονὰς τὸν Β ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν Ε· ἐναλλὰξ ἄρα ἰσάκις ἡ Α μονὰς τὸν Δ μετρεῖ καὶ ὁ Β τὸν Ε. ἡ δὲ Α μονὰς τὸν Δ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· καὶ ὁ Β ἄρα τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὥστε ὁ ἐλάσσων ὁ Β τὸν μείζονα τὸν Ε μετρεῖ κατά τινα ἀριθμὸν τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

Proposition 11

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then a lesser (number) measures a greater according to some existing (number) among the proportional numbers.



Let any multitude whatsoever of numbers, *B*, *C*, *D*, *E*, be continuously proportional, (starting) from the unit *A*. I say that, for *B*, *C*, *D*, *E*, the least (number), *B*, measures *E* according to some (one) of *C*, *D*.

For since as the unit *A* is to *B*, so *D* (is) to *E*, the unit *A* thus measures the number *B* the same number of times as *D* (measures) *E*. Thus, alternately, the unit *A* measures *D* the same number of times as *B* (measures) *E* [Prop. 7.15]. And the unit *A* measures *D* according to the units in it. Thus, *B* also measures *E* according to the units in *D*. Hence, the lesser (number) *B* measures the greater *E* according to some existing number among the proportional numbers (namely, *D*).

Πόρισμα.

Καὶ φανερόν, ὅτι ἣν ἔχει τάξιν ὁ μετρῶν ἀπὸ μονάδος, τὴν αὐτὴν ἔχει καὶ ὁ καθ᾿ ὃν μετρεῖ ἀπὸ τοῦ μετρουμένου ἐπὶ τὸ πρὸ αὐτοῦ. ὅπερ ἔδει δεῖξαι.

Corollary

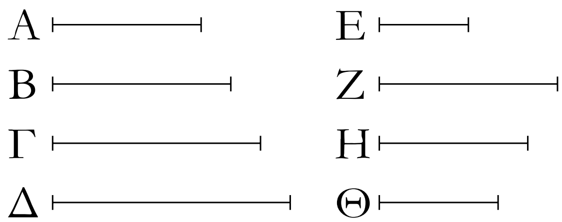
And (it is) clear that what(ever relative) place the measuring (number) has from the unit, the (number) according to which it measures has the same (relative) place from the measured (number), in (the direction of the number) before it. (Which is) the very thing it was required to show.

ιβʹ.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὑφ᾿ ὅσων ἂν ὁ ἔσχατος πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν

Proposition 12

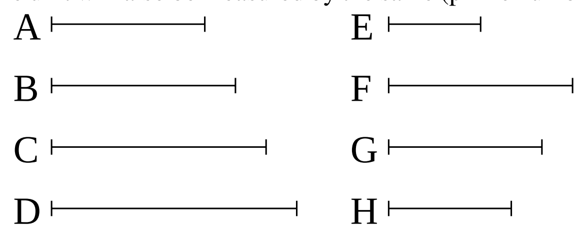
If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then however many prime *αὐτῶν καὶ ὁ παρὰ τὴν μονάδα μετρηθήσεται.*



Ἔστωσαν ἀπὸ μονάδος ὁποσοιδηποτοῦν ἀριθμοὶ ἀνάλογον οἱ Α, Β, Γ, Δ· λέγω, ὅτι ὑφ᾽ ὅσων ἂν ὁ Δ πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ Α μετρηθήσεται.

Μετρείσθω γὰρ ὁ Δ ὑπό τινος πρώτου ἀριθμοῦ τοῦ Ε· λέγω, ὅτι ὁ Ε τὸν Α μετρεῖ. μὴ γάρ· καί ἐστιν ὁ Ε πρῶτος, ἅπας δὲ πρῶτος ἀριθμὸς πρὸς ἅπαντα, ὃν μὴ μετρεῖ, πρῶτός ἐστιν· οἱ Ε, Α ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ· ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. πάλιν, ἐπεὶ ὁ Α τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, ὁ Α ἄρα τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Ε τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Γ ἴσος ἐστὶ τῷ ἐκ τῶν Ε, Ζ. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Ε, ὁ Ζ πρὸς τὸν Γ. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Ε τὸν Γ. μετρείτω αὐτὸν κατὰ τὸν Η· ὁ Ε ἄρα τὸν Η πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν διὰ τὸ πρὸ τούτου καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ ἄρα ἐκ τῶν Α, Β ἴσος ἐστὶ τῷ ἐκ τῶν Ε, Η. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Ε, ὁ Η πρὸς τὸν Β. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Ε τὸν Β. μετρείτω αὐτὸν κατὰ τὸν Θ· ὁ Ε ἄρα τὸν Θ πολλαπλασιάσας τὸν Β πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· ὁ ἄρα ἐκ τῶν Ε, Θ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Α. ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Α, ὁ Α πρὸς τὸν Θ. οἱ δὲ Α, Ε πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Ε τὸν Α ὡς ἡγούμενος ἡγούμενον. ἀλλὰ μὴν καὶ οὐ μετρεῖ· ὅπερ ἀδύνατον. οὐκ ἄρα οἱ Ε, Α πρῶτοι πρὸς ἀλλήλους εἰσίν. σύνθετοι ἄρα. οἱ δὲ σύνθετοι ὑπὸ [πρώτου] ἀριθμοῦ τινος μετροῦνται. καὶ ἐπεὶ ὁ Ε πρῶτος ὑπόκειται, ὁ δὲ πρῶτος ὑπὸ ἑτέρου ἀριθμοῦ οὐ μετρεῖται ἢ ὑφ᾽ ἑαυτοῦ, ὁ Ε ἄρα τοὺς Α, Ε μετρεῖ· ὥστε ὁ Ε τὸν Α μετρεῖ. μετρεῖ δὲ καὶ τὸν Δ· ὁ Ε ἄρα τοὺς Α, Δ μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι ὑφ᾽ ὅσων ἂν ὁ Δ πρώτων ἀριθμῶν μετρῆται, ὑπὸ τῶν αὐτῶν καὶ ὁ Α μετρηθήσεται· ὅπερ ἔδει δεῖξαι.

numbers the last (number) is measured by, the (number) next to the unit will also be measured by the same (prime numbers).



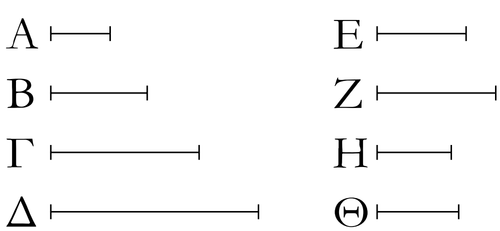
Let any multitude whatsoever of numbers, *A*, *B*, *C*, *D*, be (continuously) proportional, (starting) from a unit. I say that however many prime numbers *D* is measured by, *A* will also be measured by the same (prime numbers).

For let *D* be measured by some prime number *E*. I say that *E* measures *A*. For (suppose it does) not. *E* is prime, and every prime number is prime to every number which it does not measure [Prop. 7.29]. Thus, *E* and *A* are prime to one another. And since *E* measures *D*, let it measure it according to *F*. Thus, *E* has made *D* (by) multiplying *F*. Again, since *A* measures *D* according to the units in *C* [Prop. 9.11 corr.], *A* has thus made *D* (by) multiplying *C*. But, in fact, *E* has also made *D* (by) multiplying *F*. Thus, the (number created) from (multiplying) *A*, *C* is equal to the (number created) from (multiplying) *E*, *F*. Thus, as *A* is to *E*, (so) *F* (is) to *C* [Prop. 7.19]. And *A* and *E* (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, *E* measures *C*. Let it measure it according to *G*. Thus, *E* has made *C* (by) multiplying *G*. But, in fact, via the (proposition) before this, *A* has also made *C* (by) multiplying *B* [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) *A*, *B* is equal to the (number created) from (multiplying) *E*, *G*. Thus, as *A* is to *E*, (so) *G* (is) to *B* [Prop. 7.19]. And *A* and *E* (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, *E* measures *B*. Let it measure it according to *H*. Thus, *E* has made *B* (by) multiplying *H*. But, in fact, *A* has also made *B* (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) *E*, *H* is equal to the (square) on *A*. Thus, as *E* is to *A*, (so) *A* (is) to *H* [Prop. 7.19]. And *A* and *E* are prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures A, as the leading (measuring the) leading. But, in fact, (E) also does not measure (A). The very thing (is) impossible. Thus, E and A are not prime to one another. Thus, (they are) composite (to one another). And (numbers) composite (to one another) are (both) measured by some [prime] number [Def. 7.14]. And since E is assumed (to be) prime, and a prime (number) is not measured by another number (other) than itself [Def. 7.11], E thus measures (both) A and E. Hence, E measures A. And it also measures D. Thus, E measures (both) A and D. So, similarly, we can show that however many prime numbers D is measured by, A will also be measured by the same (prime numbers). (Which is) the very thing it was required to show.

ιγ'.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα πρῶτος ᾖ, ὁ μέγιστος ὑπ᾽ οὐδενὸς [ἄλλου] μετρηθήσεται παρὲξ τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

Ἔστωσαν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, ὁ δὲ μετὰ τὴν μονάδα ὁ Α πρῶτος ἔστω· λέγω, ὅτι ὁ μέγιστος αὐτῶν ὁ Δ ὑπ᾽ οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ.

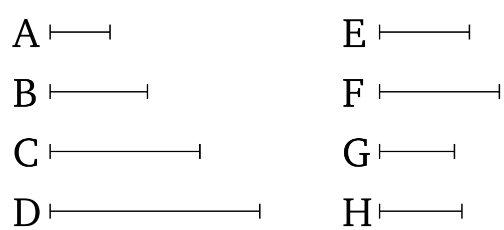


Εἰ γὰρ δυνατόν, μετρείσθω ὑπὸ τοῦ Ε, καὶ ὁ Ε μηδενὶ τῶν Α, Β, Γ ἔστω ὁ αὐτός. φανερὸν δή, ὅτι ὁ Ε πρῶτος οὐκ ἐστιν. εἰ γὰρ ὁ Ε πρῶτός ἐστι καὶ μετρεῖ τὸν Δ, καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ Ε πρῶτός ἐστιν. σύνθετος ἄρα. πᾶς δὲ σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὁ Ε ἄρα ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δή, ὅτι ὑπ᾽ οὐδενὸς ἄλλου πρώτου μετρηθήσεται πλὴν τοῦ Α. εἰ γὰρ ὑφ᾽ ἑτέρου μετρεῖται ὁ Ε, ὁ δὲ Ε τὸν Δ μετρεῖ, κἀκεῖνος ἄρα τὸν Δ μετρήσει· ὥστε καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ Α ἄρα τὸν Ε μετρεῖ. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ. λέγω, ὅτι ὁ Ζ οὐδενὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. εἰ γὰρ ὁ Ζ ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτὸς καὶ μετρεῖ τὸν Δ κατὰ τὸν Ε, καὶ εἷς ἄρα τῶν Α, Β, Γ τὸν Δ μετρεῖ κατὰ τὸν Ε. ἀλλὰ εἷς τῶν Α, Β, Γ τὸν Δ μετρεῖ κατά τινα τῶν Α, Β, Γ· καὶ ὁ Ε ἄρα ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ὁ Ζ ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. ὁμοίως δὴ δείξομεν, ὅτι μετρεῖται

Proposition 13

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is prime, then the greatest (number) will be measured by no [other] (numbers) except (numbers) existing among the proportional numbers.

Let any multitude whatsoever of numbers, A, B, C, D, be continuously proportional, (starting) from a unit. And let the (number) after the unit, A, be prime. I say that the greatest of them, D, will be measured by no other (numbers) except A, B, C.



For, if possible, let it be measured by E, and let E not be the same as one of A, B, C. So it is clear that E is not prime. For if E is prime, and measures D, then it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, E is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, E is measured by some prime number. So I say that it will be measured by no other prime number than A. For if E is measured by another (prime number), and E measures D, then this (prime number) will thus also measure D. Hence, it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, A measures E. And since E measures D, let it measure it according to F. I say that F is not the same as one of A, B, C. For if F is the same as one of A, B, C, and measures D according to E, then one of A, B, C thus also measures D according to E. But one of A, B, C (only) measures D according to some *ὁ Ζ ὑπὸ τοῦ Α, δεικνύντες πάλιν, ὅτι ὁ Ζ οὔκ ἐστι πρῶτος. εἰ γάρ, καὶ μετρεῖ τὸν Δ, καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα πρῶτός ἐστιν ὁ Ζ· σύνθετος ἄρα. ἅπας δὲ σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται· ὁ Ζ ἄρα ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δή, ὅτι ὑφ' ἑτέρου πρώτου οὐ μετρηθήσεται πλὴν τοῦ Α. εἰ γὰρ ἕτερός τις πρῶτος τὸν Ζ μετρεῖ, ὁ δὲ Ζ τὸν Δ μετρεῖ, κἀκεῖνος ἄρα τὸν Δ μετρήσει· ὥστε καὶ τὸν Α μετρήσει πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ Α ἄρα τὸν Ζ μετρεῖ. καὶ ἐπεὶ ὁ Ε τὸν Δ μετρεῖ κατὰ τὸν Ζ, ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Γ ἴσος ἐστὶ τῷ ἐκ τῶν Ε, Ζ. ἀνάλογον ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Ε, οὕτως ὁ Ζ πρὸς τὸν Γ. ὁ δὲ Α τὸν Ε μετρεῖ· καὶ ὁ Ζ ἄρα τὸν Γ μετρεῖ. μετρείτω αὐτὸν κατὰ τὸν Η. ὁμοίως δὴ δείξομεν, ὅτι ὁ Η οὐδενὶ τῶν Α, Β ἐστιν ὁ αὐτός, καὶ ὅτι μετρεῖται ὑπὸ τοῦ Α. καὶ ἐπεὶ ὁ Ζ τὸν Γ μετρεῖ κατὰ τὸν Η, ὁ Ζ ἄρα τὸν Η πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Β ἴσος ἐστὶ τῷ ἐκ τῶν Ζ, Η. ἀνάλογον ἄρα ὡς ὁ Α πρὸς τὸν Ζ, ὁ Η πρὸς τὸν Β. μετρεῖ δὲ ὁ Α τὸν Ζ· μετρεῖ ἄρα καὶ ὁ Η τὸν Β. μετρείτω αὐτὸν κατὰ τὸν Θ. ὁμοίως δὴ δείξομεν, ὅτι ὁ Θ τῷ Α οὐκ ἔστιν ὁ αὐτός. καὶ ἐπεὶ ὁ Η τὸν Β μετρεῖ κατὰ τὸν Θ, ὁ Η ἄρα τὸν Θ πολλαπλασιάσας τὸν Β πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· ὁ ἄρα ὑπὸ Θ, Η ἴσος ἐστὶ τῷ ἀπὸ τοῦ Α τετραγώνῳ· ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Α, ὁ Α πρὸς τὸν Η. μετρεῖ δὲ ὁ Α τὸν Η· μετρεῖ ἄρα καὶ ὁ Θ τὸν Α πρῶτον ὄντα μὴ ὢν αὐτῷ ὁ αὐτός· ὅπερ ἄτοπον. οὐκ ἄρα ὁ μέγιστος ὁ Δ ὑπὸ ἑτέρου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Α, Β, Γ· ὅπερ ἔδει δεῖξαι.*

(one) of *A*, *B*, *C* [Prop. 9.11]. And thus *E* is the same as one of *A*, *B*, *C*. The very opposite thing was assumed. Thus, *F* is not the same as one of *A*, *B*, *C*. Similarly, we can show that *F* is measured by *A*, (by) again showing that *F* is not prime. For if (*F* is prime), and measures *D*, then it will also measure *A*, (despite *A*) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, *F* is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, *F* is measured by some prime number. So I say that it will be measured by no other prime number than *A*. For if some other prime (number) measures *F*, and *F* measures *D*, then this (prime number) will thus also measure *D*. Hence, it will also measure *A*, (despite *A*) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, *A* measures *F*. And since *E* measures *D* according to *F*, *E* has thus made *D* (by) multiplying *F*. But, in fact, *A* has also made *D* (by) multiplying *C* [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) *A*, *C* is equal to the (number created) from (multiplying) *E*, *F*. Thus, proportionally, as *A* is to *E*, so *F* (is) to *C* [Prop. 7.19]. And *A* measures *E*. Thus, *F* also measures *C*. Let it measure it according to *G*. So, similarly, we can show that *G* is not the same as one of *A*, *B*, and that it is measured by *A*. And since *F* measures *C* according to *G*, *F* has thus made *C* (by) multiplying *G*. But, in fact, *A* has also made *C* (by) multiplying *B* [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) *A*, *B* is equal to the (number created) from (multiplying) *F*, *G*. Thus, proportionally, as *A* (is) to *F*, so *G* (is) to *B* [Prop. 7.19]. And *A* measures *F*. Thus, *G* also measures *B*. Let it measure it according to *H*. So, similarly, we can show that *H* is not the same as *A*. And since *G* measures *B* according to *H*, *G* has thus made *B* (by) multiplying *H*. But, in fact, *A* has also made *B* (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) *H*, *G* is equal to the square on *A*. Thus, as *H* is to *A*, (so) *A* (is) to *G* [Prop. 7.19]. And *A* measures *G*. Thus, *H* also measures *A*, (despite *A*) being prime (and) not being the same as it. The very thing (is) absurd. Thus, the greatest (number) *D* cannot be measured by another (number) except (one of) *A*, *B*, *C*. (Which is) the very thing it was required to show.

ιδ'.

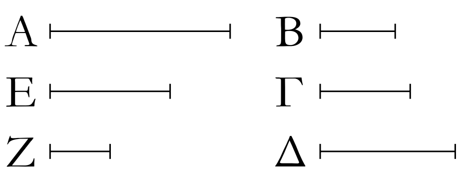
Ἐὰν ἐλάχιστος ἀριθμὸς ὑπὸ πρώτων ἀριθμῶν μετρῆται, ὑπ' οὐδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν ἐξ ἀρχῆς μετρούντων.

Ἐλάχιστος γὰρ ἀριθμὸς ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν Β, Γ, Δ μετρείσθω· λέγω, ὅτι ὁ Α ὑπ' οὐδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Β, Γ, Δ.

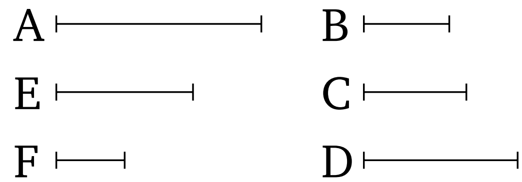
Proposition 14

If a least number is measured by (some) prime numbers then it will not be measured by any other prime number except (one of) the original measuring (numbers).

For let *A* be the least number measured by the prime numbers *B*, *C*, *D*. I say that *A* will not be measured by any other prime number except (one of) *B*, *C*, *D*.



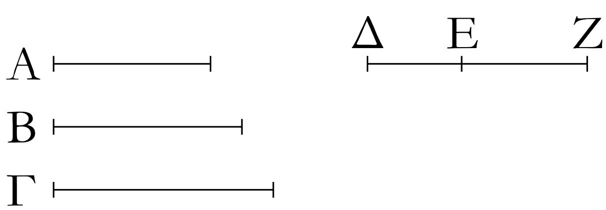
Εἰ γὰρ δυνατόν, μετρείσθω ὑπὸ πρώτου τοῦ Ε, καὶ ὁ Ε μηδενὶ τῶν Β, Γ, Δ ἔστω ὁ αὐτός. καὶ ἐπεὶ ὁ Ε τὸν Α μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Ζ· ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ μετρεῖται ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν Β, Γ, Δ. ἐὰν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γενόμενον ἐξ αὐτῶν μετρῇ τις πρῶτος ἀριθμός, καὶ ἕνα τῶν ἐξ ἀρχῆς μετρήσει· οἱ Β, Γ, Δ ἄρα ἕνα τῶν Ε, Ζ μετρήσουσιν. τὸν μὲν οὖν Ε οὐ μετρήσουσιν· ὁ γὰρ Ε πρῶτός ἐστι καὶ οὐδενὶ τῶν Β, Γ, Δ ὁ αὐτός. τὸν Ζ ἄρα μετροῦσιν ἐλάσσονα ὄντα τοῦ Α· ὅπερ ἀδύνατον. ὁ γὰρ Α ὑπόκειται ἐλάχιστος ὑπὸ τῶν Β, Γ, Δ μετρούμενος. οὐκ ἄρα τὸν Α μετρήσει πρῶτος ἀριθμὸς παρὲξ τῶν Β, Γ, Δ· ὅπερ ἔδει δεῖξαι.



For, if possible, let it be measured by the prime (number) *E*. And let *E* not be the same as one of *B*, *C*, *D*. And since *E* measures *A*, let it measure it according to *F*. Thus, *E* has made *A* (by) multiplying *F*. And *A* is measured by the prime numbers *B*, *C*, *D*. And if two numbers make some (number by) multiplying one another, and some prime number measures the number created from them, then (the prime number) will also measure one of the original (numbers) [Prop. 7.30]. Thus, *B*, *C*, *D* will measure one of *E*, *F*. In fact, they do not measure *E*. For *E* is prime, and not the same as one of *B*, *C*, *D*. Thus, they (all) measure *F*, which is less than *A*. The very thing (is) impossible. For *A* was assumed (to be) the least (number) measured by *B*, *C*, *D*. Thus, no prime number can measure *A* except (one of) *B*, *C*, *D*. (Which is) the very thing it was required to show.

ιε'.

Ἐὰν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ὦσιν ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, δύο ὁποιοιοῦν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοί εἰσιν.

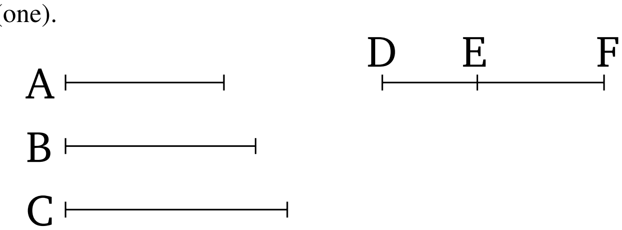


Ἔστωσαν τρεῖς ἀριθμοὶ ἑξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ Α, Β, Γ· λέγω, ὅτι τῶν Α, Β, Γ δύο ὁποιοιοῦν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοι εἰσιν, οἱ μὲν Α, Β πρὸς τὸν Γ, οἱ δὲ Β, Γ πρὸς τὸν Α καὶ ἔτι οἱ Α, Γ πρὸς τὸν Β.

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ δύο οἱ ΔΕ, ΕΖ. φανερὸν δή, ὅτι ὁ μὲν ΔΕ ἑαυτὸν πολλαπλασιάσας τὸν Α πεποίηκεν, τὸν δὲ ΕΖ πολλαπλασιάσας τὸν Β πεποίηκεν, καὶ ἔτι ὁ ΕΖ ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν. καὶ ἐπεὶ οἱ ΔΕ, ΕΖ ἐλάχιστοί εἰσιν, πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ συναμφότερος πρὸς ἑκάτερον πρῶτός ἐστιν· καὶ ὁ ΔΖ ἄρα πρὸς ἑκάτερον τῶν ΔΕ, ΕΖ πρῶτός ἐστιν. ἀλλὰ μὴν καὶ ὁ ΔΕ πρὸς τὸν ΕΖ πρῶτός ἐστιν· οἱ ΔΖ, ΔΕ ἄρα πρὸς τὸν ΕΖ πρῶτοί εἰσιν. ἐὰν δὲ δύο ἀριθμοὶ πρός τινα ἀριθμὸν πρῶτοι ὦσιν, καὶ ὁ ἐξ αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν· ὥστε ὁ ἐκ τῶν ΖΔ, ΔΕ πρὸς τὸν ΕΖ

Proposition 15

If three continuously proportional numbers are the least of those (numbers) having the same ratio as them then two (of them) added together in any way are prime to the remaining (one).



Let *A*, *B*, *C* be three continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that two of *A*, *B*, *C* added together in any way are prime to the remaining (one), (that is) *A* and *B* (prime) to *C*, *B* and *C* to *A*, and, further, *A* and *C* to *B*.

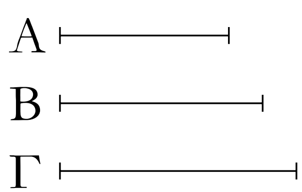
Let the two least numbers, *DE* and *EF*, having the same ratio as *A*, *B*, *C*, be taken [Prop. 8.2]. So it is clear that *DE* has made *A* (by) multiplying itself, and has made *B* (by) multiplying *EF*, and, further, *EF* has made *C* (by) multiplying itself [Prop. 8.2]. And since *DE*, *EF* are the least (of those numbers having the same ratio as them), they are prime to one another [Prop. 7.22]. And if two numbers are prime to one another then the sum (of them) is also prime to each [Prop. 7.28]. Thus, *DF* is also prime to each of *DE*, *EF*. But, in fact, *DE* is also prime to *EF*. Thus, *DF*, *DE* are (both) prime to *EF*. And if two numbers are (both) prime to some number then the (number) created from (multiplying) them is also prime to *πρῶτός ἐστιν· ὥστε καὶ ὁ ἐκ τῶν ΖΔ, ΔΕ πρὸς τὸν ἀπὸ τοῦ ΕΖ πρῶτός ἐστιν. [ἐὰν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ ἐκ τοῦ ἑνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν]. ἀλλ᾿ ὁ ἐκ τῶν ΖΔ, ΔΕ ὁ ἀπὸ τοῦ ΔΕ ἐστι μετὰ τοῦ ἐκ τῶν ΔΕ, ΕΖ· ὁ ἄρα ἀπὸ τοῦ ΔΕ μετὰ τοῦ ἐκ τῶν ΔΕ, ΕΖ πρὸς τὸν ἀπὸ τοῦ ΕΖ πρῶτός ἐστιν. καί ἐστιν ὁ μὲν ἀπὸ τοῦ ΔΕ ὁ Α, ὁ δὲ ἐκ τῶν ΔΕ, ΕΖ ὁ Β, ὁ δὲ ἀπὸ τοῦ ΕΖ ὁ Γ· οἱ Α, Β ἄρα συντεθέντες πρὸς τὸν Γ πρῶτοί εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ Β, Γ πρὸς τὸν Α πρῶτοί εἰσιν. λέγω δή, ὅτι καὶ οἱ Α, Γ πρὸς τὸν Β πρῶτοί εἰσιν. ἐπεὶ γὰρ ὁ ΔΖ πρὸς ἑκάτερον τῶν ΔΕ, ΕΖ πρῶτός ἐστιν, καὶ ὁ ἀπὸ τοῦ ΔΖ πρὸς τὸν ἐκ τῶν ΔΕ, ΕΖ πρῶτός ἐστιν. ἀλλὰ τῷ ἀπὸ τοῦ ΔΖ ἴσοι εἰσὶν οἱ ἀπὸ τῶν ΔΕ, ΕΖ μετὰ τοῦ δὶς ἐκ τῶν ΔΕ, ΕΖ· καὶ οἱ ἀπὸ τῶν ΔΕ, ΕΖ ἄρα μετὰ τοῦ δὶς ὑπὸ τῶν ΔΕ, ΕΖ πρὸς τὸν ὑπὸ τῶν ΔΕ, ΕΖ πρῶτοι [εἰσι]. διελόντι οἱ ἀπὸ τῶν ΔΕ, ΕΖ μετὰ τοῦ ἅπαξ ὑπὸ ΔΕ, ΕΖ πρὸς τὸν ὑπὸ ΔΕ, ΕΖ πρῶτοί εἰσιν. ἔτι διελόντι οἱ ἀπὸ τῶν ΔΕ, ΕΖ ἄρα πρὸς τὸν ὑπὸ ΔΕ, ΕΖ πρῶτοί εἰσιν. καί ἐστιν ὁ μὲν ἀπὸ τοῦ ΔΕ ὁ Α, ὁ δὲ ὑπὸ τῶν ΔΕ, ΕΖ ὁ Β, ὁ δὲ ἀπὸ τοῦ ΕΖ ὁ Γ. οἱ Α, Γ ἄρα συντεθέντες πρὸς τὸν Β πρῶτοί εἰσιν· ὅπερ ἔδει δεῖξαι.*

the remaining (number) [Prop. 7.24]. Hence, the (number created) from (multiplying) FD, DE is prime to EF. Hence, the (number created) from (multiplying) FD, DE is also prime to the (square) on EF [Prop. 7.25]. [For if two numbers are prime to one another then the (number) created from (squaring) one of them is prime to the remaining (number).] But the (number created) from (multiplying) FD, DE is the (square) on DE plus the (number created) from (multiplying) DE, EF [Prop. 2.3]. Thus, the (square) on DE plus the (number created) from (multiplying) DE, EF is prime to the (square) on EF. And the (square) on DE is A, and the (number created) from (multiplying) DE, EF (is) B, and the (square) on EF (is) C. Thus, A, B summed is prime to C. So, similarly, we can show that B, C (summed) is also prime to A. So I say that A, C (summed) is also prime to B. For since DF is prime to each of DE, EF then the (square) on DF is also prime to the (number created) from (multiplying) DE, EF [Prop. 7.25]. But, the (sum of the squares) on DE, EF plus twice the (number created) from (multiplying) DE, EF is equal to the (square) on DF [Prop. 2.4]. And thus the (sum of the squares) on DE, EF plus twice the (rectangle contained) by DE, EF [is] prime to the (rectangle contained) by DE, EF. By separation, the (sum of the squares) on DE, EF plus once the (rectangle contained) by DE, EF is prime to the (rectangle contained) by DE, EF.[†] Again, by separation, the (sum of the squares) on DE, EF is prime to the (rectangle contained) by DE, EF. And the (square) on DE is A, and the (rectangle contained) by DE, EF (is) B, and the (square) on EF (is) C. Thus, A, C summed is prime to B. (Which is) the very thing it was required to show.

[†] Since if $\alpha\beta$ measures $\alpha^2+\beta^2+2\,\alpha\beta$ then it also measures $\alpha^2+\beta^2+\alpha\beta$, and vice versa.

ιϛ'.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ δεύτερος πρὸς ἄλλον τινά.

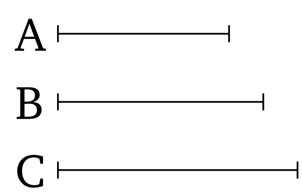


Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐκ ἔστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς ἄλλον τινά.

Εἰ γὰρ δυνατόν, ἔστω ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· μετρεῖ ἄρα ὁ Α τὸν Β ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτόν· ὁ Α ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους·

Proposition 16

If two numbers are prime to one another then as the first is to the second, so the second (will) not (be) to some other (number).



For let the two numbers A and B be prime to one another. I say that as A is to B, so B is not to some other (number).

For, if possible, let it be that as A (is) to B, (so) B (is) to C. And A and B (are) prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A

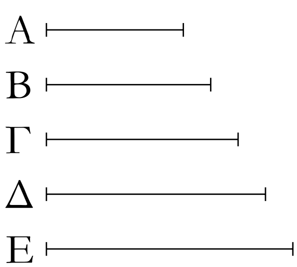
ὅπερ ἄτοπον. οὐκ ἄρα ἔσται ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· ὅπερ ἔδει δεῖξαι.

measures *B*, as the leading (measuring) the leading. And (*A*) also measures itself. Thus, *A* measures *A* and *B*, which are prime to one another. The very thing (is) absurd. Thus, as *A* (is) to *B*, so *B* cannot be to *C*. (Which is) the very thing it was required to show.

ιζʹ.

Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ ἔσχατος πρὸς ἄλλον τινά.

Ἔστωσαν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, Β, Γ, Δ, οἱ δὲ ἄκροι αὐτῶν οἱ Α, Δ πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐκ ἔστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς ἄλλον τινά.

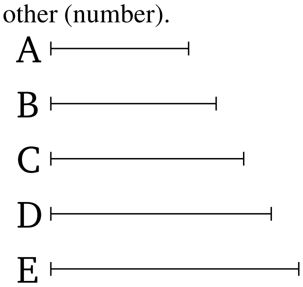


Εἰ γὰρ δυνατόν, ἔστω ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς τὸν Ε· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, ὁ Β πρὸς τὸν Ε. οἱ δὲ Α, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Β. καί ἐστιν ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. καὶ ὁ Β ἄρα τὸν Γ μετρεῖ· ὥστε καὶ ὁ Α τὸν Γ μετρεῖ. καὶ ἐπεί ἐστιν ὡς ὁ Β πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ, μετρεῖ δὲ ὁ Β τὸν Γ, μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ. ἀλλ᾽ ὁ Α τὸν Γ ἐμέτρει· ὥστε ὁ Α καὶ τὸν Δ μετρεῖ. μετρεῖ δὲ καὶ ἑαυτόν. ὁ Α ἄρα τοὺς Α, Δ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσται ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Δ πρὸς ἄλλον τινά· ὅπερ ἔδει δεῖξαι.

Proposition 17

If any multitude whatsoever of numbers is continuously proportional, and the outermost of them are prime to one another, then as the first (is) to the second, so the last will not be to some other (number).

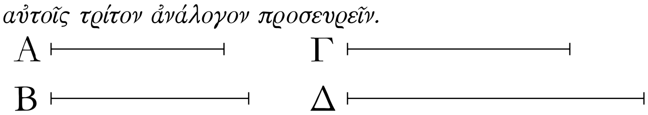
Let *A*, *B*, *C*, *D* be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, *A* and *D*, be prime to one another. I say that as *A* is to *B*, so *D* (is) not to some other (number).



For, if possible, let it be that as *A* (is) to *B*, so *D* (is) to *E*. Thus, alternately, as *A* is to *D*, (so) *B* (is) to *E* [Prop. 7.13]. And *A* and *D* are prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, *A* measures *B*. And as *A* is to *B*, (so) *B* (is) to *C*. Thus, *B* also measures *C*. And hence *A* measures *C* [Def. 7.20]. And since as *B* is to *C*, (so) *C* (is) to *D*, and *B* measures *C*, *C* thus also measures *D* [Def. 7.20]. But, *A* was (found to be) measuring *C*. And hence *A* also measures *D*. And (*A*) also measures itself. Thus, *A* measures *A* and *D*, which are prime to one another. The very thing is impossible. Thus, as *A* (is) to *B*, so *D* cannot be to some other (number). (Which is) the very thing it was required to show.

ιηʹ.

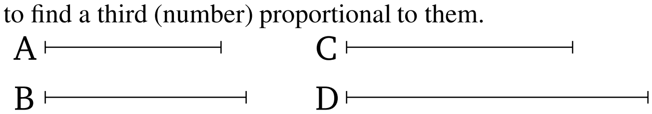
Δύο ἀριθμῶν δοθέντων ἐπισκέψασθαι, εἰ δυνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.



Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β, καὶ δέον ἔστω ἐπισκέψασθαι, εἰ δυνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.

Proposition 18

For two given numbers, to investigate whether it is possible to find a third (number) proportional to them.



Let *A* and *B* be the two given numbers. And let it be required to investigate whether it is possible to find a third (number) proportional to them.

Οἱ δὴ Α, Β ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὔ. καὶ εἰ πρῶτοι πρὸς ἀλλήλους εἰσίν, δέδεικται, ὅτι ἀδύνατόν ἐστιν αὐτοῖς τρίτον ἀνάλογον προσευρεῖν.

Ἀλλὰ δὴ μὴ ἔστωσαν οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Β ἑαυτον πολλαπλασιάσας τὸν Γ ποιείτω. ὁ Α δὴ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον κατὰ τὸν Δ· ὁ Α ἄρα τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλα μὴν καὶ ὁ Β ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Β. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Δ· τοῖς Α, Β ἄρα τρίτος ἀριθμὸς ἀνάλογον προσηύρηται ὁ Δ.

Ἀλλὰ δὴ μὴ μετρείτω ὁ Α τὸν Γ· λέγω, ὅτι τοῖς Α, Β ἀδύνατόν ἐστι τρίτον ἀνάλογον προσευρεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσηυρήσθω ὁ Δ. ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ Β. ὁ δὲ ἀπὸ τοῦ Β ἐστιν ὁ Γ· ὁ ἄρα ἐκ τῶν Α, Δ ἴσος ἐστὶ τῷ Γ. ὥστε ὁ Α τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ Α ἄρα τὸν Γ μετρεῖ κατὰ τὸν Δ. ἀλλα μὴν ὑπόκειται καὶ μὴ μετρῶν· ὅπερ ἄτοπον. οὐκ ἄρα δυνατόν ἐστι τοῖς Α, Β τρίτον ἀνάλογον προσευρεῖν ἀριθμὸν, ὅταν ὁ Α τὸν Γ μὴ μετρῇ· ὅπερ ἔδει δεῖξαι.

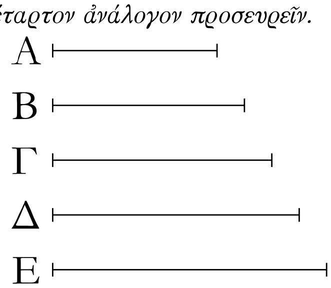
So *A* and *B* are either prime to one another, or not. And if they are prime to one another then it has (already) been show that it is impossible to find a third (number) proportional to them [Prop. 9.16].

And so let *A* and *B* not be prime to one another. And let *B* make *C* (by) multiplying itself. So *A* either measures, or does not measure, *C*. Let it first of all measure (*C*) according to *D*. Thus, *A* has made *C* (by) multiplying *D*. But, in fact, *B* has also made *C* (by) multiplying itself. Thus, the (number created) from (multiplying) *A*, *D* is equal to the (square) on *B*. Thus, as *A* is to *B*, (so) *B* (is) to *D* [Prop. 7.19]. Thus, a third number has been found proportional to *A*, *B*, (namely) *D*.

And so let *A* not measure *C*. I say that it is impossible to find a third number proportional to *A*, *B*. For, if possible, let it be found, (and let it be) *D*. Thus, the (number created) from (multiplying) *A*, *D* is equal to the (square) on *B* [Prop. 7.19]. And the (square) on *B* is *C*. Thus, the (number created) from (multiplying) *A*, *D* is equal to *C*. Hence, *A* has made *C* (by) multiplying *D*. Thus, *A* measures *C* according to *D*. But (*A*) was, in fact, also assumed (to be) not measuring (*C*). The very thing (is) absurd. Thus, it is not possible to find a third number proportional to *A*, *B* when *A* does not measure *C*. (Which is) the very thing it was required to show.

ιθ'.

Τριῶν ἀριθμῶν δοθέντων ἐπισκέψασθαι, πότε δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν.



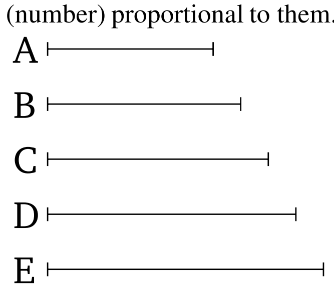
Ἔστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ Α, Β, Γ, καὶ δέον ἔστω ἐπισκέψασθαι, πότε δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν.

Ἤτοι οὖν οὔκ εἰσιν ἑξῆς ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οὔκ εἰσι πρῶτοι πρὸς ἀλλήλους, ἢ οὔτε ἑξῆς εἰσιν ἀνάλογον, οὔτε οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ καὶ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ μὲν οὖν οἱ Α, Β, Γ ἑξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οἱ Α, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν, δέδεικται, ὅτι ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν. μὴ ἔστωσαν δὴ οἱ Α, Β, Γ ἑξῆς ἀνάλογον τῶν ἄκρων πάλιν ὄντων πρώτων πρὸς ἀλλήλους. λέγω, ὅτι καὶ οὕτως ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν. εἰ γὰρ δυνατόν,

Proposition 19[†]

For three given numbers, to investigate when it is possible to find a fourth (number) proportional to them.



Let *A*, *B*, *C* be the three given numbers. And let it be required to investigate when it is possible to find a fourth (number) proportional to them.

In fact, (*A*, *B*, *C*) are either not continuously proportional and the outermost of them are prime to one another, or are continuously proportional and the outermost of them are not prime to one another, or are neither continuously proportional nor are the outermost of them prime to one another, or are continuously proportional and the outermost of them are prime to one another.

In fact, if *A*, *B*, *C* are continuously proportional, and the outermost of them, *A* and *C*, are prime to one another, (then) it has (already) been shown that it is impossible to find a fourth number proportional to them [Prop. 9.17]. So let *A*, *B*, *C* not be continuously proportional, (with) the outermost of them again *προσευρήσθω ὁ Δ, ὥστε εἶναι ὡς τὸν Α πρὸς τὸν Β, τὸν Γ πρὸς τὸν Δ, καὶ γεγονέτω ὡς ὁ Β πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Ε. καὶ ἐπεί ἐστιν ὡς μὲν ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Δ, ὡς δὲ ὁ Β πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Ε, δι᾿ ἴσου ἄρα ὡς ὁ Α πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Ε. οἱ δὲ Α, Γ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον. μετρεῖ ἄρα ὁ Α τὸν Γ ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτόν· ὁ Α ἄρα τοὺς Α, Γ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοῖς Α, Β, Γ δυνατόν ἐστι τέταρτον ἀνάλογον προσευρεῖν.*

Ἀλλὰ δὴ πάλιν ἔστωσαν οἱ Α, Β, Γ ἑξῆς ἀνάλογον, οἱ δὲ Α, Γ μὴ ἔστωσαν πρῶτοι πρὸς ἀλλήλους. λέγω, ὅτι δυνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσευρεῖν. ὁ γὰρ Β τὸν Γ πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Α ἄρα τὸν Δ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω αὐτὸν πρότερον κατὰ τὸν Ε· ὁ Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν Α, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Β, Γ. ἀνάλογον ἄρα [ἐστὶν] ὡς ὁ Α πρὸς τὸν Β, ὁ Γ πρὸς τὸν Ε· τοῖς Α, Β, Γ ἄρα τέταρτος ἀνάλογον προσηύρηται ὁ Ε.

Ἀλλὰ δὴ μὴ μετρείτω ὁ Α τὸν Δ· λέγω, ὅτι ἀδύνατόν ἐστι τοῖς Α, Β, Γ τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσευρήσθω ὁ Ε· ὁ ἄρα ἐκ τῶν Α, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Β, Γ. ἀλλὰ ὁ ἐκ τῶν Β, Γ ἐστιν ὁ Δ· καὶ ὁ ἐκ τῶν Α, Ε ἄρα ἴσος ἐστὶ τῷ Δ. ὁ Α ἄρα τὸν Ε πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ Α ἄρα τὸν Δ μετρεῖ κατὰ τὸν Ε· ὥστε μετρεῖ ὁ Α τὸν Δ. ἀλλὰ καὶ οὐ μετρεῖ· ὅπερ ἄτοπον. οὐκ ἄρα δυνάτον ἐστι τοῖς Α, Β, Γ τέταρτον ἀνάλογον προσευρεῖν ἀριθμόν, ὅταν ὁ Α τὸν Δ μὴ μετρῇ. ἀλλὰ δὴ οἱ Α, Β, Γ μήτε ἑξῆς ἔστωσαν ἀνάλογον μήτε οἱ ἄκροι πρῶτοι πρὸς ἀλλήλους. καὶ ὁ Β τὸν Γ πολλαπλασιάσας τὸν Δ ποιείτω. ὁμοίως δὴ δειχθήσεται, ὅτι εἰ μὲν μετρεῖ ὁ Α τὸν Δ, δυνατόν ἐστιν αὐτοῖς ἀνάλογον προσευρεῖν, εἰ δὲ οὐ μετρεῖ, ἀδύνατον· ὅπερ ἔδει δεῖξαι.

being prime to one another. I say that, in this case, it is also impossible to find a fourth (number) proportional to them. For, if possible, let it be found, (and let it be) D. Hence, it will be that as A (is) to B, (so) C (is) to D. And let it be contrived that as B (is) to C, (so) D (is) to E. And since as A is to B, (so) C (is) to D, and as B (is) to C, (so) D (is) to E, thus, via equality, as A (is) to C, (so) C (is) to E [Prop. 7.14]. And A and C (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least (numbers) measure those numbers having the same ratio as them (the same number of times), the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures C, (as) the leading (measuring) the leading. And it also measures itself. Thus, A measures A and C, which are prime to one another. The very thing is impossible. Thus, it is not possible to find a fourth (number) proportional to A, B, C.

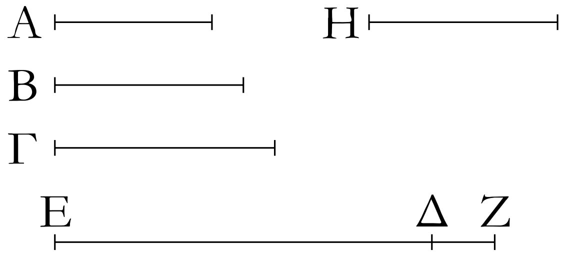
And so let A, B, C again be continuously proportional, and let A and C not be prime to one another. I say that it is possible to find a fourth (number) proportional to them. For let B make D (by) multiplying C. Thus, A either measures or does not measure D. Let it, first of all, measure (D) according to E. Thus, A has made D (by) multiplying E. But, in fact, B has also made D (by) multiplying C. Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C. Thus, proportionally, as A [is] to B, (so) C (is) to E [Prop. 7.19]. Thus, a fourth (number) proportional to A, B, C has been found, (namely) E.

And so let A not measure D. I say that it is impossible to find a fourth number proportional to A, B, C. For, if possible, let it be found, (and let it be) E. Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C. But, the (number created) from (multiplying) B, C is D. And thus the (number created) from (multiplying) A, E is equal to D. Thus, A has made D (by) multiplying E. Thus, A measures D according to E. Hence, A measures D. But, it also does not measure (D). The very thing (is) absurd. Thus, it is not possible to find a fourth number proportional to A, B, C when A does not measure D. And so (let) A, B, C (be) neither continuously proportional, nor (let) the outermost of them (be) prime to one another. And let B make D (by) multiplying C. So, similarly, it can be show that if A measures D then it is possible to find a fourth (number) proportional to (A, B, C), and impossible if (A) does not measure (D). (Which is) the very thing it was required to show.

† The proof of this proposition is incorrect. There are, in fact, only two cases. Either A, B, C are continuously proportional, with A and C prime to one another, or not. In the first case, it is impossible to find a fourth proportional number. In the second case, it is possible to find a fourth proportional number provided that A measures B times C. Of the four cases considered by Euclid, the proof given in the second case is incorrect, since it only demonstrates that if $A : B :: C : D$ then a number E cannot be found such that $B : C :: D : E$. The proofs given in the other three cases are correct.

κʹ.

Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν.



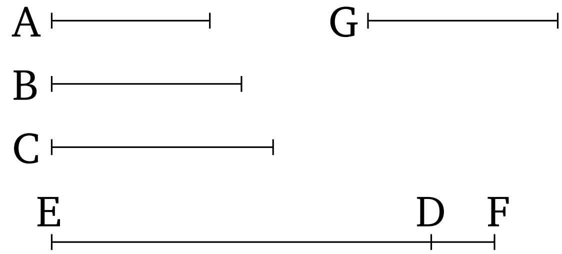
Ἔστωσαν οἱ προτεθέντες πρῶτοι ἀριθμοὶ οἱ Α, Β, Γ· λέγω, ὅτι τῶν Α, Β, Γ πλείους εἰσὶ πρῶτοι ἀριθμοί.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν Α, Β, Γ ἐλάχιστος μετρούμενος καὶ ἔστω ΔΕ, καὶ προσκείσθω τῷ ΔΕ μονὰς ἡ ΔΖ. ὁ δὴ ΕΖ ἤτοι πρῶτός ἐστιν ἢ οὔ. ἔστω πρότερον πρῶτος· εὑρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ οἱ Α, Β, Γ, ΕΖ πλείους τῶν Α, Β, Γ.

Ἀλλὰ δὴ μὴ ἔστω ὁ ΕΖ πρῶτος· ὑπὸ πρώτου ἄρα τινὸς ἀριθμοῦ μετρεῖται. μετρείσθω ὑπὸ πρώτου τοῦ Η· λέγω, ὅτι ὁ Η οὐδενὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. εἰ γὰρ δυνατόν, ἔστω. οἱ δὲ Α, Β, Γ τὸν ΔΕ μετροῦσιν· καὶ ὁ Η ἄρα τὸν ΔΕ μετρήσει. μετρεῖ δὲ καὶ τὸν ΕΖ· καὶ λοιπὴν τὴν ΔΖ μονάδα μετρήσει ὁ Η ἀριθμὸς ὤν· ὅπερ ἄτοπον. οὐκ ἄρα ὁ Η ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. καὶ ὑπόκειται πρῶτος. εὑρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ πλείους τοῦ προτεθέντος πλήθους τῶν Α, Β, Γ οἱ Α, Β, Γ, Η· ὅπερ ἔδει δεῖξαι.

Proposition 20

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.



Let *A*, *B*, *C* be the assigned prime numbers. I say that the (set of all) primes numbers is more numerous than *A*, *B*, *C*.

For let the least number measured by *A*, *B*, *C* be taken, and let it be *DE* [Prop. 7.36]. And let the unit *DF* be added to *DE*. So *EF* is either prime, or not. Let it, first of all, be prime. Thus, the (set of) prime numbers *A*, *B*, *C*, *EF*, (which is) more numerous than *A*, *B*, *C*, has been found.

And so let *EF* not be prime. Thus, it is measured by some prime number [Prop. 7.31]. Let it be measured by the prime (number) *G*. I say that *G* is not the same as any of *A*, *B*, *C*. For, if possible, let it be (the same). And *A*, *B*, *C* (all) measure *DE*. Thus, *G* will also measure *DE*. And it also measures *EF*. (So) *G* will also measure the remainder, unit *DF*, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, *G* is not the same as one of *A*, *B*, *C*. And it was assumed (to be) prime. Thus, the (set of) prime numbers *A*, *B*, *C*, *G*, (which is) more numerous than the assigned multitude (of prime numbers), *A*, *B*, *C*, has been found. (Which is) the very thing it was required to show.

καʹ.

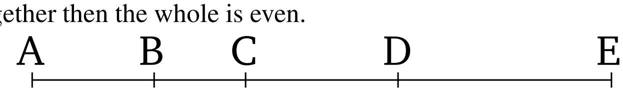
Ἐὰν ἄρτιοι ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, ὁ ὅλος ἄρτιός ἐστιν.

Συγκείσθωσαν γὰρ ἄρτιοι ἀριθμοὶ ὁποσοιοῦν οἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ· λέγω, ὅτι ὅλος ὁ ΑΕ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ἕκαστος τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ ἄρτιός ἐστιν, ἔχει μέρος ἥμισυ· ὥστε καὶ ὅλος ὁ ΑΕ ἔχει μέρος ἥμισυ. ἄρτιος δὲ ἀριθμός ἐστιν ὁ δίχα διαιρούμενος· ἄρτιος ἄρα ἐστὶν ὁ ΑΕ· ὅπερ ἔδει δεῖξαι.

Proposition 21

If any multitude whatsoever of even numbers is added together then the whole is even.



For let any multitude whatsoever of even numbers, *AB*, *BC*, *CD*, *DE*, lie together. I say that the whole, *AE*, is even.

For since everyone of *AB*, *BC*, *CD*, *DE* is even, it has a half part [Def. 7.6]. And hence the whole *AE* has a half part. And an even number is one (which can be) divided in half [Def. 7.6]. Thus, *AE* is even. (Which is) the very thing it was required to show.

κβʹ.

Ἐὰν περισσοὶ ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν ἄρτιον ᾖ, ὁ ὅλος ἄρτιος ἔσται.

Proposition 22

If any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.

Συγκείσθωσαν γὰρ περισσοὶ ἀριθμοὶ ὁσοιδηποτοῦν ἄρτιοι τὸ πλῆθος οἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ· λέγω, ὅτι ὅλος ὁ ΑΕ ἄρτιός ἐστιν.

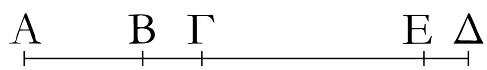
Ἐπεὶ γὰρ ἕκαστος τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ περιττός ἐστιν, ἀφαιρεθείσης μονάδος ἀφ᾽ ἑκάστου ἕκαστος τῶν λοιπῶν ἄρτιος ἔσται· ὥστε καὶ ὁ συγκείμενος ἐξ αὐτῶν ἄρτιος ἔσται. ἔστι δὲ καὶ τὸ πλῆθος τῶν μονάδων ἄρτιον. καὶ ὅλος ἄρα ὁ ΑΕ ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

For let any even multitude whatsoever of odd numbers, AB, BC, CD, DE, lie together. I say that the whole, AE, is even.

For since everyone of AB, BC, CD, DE is odd then, a unit being subtracted from each, everyone of the remainders will be (made) even [Def. 7.7]. And hence the sum of them will be even [Prop. 9.21]. And the multitude of the units is even. Thus, the whole AE is also even [Prop. 9.21]. (Which is) the very thing it was required to show.

κγʹ.

Ἐὰν περισσοὶ ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν περισσὸν ᾖ, καὶ ὁ ὅλος περισσὸς ἔσται.

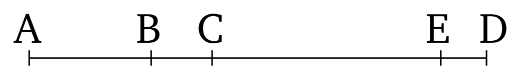


Συγκείσθωσαν γὰρ ὁποσοιοῦν περισσοὶ ἀριθμοί, ὧν τὸ πλῆθος περισσὸν ἔστω, οἱ ΑΒ, ΒΓ, ΓΔ· λέγω, ὅτι καὶ ὅλος ὁ ΑΔ περισσός ἐστιν.

Ἀφῃρήσθω ἀπὸ τοῦ ΓΔ μονὰς ἡ ΔΕ· λοιπὸς ἄρα ὁ ΓΕ ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΓΑ ἄρτιος· καὶ ὅλος ἄρα ὁ ΑΕ ἄρτιός ἐστιν. καί ἐστι μονὰς ἡ ΔΕ. περισσὸς ἄρα ἐστὶν ὁ ΑΔ· ὅπερ ἔδει δεῖξαι.

Proposition 23

If any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.

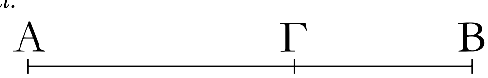


For let any multitude whatsoever of odd numbers, AB, BC, CD, lie together, and let the multitude of them be odd. I say that the whole, AD, is also odd.

For let the unit DE be subtracted from CD. The remainder, CE, is thus even [Def. 7.7]. And CA is also even [Prop. 9.22]. Thus, the whole, AE, is also even [Prop. 9.21]. And DE is a unit. Thus, AD is odd [Def. 7.7]. (Which is) the very thing it was required to show.

κδʹ.

Ἐὰν ἀπὸ ἀρτίου ἀριθμοῦ ἄρτιος ἀφαιρεθῇ, ὁ λοιπὸς ἄρτιος ἔσται.

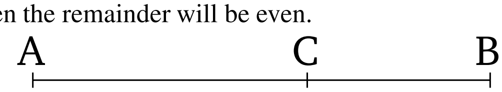


Ἀπὸ γὰρ ἀρτίου τοῦ ΑΒ ἄρτιος ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ ΑΒ ἄρτιός ἐστιν, ἔχει μέρος ἥμισυ. διὰ τὰ αὐτὰ δὴ καὶ ὁ ΒΓ ἔχει μέρος ἥμισυ· ὥστε καὶ λοιπὸς [ὁ ΓΑ ἔχει μέρος ἥμισυ] ἄρτιος [ἄρα] ἐστὶν ὁ ΑΓ· ὅπερ ἔδει δεῖξαι.

Proposition 24

If an even (number) is subtracted from an(other) even number then the remainder will be even.

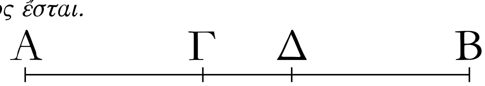


For let the even (number) BC be subtracted from the even number AB. I say that the remainder, CA, is even.

For since AB is even, it has a half part [Def. 7.6]. So, for the same (reasons), BC also has a half part. And hence the remainder [CA has a half part]. [Thus,] AC is even. (Which is) the very thing it was required to show.

κεʹ.

Ἐὰν ἀπὸ ἀρτίου ἀριθμοῦ περισσὸς ἀφαιρεθῇ, ὁ λοιπὸς περισσὸς ἔσται.

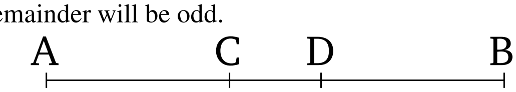


Ἀπὸ γὰρ ἀρτίου τοῦ ΑΒ περισσὸς ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσός ἐστιν.

Ἀφῃρήσθω γὰρ ἀπὸ τοῦ ΒΓ μονὰς ἡ ΓΔ· ὁ ΔΒ ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΑΒ ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΑΔ ἄρτιός ἐστιν. καί ἐστι μονὰς ἡ ΓΔ· ὁ ΓΑ ἄρα περισσός ἐστιν·

Proposition 25

If an odd (number) is subtracted from an even number then the remainder will be odd.



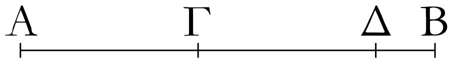
For let the odd (number) BC be subtracted from the even number AB. I say that the remainder CA is odd.

For let the unit CD be subtracted from BC. DB is thus even [Def. 7.7]. And AB is also even. And thus the remainder AD is even [Prop. 9.24]. And CD is a unit. Thus, CA is odd ὅπερ ἔδει δεῖξαι.

[Def. 7.7]. (Which is) the very thing it was required to show.

κϛ'.

Ἐὰν ἀπὸ περισσοῦ ἀριθμοῦ περισσὸς ἀφαιρεθῇ, ὁ λοιπὸς ἄρτιος ἔσται.

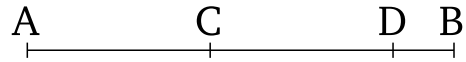


Ἀπὸ γὰρ περισσοῦ τοῦ ΑΒ περισσὸς ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ ΑΒ περισσός ἐστιν, ἀφῃρήσθω μονὰς ἡ ΒΔ· λοιπὸς ἄρα ὁ ΑΔ ἄρτιός ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ὁ ΓΔ ἄρτιός ἐστιν· ὥστε καὶ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

Proposition 26

If an odd (number) is subtracted from an odd number then the remainder will be even.



For let the odd (number) *BC* be subtracted from the odd (number) *AB*. I say that the remainder, *CA*, is even.

For since *AB* is odd, let the unit *BD* be subtracted (from it). Thus, the remainder *AD* is even [Def. 7.7]. So, for the same (reasons), *CD* is also even. And hence the remainder *CA* is even [Prop. 9.24]. (Which is) the very thing it was required to show.

κζ'.

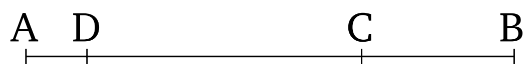
Ἐὰν ἀπὸ περισσοῦ ἀριθμοῦ ἄρτιος ἀφαιρεθῇ, ὁ λοιπὸς περισσὸς ἔσται.

Ἀπὸ γὰρ περισσοῦ τοῦ ΑΒ ἄρτιος ἀφῃρήσθω ὁ ΒΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσός ἐστιν.

Ἀφῃρήσθω [γὰρ] μονὰς ἡ ΑΔ· ὁ ΔΒ ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΒΓ ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΓΔ ἄρτιός ἐστιν. περισσὸς ἄρα ὁ ΓΑ· ὅπερ ἔδει δεῖξαι.

Proposition 27

If an even (number) is subtracted from an odd number then the remainder will be odd.

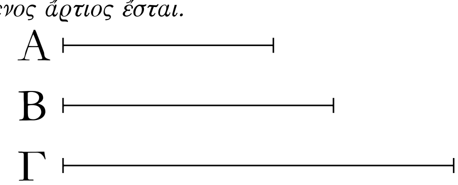


For let the even (number) *BC* be subtracted from the odd (number) *AB*. I say that the remainder, *CA*, is odd.

[For] let the unit *AD* be subtracted (from *AB*). *DB* is thus even [Def. 7.7]. And *BC* is also even. Thus, the remainder *CD* is also even [Prop. 9.24]. *CA* (is) thus odd [Def. 7.7]. (Which is) the very thing it was required to show.

κη'.

Ἐὰν περισσὸς ἀριθμὸς ἄρτιον πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος ἄρτιος ἔσται.

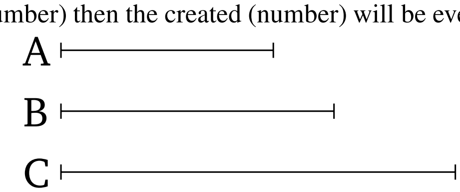


Περισσὸς γὰρ ἀριθμὸς ὁ Α ἄρτιον τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα σύγκειται ἐκ τοσούτων ἴσων τῷ Β, ὅσαι εἰσὶν ἐν τῷ Α μονάδες. καί ἐστιν ὁ Β ἄρτιος· ὁ Γ ἄρα σύγκειται ἐξ ἀρτίων. ἐὰν δὲ ἄρτιοι ἀριθμοὶ ὁποσοιοῦν συντεθῶσιν, ὁ ὅλος ἄρτιός ἐστιν. ἄρτιος ἄρα ἐστὶν ὁ Γ· ὅπερ ἔδει δεῖξαι.

Proposition 28

If an odd number makes some (number by) multiplying an even (number) then the created (number) will be even.



For let the odd number *A* make *C* (by) multiplying the even (number) *B*. I say that *C* is even.

For since *A* has made *C* (by) multiplying *B*, *C* is thus composed out of so many (magnitudes) equal to *B*, as many as (there) are units in *A* [Def. 7.15]. And *B* is even. Thus, *C* is composed out of even (numbers). And if any multitude whatsoever of even numbers is added together then the whole is even [Prop. 9.21]. Thus, *C* is even. (Which is) the very thing it was required to show.

κθ'.

Ἐὰν περισσὸς ἀριθμὸς περισσὸν ἀριθμὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος περισσὸς ἔσται.

Proposition 29

If an odd number makes some (number by) multiplying an odd (number) then the created (number) will be odd.

Περισσὸς γὰρ ἀριθμὸς ὁ Α περισσὸν τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ περισσός ἐστιν.

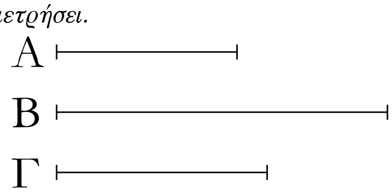
Ἐπεὶ γὰρ ὁ Α τὸν Β πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα σύγκειται ἐκ τοσούτων ἴσων τῷ Β, ὅσαι εἰσὶν ἐν τῷ Α μονάδες. καί ἐστιν ἑκάτερος τῶν Α, Β περισσός· ὁ Γ ἄρα σύγκειται ἐκ περισσῶν ἀριθμῶν, ὧν τὸ πλῆθος περισσόν ἐστιν. ὥστε ὁ Γ περισσός ἐστιν· ὅπερ ἔδει δεῖξαι.

For let the odd number A make C (by) multiplying the odd (number) B. I say that C is odd.

For since A has made C (by) multiplying B, C is thus composed out of so many (magnitudes) equal to B, as many as (there) are units in A [Def. 7.15]. And each of A, B is odd. Thus, C is composed out of odd (numbers), (and) the multitude of them is odd. Hence C is odd [Prop. 9.23]. (Which is) the very thing it was required to show.

λ'.

Ἐὰν περισσὸς ἀριθμὸς ἄρτιον ἀριθμὸν μετρῇ, καὶ τὸν ἥμισυν αὐτοῦ μετρήσει.

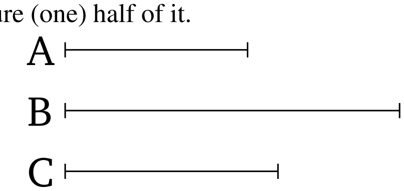


Περισσὸς γὰρ ἀριθμὸς ὁ Α ἄρτιον τὸν Β μετρείτω· λέγω, ὅτι καὶ τὸν ἥμισυν αὐτοῦ μετρήσει.

Ἐπεὶ γὰρ ὁ Α τὸν Β μετρεῖ, μετρείτω αὐτὸν κατὰ τὸν Γ· λέγω, ὅτι ὁ Γ οὐκ ἔστι περισσός. εἰ γὰρ δυνατόν, ἔστω. καὶ ἐπεὶ ὁ Α τὸν Β μετρεῖ κατὰ τὸν Γ, ὁ Α ἄρα τὸν Γ πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ Β ἄρα σύγκειται ἐκ περισσῶν ἀριθμῶν, ὧν τὸ πλῆθος περισσόν ἐστιν. ὁ Β ἄρα περισσός ἐστιν· ὅπερ ἄτοπον· ὑπόκειται γὰρ ἄρτιος. οὐκ ἄρα ὁ Γ περισσός ἐστιν· ἄρτιος ἄρα ἐστὶν ὁ Γ. ὥστε ὁ Α τὸν Β μετρεῖ ἀρτιάκις. διὰ δὴ τοῦτο καὶ τὸν ἥμισυν αὐτοῦ μετρήσει· ὅπερ ἔδει δεῖξαι.

Proposition 30

If an odd number measures an even number then it will also measure (one) half of it.

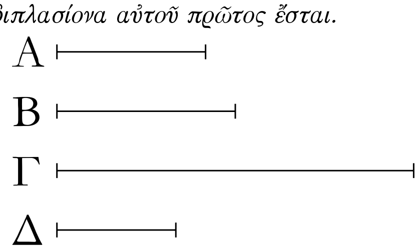


For let the odd number A measure the even (number) B. I say that (A) will also measure (one) half of (B).

For since A measures B, let it measure it according to C. I say that C is not odd. For, if possible, let it be (odd). And since A measures B according to C, A has thus made B (by) multiplying C. Thus, B is composed out of odd numbers, (and) the multitude of them is odd. B is thus odd [Prop. 9.23]. The very thing (is) absurd. For (B) was assumed (to be) even. Thus, C is not odd. Thus, C is even. Hence, A measures B an even number of times. So, on account of this, (A) will also measure (one) half of (B). (Which is) the very thing it was required to show.

λα'.

Ἐὰν περισσὸς ἀριθμὸς πρός τινα ἀριθμὸν πρῶτος ᾖ, καὶ πρὸς τὸν διπλασίονα αὐτοῦ πρῶτος ἔσται.

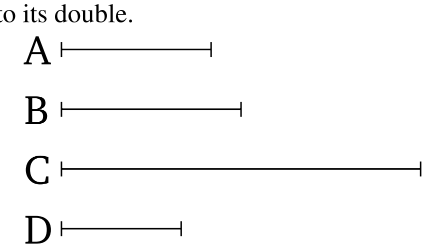


Περισσὸς γὰρ ἀριθμὸς ὁ Α πρός τινα ἀριθμὸν τὸν Β πρῶτος ἔστω, τοῦ δὲ Β διπλασίων ἔστω ὁ Γ· λέγω, ὅτι ὁ Α [καὶ] πρὸς τὸν Γ πρῶτός ἐστιν.

Εἰ γὰρ μή εἰσιν [οἱ Α, Γ] πρῶτοι, μετρήσει τις αὐτοὺς

Proposition 31

If an odd number is prime to some number then it will also be prime to its double.



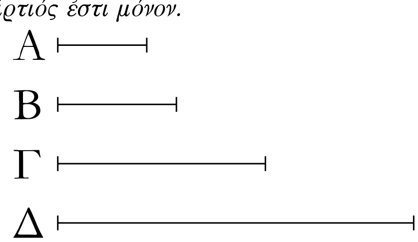
For let the odd number A be prime to some number B. And let C be double B. I say that A is [also] prime to C.

For if [A and C] are not prime (to one another) then some number will measure them. Let it measure (them), and let it be *ἀριθμός. μετρείτω, καὶ ἔστω ὁ Δ. καί ἐστιν ὁ Α περισσός· περισσὸς ἄρα καὶ ὁ Δ. καὶ ἐπεὶ ὁ Δ περισσὸς ὢν τὸν Γ μετρεῖ, καί ἐστιν ὁ Γ ἄρτιος, καὶ τὸν ἥμισυν ἄρα τοῦ Γ μετρήσει [ὁ Δ]. τοῦ δὲ Γ ἥμισύ ἐστιν ὁ Β· ὁ Δ ἄρα τὸν Β μετρεῖ. μετρεῖ δὲ καὶ τὸν Α. ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ Α πρὸς τὸν Γ πρῶτος οὔκ ἐστιν. οἱ Α, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.*

D. And *A* is odd. Thus, *D* (is) also odd. And since *D*, which is odd, measures *C*, and *C* is even, [*D*] will thus also measure half of *C* [Prop. 9.30]. And *B* is half of *C*. Thus, *D* measures *B*. And it also measures *A*. Thus, *D* measures (both) *A* and *B*, (despite) them being prime to one another. The very thing is impossible. Thus, *A* is not unprime to *C*. Thus, *A* and *C* are prime to one another. (Which is) the very thing it was required to show.

λβʹ.

Τῶν ἀπὸ δύαδος διπλασιαζομένων ἀριθμων ἕκαστος ἀρτιάκις ἄρτιός ἐστι μόνον.

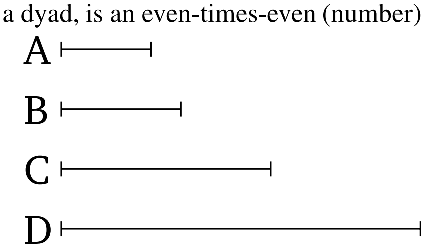


Ἀπὸ γὰρ δύαδος τῆς Α δεδιπλασιάσθωσαν ὁσοιδηποτοῦν ἀριθμοὶ οἱ Β, Γ, Δ· λέγω, ὅτι οἱ Β, Γ, Δ ἀρτιάκις ἄρτιοί εἰσι μόνον.

Ὅτι μὲν οὖν ἕκαστος [τῶν Β, Γ, Δ] ἀρτιάκις ἄρτιός ἐστιν, φανερόν· ἀπὸ γὰρ δυάδος ἐστὶ διπλασιασθείς. λέγω, ὅτι καὶ μόνον. ἐκκείσθω γὰρ μονάς. ἐπεὶ οὖν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἀνάλογόν εἰσιν, ὁ δὲ μετὰ τὴν μονάδα ὁ Α πρῶτός ἐστιν, ὁ μέγιστος τῶν Α, Β, Γ, Δ ὁ Δ ὑπ᾽ οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ. καί ἐστιν ἕκαστος τῶν Α, Β, Γ ἄρτιος· ὁ Δ ἄρα ἀρτιάκις ἄρτιός ἐστι μόνον. ὁμοίως δὴ δείξομεν, ὅτι [καὶ] ἑκάτερος τῶν Β, Γ ἀρτιάκις ἄρτιός ἐστι μόνον· ὅπερ ἔδει δεῖξαι.

Proposition 32

Each of the numbers (which is continually) doubled, (starting) from a dyad, is an even-times-even (number) only.

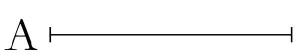


For let any multitude of numbers whatsoever, *B*, *C*, *D*, be (continually) doubled, (starting) from the dyad *A*. I say that *B*, *C*, *D* are even-times-even (numbers) only.

In fact, (it is) clear that each [of *B*, *C*, *D*] is an even-times-even (number). For it is doubled from a dyad [Def. 7.8]. I also say that (they are even-times-even numbers) only. For let a unit be laid down. Therefore, since any multitude of numbers whatsoever are continuously proportional, starting from a unit, and the (number) *A* after the unit is prime, the greatest of *A*, *B*, *C*, *D*, (namely) *D*, will not be measured by any other (numbers) except *A*, *B*, *C* [Prop. 9.13]. And each of *A*, *B*, *C* is even. Thus, *D* is an even-time-even (number) only [Def. 7.8]. So, similarly, we can show that each of *B*, *C* is [also] an even-time-even (number) only. (Which is) the very thing it was required to show.

λγʹ.

Ἐὰν ἀριθμὸς τὸν ἥμισυν ἔχῃ περισσόν, ἀρτιάκις περισσός ἐστι μόνον.

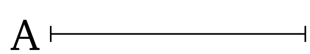


Ἀριθμὸς γὰρ ὁ Α τὸν ἥμισυν ἐχέτω περισσόν· λέγω, ὅτι ὁ Α ἀρτιάκις περισσός ἐστι μόνον.

Ὅτι μὲν οὖν ἀρτιάκις περισσός ἐστιν, φανερόν· ὁ γὰρ ἥμισυς αὐτοῦ περισσὸς ὢν μετρεῖ αὐτὸν ἀρτιάκις, λέγω δή, ὅτι καὶ μόνον. εἰ γὰρ ἔσται ὁ Α καὶ ἀρτιάκις ἄρτιος, μετρηθήσεται ὑπὸ ἀρτίου κατὰ ἄρτιον ἀριθμόν· ὥστε καὶ ὁ ἥμισυς αὐτοῦ μετρηθήσεται ὑπὸ ἀρτίου ἀριθμοῦ περισσὸς ὤν· ὅπερ ἐστὶν ἄτοπον. ὁ Α ἄρα ἀρτιάκις περισσός ἐστι μόνον· ὅπερ ἔδει δεῖξαι.

Proposition 33

If a number has an odd half then it is an even-time-odd (number) only.

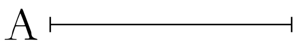


For let the number *A* have an odd half. I say that *A* is an even-times-odd (number) only.

In fact, (it is) clear that (*A*) is an even-times-odd (number). For its half, being odd, measures it an even number of times [Def. 7.9]. So I also say that (it is an even-times-odd number) only. For if *A* is also an even-times-even (number) then it will be measured by an even (number) according to an even number [Def. 7.8]. Hence, its half will also be measured by an even number, (despite) being odd. The very thing is absurd. Thus, *A* is an even-times-odd (number) only. (Which is) the very thing it was required to show.

λδ'.

Ἐὰν ἀριθμὸς μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ᾖ, μήτε τὸν ἥμισυν ἔχῃ περισσόν, ἀρτιάκις τε ἄρτιός ἐστι καὶ ἀρτιάκις περισσός.

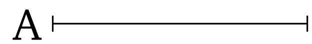


Ἀριθμὸς γὰρ ὁ Α μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ἔστω μήτε τὸν ἥμισυν ἐχέτω περισσόν· λέγω, ὅτι ὁ Α ἀρτιάκις τέ ἐστιν ἄρτιος καὶ ἀρτιάκις περισσός.

Ὅτι μὲν οὖν ὁ Α ἀρτιάκις ἐστὶν ἄρτιος, φανερόν· τὸν γὰρ ἥμισυν οὐκ ἔχει περισσόν. λέγω δή, ὅτι καὶ ἀρτιάκις περισσός ἐστιν. ἐὰν γὰρ τὸν Α τέμνωμεν δίχα καὶ τὸν ἥμισυν αὐτοῦ δίχα καὶ τοῦτο ἀεὶ ποιῶμεν, καταντήσομεν εἴς τινα ἀριθμὸν περισσόν, ὃς μετρήσει τὸν Α κατὰ ἄρτιον ἀριθμόν. εἰ γὰρ οὔ, καταντήσομεν εἰς δυάδα, καὶ ἔσται ὁ Α τῶν ἀπὸ δυάδος διπλασιαζομένων· ὅπερ οὐχ ὑπόκειται. ὥστε ὁ Α ἀρτιάκις περισσόν ἐστιν. ἐδείχθη δὲ καὶ ἀρτιάκις ἄρτιος. ὁ Α ἄρα ἀρτιάκις τε ἄρτιός ἐστι καὶ ἀρτιάκις περισσός· ὅπερ ἔδει δεῖξαι.

Proposition 34

If a number is neither (one) of the (numbers) doubled from a dyad, nor has an odd half, then it is (both) an even-times-even and an even-times-odd (number).

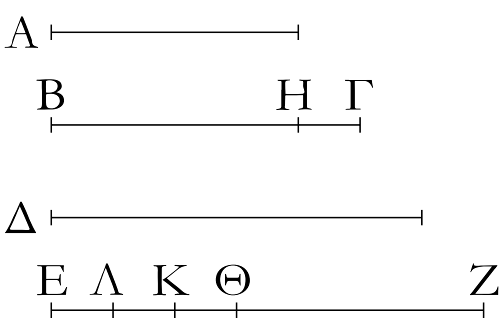


For let the number A neither be (one) of the (numbers) doubled from a dyad, nor let it have an odd half. I say that A is (both) an even-times-even and an even-times-odd (number).

In fact, (it is) clear that A is an even-times-even (number) [Def. 7.8]. For it does not have an odd half. So I say that it is also an even-times-odd (number). For if we cut A in half, and (then cut) its half in half, and we do this continually, then we will arrive at some odd number which will measure A according to an even number. For if not, we will arrive at a dyad, and A will be (one) of the (numbers) doubled from a dyad. The very opposite thing (was) assumed. Hence, A is an even-times-odd (number) [Def. 7.9]. And it was also shown (to be) an even-times-even (number). Thus, A is (both) an even-times-even and an even-times-odd (number). (Which is) the very thing it was required to show.

λε'.

Ἐὰν ὦσιν ὁσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον, ἀφαιρεθῶσι δὲ ἀπό τε τοῦ δευτέρου καὶ τοῦ ἐσχάτου ἴσοι τῷ πρώτῳ, ἔσται ὡς ἡ τοῦ δευτέρου ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας.

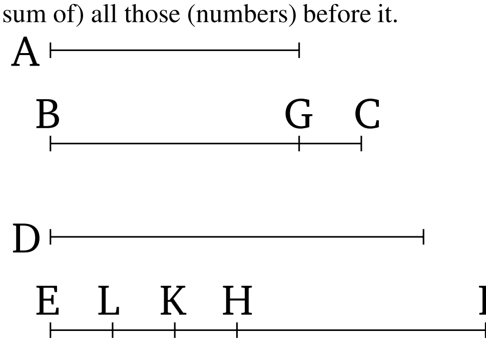


Ἔστωσαν ὁποσοιδηποτοῦν ἀριθμοὶ ἑξῆς ἀνάλογον οἱ Α, ΒΓ, Δ, ΕΖ ἀρχόμενοι ἀπὸ ἐλαχίστου τοῦ Α, καὶ ἀφῃρήσθω ἀπὸ τοῦ ΒΓ καὶ τοῦ ΕΖ τῷ Α ἴσος ἑκάτερος τῶν ΒΗ, ΖΘ· λέγω, ὅτι ἐστὶν ὡς ὁ ΗΓ πρὸς τὸν Α, οὕτως ὁ ΕΘ πρὸς τοὺς Α, ΒΓ, Δ.

Κείσθω γὰρ τῷ μὲν ΒΓ ἴσος ὁ ΖΚ, τῷ δὲ Δ ἴσος ὁ ΖΛ. καὶ ἐπεὶ ὁ ΖΚ τῷ ΒΓ ἴσος ἐστίν, ὧν ὁ ΖΘ τῷ ΒΗ ἴσος ἐστίν, λοιπὸς ἄρα ὁ ΘΚ λοιπῷ τῷ ΗΓ ἐστιν ἴσος. καὶ ἐπεί ἐστιν ὡς ὁ ΕΖ πρὸς τὸν Δ, οὕτως ὁ Δ πρὸς τὸν ΒΓ καὶ ὁ ΒΓ πρὸς τὸν Α, ἴσος δὲ ὁ μὲν Δ τῷ ΖΛ, ὁ δὲ ΒΓ τῷ ΖΚ, ὁ δὲ Α τῷ ΖΘ,

Proposition 35[†]

If there is any multitude whatsoever of continually proportional numbers, and (numbers) equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those (numbers) before it.



Let A, BC, D, EF be any multitude whatsoever of continuously proportional numbers, beginning from the least A. And let BG and FH, each equal to A, be subtracted from BC and EF (respectively). I say that as GC is to A, so EH is to A, BC, D.

For let FK be made equal to BC, and FL to D. And since FK is equal to BC, of which FH is equal to BG, the remainder, HK, is thus equal to the remainder, GC. And since as EF is to D, so D (is) to BC, and BC to A [Prop. 7.13], and D (is) equal to FL, and BC to FK, and A to FH, thus as EF is to FL, so *ἔστιν ἄρα ὡς ὁ ΕΖ πρὸς τὸν ΖΛ, οὕτως ὁ ΛΖ πρὸς τὸν ΖΚ καὶ ὁ ΖΚ πρὸς τὸν ΖΘ. διελόντι, ὡς ὁ ΕΛ πρὸς τὸν ΛΖ, οὕτως ὁ ΛΚ πρὸς τὸν ΖΚ καὶ ὁ ΚΘ πρὸς τὸν ΖΘ. ἔστιν ἄρα καὶ ὡς εἷς τῶν ἡγουμένων πρὸς ἕνα τῶν ἑπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἑπομένους· ἔστιν ἄρα ὡς ὁ ΚΘ πρὸς τὸν ΖΘ, οὕτως οἱ ΕΛ, ΛΚ, ΚΘ πρὸς τοὺς ΛΖ, ΖΚ, ΘΖ. ἴσος δὲ ὁ μὲν ΚΘ τῷ ΓΗ, ὁ δὲ ΖΘ τῷ Α, οἱ δὲ ΛΖ, ΖΚ, ΘΖ τοῖς Δ, ΒΓ, Α· ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν Α, οὕτως ὁ ΕΘ πρὸς τοὺς Δ, ΒΓ, Α. ἔστιν ἄρα ὡς ἡ τοῦ δευτέρου ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας· ὅπερ ἔδει δεῖξαι.*

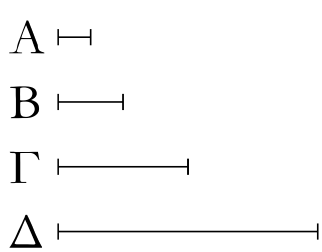
LF (is) to *FK*, and *FK* to *FH*. By separation, as *EL* (is) to *LF*, so *LK* (is) to *FK*, and *KH* to *FH* [Props. 7.11, 7.13]. And thus as one of the leading (numbers) is to one of the following, so (the sum of) all of the leading (numbers is) to (the sum of) all of the following [Prop. 7.12]. Thus, as *KH* is to *FH*, so *EL*, *LK*, *KH* (are) to *LF*, *FK*, *HF*. And *KH* (is) equal to *CG*, and *FH* to *A*, and *LF*, *FK*, *HF* to *D*, *BC*, *A*. Thus, as *CG* is to *A*, so *EH* (is) to *D*, *BC*, *A*. Thus, as the excess of the second (number) is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it. (Which is) the very thing it was required to show.

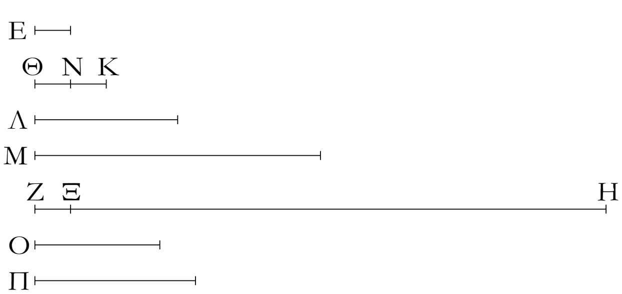
† This proposition allows us to sum a geometric series of the form a, $a\,r$, $a\,r^2$, $a\,r^3$, $\cdots a\,r^{n-1}$. According to Euclid, the sum, S_n, satisfies $(a\,r-a)/a = (a\,r^n-a)/S_n$. Hence, $S_n = a\,(r^n-1)/(r-1)$.

λϛ΄.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἑξῆς ἐκτεθῶσιν ἐν τῇ διπλασίονι ἀναλογίᾳ, ἕως οὗ ὁ σύμπας συντεθεὶς πρῶτος γένηται, καὶ ὁ σύμπας ἐπὶ τὸν ἔσχατον πολλαπλασιασθεὶς ποιῇ τινα, ὁ γενόμενος τέλειος ἔσται.

Ἀπὸ γὰρ μονάδος ἐκκείσθωσαν ὁσοιδηποτοῦν ἀριθμοὶ ἐν τῇ διπλασίονι ἀναλογίᾳ, ἕως οὗ ὁ σύμπας συντεθεὶς πρῶτος γένηται, οἱ Α, Β, Γ, Δ, καὶ τῷ σύμπαντι ἴσος ἔστω ὁ Ε, καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν ΖΗ ποιείτω. λέγω, ὅτι ὁ ΖΗ τέλειός ἐστιν.



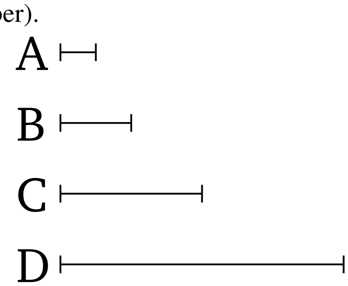


Ὅσοι γάρ εἰσιν οἱ Α, Β, Γ, Δ τῷ πλήθει, τοσοῦτοι ἀπὸ τοῦ Ε εἰλήφθωσαν ἐν τῇ διπλασίονι ἀναλογίᾳ οἱ Ε, ΘΚ, Λ, Μ· δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Μ. ὁ ἄρα ἐκ τῶν Ε, Δ ἴσος ἐστὶ τῷ ἐκ τῶν Α, Μ. καί ἐστιν ὁ ἐκ τῶν Ε, Δ ὁ ΖΗ· καὶ ὁ ἐκ τῶν Α, Μ ἄρα ἐστὶν ὁ

Proposition 36†

If any multitude whatsoever of numbers is set out continuously in a double proportion, (starting) from a unit, until the whole sum added together becomes prime, and the sum multiplied into the last (number) makes some (number), then the (number so) created will be perfect.

For let any multitude of numbers, *A*, *B*, *C*, *D*, be set out (continuouly) in a double proportion, until the whole sum added together is made prime. And let *E* be equal to the sum. And let *E* make *FG* (by) multiplying *D*. I say that *FG* is a perfect (number).



For as many as is the multitude of *A*, *B*, *C*, *D*, let so many (numbers), *E*, *HK*, *L*, *M*, be taken in a double proportion, (starting) from *E*. Thus, via equality, as *A* is to *D*, so *E* (is) to *M* [Prop. 7.14]. Thus, the (number created) from (multiplying) *E*, *D* is equal to the (number created) from (multiplying)

ΖΗ. ὁ Α ἄρα τὸν Μ πολλαπλασιάσας τὸν ΖΗ πεποίηκεν· ὁ Μ ἄρα τὸν ΖΗ μετρεῖ κατὰ τὰς ἐν τῷ Α μονάδας. καί ἐστι δυὰς ὁ Α· διπλάσιος ἄρα ἐστὶν ὁ ΖΗ τοῦ Μ. εἰσὶ δὲ καὶ οἱ Μ, Λ, ΘΚ, Ε ἑξῆς διπλάσιοι ἀλλήλων· οἱ Ε, ΘΚ, Λ, Μ, ΖΗ ἄρα ἑξῆς ἀνάλογόν εἰσιν ἐν τῇ διπλασίονι ἀναλογίᾳ. ἀφῃρήσθω δὴ ἀπὸ τοῦ δευτέρου τοῦ ΘΚ καὶ τοῦ ἐσχάτου τοῦ ΖΗ τῷ πρώτῳ τῷ Ε ἴσος ἑκάτερος τῶν ΘΝ, ΖΞ· ἔστιν ἄρα ὡς ἡ τοῦ δευτέρου ἀριθμοῦ ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας. ἔστιν ἄρα ὡς ὁ ΝΚ πρὸς τὸν Ε, οὕτως ὁ ΞΗ πρὸς τοὺς Μ, Λ, ΚΘ, Ε. καί ἐστιν ὁ ΝΚ ἴσος τῷ Ε· καὶ ὁ ΞΗ ἄρα ἴσος ἐστὶ τοῖς Μ, Λ, ΘΚ, Ε. ἔστι δὲ καὶ ὁ ΖΞ τῷ Ε ἴσος, ὁ δὲ Ε τοῖς Α, Β, Γ, Δ καὶ τῇ μονάδι. ὅλος ἄρα ὁ ΖΗ ἴσος ἐστὶ τοῖς τε Ε, ΘΚ, Λ, Μ καὶ τοῖς Α, Β, Γ, Δ καὶ τῇ μονάδι· καὶ μετρεῖται ὑπ᾽ αὐτῶν. λέγω, ὅτι καὶ ὁ ΖΗ ὑπ᾽ οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῆς μονάδος. εἰ γὰρ δυνατόν, μετρείτω τις τὸν ΖΗ ὁ Ο, καὶ ὁ Ο μηδενὶ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ ἔστω ὁ αὐτός. καὶ ὁσάκις ὁ Ο τὸν ΖΗ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Π· ὁ Π ἄρα τὸν Ο πολλαπλασιάσας τὸν ΖΗ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν ΖΗ πεποίηκεν· ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Π, ὁ Ο πρὸς τὸν Δ. καὶ ἐπεὶ ἀπὸ μονάδος ἑξῆς ἀνάλογόν εἰσιν οἱ Α, Β, Γ, Δ, ὁ Δ ἄρα ὑπ᾽ οὐδενὸς ἄλλου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Α, Β, Γ. καὶ ὑπόκειται ὁ Ο οὐδενὶ τῶν Α, Β, Γ ὁ αὐτός· οὐκ ἄρα μετρήσει ὁ Ο τὸν Δ. ἀλλ᾽ ὡς ὁ Ο πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Π· οὐδὲ ὁ Ε ἄρα τὸν Π μετρεῖ. καί ἐστιν ὁ Ε πρῶτος· πᾶς δὲ πρῶτος ἀριθμὸς πρὸς ἅπαντα, ὃν μὴ μετρεῖ, πρῶτός [ἐστιν]. οἱ Ε, Π ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἑπόμενος τὸν ἑπόμενον· καί ἐστιν ὡς ὁ Ε πρὸς τὸν Π, ὁ Ο πρὸς τὸν Δ· ἰσάκις ἄρα ὁ Ε τὸν Ο μετρεῖ καὶ ὁ Π τὸν Δ. ὁ δὲ Δ ὑπ᾽ οὐδενὸς ἄλλου μετρεῖται παρὲξ τῶν Α, Β, Γ· ὁ Π ἄρα ἑνὶ τῶν Α, Β, Γ ἐστιν ὁ αὐτός. ἔστω τῷ Β ὁ αὐτός. καὶ ὅσοι εἰσὶν οἱ Β, Γ, Δ τῷ πλήθει τοσοῦτοι εἰλήφθωσαν ἀπὸ τοῦ Ε οἱ Ε, ΘΚ, Λ. καί εἰσιν οἱ Ε, ΘΚ, Λ τοῖς Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ· δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Β πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Λ. ὁ ἄρα ἐκ τῶν Β, Λ ἴσος ἐστὶ τῷ ἐκ τῶν Δ, Ε· ἀλλ᾽ ὁ ἐκ τῶν Δ, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Π, Ο· καὶ ὁ ἐκ τῶν Π, Ο ἄρα ἴσος ἐστὶ τῷ ἐκ τῶν Β, Λ. ἔστιν ἄρα ὡς ὁ Π πρὸς τὸν Β, ὁ Λ πρὸς τὸν Ο. καί ἐστιν ὁ Π τῷ Β ὁ αὐτός· καὶ ὁ Λ ἄρα τῳ Ο ἐστιν ὁ αὐτός· ὅπερ ἀδύνατον· ὁ γὰρ Ο ὑπόκειται μηδενὶ τῶν ἐκκειμένων ὁ αὐτός· οὐκ ἄρα τὸν ΖΗ μετρήσει τις ἀριθμὸς παρὲξ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῆς μονάδος. καὶ ἐδείχη ὁ ΖΗ τοῖς Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῇ μονάδι ἴσος. τέλειος δὲ ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν· τέλειος ἄρα ἐστὶν ὁ ΖΗ· ὅπερ ἔδει δεῖξαι.

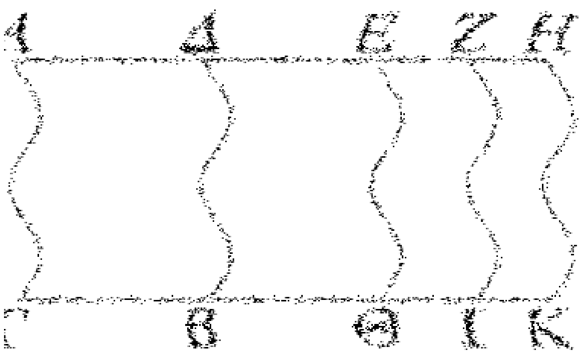
A, M. And FG is the (number created) from (multiplying) E, D. Thus, FG is also the (number created) from (multiplying) A, M [Prop. 7.19]. Thus, A has made FG (by) multiplying M. Thus, M measures FG according to the units in A. And A is a dyad. Thus, FG is double M. And M, L, HK, E are also continuously double one another. Thus, E, HK, L, M, FG are continuously proportional in a double proportion. So let HN and FO, each equal to the first (number) E, be subtracted from the second (number) HK and the last FG (respectively). Thus, as the excess of the second number is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it [Prop. 9.35]. Thus, as NK is to E, so OG (is) to M, L, KH, E. And NK is equal to E. And thus OG is equal to M, L, HK, E. And FO is also equal to E, and E to A, B, C, D, and a unit. Thus, the whole of FG is equal to E, HK, L, M, and A, B, C, D, and a unit. And it is measured by them. I also say that FG will be measured by no other (numbers) except A, B, C, D, E, HK, L, M, and a unit. For, if possible, let some (number) P measure FG, and let P not be the same as any of A, B, C, D, E, HK, L, M. And as many times as P measures FG, so many units let there be in Q. Thus, Q has made FG (by) multiplying P. But, in fact, E has also made FG (by) multiplying D. Thus, as E is to Q, so P (is) to D [Prop. 7.19]. And since A, B, C, D are continually proportional, (starting) from a unit, D will thus not be measured by any other numbers except A, B, C [Prop. 9.13]. And P was assumed not (to be) the same as any of A, B, C. Thus, P does not measure D. But, as P (is) to D, so E (is) to Q. Thus, E does not measure Q either [Def. 7.20]. And E is a prime (number). And every prime number [is] prime to every (number) which it does not measure [Prop. 7.29]. Thus, E and Q are prime to one another. And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. And as E is to Q, (so) P (is) to D. Thus, E measures P the same number of times as Q (measures) D. And D is not measured by any other (numbers) except A, B, C. Thus, Q is the same as one of A, B, C. Let it be the same as B. And as many as is the multitude of B, C, D, let so many (of the set out numbers) be taken, (starting) from E, (namely) E, HK, L. And E, HK, L are in the same ratio as B, C, D. Thus, via equality, as B (is) to D, (so) E (is) to L [Prop. 7.14]. Thus, the (number created) from (multiplying) B, L is equal to the (number created) from multiplying D, E [Prop. 7.19]. But, the (number created) from (multiplying) D, E is equal to the (number created) from (multiplying) Q, P. Thus, the (number created) from (multiplying) Q, P is equal to the (number created) from (multiplying) B, L. Thus, as Q is to B, (so) L (is) to P [Prop. 7.19]. And Q is the same as B. Thus, L is also the same as P. The very thing (is) impossible. For P

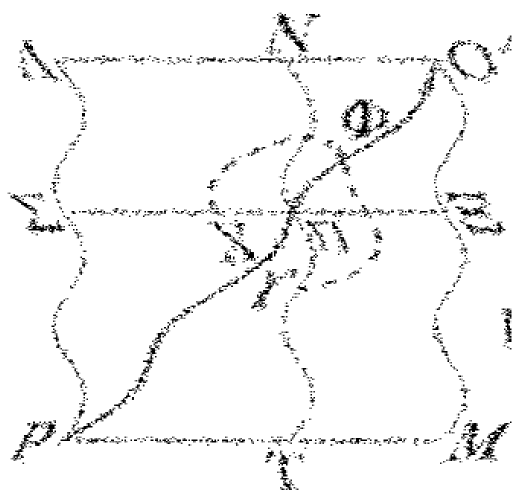
was assumed not (to be) the same as any of the (numbers) set out. Thus, FG cannot be measured by any number except A, B, C, D, E, HK, L, M, and a unit. And FG was shown (to be) equal to (the sum of) A, B, C, D, E, HK, L, M, and a unit. And a perfect number is one which is equal to (the sum of) its own parts [Def. 7.22]. Thus, FG is a perfect (number). (Which is) the very thing it was required to show.

† This proposition demonstrates that perfect numbers take the form $2^{n-1}(2^n - 1)$ provided that $2^n - 1$ is a prime number. The ancient Greeks knew of four perfect numbers: 6, 28, 496, and 8128, which correspond to $n = 2, 3, 5$, and 7, respectively.

ELEMENTS BOOK 10

Incommensurable Magnitudes†





---

†The theory of incommensurable magntidues set out in this book is generally attributed to Theaetetus of Athens. In the footnotes throughout this book, k, k', etc. stand for distinct ratios of positive integers.

Ὅροι.

α΄. Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μετρῳ μετρούμενα, ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.

β΄. Εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ᾿ αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ᾿ αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον κοινὸν μέτρον γενέσθαι.

γ΄. Τούτων ὑποκειμένων δείκνυται, ὅτι τῇ προτεθείση εὐθείᾳ ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήκει μόνον, αἱ δὲ καὶ δυνάμει. καλείσθω οὖν ἡ μὲν προτεθεῖσα εὐθεῖα ῥητή, καὶ αἱ ταύτη σύμμετροι εἴτε μήκει καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτη ἀσύμμετροι ἄλογοι καλείσθωσαν.

δ΄. Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλογα καλείσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερά τινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

Definitions

1. Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable.†

2. (Two) straight-lines are commensurable in square‡ when the squares on them are measured by the same area, but (are) incommensurable (in square) when no area admits to be a common measure of the squares on them.§

3. These things being assumed, it is proved that there exist an infinite multitude of straight-lines commensurable and incommensurable with an assigned straight-line—those (incommensurable) in length only, and those also (commensurable or incommensurable) in square.¶ Therefore, let the assigned straight-line be called rational. And (let) the (straight-lines) commensurable with it, either in length and square, or in square only, (also be called) rational. But let the (straight-lines) incommensurable with it be called irrational.*

4. And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their square-roots$ (also be called) irrational—the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure).♭

† In other words, two magnitudes α and β are commensurable if $\alpha : \beta :: 1 : k$, and incommensurable otherwise.

‡ Literally, "in power".

§ In other words, two straight-lines of length α and β are commensurable in square if $\alpha : \beta :: 1 : k^{1/2}$, and incommensurable in square otherwise. Likewise, the straight-lines are commensurable in length if $\alpha : \beta :: 1 : k$, and incommensurable in length otherwise.

¶ To be more exact, straight-lines can either be commensurable in square only, incommensurable in length only, or commenusrable/incommensurable in both length and square, with an assigned straight-line.

* Let the length of the assigned straight-line be unity. Then rational straight-lines have lengths expressible as k or $k^{1/2}$, depending on whether the lengths are commensurable in length, or in square only, respectively, with unity. All other straight-lines are irrational.

$ The square-root of an area is the length of the side of an equal area square.

♭ The area of the square on the assigned straight-line is unity. Rational areas are expressible as k. All other areas are irrational. Thus, squares whose sides are of rational length have rational areas, and vice versa.

α΄.

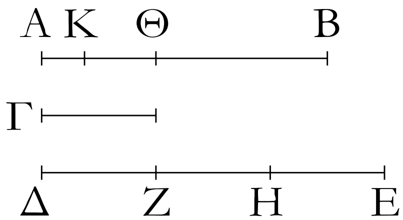
Δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους.

Ἔστω δύο μεγέθη ἄνισα τὰ ΑΒ, Γ, ὧν μεῖζον τὸ ΑΒ· λέγω, ὅτι, ἐὰν ἀπὸ τοῦ ΑΒ ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ Γ μεγέθους.

Proposition 1†

If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude.

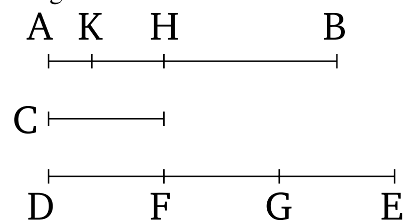
Let AB and C be two unequal magnitudes, of which (let) AB (be) the greater. I say that if (a part) greater than half is subtracted from AB, and (if a part) greater than half (is subtracted) from the remainder, and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the magnitude *C*.



Τὸ Γ γὰρ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ ΑΒ μεῖζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΔΕ τοῦ μὲν Γ πολλαπλάσιον, τοῦ δὲ ΑΒ μεῖζον, καὶ διῃρήσθω τὸ ΔΕ εἰς τὰ τῷ Γ ἴσα τὰ ΔΖ, ΖΗ, ΗΕ, καὶ ἀφῃρήσθω ἀπὸ μὲν τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, ἀπὸ δὲ τοῦ ΑΘ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, καὶ τοῦτο ἀεὶ γιγνέσθω, ἕως ἂν αἱ ἐν τῷ ΑΒ διαιρέσεις ἰσοπληθεῖς γένωνται ταῖς ἐν τῷ ΔΕ διαιρέσεσιν.

Ἔστωσαν οὖν αἱ ΑΚ, ΚΘ, ΘΒ διαιρέσεις ἰσοπληθεῖς οὖσαι ταῖς ΔΖ, ΖΗ, ΗΕ· καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΔΕ τοῦ ΑΒ, καὶ ἀφῄρηται ἀπὸ μὲν τοῦ ΔΕ ἔλασσον τοῦ ἡμίσεως τὸ ΕΗ, ἀπὸ δὲ τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, λοιπὸν ἄρα τὸ ΗΔ λοιποῦ τοῦ ΘΑ μεῖζόν ἐστιν. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΔ τοῦ ΘΑ, καὶ ἀφῄρηται τοῦ μὲν ΗΔ ἥμισυ τὸ ΗΖ, τοῦ δὲ ΘΑ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, λοιπὸν ἄρα τὸ ΔΖ λοιποῦ τοῦ ΑΚ μεῖζόν ἐστιν. ἴσον δὲ τὸ ΔΖ τῷ Γ· καὶ τὸ Γ ἄρα τοῦ ΑΚ μεῖζόν ἐστιν. ἔλασσον ἄρα τὸ ΑΚ τοῦ Γ.

Καταλείπεται ἄρα ἀπὸ τοῦ ΑΒ μεγέθους τὸ ΑΚ μέγεθος ἔλασσον ὂν τοῦ ἐκκειμένου ἐλάσσονος μεγέθους τοῦ Γ· ὅπερ ἔδει δεῖξαι.—ὁμοίως δὲ δειχθήσεται, κἂν ἡμίση ᾖ τὰ ἀφαιρούμενα.



For *C*, when multiplied (by some number), will sometimes be greater than *AB* [Def. 5.4]. Let it be (so) multiplied. And let *DE* be (both) a multiple of *C*, and greater than *AB*. And let *DE* be divided into the (divisions) *DF*, *FG*, *GE*, equal to *C*. And let *BH*, (which is) greater than half, be subtracted from *AB*. And (let) *HK*, (which is) greater than half, (be subtracted) from *AH*. And let this happen continually, until the divisions in *AB* become equal in number to the divisions in *DE*.

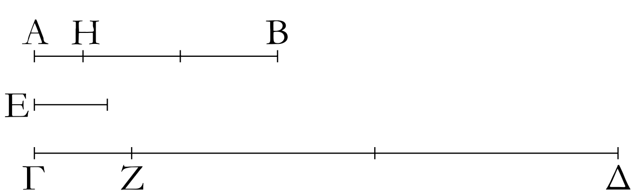
Therefore, let the divisions (in *AB*) be *AK*, *KH*, *HB*, being equal in number to *DF*, *FG*, *GE*. And since *DE* is greater than *AB*, and *EG*, (which is) less than half, has been subtracted from *DE*, and *BH*, (which is) greater than half, from *AB*, the remainder *GD* is thus greater than the remainder *HA*. And since *GD* is greater than *HA*, and the half *GF* has been subtracted from *GD*, and *HK*, (which is) greater than half, from *HA*, the remainder *DF* is thus greater than the remainder *AK*. And *DF* (is) equal to *C*. *C* is thus also greater than *AK*. Thus, *AK* (is) less than *C*.

Thus, the magnitude *AK*, which is less than the lesser laid out magnitude *C*, is left over from the magnitude *AB*. (Which is) the very thing it was required to show. — (The theorem) can similarly be proved even if the (parts) subtracted are halves.

† This theorem is the basis of the so-called method of exhaustion, and is generally attributed to Eudoxus of Cnidus.

β'.

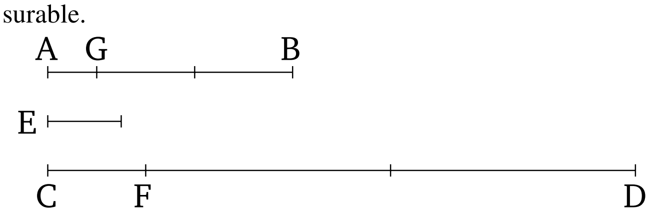
Ἐὰν δύο μεγεθῶν [ἐκκειμένων] ἀνίσων ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ καταλειπόμενον μηδέποτε καταμετρῇ τὸ πρὸ ἑαυτοῦ, ἀσύμμετρα ἔσται τὰ μεγέθη.



Δύο γὰρ μεγεθῶν ὄντων ἀνίσων τῶν ΑΒ, ΓΔ καὶ ἐλάσσονος τοῦ ΑΒ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ περιλειπόμενον μηδέποτε καταμετρείτω τὸ πρὸ ἑαυτοῦ· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ ΑΒ, ΓΔ μεγέθη.

Proposition 2

If the remainder of two unequal magnitudes (which are) [laid out] never measures the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater, then the (original) magnitudes will be incommensurable.



For, *AB* and *CD* being two unequal magnitudes, and *AB* (being) the lesser, let the remainder never measure the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater. I say that the magnitudes

Εἰ γὰρ ἐστι σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, εἰ δυνατόν, καὶ ἔστω τὸ Ε· καὶ τὸ μὲν ΑΒ τὸ ΖΔ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΓΖ, τὸ δὲ ΓΖ τὸ ΒΗ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΑΗ, καὶ τοῦτο ἀεὶ γινέσθω, ἕως οὗ λειφθῇ τι μέγεθος, ὅ ἐστιν ἔλασσον τοῦ Ε. γεγονέτω, καὶ λελείφθω τὸ ΑΗ ἔλασσον τοῦ Ε. ἐπεὶ οὖν τὸ Ε τὸ ΑΒ μετρεῖ, ἀλλὰ τὸ ΑΒ τὸ ΔΖ μετρεῖ, καὶ τὸ Ε ἄρα τὸ ΖΔ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ· καὶ λοιπὸν ἄρα τὸ ΓΖ μετρήσει. ἀλλὰ τὸ ΓΖ τὸ ΒΗ μετρεῖ· καὶ τὸ Ε ἄρα τὸ ΒΗ μετρεῖ. μετρεῖ δὲ καὶ ὅλον τὸ ΑΒ· καὶ λοιπὸν ἄρα τὸ ΑΗ μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΑΒ, ΓΔ μεγέθη μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΓΔ μεγέθη.

Ἐὰν ἄρα δύο μεγεθῶν ἀνίσων, καὶ τὰ ἑξῆς.

AB and *CD* are incommensurable.

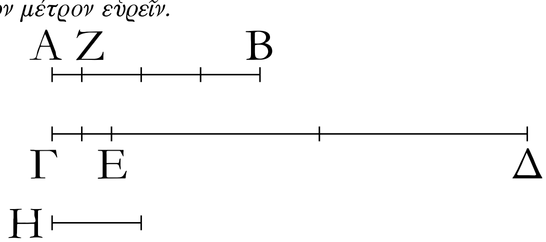
For if they are commensurable then some magnitude will measure them (both). If possible, let it (so) measure (them), and let it be *E*. And let *AB* leave *CF* less than itself (in) measuring *FD*, and let *CF* leave *AG* less than itself (in) measuring *BG*, and let this happen continually, until some magnitude which is less than *E* is left. Let (this) have occurred,† and let *AG*, (which is) less than *E*, be left. Therefore, since *E* measures *AB*, but *AB* measures *DF*, *E* will thus also measure *FD*. And it also measures the whole (of) *CD*. Thus, it will also measure the remainder *CF*. But, *CF* measures *BG*. Thus, *E* also measures *BG*. And it also measures the whole (of) *AB*. Thus, it will also measure the remainder *AG*, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude cannot measure (both) the magnitudes *AB* and *CD*. Thus, the magnitudes *AB* and *CD* are incommensurable [Def. 10.1].

Thus, if ... of two unequal magnitudes, and so on ....

† The fact that this will eventually occur is guaranteed by Prop. 10.1.

γ΄.

Δύο μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.



Ἔστω τὰ δοθέντα δύο μεγέθη σύμμετρα τὰ ΑΒ, ΓΔ, ὧν ἔλασσον τὸ ΑΒ· δεῖ δὴ τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

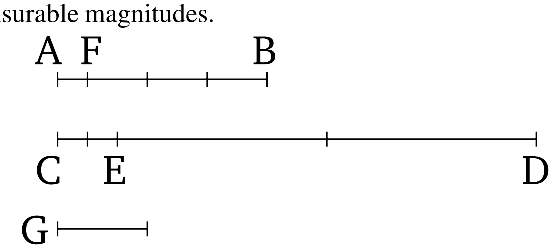
Τὸ ΑΒ γὰρ μέγεθος ἤτοι μετρεῖ τὸ ΓΔ ἢ οὔ. εἰ μὲν οὖν μετρεῖ, μετρεῖ δὲ καὶ ἑαυτό, τὸ ΑΒ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν· καὶ φανερόν, ὅτι καὶ μέγιστον. μεῖζον γὰρ τοῦ ΑΒ μεγέθους τὸ ΑΒ οὐ μετρήσει.

μὴ μετρείτω δὴ τὸ ΑΒ τὸ ΓΔ. καὶ ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, τὸ περιλειπόμενον μετρήσει ποτὲ τὸ πρὸ ἑαυτοῦ διὰ τὸ μὴ εἶναι ἀσύμμετρα τὰ ΑΒ, ΓΔ· καὶ τὸ μὲν ΑΒ τὸ ΕΔ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΕΓ, τὸ δὲ ΕΓ τὸ ΖΒ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΑΖ, τὸ δὲ ΑΖ τὸ ΓΕ μετρείτω.

Ἐπεὶ οὖν τὸ ΑΖ τὸ ΓΕ μετρεῖ, ἀλλὰ τὸ ΓΕ τὸ ΖΒ μετρεῖ, καὶ τὸ ΑΖ ἄρα τὸ ΖΒ μετρήσει. μετρεῖ δὲ καὶ ἑαυτό· καὶ ὅλον ἄρα τὸ ΑΒ μετρήσει τὸ ΑΖ. ἀλλὰ τὸ ΑΒ τὸ ΔΕ μετρεῖ· καὶ τὸ ΑΖ ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ τὸ ΓΕ· καί ὅλον ἄρα τὸ ΓΔ μετρεῖ· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ κοινὸν μέτρον ἐστίν. λέγω δή, ὅτι καὶ μέγιστον. εἰ γὰρ μή, ἔσται τι μέγεθος μεῖζον τοῦ ΑΖ, ὃ μετρήσει τὰ ΑΒ, ΓΔ. ἔστω τὸ Η. ἐπεὶ οὖν

Proposition 3

To find the greatest common measure of two given commensurable magnitudes.



Let *AB* and *CD* be the two given magnitudes, of which (let) *AB* (be) the lesser. So, it is required to find the greatest common measure of *AB* and *CD*.

For the magnitude *AB* either measures, or (does) not (measure), *CD*. Therefore, if it measures (*CD*), and (since) it also measures itself, *AB* is thus a common measure of *AB* and *CD*. And (it is) clear that (it is) also (the) greatest. For a (magnitude) greater than magnitude *AB* cannot measure *AB*.

So let *AB* not measure *CD*. And continually subtracting in turn the lesser (magnitude) from the greater, the remaining (magnitude) will (at) some time measure the (magnitude) before it, on account of *AB* and *CD* not being incommensurable [Prop. 10.2]. And let *AB* leave *EC* less than itself (in) measuring *ED*, and let *EC* leave *AF* less than itself (in) measuring *FB*, and let *AF* measure *CE*.

Therefore, since *AF* measures *CE*, but *CE* measures *FB*, *AF* will thus also measure *FB*. And it also measures itself. Thus, *AF* will also measure the whole (of) *AB*. But, *AB* measures *DE*. Thus, *AF* will also measure *ED*. And it also measures *CE*. Thus, it also measures the whole of *CD*. Thus, *AF* *τὸ Η τὸ ΑΒ μετρεῖ, ἀλλὰ τὸ ΑΒ τὸ ΕΔ μετρεῖ, καὶ τὸ Η ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ· καὶ λοιπὸν ἄρα τὸ ΓΕ μετρήσει τὸ Η. ἀλλὰ τὸ ΓΕ τὸ ΖΒ μετρεῖ· καὶ τὸ Η ἄρα τὸ ΖΒ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΑΒ, καὶ λοιπὸν τὸ ΑΖ μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μεῖζόν τι μέγεθος τοῦ ΑΖ τὰ ΑΒ, ΓΔ μετρήσει· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον ἐστίν.*

Δύο ἄρα μεγεθῶν συμμέτρων δοθέντων τῶν ΑΒ, ΓΔ τὸ μέγιστον κοινὸν μέτρον ηὕρηται· ὅπερ ἔδει δεῖξαι.

is a common measure of *AB* and *CD*. So I say that (it is) also (the) greatest (common measure). For, if not, there will be some magnitude, greater than *AF*, which will measure (both) *AB* and *CD*. Let it be *G*. Therefore, since *G* measures *AB*, but *AB* measures *ED*, *G* will thus also measure *ED*. And it also measures the whole of *CD*. Thus, *G* will also measure the remainder *CE*. But *CE* measures *FB*. Thus, *G* will also measure *FB*. And it also measures the whole (of) *AB*. And (so) it will measure the remainder *AF*, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude greater than *AF* cannot measure (both) *AB* and *CD*. Thus, *AF* is the greatest common measure of *AB* and *CD*.

Thus, the greatest common measure of two given commensurable magnitudes, *AB* and *CD*, has been found. (Which is) the very thing it was required to show.

Πόρισμα.

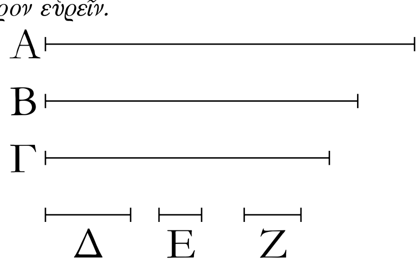
Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν μέγεθος δύο μεγέθη μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

Corollary

So (it is) clear, from this, that if a magnitude measures two magnitudes then it will also measure their greatest common measure.

δ΄.

Τριῶν μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.



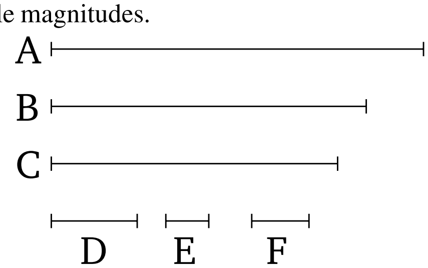
Ἔστω τὰ δοθέντα τρία μεγέθη σύμμετρα τὰ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰλήφθω γὰρ δύο τῶν Α, Β τὸ μέγιστον κοινὸν μέτρον, καὶ ἔστω τὸ Δ· τὸ δὴ Δ τὸ Γ ἤτοι μετρεῖ ἢ οὔ [μετρεῖ]. μετρείτω πρότερον. ἐπεὶ οὖν τὸ Δ τὸ Γ μετρεῖ, μετρεῖ δὲ καὶ τὰ Α, Β, τὸ Δ ἄρα τὰ Α, Β, Γ μετρεῖ· τὸ Δ ἄρα τῶν Α, Β, Γ κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· μεῖζον γὰρ τοῦ Δ μεγέθους τὰ Α, Β οὐ μετρεῖ.

μὴ μετρείτω δὴ τὸ Δ τὸ Γ. λέγω πρῶτον, ὅτι σύμμετρά ἐστι τὰ Γ, Δ. ἐπεὶ γὰρ σύμμετρά ἐστι τὰ Α, Β, Γ, μετρήσει τι αὐτὰ μέγεθος, ὃ δηλαδὴ καὶ τὰ Α, Β μετρήσει· ὥστε καὶ τὸ τῶν Α, Β μέγιστον κοινὸν μέτρον τὸ Δ μετρήσει. μετρεῖ δὲ καὶ τὸ Γ· ὥστε τὸ εἰρημένον μέγεθος μετρήσει τὰ Γ, Δ· σύμμετρα ἄρα ἐστὶ τὰ Γ, Δ. εἰλήφθω οὖν αὐτῶν τὸ μέγιστον κοινὸν μέτρον, καὶ ἔστω τὸ Ε. ἐπεὶ οὖν τὸ Ε τὸ Δ μετρεῖ, ἀλλὰ τὸ Δ τὰ Α, Β μετρεῖ, καὶ τὸ Ε ἄρα τὰ Α, Β μετρήσει. μετρεῖ δὲ καὶ τὸ Γ. τὸ Ε ἄρα τὰ Α, Β, Γ μετρεῖ· τὸ Ε ἄρα τῶν Α, Β, Γ κοινόν ἐστι μέτρον. λέγω δή, ὅτι καὶ μέγιστον. εἰ

Proposition 4

To find the greatest common measure of three given commensurable magnitudes.



Let *A*, *B*, *C* be the three given commensurable magnitudes. So it is required to find the greatest common measure of *A*, *B*, *C*.

For let the greatest common measure of the two (magnitudes) *A* and *B* be taken [Prop. 10.3], and let it be *D*. So *D* either measures, or [does] not [measure], *C*. Let it, first of all, measure (*C*). Therefore, since *D* measures *C*, and it also measures *A* and *B*, *D* thus measures *A*, *B*, *C*. Thus, *D* is a common measure of *A*, *B*, *C*. And (it is) clear that (it is) also (the) greatest (common measure). For no magnitude larger than *D* measures (both) *A* and *B*.

So let *D* not measure *C*. I say, first, that *C* and *D* are commensurable. For if *A*, *B*, *C* are commensurable then some magnitude will measure them which will clearly also measure *A* and *B*. Hence, it will also measure *D*, the greatest common measure of *A* and *B* [Prop. 10.3 corr.]. And it also measures *C*. Hence, the aforementioned magnitude will measure (both) *C* and *D*. Thus, *C* and *D* are commensurable [Def. 10.1]. There-

γὰρ δυνατόν, ἔστω τι τοῦ Ε μεῖζον μέγεθος τὸ Ζ, καὶ μετρείτω τὰ Α, Β, Γ. καὶ ἐπεὶ τὸ Ζ τὰ Α, Β, Γ μετρεῖ, καὶ τὰ Α, Β ἄρα μετρήσει καὶ τὸ τῶν Α, Β μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Α, Β μέγιστον κοινὸν μέτρον ἐστὶ τὸ Δ· τὸ Ζ ἄρα τὸ Δ μετρεῖ. μετρεῖ δὲ καὶ τὸ Γ· τὸ Ζ ἄρα τὰ Γ, Δ μετρεῖ· καὶ τὸ τῶν Γ, Δ ἄρα μέγιστον κοινὸν μέτρον μετρήσει τὸ Ζ. ἔστι δὲ τὸ Ε· τὸ Ζ ἄρα τὸ Ε μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μεῖζόν τι τοῦ Ε μεγέθους [μέγεθος] τὰ Α, Β, Γ μετρεῖ· τὸ Ε ἄρα τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον ἐστίν, ἐὰν μὴ μετρῇ τὸ Δ τὸ Γ, ἐὰν δὲ μετρῇ, αὐτὸ τὸ Δ.

Τριῶν ἄρα μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον κοινὸν μέτρον ηὕρηται [ὅπερ ἔδει δεῖξαι].

fore, let their greatest common measure be taken [Prop. 10.3], and let it be E. Therefore, since E measures D, but D measures (both) A and B, E will thus also measure A and B. And it also measures C. Thus, E measures A, B, C. Thus, E is a common measure of A, B, C. So I say that (it is) also (the) greatest (common measure). For, if possible, let F be some magnitude greater than E, and let it measure A, B, C. And since F measures A, B, C, it will thus also measure A and B, and will (thus) measure the greatest common measure of A and B [Prop. 10.3 corr.]. And D is the greatest common measure of A and B. Thus, F measures D. And it also measures C. Thus, F measures (both) C and D. Thus, F will also measure the greatest common measure of C and D [Prop. 10.3 corr.]. And it is E. Thus, F will measure E, the greater (measuring) the lesser. The very thing is impossible. Thus, some [magnitude] greater than the magnitude E cannot measure A, B, C. Thus, if D does not measure C then E is the greatest common measure of A, B, C. And if it does measure (C) then D itself (is the greatest common measure).

Thus, the greatest common measure of three given commensurable magnitudes has been found. [(Which is) the very thing it was required to show.]

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν μέγεθος τρία μεγέθη μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

Ὁμοίως δὴ καὶ ἐπὶ πλειόνων τὸ μέγιστον κοινὸν μέτρον ληφθήσεται, καὶ τὸ πόρισμα προχωρήσει. ὅπερ ἔδει δεῖξαι.

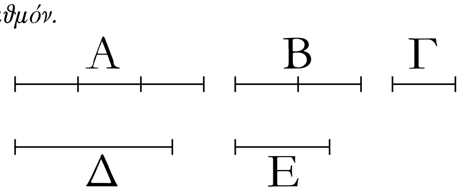
Corollary

So (it is) clear, from this, that if a magnitude measures three magnitudes then it will also measure their greatest common measure.

So, similarly, the greatest common measure of more (magnitudes) can also be taken, and the (above) corollary will go forward. (Which is) the very thing it was required to show.

ε΄.

Τὰ σύμμετρα μεγέθη πρὸς ἄλληλα λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.



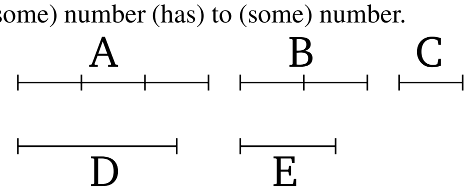
Ἔστω σύμμετρα μεγέθη τὰ Α, Β· λέγω, ὅτι τὸ Α πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ Α, Β, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Γ. καὶ ὁσάκις τὸ Γ τὸ Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ, ὁσάκις δὲ τὸ Γ τὸ Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.

Ἐπεὶ οὖν τὸ Γ τὸ Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν Δ κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ μονὰς τὸν Δ μετρεῖ ἀριθμὸν καὶ τὸ Γ μέγεθος τὸ Α· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως ἡ μονὰς πρὸς τὸν Δ· ἀνάπαλιν ἄρα, ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς

Proposition 5

Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.



Let A and B be commensurable magnitudes. I say that A has to B the ratio which (some) number (has) to (some) number.

For if A and B are commensurable (magnitudes) then some magnitude will measure them. Let it (so) measure (them), and let it be C. And as many times as C measures A, so many units let there be in D. And as many times as C measures B, so many units let there be in E.

Therefore, since C measures A according to the units in D, and a unit also measures D according to the units in it, a unit thus measures the number D as many times as the mag-

τὴν μονάδα. πάλιν ἐπεὶ τὸ Γ τὸ Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν Ε κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ μονὰς τὸν Ε μετρεῖ καὶ τὸ Γ τὸ Β· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Β, οὕτως ἡ μονὰς πρὸς τὸν Ε. ἐδείχθη δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, ὁ Δ πρὸς τὴν μονάδα· δι᾽ ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως ὁ Δ ἀριθμὸς πρὸς τὸν Ε.

Τὰ ἄρα σύμμετρα μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον ἔχει, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν Ε· ὅπερ ἔδει δεῖξαι.

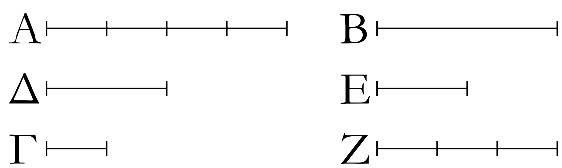
nitude C (measures) A. Thus, as C is to A, so a unit (is) to D [Def. 7.20].† Thus, inversely, as A (is) to C, so D (is) to a unit [Prop. 5.7 corr.]. Again, since C measures B according to the units in E, and a unit also measures E according to the units in it, a unit thus measures E the same number of times that C (measures) B. Thus, as C is to B, so a unit (is) to E [Def. 7.20]. And it was also shown that as A (is) to C, so D (is) to a unit. Thus, via equality, as A is to B, so the number D (is) to the (number) E [Prop. 5.22].

Thus, the commensurable magnitudes A and B have to one another the ratio which the number D (has) to the number E. (Which is) the very thing it was required to show.

† There is a slight logical gap here, since Def. 7.20 applies to four numbers, rather than two number and two magnitudes.

ϛ´.

Ἐὰν δύο μεγέθη πρὸς ἄλληλα λόγον ἔχῃ, ὃν ἀριθμὸς πρὸς ἀριθμόν, σύμμετρα ἔσται τὰ μεγέθη.



Δύο γὰρ μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον ἐχέτω, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν Ε· λέγω, ὅτι σύμμετρά ἐστι τὰ Α, Β μεγέθη.

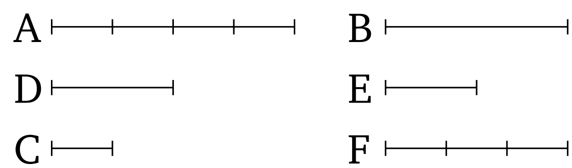
Ὅσαι γάρ εἰσιν ἐν τῷ Δ μονάδες, εἰς τοσαῦτα ἴσα διῃρήσθω τὸ Α, καὶ ἑνὶ αὐτῶν ἴσον ἔστω τὸ Γ· ὅσαι δέ εἰσιν ἐν τῷ Ε μονάδες, ἐκ τοσούτων μεγεθῶν ἴσων τῷ Γ συγκείσθω τὸ Ζ.

Ἐπεὶ οὖν, ὅσαι εἰσὶν ἐν τῷ Δ μονάδες, τοσαῦτά εἰσι καὶ ἐν τῷ Α μεγέθη ἴσα τῷ Γ, ὃ ἄρα μέρος ἐστὶν ἡ μονὰς τοῦ Δ, τὸ αὐτὸ μέρος ἐστὶ καὶ τὸ Γ τοῦ Α· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Α, οὕτως ἡ μονὰς πρὸς τὸν Δ. μετρεῖ δὲ ἡ μονὰς τὸν Δ ἀριθμόν· μετρεῖ ἄρα καὶ τὸ Γ τὸ Α. καὶ ἐπεί ἐστιν ὡς τὸ Γ πρὸς τὸ Α, οὕτως ἡ μονὰς πρὸς τὸν Δ [ἀριθμόν], ἀνάπαλιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ ἀριθμὸς πρὸς τὴν μονάδα. πάλιν ἐπεί, ὅσαι εἰσὶν ἐν τῷ Ε μονάδες, τοσαῦτά εἰσι καὶ ἐν τῷ Ζ ἴσα τῷ Γ, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ζ, οὕτως ἡ μονὰς πρὸς τὸν Ε [ἀριθμόν]. ἐδείχθη δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς τὴν μονάδα· δι᾽ ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Ζ, οὕτως ὁ Δ πρὸς τὸν Ε. ἀλλ᾽ ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ἐστὶ τὸ Α πρὸς τὸ Β· καὶ ὡς ἄρα τὸ Α πρὸς τὸ Β, οὕτως καὶ πρὸς τὸ Ζ. τὸ Α ἄρα πρὸς ἑκάτερον τῶν Β, Ζ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ Β τῷ Ζ. μετρεῖ δὲ τὸ Γ τὸ Ζ· μετρεῖ ἄρα καὶ τὸ Β. ἀλλὰ μὴν καὶ τὸ Α· τὸ Γ ἄρα τὰ Α, Β μετρεῖ. σύμμετρον ἄρα ἐστὶ τὸ Α τῷ Β.

Ἐὰν ἄρα δύο μεγέθη πρὸς ἄλληλα, καὶ τὰ ἑξῆς.

Proposition 6

If two magnitudes have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be commensurable.



For let the two magnitudes A and B have to one another the ratio which the number D (has) to the number E. I say that the magnitudes A and B are commensurable.

For, as many units as there are in D, let A be divided into so many equal (divisions). And let C be equal to one of them. And as many units as there are in E, let F be the sum of so many magnitudes equal to C.

Therefore, since as many units as there are in D, so many magnitudes equal to C are also in A, therefore whichever part a unit is of D, C is also the same part of A. Thus, as C is to A, so a unit (is) to D [Def. 7.20]. And a unit measures the number D. Thus, C also measures A. And since as C is to A, so a unit (is) to the [number] D, thus, inversely, as A (is) to C, so the number D (is) to a unit [Prop. 5.7 corr.]. Again, since as many units as there are in E, so many (magnitudes) equal to C are also in F, thus as C is to F, so a unit (is) to the [number] E [Def. 7.20]. And it was also shown that as A (is) to C, so D (is) to a unit. Thus, via equality, as A is to F, so D (is) to E [Prop. 5.22]. But, as D (is) to E, so A is to B. And thus as A (is) to B, so (it) also is to F [Prop. 5.11]. Thus, A has the same ratio to each of B and F. Thus, B is equal to F [Prop. 5.9]. And C measures F. Thus, it also measures B. But, in fact, (it) also (measures) A. Thus, C measures (both) A and B. Thus, A is commensurable with B [Def. 10.1].

Thus, if two magnitudes … to one another, and so on ….

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο ἀριθμοί, ὡς οἱ Δ, Ε, καὶ εὐθεῖα, ὡς ἡ Α, δύνατόν ἐστι ποιῆσαι ὡς ὁ Δ ἀριθμὸς πρὸς τὸν Ε ἀριθμόν, οὕτως τὴν εὐθεῖαν πρὸς εὐθεῖαν. ἐὰν δὲ καὶ τῶν Α, Ζ μέση ἀνάλογον ληφθῇ, ὡς ἡ Β, ἔσται ὡς ἡ Α πρὸς τὴν Ζ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β, τουτέστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ἀλλ᾽ ὡς ἡ Α πρὸς τὴν Ζ, οὕτως ἐστὶν ὁ Δ ἀριθμος πρὸς τὸν Ε ἀριθμόν· γέγονεν ἄρα καὶ ὡς ὁ Δ ἀριθμὸς πρὸς τὸν Ε ἀριθμόν, οὕτως τὸ ἀπὸ τῆς Α εὐθείας πρὸς τὸ ἀπὸ τῆς Β εὐθείας· ὅπερ ἔδει δεῖξαι.

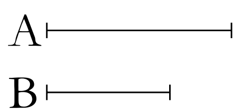
Corollary

So it is clear, from this, that if there are two numbers, like *D* and *E*, and a straight-line, like *A*, then it is possible to contrive that as the number *D* (is) to the number *E*, so the straight-line (is) to (another) straight-line (i.e., *F*). And if the mean proportion, (say) *B*, is taken of *A* and *F*, then as *A* is to *F*, so the (square) on *A* (will be) to the (square) on *B*. That is to say, as the first (is) to the third, so the (figure) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. But, as *A* (is) to *F*, so the number *D* is to the number *E*. Thus, it has also been contrived that as the number *D* (is) to the number *E*, so the (figure) on the straight-line *A* (is) to the (similar figure) on the straight-line *B*. (Which is) the very thing it was required to show.

ζ′.

Τὰ ἀσύμμετρα μεγέθη πρὸς ἄλληλα λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Ἔστω ἀσύμμετρα μεγέθη τὰ Α, Β· λέγω, ὅτι τὸ Α πρὸς τὸ Β λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.



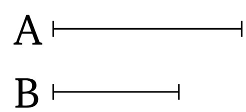
Εἰ γὰρ ἔχει τὸ Α πρὸς τὸ Β λόγον, ὃν ἀριθμὸς πρὸς ἀριθμόν, σύμμετρον ἔσται τὸ Α τῷ Β. οὐκ ἔστι δέ· οὐκ ἄρα τὸ Α πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν.

Τὰ ἄρα ἀσύμμετρα μεγέθη πρὸς ἄλληλα λόγον οὐκ ἔχει, καὶ τὰ ἑξῆς.

Proposition 7

Incommensurable magnitudes do not have to one another the ratio which (some) number (has) to (some) number.

Let *A* and *B* be incommensurable magnitudes. I say that *A* does not have to *B* the ratio which (some) number (has) to (some) number.

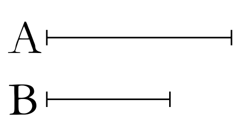


For if *A* has to *B* the ratio which (some) number (has) to (some) number then *A* will be commensurable with *B* [Prop. 10.6]. But it is not. Thus, *A* does not have to *B* the ratio which (some) number (has) to (some) number.

Thus, incommensurable numbers do not have to one another, and so on ....

η′.

Ἐὰν δύο μεγέθη πρὸς ἄλληλα λόγον μὴ ἔχῃ, ὃν ἀριθμὸς πρὸς ἀριθμόν, ἀσύμμετρα ἔσται τὰ μεγέθη.



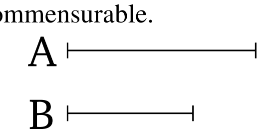
Δύο γὰρ μεγέθη τὰ Α, Β πρὸς ἄλληλα λόγον μὴ ἐχέτω, ὃν ἀριθμὸς πρὸς ἀριθμόν· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ Α, Β μεγέθη.

Εἰ γὰρ ἔσται σύμμετρα, τὸ Α πρὸς τὸ Β λόγον ἕξει, ὃν ἀριθμὸς πρὸς ἀριθμόν. οὐκ ἔχει δέ. ἀσύμμετρα ἄρα ἐστὶ τὰ Α, Β μεγέθη.

Ἐὰν ἄρα δύο μεγέθη πρὸς ἄλληλα, καὶ τὰ ἑξῆς.

Proposition 8

If two magnitudes do not have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be incommensurable.



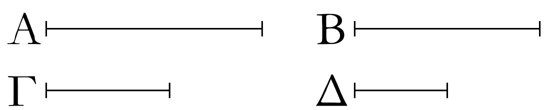
For let the two magnitudes *A* and *B* not have to one another the ratio which (some) number (has) to (some) number. I say that the magnitudes *A* and *B* are incommensurable.

For if they are commensurable, *A* will have to *B* the ratio which (some) number (has) to (some) number [Prop. 10.5]. But it does not have (such a ratio). Thus, the magnitudes *A* and *B* are incommensurable.

Thus, if two magnitudes ... to one another, and so on ....

ϑ′.

Τὰ ἀπὸ τῶν μήκει συμμέτρων εὐθειῶν τετράγωνα πρὸς ἄλληλα λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὰ τετράγωνα τὰ πρὸς ἄλληλα λόγον ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ τὰς πλευρὰς ἕξει μήκει συμμέτρους. τὰ δὲ ἀπὸ τῶν μήκει ἀσυμμέτρων εὐθειῶν τετράγωνα πρὸς ἄλληλα λόγον οὐκ ἔχει, ὅνπερ τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὰ τετράγωνα τὰ πρὸς ἄλληλα λόγον μὴ ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὰς πλευρὰς ἕξει μήκει συμμέτρους.



Ἔστωσαν γὰρ αἱ Α, Β μήκει σύμμετροι· λέγω, ὅτι τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β τετράγωνον λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ Α τῇ Β μήκει, ἡ Α ἄρα πρὸς τὴν Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Γ πρὸς τὸν Δ. ἐπεὶ οὖν ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ὁ Γ πρὸς τὸν Δ, ἀλλὰ τοῦ μὲν τῆς Α πρὸς τὴν Β λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς Α τετραγώνου πρὸς τὸ ἀπὸ τῆς Β τετράγωνον· τὰ γὰρ ὅμοια σχήματα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· τοῦ δὲ τοῦ Γ [ἀριθμοῦ] πρὸς τὸν Δ [ἀριθμὸν] λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τοῦ Γ τετραγώνου πρὸς τὸν ἀπὸ τοῦ Δ τετράγωνον· δύο γὰρ τετραγώνων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον [ἀριθμὸν] διπλασίονα λόγον ἔχει, ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν· ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β τετράγωνον, οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος [ἀριθμὸς] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμόν].

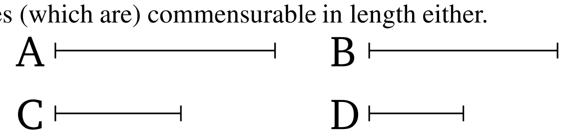
Ἀλλὰ δὴ ἔστω ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β, οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον]· λέγω, ὅτι σύμμετρός ἐστιν ἡ Α τῇ Β μήκει.

Ἐπεὶ γάρ ἐστιν ὡς τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον], οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον], ἀλλ᾽ ὁ μὲν τοῦ ἀπὸ τῆς Α τετραγώνου πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγος διπλασίων ἐστὶ τοῦ τῆς Α πρὸς τὴν Β λόγου, ὁ δὲ τοῦ ἀπὸ τοῦ Γ [ἀριθμοῦ] τετραγώνου [ἀριθμοῦ] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμὸν] λόγος διπλασίων ἐστὶ τοῦ τοῦ Γ [ἀριθμοῦ] πρὸς τὸν Δ [ἀριθμὸν] λόγου, ἔστιν ἄρα καὶ ὡς ἡ Α πρὸς τὴν Β, οὕτως ὁ Γ [ἀριθμὸς] πρὸς τὸν Δ [ἀριθμόν]. ἡ Α ἄρα πρὸς τὴν Β λόγον ἔχει, ὃν ἀριθμὸς ὁ Γ πρὸς ἀριθμὸν τὸν Δ· σύμμετρος ἄρα ἐστὶν ἡ Α τῇ Β μήκει.

Ἀλλὰ δὴ ἀσύμμετρος ἔστω ἡ Α τῇ Β μήκει· λέγω, ὅτι τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Proposition 9

Squares on straight-lines (which are) commensurable in length have to one another the ratio which (some) square number (has) to (some) square number. And squares having to one another the ratio which (some) square number (has) to (some) square number will also have sides (which are) commensurable in length. But squares on straight-lines (which are) incommensurable in length do not have to one another the ratio which (some) square number (has) to (some) square number. And squares not having to one another the ratio which (some) square number (has) to (some) square number will not have sides (which are) commensurable in length either.



For let A and B be (straight-lines which are) commensurable in length. I say that the square on A has to the square on B the ratio which (some) square number (has) to (some) square number.

For since A is commensurable in length with B, A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (that) which C (has) to D. Therefore, since as A is to B, so C (is) to D. But the (ratio) of the square on A to the square on B is the square of the ratio of A to B. For similar figures are in the squared ratio of (their) corresponding sides [Prop. 6.20 corr.]. And the (ratio) of the square on C to the square on D is the square of the ratio of the [number] C to the [number] D. For there exits one number in mean proportion to two square numbers, and (one) square (number) has to the (other) square [number] a squared ratio with respect to (that) the side (of the former has) to the side (of the latter) [Prop. 8.11]. And, thus, as the square on A is to the square on B, so the square [number] on the (number) C (is) to the square [number] on the [number] D.†

And so let the square on A be to the (square) on B as the square (number) on C (is) to the [square] (number) on D. I say that A is commensurable in length with B.

For since as the square on A is to the [square] on B, so the square (number) on C (is) to the [square] (number) on D. But, the ratio of the square on A to the (square) on B is the square of the (ratio) of A to B [Prop. 6.20 corr.]. And the (ratio) of the square [number] on the [number] C to the square [number] on the [number] D is the square of the ratio of the [number] C to the [number] D [Prop. 8.11]. Thus, as A is to B, so the [number] C also (is) to the [number] D. A, thus, has to B the ratio which the number C has to the number D. Thus, A is commensurable in length with B [Prop. 10.6].‡

And so let A be incommensurable in length with B. I say that the square on A does not have to the [square] on B the ratio which (some) square number (has) to (some) square number.

Εἰ γὰρ ἔχει τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, σύμμετρος ἔσται ἡ Α τῇ Β. οὐκ ἔστι δέ· οὐκ ἄρα τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν.

Πάλιν δὴ τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Β [τετράγωνον] λόγον μὴ ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· λέγω, ὅτι ἀσύμμετρός ἐστιν ἡ Α τῇ Β μήκει.

Εἰ γάρ ἐστι σύμμετρος ἡ Α τῇ Β, ἕξει τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. οὐκ ἔχει δέ· οὐκ ἄρα σύμμετρός ἐστιν ἡ Α τῇ Β μήκει.

Τὰ ἄρα ἀπὸ τῶν μήκει συμμέτρων, καὶ τὰ ἑξῆς.

For if the square on *A* has to the [square] on *B* the ratio which (some) square number (has) to (some) square number then *A* will be commensurable (in length) with *B*. But it is not. Thus, the square on *A* does not have to the [square] on the *B* the ratio which (some) square number (has) to (some) square number.

So, again, let the square on *A* not have to the [square] on *B* the ratio which (some) square number (has) to (some) square number. I say that *A* is incommensurable in length with *B*.

For if *A* is commensurable (in length) with *B* then the (square) on *A* will have to the (square) on *B* the ratio which (some) square number (has) to (some) square number. But it does not have (such a ratio). Thus, *A* is not commensurable in length with *B*.

Thus, (squares) on (straight-lines which are) commensurable in length, and so on ....

Πόρισμα.

Καὶ φανερὸν ἐκ τῶν δεδειγμένων ἔσται, ὅτι αἱ μήκει σύμμετροι πάντως καὶ δυνάμει, αἱ δὲ δυνάμει οὐ πάντως καὶ μήκει.

Corollary

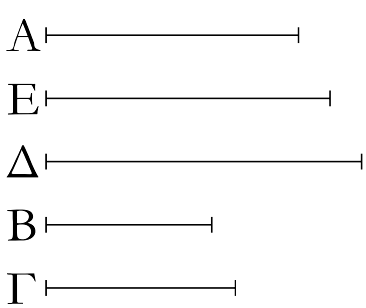
And it will be clear, from (what) has been demonstrated, that (straight-lines) commensurable in length (are) always also (commensurable) in square, but (straight-lines commensurable) in square (are) not always also (commensurable) in length.

† There is an unstated assumption here that if $\alpha : \beta :: \gamma : \delta$ then $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$.

‡ There is an unstated assumption here that if $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$ then $\alpha : \beta :: \gamma : \delta$.

ι΄.

Τῇ προτεθείσῃ εὐθείᾳ προσευρεῖν δύο εὐθείας ἀσυμμέτρους, τὴν μὲν μήκει μόνον, τὴν δὲ καὶ δυνάμει.

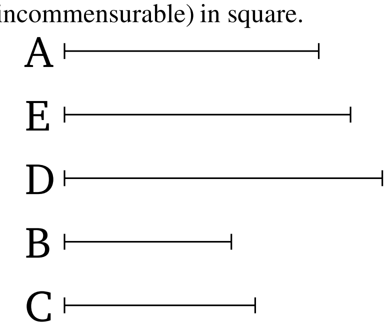


Ἔστω ἡ προτεθεῖσα εὐθεῖα ἡ Α· δεῖ δὴ τῇ Α προσευρεῖν δύο εὐθείας ἀσυμμέτρους, τὴν μὲν μήκει μόνον, τὴν δὲ καὶ δυνάμει.

Ἐκκείσθωσαν γὰρ δύο αριθμοὶ οἱ Β, Γ πρὸς ἀλλήλους λόγον μὴ ἔχοντες, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, τουτέστι μὴ ὅμοιοι ἐπίπεδοι, καὶ γεγονέτω ὡς ὁ Β πρὸς τὸν Γ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Δ τετράγωνον· ἐμάθομεν γάρ· σύμμετρον ἄρα τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς Δ. καὶ ἐπεὶ ὁ Β πρὸς τὸν Γ λόγον οὐκ ἔχει,

Proposition 10†

To find two straight-lines incommensurable with a given straight-line, the one (incommensurable) in length only, the other also (incommensurable) in square.



Let *A* be the given straight-line. So it is required to find two straight-lines incommensurable with *A*, the one (incommensurable) in length only, the other also (incommensurable) in square.

For let two numbers, *B* and *C*, not having to one another the ratio which (some) square number (has) to (some) square number—that is to say, not (being) similar plane (numbers)—be taken. And let it be contrived that as *B* (is) to *C*, so the square on *A* (is) to the square on *D*. For we learned (how to *ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Δ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ Δ μήκει. εἰλήφθω τῶν Α, Δ μέση ἀνάλογον ἡ Ε· ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Δ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς Ε. ἀσύμμετρος δέ ἐστιν ἡ Α τῇ Δ μήκει· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς Α τετράγωνον τῷ ἀπὸ τῆς Ε τετραγώνῳ· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ Ε δυνάμει.*

Τῇ ἄρα προτεθείσῃ εὐθείᾳ τῇ Α προσεύρηνται δύο εὐθεῖαι ἀσύμμετροι αἱ Δ, Ε, μήκει μὲν μόνον ἡ Δ, δυνάμει δὲ καὶ μήκει δηλαδὴ ἡ Ε [ὅπερ ἔδει δεῖξαι].

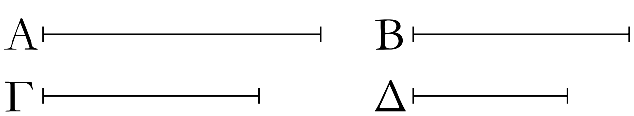
do this) [Prop. 10.6 corr.]. Thus, the (square) on *A* (is) commensurable with the (square) on *D* [Prop. 10.6]. And since *B* does not have to *C* the ratio which (some) square number (has) to (some) square number, the (square) on *A* thus does not have to the (square) on *D* the ratio which (some) square number (has) to (some) square number either. Thus, *A* is incommensurable in length with *D* [Prop. 10.9]. Let the (straight-line) *E* (which is) in mean proportion to *A* and *D* be taken [Prop. 6.13]. Thus, as *A* is to *D*, so the square on *A* (is) to the (square) on *E* [Def. 5.9]. And *A* is incommensurable in length with *D*. Thus, the square on *A* is also incommensurble with the square on *E* [Prop. 10.11]. Thus, *A* is incommensurable in square with *E*.

Thus, two straight-lines, *D* and *E*, (which are) incommensurable with the given straight-line *A*, be found, the one, *D*, (incommensurable) in length only, the other, *E*, (incommensurable) in square, and, clearly, also in length. [(Which is) the very thing it was required to show.]

† This whole proposition is regarded by Heiberg as an interpolation into the original text.

ια΄.

Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ δὲ πρῶτον τῷ δευτέρῳ σύμμετρον ᾖ, καὶ τὸ τρίτον τῷ τετάρτῳ σύμμετρον ἔσται· κἂν τὸ πρῶτον τῷ δευτέρῳ ἀσύμμετρον ᾖ, καὶ τὸ τρίτον τῷ τετάρτῳ ἀσύμμετρον ἔσται.



Ἔστωσαν τέσσαρα μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, τὸ Α δὲ τῷ Β σύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ Γ τῷ Δ σύμμετρον ἔσται.

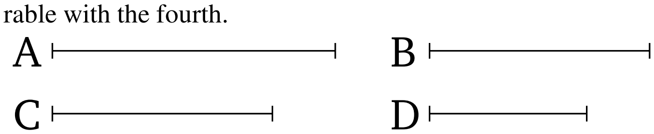
Ἐπεὶ γὰρ σύμμετρόν ἐστι τὸ Α τῷ Β, τὸ Α ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ τὸ Γ ἄρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· σύμμετρον ἄρα ἐστὶ τὸ Γ τῷ Δ.

Ἀλλὰ δὴ τὸ Α τῷ Β ἀσύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ Γ τῷ Δ ἀσύμμετρον ἔσται. ἐπεὶ γὰρ ἀσύμμετρόν ἐστι τὸ Α τῷ Β, τὸ Α ἄρα πρὸς τὸ Β λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καί ἐστιν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· οὐδὲ τὸ Γ ἄρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· ἀσύμμετρον ἄρα ἐστὶ τὸ Γ τῷ Δ.

Ἐὰν ἄρα τέσσαρα μεγέθη, καὶ τὰ ἑξῆς.

Proposition 11

If four magnitudes are proportional, and the first is commensurable with the second, then the third will also be commensurable with the fourth. And if the first is incommensurable with the second, then the third will also be incommensurable with the fourth.



Let *A*, *B*, *C*, *D* be four proportional magnitudes, (such that) as *A* (is) to *B*, so *C* (is) to *D*. And let *A* be commensurable with *B*. I say that *C* will also be commensurable with *D*.

For since *A* is commensurable with *B*, *A* thus has to *B* the ratio which (some) number (has) to (some) number [Prop. 10.5]. And as *A* is to *B*, so *C* (is) to *D*. Thus, *C* also has to *D* the ratio which (some) number (has) to (some) number. Thus, *C* is commensurable with *D* [Prop. 10.6].

And so let *A* be incommensurable with *B*. I say that *C* will also be incommensurable with *D*. For since *A* is incommensurable with *B*, *A* thus does not have to *B* the ratio which (some) number (has) to (some) number [Prop. 10.7]. And as *A* is to *B*, so *C* (is) to *D*. Thus, *C* does not have to *D* the ratio which (some) number (has) to (some) number either. Thus, *C* is incommensurable with *D* [Prop. 10.8].

Thus, if four magnitudes, and so on ....

ιβ΄.

Τὰ τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ἐστὶ σύμμετρα.

Ἑκάτερον γὰρ τῶν Α, Β τῷ Γ ἔστω σύμμετρον. λέγω, ὅτι καὶ τὸ Α τῷ Β ἐστι σύμμετρον.

Proposition 12

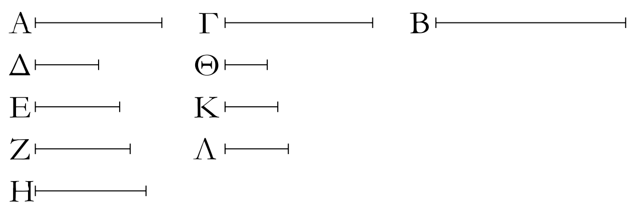
(Magnitudes) commensurable with the same magnitude are also commensurable with one another.

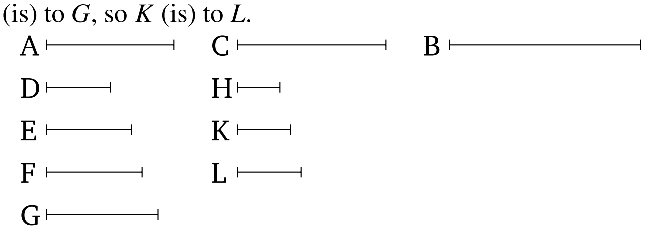
For let *A* and *B* each be commensurable with *C*. I say that

Ἐπεὶ γὰρ σύμμετρόν ἐστι τὸ Α τῷ Γ, τὸ Α ἄρα πρὸς τὸ Γ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Δ πρὸς τὸν Ε. πάλιν, ἐπεὶ σύμμετρόν ἐστι τὸ Γ τῷ Β, τὸ Γ ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Ζ πρὸς τὸν Η. καὶ λόγων δοθέντων ὁποσωνοῦν τοῦ τε, ὃν ἔχει ὁ Δ πρὸς τὸν Ε, καὶ ὁ Ζ πρὸς τὸν Η εἰλήφθωσαν ἀριθμοὶ ἑξῆς ἐν τοῖς δοθεῖσι λόγοις οἱ Θ, Κ, Λ· ὥστε εἶναι ὡς μὲν τὸν Δ πρὸς τὸν Ε, οὕτως τὸν Θ πρὸς τὸν Κ, ὡς δὲ τὸν Ζ πρὸς τὸν Η, οὕτως τὸν Κ πρὸς τὸν Λ.

A is also commensurable with *B*.

For since *A* is commensurable with *C*, *A* thus has to *C* the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which *D* (has) to *E*. Again, since *C* is commensurable with *B*, *C* thus has to *B* the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which *F* (has) to *G*. And for any multitude whatsoever of given ratios—(namely,) those which *D* has to *E*, and *F* to *G*—let the numbers *H*, *K*, *L* (which are) continuously (proportional) in the(se) given ratios be taken [Prop. 8.4]. Hence, as *D* is to *E*, so *H* (is) to *K*, and as *F* (is) to *G*, so *K* (is) to *L*.





Ἐπεὶ οὖν ἐστιν ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς τὸν Ε, ἀλλ' ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Θ πρὸς τὸν Κ, ἔστιν ἄρα καὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεί ἐστιν ὡς τὸ Γ πρὸς τὸ Β, οὕτως ὁ Ζ πρὸς τὸν Η, ἀλλ' ὡς ὁ Ζ πρὸς τὸν Η, [οὕτως] ὁ Κ πρὸς τὸν Λ, καὶ ὡς ἄρα τὸ Γ πρὸς τὸ Β, οὕτως ὁ Κ πρὸς τὸν Λ. ἔστι δὲ καὶ ὡς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ· δι' ἴσου ἄρα ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως ὁ Θ πρὸς τὸν Λ. τὸ Α ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς ὁ Θ πρὸς ἀριθμὸν τὸν Λ· σύμμετρον ἄρα ἐστὶ τὸ Α τῷ Β.

Τὰ ἄρα τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ἐστὶ σύμμετρα· ὅπερ ἔδει δεῖξαι.

Therefore, since as *A* is to *C*, so *D* (is) to *E*, but as *D* (is) to *E*, so *H* (is) to *K*, thus also as *A* is to *C*, so *H* (is) to *K* [Prop. 5.11]. Again, since as *C* is to *B*, so *F* (is) to *G*, but as *F* (is) to *G*, [so] *K* (is) to *L*, thus also as *C* (is) to *B*, so *K* (is) to *L* [Prop. 5.11]. And also as *A* is to *C*, so *H* (is) to *K*. Thus, via equality, as *A* is to *B*, so *H* (is) to *L* [Prop. 5.22]. Thus, *A* has to *B* the ratio which the number *H* (has) to the number *L*. Thus, *A* is commensurable with *B* [Prop. 10.6].

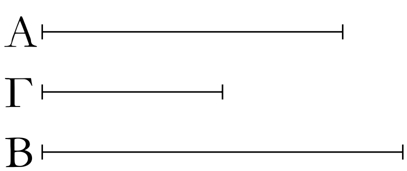
Thus, (magnitudes) commensurable with the same magnitude are also commensurable with one another. (Which is) the very thing it was required to show.

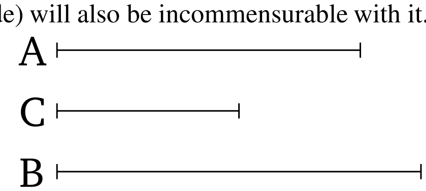
ιγ'.

Ἐὰν ᾖ δύο μεγέθη σύμμετρα, τὸ δὲ ἕτερον αὐτῶν μεγέθει τινὶ ἀσύμμετρον ᾖ, καὶ τὸ λοιπὸν τῷ αὐτῷ ἀσύμμετρον ἔσται.

Proposition 13

If two magnitudes are commensurable, and one of them is incommensurable with some magnitude, then the remaining (magnitude) will also be incommensurable with it.





Ἔστω δύο μεγέθη σύμμετρα τὰ Α, Β, τὸ δὲ ἕτερον αὐτῶν τὸ Α ἄλλῳ τινὶ τῷ Γ ἀσύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ λοιπὸν τὸ Β τῷ Γ ἀσύμμετρόν ἐστιν.

Εἰ γάρ ἐστι σύμμετρον τὸ Β τῷ Γ, ἀλλὰ καὶ τὸ Α τῷ Β σύμμετρόν ἐστιν, καὶ τὸ Α ἄρα τῷ Γ σύμμετρόν ἐστιν. ἀλλὰ καὶ ἀσύμμετρον· ὅπερ ἀδύνατον. οὐκ ἄρα σύμμετρόν ἐστι τὸ Β τῷ Γ· ἀσύμμετρον ἄρα.

Ἐὰν ἄρα ᾖ δύο μεγέθη σύμμετρα, καὶ τὰ ἑξῆς.

Let *A* and *B* be two commensurable magnitudes, and let one of them, *A*, be incommensurable with some other (magnitude), *C*. I say that the remaining (magnitude), *B*, is also incommensurable with *C*.

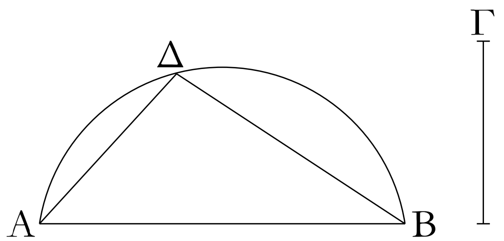
For if *B* is commensurable with *C*, but *A* is also commensurable with *B*, *A* is thus also commensurable with *C* [Prop. 10.12]. But, (it is) also incommensurable (with *C*). The very thing (is) impossible. Thus, *B* is not commensurable with *C*. Thus, (it is) incommensurable.

Thus, if two magnitudes are commensurable, and so on

....

Λῆμμα.

Δύο δοθεισῶν εὐθειῶν ἀνίσων εὑρεῖν, τίνι μεῖζον δύναται ἡ μείζων τῆς ἐλάσσονος.



Ἔστωσαν αἱ δοθεῖσαι δύο ἄνισοι εὐθεῖαι αἱ ΑΒ, Γ, ὧν μείζων ἔστω ἡ ΑΒ· δεῖ δὴ εὑρεῖν, τίνι μεῖζον δύναται ἡ ΑΒ τῆς Γ.

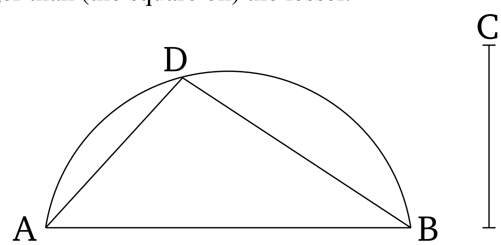
Γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ εἰς αὐτὸ ἐνηρμόσθω τῇ Γ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΒ. φανερὸν δή, ὅτι ὀρθή ἐστιν ἡ ὑπὸ ΑΔΒ γωνία, καὶ ὅτι ἡ ΑΒ τῆς ΑΔ, τουτέστι τῆς Γ, μεῖζον δύναται τῇ ΔΒ.

Ὁμοίως δὲ καὶ δύο δοθεισῶν εὐθειῶν ἡ δυναμένη αὐτὰς εὑρίσκεται οὕτως.

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΔ, ΔΒ, καὶ δέον ἔστω εὑρεῖν τὴν δυναμένην αὐτάς. κείσθωσαν γάρ, ὥστε ὀρθὴν γωνίαν περιέχειν τὴν ὑπὸ ΑΔ, ΔΒ, καὶ ἐπεζεύχθω ἡ ΑΒ· φανερὸν πάλιν, ὅτι ἡ τὰς ΑΔ, ΔΒ δυναμένη ἐστὶν ἡ ΑΒ· ὅπερ ἔδει δεῖξαι.

Lemma

For two given unequal straight-lines, to find by (the square on) which (straight-line) the square on the greater (straight-line is) larger than (the square on) the lesser.†



Let *AB* and *C* be the two given unequal straight-lines, and let *AB* be the greater of them. So it is required to find by (the square on) which (straight-line) the square on *AB* (is) greater than (the square on) *C*.

Let the semi-circle *ADB* be described on *AB*. And let *AD*, equal to *C*, be inserted into it [Prop. 4.1]. And let *DB* be joined. So (it is) clear that the angle *ADB* is a right-angle [Prop. 3.31], and that the square on *AB* (is) greater than (the square on) *AD*—that is to say, (the square on) *C*—by (the square on) *DB* [Prop. 1.47].

And, similarly, the square-root of (the sum of the squares on) two given straight-lines is also found likeso.

Let *AD* and *DB* be the two given straight-lines. And let it be necessary to find the square-root of (the sum of the squares on) them. For let them be laid down such as to encompass a right-angle—(namely), that (angle encompassed) by *AD* and *DB*. And let *AB* be joined. (It is) again clear that *AB* is the square-root of (the sum of the squares on) *AD* and *DB* [Prop. 1.47]. (Which is) the very thing it was required to show.

† That is, if α and β are the lengths of two given straight-lines, with α being greater than β, to find a straight-line of length γ such that $\alpha^2 = \beta^2 + \gamma^2$. Similarly, we can also find γ such that $\gamma^2 = \alpha^2 + \beta^2$.

ιδ΄.

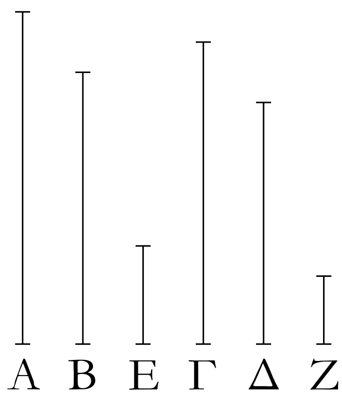
Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, δύνηται δὲ ἡ πρώτη τῆς δευτέρας μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῇ [μήκει], καὶ ἡ τρίτη τῆς τετάρτης μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ [μήκει]. καὶ ἐὰν ἡ πρώτη τῆς δευτέρας μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει], καὶ ἡ τρίτη τῆς τετάρτης μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει].

Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, Δ, ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, καὶ ἡ Α μὲν τῆς Β μεῖζον δυνάσθω τῷ ἀπὸ τῆς Ε, ἡ δὲ Γ τῆς Δ μεῖζον δυνάσθω τῷ ἀπὸ τῆς Ζ· λέγω, ὅτι, εἴτε σύμμετρός ἐστιν ἡ Α τῇ Ε, σύμμετρός ἐστι καὶ ἡ Γ τῇ Ζ, εἴτε ἀσύμμετρός ἐστιν ἡ Α τῇ Ε, ἀσύμμετρός ἐστι καὶ ὁ Γ τῇ Ζ.

Proposition 14

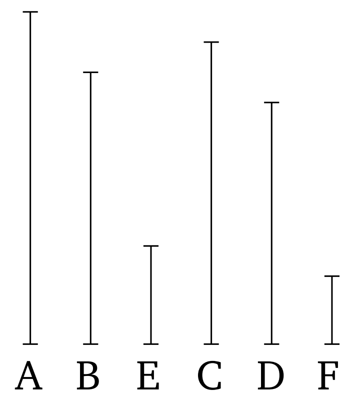
If four straight-lines are proportional, and the square on the first is greater than (the square on) the second by the (square) on (some straight-line) commensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) commensurable [in length] with the third. And if the square on the first is greater than (the square on) the second by the (square) on (some straight-line) incommensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) incommensurable [in length] with the third.

Let *A*, *B*, *C*, *D* be four proportional straight-lines, (such that) as A (is) to B, so C (is) to D. And let the square on A be greater than (the square on) B by the (square) on E, and let the square on C be greater than (the square on) D by the (square) on F. I say that A is either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F.



Ἐπεὶ γάρ ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ἀπὸ τῆς Δ. ἀλλὰ τῷ μὲν ἀπὸ τῆς Α ἴσα ἐστὶ τὰ ἀπὸ τῶν Ε, Β, τῷ δὲ ἀπὸ τῆς Γ ἴσα ἐστὶ τὰ ἀπὸ τῶν Δ, Ζ. ἔστιν ἄρα ὡς τὰ ἀπὸ τῶν Ε, Β πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὰ ἀπὸ τῶν Δ, Ζ πρὸς τὸ ἀπὸ τῆς Δ· διελόντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς Β, οὕτως τὸ ἀπὸ τῆς Ζ πρὸς τὸ ἀπὸ τῆς Δ· ἔστιν ἄρα καὶ ὡς ἡ Ε πρὸς τὴν Β, οὕτως ἡ Ζ πρὸς τὴν Δ· ἀνάπαλιν ἄρα ἐστὶν ὡς ἡ Β πρὸς τὴν Ε, οὕτως ἡ Δ πρὸς τὴν Ζ. ἔστι δὲ καὶ ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ· δι᾽ ἴσου ἄρα ἐστὶν ὡς ἡ Α πρὸς τὴν Ε, οὕτως ἡ Γ πρὸς τὴν Ζ. εἴτε οὖν σύμμετρός ἐστιν ἡ Α τῇ Ε, συμμετρός ἐστι καὶ ἡ Γ τῇ Ζ, εἴτε ἀσύμμετρός ἐστιν ἡ Α τῇ Ε, ἀσύμμετρός ἐστι καὶ ἡ Γ τῇ Ζ.

Ἐὰν ἄρα, καὶ τὰ ἑξῆς.



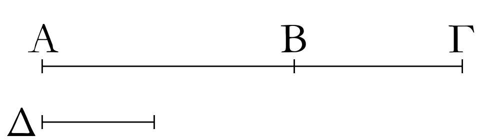
For since as A is to B, so C (is) to D, thus as the (square) on A is to the (square) on B, so the (square) on C (is) to the (square) on D [Prop. 6.22]. But the (sum of the squares) on E and B is equal to the (square) on A, and the (sum of the squares) on D and F is equal to the (square) on C. Thus, as the (sum of the squares) on E and B is to the (square) on B, so the (sum of the squares) on D and F (is) to the (square) on D. Thus, via separation, as the (square) on E is to the (square) on B, so the (square) on F (is) to the (square) on D [Prop. 5.17]. Thus, also, as E is to B, so F (is) to D [Prop. 6.22]. Thus, inversely, as B is to E, so D (is) to F [Prop. 5.7 corr.]. But, as A is to B, so C also (is) to D. Thus, via equality, as A is to E, so C (is) to F [Prop. 5.22]. Therefore, A is either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F [Prop. 10.11].

Thus, if, and so on ....

ιε΄.

Ἐὰν δύο μεγέθη σύμμετρα συντεθῇ, καὶ τὸ ὅλον ἑκατέρῳ αὐτῶν σύμμετρον ἔσται· κἂν τὸ ὅλον ἑνὶ αὐτῶν σύμμετρον ᾖ, καὶ τὰ ἐξ ἀρχῆς μεγέθη σύμμετρα ἔσται.

Συγκείσθω γὰρ δύο μεγέθη σύμμετρα τὰ ΑΒ, ΒΓ· λέγω, ὅτι καὶ ὅλον τὸ ΑΓ ἑκατέρῳ τῶν ΑΒ, ΒΓ ἐστι σύμμετρον.

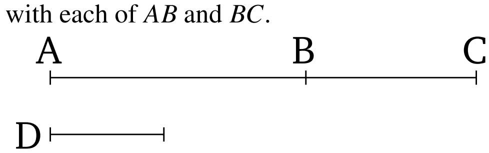


Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ ΑΒ, ΒΓ, μετρήσει τι αὐτὰ

Proposition 15

If two commensurable magnitudes are added together then the whole will also be commensurable with each of them. And if the whole is commensurable with one of them then the original magnitudes will also be commensurable (with one another).

For let the two commensurable magnitudes AB and BC be laid down together. I say that the whole AC is also commensurable with each of AB and BC.



For since AB and BC are commensurable, some magni-

μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΑΒ, ΒΓ μετρεῖ, καὶ ὅλον τὸ ΑΓ μετρήσει. μετρεῖ δὲ καὶ τὰ ΑΒ, ΒΓ. τὸ Δ ἄρα τὰ ΑΒ, ΒΓ, ΑΓ μετρεῖ· σύμμετρον ἄρα ἐστὶ τὸ ΑΓ ἑκατέρῳ τῶν ΑΒ, ΒΓ.

Ἀλλὰ δὴ τὸ ΑΓ ἔστω σύμμετρον τῷ ΑΒ· λέγω δή, ὅτι καὶ τὰ ΑΒ, ΒΓ σύμμετρά ἐστιν.

Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ ΑΓ, ΑΒ, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸ ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἄρα τὰ ΑΒ, ΒΓ μετρήσει· σύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ.

Ἐὰν ἄρα δύο μεγέθη, καὶ τὰ ἑξῆς.

tude will measure them. Let it (so) measure (them), and let it be *D*. Therefore, since *D* measures (both) *AB* and *BC*, it will also measure the whole *AC*. And it also measures *AB* and *BC*. Thus, *D* measures *AB*, *BC*, and *AC*. Thus, *AC* is commensurable with each of *AB* and *BC* [Def. 10.1].

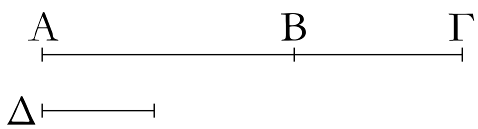
And so let *AC* be commensurable with *AB*. I say that *AB* and *BC* are also commensurable.

For since *AC* and *AB* are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be *D*. Therefore, since *D* measures (both) *CA* and *AB*, it will thus also measure the remainder *BC*. And it also measures *AB*. Thus, *D* will measure (both) *AB* and *BC*. Thus, *AB* and *BC* are commensurable [Def. 10.1].

Thus, if two magnitudes, and so on . . . .

ις′.

Ἐὰν δύο μεγέθη ἀσύμμετρα συντεθῇ, καὶ τὸ ὅλον ἑκατέρῳ αὐτῶν ἀσύμμετρον ἔσται· κἂν τὸ ὅλον ἑνὶ αὐτῶν ἀσύμμετρον ᾖ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται.



Συγκείσθω γὰρ δύο μεγέθη ἀσύμμετρα τὰ ΑΒ, ΒΓ· λέγω, ὅτι καὶ ὅλον τὸ ΑΓ ἑκατέρῳ τῶν ΑΒ, ΒΓ ἀσύμμετρόν ἐστιν.

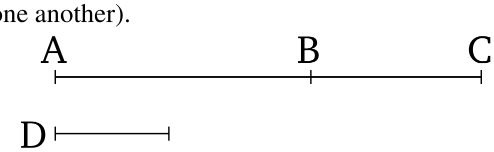
Εἰ γὰρ μή ἐστιν ἀσύμμετρα τὰ ΓΑ, ΑΒ, μετρήσει τι [αὐτὰ] μέγεθος. μετρείτω, εἰ δυνατόν, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΓΑ, ΑΒ μετρεῖ, καὶ λοιπὸν ἄρα τὸ ΒΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἄρα τὰ ΑΒ, ΒΓ μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ· ὑπέκειντο δὲ καὶ ἀσύμμετρα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΓΑ, ΑΒ μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΓΑ, ΑΒ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τὰ ΑΓ, ΓΒ ἀσύμμετρά ἐστιν. τὸ ΑΓ ἄρα ἑκατέρῳ τῶν ΑΒ, ΒΓ ἀσύμμετρόν ἐστιν.

Ἀλλὰ δὴ τὸ ΑΓ ἑνὶ τῶν ΑΒ, ΒΓ ἀσύμμετρον ἔστω. ἔστω δὴ πρότερον τῷ ΑΒ· λέγω, ὅτι καὶ τὰ ΑΒ, ΒΓ ἀσύμμετρά ἐστιν. εἰ γὰρ ἔσται σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρείτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ ΑΒ, ΒΓ μετρεῖ, καὶ ὅλον ἄρα τὸ ΑΓ μετρήσει. μετρεῖ δὲ καὶ τὸ ΑΒ· τὸ Δ ἄρα τὰ ΓΑ, ΑΒ μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ ΓΑ, ΑΒ· ὑπέκειτο δὲ καὶ ἀσύμμετρα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΑΒ, ΒΓ μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ.

Ἐὰν ἄρα δύο μεγέθη, καὶ τὰ ἑξῆς.

Proposition 16

If two incommensurable magnitudes are added together then the whole will also be incommensurable with each of them. And if the whole is incommensurable with one of them then the original magnitudes will also be incommensurable (with one another).



For let the two incommensurable magnitudes *AB* and *BC* be laid down together. I say that that the whole *AC* is also incommensurable with each of *AB* and *BC*.

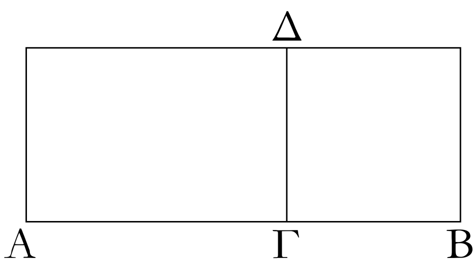
For if *CA* and *AB* are not incommensurable then some magnitude will measure [them]. If possible, let it (so) measure (them), and let it be *D*. Therefore, since *D* measures (both) *CA* and *AB*, it will thus also measure the remainder *BC*. And it also measures *AB*. Thus, *D* measures (both) *AB* and *BC*. Thus, *AB* and *BC* are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) *CA* and *AB*. Thus, *CA* and *AB* are incommensurable [Def. 10.1]. So, similarly, we can show that *AC* and *CB* are also incommensurable. Thus, *AC* is incommensurable with each of *AB* and *BC*.

And so let *AC* be incommensurable with one of *AB* and *BC*. So let it, first of all, be incommensurable with *AB*. I say that *AB* and *BC* are also incommensurable. For if they are commensurable then some magnitude will measure them. Let it (so) measure (them), and let it be *D*. Therefore, since *D* measures (both) *AB* and *BC*, it will thus also measure the whole *AC*. And it also measures *AB*. Thus, *D* measures (both) *CA* and *AB*. Thus, *CA* and *AB* are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) AB and BC. Thus, AB and BC are incommensurable [Def. 10.1].

Thus, if two... magnitudes, and so on ....

Λῆμμα.

Ἐὰν παρά τινα εὐθεῖαν παραβληθῇ παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ, τὸ παραβληθὲν ἴσον ἐστὶ τῷ ὑπὸ τῶν ἐκ τῆς παραβολῆς γενομένων τμημάτων τῆς εὐθείας.



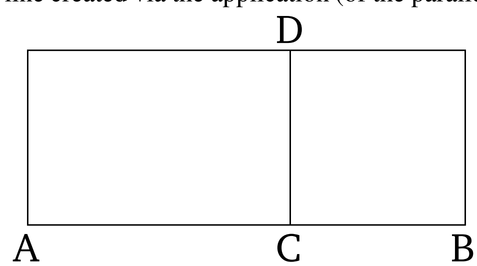
Παρὰ γὰρ εὐθεῖαν τὴν ΑΒ παραβεβλήσθω παραλληλόγραμμον τὸ ΑΔ ἐλλεῖπον εἴδει τετραγώνῳ τῷ ΔΒ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΔ τῷ ὑπὸ τῶν ΑΓ, ΓΒ.

Καί ἐστιν αὐτόθεν φανερόν· ἐπεὶ γὰρ τετράγωνόν ἐστι τὸ ΔΒ, ἴση ἐστὶν ἡ ΔΓ τῇ ΓΒ, καί ἐστι τὸ ΑΔ τὸ ὑπὸ τῶν ΑΓ, ΓΔ, τουτέστι τὸ ὑπὸ τῶν ΑΓ, ΓΒ.

Ἐὰν ἄρα παρά τινα εὐθεῖαν, καὶ τὰ ἑξῆς.

Lemma

If a parallelogram,† falling short by a square figure, is applied to some straight-line then the applied (parallelogram) is equal (in area) to the (rectangle contained) by the pieces of the straight-line created via the application (of the parallelogram).



For let the parallelogram AD, falling short by the square figure DB, be applied to the straight-line AB. I say that AD is equal to the (rectangle contained) by AC and CB.

And it is immediately obvious. For since DB is a square, DC is equal to CB. And AD is the (rectangle contained) by AC and CD—that is to say, by AC and CB.

Thus, if ... to some straight-line, and so on ....

† Note that this lemma only applies to rectangular parallelograms.

ιζ´.

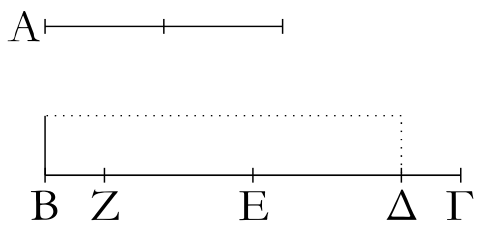
Ἐὰν ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετράτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρῇ μήκει, ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ συμμέτου ἑαυτῇ [μήκει]. καὶ ἐὰν ἡ μείζων τῆς ἐλάσσονος μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ [μήκει], τῷ δὲ τετράρτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ μήκει.

Ἔστωσαν δύο εὐθεῖαι ἄνισοι αἱ Α, ΒΓ, ὧν μείζων ἡ ΒΓ, τῷ δὲ τετράρτῳ μέρει τοῦ ἀπὸ ἐλάσσονος τῆς Α, τουτέστι τῷ ἀπὸ τῆς ἡμισείας τῆς Α, ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ, σύμμετρος δὲ ἔστω ἡ ΒΔ τῇ ΔΓ μήκει· λέγω, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Proposition 17†

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable in length, then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater. And if the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) commensurable in length.

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater. And let a (rectangle) equal to the fourth part of the (square) on the lesser, A—that is, (equal) to the (square) on half of A—falling short by a square figure, be applied to BC. And let it be the (rectangle contained) by BD and DC [see previous lemma]. And let BD be commensurable in length with DC. I say that that the square on BC is greater than the (square on) A by (the square on some straight-line) commensurable (in length) with (BC).

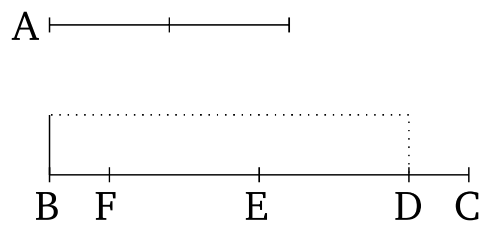


Τετμήσθω γὰρ ἡ ΒΓ δίχα κατὰ τὸ Ε σημεῖον, καὶ κείσθω τῇ ΔΕ ἴση ἡ ΕΖ. λοιπὴ ἄρα ἡ ΔΓ ἴση ἐστὶ τῇ ΒΖ. καὶ ἐπεὶ εὐθεῖα ἡ ΒΓ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Ε, εἰς δὲ ἄνισα κατὰ τὸ Δ, τὸ ἄρα ὑπὸ ΒΔ, ΔΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΓ τετραγώνῳ· καὶ τὰ τετραπλάσια· τὸ ἄρα τετράκις ὑπὸ τῶν ΒΔ, ΔΓ μετὰ τοῦ τετραπλασίου τοῦ ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ τετράκις ἀπὸ τῆς ΕΓ τετραγώνῳ. ἀλλὰ τῷ μὲν τετραπλασίῳ τοῦ ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς Α τετράγωνον, τῷ δὲ τετραπλασίῳ τοῦ ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΔΖ τετράγωνον· διπλασίων γάρ ἐστιν ἡ ΔΖ τῆς ΔΕ. τῷ δὲ τετραπλασίῳ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΓ τετράγωνον· διπλασίων γάρ ἐστι πάλιν ἡ ΒΓ τῆς ΓΕ. τὰ ἄρα ἀπὸ τῶν Α, ΔΖ τετράγωνα ἴσα ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· ὥστε τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς Α μεῖζόν ἐστι τῷ ἀπὸ τῆς ΔΖ· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῇ ΔΖ. δεικτέον, ὅτι καὶ σύμμετρός ἐστιν ἡ ΒΓ τῇ ΔΖ. ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει, σύμμετρος ἄρα ἐστὶ καὶ ἡ ΒΓ τῇ ΓΔ μήκει. ἀλλὰ ἡ ΓΔ ταῖς ΓΔ, ΒΖ ἐστι σύμμετρος μήκει· ἴση γάρ ἐστιν ἡ ΓΔ τῇ ΒΖ. καὶ ἡ ΒΓ ἄρα σύμμετρός ἐστι ταῖς ΒΖ, ΓΔ μήκει· ὥστε καὶ λοιπῇ τῇ ΖΔ σύμμετρός ἐστιν ἡ ΒΓ μήκει· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Ἀλλὰ δὴ ἡ ΒΓ τῆς Α μεῖζον δυνάσθω τῷ ἀπὸ συμμέτρου ἑαυτῇ, τῷ δὲ τετράτρῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ. δεικτέον, ὅτι σύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. δύναται δὲ ἡ ΒΓ τῆς Α μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῇ. σύμμετρος ἄρα ἐστὶν ἡ ΒΓ τῇ ΖΔ μήκει· ὥστε καὶ λοιπῇ συναμφοτέρῳ τῇ ΒΖ, ΔΓ σύμμετρός ἐστιν ἡ ΒΓ μήκει. ἀλλὰ συναμφότερος ἡ ΒΖ, ΔΓ σύμμετρός ἐστι τῇ ΔΓ [μήκει]. ὥστε καὶ ἡ ΒΓ τῇ ΓΔ σύμμετρός ἐστι μήκει· καὶ διελόντι ἄρα ἡ ΒΔ τῇ ΔΓ ἐστι σύμμετρος μήκει.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι ἄνισοι, καὶ τὰ ἑξῆς.



For let *BC* be cut in half at the point *E* [Prop. 1.10]. And let *EF* be made equal to *DE* [Prop. 1.3]. Thus, the remainder *DC* is equal to *BF*. And since the straight-line *BC* has been cut into equal (pieces) at *E*, and into unequal (pieces) at *D*, the rectangle contained by *BD* and *DC*, plus the square on *ED*, is thus equal to the square on *EC* [Prop. 2.5]. (The same) also (for) the quadruples. Thus, four times the (rectangle contained) by *BD* and *DC*, plus the quadruple of the (square) on *DE*, is equal to four times the square on *EC*. But, the square on *A* is equal to the quadruple of the (rectangle contained) by *BD* and *DC*, and the square on *DF* is equal to the quadruple of the (square) on *DE*. For *DF* is double *DE*. And the square on *BC* is equal to the quadruple of the (square) on *EC*. For, again, *BC* is double *CE*. Thus, the (sum of the) squares on *A* and *DF* is equal to the square on *BC*. Hence, the (square) on *BC* is greater than the (square) on *A* by the (square) on *DF*. Thus, *BC* is greater in square than *A* by *DF*. It must also be shown that *BC* is commensurable (in length) with *DF*. For since *BD* is commensurable in length with *DC*, *BC* is thus also commensurable in length with *CD* [Prop. 10.15]. But, *CD* is commensurable in length with *CD* plus *BF*. For *CD* is equal to *BF* [Prop. 10.6]. Thus, *BC* is also commensurable in length with *BF* plus *CD* [Prop. 10.12]. Hence, *BC* is also commensurable in length with the remainder *FD* [Prop. 10.15]. Thus, the square on *BC* is greater than (the square on) *A* by the (square) on (some straight-line) commensurable (in length) with (*BC*).

And so let the square on *BC* be greater than the (square on) *A* by the (square) on (some straight-line) commensurable (in length) with (*BC*). And let a (rectangle) equal to the fourth (part) of the (square) on *A*, falling short by a square figure, be applied to *BC*. And let it be the (rectangle contained) by *BD* and *DC*. It must be shown that *BD* is commensurable in length with *DC*.

For, similarly, by the same construction, we can show that the square on *BC* is greater than the (square on) *A* by the (square) on *FD*. And the square on *BC* is greater than the (square on) *A* by the (square) on (some straight-line) commensurable (in length) with (*BC*). Thus, *BC* is commensurable in length with *FD*. Hence, *BC* is also commensurable in length with the remaining sum of *BF* and *DC* [Prop. 10.15]. But, the sum of *BF* and *DC* is commensurable [in length] with *DC* [Prop. 10.6]. Hence, *BC* is also commensurable in length with *CD* [Prop. 10.12]. Thus, via separation, *BD* is also commensurable in length with *DC* [Prop. 10.15].

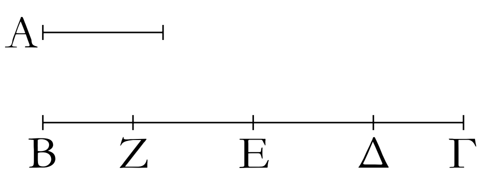
Thus, if there are two unequal straight-lines, and so on ....

† This proposition states that if $\alpha x - x^2 = \beta^2/4$ (where $\alpha = BC$, $x = DC$, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are commensurable when $\alpha - x$ are x are commensurable, and vice versa.

ιη΄.

Ἐὰν ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, καὶ εἰς ἀσυμμετρα αὐτὴν διαιρῇ [μήκει], ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ ἐὰν ἡ μείζων τῆς ἐλάσσονος μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετράρτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διαιρεῖ [μήκει].

Ἔστωσαν δύο εὐθεῖαι ἄνισοι αἱ Α, ΒΓ, ὧν μείζων ἡ ΒΓ, τῷ δὲ τετάρτῳ [μέρει] τοῦ ἀπὸ τῆς ἐλάσσονος τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔΓ, ἀσύμμετρος δὲ ἔστω ἡ ΒΔ τῇ ΔΓ μήκει· λέγω, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.



Τῶν γὰρ αὐτῶν κατασκευασθέντων τῷ πρότερον ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. δεικτέον [οὖν], ὅτι ἀσύμμετρός ἐστιν ἡ ΒΓ τῇ ΔΖ μήκει. ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΒΓ τῇ ΓΔ μήκει. ἀλλὰ ἡ ΔΓ σύμμετρός ἐστι συναμφοτέραις ταῖς ΒΖ, ΔΓ· καὶ ἡ ΒΓ ἄρα ἀσύμμετρός ἐστι συναμφοτέραις ταῖς ΒΖ, ΔΓ. ὥστε καὶ λοιπῇ τῇ ΖΔ ἀσύμμετρός ἐστιν ἡ ΒΓ μήκει. καὶ ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ· ἡ ΒΓ ἄρα τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

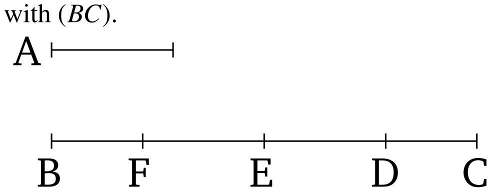
Δυνάσθω δὴ πάλιν ἡ ΒΓ τῆς Α μεῖζον τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ. δεικτέον, ὅτι ἀσύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ τῆς ΖΔ. ἀλλὰ ἡ ΒΓ τῆς Α μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. ἀσύμμετρος ἄρα

Proposition 18†

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable [in length], then the square on the greater will be larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater. And if the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) incommensurable [in length].

Let *A* and *BC* be two unequal straight-lines, of which (let) *BC* (be) the greater. And let a (rectangle) equal to the fourth [part] of the (square) on the lesser, *A*, falling short by a square figure, be applied to *BC*. And let it be the (rectangle contained) by *BDC*. And let *BD* be incommensurable in length with *DC*. I say that that the square on *BC* is greater than the (square on) *A* by the (square) on (some straight-line) incommensurable (in length) with (*BC*).



For, similarly, by the same construction as before, we can show that the square on *BC* is greater than the (square on) *A* by the (square) on *FD*. [Therefore] it must be shown that *BC* is incommensurable in length with *DF*. For since *BD* is incommensurable in length with *DC*, *BC* is thus also incommensurable in length with *CD* [Prop. 10.16]. But, *DC* is commensurable (in length) with the sum of *BF* and *DC* [Prop. 10.6]. And, thus, *BC* is incommensurable (in length) with the sum of *BF* and *DC* [Prop. 10.13]. Hence, *BC* is also incommensurable in length with the remainder *FD* [Prop. 10.16]. And the square on *BC* is greater than the (square on) *A* by the (square) on *FD*. Thus, the square on *BC* is greater than the (square on) *A* by the (square) on (some straight-line) incommensurable (in length) with (*BC*).

So, again, let the square on *BC* be greater than the (square on) *A* by the (square) on (some straight-line) incommensurable (in length) with (*BC*). And let a (rectangle) equal to the fourth [part] of the (square) on *A*, falling short by a square figure, *ἐστὶν ἡ ΒΓ τῇ ΖΔ μήκει· ὥστε καὶ λοιπῇ συναμφοτέρῳ τῇ ΒΖ, ΔΓ ἀσύμμετρός ἐστιν ἡ ΒΓ. ἀλλὰ συναμφότερος ἡ ΒΖ, ΔΓ τῇ ΔΓ σύμμετρός ἐστι μήκει· καὶ ἡ ΒΓ ἄρα τῇ ΔΓ ἀσύμμετρός ἐστι μήκει· ὥστε καὶ διελόντι ἡ ΒΔ τῇ ΔΓ ἀσύμμετρός ἐστι μήκει.*

Ἐὰν ἄρα ᾽ὦσι δύο εὐθεῖαι, καὶ τὰ ἑξῆς.

be applied to *BC*. And let it be the (rectangle contained) by *BD* and *DC*. It must be shown that *BD* is incommensurable in length with *DC*.

For, similarly, by the same construction, we can show that the square on *BC* is greater than the (square) on *A* by the (square) on *FD*. But, the square on *BC* is greater than the (square) on *A* by the (square) on (some straight-line) incommensurable (in length) with (*BC*). Thus, *BC* is incommensurable in length with *FD*. Hence, *BC* is also incommensurable (in length) with the remaining sum of *BF* and *DC* [Prop. 10.16]. But, the sum of *BF* and *DC* is commensurable in length with *DC* [Prop. 10.6]. Thus, *BC* is also incommensurable in length with *DC* [Prop. 10.13]. Hence, via separation, *BD* is also incommensurable in length with *DC* [Prop. 10.16].

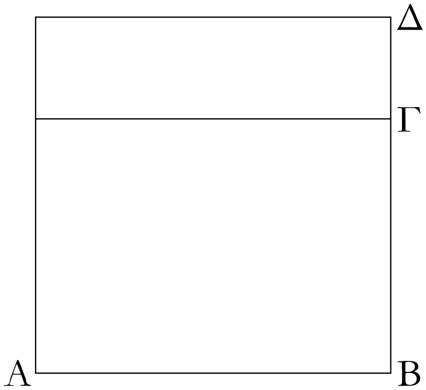
Thus, if there are two . . . straight-lines, and so on . . . .

† This proposition states that if $\alpha x - x^2 = \beta^2/4$ (where $\alpha = BC$, $x = DC$, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are incommensurable when $\alpha - x$ are x are incommensurable, and vice versa.

ιθ´.

Τὸ ὑπὸ ῥητῶν μήκει συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ῥητόν ἐστιν.

Ὑπὸ γὰρ ῥητῶν μήκει συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι ῥητόν ἐστι τὸ ΑΓ.



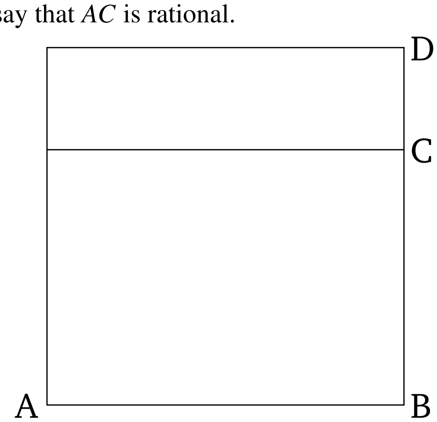
Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητὸν ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, ἴση δέ ἐστιν ἡ ΑΒ τῇ ΒΔ, σύμμετρος ἄρα ἐστὶν ἡ ΒΔ τῇ ΒΓ μήκει. καί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΒΓ, οὕτως τὸ ΔΑ πρὸς τὸ ΑΓ. σύμμετρον ἄρα ἐστὶ τὸ ΔΑ τῷ ΑΓ. ῥητὸν δὲ τὸ ΔΑ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ΑΓ.

Τὸ ἄρα ὑπὸ ῥητῶν μήκει συμμέτρων, καὶ τὰ ἑξῆς.

Proposition 19

The rectangle contained by rational straight-lines (which are) commensurable in length is rational.

For let the rectangle *AC* be enclosed by the rational straight-lines *AB* and *BC* (which are) commensurable in length. I say that *AC* is rational.



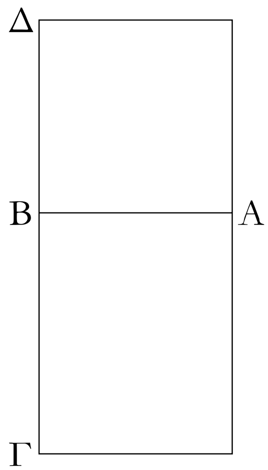
For let the square *AD* be described on *AB*. *AD* is thus rational [Def. 10.4]. And since *AB* is commensurable in length with *BC*, and *AB* is equal to *BD*, *BD* is thus commensurable in length with *BC*. And as *BD* is to *BC*, so *DA* (is) to *AC* [Prop. 6.1]. Thus, *DA* is commensurable with *AC* [Prop. 10.11]. And *DA* (is) rational. Thus, *AC* is also rational [Def. 10.4]. Thus, the . . . by rational straight-lines . . . commensurable, and so on . . . .

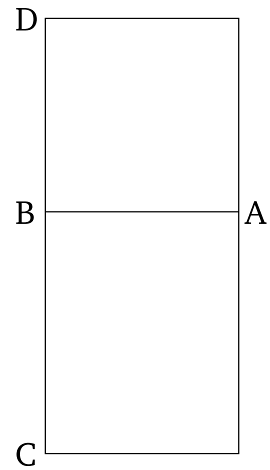
κ´.

Ἐὰν ῥητὸν παρὰ ῥητὴν παραβληθῇ, πλάτος ποιεῖ ῥητὴν καὶ σύμμετρον τῇ, παρ᾽ ἣν παράκειται, μήκει.

Proposition 20

If a rational (area) is applied to a rational (straight-line) then it produces as breadth a (straight-line which is) rational, and commensurable in length with the (straight-line) to which it is applied.





Ῥητὸν γὰρ τὸ ΑΓ παρὰ ῥητὴν τὴν ΑΒ παραβεβλήσθω πλάτος ποιοῦν τὴν ΒΓ· λέγω, ὅτι ῥητή ἐστιν ἡ ΒΓ καὶ σύμμετρος τῇ ΒΑ μήκει.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητὸν ἄρα ἐστὶ τὸ ΑΔ. ῥητὸν δὲ καὶ τὸ ΑΓ· σύμμετρον ἄρα ἐστὶ τὸ ΔΑ τῷ ΑΓ. καί ἐστιν ὡς τὸ ΔΑ πρὸς τὸ ΑΓ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΓ. σύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ· ἴση δὲ ἡ ΔΒ τῇ ΒΑ· σύμμετρος ἄρα καὶ ἡ ΑΒ τῇ ΒΓ. ῥητὴ δέ ἐστιν ἡ ΑΒ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΓ καὶ σύμμετρος τῇ ΑΒ μήκει.

Ἐὰν ἄρα ῥητὸν παρὰ ῥητὴν παραβληθῇ, καὶ τὰ ἑξῆς.

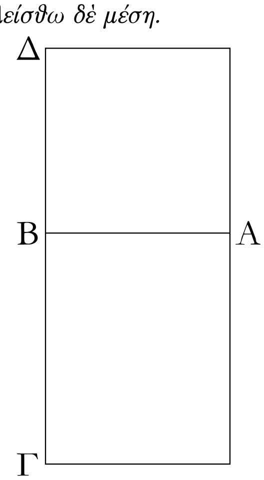
For let the rational (area) AC be applied to the rational (straight-line) AB, producing the (straight-line) BC as breadth. I say that BC is rational, and commensurable in length with BA.

For let the square AD be described on AB. AD is thus rational [Def. 10.4]. And AC (is) also rational. DA is thus commensurable with AC. And as DA is to AC, so DB (is) to BC [Prop. 6.1]. Thus, DB is also commensurable (in length) with BC [Prop. 10.11]. And DB (is) equal to BA. Thus, AB (is) also commensurable (in length) with BC. And AB is rational. Thus, BC is also rational, and commensurable in length with AB [Def. 10.3].

Thus, if a rational (area) is applied to a rational (straight-line), and so on ....

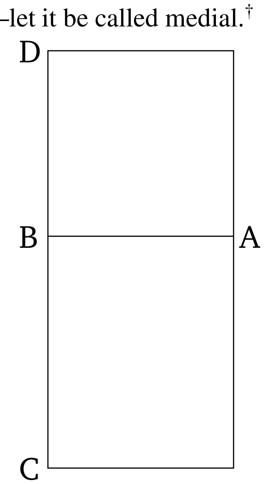
κα΄.

Τὸ ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλείσθω δὲ μέση.



Proposition 21

The rectangle contained by rational straight-lines (which are) commensurable in square only is irrational, and its square-root is irrational—let it be called medial.†



Ὑπὸ γὰρ ῥητῶν δυνάμει μόνον συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι ἄλογόν ἐστι τὸ ΑΓ, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλείσθω δὲ μέση.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητὸν ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει· δυνάμει γὰρ μόνον ὑπόκεινται σύμμετροι· ἴση δὲ ἡ ΑΒ τῇ ΒΔ, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ μήκει. καί ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΑΔ πρὸς τὸ ΑΓ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔΑ τῷ ΑΓ. ῥητὸν δὲ τὸ ΔΑ· ἄλογον ἄρα ἐστὶ τὸ ΑΓ· ὥστε καὶ ἡ δυναμένη τὸ ΑΓ [τουτέστιν ἡ ἴσον αὐτῷ τετράγωνον δυναμένη] ἄλογός ἐστιν, καλείσθω δε μέση· ὅπερ ἔδει δεῖξαι.

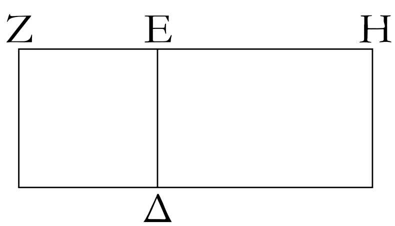
For let the rectangle AC be contained by the rational straight-lines AB and BC (which are) commensurable in square only. I say that AC is irrational, and its square-root is irrational—let it be called medial.

For let the square AD be described on AB. AD is thus rational [Def. 10.4]. And since AB is incommensurable in length with BC. For they were assumed to be commensurable in square only. And AB (is) equal to BD. DB is thus also incommensurable in length with BC. And as DB is to BC, so AD (is) to AC [Prop. 6.1]. Thus, DA [is] incommensurable with AC [Prop. 10.11]. And DA (is) rational. Thus, AC is irrational [Def. 10.4]. Hence, its square-root [that is to say, the square-root of the square equal to it] is also irrational [Def. 10.4]—let it be called medial. (Which is) the very thing it was required to show.

† Thus, a medial straight-line has a length expressible as $k^{1/4}$.

Λῆμμα.

Ἐὰν ὦσι δύο εὐθεῖαι, ἔστιν ὡς ἡ πρώτη πρὸς τὴν δευτέραν, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ὑπὸ τῶν δύο εὐθειῶν.

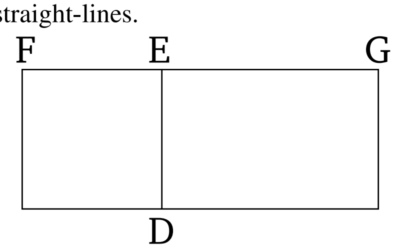


Ἔστωσαν δύο εὐθεῖαι αἱ ΖΕ, ΕΗ. λέγω, ὅτι ἐστὶν ὡς ἡ ΖΕ πρὸς τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΖΕ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΖΕ τετράγωνον τὸ ΔΖ, καὶ συμπεπληρώσθω τὸ ΗΔ. ἐπεὶ οὖν ἐστιν ὡς ἡ ΖΕ πρὸς τὴν ΕΗ, οὕτως τὸ ΖΔ πρὸς τὸ ΔΗ, καί ἐστι τὸ μὲν ΖΔ τὸ ἀπὸ τῆς ΖΕ, τὸ δὲ ΔΗ τὸ ὑπὸ τῶν ΔΕ, ΕΗ, τουτέστι τὸ ὑπὸ τῶν ΖΕ, ΕΗ, ἔστιν ἄρα ὡς ἡ ΖΕ πρὸς τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΖΕ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ. ὁμοίως δὲ καὶ ὡς τὸ ὑπὸ τῶν ΗΕ, ΕΖ πρὸς τὸ ἀπὸ τῆς ΕΖ, τουτέστιν ὡς τὸ ΗΔ πρὸς τὸ ΖΔ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

Lemma

If there are two straight-lines then as the first is to the second, so the (square) on the first (is) to the (rectangle contained) by the two straight-lines.



Let FE and EG be two straight-lines. I say that as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG.

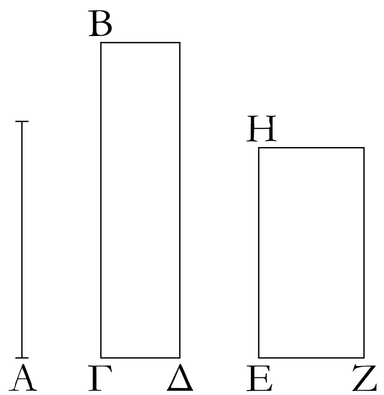
For let the square DF be described on FE. And let GD be completed. Therefore, since as FE is to EG, so FD (is) to DG [Prop. 6.1], and FD is the (square) on FE, and DG the (rectangle contained) by DE and EG—that is to say, the (rectangle contained) by FE and EG—thus as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG. And also, similarly, as the (rectangle contained) by GE and EF is to the (square on) EF—that is to say, as GD (is) to FD—so GE (is) to EF. (Which is) the very thing it was required to show.

κβ'.

Proposition 22

Τὸ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ, παρ᾿ ἣν παράκειται, μήκει.

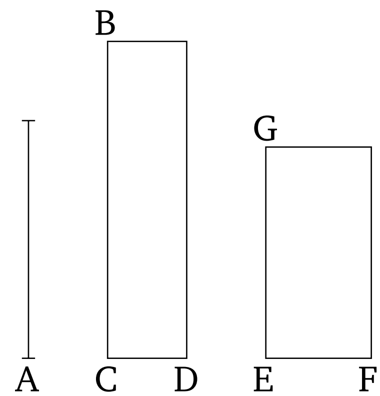
Ἔστω μέση μὲν ἡ Α, ῥητὴ δὲ ἡ ΓΒ, καὶ τῷ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΒΔ πλάτος ποιοῦν τὴν ΓΔ· λέγω, ὅτι ῥητή ἐστιν ἡ ΓΔ καὶ ἀσύμμετρος τῇ ΓΒ μήκει.



Ἐπεὶ γὰρ μέση ἐστὶν ἡ Α, δύναται χωρίον περιεχόμενον ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων. δυνάσθω τὸ ΗΖ. δύναται δὲ καὶ τὸ ΒΔ· ἴσον ἄρα ἐστὶ τὸ ΒΔ τῷ ΗΖ. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον· τῶν δὲ ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΖ πρὸς τὴν ΓΔ. ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΓΔ. σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς ΓΒ τῷ ἀπὸ τῆς ΕΗ· ῥητὴ γάρ ἐστιν ἑκατέρα αὐτῶν· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΓΔ. ῥητὸν δέ ἐστι τὸ ἀπὸ τῆς ΕΖ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ· ῥητὴ ἄρα ἐστὶν ἡ ΓΔ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΖ τῇ ΕΗ μήκει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ ΕΖ πρὸς τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ, ἀσύμμετρον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς ΕΖ τῷ ὑπὸ τῶν ΖΕ, ΕΗ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΖ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΓΔ· ῥηταὶ γάρ εἰσι δυνάμει· τῷ δὲ ὑπὸ τῶν ΖΕ, ΕΗ σύμμετρόν ἐστι τὸ ὑπὸ τῶν ΔΓ, ΓΒ· ἴσα γάρ ἐστι τῷ ἀπὸ τῆς Α· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ τῷ ὑπὸ τῶν ΔΓ, ΓΒ. ὡς δὲ τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ὑπὸ τῶν ΔΓ, ΓΒ, οὕτως ἐστὶν ἡ ΔΓ πρὸς τὴν ΓΒ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΓ τῇ ΓΒ μήκει. ῥητὴ ἄρα ἐστὶν ἡ ΓΔ καὶ ἀσύμμετρος τῇ ΓΒ μήκει· ὅπερ ἔδει δεῖξαι.

The square on a medial (straight-line), being applied to a rational (straight-line), produces as breadth a (straight-line which is) rational, and incommensurable in length with the (straight-line) to which it is applied.

Let A be a medial (straight-line), and CB a rational (straight-line), and let the rectangular area BD, equal to the (square) on A, be applied to BC, producing CD as breadth. I say that CD is rational, and incommensurable in length with CB.



For since A is medial, the square on it is equal to a (rectangular) area contained by rational (straight-lines which are) commensurable in square only [Prop. 10.21]. Let the square on (A) be equal to GF. And the square on (A) is also equal to BD. Thus, BD is equal to GF. And (BD) is also equiangular with (GF). And for equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, proportionally, as BC is to EG, so EF (is) to CD. And, also, as the (square) on BC is to the (square) on EG, so the (square) on EF (is) to the (square) on CD [Prop. 6.22]. And the (square) on CB is commensurable with the (square) on EG. For they are each rational. Thus, the (square) on EF is also commensurable with the (square) on CD [Prop. 10.11]. And the (square) on EF is rational. Thus, the (square) on CD is also rational [Def. 10.4]. Thus, CD is rational. And since EF is incommensurable in length with EG. For they are commensurable in square only. And as EF (is) to EG, so the (square) on EF (is) to the (rectangle contained) by FE and EG [see previous lemma]. The (square) on EF [is] thus incommensurable with the (rectangle contained) by FE and EG [Prop. 10.11]. But, the (square) on CD is commensurable with the (square) on EF. For they are rational in square. And the (rectangle contained) by DC and CB is commensurable with the (rectangle contained) by FE and EG. For they are (both) equal to the (square) on A. Thus, the (square) on CD is also incommensurable with the (rectangle contained) by DC and CB [Prop. 10.13]. And as the (square) on CD (is) to the (rectangle contained) by DC and CB, so DC is to CB [see previous lemma]. Thus, DC is incommensurable in length with

CB [Prop. 10.11]. Thus, *CD* is rational, and incommensurable in length with *CB*. (Which is) the very thing it was required to show.

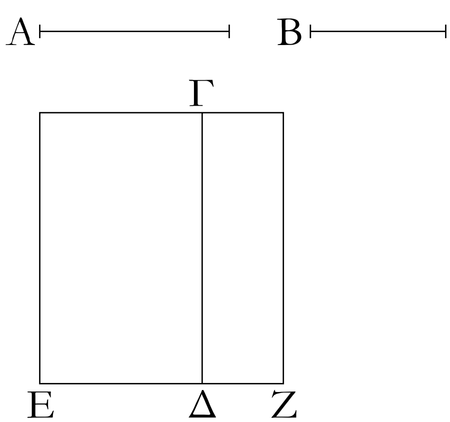
† Literally, "rational".

κγ΄.

Ἡ τῇ μέσῃ σύμμετρος μέση ἐστίν.

Ἔστω μέση ἡ Α, καὶ τῇ Α σύμμετρος ἔστω ἡ Β· λέγω, ὅτι καὶ ἡ Β μέση ἐστίν.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΓΔ, καὶ τῷ μὲν ἀπὸ τῆς Α ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΓΕ πλάτος ποιοῦν τὴν ΕΔ· ῥητὴ ἄρα ἐστὶν ἡ ΕΔ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. τῷ δὲ ἀπὸ τῆς Β ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΓΖ πλάτος ποιοῦν τὴν ΔΖ. ἐπεὶ οὖν σύμμετρός ἐστιν ἡ Α τῇ Β, σύμμετρόν ἐστι καὶ τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς Β. ἀλλὰ τῷ μὲν ἀπὸ τῆς Α ἴσον ἐστὶ τὸ ΕΓ, τῷ δὲ ἀπὸ τῆς Β ἴσον ἐστὶ τὸ ΓΖ· σύμμετρον ἄρα ἐστὶ τὸ ΕΓ τῷ ΓΖ. καί ἐστιν ὡς τὸ ΕΓ πρὸς τὸ ΓΖ, οὕτως ἡ ΕΔ πρὸς τὴν ΔΖ· σύμμετρος ἄρα ἐστὶν ἡ ΕΔ τῇ ΔΖ μήκει. ῥητὴ δέ ἐστιν ἡ ΕΔ καὶ ἀσύμμετρος τῇ ΔΓ μήκει· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΔΖ καὶ ἀσύμμετρος τῇ ΔΓ μήκει· αἱ ΓΔ, ΔΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἡ δὲ τὸ ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων δυναμένη μέση ἐστίν. ἡ ἄρα τὸ ὑπὸ τῶν ΓΔ, ΔΖ δυναμένη μέση ἐστίν· καὶ δύναται τὸ ὑπὸ τῶν ΓΔ, ΔΖ ἡ Β· μέση ἄρα ἐστὶν ἡ Β.



Πόρισμα.

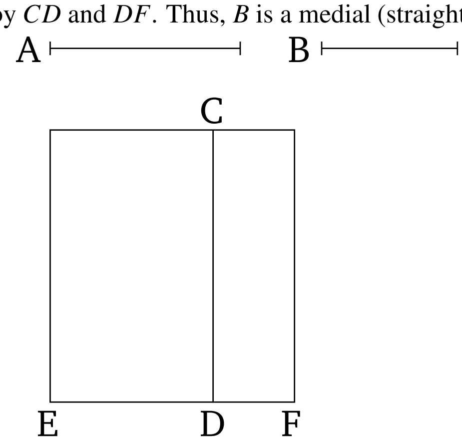
Ἐκ δὴ τούτου φανερόν, ὅτι τὸ τῷ μέσῳ χωρίῳ σύμμετρον μέσον ἐστίν.

Proposition 23

A (straight-line) commensurable with a medial (straight-line) is medial.

Let *A* be a medial (straight-line), and let *B* be commensurable with *A*. I say that *B* is also a medial (staight-line).

Let the rational (straight-line) *CD* be set out, and let the rectangular area *CE*, equal to the (square) on *A*, be applied to *CD*, producing *ED* as width. *ED* is thus rational, and incommensurable in length with *CD* [Prop. 10.22]. And let the rectangular area *CF*, equal to the (square) on *B*, be applied to *CD*, producing *DF* as width. Therefore, since *A* is commensurable with *B*, the (square) on *A* is also commensurable with the (square) on *B*. But, *EC* is equal to the (square) on *A*, and *CF* is equal to the (square) on *B*. Thus, *EC* is commensurable with *CF*. And as *EC* is to *CF*, so *ED* (is) to *DF* [Prop. 6.1]. Thus, *ED* is commensurable in length with *DF* [Prop. 10.11]. And *ED* is rational, and incommensurable in length with *CD*. *DF* is thus also rational [Def. 10.3], and incommensurable in length with *DC* [Prop. 10.13]. Thus, *CD* and *DF* are rational, and commensurable in square only. And the square-root of a (rectangle contained) by rational (straight-lines which are) commensurable in square only is medial [Prop. 10.21]. Thus, the square-root of the (rectangle contained) by *CD* and *DF* is medial. And the square on *B* is equal to the (rectangle contained) by *CD* and *DF*. Thus, *B* is a medial (straight-line).



Corollary

And (it is) clear, from this, that an (area) commensurable with a medial area† is medial.

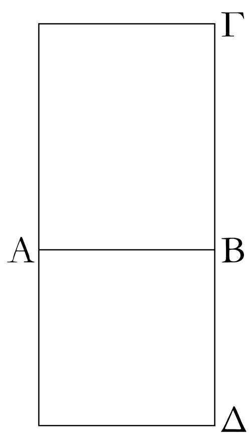
† A medial area is equal to the square on some medial straight-line. Hence, a medial area is expressible as $k^{1/2}$.

κδ′.

Τὸ ὑπὸ μέσων μήκει συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον μέσον ἐστίν.

Ὑπὸ γὰρ μέσων μήκει συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ περιεχέσθω ὀρθογώνιον τὸ ΑΓ· λέγω, ὅτι τὸ ΑΓ μέσον ἐστίν.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· μέσον ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, ἴση δὲ ἡ ΑΒ τῇ ΒΔ, σύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ μήκει· ὥστε καὶ τὸ ΔΑ τῷ ΑΓ σύμμετρόν ἐστιν. μέσον δὲ τὸ ΔΑ· μέσον ἄρα καὶ τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

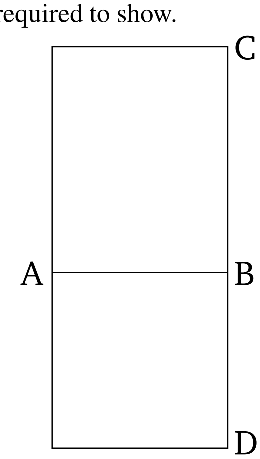


Proposition 24

A rectangle contained by medial straight-lines (which are) commensurable in length is medial.

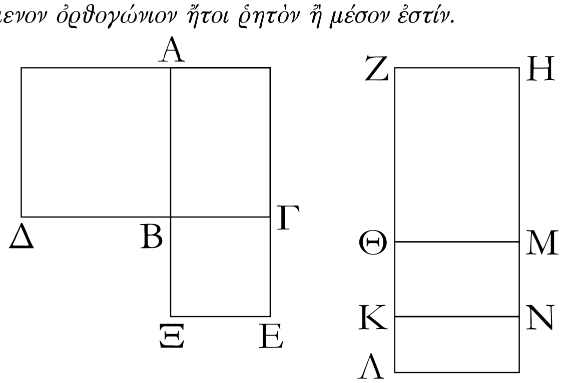
For let the rectangle *AC* be contained by the medial straight-lines *AB* and *BC* (which are) commensurable in length. I say that *AC* is medial.

For let the square *AD* be described on *AB*. *AD* is thus medial [see previous footnote]. And since *AB* is commensurable in length with *BC*, and *AB* (is) equal to *BD*, *DB* is thus also commensurable in length with *BC*. Hence, *DA* is also commensurable with *AC* [Props. 6.1, 10.11]. And *DA* (is) medial. Thus, *AC* (is) also medial [Prop. 10.23 corr.]. (Which is) the very thing it was required to show.



κε′.

Τὸ ὑπὸ μέσων δυνάμει μόνον συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἤτοι ῥητὸν ἢ μέσον ἐστίν.

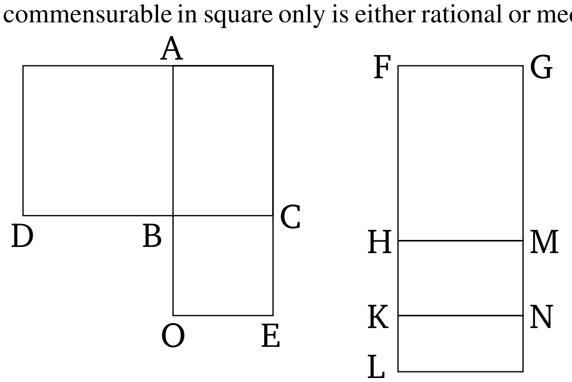


Ὑπὸ γὰρ μέσων δυνάμει μόνον συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι τὸ ΑΓ ἤτοι ῥητὸν ἢ μέσον ἐστίν.

Ἀναγεγράφθω γὰρ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα τὰ ΑΔ, ΒΕ· μέσον ἄρα ἐστὶν ἑκάτερον τῶν ΑΔ, ΒΕ. καὶ ἐκκείσθω ῥητὴ ἡ ΖΗ, καὶ τῷ μὲν ΑΔ ἴσον παρὰ τὴν ΖΗ παραβεβλήσθω

Proposition 25

The rectangle contained by medial straight-lines (which are) commensurable in square only is either rational or medial.



For let the rectangle *AC* be contained by the medial straight-lines *AB* and *BC* (which are) commensurable in square only. I say that *AC* is either rational or medial.

For let the squares *AD* and *BE* be described on (the straight-lines) *AB* and *BC* (respectively). *AD* and *BE* are thus each medial. And let the rational (straight-line) *FG* be laid ὀρθογώνιον παραλληλόγραμμον τὸ ΗΘ πλάτος ποιοῦν τὴν ΖΘ, τῷ δὲ ΑΓ ἴσον παρὰ τὴν ΘΜ παραβεβλήσθω ὀρθογώνιον παραλληλόγραμμον τὸ ΜΚ πλάτος ποιοῦν τὴν ΘΚ, καὶ ἔτι τῷ ΒΕ ἴσον ὁμοίως παρὰ τὴν ΚΝ παραβεβλήσθω τὸ ΝΛ πλάτος ποιοῦν τὴν ΚΛ· ἐπ᾿ εὐθείας ἄρα εἰσὶν αἱ ΖΘ, ΘΚ, ΚΛ. ἐπεὶ οὖν μέσον ἐστὶν ἑκάτερον τῶν ΑΔ, ΒΕ, καί ἐστιν ἴσον τὸ μὲν ΑΔ τῷ ΗΘ, τὸ δὲ ΒΕ τῷ ΝΛ, μέσον ἄρα καὶ ἑκάτερον τῶν ΗΘ, ΝΛ. καὶ παρὰ ῥητὴν τὴν ΖΗ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΖΘ, ΚΛ καὶ ἀσύμμετρος τῇ ΖΗ μήκει. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ΑΔ τῷ ΒΕ, σύμμετρον ἄρα ἐστὶ καὶ τὸ ΗΘ τῷ ΝΛ. καί ἐστιν ὡς τὸ ΗΘ πρὸς τὸ ΝΛ, οὕτως ἡ ΖΘ πρὸς τὴν ΚΛ· σύμμετρος ἄρα ἐστὶν ἡ ΖΘ τῇ ΚΛ μήκει. αἱ ΖΘ, ΚΛ ἄρα ῥηταί εἰσι μήκει σύμμετροι· ῥητὸν ἄρα ἐστὶ τὸ ὑπὸ τῶν ΖΘ, ΚΛ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΑ, ἡ δὲ ΞΒ τῇ ΒΓ, ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΑΒ πρὸς τὴν ΒΞ. ἀλλ᾿ ὡς μὲν ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΔΑ πρὸς τὸ ΑΓ· ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΞ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΞ· ἔστιν ἄρα ὡς τὸ ΔΑ πρὸς τὸ ΑΓ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΞ. ἴσον δέ ἐστι τὸ μὲν ΑΔ τῷ ΗΘ, τὸ δὲ ΑΓ τῷ ΜΚ, τὸ δὲ ΓΞ τῷ ΝΛ· ἔστιν ἄρα ὡς τὸ ΗΘ πρὸς τὸ ΜΚ, οὕτως τὸ ΜΚ πρὸς τὸ ΝΛ· ἔστιν ἄρα καὶ ὡς ἡ ΖΘ πρὸς τὴν ΘΚ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΛ· τὸ ἄρα ὑπὸ τῶν ΖΘ, ΚΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΘΚ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΖΘ, ΚΛ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘΚ· ῥητὴ ἄρα ἐστὶν ἡ ΘΚ. καὶ εἰ μὲν σύμμετρός ἐστι τῇ ΖΗ μήκει, ῥητόν ἐστι τὸ ΘΝ· εἰ δὲ ἀσύμμετρός ἐστι τῇ ΖΗ μήκει, αἱ ΚΘ, ΘΜ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΘΝ. τὸ ΘΝ ἄρα ἤτοι ῥητὸν ἢ μέσον ἐστίν. ἴσον δὲ τὸ ΘΝ τῷ ΑΓ· τὸ ΑΓ ἄρα ἤτοι ῥητὸν ἢ μέσον ἐστίν.

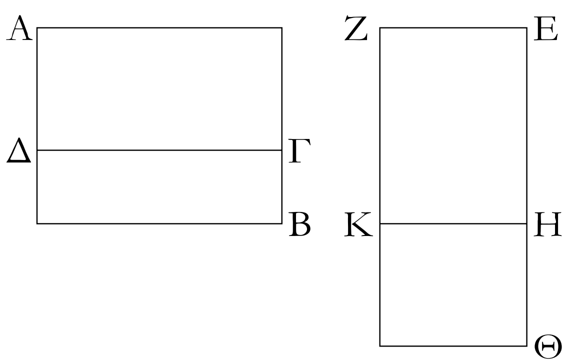
Τὸ ἄρα ὑπὸ μέσων δυνάμει μόνον συμμέτρων, καὶ τὰ ἑξῆς.

out. And let the rectangular parallelogram GH, equal to AD, be applied to FG, producing FH as breadth. And let the rectangular parallelogram MK, equal to AC, be applied to HM, producing HK as breadth. And, finally, let NL, equal to BE, have similarly been applied to KN, producing KL as breadth. Thus, FH, HK, and KL are in a straight-line. Therefore, since AD and BE are each medial, and AD is equal to GH, and BE to NL, GH and NL (are) thus each also medial. And they are applied to the rational (straight-line) FG. FH and KL are thus each rational, and incommensurable in length with FG [Prop. 10.22]. And since AD is commensurable with BE, GH is thus also commensurable with NL. And as GH is to NL, so FH (is) to KL [Prop. 6.1]. Thus, FH is commensurable in length with KL [Prop. 10.11]. Thus, FH and KL are rational (straight-lines which are) commensurable in length. Thus, the (rectangle contained) by FH and KL is rational [Prop. 10.19]. And since DB is equal to BA, and OB to BC, thus as DB is to BC, so AB (is) to BO. But, as DB (is) to BC, so DA (is) to AC [Props. 6.1]. And as AB (is) to BO, so AC (is) to CO [Prop. 6.1]. Thus, as DA is to AC, so AC (is) to CO. And AD is equal to GH, and AC to MK, and CO to NL. Thus, as GH is to MK, so MK (is) to NL. Thus, also, as FH is to HK, so HK (is) to KL [Props. 6.1, 5.11]. Thus, the (rectangle contained) by FH and KL is equal to the (square) on HK [Prop. 6.17]. And the (rectangle contained) by FH and KL (is) rational. Thus, the (square) on HK is also rational. Thus, HK is rational. And if it is commensurable in length with FG then HN is rational [Prop. 10.19]. And if it is incommensurable in length with FG then KH and HM are rational (straight-lines which are) commensurable in square only: thus, HN is medial [Prop. 10.21]. Thus, HN is either rational or medial. And HN (is) equal to AC. Thus, AC is either rational or medial.

Thus, the ... by medial straight-lines (which are) commensurable in square only, and so on ....

κϛ´.

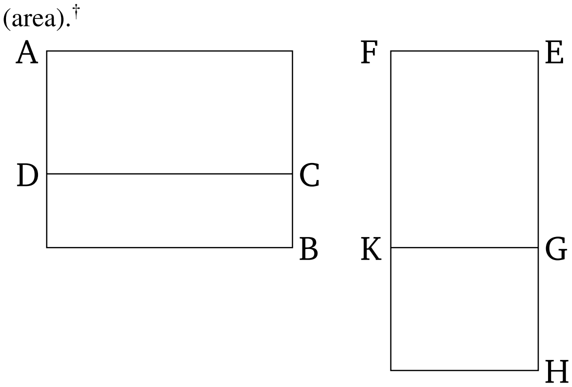
Μέσον μέσου οὐχ ὑπερέχει ῥητῷ.



Εἰ γὰρ δυνατόν, μέσον τὸ ΑΒ μέσου τοῦ ΑΓ ὑπερεχέτω

Proposition 26

A medial (area) does not exceed a medial (area) by a rational (area).†



For, if possible, let the medial (area) AB exceed the me-

ῥητῷ τῷ ΔΒ, καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τῷ ΑΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω παραλληλόγραμμον ὀρθογώνιον τὸ ΖΘ πλάτος ποιοῦν τὴν ΕΘ, τῷ δὲ ΑΓ ἴσον ἀφῃρήσθω τὸ ΖΗ· λοιπὸν ἄρα τὸ ΒΔ λοιπῷ τῷ ΚΘ ἐστιν ἴσον. ῥητὸν δέ ἐστι τὸ ΔΒ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ΚΘ. ἐπεὶ οὖν μέσον ἐστὶν ἑκάτερον τῶν ΑΒ, ΑΓ, καί ἐστι τὸ μὲν ΑΒ τῷ ΖΘ ἴσον, τὸ δὲ ΑΓ τῷ ΖΗ, μέσον ἄρα καὶ ἑκάτερον τῶν ΖΘ, ΖΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΘΕ, ΕΗ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ῥητόν ἐστι τὸ ΔΒ καί ἐστιν ἴσον τῷ ΚΘ, ῥητὸν ἄρα ἐστὶ καὶ τὸ ΚΘ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΗΘ καὶ σύμμετρος τῇ ΕΖ μήκει. ἀλλά καὶ ἡ ΕΗ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΗ τῇ ΗΘ μήκει. καί ἐστιν ὡς ἡ ΕΗ πρὸς τὴν ΗΘ, οὕτως τὸ ἀπὸ τῆς ΕΗ πρὸς τὸ ὑπὸ τῶν ΕΗ, ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΗ τῷ ὑπὸ τῶν ΕΗ, ΗΘ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΕΗ, ΗΘ τετράγωνα· ῥητὰ γὰρ ἀμφότερα· τῷ δὲ ὑπὸ τῶν ΕΗ, ΗΘ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΕΗ, ΗΘ· διπλάσιον γάρ ἐστιν αὐτοῦ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΕΗ, ΗΘ τῷ δὶς ὑπὸ τῶν ΕΗ, ΗΘ· καὶ συναμφότερα ἄρα τά τε ἀπὸ τῶν ΕΗ, ΗΘ καὶ τὸ δὶς ὑπὸ τῶν ΕΗ, ΗΘ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΕΘ, ἀσύμμετρόν ἐστι τοῖς ἀπὸ τῶν ΕΗ, ΗΘ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΕΗ, ΗΘ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΕΘ. ἄλογος ἄρα ἐστὶν ἡ ΕΘ. ἀλλὰ καὶ ῥηρή· ὅπερ ἐστὶν ἀδύνατον.

Μέσον ἄρα μέσου οὐχ ὑπερέχει ῥητῷ· ὅπερ ἔδει δεῖξαι.

dial (area) AC by the rational (area) DB. And let the rational (straight-line) EF be laid down. And let the rectangular parallelogram FH, equal to AB, be applied to to EF, producing EH as breadth. And let FG, equal to AC, be cut off (from FH). Thus, the remainder BD is equal to the remainder KH. And DB is rational. Thus, KH is also rational. Therefore, since AB and AC are each medial, and AB is equal to FH, and AC to FG, FH and FG are thus each also medial. And they are applied to the rational (straight-line) EF. Thus, HE and EG are each rational, and incommensurable in length with EF [Prop. 10.22]. And since DB is rational, and is equal to KH, KH is thus also rational. And (KH) is applied to the rational (straight-line) EF. GH is thus rational, and commensurable in length with EF [Prop. 10.20]. But, EG is also rational, and incommensurable in length with EF. Thus, EG is incommensurable in length with GH [Prop. 10.13]. And as EG is to GH, so the (square) on EG (is) to the (rectangle contained) by EG and GH [Prop. 10.13 lem.]. Thus, the (square) on EG is incommensurable with the (rectangle contained) by EG and GH [Prop. 10.11]. But, the (sum of the) squares on EG and GH is commensurable with the (square) on EG. For (EG and GH are) both rational. And twice the (rectangle contained) by EG and GH is commensurable with the (rectangle contained) by EG and GH [Prop. 10.6]. For (the former) is double the latter. Thus, the (sum of the squares) on EG and GH is incommensurable with twice the (rectangle contained) by EG and GH [Prop. 10.13]. And thus the sum of the (squares) on EG and GH plus twice the (rectangle contained) by EG and GH, that is the (square) on EH [Prop. 2.4], is incommensurable with the (sum of the squares) on EG and GH [Prop. 10.16]. And the (sum of the squares) on EG and GH (is) rational. Thus, the (square) on EH is irrational [Def. 10.4]. Thus, EH is irrational [Def. 10.4]. But, (it is) also rational. The very thing is impossible.

Thus, a medial (area) does not exceed a medial (area) by a rational (area). (Which is) the very thing it was required to show.

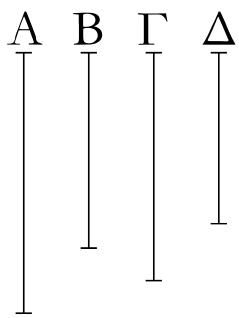
† In other words, $\sqrt{k} - \sqrt{k'} \neq k''$.

κζ΄.

Μέσας εὑρεῖν δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας.

Proposition 27

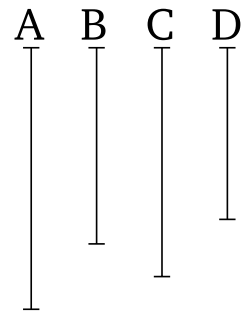
To find (two) medial (straight-lines), containing a rational (area), (which are) commensurable in square only.



Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Γ, καὶ γεγονέτω ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ.

Καὶ ἐπεὶ αἱ Α, Β ῥηταί εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν Α, Β, τουτέστι τὸ ἀπὸ τῆς Γ, μέσον ἐστίν. μέση ἄρα ἡ Γ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, [οὕτως] ἡ Γ πρὸς τὴν Δ, αἱ δὲ Α, Β δυνάμει μόνον [εἰσὶ] σύμμετροι, καὶ αἱ Γ, Δ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. αἱ Γ, Δ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω, ὅτι καὶ ῥητὸν περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ Α πρὸς τὴν Γ, ἡ Β πρὸς τὴν Δ. ἀλλ᾽ ὡς ἡ Α πρὸς τὴν Γ, ἡ Γ πρὸς τὴν Β· καὶ ὡς ἄρα ἡ Γ πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Δ· τὸ ἄρα ὑπὸ τῶν Γ, Δ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β. ῥητὸν δὲ τὸ ἀπὸ τῆς Β· ῥητὸν ἄρα [ἐστὶ] καὶ τὸ ὑπὸ τῶν Γ, Δ.

Εὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ὅπερ ἔδει δεῖξαι.



Let the two rational (straight-lines) *A* and *B*, (which are) commensurable in square only, be laid down. And let *C*—the mean proportional (straight-line) to *A* and *B*—be taken [Prop. 6.13]. And let it be contrived that as *A* (is) to *B*, so *C* (is) to *D* [Prop. 6.12].

And since the rational (straight-lines) *A* and *B* are commensurable in square only, the (rectangle contained) by *A* and *B*—that is to say, the (square) on *C* [Prop. 6.17]—is thus medial [Prop 10.21]. Thus, *C* is medial [Prop. 10.21]. And since as *A* is to *B*, [so] *C* (is) to *D*, and *A* and *B* [are] commensurable in square only, *C* and *D* are thus also commensurable in square only [Prop. 10.11]. And *C* is medial. Thus, *D* is also medial [Prop. 10.23]. Thus, *C* and *D* are medial (straight-lines which are) commensurable in square only. I say that they also contain a rational (area). For since as *A* is to *B*, so *C* (is) to *D*, thus, alternately, as *A* is to *C*, so *B* (is) to *D* [Prop. 5.16]. But, as *A* (is) to *C*, (so) *C* (is) to *B*. And thus as *C* (is) to *B*, so *B* (is) to *D* [Prop. 5.11]. Thus, the (rectangle contained) by *C* and *D* is equal to the (square) on *B* [Prop. 6.17]. And the (square) on *B* (is) rational. Thus, the (rectangle contained) by *C* and *D* [is] also rational.

Thus, (two) medial (straight-lines, *C* and *D*), containing a rational (area), (which are) commensurable in square only, be found.† (Which is) the very thing it was required to show.

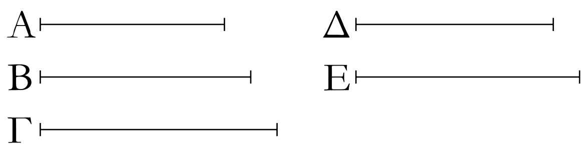
† *C* and *D* have lengths $k^{1/4}$ and $k^{3/4}$ times that of *A*, respectively, where the length of *B* is $k^{1/2}$ times that of *A*.

κη΄.

Μέσας εὑρεῖν δυνάμει μόνον συμμέτρους μέσον περιεχούσας.

Proposition 28

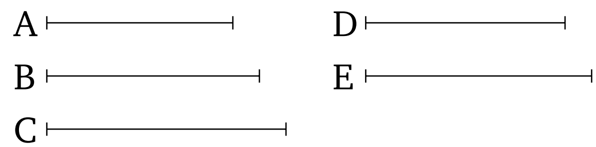
To find (two) medial (straight-lines), containing a medial (area), (which are) commensurable in square only.



Ἐκκείσθωσαν [τρεῖς] ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, Γ, καὶ εἰλήφθω τῶν Α, Β μέση ἀνάλογον ἡ Δ, καὶ γεγονέτω ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε.

Ἐπεὶ αἱ Α, Β ῥηταί εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν Α, Β, τουτέστι τὸ ἀπὸ τῆς Δ, μέσον ἐστίν. μέση ἄρα ἡ Δ. καὶ ἐπεὶ αἱ Β, Γ δυνάμει μόνον εἰσὶ σύμμετροι, καί ἐστιν ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, καὶ αἱ Δ, Ε ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. μέση δὲ ἡ Δ· μέση ἄρα καὶ ἡ Ε· αἱ Δ, Ε ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι καὶ μέσον περιέχουσιν. ἐπεὶ γάρ ἐστιν ὡς ἡ Β πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, ἐναλλὰξ ἄρα ὡς ἡ Β πρὸς τὴν Δ, ἡ Γ πρὸς τὴν Ε. ὡς δὲ ἡ Β πρὸς τὴν Δ, ἡ Δ πρὸς τὴν Α· καὶ ὡς ἄρα ἡ Δ πρὸς τὴν Α, ἡ Γ πρὸς τὴν Ε· τὸ ἄρα ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Δ, Ε. μέσον δὲ τὸ ὑπὸ τῶν Α, Γ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ, Ε.

Εὕρηνται ἄρα μέσαι δυνάμει μόνον σύμμετροι μέσον περιέχουσαι· ὅπερ ἔδει δεῖξαι.



Let the [three] rational (straight-lines) *A*, *B*, and *C*, (which are) commensurable in square only, be laid down. And let, *D*, the mean proportional (straight-line) to *A* and *B*, be taken [Prop. 6.13]. And let it be contrived that as *B* (is) to *C*, (so) *D* (is) to *E* [Prop. 6.12].

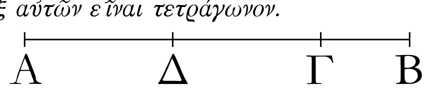
Since the rational (straight-lines) *A* and *B* are commensurable in square only, the (rectangle contained) by *A* and *B*—that is to say, the (square) on *D* [Prop. 6.17]—is medial [Prop. 10.21]. Thus, *D* (is) medial [Prop. 10.21]. And since *B* and *C* are commensurable in square only, and as *B* is to *C*, (so) *D* (is) to *E*, *D* and *E* are thus commensurable in square only [Prop. 10.11]. And *D* (is) medial. *E* (is) thus also medial [Prop. 10.23]. Thus, *D* and *E* are medial (straight-lines which are) commensurable in square only. So, I say that they also enclose a medial (area). For since as *B* is to *C*, (so) *D* (is) to *E*, thus, alternately, as *B* (is) to *D*, (so) *C* (is) to *E* [Prop. 5.16]. And as *B* (is) to *D*, (so) *D* (is) to *A*. And thus as *D* (is) to *A*, (so) *C* (is) to *E*. Thus, the (rectangle contained) by *A* and *C* is equal to the (rectangle contained) by *D* and *E* [Prop. 6.16]. And the (rectangle contained) by *A* and *C* is medial [Prop. 10.21]. Thus, the (rectangle contained) by *D* and *E* (is) also medial.

Thus, (two) medial (straight-lines, *D* and *E*), containing a medial (area), (which are) commensurable in square only, be found. (Which is) the very thing it was required to show.

† *D* and *E* have lengths $k^{1/4}$ and $k'^{1/2}/k^{1/4}$ times that of *A*, respectively, where the lengths of *B* and *C* are $k^{1/2}$ and $k'^{1/2}$ times that of *A*, respectively.

Λῆμμα α′.

Εὑρεῖν δύο τετραγώνους ἀριθμούς, ὥστε καὶ τὸν συγκείμενον ἐξ αὐτῶν εἶναι τετράγωνον.



Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΒ, ΒΓ, ἔστωσαν δὲ ἤτοι ἄρτιοι ἢ περιττοί. καὶ ἐπεί, ἐάν τε ἀπὸ ἀρτίου ἄρτιος ἀφαιρεθῇ, ἐάν τε ἀπὸ περισσοῦ περισσός, ὁ λοιπὸς ἄρτιός ἐστιν, ὁ λοιπὸς ἄρα ὁ ΑΓ ἄρτιός ἐστιν. τετμήσθω ὁ ΑΓ δίχα κατὰ τὸ Δ. ἔστωσαν δὲ καὶ οἱ ΑΒ, ΒΓ ἤτοι ὅμοιοι ἐπίπεδοι ἢ τετράγωνοι, οἳ καὶ αὐτοὶ ὅμοιοί εἰσιν ἐπίπεδοι· ὁ ἄρα ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ τοῦ ΒΔ τετραγώνῳ. καί ἐστι τετράγωνος ὁ ἐκ τῶν ΑΒ, ΒΓ, ἐπειδήπερ ἐδείχθη, ὅτι, ἐὰν δύο ὅμοιοι ἐπίπεδοι πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος τετράγωνός ἐστιν. εὕρηνται ἄρα δύο τετράγωνοι ἀριθμοὶ ὅ τε ἐκ τῶν ΑΒ, ΒΓ καὶ ὁ ἀπὸ τοῦ ΓΔ, οἳ συντεθέντες ποιοῦσι τὸν ἀπὸ τοῦ ΒΔ τετράγωνον.

Lemma I

To find two square numbers such that the sum of them is also square.

Let the two numbers *AB* and *BC* be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder *AC* is thus even. Let *AC* be cut in half at *D*. And let *AB* and *BC* also be either similar plane (numbers), or square (numbers)—which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) *AB* and *BC*, plus the square on *CD*, is equal to the square on *BD* [Prop. 2.6]. And the (number created) from (multiplying) *AB* and *BC* is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so)

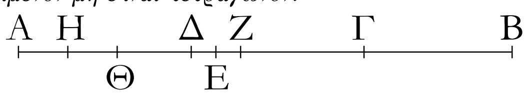
Καὶ φανερόν, ὅτι εὕρηνται πάλιν δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΓΔ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ὑπὸ ΑΒ, ΒΓ εἶναι τετράγωνον, ὅταν οἱ ΑΒ, ΒΓ ὅμοιοι ὦσιν ἐπίπεδοι. ὅταν δὲ μὴ ὦσιν ὅμοιοι ἐπίπεδοι, εὕρηνται δύο τετράγωνοι ὅ τε ἀπὸ τοῦ ΒΔ καὶ ὁ ἀπὸ τοῦ ΔΓ, ὧν ἡ ὑπεροχὴ ὁ ὑπὸ τῶν ΑΒ, ΒΓ οὐκ ἔστι τετράγωνος· ὅπερ ἔδει δεῖξαι.

created is square [Prop. 9.1]. Thus, two square numbers be found—(namely,) the (number created) from (multiplying) *AB* and *BC*, and the (square) on *CD*—which, (when) added (together), make the square on *BD*.

And (it is) clear that two square (numbers) have again been found—(namely,) the (square) on *BD*, and the (square) on *CD*—such that their difference—(namely,) the (rectangle) contained by *AB* and *BC*—is square whenever *AB* and *BC* are similar plane (numbers). But, when they are not similar plane numbers, two square (numbers) be found—(namely,) the (square) on *BD*, and the (square) on *DC*—between which the difference—(namely,) the (rectangle) contained by *AB* and *BC*—is not square. (Which is) the very thing it was required to show.

Λῆμμα β´.

Εὑρεῖν δύο τετραγώνους ἀριθμούς, ὥστε τὸν ἐξ αὐτῶν συγκείμενον μὴ εἶναι τετράγωνον.

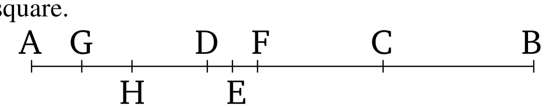


Ἔστω γὰρ ὁ ἐκ τῶν ΑΒ, ΒΓ, ὡς ἔφαμεν, τετράγωνος, καὶ ἄρτιος ὁ ΓΑ, καὶ τετμήσθω ὁ ΓΑ δίχα τῷ Δ. φανερὸν δή, ὅτι ὁ ἐκ τῶν ΑΒ, ΒΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΔ τετραγώνῳ. ἀφῃρήσθω μονὰς ἡ ΔΕ· ὁ ἄρα ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἐλάσσων ἐστὶ τοῦ ἀπὸ [τοῦ] ΒΔ τετραγώνου. λέγω οὖν, ὅτι ὁ ἐκ τῶν ΑΒ, ΒΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ οὐκ ἔσται τετράγωνος.

Εἰ γὰρ ἔσται τετράγωνος, ἤτοι ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΕ ἢ ἐλάσσων τοῦ ἀπὸ [τοῦ] ΒΕ, οὐκέτι δὲ καὶ μείζων, ἵνα μὴ τμηθῇ ἡ μονάς. ἔστω, εἰ δυνατόν, πρότερον ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΕ, καὶ ἔστω τῆς ΔΕ μονάδος διπλασίων ὁ ΗΑ. ἐπεὶ οὖν ὅλος ὁ ΑΓ ὅλου τοῦ ΓΔ ἐστι διπλασίων, ὧν ὁ ΑΗ τοῦ ΔΕ ἐστι διπλασίων, καὶ λοιπὸς ἄρα ὁ ΗΓ λοιποῦ τοῦ ΕΓ ἐστι διπλασίων· δίχα ἄρα τέτμηται ὁ ΗΓ τῷ Ε. ὁ ἄρα ἐκ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ τετραγώνῳ. ἀλλὰ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ὑπόκειται τῷ ἀπὸ [τοῦ] ΒΕ τετραγώνῳ· ὁ ἄρα ἐκ τῶν ΗΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ. καὶ κοινοῦ ἀφαιρεθέντος τοῦ ἀπὸ ΓΕ συνάγεται ὁ ΑΒ ἴσος τῷ ΗΒ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ. λέγω δή, ὅτι οὐδὲ ἐλάσσων τοῦ ἀπὸ ΒΕ. εἰ γὰρ δυνατόν, ἔστω τῷ ἀπὸ ΒΖ ἴσος, καὶ τοῦ ΔΖ διπλασίων ὁ ΘΑ. καὶ συναχθήσεται πάλιν διπλασίων ὁ ΘΓ τοῦ ΓΖ· ὥστε καὶ τὸν ΓΘ δίχα τετμῆσθαι κατὰ τὸ Ζ, καὶ διὰ τοῦτο τὸν ἐκ τῶν ΘΒ, ΒΓ μετὰ τοῦ ἀπὸ ΖΓ ἴσον γίνεσθαι τῷ ἀπὸ ΒΖ. ὑπόκειται δὲ καὶ ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΖ. ὥστε καὶ ὁ ἐκ τῶν ΘΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΖ ἴσος ἔσται τῷ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ

Lemma II

To find two square numbers such that the sum of them is not square.



For let the (number created) from (multiplying) *AB* and *BC*, as we said, be square. And (let) *CA* (be) even. And let *CA* be cut in half at *D*. So it is clear that the square (number created) from (multiplying) *AB* and *BC*, plus the square on *CD*, is equal to the square on *BD* [see previous lemma]. Let the unit *DE* be subtracted (from *BD*). Thus, the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*, is less than the square on *BD*. I say, therefore, that the square (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*, is not square.

For if it is square, it is either equal to the (square) on *BE*, or less than the (square) on *BE*, but cannot any more be greater (than the square on *BE*), lest the unit be divided. First of all, if possible, let the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*, be equal to the (square) on *BE*. And let *GA* be double the unit *DE*. Therefore, since the whole of *AC* is double the whole of *CD*, of which *AG* is double *DE*, the remainder *GC* is thus double the remainder *EC*. Thus, *GC* has been cut in half at *E*. Thus, the (number created) from (multiplying) *GB* and *BC*, plus the (square) on *CE*, is equal to the square on *BE* [Prop. 2.6]. But, the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*, was also assumed (to be) equal to the square on *BE*. Thus, the (number created) from (multiplying) *GB* and *BC*, plus the (square) on *CE*, is equal to the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*. And subtracting the (square) on *CE* from both, *AB* is inferred (to be) equal to *GB*. The very thing is absurd. Thus, the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*, is not equal to the (square) on *BE*. So I say that (it is) not less than the (square) *ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ [τῷ] ἐλάσσονι τοῦ ἀπὸ ΒΕ. ἐδείχθη δέ, ὅτι οὐδὲ [αὐτῷ] τῷ ἀπὸ ΒΕ. οὐκ ἄρα ὁ ἐκ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΕ τετράγωνός ἐστιν. ὅπερ ἔδει δεῖξαι.*

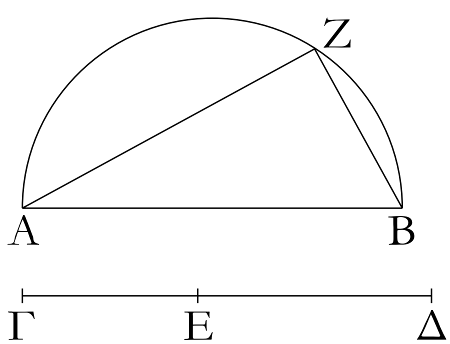
on *BE* either. For, if possible, let it be equal to the (square) on *BF*. And (let) *HA* (be) double *DF*. And it can again be inferred that *HC* (is) double *CF*. Hence, *CH* has also been cut in half at *F*. And, on account of this, the (number created) from (multiplying) *HB* and *BC*, plus the (square) on *FC*, becomes equal to the (square) on *BF* [Prop. 2.6]. And the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*, was also assumed (to be) equal to the (square) on *BF*. Hence, the (number created) from (multiplying) *HB* and *BC*, plus the (square) on *CF*, will also be equal to the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*. The very thing is absurd. Thus, the (number created) from (multiplying) *AB* and *BC*, plus the (square) on *CE*, is not equal to less than the (square) on *BE*. And it was shown that (is it) not equal to the (square) on *BE* either. Thus, the (number created) from (multiplying) *AB* and *BC*, plus the square on *CE*, is not square. (Which is) the very thing it was required to show.

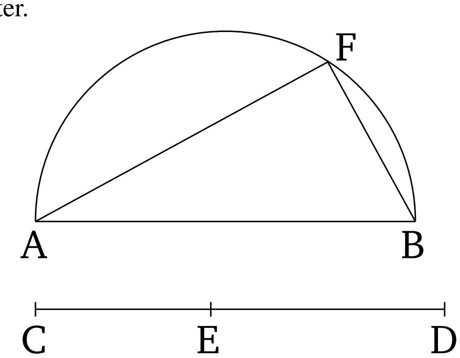
κθ'.

Εὑρεῖν δύο ῥητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

Proposition 29

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line which is) commensurable in length with the greater.





Ἐκκείσθω γάρ τις ῥητὴ ἡ ΑΒ καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΔ, ΔΕ, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ΓΕ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΑΖ τετράγωνον, καὶ ἐπεζεύχθω ἡ ΖΒ.

Ἐπεὶ [οὖν] ἐστιν ὡς τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, οὕτως ὁ ΔΓ πρὸς τὸν ΓΕ, τὸ ἀπὸ τῆς ΒΑ ἄρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὃν ἀριθμὸς ὁ ΔΓ πρὸς ἀριθμὸν τὸν ΓΕ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΖ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΑΒ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΑΖ· ῥητὴ ἄρα καὶ ἡ ΑΖ. καὶ ἐπεὶ ὁ ΔΓ πρὸς τὸν ΓΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΒΑ ἄρα πρὸς τὸ ἀπὸ τῆς ΑΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΑΖ μήκει· αἱ ΒΑ, ΑΖ ἄρα ῥηταί εἰσι δυνάμει μόνον

For let some rational (straight-line) *AB* be laid down, and two square numbers, *CD* and *DE*, such that the difference between them, *CE*, is not square [Prop. 10.28 lem. I]. And let the semi-circle *AFB* be drawn on *AB*. And let it be contrived that as *DC* (is) to *CE*, so the square on *BA* (is) to the square on *AF* [Prop. 10.6 corr.]. And let *FB* be joined.

[Therefore,] since as the (square) on *BA* is to the (square) on *AF*, so *DC* (is) to *CE*, the (square) on *BA* thus has to the (square) on *AF* the ratio which the number *DC* (has) to the number *CE*. Thus, the (square) on *BA* is commensurable with the (square) on *AF* [Prop. 10.6]. And the (square) on *AB* (is) rational [Def. 10.4]. Thus, the (square) on *AF* (is) also rational. Thus, *AF* (is) also rational. And since *DC* does not have to *CE* the ratio which (some) square number (has) to (some) square number, the (square) on *BA* thus does not have to the (square) on *AF* the ratio which (some) square number has to *σύμμετροι. καὶ ἐπεί [ἐστιν] ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ἀναστρέψαντι ἄρα ὡς ὁ ΓΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΑΒ ἄρα πρὸς τὸ ἀπὸ τῆς ΒΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΖ μήκει. καί ἐστι τὸ ἀπὸ τῆς ΑΒ ἴσον τοῖς ἀπὸ τῶν ΑΖ, ΖΒ· ἡ ΑΒ ἄρα τῆς ΑΖ μεῖζον δύναται τῇ ΒΖ συμμέτρῳ ἑαυτῇ.*

Εὕρηνται ἄρα δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΒΑ, ΑΖ, ὥστε τὴν μείζονα τὴν ΑΒ τῆς ἐλάσσονος τῆς ΑΖ μεῖζον δύνασθαι τῷ ἀπὸ τῆς ΒΖ συμμέτρου ἑαυτῇ μήκει· ὅπερ ἔδει δεῖξαι.

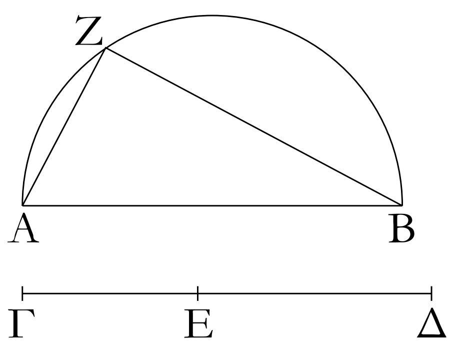
(some) square number either. Thus, *AB* is incommensurable in length with *AF* [Prop. 10.9]. Thus, the rational (straight-lines) *BA* and *AF* are commensurable in square only. And since as *DC* [is] to *CE*, so the (square) on *BA* (is) to the (square) on *AF*, thus, via conversion, as *CD* (is) to *DE*, so the (square) on *AB* (is) to the (square) on *BF* [Props. 5.19 corr., 3.31, 1.47]. And *CD* has to *DE* the ratio which (some) square number (has) to (some) square number. Thus, the (square) on *AB* also has to the (square) on *BF* the ratio which (some) square number has to (some) square number. *AB* is thus commensurable in length with *BF* [Prop. 10.9]. And the (square) on *AB* is equal to the (sum of the squares) on *AF* and *FB* [Prop. 1.47]. Thus, the square on *AB* is greater than (the square on) *AF* by (the square on) *BF*, (which is) commensurable (in length) with (*AB*).

Thus, two rational (straight-lines), *BA* and *AF*, commensurable in square only, be found such that the square on the greater, *AB*, is larger than (the square on) the lesser, *AF*, by the (square) on *BF*, (which is) commensurable in length with (*AB*).† (Which is) the very thing it was required to show.

† *BA* and *AF* have lengths 1 and $\sqrt{1-k^2}$ times that of *AB*, respectively, where $k = \sqrt{DE/CD}$.

λ΄.

Εὑρεῖν δύο ῥητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει.

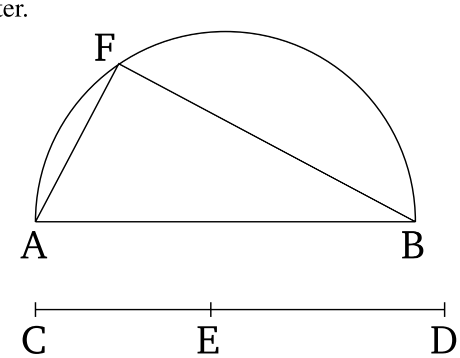


Ἐκκείσθω ῥητὴ ἡ ΑΒ καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓΕ, ΕΔ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΓΔ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ πεποιήσθω ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, καὶ ἐπεζεύχθω ἡ ΖΒ.

Ὁμοίως δὴ δείξομεν τῷ πρὸ τούτου, ὅτι αἱ ΒΑ, ΑΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί ἐστιν ὡς ὁ ΔΓ πρὸς τὸν ΓΕ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ, ἀναστρέψαντι ἄρα ὡς ὁ ΓΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ. ὁ δὲ ΓΔ πρὸς τὸν ΔΕ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ λόγον ἔχει, ὃν τετράγωνος

Proposition 30

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (the square on) lesser by the (square) on (some straight-line which is) incommensurable in length with the greater.



Let the rational (straight-line) *AB* be laid out, and the two square numbers, *CE* and *ED*, such that the sum of them, *CD*, is not square [Prop. 10.28 lem. II]. And let the semi-circle *AFB* be drawn on *AB*. And let it be contrived that as *DC* (is) to *CE*, so the (square) on *BA* (is) to the (square) on *AF* [Prop. 10.6 corr]. And let *FB* be joined.

So, similarly to the (proposition) before this, we can show that *BA* and *AF* are rational (straight-lines which are) commensurable in square only. And since as *DC* is to *CE*, so the (square) on *BA* (is) to the (square) on *AF*, thus, via conversion, as *CD* (is) to *DE*, so the (square) on *AB* (is) to the (square) on *BF* [Props. 5.19 corr., 3.31, 1.47]. And *CD* does not have *ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΖ μήκει. καὶ δύναται ἡ ΑΒ τῆς ΑΖ μεῖζον τῷ ἀπὸ τῆς ΖΒ ἀσυμμέτρου ἑαυτῇ.*

Αἱ ΑΒ, ΑΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΒ τῆς ΑΖ μεῖζον δύναται τῷ ἀπὸ τῆς ΖΒ ἀσυμμέτρου ἑαυτῇ μήκει· ὅπερ ἔδει δεῖξαι.

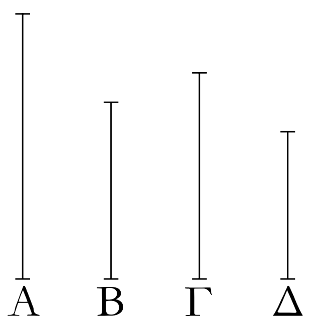
to DE the ratio which (some) square number (has) to (some) square number. Thus, the (square) on AB does not have to the (square) on BF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with BF [Prop. 10.9]. And the square on AB is greater than the (square on) AF by the (square) on FB [Prop. 1.47], (which is) incommensurable (in length) with (AB).

Thus, AB and AF are rational (straight-lines which are) commensurable in square only, and the square on AB is greater than (the square on) AF by the (square) on FB, (which is) incommensurable (in length) with (AB).† (Which is) the very thing it was required to show.

† AB and AF have lengths 1 and $1/\sqrt{1+k^2}$ times that of AB, respectively, where $k = \sqrt{DE/CE}$.

λα´.

Εὑρεῖν δύο μέσας δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

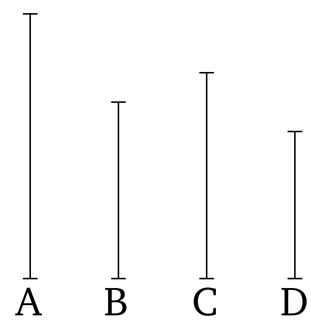


Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, ὥστε τὴν Α μείζονα οὖσαν τῆς ἐλάσσονος τῆς Β μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καὶ τῷ ὑπὸ τῶν Α, Β ἴσον ἔστω τὸ ἀπὸ τῆς Γ. μέσον δὲ τὸ ὑπὸ τῶν Α, Β· μέσον ἄρα καὶ τὸ ἀπὸ τῆς Γ· μέση ἄρα καὶ ἡ Γ. τῷ δὲ ἀπὸ τῆς Β ἴσον ἔστω τὸ ὑπὸ τῶν Γ, Δ· ῥητὸν δὲ τὸ ἀπὸ τῆς Β· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν Γ, Δ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως τὸ ὑπὸ τῶν Α, Β πρὸς τὸ ἀπὸ τῆς Β, ἀλλὰ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἐστὶ τὸ ἀπὸ τῆς Γ, τῷ δὲ ἀπὸ τῆς Β ἴσον τὸ ὑπὸ τῶν Γ, Δ, ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ. ὡς δὲ τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ, Δ, οὕτως ἡ Γ πρὸς τὴν Δ· καὶ ὡς ἄρα ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ. σύμμετρος δὲ ἡ Α τῇ Β δυνάμει μόνον· σύμμετρος ἄρα καὶ ἡ Γ τῇ Δ δυνάμει μόνον. καί ἐστι μέση ἡ Γ· μέση ἄρα καὶ ἡ Δ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, ἡ Γ πρὸς τὴν Δ, ἡ δὲ Α τῆς Β μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ Γ ἄρα τῆς Δ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Γ,

Proposition 31

To find two medial (straight-lines), commensurable in square only, (and) containing a rational (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable in length with the greater.



Let two rational (straight-lines), A and B, commensurable in square only, be laid out, such that the square on the greater A is larger than the (square on the) lesser B by the (square) on (some straight-line) commensurable in length with (A) [Prop. 10.29]. And let the (square) on C be equal to the (rectangle contained) by A and B. And the (rectangle contained by) A and B (is) medial [Prop. 10.21]. Thus, the (square) on C (is) also medial. Thus, C (is) also medial [Prop. 10.21]. And let the (rectangle contained) by C and D be equal to the (square) on B. And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D (is) also rational. And since as A is to B, so the (rectangle contained) by A and B (is) to the (square) on B [Prop. 10.21 lem.], but the (square) on C is equal to the (rectangle contained) by A and B, and the (rectangle contained) by C and D to the (square) on B, thus as A (is) to B, so the (square) on C (is) to the (rectangle contained) by C and D. And as the (square) on C (is) to the (rectangle contained) by C and D, so C (is) to D [Prop. 10.21 lem.]. And thus as A (is) to B, so C (is) to D. And A is commensurable in square only

Δ ῥητὸν περιέχουσαι, καὶ ἡ Γ τῆς Δ μεῖζον δυνάται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

Ὁμοίως δὴ δειχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Β μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

with *B*. Thus, *C* (is) also commensurable in square only with *D* [Prop. 10.11]. And *C* is medial. Thus, *D* (is) also medial [Prop. 10.23]. And since as *A* is to *B*, (so) *C* (is) to *D*, and the square on *A* is greater than (the square on) *B* by the (square) on (some straight-line) commensurable (in length) with (*A*), the square on *C* is thus also greater than (the square on) *D* by the (square) on (some straight-line) commensurable (in length) with (*C*) [Prop. 10.14].

Thus, two medial (straight-lines), *C* and *D*, commensurable in square only, (and) containing a rational (area), be found. And the square on *C* is greater than (the square on) *D* by the (square) on (some straight-line) commensurable in length with (*C*).†

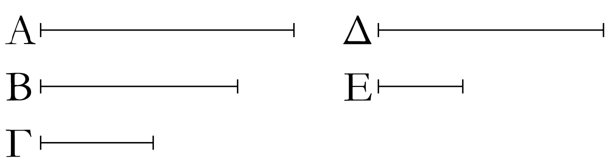
So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with *C*), provided that the square on *A* is greater than (the square on *B*) by the (square) on (some straight-line) incommensurable (in length) with (*A*) [Prop. 10.30].‡

† *C* and *D* have lengths $(1-k^2)^{1/4}$ and $(1-k^2)^{3/4}$ times that of *A*, respectively, where *k* is defined in the footnote to Prop. 10.29.

‡ *C* and *D* would have lengths $1/(1+k^2)^{1/4}$ and $1/(1+k^2)^{3/4}$ times that of *A*, respectively, where *k* is defined in the footnote to Prop. 10.30.

λβ′.

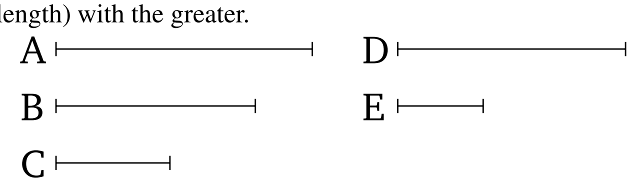
Εὑρεῖν δύο μέσας δυνάμει μόνον συμμέτρους μέσον περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ.



Ἐκκείσθωσαν τρεῖς ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, Γ, ὥστε τὴν Α τῆς Γ μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἔστω τὸ ἀπὸ τῆς Δ. μέσον ἄρα τὸ ἀπὸ τῆς Δ· καὶ ἡ Δ ἄρα μέση ἐστίν. τῷ δὲ ὑπὸ τῶν Β, Γ ἴσον ἔστω τὸ ὑπὸ τῶν Δ, Ε. καὶ ἐπεί ἐστιν ὡς τὸ ὑπὸ τῶν Α, Β πρὸς τὸ ὑπὸ τῶν Β, Γ, οὕτως ἡ Α πρὸς τὴν Γ, ἀλλὰ τῷ μὲν ὑπὸ τῶν Α, Β ἴσον ἐστὶ τὸ ἀπὸ τῆς Δ, τῷ δὲ ὑπὸ τῶν Β, Γ ἴσον τὸ ὑπὸ τῶν Δ, Ε, ἔστιν ἄρα ὡς ἡ Α πρὸς τὴν Γ, οὕτως τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε. ὡς δὲ τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ, Ε, οὕτως ἡ Δ πρὸς τὴν Ε· καὶ ὡς ἄρα ἡ Α πρὸς τὴν Γ, οὕτως ἡ Δ πρὸς τὴν Ε. σύμμετρος δὲ ἡ Α τῇ Γ δυνάμει [μόνον]. σύμμετρος ἄρα καὶ ἡ Δ τῇ Ε δυνάμει μόνον. μέση δὲ ἡ Δ· μέση ἄρα καὶ ἡ Ε. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Γ, ἡ Δ πρὸς τὴν Ε, ἡ δὲ Α τῆς Γ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ Δ ἄρα τῆς Ε μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. λέγω δή, ὅτι καὶ μέσον ἐστὶ τὸ ὑπὸ τῶν Δ, Ε. ἐπεὶ γὰρ ἴσον ἐστὶ τὸ

Proposition 32

To find two medial (straight-lines), commensurable in square only, (and) containing a medial (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.



Let three rational (straight-lines), *A*, *B* and *C*, commensurable in square only, be laid out such that the square on *A* is greater than (the square on *C*) by the (square) on (some straight-line) commensurable (in length) with (*A*) [Prop. 10.29]. And let the (square) on *D* be equal to the (rectangle contained) by *A* and *B*. Thus, the (square) on *D* (is) medial. Thus, *D* is also medial [Prop. 10.21]. And let the (rectangle contained) by *D* and *E* be equal to the (rectangle contained) by *B* and *C*. And since as the (rectangle contained) by *A* and *B* is to the (rectangle contained) by *B* and *C*, so *A* (is) to *C* [Prop. 10.21 lem.], but the (square) on *D* is equal to the (rectangle contained) by *A* and *B*, and the (rectangle contained) by *D* and *E* to the (rectangle contained) by *B* and *C*, thus as *A* is to *C*, so the (square) on *D* (is) to the (rectangle contained) by *D* and *E*. And as the (square) on *D* (is) to the (rectangle contained) by *D* and *E*, so *D* (is) to *E* [Prop. 10.21 lem.]. And thus as *A* (is) to *C*, so *D* (is) to *E*. And *A* (is) commensurable *ὑπὸ τῶν Β, Γ τῷ ὑπὸ τῶν Δ, Ε, μέσον δὲ τὸ ὑπὸ τῶν Β, Γ [αἱ γὰρ Β, Γ ῥηταί εἰσι δυνάμει μόνον σύμμετροι], μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ, Ε.*

Εὕρηνται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Δ, Ε μέσον περιέχουσαι, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Ὁμοίως δὴ πάλιν διεχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ Α τῆς Γ μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῃ.

in square [only] with C. Thus, D (is) also commensurable in square only with E [Prop. 10.11]. And D (is) medial. Thus, E (is) also medial [Prop. 10.23]. And since as A is to C, (so) D (is) to E, and the square on A is greater than (the square on) C by the (square) on (some straight-line) commensurable (in length) with (A), the square on D will thus also be greater than (the square on) E by the (square) on (some straight-line) commensurable (in length) with (D) [Prop. 10.14]. So, I also say that the (rectangle contained) by D and E is medial. For since the (rectangle contained) by B and C is equal to the (rectangle contained) by D and E, and the (rectangle contained) by B and C (is) medial [for B and C are rational (straight-lines which are) commensurable in square only] [Prop. 10.21], the (rectangle contained) by D and E (is) thus also medial.

Thus, two medial (straight-lines), D and E, commensurable in square only, (and) containing a medial (area), be found such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.†.

So, similarly, (the proposition) can again also be demonstrated for (some straight-line) incommensurable (in length with the greater), provided that the square on A is greater than (the square on) C by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].‡

† D and E have lengths $k'^{1/4}$ and $k'^{1/4}\sqrt{1-k^2}$ times that of A, respectively, where the length of B is $k'^{1/2}$ times that of A, and k is defined in the footnote to Prop. 10.29.

‡ D and E would have lengths $k'^{1/4}$ and $k'^{1/4}/\sqrt{1+k^2}$ times that of A, respectively, where the length of B is $k'^{1/2}$ times that of A, and k is defined in the footnote to Prop. 10.30.

Λῆμμα.

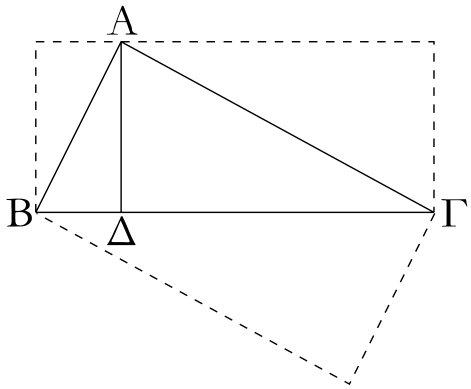
Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν Α, καὶ ἤχθω κάθετος ἡ ΑΔ· λέγω, ὅτι τὸ μὲν ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΑ, τὸ δὲ ὑπὸ τῶν ΒΓΑ ἴσον τῷ ἀπὸ τῆς ΓΑ, καὶ τὸ ὑπὸ τῶν ΒΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΑΔ, καὶ ἔτι τὸ ὑπὸ τῶν ΒΓ, ΑΔ ἴσον [ἐστὶ] τῷ ὑπὸ τῶν ΒΑ, ΑΓ.

Καὶ πρῶτον, ὅτι τὸ ὑπὸ τῶν ΓΒΔ ἴσον [ἐστὶ] τῷ ἀπὸ τῆς ΒΑ.

Lemma

Let ABC be a right-angled triangle having the (angle) A a right-angle. And let the perpendicular AD be drawn. I say that the (rectangle contained) by CBD is equal to the (square) on BA, and the (rectangle contained) by BCD (is) equal to the (square) on CA, and the (rectangle contained) by BD and DC (is) equal to the (square) on AD, and, further, the (rectangle contained) by BC and AD [is] equal to the (rectangle contained) by BA and AC.

And, first of all, (let us prove) that the (rectangle contained) by CBD [is] equal to the (square) on BA.

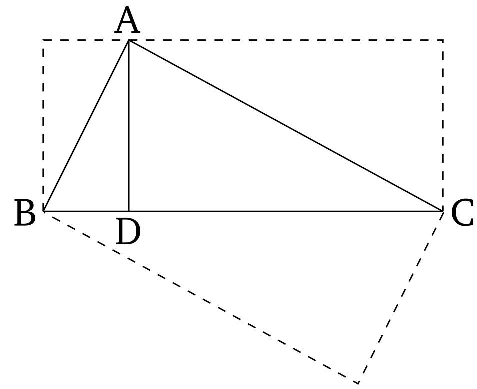


Ἐπεὶ γὰρ ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΑΔ, τὰ ΑΒΔ, ΑΔΓ ἄρα τρίγωνα ὅμοιά ἐστι τῷ τε ὅλῳ τῷ ΑΒΓ καὶ ἀλλήλοις. καὶ ἐπεὶ ὅμοιόν ἐστι τὸ ΑΒΓ τρίγωνον τῷ ΑΒΔ τριγώνῳ, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΒΔ· τὸ ἄρα ὑπὸ τῶν ΓΒΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ.

Διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν ΒΓΔ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ.

Καὶ ἐπεί, ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΑΔ πρὸς τὴν ΔΓ· τὸ ἄρα ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΑ.

Λέγω, ὅτι καὶ τὸ ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ. ἐπεὶ γάρ, ὡς ἔφαμεν, ὅμοιόν ἐστι τὸ ΑΒΓ τῷ ΑΒΔ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΔ. τὸ ἄρα ὑπὸ τῶν ΒΓ, ΑΔ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΑ, ΑΓ· ὅπερ ἔδει δεῖξαι.



For since *AD* has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, *ABD* and *ADC* are thus triangles (which are) similar to the whole, *ABC*, and to one another [Prop. 6.8]. And since triangle *ABC* is similar to triangle *ABD*, thus as *CB* is to *BA*, so *BA* (is) to *BD* [Prop. 6.4]. Thus, the (rectangle contained) by *CBD* is equal to the (square) on *AB* [Prop. 6.17].

So, for the same (reasons), the (rectangle contained) by *BCD* is also equal to the (square) on *AC*.

And since if a (straight-line) is drawn from the right-angle in a right-angled triangle, perpendicular to the base, the (straight-line so) drawn is the mean proportional to the pieces of the base [Prop. 6.8 corr.], thus as *BD* is to *DA*, so *AD* (is) to *DC*. Thus, the (rectangle contained) by *BD* and *DC* is equal to the (square) on *DA* [Prop. 6.17].

I also say that the (rectangle contained) by *BC* and *AD* is equal to the (rectangle contained) by *BA* and *AC*. For since, as we said, *ABC* is similar to *ABD*, thus as *BC* is to *CA*, so *BA* (is) to *AD* [Prop. 6.4]. Thus, the (rectangle contained) by *BC* and *AD* is equal to the (rectangle contained) by *BA* and *AC* [Prop. 6.16]. (Which is) the very thing it was required to show.

λγʹ.

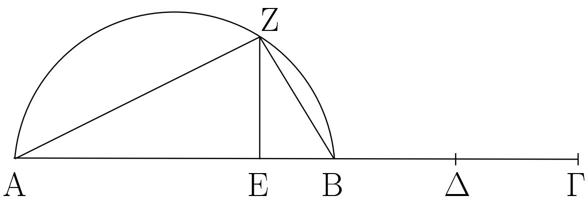
Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπʼ αὐτῶν τετραγώνων ῥητόν, τὸ δʼ ὑπʼ αὐτῶν μέσον.

Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ, ὥστε τὴν μείζονα τὴν ΑΒ τῆς ἐλάσσονος τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Δ, καὶ τῷ ἀφʼ ὁποτέρας τῶν ΒΔ, ΔΓ ἴσον παρὰ τὴν ΑΒ παραβεβλήσθω παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΕΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΖΒ, καὶ ἤχθω τῇ ΑΒ πρὸς ὀρθὰς ἡ ΕΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ.

Proposition 33

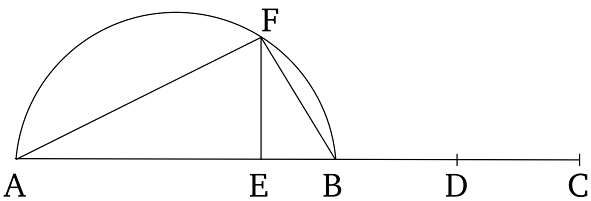
To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Let the two rational (straight-lines) *AB* and *BC*, (which are) commensurable in square only, be laid out such that the square on the greater, *AB*, is larger than (the square on) the lesser, *BC*, by the (square) on (some straight-line which is) incommensurable (in length) with (*AB*) [Prop. 10.30]. And let *BC* be cut in half at *D*. And let a parallelogram equal to the (square) on either of *BD* or *DC*, (and) falling short by a square figure, be applied to *AB* [Prop. 6.28], and let it be the (rectangle contained) by *AEB*. And let the semi-circle *AFB* be drawn on *AB*. And let *EF* be drawn at right-angles to *AB*. And let *AF* and *FB* be joined.



Καὶ ἐπεὶ [δύο] εὐθεῖαι ἄνισοί εἰσιν αἱ ΑΒ, ΒΓ, καὶ ἡ ΑΒ τῆς ΒΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς ΒΓ, τουτέστι τῷ ἀπὸ τῆς ἡμισείας αὐτῆς, ἴσον παρὰ τὴν ΑΒ παραβέβληται παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ καὶ ποιεῖ τὸ ὑπὸ τῶν ΑΕΒ, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΕ τῇ ΕΒ. καί ἐστιν ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΕ, ἴσον δὲ τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΕ τῷ ἀπὸ τῆς ΑΖ, τὸ δὲ ὑπὸ τῶν ΑΒ, ΒΕ τῷ ἀπὸ τῆς ΒΖ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΖ τῷ ἀπὸ τῆς ΖΒ· αἱ ΑΖ, ΖΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ ΑΒ ῥητή ἐστιν, ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΒ· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΖ, ΖΒ ῥητόν ἐστιν. καὶ ἐπεὶ πάλιν τὸ ὑπὸ τῶν ΑΕ, ΕΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ, ὑπόκειται δὲ τὸ ὑπὸ τῶν ΑΕ, ΕΒ καὶ τῷ ἀπὸ τῆς ΒΔ ἴσον, ἴση ἄρα ἐστὶν ἡ ΖΕ τῇ ΒΔ· διπλῆ ἄρα ἡ ΒΓ τὴς ΖΕ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τῷ ὑπὸ τῶν ΑΒ, ΕΖ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΕΖ. ἴσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΕΖ τῷ ὑπὸ τῶν ΑΖ, ΖΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΖ, ΖΒ. ἐδείχθη δὲ καὶ ῥητὸν τὸ συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΖ, ΖΒ ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ᾽ αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.



And since *AB* and *BC* are [two] unequal straight-lines, and the square on *AB* is greater than (the square on) *BC* by the (square) on (some straight-line which is) incommensurable (in length) with (*AB*). And a parallelogram, equal to one quarter of the (square) on *BC*—that is to say, (equal) to the (square) on half of it—(and) falling short by a square figure, has been applied to *AB*, and makes the (rectangle contained) by *AEB*. *AE* is thus incommensurable (in length) with *EB* [Prop. 10.18]. And as *AE* is to *EB*, so the (rectangle contained) by *BA* and *AE* (is) to the (rectangle contained) by *AB* and *BE*. And the (rectangle contained) by *BA* and *AE* (is) equal to the (square) on *AF*, and the (rectangle contained) by *AB* and *BE* to the (square) on *BF* [Prop. 10.32 lem.]. The (square) on *AF* is thus incommensurable with the (square) on *FB* [Prop. 10.11]. Thus, *AF* and *FB* are incommensurable in square. And since *AB* is rational, the (square) on *AB* is also rational. Hence, the sum of the (squares) on *AF* and *FB* is also rational [Prop. 1.47]. And, again, since the (rectangle contained) by *AE* and *EB* is equal to the (square) on *EF*, and the (rectangle contained) by *AE* and *EB* was assumed (to be) equal to the (square) on *BD*, *FE* is thus equal to *BD*. Thus, *BC* is double *FE*. And hence the (rectangle contained) by *AB* and *BC* is commensurable with the (rectangle contained) by *AB* and *EF* [Prop. 10.6]. And the (rectangle contained) by *AB* and *BC* (is) medial [Prop. 10.21]. Thus, the (rectangle contained) by *AB* and *EF* (is) also medial [Prop. 10.23 corr.]. And the (rectangle contained) by *AB* and *EF* (is) equal to the (rectangle contained) by *AF* and *FB* [Prop. 10.32 lem.]. Thus, the (rectangle contained) by *AF* and *FB* (is) also medial. And the sum of the squares on them was also shown (to be) rational.

Thus, the two straight-lines, *AF* and *FB*, (which are) incommensurable in square, be found, making the sum of the squares on them rational, and the (rectangle contained) by them medial. (Which is) the very thing it was required to show.

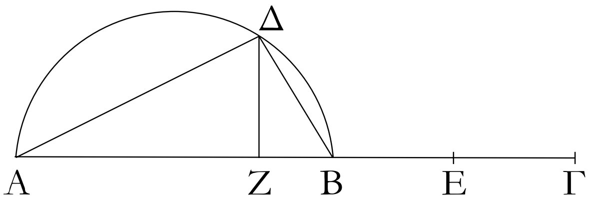
† *AF* and *FB* have lengths $\sqrt{[1 + k/(1 + k^2)^{1/2}]/2}$ and $\sqrt{[1 - k/(1 + k^2)^{1/2}]/2}$ times that of *AB*, respectively, where k is defined in the footnote to Prop. 10.30.

λδ′.

Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν.

Proposition 34

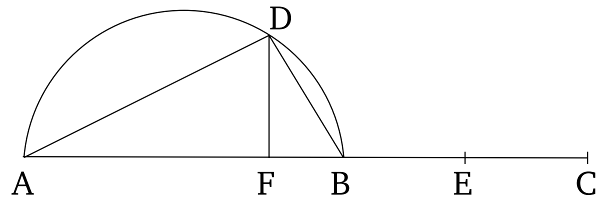
To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.



Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι τὸ ὑπ᾽ αὐτῶν, ὥστε τὴν ΑΒ τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ τὸ ΑΔΒ ἡμικύκλιον, καὶ τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε, καὶ παραβεβλήσθω παρὰ τὴν ΑΒ τῷ ἀπὸ τῆς ΒΕ ἴσον παραλληλόγραμμον ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΑΖΒ· ἀσύμμετρος ἄρα [ἐστὶν] ἡ ΑΖ τῇ ΖΒ μήκει. καὶ ἤχθω ἀπὸ τοῦ Ζ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΖΔ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΒ.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΒ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΒΑ, ΑΖ τῷ ὑπὸ τῶν ΑΒ, ΒΖ. ἴσον δὲ τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΖ τῷ ἀπὸ τῆς ΑΔ, τὸ δὲ ὑπὸ τῶν ΑΒ, ΒΖ τῷ ἀπὸ τῆς ΔΒ· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΔ τῷ ἀπὸ τῆς ΔΒ. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς ΑΒ, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΓ τῆς ΔΖ, διπλάσιον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ τοῦ ὑπὸ τῶν ΑΒ, ΖΔ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ. τὸ δὲ ὑπὸ τῶν ΑΒ, ΖΔ ἴσον τῷ ὑπὸ τῶν ΑΔ, ΔΒ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ ῥητόν ἐστιν.

Εὕρηνται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΔ, ΔΒ ποιοῦσαι τὸ [μὲν] συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν· ὅπερ ἔδει δεῖξαι.



Let the two medial (straight-lines) *AB* and *BC*, (which are) commensurable in square only, be laid out having the (rectangle contained) by them rational, (and) such that the square on *AB* is greater than (the square on) *BC* by the (square) on (some straight-line) incommensurable (in length) with (*AB*) [Prop. 10.31]. And let the semi-circle *ADB* be drawn on *AB*. And let *BC* be cut in half at *E*. And let a (rectangular) parallelogram equal to the (square) on *BE*, (and) falling short by a square figure, be applied to *AB*, (and let it be) the (rectangle contained by) *AFB* [Prop. 6.28]. Thus, *AF* [is] incommensurable in length with *FB* [Prop. 10.18]. And let *FD* be drawn from *F* at right-angles to *AB*. And let *AD* and *DB* be joined.

Since *AF* is incommensurable (in length) with *FB*, the (rectangle contained) by *BA* and *AF* is thus also incommensurable with the (rectangle contained) by *AB* and *BF* [Prop. 10.11]. And the (rectangle contained) by *BA* and *AF* (is) equal to the (square) on *AD*, and the (rectangle contained) by *AB* and *BF* to the (square) on *DB* [Prop. 10.32 lem.]. Thus, the (square) on *AD* is also incommensurable with the (square) on *DB*. And since the (square) on *AB* is medial, the sum of the (squares) on *AD* and *DB* (is) thus also medial [Props. 3.31, 1.47]. And since *BC* is double *DF* [see previous proposition], the (rectangle contained) by *AB* and *BC* (is) thus also double the (rectangle contained) by *AB* and *FD*. And the (rectangle contained) by *AB* and *BC* (is) rational. Thus, the (rectangle contained) by *AB* and *FD* (is) also rational [Prop. 10.6, Def. 10.4]. And the (rectangle contained) by *AB* and *FD* (is) equal to the (rectangle contained) by *AD* and *DB* [Prop. 10.32 lem.]. And hence the (rectangle contained) by *AD* and *DB* is rational.

Thus, two straight-lines, *AD* and *DB*, (which are) incommensurable in square, be found, making the sum of the squares on them medial, and the (rectangle contained) by them rational.† (Which is) the very thing it was required to show.

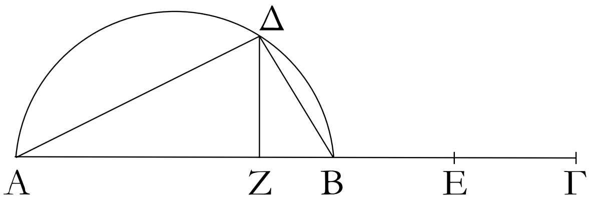
† *AD* and *DB* have lengths $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]}$ and $\sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]}$ times that of *AB*, respectively, where *k* is defined in the footnote to Prop. 10.29.

λε΄.

Εὑρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνῳ.

Proposition 35

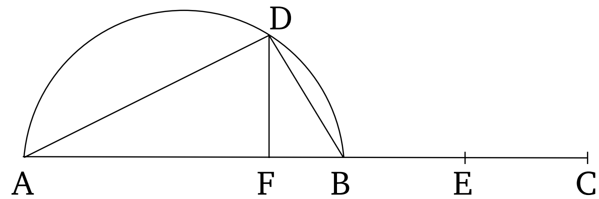
To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.



Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ μέσον περιέχουσαι, ὥστε τὴν ΑΒ τῆς ΒΓ μεῖζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ τὰ λοιπὰ γεγονέτω τοῖς ἐπάνω ὁμοίως.

Καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΒ μήκει, ἀσύμμετρός ἐστι καὶ ἡ ΑΔ τῇ ΔΒ δυνάμει. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς ΑΒ, μέσον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΖ, ΖΒ ἴσον ἐστὶ τῷ ἀφ᾽ ἑκατέρας τῶν ΒΕ, ΔΖ, ἴση ἄρα ἐστὶν ἡ ΒΕ τῇ ΔΖ· διπλῆ ἄρα ἡ ΒΓ τῆς ΖΔ· ὥστε καὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ διπλάσιόν ἐστι τοῦ ὑπὸ τῶν ΑΒ, ΖΔ. μέσον δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ. καί ἐστιν ἴσον τῷ ὑπὸ τῶν ΑΔ, ΔΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, σύμμετρος δὲ ἡ ΓΒ τῇ ΒΕ, ἀσύμμετρος ἄρα καὶ ἡ ΑΒ τῇ ΒΕ μήκει· ὥστε καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΕ ἀσύμμετρόν ἐστιν. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΕ ἴσον ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΖΔ, τουτέστι τὸ ὑπὸ τῶν ΑΔ, ΔΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ τῷ ὑπὸ τῶν ΑΔ, ΔΒ.

Εὕρηνται ἄρα δύο εὐθεῖαι αἱ ΑΔ, ΔΒ δυνάμει ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων· ὅπερ ἔδει δεῖξαι.



Let the two medial (straight-lines) *AB* and *BC*, (which are) commensurable in square only, be laid out containing a medial (area), such that the square on *AB* is greater than (the square on) *BC* by the (square) on (some straight-line) incommensurable (in length) with (*AB*) [Prop. 10.32]. And let the semicircle *ADB* be drawn on *AB*. And let the remainder (of the figure) be generated similarly to the above (proposition).

And since *AF* is incommensurable in length with *FB* [Prop. 10.18], *AD* is also incommensurable in square with *DB* [Prop. 10.11]. And since the (square) on *AB* is medial, the sum of the (squares) on *AD* and *DB* (is) thus also medial [Props. 3.31, 1.47]. And since the (rectangle contained) by *AF* and *FB* is equal to the (square) on each of *BE* and *DF*, *BE* is thus equal to *DF*. Thus, *BC* (is) double *FD*. And hence the (rectangle contained) by *AB* and *BC* is double the (rectangle) contained by *AB* and *FD*. And the (rectangle contained) by *AB* and *BC* (is) medial. Thus, the (rectangle contained) by *AB* and *FD* (is) also medial. And it is equal to the (rectangle contained) by *AD* and *DB* [Prop. 10.32 lem.]. Thus, the (rectangle contained) by *AD* and *DB* (is) also medial. And since *AB* is incommensurable in length with *BC*, and *CB* (is) commensurable (in length) with *BE*, *AB* (is) thus also incommensurable in length with *BE* [Prop. 10.13]. And hence the (square) on *AB* is also incommensurable with the (rectangle contained) by *AB* and *BE* [Prop. 10.11]. But the (sum of the squares) on *AD* and *DB* is equal to the (square) on *AB* [Prop. 1.47]. And the (rectangle contained) by *AB* and *FD*—that is to say, the (rectangle contained) by *AD* and *DB*—is equal to the (rectangle contained) by *AB* and *BE*. Thus, the sum of the (squares) on *AD* and *DB* is incommensurable with the (rectangle contained) by *AD* and *DB*.

Thus, two straight-lines, *AD* and *DB*, (which are) incommensurable in square, be found, making the sum of the (squares) on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.† (Which is) the very thing it was required to show.

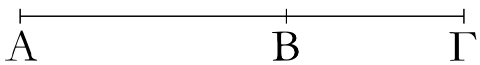
† *AD* and *DB* have lengths $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2}$ and $k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2}$ times that of *AB*, respectively, where k and k' are defined in the footnote to Prop. 10.32.

λϛ΄.

Ἐὰν δύο ῥηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων.

Proposition 36

If two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is



Συγκείσθωσαν γὰρ δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ὑπὸ τῶν ΑΒΓ πρὸς τὸ ἀπὸ τῆς ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ τῷ ἀπὸ τῆς ΒΓ. ἀλλὰ τῷ μὲν ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, τῷ δὲ ἀπὸ τῆς ΒΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ· αἱ γὰρ ΑΒ, ΒΓ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ. καὶ συνθέντι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μετὰ τῶν ἀπὸ τῶν ΑΒ, ΒΓ, τουτέστι τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ· ῥητὸν δὲ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς ΑΓ· ὥστε καὶ ἡ ΑΓ ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι.

irrational—let it be called a binomial (straight-line).†

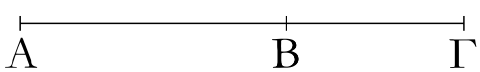
For let the two rational (straight-lines), *AB* and *BC*, (which are) commensurable in square only, be laid down together. I say that the whole (straight-line), *AC*, is irrational. For since *AB* is incommensurable in length with *BC*—for they are commensurable in square only—and as *AB* (is) to *BC*, so the (rectangle contained) by *ABC* (is) to the (square) on *BC*, the (rectangle contained) by *AB* and *BC* is thus incommensurable with the (square) on *BC* [Prop. 10.11]. But, twice the (rectangle contained) by *AB* and *BC* is commensurable with the (rectangle contained) by *AB* and *BC* [Prop. 10.6]. And (the sum of) the (squares) on *AB* and *BC* is commensurable with the (square) on *BC*—for the rational (straight-lines) *AB* and *BC* are commensurable in square only [Prop. 10.15]. Thus, twice the (rectangle contained) by *AB* and *BC* is incommensurable with (the sum of) the (squares) on *AB* and *BC* [Prop. 10.13]. And, via composition, twice the (rectangle contained) by *AB* and *BC*, plus (the sum of) the (squares) on *AB* and *BC*—that is to say, the (square) on *AC* [Prop. 2.4]—is incommensurable with the sum of the (squares) on *AB* and *BC* [Prop. 10.16]. And the sum of the (squares) on *AB* and *BC* (is) rational. Thus, the (square) on *AC* [is] irrational [Def. 10.4]. Hence, *AC* is also irrational [Def. 10.4]—let it be called a binomial (straight-line).‡ (Which is) the very thing it was required to show.

† Literally, “from two names”.

‡ Thus, a binomial straight-line has a length expressible as $1 + k^{1/2}$ [or, more generally, $\rho\,(1 + k^{1/2})$, where ρ is rational—the same proviso applies to the definitions in the following propositions]. The binomial and the corresponding apotome, whose length is expressible as $1 - k^{1/2}$ (see Prop. 10.73), are the positive roots of the quartic $x^4 - 2\,(1 + k)\,x^2 + (1 - k)^2 = 0$.

λζ΄.

Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ῥητὸν περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων πρώτη.

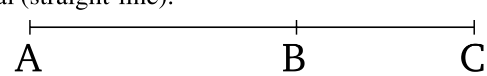


Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι· λέγω, ὅτι ὅλη ἡ ΑΓ ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ ἄρα ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ συνθέντι τὰ ἀπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ ὑπὸ τῶν ΑΒ, ΒΓ· ὑπόκεινται γὰρ αἱ ΑΒ, ΒΓ ῥητὸν περιέχουσαι· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· ἄλογος ἄρα ἡ ΑΓ, καλείσθω δὲ ἐκ δύο μέσων πρώτη· ὅπερ ἔδει δεῖξαι.

Proposition 37

If two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together then the whole (straight-line) is irrational—let it be called a first bimedial (straight-line).†



For let the two medial (straight-lines), *AB* and *BC*, commensurable in square only, (and) containing a rational (area), be laid down together. I say that the whole (straight-line), *AC*, is irrational.

For since *AB* is incommensurable in length with *BC*, (the sum of) the (squares) on *AB* and *BC* is thus also incommensurable with twice the (rectangle contained) by *AB* and *BC* [see previous proposition]. And, via composition, (the sum of) the (squares) on *AB* and *BC*, plus twice the (rectangle contained) by *AB* and *BC*—that is, the (square) on *AC* [Prop. 2.4]—is incommensurable with the (rectangle contained) by *AB* and *BC*

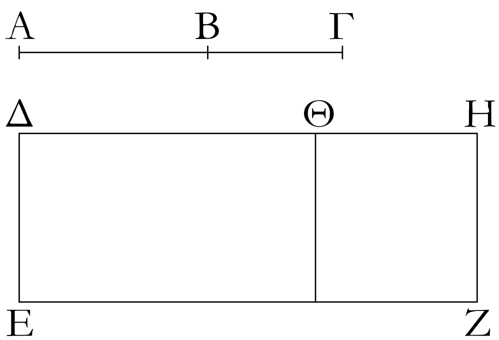
[Prop. 10.16]. And the (rectangle contained) by *AB* and *BC* (is) rational—for *AB* and *BC* were assumed to enclose a rational (area). Thus, the (square) on *AC* (is) irrational. Thus, *AC* (is) irrational [Def. 10.4]—let it be called a first bimedial (straight-line).‡ (Which is) the very thing it was required to show.

† Literally, "first from two medials".

‡ Thus, a first bimedial straight-line has a length expressible as $k^{1/4} + k^{3/4}$. The first bimedial and the corresponding first apotome of a medial, whose length is expressible as $k^{1/4} - k^{3/4}$ (see Prop. 10.74), are the positive roots of the quartic $x^4 - 2\sqrt{k}\,(1+k)\,x^2 + k\,(1-k)^2 = 0$.

λη'.

Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι μέσον περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων δυετέρα.

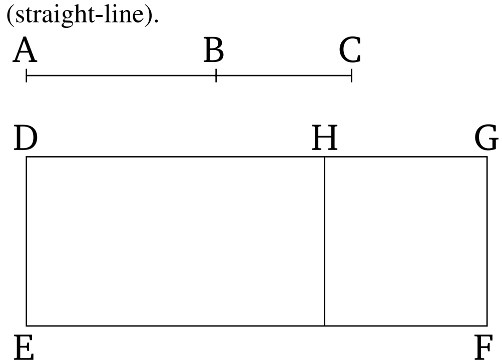


Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ ΑΒ, ΒΓ μέσον περιέχουσαι· λέγω, ὅτι ἄλογός ἐστιν ἡ ΑΓ.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΓ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΑΓ ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΒ, ΒΓ καὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, παραβεβλήσθω δὴ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ παρὰ τὴν ΔΕ ἴσον τὸ ΕΘ· λοιπὸν ἄρα τὸ ΘΖ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐπεὶ μέση ἐστὶν ἑκατέρα τῶν ΑΒ, ΒΓ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ. μέσον δὲ ὑπόκειται καὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΖΘ· μέσον ἄρα ἑκάτερον τῶν ΕΘ, ΘΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΘ, ΘΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. ἐπεὶ οὖν ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΕΘ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΘΖ. ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΘ τῷ ΘΖ· ὥστε καὶ ἡ ΔΘ τῇ ΘΗ ἐστιν ἀσύμμετρος μήκει. αἱ ΔΘ,

Proposition 38

If two medial (straight-lines), commensurable in square only, which contain a medial (area), are added together then the whole (straight-line) is irrational—let it be called a second bimedial (straight-line).



For let the two medial (straight-lines), *AB* and *BC*, commensurable in square only, (and) containing a medial (area), be laid down together [Prop. 10.28]. I say that *AC* is irrational.

For let the rational (straight-line) *DE* be laid down, and let (the rectangle) *DF*, equal to the (square) on *AC*, be applied to *DE*, making *DG* as breadth [Prop. 1.44]. And since the (square) on *AC* is equal to (the sum of) the (squares) on *AB* and *BC*, plus twice the (rectangle contained) by *AB* and *BC* [Prop. 2.4], so let (the rectangle) *EH*, equal to (the sum of) the squares on *AB* and *BC*, be applied to *DE*. The remainder *HF* is thus equal to twice the (rectangle contained) by *AB* and *BC*. And since *AB* and *BC* are each medial, (the sum of) the squares on *AB* and *BC* is thus also medial.‡ And twice the (rectangle contained) by *AB* and *BC* was also assumed (to be) medial. And *EH* is equal to (the sum of) the squares on *AB* and *BC*, and *FH* (is) equal to twice the (rectangle contained) by *AB* and *BC*. Thus, *EH* and *HF* (are) each medial. And they were applied to the rational (straight-line) *DE*. Thus, *DH* and *HG* are each rational, and incommensurable in length with *DE* [Prop. 10.22]. Therefore, since *AB* is incommensurable in length with *BC*, and as *AB* is to *BC*, so the (square) on *AB* (is) to the (rectangle contained) by *AB* and *BC* [Prop. 10.21 lem.], the (square) on *AB* is thus incommensurable with the (rectangle contained) by *AB* and *BC* [Prop. 10.11]. But, the sum of

ΘΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ὥστε ἡ ΔΗ ἄλογός ἐστιν. ῥητὴ δὲ ἡ ΔΕ· τὸ δὲ ὑπὸ ἀλόγου καὶ ῥητῆς περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν· ἄλογον ἄρα ἐστὶ τὸ ΔΖ χωρίον, καὶ ἡ δυναμένη [αὐτὸ] ἄλογός ἐστιν. δύναται δὲ τὸ ΔΖ ἡ ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ, καλείσθω δὲ ἐκ δύο μέσων δευτέρα. ὅπερ ἔδει δεῖξαι.

the squares on *AB* and *BC* is commensurable with the (square) on *AB* [Prop. 10.15], and twice the (rectangle contained) by *AB* and *BC* is commensurable with the (rectangle contained) by *AB* and *BC* [Prop. 10.6]. Thus, the sum of the (squares) on *AB* and *BC* is incommensurable with twice the (rectangle contained) by *AB* and *BC* [Prop. 10.13]. But, *EH* is equal to (the sum of) the squares on *AB* and *BC*, and *HF* is equal to twice the (rectangle) contained by *AB* and *BC*. Thus, *EH* is incommensurable with *HF*. Hence, *DH* is also incommensurable in length with *HG* [Props. 6.1, 10.11]. Thus, *DH* and *HG* are rational (straight-lines which are) commensurable in square only. Hence, *DG* is irrational [Prop. 10.36]. And *DE* (is) rational. And the rectangle contained by irrational and rational (straight-lines) is irrational [Prop. 10.20]. The area *DF* is thus irrational, and (so) the square-root [of it] is irrational [Def. 10.4]. And *AC* is the square-root of *DF*. *AC* is thus irrational—let it be called a second bimedial (straight-line).[§] (Which is) the very thing it was required to show.

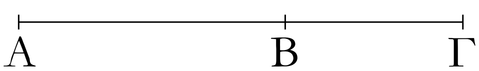
[†] Literally, "second from two medials".

[‡] Since, by hypothesis, the squares on *AB* and *BC* are commensurable—see Props. 10.15, 10.23.

[§] Thus, a second bimedial straight-line has a length expressible as $k^{1/4} + k'^{1/2}/k^{1/4}$. The second bimedial and the corresponding second apotome of a medial, whose length is expressible as $k^{1/4} - k'^{1/2}/k^{1/4}$ (see Prop. 10.75), are the positive roots of the quartic $x^4 - 2\,[(k+k')/\sqrt{k}\,]\,x^2 + [(k-k')^2/k] = 0$.

λθ'.

Ἐὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ μείζων.



Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΒ, ΒΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ ΑΓ.

Ἐπεὶ γὰρ τὸ ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν, καὶ τὸ δὶς [ἄρα] ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν. τὸ δὲ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ ῥητόν· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ· ὥστε καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΑΓ, ἀσύμμετρόν ἐστι τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ [ῥητὸν δὲ τὸ συγμείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ]· ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ. ὥστε καὶ ἡ ΑΓ ἄλογός ἐστιν, καλείσθω δὲ μείζων. ὅπερ ἔδει δεῖξαι.

Proposition 39

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together then the whole straight-line is irrational—let it be called a major (straight-line).

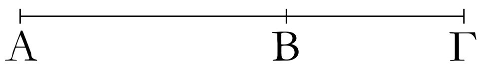
For let the two straight-lines, *AB* and *BC*, incommensurable in square, and fulfilling the prescribed (conditions), be laid down together [Prop. 10.33]. I say that *AC* is irrational.

For since the (rectangle contained) by *AB* and *BC* is medial, twice the (rectangle contained) by *AB* and *BC* is [thus] also medial [Props. 10.6, 10.23 corr.]. And the sum of the (squares) on *AB* and *BC* (is) rational. Thus, twice the (rectangle contained) by *AB* and *BC* is incommensurable with the sum of the (squares) on *AB* and *BC* [Def. 10.4]. Hence, (the sum of) the squares on *AB* and *BC*, plus twice the (rectangle contained) by *AB* and *BC*—that is, the (square) on *AC* [Prop. 2.4]—is also incommensurable with the sum of the (squares) on *AB* and *BC* [Prop. 10.16] [and the sum of the (squares) on *AB* and *BC* (is) rational]. Thus, the (square) on *AC* is irrational. Hence, *AC* is also irrational [Def. 10.4]—let it be called a major (straight-line).[†] (Which is) the very thing it was required to show.

† Thus, a major straight-line has a length expressible as $\sqrt{[1+k/(1+k^2)^{1/2}]/2} + \sqrt{[1-k/(1+k^2)^{1/2}]/2}$. The major and the corresponding minor, whose length is expressible as $\sqrt{[1+k/(1+k^2)^{1/2}]/2} - \sqrt{[1-k/(1+k^2)^{1/2}]/2}$ (see Prop. 10.76), are the positive roots of the quartic $x^4 - 2\,x^2 + k^2/(1+k^2) = 0$.

μ´.

Ἐὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ ῥητὸν καὶ μέσον δυναμένη.



Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΒ, ΒΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ ΑΓ.

Ἐπεὶ γὰρ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν, τὸ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ὥστε καὶ τὸ ἀπὸ τῆς ΑΓ ἀσύμμετρόν ἐστι τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ. ἄλογος ἄρα ἡ ΑΓ, καλείσθω δὲ ῥητὸν καὶ μέσον δυναμένη. ὅπερ ἔδει δεῖξαι.

Proposition 40

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational, are added together then the whole straight-line is irrational—let it be called the square-root of a rational plus a medial (area).

For let the two straight-lines, *AB* and *BC*, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.34]. I say that *AC* is irrational.

For since the sum of the (squares) on *AB* and *BC* is medial, and twice the (rectangle contained) by *AB* and *BC* (is) rational, the sum of the (squares) on *AB* and *BC* is thus incommensurable with twice the (rectangle contained) by *AB* and *BC*. Hence, the (square) on *AC* is also incommensurable with twice the (rectangle contained) by *AB* and *BC* [Prop. 10.16]. And twice the (rectangle contained) by *AB* and *BC* (is) rational. The (square) on *AC* (is) thus irrational. Thus, *AC* (is) irrational [Def. 10.4]—let it be called the square-root of a rational plus a medial (area).† (Which is) the very thing it was required to show.

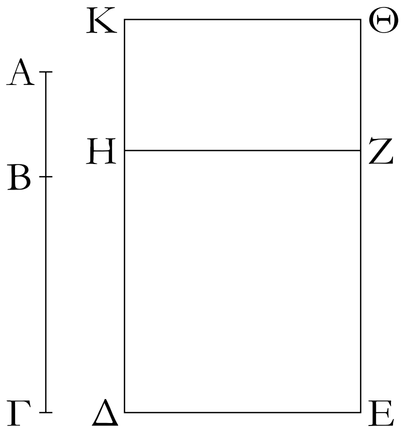
† Thus, the square-root of a rational plus a medial (area) has a length expressible as $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]} + \sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]}$. This and the corresponding irrational with a minus sign, whose length is expressible as $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]} - \sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]}$ (see Prop. 10.77), are the positive roots of the quartic $x^4 - (2/\sqrt{1+k^2})\,x^2 + k^2/(1+k^2)^2 = 0$.

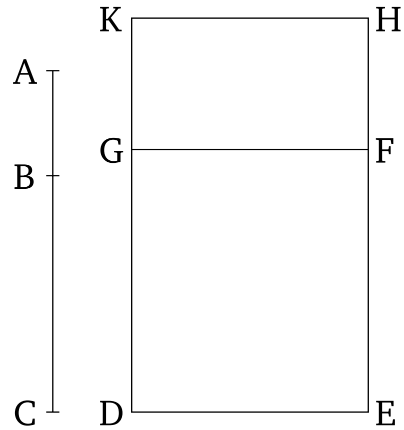
μα´.

Ἐὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ δύο μέσα δυναμένη.

Proposition 41

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them, are added together then the whole straight-line is irrational—let it be called the square-root of (the sum of) two medial (areas).





Συγκείσθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ ΑΒ, ΒΓ ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι ἡ ΑΓ ἄλογός ἐστιν.

Ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ παραβεβλήσθω παρὰ τὴν ΔΕ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΔΖ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΗΘ· ὅλον ἄρα τὸ ΔΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ. καὶ ἐπεὶ μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ, καί ἐστιν ἴσον τῷ ΔΖ, μέσον ἄρα ἐστὶ καὶ τὸ ΔΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΗΚ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΗΖ, τουτέστι τῇ ΔΕ, μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρόν ἐστι τὸ ΔΖ τῷ ΗΘ· ὥστε καὶ ἡ ΔΗ τῇ ΗΚ ἀσύμμετρός ἐστιν. καὶ εἰσι ῥηταί· αἱ ΔΗ, ΗΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἄλογος ἄρα ἐστὶν ἡ ΔΚ ἡ καλουμένη ἐκ δύο ὀνομάτων. ῥητὴ δὲ ἡ ΔΕ· ἄλογον ἄρα ἐστὶ τὸ ΔΘ καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. δύναται δὲ τὸ ΘΔ ἡ ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ, καλείσθω δὲ δύο μέσα δυναμένη. ὅπερ ἔδει δεῖξαι.

For let the two straight-lines, *AB* and *BC*, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.35]. I say that *AC* is irrational.

Let the rational (straight-line) *DE* be laid out, and let (the rectangle) *DF*, equal to (the sum of) the (squares) on *AB* and *BC*, and (the rectangle) *GH*, equal to twice the (rectangle contained) by *AB* and *BC*, be applied to *DE*. Thus, the whole of *DH* is equal to the square on *AC* [Prop. 2.4]. And since the sum of the (squares) on *AB* and *BC* is medial, and is equal to *DF*, *DF* is thus also medial. And it is applied to the rational (straight-line) *DE*. Thus, *DG* is rational, and incommensurable in length with *DE* [Prop. 10.22]. So, for the same (reasons), *GK* is also rational, and incommensurable in length with *GF*—that is to say, *DE*. And since (the sum of) the (squares) on *AB* and *BC* is incommensurable with twice the (rectangle contained) by *AB* and *BC*, *DF* is incommensurable with *GH*. Hence, *DG* is also incommensurable (in length) with *GK* [Props. 6.1, 10.11]. And they are rational. Thus, *DG* and *GK* are rational (straight-lines which are) commensurable in square only. Thus, *DK* is irrational, and that (straight-line which is) called binomial [Prop. 10.36]. And *DE* (is) rational. Thus, *DH* is irrational, and its square-root is irrational [Def. 10.4]. And *AC* (is) the square-root of *HD*. Thus, *AC* is irrational—let it be called the square-root of (the sum of) two medial (areas).† (Which is) the very thing it was required to show.

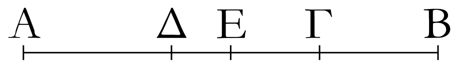
† Thus, the square-root of (the sum of) two medial (areas) has a length expressible as $k'^{1/4}\left(\sqrt{[1+k/(1+k^2)^{1/2}]/2}+\sqrt{[1-k/(1+k^2)^{1/2}]/2}\right)$. This and the corresponding irrational with a minus sign, whose length is expressible as $k'^{1/4}\left(\sqrt{[1+k/(1+k^2)^{1/2}]/2}-\sqrt{[1-k/(1+k^2)^{1/2}]/2}\right)$ (see Prop. 10.78), are the positive roots of the quartic $x^4 - 2\,k'^{1/2}\,x^2 + k'\,k^2/(1+k^2) = 0$.

Λῆμμα.

Ὅτι δὲ αἱ εἰρημέναι ἄλογοι μοναχῶς διαιροῦνται εἰς τὰς εὐθείας, ἐξ ὧν σύγκεινται ποιουσῶν τὰ προκείμενα εἴδη, δείξομεν ἤδη προεκθέμενοι λημμάτιον τοιοῦτον·

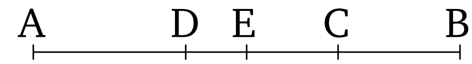
Lemma

We will now demonstrate that the aforementioned irrational (straight-lines) are uniquely divided into the straight-lines of which they are the sum, and which produce the prescribed types, (after) setting forth the following lemma.



Ἐκκείσθω εὐθεῖα ἡ ΑΒ καὶ τετμήσθω ἡ ὅλη εἰς ἄνισα καθ᾽ ἑκάτερον τῶν Γ, Δ, ὑποκείσθω δὲ μείζων ἡ ΑΓ τῆς ΔΒ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τῶν ἀπὸ τῶν ΑΔ, ΔΒ.

Τετμήσθω γὰρ ἡ ΑΒ δίχα κατὰ τὸ Ε. καὶ ἐπεὶ μείζων ἐστὶν ἡ ΑΓ τῆς ΔΒ, κοινὴ ἀφῃρήσθω ἡ ΔΓ· λοιπὴ ἄρα ἡ ΑΔ λοιπῆς τῆς ΓΒ μείζων ἐστίν. ἴση δὲ ἡ ΑΕ τῇ ΕΒ· ἐλάττων ἄρα ἡ ΔΕ τῆς ΕΓ· τὰ Γ, Δ ἄρα σημεῖα οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΒ, ἀλλὰ μὴν καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ μετὰ τοῦ ἀπὸ ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΒ, τὸ ἄρα ὑπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ μετὰ τοῦ ἀπὸ τῆς ΔΕ· ὧν τὸ ἀπὸ τῆς ΔΕ ἔλασσόν ἐστι τοῦ ἀπὸ τῆς ΕΓ· καὶ λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΓ, ΓΒ ἔλασσόν ἐστι τοῦ ὑπὸ τῶν ΑΔ, ΔΒ. ὥστε καὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἔλασσόν ἐστι τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ. καὶ λοιπὸν ἄρα τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μεῖζόν ἐστι τοῦ συγκειμένου ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ. ὅπερ ἔδει δεῖξαι.



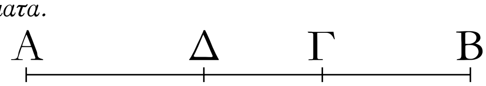
Let the straight-line *AB* be laid out, and let the whole (straight-line) be cut into unequal parts at each of the (points) *C* and *D*. And let *AC* be assumed (to be) greater than *DB*. I say that (the sum of) the (squares) on *AC* and *CB* is greater than (the sum of) the (squares) on *AD* and *DB*.

For let *AB* be cut in half at *E*. And since *AC* is greater than *DB*, let *DC* be subtracted from both. Thus, the remainder *AD* is greater than the remainder *CB*. And *AE* (is) equal to *EB*. Thus, *DE* (is) less than *EC*. Thus, points *C* and *D* are not equally far from the point of bisection. And since the (rectangle contained) by *AC* and *CB*, plus the (square) on *EC*, is equal to the (square) on *EB* [Prop. 2.5], but, moreover, the (rectangle contained) by *AD* and *DB*, plus the (square) on *DE*, is also equal to the (square) on *EB* [Prop. 2.5], the (rectangle contained) by *AC* and *CB*, plus the (square) on *EC*, is thus equal to the (rectangle contained) by *AD* and *DB*, plus the (square) on *DE*. And, of these, the (square) on *DE* is less than the (square) on *EC*. And, thus, the remaining (rectangle contained) by *AC* and *CB* is less than the (rectangle contained) by *AD* and *DB*. And, hence, twice the (rectangle contained) by *AC* and *CB* is less than twice the (rectangle contained) by *AD* and *DB*. And thus the remaining sum of the (squares) on *AC* and *CB* is greater than the sum of the (squares) on *AD* and *DB*.† (Which is) the very thing it was required to show.

† Since, $AC^2 + CB^2 + 2\,AC\,CB = AD^2 + DB^2 + 2\,AD\,DB = AB^2$.

μβ΄.

Ἡ ἐκ δύο ὀνομάτων κατὰ ἓν μόνον σημεῖον διαιρεῖται εἰς τὰ ὀνόματα.



Ἔστω ἐκ δύο ὀνομάτων ἡ ΑΒ διῃρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ· αἱ ΑΓ, ΓΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται εἰς δύο ῥητὰς δυνάμει μόνον συμμέτρους.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ ῥητὰς εἶναι δυνάμει μόνον συμμέτρους. φανερὸν δή, ὅτι ἡ ΑΓ τῇ ΔΒ οὐκ ἔστιν ἡ αὐτή. εἰ γὰρ δυνατόν, ἔστω. ἔσται δὴ καὶ ἡ ΑΔ τῇ ΓΒ ἡ αὐτή· καὶ ἔσται ὡς ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ, καὶ ἔσται ἡ ΑΒ κατὰ τὸ αὐτὸ τῇ κατὰ τὸ Γ διαιρέσει διαιρεθεῖσα καὶ κατὰ τὸ Δ· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ἡ ΑΓ τῇ ΔΒ ἐστιν ἡ αὐτή. διὰ δὴ τοῦτο καὶ τὰ Γ, Δ σημεῖα οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. ᾧ ἄρα διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ, τούτῳ διαφέρει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ διὰ τὸ καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ μετὰ τοῦ δὶς ὑπὸ τῶν ΑΔ,

Proposition 42

A binomial (straight-line) can be divided into its (component) terms at one point only.†

Let *AB* be a binomial (straight-line) which has been divided into its (component) terms at *C*. *AC* and *CB* are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. I say that *AB* cannot be divided at another point into two rational (straight-lines which are) commensurable in square only.

For, if possible, let it also be divided at *D*, such that *AD* and *DB* are also rational (straight-lines which are) commensurable in square only. So, (it is) clear that *AC* is not the same as *DB*. For, if possible, let it be (the same). So, *AD* will also be the same as *CB*. And as *AC* will be to *CB*, so *BD* (will be) to *DA*. And *AB* will (thus) also be divided at *D* in the same (manner) as the division at *C*. The very opposite was assumed. Thus, *AC* is not the same as *DB*. So, on account of this, points *C* and *D* are not equally far from the point of bisection. Thus, by whatever (amount the sum of) the (squares) on

ΔΒ ἴσα εἶναι τῷ ἀπὸ τῆς ΑΒ. ἀλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ διαφέρει ῥητῷ· ῥητὰ γὰρ ἀμφότερα· καὶ τὸ δὶς ἄρα ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ διαφέρει ῥητῷ μέσα ὄντα· ὅπερ ἄτοπον· μέσον γὰρ μέσου οὐχ ὑπερέχει ῥητῷ.

Οὐχ ἄρα ἡ ἐκ δύο ὀνομάτων κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· καθ᾽ ἓν ἄρα μόνον· ὅπερ ἔδει δεῖξαι.

AC and *CB* differs from (the sum of) the (squares) on *AD* and *DB*, twice the (rectangle contained) by *AD* and *DB* also differs from twice the (rectangle contained) by *AC* and *CB* by this (same amount)—on account of both (the sum of) the (squares) on *AC* and *CB*, plus twice the (rectangle contained) by *AC* and *CB*, and (the sum of) the (squares) on *AD* and *DB*, plus twice the (rectangle contained) by *AD* and *DB*, being equal to the (square) on *AB* [Prop. 2.4]. But, (the sum of) the (squares) on *AC* and *CB* differs from (the sum of) the (squares) on *AD* and *DB* by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle contained) by *AD* and *DB* also differs from twice the (rectangle contained) by *AC* and *CB* by a rational (area, despite both) being medial (areas) [Prop. 10.21]. The very thing is absurd. For a medial (area) cannot exceed a medial (area) by a rational (area) [Prop. 10.26].

Thus, a binomial (straight-line) cannot be divided (into its component terms) at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show.

† In other words, $k + k'^{1/2} = k'' + k'''^{1/2}$ has only one solution: i.e., $k'' = k$ and $k''' = k'$. Likewise, $k^{1/2} + k'^{1/2} = k''^{1/2} + k'''^{1/2}$ has only one solution: i.e., $k'' = k$ and $k''' = k'$ (or, equivalently, $k'' = k'$ and $k''' = k$).

μγ΄.

Ἡ ἐκ δύο μέσων πρώτη καθ᾽ ἓν μόνον σημεῖον διαιρεῖται.

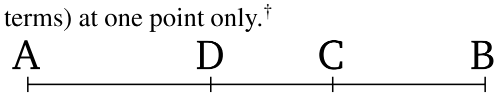
Ἔστω ἐκ δύο μέσων πρώτη ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ μέσας εἶναι δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας· λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατὸν διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ μέσας εἶναι δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας. ἐπεὶ οὖν, ᾧ διαφέρει τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τούτῳ διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ, ῥητῷ δὲ διαφέρει τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· ῥητὰ γὰρ ἀμφότερα· ῥητῷ ἄρα διαφέρει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ μέσα ὄντα· ὅπερ ἄτοπον.

Οὐκ ἄρα ἡ ἐκ δύο μέσων πρώτη κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται εἰς τὰ ὀνόματα· καθ᾽ ἓν ἄρα μόνον· ὅπερ ἔδει δεῖξαι.

Proposition 43

A first bimedial (straight-line) can be divided (into its component terms) at one point only.†



Let *AB* be a first bimedial (straight-line) which has been divided at *C*, such that *AC* and *CB* are medial (straight-lines), commensurable in square only, (and) containing a rational (area) [Prop. 10.37]. I say that *AB* cannot be (so) divided at another point.

For, if possible, let it also be divided at *D*, such that *AD* and *DB* are also medial (straight-lines), commensurable in square only, (and) containing a rational (area). Since, therefore, by whatever (amount) twice the (rectangle contained) by *AD* and *DB* differs from twice the (rectangle contained) by *AC* and *CB*, (the sum of) the (squares) on *AC* and *CB* differs from (the sum of) the (squares) on *AD* and *DB* by this (same amount) [Prop. 10.41 lem.]. And twice the (rectangle contained) by *AD* and *DB* differs from twice the (rectangle contained) by *AC* and *CB* by a rational (area). For (they are) both rational (areas). (The sum of) the (squares) on *AC* and *CB* thus differs from (the sum of) the (squares) on *AD* and *DB* by a rational (area, despite both) being medial (areas). The very thing is absurd [Prop. 10.26].

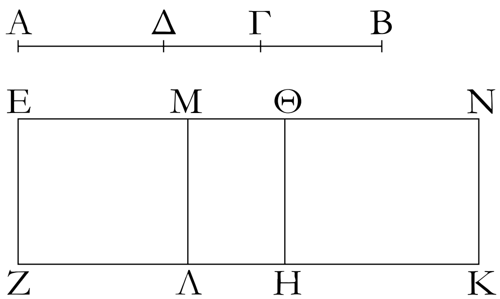
Thus, a first bimedial (straight-line) cannot be divided into its (component) terms at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show.

† In other words, $k^{1/4} + k^{3/4} = k'^{1/4} + k'^{3/4}$ has only one solution: i.e., $k' = k$.

μδ'.

Ἡ ἐκ δύο μέσων δευτέρα καθ᾽ ἓν μόνον σημεῖον διαιρεῖται.

Ἔστω ἐκ δύο μέσων δευτέρα ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ μέσας εἶναι δυνάμει μόνον συμμέτρους μέσον περιεχούσας· φανερὸν δή, ὅτι τὸ Γ οὐκ ἔστι κατὰ τῆς διχοτομίας, ὅτι οὐκ εἰσὶ μήκει σύμμετροι. λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται.

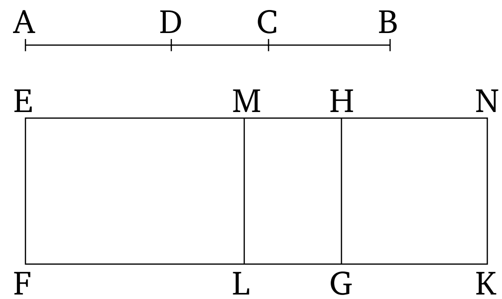


Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε τὴν ΑΓ τῇ ΔΒ μὴ εἶναι τὴν αὐτήν, ἀλλὰ μείζονα καθ᾽ ὑπόθεσιν τὴν ΑΓ· δῆλον δή, ὅτι καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ, ὡς ἐπάνω ἐδείξαμεν, ἐλάσσονα τῶν ἀπὸ τῶν ΑΓ, ΓΒ· καὶ τὰς ΑΔ, ΔΒ μέσας εἶναι δυνάμει μόνον συμμέτρους μέσον περιεχούσας. καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τῷ μὲν ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΕΖ παραλληλόγραμμον ὀρθογώνιον παραβεβλήσθω τὸ ΕΚ, τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΒ ἴσον ἀφῃρήσθω τὸ ΕΗ· λοιπὸν ἄρα τὸ ΘΚ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. πάλιν δὴ τοῖς ἀπὸ τῶν ΑΔ, ΔΒ, ἅπερ ἐλάσσονα ἐδείχθη τῶν ἀπὸ τῶν ΑΓ, ΓΒ, ἴσον ἀφῃρήσθω τὸ ΕΛ· καὶ λοιπὸν ἄρα τὸ ΜΚ ἴσον τῷ δὶς ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ μέσα ἐστὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα [καὶ] τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΕΘ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΝ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ αἱ ΑΓ, ΓΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΓ τῇ ΓΒ μήκει. ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ· δυνάμει γάρ εἰσι σύμμετροι αἱ ΑΓ, ΓΒ. τῷ δὲ ὑπὸ τῶν ΑΓ, ΓΒ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ ἄρα ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον ἐστὶ τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΘΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΗ τῷ ΘΚ· ὥστε καὶ ἡ ΕΘ τῇ ΘΝ ἀσύμμετρός ἐστι μήκει. καί εἰσι ῥηταί· αἱ ΕΘ, ΘΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ δύο ῥη-

Proposition 44

A second bimedial (straight-line) can be divided (into its component terms) at one point only.†

Let AB be a second bimedial (straight-line) which has been divided at C, so that AC and BC are medial (straight-lines), commensurable in square only, (and) containing a medial (area) [Prop. 10.38]. So, (it is) clear that C is not (located) at the point of bisection, since (AC and BC) are not commensurable in length. I say that AB cannot be (so) divided at another point.



For, if possible, let it also have been (so) divided at D, so that AC is not the same as DB, but AC (is), by hypothesis, greater. So, (it is) clear that (the sum of) the (squares) on AD and DB is also less than (the sum of) the (squares) on AC and CB, as we showed above [Prop. 10.41 lem.]. And AD and DB are medial (straight-lines), commensurable in square only, (and) containing a medial (area). And let the rational (straight-line) EF be laid down. And let the rectangular parallelogram EK, equal to the (square) on AB, be applied to EF. And let EG, equal to (the sum of) the (squares) on AC and CB, be cut off (from EK). Thus, the remainder, HK, is equal to twice the (rectangle contained) by AC and CB [Prop. 2.4]. So, again, let EL, equal to (the sum of) the (squares) on AD and DB—which was shown (to be) less than (the sum of) the (squares) on AC and CB—be cut off (from EK). And, thus, the remainder, MK, (is) equal to twice the (rectangle contained) by AD and DB. And since (the sum of) the (squares) on AC and CB is medial, EG (is) thus [also] medial. And it is applied to the rational (straight-line) EF. Thus, EH is rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF. And since AC and CB are medial (straight-lines which are) commensurable in square only, AC is thus incommensurable in length with CB. And as AC (is) to CB, so the (square) on AC (is) to the (rectangle contained) by AC and CB [Prop. 10.21 lem.]. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and ται δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων· ἡ ΕΝ ἄρα ἐκ δύο ὀνομάτων ἐστὶ διῃρημένη κατὰ τὸ Θ. κατὰ τὰ αὐτὰ δὴ δειχθήσονται καὶ αἱ ΕΜ, ΜΝ ῥηταὶ δυνάμει μόνον σύμμετροι· καὶ ἔσται ἡ ΕΝ ἐκ δύο ὀνομάτων κατ᾽ ἄλλο καὶ ἄλλο διῃρημένη τό τε Θ καὶ τὸ Μ, καὶ οὐκ ἔστιν ἡ ΕΘ τῇ ΜΝ ἡ αὐτή, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τῶν ἀπὸ τῶν ΑΔ, ΔΒ. ἀλλὰ τὰ ἀπὸ τῶν ΑΔ, ΔΒ μείζονά ἐστι τοῦ δὶς ὑπὸ ΑΔ, ΔΒ· πολλῷ ἄρα καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΕΗ, μεῖζόν ἐστι τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τουτέστι τοῦ ΜΚ· ὥστε καὶ ἡ ΕΘ τῆς ΜΝ μείζων ἐστίν. ἡ ἄρα ΕΘ τῇ ΜΝ οὐκ ἔστιν ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

CB [Prop. 10.11]. But, (the sum of) the (squares) on *AC* and *CB* is commensurable with the (square) on *AC*. For, *AC* and *CB* are commensurable in square [Prop. 10.15]. And twice the (rectangle contained) by *AC* and *CB* is commensurable with the (rectangle contained) by *AC* and *CB* [Prop. 10.6]. And thus (the sum of) the squares on *AC* and *CB* is incommensurable with twice the (rectangle contained) by *AC* and *CB* [Prop. 10.13]. But, *EG* is equal to (the sum of) the (squares) on *AC* and *CB*, and *HK* equal to twice the (rectangle contained) by *AC* and *CB*. Thus, *EG* is incommensurable with *HK*. Hence, *EH* is also incommensurable in length with *HN* [Props. 6.1, 10.11]. And (they are) rational (straight-lines). Thus, *EH* and *HN* are rational (straight-lines which are) commensurable in square only. And if two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is that irrational called binomial [Prop. 10.36]. Thus, *EN* is a binomial (straight-line) which has been divided (into its component terms) at *H*. So, according to the same (reasoning), *EM* and *MN* can be shown (to be) rational (straight-lines which are) commensurable in square only. And *EN* will (thus) be a binomial (straight-line) which has been divided (into its component terms) at the different (points) *H* and *M* (which is absurd [Prop. 10.42]). And *EH* is not the same as *MN*, since (the sum of) the (squares) on *AC* and *CB* is greater than (the sum of) the (squares) on *AD* and *DB*. But, (the sum of) the (squares) on *AD* and *DB* is greater than twice the (rectangle contained) by *AD* and *DB* [Prop. 10.59 lem.]. Thus, (the sum of) the (squares) on *AC* and *CB*—that is to say, *EG*—is also much greater than twice the (rectangle contained) by *AD* and *DB*—that is to say, *MK*. Hence, *EH* is also greater than *MN* [Prop. 6.1]. Thus, *EH* is not the same as *MN*. (Which is) the very thing it was required to show.

† In other words, $k^{1/4} + k'^{1/2}/k^{1/4} = k''^{1/4} + k'''^{1/2}/k''^{1/4}$ has only one solution: i.e., $k'' = k$ and $k''' = k'$.

με´.

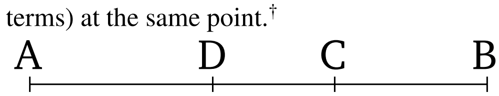
Ἡ μείζων κατὰ τὸ αὐτὸ μόνον σημεῖον διαιρεῖται.

Ἔστω μείζων ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνων ῥητόν, τὸ δ᾽ ὑπὸ τῶν ΑΓ, ΓΒ μέσον· λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον. καὶ ἐπεί, ᾧ διαφέρει τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ

Proposition 45

A major (straight-line) can only be divided (into its component terms) at the same point.†



Let *AB* be a major (straight-line) which has been divided at *C*, so that *AC* and *CB* are incommensurable in square, making the sum of the squares on *AC* and *CB* rational, and the (rectangle contained) by *AC* and *CD* medial [Prop. 10.39]. I say that *AB* cannot be (so) divided at another point.

For, if possible, let it also be divided at *D*, such that *AD* and *DB* are also incommensurable in square, making the sum of the (squares) on *AD* and *DB* rational, and the (rectangle contained) by them medial. And since, by whatever (amount τῶν ΑΔ, ΔΒ, τούτῳ διαφέρει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῶν ἀπὸ τῶν ΑΔ, ΔΒ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφότερα· καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἄρα τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ μείζων κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· κατὰ τὸ αὐτὸ ἄρα μόνον διαιρεῖται· ὅπερ ἔδει δεῖξαι.

the sum of) the (squares) on *AC* and *CB* differs from (the sum of) the (squares) on *AD* and *DB*, twice the (rectangle contained) by *AD* and *DB* also differs from twice the (rectangle contained) by *AC* and *CB* by this (same amount). But, (the sum of) the (squares) on *AC* and *CB* exceeds (the sum of) the (squares) on *AD* and *DB* by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle contained) by *AD* and *DB* also exceeds twice the (rectangle contained) by *AC* and *CB* by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, a major (straight-line) cannot be divided (into its component terms) at different points. Thus, it can only be (so) divided at the same (point). (Which is) the very thing it was required to show.

† In other words, $\sqrt{[1+k/(1+k^2)^{1/2}]/2} + \sqrt{[1-k/(1+k^2)^{1/2}]/2} = \sqrt{[1+k'/(1+k'^2)^{1/2}]/2} + \sqrt{[1-k'/(1+k'^2)^{1/2}]/2}$ has only one solution: i.e., $k' = k$.

μϛ´.

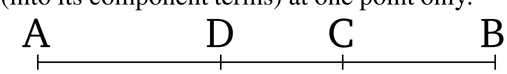
Ἡ ῥητὸν καὶ μέσον δυναμένη καθ᾽ ἓν μόνον σημεῖον διαιρεῖται.

Ἔστω ῥητὸν καὶ μέσον δυναμένη ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον, τὸ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ῥητόν· λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διῃρήσθω καὶ κατὰ τὸ Δ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΔ, ΔΒ μέσον, τὸ δὲ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ῥητόν. ἐπεὶ οὖν, ᾧ διαφέρει τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ διαφέρει καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τὸ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ὑπερέχει ῥητῷ, καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ῥητὸν καὶ μέσον δυναμένη κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται. κατὰ ἓν ἄρα σημεῖον διαιρεῖται· ὅπερ ἔδει δεῖξαι.

Proposition 46

The square-root of a rational plus a medial (area) can be divided (into its component terms) at one point only.†



Let *AB* be the square-root of a rational plus a medial (area) which has been divided at *C*, so that *AC* and *CB* are incommensurable in square, making the sum of the (squares) on *AC* and *CB* medial, and twice the (rectangle contained) by *AC* and *CB* rational [Prop. 10.40]. I say that *AB* cannot be (so) divided at another point.

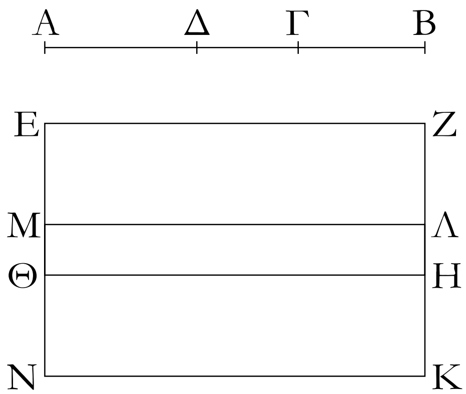
For, if possible, let it also be divided at *D*, so that *AD* and *DB* are also incommensurable in square, making the sum of the (squares) on *AD* and *DB* medial, and twice the (rectangle contained) by *AD* and *DB* rational. Therefore, since by whatever (amount) twice the (rectangle contained) by *AC* and *CB* differs from twice the (rectangle contained) by *AD* and *DB*, (the sum of) the (squares) on *AD* and *DB* also differs from (the sum of) the (squares) on *AC* and *CB* by this (same amount). And twice the (rectangle contained) by *AC* and *CB* exceeds twice the (rectangle contained) by *AD* and *DB* by a rational (area). (The sum of) the (squares) on *AD* and *DB* thus also exceeds (the sum of) the (squares) on *AC* and *CB* by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, the square-root of a rational plus a medial (area) cannot be divided (into its component terms) at different points. Thus, it can be (so) divided at one point (only). (Which is) the very thing it was required to show.

† In other words, $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]} + \sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]} = \sqrt{[(1+k'^2)^{1/2}+k']/[2\,(1+k'^2)]}$ $+ \sqrt{[(1+k'^2)^{1/2}-k']/[2\,(1+k'^2)]}$ has only one solution: i.e., $k' = k$.

μζ´.

Proposition 47

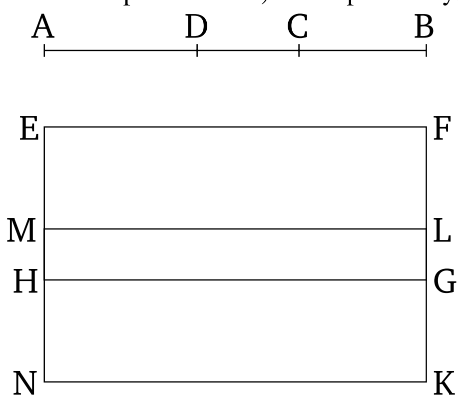
Ἡ δύο μέσα δυναμένη καθ᾽ ἓν μόνον σημεῖον διαιρεῖται.



Ἔστω [δύο μέσα δυναμένη] ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε τὰς ΑΓ, ΓΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τό τε συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον καὶ τὸ ὑπὸ τῶν ΑΓ, ΓΒ μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν. λέγω, ὅτι ἡ ΑΒ κατ᾽ ἄλλο σημεῖον οὐ διαιρεῖται ποιοῦσα τὰ προκείμενα.

Εἰ γὰρ δυνατόν, διῃρήσθω κατὰ τὸ Δ, ὥστε πάλιν δηλονότι τὴν ΑΓ τῇ ΔΒ μὴ εἶναι τὴν αὐτήν, ἀλλὰ μείζονα καθ᾽ ὑπόθεσιν τὴν ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ παραβεβλήσθω παρὰ τὴν ΕΖ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΘΚ· ὅλον ἄρα τὸ ΕΚ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. πάλιν δὴ παραβεβλήσθω παρὰ τὴν ΕΖ τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον τὸ ΕΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ λοιπῷ τῷ ΜΚ ἴσον ἐστίν. καὶ ἐπεὶ μέσον ὑπόκειται τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα ἐστὶ καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΘΕ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΝ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, καὶ τὸ ΕΗ ἄρα τῷ ΗΝ ἀσύμμετρόν ἐστιν· ὥστε καὶ ἡ ΕΘ τῇ ΘΝ ἀσύμμετρός ἐστιν. καί εἰσι ῥηταί· αἱ ΕΘ, ΘΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΕΝ ἄρα ἐκ δύο ὀνομάτων ἐστὶ διῃρημένη κατὰ τὸ Θ. ὁμοίως δὴ δείξομεν, ὅτι καὶ κατὰ τὸ Μ διῄρηται. καὶ οὐκ ἔστιν ἡ ΕΘ τῇ ΜΝ ἡ αὐτή· ἡ ἄρα ἐκ δύο ὀνομάτων κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διῄρηται· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἡ δύο μέσα δυναμένη κατ᾽ ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· καθ᾽ ἓν ἄρα μόνον [σημεῖον] διαιρεῖται.

The square-root of (the sum of) two medial (areas) can be divided (into its component terms) at one point only.†



Let AB be [the square-root of (the sum of) two medial (areas)] which has been divided at C, such that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and the (rectangle contained) by AC and CB medial, and, moreover, incommensurable with the sum of the (squares) on (AC and CB) [Prop. 10.41]. I say that AB cannot be divided at another point fulfilling the prescribed (conditions).

For, if possible, let it be divided at D, such that AC is again manifestly not the same as DB, but AC (is), by hypothesis, greater. And let the rational (straight-line) EF be laid down. And let EG, equal to (the sum of) the (squares) on AC and CB, and HK, equal to twice the (rectangle contained) by AC and CB, be applied to EF. Thus, the whole of EK is equal to the square on AB [Prop. 2.4]. So, again, let EL, equal to (the sum of) the (squares) on AD and DB, be applied to EF. Thus, the remainder—twice the (rectangle contained) by AD and DB—is equal to the remainder, MK. And since the sum of the (squares) on AC and CB was assumed (to be) medial, EG is also medial. And it is applied to the rational (straight-line) EF. HE is thus rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is thus also incommensurable with GN. Hence, EH is also incommensurable with HN [Props. 6.1, 10.11]. And they are (both) rational (straight-lines). Thus, EH and HN are rational (straight-lines which are) commensurable in square only. Thus, EN is a binomial (straight-line) which has been divided (into its component terms) at H [Prop. 10.36]. So, similarly, we can show that it has also been (so) divided at M. And EH is not the same as MN. Thus, a binomial (straight-line) has been divided (into its component terms) at different points. The very thing is absurd [Prop. 10.42]. Thus, the square-root of (the sum of) two medial (areas) cannot be divided (into its component terms) at different points. Thus, it can be (so) divided at one [point] only.

† In other words, $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2}+k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2}=k'''^{1/4}\sqrt{[1+k''/(1+k''^2)^{1/2}]/2}$ $+k'''^{1/4}\sqrt{[1-k''/(1+k''^2)^{1/2}]/2}$ has only one solution: i.e., $k''=k$ and $k'''=k'$.

Ὅροι δεύτεροι.

εʹ. Ὑποκειμένης ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων διῃρημένης εἰς τὰ ὀνόματα, ἧς τὸ μεῖζον ὄνομα τοῦ ἐλάσσονος μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει, ἐὰν μὲν τὸ μεῖζον ὄνομα σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω [ἡ ὅλη] ἐκ δύο ὀνομάτων πρώτη.

ϛʹ. Ἐὰν δὲ τὸ ἔλασσον ὄνομα σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω ἐκ δύο ὀνομάτων δευτέρα.

ζʹ. Ἐὰν δὲ μηδέτερον τῶν ὀνομάτων σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω ἐκ δύο ὀνομάτων τρίτη.

ηʹ. Πάλιν δὴ ἐὰν τὸ μεῖζον ὄνομα [τοῦ ἐλάσσονος] μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει, ἐὰν μὲν τὸ μεῖζον ὄνομα σύμμετρον ᾖ μήκει τῇ ἐκκειμένῃ ῥητῇ, καλείσθω ἐκ δύο ὀνομάτων τετάρτη.

θʹ. Ἐὰν δὲ τὸ ἔλασσον, πέμπτη.

ιʹ. Ἐὰν δὲ μηδέτερον, ἕκτη.

Definitions II

5. Given a rational (straight-line), and a binomial (straight-line) which has been divided into its (component) terms, of which the square on the greater term is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let [the whole] (straight-line) be called a first binomial (straight-line).

6. And if the lesser term is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a second binomial (straight-line).

7. And if neither of the terms is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a third binomial (straight-line).

8. So, again, if the square on the greater term is larger than (the square on) [the lesser] by the (square) on (some straight-line) incommensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let (the whole straight-line) be called a fourth binomial (straight-line).

9. And if the lesser (term is commensurable), a fifth (binomial straight-line).

10. And if neither (term is commensurable), a sixth (binomial straight-line).

μηʹ.

Εὑρεῖν τὴν ἐκ δύο ὀνομάτων πρώτην.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΓΑ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐκκείσθω τις ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω μήκει ἡ ΕΖ. ῥητὴ ἄρα ἐστὶ καὶ ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ. ὁ δὲ ΑΒ πρὸς τὸν ΑΓ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· ὥστε σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ. καὶ ἔστι ῥητὴ ἡ ΕΖ· ῥητὴ ἄρα καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ

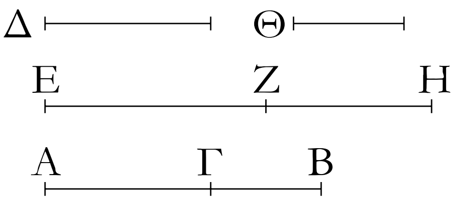
Proposition 48

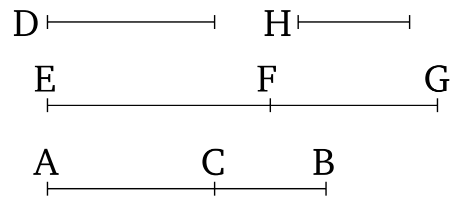
To find a first binomial (straight-line).

Let two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to CA the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let some rational (straight-line) D be laid down. And let EF be commensurable in length with D. EF is thus also rational [Def. 10.3]. And let it be contrived that as the number BA (is) to AC, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And AB has to AC the ratio which (some) number (has) to (some) number. Thus, the (square) on EF also has to the (square) on FG the ratio which (some) number (has) to (some) number. Hence, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. And EF is rational. Thus, FG (is) also rational. And since BA does not have to AC the ratio which (some) square num-

ΕΖ τῇ ΖΗ μήκει. αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ. λέγω, ὅτι καὶ πρώτη.

ber (has) to (some) square number, thus the (square) on *EF* does not have to the (square) on *FG* the ratio which (some) square number (has) to (some) square number either. Thus, *EF* is incommensurable in length with *FG* [Prop 10.9]. *EF* and *FG* are thus rational (straight-lines which are) commensurable in square only. Thus, *EG* is a binomial (straight-line) [Prop. 10.36]. I say that (it is) also a first (binomial straight-line).





Ἐπεὶ γάρ ἐστιν ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, μείζων δὲ ὁ ΒΑ τοῦ ΑΓ, μεῖζον ἄρα καὶ τὸ ἀπὸ τῆς ΕΖ τοῦ ἀπὸ τῆς ΖΗ. ἔστω ο᾿ὖν τῷ ἀπὸ τῆς ΕΖ ἴσα τὰ ἀπὸ τῶν ΖΗ, Θ. καὶ ἐπεί ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΑΒ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. σύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ Θ μήκει· ἡ ΕΖ ἄρα τῆς ΖΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσι ῥηταὶ αἱ ΕΖ, ΖΗ, καὶ σύμμετρος ἡ ΕΖ τῇ Δ μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

For since as the number *BA* is to *AC*, so the (square) on *EF* (is) to the (square) on *FG*, and *BA* (is) greater than *AC*, the (square) on *EF* (is) thus also greater than the (square) on *FG* [Prop. 5.14]. Therefore, let (the sum of) the (squares) on *FG* and *H* be equal to the (square) on *EF*. And since as *BA* is to *AC*, so the (square) on *EF* (is) to the (square) on *FG*, thus, via conversion, as *AB* is to *BC*, so the (square) on *EF* (is) to the (square) on *H* [Prop. 5.19 corr.]. And *AB* has to *BC* the ratio which (some) square number (has) to (some) square number. Thus, the (square) on *EF* also has to the (square) on *H* the ratio which (some) square number (has) to (some) square number. Thus, *EF* is commensurable in length with *H* [Prop. 10.9]. Thus, the square on *EF* is greater than (the square on) *FG* by the (square) on (some straight-line) commensurable (in length) with (*EF*). And *EF* and *FG* are rational (straight-lines). And *EF* (is) commensurable in length with *D*.

Thus, *EG* is a first binomial (straight-line) [Def. 10.5].†
(Which is) the very thing it was required to show.

†If the rational straight-line has unit length then the length of a first binomial straight-line is $k + k\sqrt{1-k'^2}$. This, and the first apotome, whose length is $k - k\sqrt{1-k'^2}$ [Prop. 10.85], are the roots of $x^2 - 2\,k\,x + k^2\,k'^2 = 0$.

μθʹ.

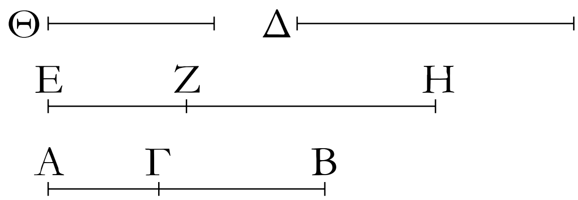
Εὑρεῖν τὴν ἐκ δύο ὀνομάτων δευτέραν.

Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐκκείσθω ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω ἡ ΕΖ μήκει· ῥητὴ ἄρα ἐστὶν ἡ ΕΖ. γεγονέτω δὴ καὶ ὡς ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ. ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος

Proposition 49

To find a second binomial (straight-line).

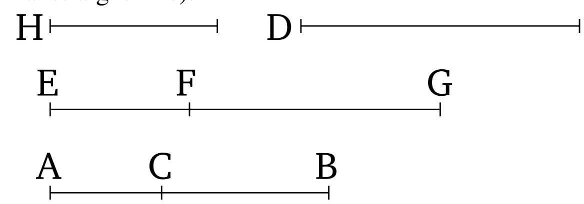
Let the two numbers *AC* and *CB* be laid down such that their sum *AB* has to *BC* the ratio which (some) square number (has) to (some) square number, and does not have to *AC* the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) *D* be laid down. And let *EF* be commensurable in length with *D*. *EF* is thus a rational (straight-line). So, let it also be contrived that as the number *CA* (is) to *AB*, so the (square) on *EF* (is) to the (square) on *FG* [Prop. 10.6 corr.]. Thus, the (square) on *EF* is commensurable with the (square) on *FG* [Prop. 10.6]. Thus, *FG* is also a rational (straight-line). And since the number *CA* does not have to *AB* the ratio which (some) square *ἄρα ἐστὶν ἡ ΕΖ τῇ ΖΗ μήκει· αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ. δεικτέον δή, ὅτι καὶ δευτέρα.*



Ἐπεὶ γὰρ ἀνάπαλίν ἐστιν ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΗΖ πρὸς τὸ ἀπὸ τῆς ΖΕ, μείζων δὲ ὁ ΒΑ τοῦ ΑΓ, μεῖζον ἄρα [καὶ] τὸ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω τῷ ἀπὸ τῆς ΗΖ ἴσα τὰ ἀπὸ τῶν ΕΖ, Θ· ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΑΒ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Θ. ἀλλ᾽ ὁ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. σύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Θ μήκει· ὥστε ἡ ΖΗ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσι ῥηταὶ αἱ ΖΗ, ΖΕ δυνάμει μόνον σύμμετροι, καὶ τὸ ΕΖ ἔλασσον ὄνομα τῇ ἐκκειμένῃ ῥητῇ σύμμετρόν ἐστι τῇ Δ μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα· ὅπερ ἔδει δεῖξαι.

number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. EF and FG are thus rational (straight-lines which are) commensurable in square only. Thus, EG is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).



For since, inversely, as the number BA is to AC, so the (square) on GF (is) to the (square) on FE [Prop. 5.7 corr.], and BA (is) greater than AC, the (square) on GF (is) thus [also] greater than the (square) on FE [Prop. 5.14]. Let (the sum of) the (squares) on EF and H be equal to the (square) on GF. Thus, via conversion, as AB is to BC, so the (square) on FG (is) to the (square) on H [Prop. 5.19 corr.]. But, AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, FG is commensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) commensurable in length with (FG). And FG and FE are rational (straight-lines which are) commensurable in square only. And the lesser term EF is commensurable in length with the rational (straight-line) D (previously) laid down.

Thus, EG is a second binomial (straight-line) [Def. 10.6].† (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then the length of a second binomial straight-line is $k/\sqrt{1-k'^2}+k$. This, and the second apotome, whose length is $k/\sqrt{1-k'^2}-k$ [Prop. 10.86], are the roots of $x^2-(2\,k/\sqrt{1-k'^2})\,x+k^2\,[k'^2/(1-k'^2)]=0$.

ν΄.

Εὑρεῖν τὴν ἐκ δύο ὀνομάτων τρίτην.

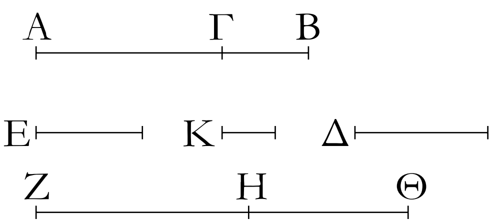
Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἐκκείσθω δέ τις καὶ ἄλλος μὴ τετράγωνος ἀριθμὸς ὁ Δ, καὶ πρὸς ἑκάτερον τῶν ΒΑ, ΑΓ λόγον μὴ ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ ἐκκείσθω τις ῥητὴ εὐθεῖα ἡ Ε, καὶ γεγονέτω ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Ε τῷ ἀπὸ τῆς ΖΗ. καί ἐστι ῥητὴ ἡ Ε· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ Δ πρὸς τὸν ΑΒ λόγον οὐκ

Proposition 50

To find a third binomial (straight-line).

Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number. And let some other non-square number D also be laid down, and let it not have to each of BA and AC the ratio which (some) square number (has) to (some) square number. And let some rational straight-line E be laid down, and let it be contrived that as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E is commensurable with the (square) on FG [Prop. 10.6].

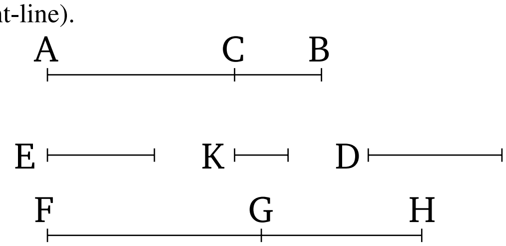
ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΖΗ μήκει. γεγονέτω δὴ πάλιν ὡς ἡ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὴ δὲ ἡ ΖΗ· ῥητὴ ἄρα καὶ ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΘΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστίν. λέγω δή, ὅτι καὶ τρίτη.



Ἐπεὶ γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Δ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Δ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΗΘ μήκει. καὶ ἐπεί ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, μεῖζον ἄρα τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ. ἔστω οὖν τῷ ἀπὸ τῆς ΖΗ ἴσα τὰ ἀπὸ τῶν ΗΘ, Κ· ἀναστρέψαντι ἄρα [ἐστὶν] ὡς ὁ ΑΒ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα [ἐστὶν] ἡ ΖΗ τῇ Κ μήκει. ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΖΗ, ΗΘ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ Ε μήκει.

Ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

And *E* is a rational (straight-line). Thus, *FG* is also a rational (straight-line). And since *D* does not have to *AB* the ratio which (some) square number has to (some) square number, the (square) on *E* does not have to the (square) on *FG* the ratio which (some) square number (has) to (some) square number either. *E* is thus incommensurable in length with *FG* [Prop. 10.9]. So, again, let it be contrived that as the number *BA* (is) to *AC*, so the (square) on *FG* (is) to the (square) on *GH* [Prop. 10.6 corr.]. Thus, the (square) on *FG* is commensurable with the (square) on *GH* [Prop. 10.6]. And *FG* (is) a rational (straight-line). Thus, *GH* (is) also a rational (straight-line). And since *BA* does not have to *AC* the ratio which (some) square number (has) to (some) square number, the (square) on *FG* does not have to the (square) on *HG* the ratio which (some) square number (has) to (some) square number either. Thus, *FG* is incommensurable in length with *GH* [Prop. 10.9]. *FG* and *GH* are thus rational (straight-lines which are) commensurable in square only. Thus, *FH* is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a third (binomial straight-line).



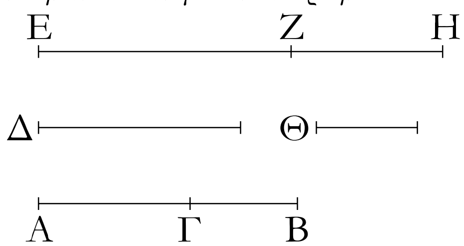
For since as *D* is to *AB*, so the (square) on *E* (is) to the (square) on *FG*, and as *BA* (is) to *AC*, so the (square) on *FG* (is) to the (square) on *GH*, thus, via equality, as *D* (is) to *AC*, so the (square) on *E* (is) to the (square) on *GH* [Prop. 5.22]. And *D* does not have to *AC* the ratio which (some) square number (has) to (some) square number. Thus, the (square) on *E* does not have to the (square) on *GH* the ratio which (some) square number (has) to (some) square number either. Thus, *E* is incommensurable in length with *GH* [Prop. 10.9]. And since as *BA* is to *AC*, so the (square) on *FG* (is) to the (square) on *GH*, the (square) on *FG* (is) thus greater than the (square) on *GH* [Prop. 5.14]. Therefore, let (the sum of) the (squares) on *GH* and *K* be equal to the (square) on *FG*. Thus, via conversion, as *AB* [is] to *BC*, so the (square) on *FG* (is) to the (square) on *K* [Prop. 5.19 corr.]. And *AB* has to *BC* the ratio which (some) square number (has) to (some) square number. Thus, the (square) on *FG* also has to the (square) on *K* the ratio which (some) square number (has) to (some) square number. Thus, *FG* [is] commensurable in length with *K* [Prop. 10.9]. Thus, the square on *FG* is greater than (the square on) *GH* by the (square) on (some straight-line) commensurable (in length) with (*FG*). And *FG* and *GH* are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with *E*.

Thus, FH is a third binomial (straight-line) [Def. 10.7].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a third binomial straight-line is $k^{1/2}\,(1 + \sqrt{1-k'^{\,2}})$. This, and the third apotome, whose length is $k^{1/2}\,(1 - \sqrt{1-k'^{\,2}})$ [Prop. 10.87], are the roots of $x^2 - 2\,k^{1/2}\,x + k\,k'^{\,2} = 0$.

να´.

Εὑρεῖν τὴν ἐκ δύο ὀνομάτων τετάρτην.



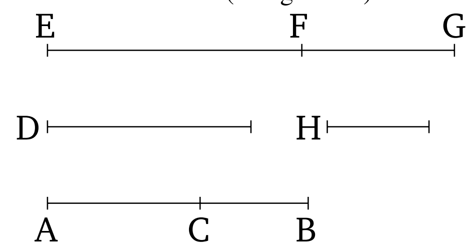
Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς τὸν ΒΓ λόγον μὴ ἔχειν μήτε μὴν πρὸς τὸν ΑΓ, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ ἐκκείσθω ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω μήκει ἡ ΕΖ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ ΖΗ μήκει. αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ὥστε ἡ ΕΗ ἐκ δύο ὀνομάτων ἐστίν. λέγω δή, ὅτι καὶ τετάρτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ [μείζων δὲ ὁ ΒΑ τοῦ ΑΓ], μεῖζον ἄρα τὸ ἀπὸ τῆς ΕΖ τοῦ ἀπὸ τῆς ΖΗ. ἔστω οὖν τῷ ἀπὸ τῆς ΕΖ ἴσα τὰ ἀπὸ τῶν ΖΗ, Θ· ἀναστρέψαντι ἄρα ὡς ὁ ΑΒ ἀριθμὸς πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ Θ μήκει· ἡ ΕΖ ἄρα τῆς ΗΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΕΖ, ΖΗ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ ΕΖ τῇ Δ σύμμετρός ἐστι μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἔδει δεῖξαι.

Proposition 51

To find a fourth binomial (straight-line).



Let the two numbers AC and CB be laid down such that AB does not have to BC, or to AC either, the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) D be laid down. And let EF be commensurable in length with D. Thus, EF is also a rational (straight-line). And let it be contrived that as the number BA (is) to AC, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. Thus, FG is also a rational (straight-line). And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. Thus, EF and FG are rational (straight-lines which are) commensurable in square only. Hence, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fourth (binomial straight-line).

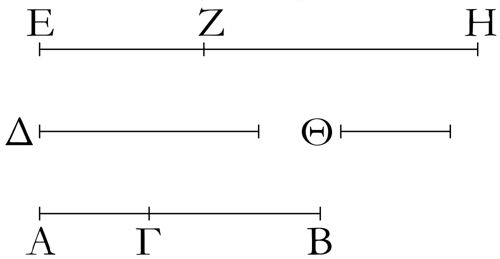
For since as BA is to AC, so the (square) on EF (is) to the (square) on FG [and BA (is) greater than AC], the (square) on EF (is) thus greater than the (square) on FG [Prop. 5.14]. Therefore, let (the sum of) the squares on FG and H be equal to the (square) on EF. Thus, via conversion, as the number AB (is) to BC, so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with H [Prop. 10.9]. Thus, the square on EF is greater than (the square on) GF by the (square) on (some straight-line) incommensurable (in length) with (EF). And EF and FG are rational (straight-lines which are) commensurable in square only. And EF is commensurable in length with D.

Thus, EG is a fourth binomial (straight-line) [Def. 10.8].†
(Which is) the very thing it was required to show.

† If the rational straight-line has unit length then the length of a fourth binomial straight-line is $k\,(1 + 1/\sqrt{1 + k'})$. This, and the fourth apotome, whose length is $k\,(1 - 1/\sqrt{1 + k'})$ [Prop. 10.88], are the roots of $x^2 - 2\,k\,x + k^2\,k'/(1 + k') = 0$.

νβ´.

Εὑρεῖν τὴν ἐκ δύο ὀνομάτων πέμπτην.



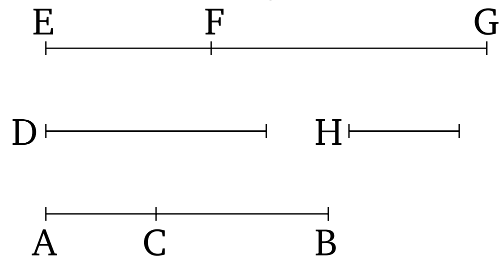
Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς ἑκάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐκκείσθω ῥητή τις εὐθεῖα ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω [μήκει] ἡ ΕΖ· ῥητὴ ἄρα ἡ ΕΖ. καὶ γεγονέτω ὡς ὁ ΓΑ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ. ὁ δὲ ΓΑ πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδὲ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. αἱ ΕΖ, ΖΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ. λέγω δή, ὅτι καὶ πέμπτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ ΓΑ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ, ἀνάπαλιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΖΕ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω οὖν τῷ ἀπὸ τῆς ΗΖ ἴσα τὰ ἀπὸ τῶν ΕΖ, Θ· ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΑΒ ἀριθμὸς πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΗΖ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Θ μήκει· ὥστε ἡ ΖΗ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΗΖ, ΖΕ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ τὸ ΕΖ ἔλαττον ὄνομα σύμμετρόν ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ Δ μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

Proposition 52

To find a fifth binomial straight-line.



Let the two numbers AC and CB be laid down such that AB does not have to either of them the ratio which (some) square number (has) to (some) square number [Prop. 10.38 lem.]. And let some rational straight-line D be laid down. And let EF be commensurable [in length] with D. Thus, EF (is) a rational (straight-line). And let it be contrived that as CA (is) to AB, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And CA does not have to AB the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF and FG are rational (straight-lines which are) commensurable in square only [Prop. 10.9]. Thus, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

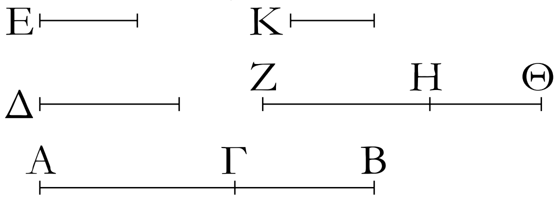
For since as CA is to AB, so the (square) on EF (is) to the (square) on FG, inversely, as BA (is) to AC, so the (square) on FG (is) to the (square) on FE [Prop. 5.7 corr.]. Thus, the (square) on GF (is) greater than the (square) on FE [Prop. 5.14]. Therefore, let (the sum of) the (squares) on EF and H be equal to the (square) on GF. Thus, via conversion, as the number AB is to BC, so the (square) on GF (is) to the (square) on H [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) incommensurable (in length) with (FG). And GF and FE are rational (straight-lines which are) commensurable in square only. And the lesser term EF is commensurable in length with the rational (straight-line previously) laid down, D.

Thus, EG is a fifth binomial (straight-line).† (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then the length of a fifth binomial straight-line is $k\,(\sqrt{1+k'}+1)$. This, and the fifth apotome, whose length is $k\,(\sqrt{1+k'}-1)$ [Prop. 10.89], are the roots of $x^2 - 2\,k\,\sqrt{1+k'}\,x + k^2\,k' = 0$.

νγ'.

Εὑρεῖν τὴν ἐκ δύο ὀνομάτων ἕκτην.

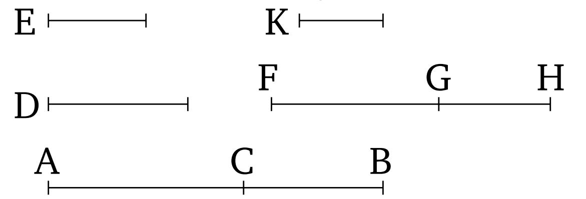


Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν ΑΒ πρὸς ἑκάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἔστω δὲ καὶ ἕτερος ἀριθμὸς ὁ Δ μὴ τετράγωνος ὢν μηδὲ πρὸς ἑκάτερον τῶν ΒΑ, ΑΓ λόγον ἔχων, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ ἐκκείσθω τις ῥητὴ εὐθεῖα ἡ Ε, καὶ γεγονέτω ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα τὸ ἀπὸ τῆς Ε τῷ ἀπὸ τῆς ΖΗ. καί ἐστι ῥητὴ ἡ Ε· ῥητὴ ἄρα καὶ ἡ ΖΗ. καὶ ἐπεὶ οὐκ ἔχει ὁ Δ πρὸς τὸν ΑΒ λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἡ Ε τῇ ΖΗ μήκει. γεγονέτω δὴ πάλιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ. σύμμετρον ἄρα τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΘΗ. ῥητὸν ἄρα τὸ ἀπὸ τῆς ΘΗ· ῥητὴ ἄρα ἡ ΘΗ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΖΘ. δεικτέον δή, ὅτι καὶ ἕκτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ, ἔστι δὲ καὶ ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Δ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Δ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΗΘ μήκει. ἐδείχθη δὲ καὶ τῇ ΖΗ ἀσύμμετρος· ἑκατέρα ἄρα τῶν ΖΗ, ΗΘ ἀσύμμετρός ἐστι τῇ Ε μήκει. καὶ ἐπεί ἐστιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, μεῖζον ἄρα τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ. ἔστω οὖν τῷ ἀπὸ [τῆς] ΖΗ ἴσα τὰ ἀπὸ τῶν ΗΘ, Κ· ἀναστρέψαντι ἄρα ὡς ὁ ΑΒ πρὸς ΒΓ, οὕτως τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς

Proposition 53

To find a sixth binomial (straight-line).



Let the two numbers AC and CB be laid down such that AB does not have to each of them the ratio which (some) square number (has) to (some) square number. And let D also be another number, which is not square, and does not have to each of BA and AC the ratio which (some) square number (has) to (some) square number either [Prop. 10.28 lem. I]. And let some rational straight-line E be laid down. And let it be contrived that as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E (is) commensurable with the (square) on FG [Prop. 10.6]. And E is rational. Thus, FG (is) also rational. And since D does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on E thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, E (is) incommensurable in length with FG [Prop. 10.9]. So, again, let it have be contrived that as BA (is) to AC, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.]. The (square) on FG (is) thus commensurable with the (square) on HG [Prop. 10.6]. The (square) on HG (is) thus rational. Thus, HG (is) rational. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. Thus, FG and GH are rational (straight-lines which are) commensurable in square only. Thus, FH is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a sixth (binomial straight-line).

For since as D is to AB, so the (square) on E (is) to the (square) on FG, and also as BA is to AC, so the (square) on FG (is) to the (square) on GH, thus, via equality, as D is to AC, so the (square) on E (is) to the (square) on GH [Prop. 5.22]. And D does not have to AC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on E does not have to the (square) on GH the ratio which (some) *πρὸς τετράγωνον ἀριθμόν· ὥστε οὐδὲ τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Κ μήκει· ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΖΗ, ΗΘ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῃ τῇ Ε.*

Ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἕκτη· ὅπερ ἔδει δεῖξαι.

square number (has) to (some) square number either. E is thus incommensurable in length with GH [Prop. 10.9]. And (E) was also shown (to be) incommensurable (in length) with FG. Thus, FG and GH are each incommensurable in length with E. And since as BA is to AC, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14]. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG. Thus, via conversion, as AB (is) to BC, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Hence, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with K [Prop. 10.9]. The square on FG is thus greater than (the square on) GH by the (square) on (some straight-line which is) incommensurable (in length) with (FG). And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with the rational (straight-line) E (previously) laid down.

Thus, FH is a sixth binomial (straight-line) [Def. 10.10].† (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then the length of a sixth binomial straight-line is $\sqrt{k} + \sqrt{k'}$. This, and the sixth apotome, whose length is $\sqrt{k} - \sqrt{k'}$ [Prop. 10.90], are the roots of $x^2 - 2\sqrt{k}\,x + (k - k') = 0$.

Λῆμμα.

Ἔστω δύο τετράγωνα τὰ ΑΒ, ΒΓ καὶ κείσθωσαν ὥστε ἐπ᾿ εὐθείας εἶναι τὴν ΔΒ τῇ ΒΕ· ἐπ᾿ εὐθείας ἄρα ἐστὶ καὶ ἡ ΖΒ τῇ ΒΗ. καὶ συμπεπληρώσθω τὸ ΑΓ παραλληλόγραμμον· λέγω, ὅτι τετράγωνόν ἐστι τὸ ΑΓ, καὶ ὅτι τῶν ΑΒ, ΒΓ μέσον ἀνάλογόν ἐστι τὸ ΔΗ, καὶ ἔτι τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν ἐστι τὸ ΔΓ.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΖ, ἡ δὲ ΒΕ τῇ ΒΗ, ὅλη ἄρα ἡ ΔΕ ὅλῃ τῇ ΖΗ ἐστιν ἴση. ἀλλ᾿ ἡ μὲν ΔΕ ἑκατέρᾳ τῶν ΑΘ, ΚΓ ἐστιν ἴση, ἡ δὲ ΖΗ ἑκατέρᾳ τῶν ΑΚ, ΘΓ ἐστιν ἴση· καὶ ἑκατέρα ἄρα τῶν ΑΘ, ΚΓ ἑκατέρᾳ τῶν ΑΚ, ΘΓ ἐστιν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΓ παραλληλόγραμμον· ἔστι δὲ καὶ ὀρθογώνιον· τετράγωνον ἄρα ἐστὶ τὸ ΑΓ.

Καὶ ἐπεὶ ἐστιν ὡς ἡ ΖΒ πρὸς τὴν ΒΗ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΕ, ἀλλ᾿ ὡς μὲν ἡ ΖΒ πρὸς τὴν ΒΗ, οὕτως τὸ ΑΒ πρὸς τὸ ΔΗ, ὡς δὲ ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως τὸ ΔΗ πρὸς τὸ ΒΓ, καὶ ὡς ἄρα τὸ ΑΒ πρὸς τὸ ΔΗ, οὕτως τὸ ΔΗ πρὸς τὸ ΒΓ. τῶν ΑΒ, ΒΓ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΔΗ.

Λέγω δή, ὅτι καὶ τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν [ἐστι] τὸ ΔΓ.

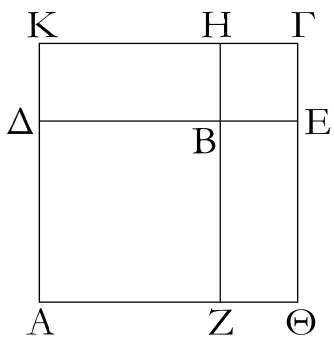
Lemma

Let AB and BC be two squares, and let them be laid down such that DB is straight-on to BE. FB is, thus, also straight-on to BG. And let the parallelogram AC be completed. I say that AC is a square, and that DG is the mean proportional to AB and BC, and, moreover, DC is the mean proportional to AC and CB.

For since DB is equal to BF, and BE to BG, the whole of DE is thus equal to the whole of FG. But DE is equal to each of AH and KC, and FG is equal to each of AK and HC [Prop. 1.34]. Thus, AH and KC are also equal to AK and HC, respectively. Thus, the parallelogram AC is equilateral. And (it is) also right-angled. Thus, AC is a square.

And since as FB is to BG, so DB (is) to BE, but as FB (is) to BG, so AB (is) to DG, and as DB (is) to BE, so DG (is) to BC [Prop. 6.1], thus also as AB (is) to DG, so DG (is) to BC [Prop. 5.11]. Thus, DG is the mean proportional to AB and BC.

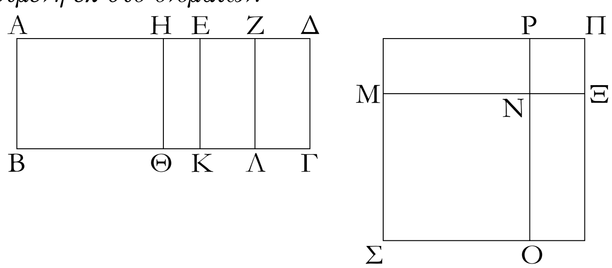
So I also say that DC [is] the mean proportional to AC and CB.



Ἐπεὶ γάρ ἐστιν ὡς ἡ ΑΔ πρὸς τὴν ΔΚ, οὕτως ἡ ΚΗ πρὸς τὴν ΗΓ· ἴση γάρ [ἐστιν] ἑκατέρα ἑκατέρᾳ· καὶ συνθέντι ὡς ἡ ΑΚ πρὸς ΚΔ, οὕτως ἡ ΚΓ πρὸς ΓΗ, ἀλλ᾿ ὡς μὲν ἡ ΑΚ πρὸς ΚΔ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΔ, ὡς δὲ ἡ ΚΓ πρὸς ΓΗ, οὕτως τὸ ΔΓ πρὸς ΓΒ, καὶ ὡς ἄρα τὸ ΑΓ πρὸς ΔΓ, οὕτως τὸ ΔΓ πρὸς τὸ ΒΓ. τῶν ΑΓ, ΓΒ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΔΓ· ἃ προέκειτο δεῖξαι.

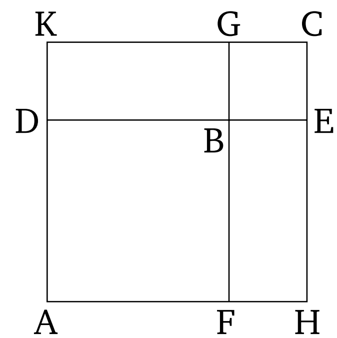
νδ΄.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πρώτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.



Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων πρώτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.

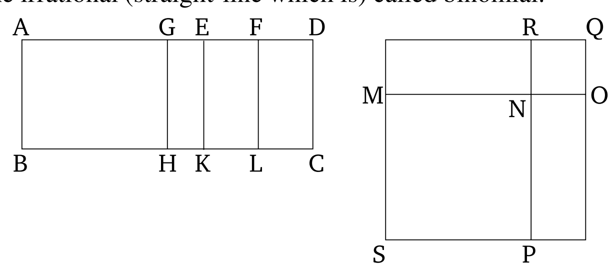
Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων ἐστὶ πρώτη ἡ ΑΔ, διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, καὶ ἔστω τὸ μεῖζον ὄνομα τὸ ΑΕ. φανερὸν δή, ὅτι αἱ ΑΕ, ΕΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτη, καὶ ἡ ΑΕ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΒ μήκει. τετμήσθω δὴ ἡ ΕΔ δίχα κατὰ τὸ Ζ σημεῖον. καὶ ἐπεὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος, τουτέστι τῷ ἀπὸ τῆς ΕΖ, ἴσον παρὰ τὴν μείζονα τὴν ΑΕ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. παραβεβλήσθω οὖν παρὰ τὴν ΑΕ τῷ ἀπὸ τῆς ΕΖ ἴσον τὸ ὑπὸ ΑΗ, ΗΕ· σύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΕΗ μήκει. καὶ ἤχθωσαν ἀπὸ τῶν Η, Ε, Ζ ὁποτέρᾳ τῶν ΑΒ, ΓΔ παράλληλοι αἱ ΗΘ, ΕΚ, ΖΛ· καὶ τῷ μὲν ΑΘ παραλληλογράμμῳ ἴσον τετράγωνον συνεστάτω τὸ ΣΝ, τῷ δὲ ΗΚ ἴσον τὸ ΝΠ, καὶ κείσθω ὥστε ἐπ᾿ εὐθείας εἶναι τὴν ΜΝ τῇ ΝΞ· ἐπ᾿ εὐθείας ἄρα ἐστὶ καὶ ἡ ΡΝ τῇ ΝΟ. καὶ συμπεπληρώσθω



For since as AD is to DK, so KG (is) to GC. For [they are] respectively equal. And, via composition, as AK (is) to KD, so KC (is) to CG [Prop. 5.18]. But as AK (is) to KD, so AC (is) to CD, and as KC (is) to CG, so DC (is) to CB [Prop. 6.1]. Thus, also, as AC (is) to DC, so DC (is) to BC [Prop. 5.11]. Thus, DC is the mean proportional to AC and CB. Which (is the very thing) it was prescribed to show.

Proposition 54

If an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called binomial.†



For let the area AC be contained by the rational (straight-line) AB and by the first binomial (straight-line) AD. I say that square-root of area AC is the irrational (straight-line which is) called binomial.

For since AD is a first binomial (straight-line), let it be divided into its (component) terms at E, and let AE be the greater term. So, (it is) clear that AE and ED are rational (straight-lines which are) commensurable in square only, and that the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and that AE is commensurable (in length) with the rational (straight-line) AB (first) laid out [Def. 10.5]. So, let ED be cut in half at point F. And since the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), thus if a (rectangle) equal to the fourth part of the (square) on the lesser (term)—that is to say, the (square) on EF—falling short by a square figure, is applied to the greater (term) AE, then it divides it into (terms which are) commensurable (in length) [Prop 10.17]. Therefore, let the (rectangle contained) by AG

τὸ ΣΠ παραλληλόγραμμον· τετράγωνον ἄρα ἐστὶ τὸ ΣΠ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΗ, ΗΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ, ἔστιν ἄρα ὡς ἡ ΑΗ πρὸς ΕΖ, οὕτως ἡ ΖΕ πρὸς ΕΗ· καὶ ὡς ἄρα τὸ ΑΘ πρὸς ΕΛ, τὸ ΕΛ πρὸς ΚΗ· τῶν ΑΘ, ΗΚ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἀλλὰ τὸ μὲν ΑΘ ἴσον ἐστὶ τῷ ΣΝ, τὸ δὲ ΗΚ ἴσον τῷ ΝΠ· τῶν ΣΝ, ΝΠ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἔστι δὲ τῶν αὐτῶν τῶν ΣΝ, ΝΠ μέσον ἀνάλογον καὶ τὸ ΜΡ· ἴσον ἄρα ἐστὶ τὸ ΕΛ τῷ ΜΡ· ὥστε καὶ τῷ ΟΞ ἴσον ἐστίν. ἔστι δὲ καὶ τὰ ΑΘ, ΗΚ τοῖς ΣΝ, ΝΠ ἴσα· ὅλον ἄρα τὸ ΑΓ ἴσον ἐστὶν ὅλῳ τῷ ΣΠ, τουτέστι τῷ ἀπὸ τῆς ΜΞ τετραγώνῳ· τὸ ΑΓ ἄρα δύναται ἡ ΜΞ. λέγω, ὅτι ἡ ΜΞ ἐκ δύο ὀνομάτων ἐστίν.

Ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ ΑΗ τῇ ΗΕ, σύμμετρός ἐστι καὶ ἡ ΑΕ ἑκατέρᾳ τῶν ΑΗ, ΗΕ. ὑπόκειται δὲ καὶ ἡ ΑΕ τῇ ΑΒ σύμμετρος· καὶ αἱ ΑΗ, ΗΕ ἄρα τῇ ΑΒ σύμμετροί εἰσιν. καί ἐστι ῥητὴ ἡ ΑΒ· ῥητὴ ἄρα ἐστὶ καὶ ἑκατέρα τῶν ΑΗ, ΗΕ· ῥητὸν ἄρα ἐστὶν ἑκάτερον τῶν ΑΘ, ΗΚ, καί ἐστι σύμμετρον τὸ ΑΘ τῷ ΗΚ. ἀλλὰ τὸ μὲν ΑΘ τῷ ΣΝ ἴσον ἐστίν, τὸ δὲ ΗΚ τῷ ΝΠ· καὶ τὰ ΣΝ, ΝΠ ἄρα, τουτέστι τὰ ἀπὸ τῶν ΜΝ, ΝΞ, ῥητά ἐστι καὶ σύμμετρα. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῇ ΕΔ μήκει, ἀλλ᾽ ἡ μὲν ΑΕ τῇ ΑΗ ἐστι σύμμετρος, ἡ δὲ ΔΕ τῇ ΕΖ σύμμετρος, ἀσύμμετρος ἄρα καὶ ἡ ΑΗ τῇ ΕΖ· ὥστε καὶ τὸ ΑΘ τῷ ΕΛ ἀσύμμετρόν ἐστιν. ἀλλὰ τὸ μὲν ΑΘ τῷ ΣΝ ἐστιν ἴσον, τὸ δὲ ΕΛ τῷ ΜΡ· καὶ τὸ ΣΝ ἄρα τῷ ΜΡ ἀσύμμετρόν ἐστιν. ἀλλ᾽ ὡς τὸ ΣΝ πρὸς ΜΡ, ἡ ΟΝ πρὸς τὴν ΝΡ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΟΝ τῇ ΝΡ. ἴση δὲ ἡ μὲν ΟΝ τῇ ΜΝ, ἡ δὲ ΝΡ τῇ ΝΞ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΜΝ τῇ ΝΞ. καί ἐστι τὸ ἀπὸ τῆς ΜΝ σύμμετρον τῷ ἀπὸ τῆς ΝΞ, καὶ ῥητὸν ἑκάτερον· αἱ ΜΝ, ΝΞ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι.

Ἡ ΜΞ ἄρα ἐκ δύο ὀνομάτων ἐστὶ καὶ δύναται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

and GE, equal to the (square) on EF, be applied to AE. AG is thus commensurable in length with EG. And let GH, EK, and FL be drawn from (points) G, E, and F (respectively), parallel to either of AB or CD. And let the square SN, equal to the parallelogram AH, be constructed, and (the square) NQ, equal to (the parallelogram) GK [Prop. 2.14]. And let MN be laid down so as to be straight-on to NO. RN is thus also straight-on to NP. And let the parallelogram SQ be completed. SQ is thus a square [Prop. 10.53 lem.]. And since the (rectangle contained) by AG and GE is equal to the (square) on EF, thus as AG is to EF, so FE (is) to EG [Prop. 6.17]. And thus as AH (is) to EL, (so) EL (is) to KG [Prop. 6.1]. Thus, EL is the mean proportional to AH and GK. But, AH is equal to SN, and GK (is) equal to NQ. EL is thus the mean proportional to SN and NQ. And MR is also the mean proportional to the same—(namely), SN and NQ [Prop. 10.53 lem.]. EL is thus equal to MR. Hence, it is also equal to PO [Prop. 1.43]. And AH plus GK is equal to SN plus NQ. Thus, the whole of AC is equal to the whole of SQ—that is to say, to the square on MO. Thus, MO (is) the square-root of (area) AC. I say that MO is a binomial (straight-line).

For since AG is commensurable (in length) with GE, AE is also commensurable (in length) with each of AG and GE [Prop. 10.15]. And AE was also assumed (to be) commensurable (in length) with AB. Thus, AG and GE are also commensurable (in length) with AB [Prop. 10.12]. And AB is rational. AG and GE are thus each also rational. Thus, AH and GK are each rational (areas), and AH is commensurable with GK [Prop. 10.19]. But, AH is equal to SN, and GK to NQ. SN and NQ—that is to say, the (squares) on MN and NO (respectively)—are thus also rational and commensurable. And since AE is incommensurable in length with ED, but AE is commensurable (in length) with AG, and DE (is) commensurable (in length) with EF, AG (is) thus also incommensurable (in length) with EF [Prop. 10.13]. Hence, AH is also incommensurable with EL [Props. 6.1, 10.11]. But, AH is equal to SN, and EL to MR. Thus, SN is also incommensurable with MR. But, as SN (is) to MR, (so) PN (is) to NR [Prop. 6.1]. PN is thus incommensurable (in length) with NR [Prop. 10.11]. And PN (is) equal to MN, and NR to NO. Thus, MN is incommensurable (in length) with NO. And the (square) on MN is commensurable with the (square) on NO, and each (is) rational. MN and NO are thus rational (straight-lines which are) commensurable in square only.

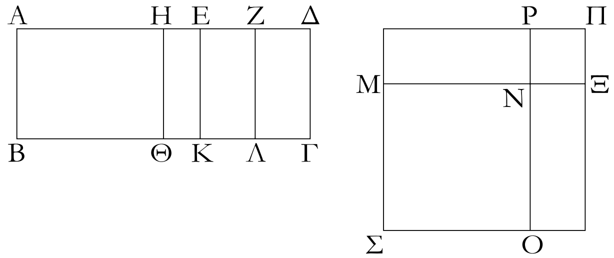
Thus, MO is (both) a binomial (straight-line) [Prop. 10.36], and the square-root of AC. (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a first binomial straight-line is a binomial straight-line: i.e., a first binomial straight-line has a length $k + k\,\sqrt{1-k'^2}$ whose square-root can be written $\rho\,(1+\sqrt{k''})$, where $\rho = \sqrt{k\,(1+k')/2}$ and $k'' = (1-k')/(1+k')$. This

is the length of a binomial straight-line (see Prop. 10.36), since ρ is rational.

νε΄.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων πρώτη.



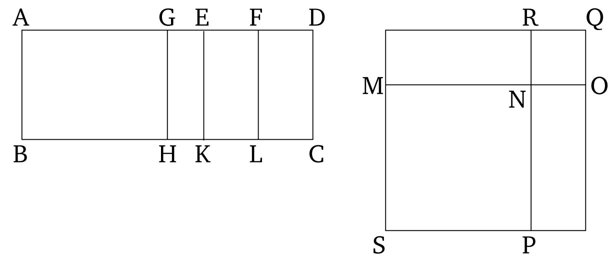
Περιεχέσθω γὰρ χωρίον τὸ ΑΒΓΔ ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων δυετέρας τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐκ δύο μέσων πρώτη ἐστίν.

Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων δευτέρα ἐστὶν ἡ ΑΔ, διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· αἱ ΑΕ, ΕΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τὸ ἔλαττον ὄνομα ἡ ΕΔ σύμμετρόν ἐστι τῇ ΑΒ μήκει. τετμήσθω ἡ ΕΔ δίχα κατὰ τὸ Ζ, καὶ τῷ ἀπὸ τῆς ΕΖ ἴσον παρὰ τὴν ΑΕ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΑΗΕ· σύμμετρος ἄρα ἡ ΑΗ τῇ ΗΕ μήκει. καὶ διὰ τῶν Η, Ε, Ζ παράλληλοι ἤχθωσαν ταῖς ΑΒ, ΓΔ αἱ ΗΘ, ΕΚ, ΖΛ, καὶ τῷ μὲν ΑΘ παραλληλογράμμῳ ἴσον τετράγωνον συνεστάτω τὸ ΣΝ, τῷ δὲ ΗΚ ἴσον τετράγωνον τὸ ΝΠ, καὶ κείσθω ὥστε ἐπ᾿ εὐθείας εἶναι τὴν ΜΝ τῇ ΝΞ· ἐπ᾿ εὐθείας ἄρα [ἐστὶ] καὶ ἡ ΡΝ τῇ ΝΟ. καὶ συμπεπληρώσθω τὸ ΣΠ τετράγωνον· φανερὸν δὴ ἐκ τοῦ προδεδειγμένου, ὅτι τὸ ΜΡ μέσον ἀνάλογόν ἐστι τῶν ΣΝ, ΝΠ, καὶ ἴσον τῷ ΕΛ, καὶ ὅτι τὸ ΑΓ χωρίον δύναται ἡ ΜΞ. δεικτέον δή, ὅτι ἡ ΜΞ ἐκ δύο μέσων ἐστὶ πρώτη.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῇ ΕΔ μήκει, σύμμετρος δὲ ἡ ΕΔ τῇ ΑΒ, ἀσύμμετρος ἄρα ἡ ΑΕ τῇ ΑΒ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΗ τῇ ΕΗ, σύμμετρός ἐστι καὶ ἡ ΑΕ ἑκατέρᾳ τῶν ΑΗ, ΗΕ. ἀλλὰ ἡ ΑΕ ἀσύμμετρος τῇ ΑΒ μήκει· καὶ αἱ ΑΗ, ΗΕ ἄρα ἀσύμμετροί εἰσι τῇ ΑΒ. αἱ ΒΑ, ΑΗ, ΗΕ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ὥστε μέσον ἐστὶν ἑκάτερον τῶν ΑΘ, ΗΚ. ὥστε καὶ ἑκάτερον τῶν ΣΝ, ΝΠ μέσον ἐστίν. καὶ αἱ ΜΝ, ΝΞ ἄρα μέσαι εἰσίν. καὶ ἐπεὶ σύμμετρος ἡ ΑΗ τῇ ΗΕ μήκει, σύμμετρόν ἐστι καὶ τὸ ΑΘ τῷ ΗΚ, τουτέστι τὸ ΣΝ τῷ ΝΠ, τουτέστι τὸ ἀπὸ τῆς ΜΝ τῷ ἀπὸ τῆς ΝΞ [ὥστε δυνάμει εἰσὶ σύμμετροι αἱ ΜΝ, ΝΞ]. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῇ ΕΔ μήκει, ἀλλ᾿ ἡ μὲν ΑΕ σύμμετρός ἐστι τῇ ΑΗ, ἡ δὲ ΕΔ τῇ ΕΖ σύμμετρος, ἀσύμμετρος ἄρα ἡ ΑΗ τῇ ΕΖ· ὥστε καὶ τὸ ΑΘ τῷ ΕΛ ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΣΝ τῷ ΜΡ, τουτέστιν ὁ ΟΝ τῇ ΝΡ, τουτέστιν ἡ ΜΝ τῇ ΝΞ ἀσύμμετρός

Proposition 55

If an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called first bimedial.†



For let the area *ABCD* be contained by the rational (straight-line) *AB* and by the second binomial (straight-line) *AD*. I say that the square-root of area *AC* is a first bimedial (straight-line).

For since *AD* is a second binomial (straight-line), let it be divided into its (component) terms at *E*, such that *AE* is the greater term. Thus, *AE* and *ED* are rational (straight-lines which are) commensurable in square only, and the square on *AE* is greater than (the square on) *ED* by the (square) on (some straight-line) commensurable (in length) with (*AE*), and the lesser term *ED* is commensurable in length with *AB* [Def. 10.6]. Let *ED* be cut in half at *F*. And let the (rectangle contained) by *AGE*, equal to the (square) on *EF*, be applied to *AE*, falling short by a square figure. *AG* (is) thus commensurable in length with *GE* [Prop. 10.17]. And let *GH*, *EK*, and *FL* be drawn through (points) *G*, *E*, and *F* (respectively), parallel to *AB* and *CD*. And let the square *SN*, equal to the parallelogram *AH*, be constructed, and the square *NQ*, equal to *GK*. And let *MN* be laid down so as to be straight-on to *NO*. Thus, *RN* [is] also straight-on to *NP*. And let the square *SQ* be completed. So, (it is) clear from what has been previously demonstrated [Prop. 10.53 lem.] that *MR* is the mean proportional to *SN* and *NQ*, and (is) equal to *EL*, and that *MO* is the square-root of the area *AC*. So, we must show that *MO* is a first bimedial (straight-line).

Since *AE* is incommensurable in length with *ED*, and *ED* (is) commensurable (in length) with *AB*, *AE* (is) thus incommensurable (in length) with *AB* [Prop. 10.13]. And since *AG* is commensurable (in length) with *EG*, *AE* is also commensurable (in length) with each of *AG* and *GE* [Prop. 10.15]. But, *AE* is incommensurable in length with *AB*. Thus, *AG* and *GE* are also (both) incommensurable (in length) with *AB* [Prop. 10.13]. Thus, *BA*, *AG*, and (*BA*, and) *GE* are (pairs of) rational (straight-lines which are) commensurable in square only. And, hence, each of *AH* and *GK* is a medial (area) [Prop. 10.21]. Hence, each of *SN* and *NQ* is also a medial *ἐστι μήκει. ἐδείχθησαν δὲ αἱ ΜΝ, ΝΞ καὶ μέσαι οὖσαι καὶ δυνάμει σύμμετροι· αἱ ΜΝ, ΝΞ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι καὶ ῥητὸν περιέχουσιν. ἐπεὶ γὰρ ἡ ΔΕ ὑπόκειται ἑκατέρᾳ τῶν ΑΒ, ΕΖ σύμμετρος, σύμμετρος ἄρα καὶ ἡ ΕΖ τῇ ΕΚ. καὶ ῥητὴ ἑκατέρα αὐτῶν· ῥητὸν ἄρα τὸ ΕΛ, τουτέστι τὸ ΜΡ· τὸ δὲ ΜΡ ἐστι τὸ ὑπὸ τῶν ΜΝΞ. ἐὰν δὲ δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ῥητὸν περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλεῖται δὲ ἐκ δύο μέσων πρώτη.*

Ἡ ἄρα ΜΞ ἐκ δύο μέσων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

(area). Thus, *MN* and *NO* are medial (straight-lines). And since *AG* (is) commensurable in length with *GE*, *AH* is also commensurable with *GK*—that is to say, *SN* with *NQ*—that is to say, the (square) on *MN* with the (square) on *NO* [hence, *MN* and *NO* are commensurable in square] [Props. 6.1, 10.11]. And since *AE* is incommensurable in length with *ED*, but *AE* is commensurable (in length) with *AG*, and *ED* commensurable (in length) with *EF*, *AG* (is) thus incommensurable (in length) with *EF* [Prop. 10.13]. Hence, *AH* is also incommensurable with *EL*—that is to say, *SN* with *MR*—that is to say, *PN* with *NR*—that is to say, *MN* is incommensurable in length with *NO* [Props. 6.1, 10.11]. But *MN* and *NO* have also been shown to be medial (straight-lines) which are commensurable in square. Thus, *MN* and *NO* are medial (straight-lines which are) commensurable in square only. So, I say that they also contain a rational (area). For since *DE* was assumed (to be) commensurable (in length) with each of *AB* and *EF*, *EF* (is) thus also commensurable with *EK* [Prop. 10.12]. And they (are) each rational. Thus, *EL*—that is to say, *MR*—(is) rational [Prop. 10.19]. And *MR* is the (rectangle contained) by *MNO*. And if two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together, then the whole is (that) irrational (straight-line which is) called first bimedial [Prop. 10.37].

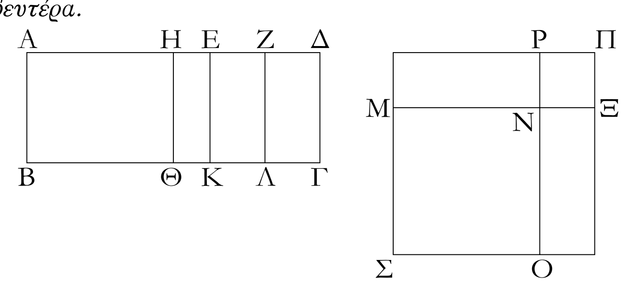
Thus, *MO* is a first bimedial (straight-line). (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a second binomial straight-line is a first bimedial straight-line: i.e., a second binomial straight-line has a length $k/\sqrt{1-k'^2}+k$ whose square-root can be written $\rho\,(k''^{1/4}+k''^{3/4})$, where $\rho = \sqrt{(k/2)\,(1+k')/(1-k')}$ and $k'' = (1-k')/(1+k')$. This is the length of a first bimedial straight-line (see Prop. 10.37), since ρ is rational.

νς΄.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.

Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τρίτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὧν μεῖζόν ἐστι τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.

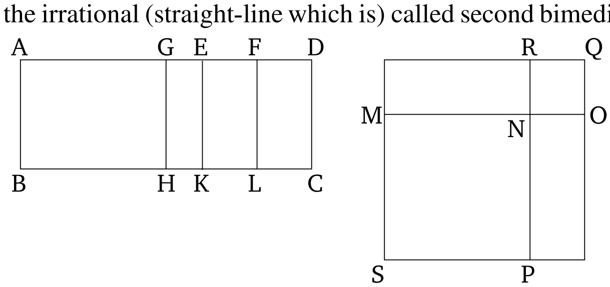


Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶ τρίτη ἡ ΑΔ, αἱ ΑΕ, ΕΔ ἄρα ῥηταί εἰσι

Proposition 56

If an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called second bimedial.†

For let the area *ABCD* be contained by the rational (straight-line) *AB* and by the third binomial (straight-line) *AD*, which has been divided into its (component) terms at *E*, of which *AE* is the greater. I say that the square-root of area *AC* is the irrational (straight-line which is) called second bimedial.



For let the same construction be made as previously. And since *AD* is a third binomial (straight-line), *AE* and *ED* are *δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ οὐδετέρα τῶν ΑΕ, ΕΔ σύμμετρός [ἐστι] τῇ ΑΒ μήκει. ὁμοίως δὴ τοῖς προδεδειγμένοις δείξομεν, ὅτι ἡ ΜΞ ἐστιν ἡ τὸ ΑΓ χωρίον δυναμένη, καὶ αἱ ΜΝ, ΝΞ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· ὥστε ἡ ΜΞ ἐκ δύο μέσων ἐστίν. δεικτέον δή, ὅτι καὶ δευτέρα.*

[Καὶ] ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΔΕ τῇ ΑΒ μήκει, τουτέστι τῇ ΕΚ, σύμμετρος δὲ ἡ ΔΕ τῇ ΕΖ, ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ ΕΚ μήκει. καί εἰσι ῥηταί· αἱ ΖΕ, ΕΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. μέσον ἄρα [ἐστὶ] τὸ ΕΛ, τουτέστι τὸ ΜΡ· καὶ περιέχεται ὑπὸ τῶν ΜΝΞ· μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΜΝΞ.

Ἡ ΜΞ ἄρα ἐκ δύο μέσων ἐστὶ δευτέρα· ὅπερ ἔδει δεῖξαι.

thus rational (straight-lines which are) commensurable in square only, and the square on *AE* is greater than (the square on) *ED* by the (square) on (some straight-line) commensurable (in length) with (*AE*), and neither of *AE* and *ED* [is] commensurable in length with *AB* [Def. 10.7]. So, similarly to that which has been previously demonstrated, we can show that *MO* is the square-root of area *AC*, and *MN* and *NO* are medial (straight-lines which are) commensurable in square only. Hence, *MO* is bimedial. So, we must show that (it is) also second (bimedial).

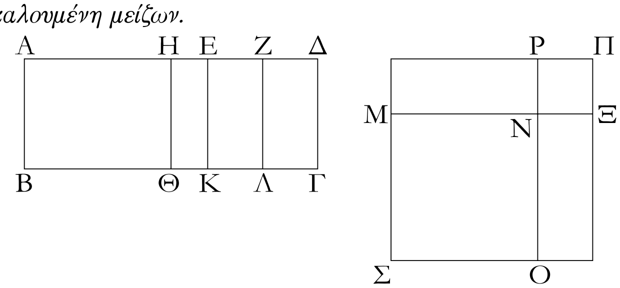
[And] since *DE* is incommensurable in length with *AB*—that is to say, with *EK*—and *DE* (is) commensurable (in length) with *EF*, *EF* is thus incommensurable in length with *EK* [Prop. 10.13]. And they are (both) rational (straight-lines). Thus, *FE* and *EK* are rational (straight-lines which are) commensurable in square only. *EL*—that is to say, *MR*—[is] thus medial [Prop. 10.21]. And it is contained by *MNO*. Thus, the (rectangle contained) by *MNO* is medial.

Thus, *MO* is a second bimedial (straight-line) [Prop. 10.38]. (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a third binomial straight-line is a second bimedial straight-line: i.e., a third binomial straight-line has a length $k^{1/2}\,(1 + \sqrt{1-k'^2})$ whose square-root can be written $\rho\,(k^{1/4} + k''^{1/2}/k^{1/4})$, where $\rho = \sqrt{(1+k')/2}$ and $k'' = k\,(1-k')/(1+k')$. This is the length of a second bimedial straight-line (see Prop. 10.38), since ρ is rational.

νζ'.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.

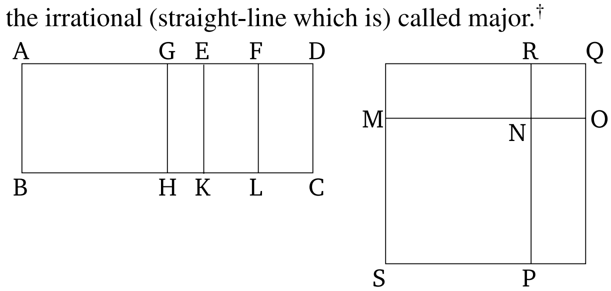


Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὧν μεῖζον ἔστω τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.

Ἐπεὶ γὰρ ἡ ΑΔ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη, αἱ ΑΕ, ΕΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΑΕ τῇ ΑΒ σύμμετρός [ἐστι] μήκει. τετμήσθω ἡ ΔΕ δίχα κατὰ τὸ Ζ, καὶ τῷ ἀπὸ τῆς ΕΖ ἴσον παρὰ τὴν ΑΕ παραβεβλήσθω παραλληλόγραμμον τὸ ὑπὸ ΑΗ, ΗΕ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΗΕ μήκει. ἤχθωσαν παράλληλοι τῇ ΑΒ αἱ ΗΘ, ΕΚ, ΖΛ, καὶ τὰ λοιπὰ τὰ αὐτὰ τοῖς πρὸ τούτου γεγονέτω· φανερὸν δή, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐστὶν ἡ ΜΞ. δεικτέον δή,

Proposition 57

If an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called major.†



For let the area *AC* be contained by the rational (straight-line) *AB* and the fourth binomial (straight-line) *AD*, which has been divided into its (component) terms at *E*, of which let *AE* be the greater. I say that the square-root of *AC* is the irrational (straight-line which is) called major.

For since *AD* is a fourth binomial (straight-line), *AE* and *ED* are thus rational (straight-lines which are) commensurable in square only, and the square on *AE* is greater than (the square on) *ED* by the (square) on (some straight-line) incommensurable (in length) with (*AE*), and *AE* [is] commensurable in length with *AB* [Def. 10.8]. Let *DE* be cut in half at *F*, and let the parallelogram (contained by) *AG* and *GE*, equal to the (square) on *EF*, (and falling short by a square figure) be ap-

ὅτι ἡ ΜΞ ἄλογός ἐστιν ἡ καλουμένη μείζων.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΗ τῇ ΕΗ μήκει, ἀσύμμετρόν ἐστι καὶ τὸ ΑΘ τῷ ΗΚ, τουτέστι τὸ ΣΝ τῷ ΝΠ· αἱ ΜΝ, ΝΞ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΕ τῇ ΑΒ μήκει, ῥητόν ἐστι τὸ ΑΚ· καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΜΝ, ΝΞ· ῥητὸν ἄρα [ἐστὶ] καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. καὶ ἐπεὶ ἀσύμμετρός [ἐστιν] ἡ ΔΕ τῇ ΑΒ μήκει, τουτέστι τῇ ΕΚ, ἀλλὰ ἡ ΔΕ σύμμετρός ἐστι τῇ ΕΖ, ἀσύμμετρος ἄρα ἡ ΕΖ τῇ ΕΚ μήκει. αἱ ΕΚ, ΕΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΛΕ, τουτέστι τὸ ΜΡ. καὶ περιέχεται ὑπὸ τῶν ΜΝ, ΝΞ· μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΜΝ, ΝΞ. καὶ ῥητὸν τὸ [συγκείμενον] ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, καί εἰσιν ἀσύμμετροι αἱ ΜΝ, ΝΞ δυνάμει. ἐὰν δὲ δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον, ἡ ὅλη ἄλογός ἐστιν, καλεῖται δὲ μείζων.

Ἡ ΜΞ ἄρα ἄλογός ἐστιν ἡ καλουμένη μείζων, καὶ δύναται τὸ ΑΓ χωρίον· ὅπερ ἔδει δεῖξαι.

plied to AE. AG is thus incommensurable in length with GE [Prop. 10.18]. Let GH, EK, and FL be drawn parallel to AB, and let the rest (of the construction) be made the same as the (proposition) before this. So, it is clear that MO is the square-root of area AC. So, we must show that MO is the irrational (straight-line which is) called major.

Since AG is incommensurable in length with EG, AH is also incommensurable with GK—that is to say, SN with NQ [Props. 6.1, 10.11]. Thus, MN and NO are incommensurable in square. And since AE is commensurable in length with AB, AK is rational [Prop. 10.19]. And it is equal to the (sum of the squares) on MN and NO. Thus, the sum of the (squares) on MN and NO [is] also rational. And since DE [is] incommensurable in length with AB [Prop. 10.13]—that is to say, with EK—but DE is commensurable (in length) with EF, EF (is) thus incommensurable in length with EK [Prop. 10.13]. Thus, EK and EF are rational (straight-lines which are) commensurable in square only. LE—that is to say, MR—(is) thus medial [Prop. 10.21]. And it is contained by MN and NO. The (rectangle contained) by MN and NO is thus medial. And the [sum] of the (squares) on MN and NO (is) rational, and MN and NO are incommensurable in square. And if two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together, then the whole is the irrational (straight-line which is) called major [Prop. 10.39].

Thus, MO is the irrational (straight-line which is) called major. And (it is) the square-root of area AC. (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a fourth binomial straight-line is a major straight-line: i.e., a fourth binomial straight-line has a length $k\,(1+1/\sqrt{1+k'})$ whose square-root can be written $\rho\,\sqrt{[1+k''/(1+k''^2)^{1/2}]/2}+\rho\,\sqrt{[1-k''/(1+k''^2)^{1/2}]/2}$, where $\rho=\sqrt{k}$ and $k''^2=k'$. This is the length of a major straight-line (see Prop. 10.39), since ρ is rational.

νη΄.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ῥητὸν καὶ μέσον δυναμένη.

Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· λέγω [δή], ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ῥητὸν καὶ μέσον δυναμένη.

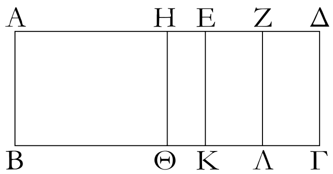
Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον δεδειγμένοις· φανερὸν δή, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐστὶν ἡ ΜΞ. δεικτέον δή, ὅτι ἡ ΜΞ ἐστιν ἡ ῥητὸν καὶ μέσον δυναμένη.

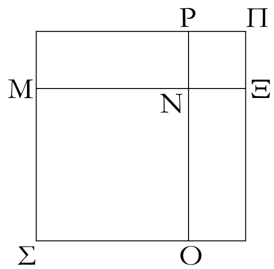
Proposition 58

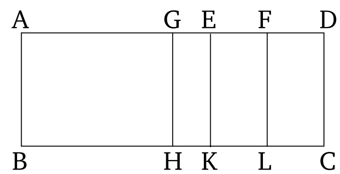
If an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).†

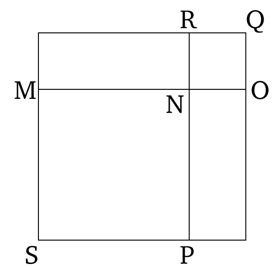
For let the area AC be contained by the rational (straight-line) AB and the fifth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term. [So] I say that the square-root of area AC is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).

For let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of area AC. So, we must show that MO is the square-root of a rational plus a medial (area).









Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΗ τῇ ΗΕ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΑΘ τῷ ΘΕ, τουτέστι τὸ ἀπὸ τῆς ΜΝ τῷ ἀπὸ τῆς ΝΞ· αἱ ΜΝ, ΝΞ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ ΑΔ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη, καί [ἐστιν] ἔλασσον αὐτῆς τμῆμα τὸ ΕΔ, σύμμετρος ἄρα ἡ ΕΔ τῇ ΑΒ μήκει. ἀλλὰ ἡ ΑΕ τῇ ΕΔ ἐστιν ἀσύμμετρος· καὶ ἡ ΑΒ ἄρα τῇ ΑΕ ἐστιν ἀσύμμετρος μήκει [αἱ ΒΑ, ΑΕ ῥηταί εἰσι δυνάμει μόνον σύμμετροι]· μέσον ἄρα ἐστὶ τὸ ΑΚ, τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΑΒ μήκει, τουτέστι τῇ ΕΚ, ἀλλὰ ἡ ΔΕ τῇ ΕΖ σύμμετρός ἐστιν, καὶ ἡ ΕΖ ἄρα τῇ ΕΚ σύμμετρός ἐστιν. καὶ ῥητὴ ἡ ΕΚ· ῥητὸν ἄρα καὶ τὸ ΕΛ, τουτέστι τὸ ΜΡ, τουτέστι τὸ ὑπὸ ΜΝΞ· αἱ ΜΝ, ΝΞ ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν.

Ἡ ΜΞ ἄρα ῥητὸν καὶ μέσον δυναμένη ἐστὶ καὶ δύναται τὸ ΑΓ χωρίον· ὅπερ ἔδει δεῖξαι.

For since *AG* is incommensurable (in length) with *GE* [Prop. 10.18], *AH* is thus also incommensurable with *HE*—that is to say, the (square) on *MN* with the (square) on *NO* [Props. 6.1, 10.11]. Thus, *MN* and *NO* are incommensurable in square. And since *AD* is a fifth binomial (straight-line), and *ED* [is] its lesser segment, *ED* (is) thus commensurable in length with *AB* [Def. 10.9]. But, *AE* is incommensurable (in length) with *ED*. Thus, *AB* is also incommensurable in length with *AE* [*BA* and *AE* are rational (straight-lines which are) commensurable in square only] [Prop. 10.13]. Thus, *AK*—that is to say, the sum of the (squares) on *MN* and *NO*—is medial [Prop. 10.21]. And since *DE* is commensurable in length with *AB*—that is to say, with *EK*—but, *DE* is commensurable (in length) with *EF*, *EF* is thus also commensurable (in length) with *EK* [Prop. 10.12]. And *EK* (is) rational. Thus, *EL*—that is to say, *MR*—that is to say, the (rectangle contained) by *MNO*—(is) also rational [Prop. 10.19]. *MN* and *NO* are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

Thus, *MO* is the square-root of a rational plus a medial (area) [Prop. 10.40]. And (it is) the square-root of area *AC*. (Which is) the very thing it was required to show.

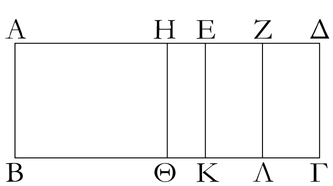
† If the rational straight-line has unit length then this proposition states that the square-root of a fifth binomial straight-line is the square root of a rational plus a medial area: i.e., a fifth binomial straight-line has a length $k\,(\sqrt{1+k'}+1)$ whose square-root can be written $\rho\,\sqrt{[(1+k''^2)^{1/2}+k'']/[2\,(1+k''^2)]}+\rho\,\sqrt{[(1+k''^2)^{1/2}-k'']/[2\,(1+k''^2)]}$, where $\rho=\sqrt{k\,(1+k''^2)}$ and $k''^2=k'$. This is the length of the square root of a rational plus a medial area (see Prop. 10.40), since ρ is rational.

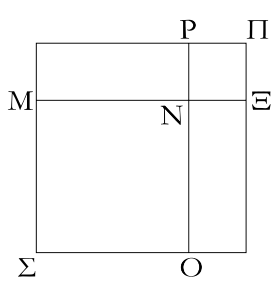
νθ´.

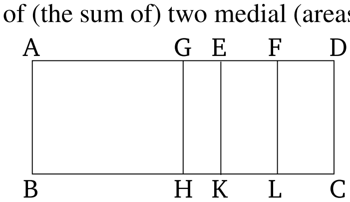
Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη δύο μέσα δυναμένη.

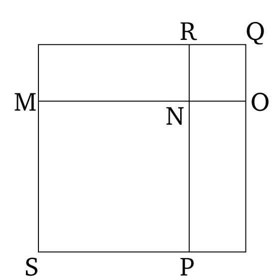
Proposition 59

If an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of (the sum of) two medial (areas).†









Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης τῆς ΑΔ διῃρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

Κατεσκευάσθω [γὰρ] τὰ αὐτὰ τοῖς προδεδειγμένοις. φανερὸν δή, ὅτι [ἡ] τὸ ΑΓ δυναμένη ἐστὶν ἡ ΜΞ, καὶ ὅτι ἀσύμμετρός ἐστιν ἡ ΜΝ τῇ ΝΞ δυνάμει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΑ τῇ ΑΒ μήκει, αἱ ΕΑ, ΑΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ ΑΚ, τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΔ τῇ ΑΒ μήκει, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΖΕ τῇ ΕΚ· αἱ ΖΕ, ΕΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ ΕΛ, τουτέστι τὸ ΜΡ, τουτέστι τὸ ὑπὸ τῶν ΜΝΞ. καὶ ἐπεὶ ἀσύμμετρος ἡ ΑΕ τῇ ΕΖ, καὶ τὸ ΑΚ τῷ ΕΛ ἀσύμμετρόν ἐστιν. ἀλλὰ τὸ μὲν ΑΚ ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, τὸ δὲ ΕΛ ἐστι τὸ ὑπὸ τῶν ΜΝΞ· ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝΞ τῷ ὑπὸ τῶν ΜΝΞ. καί ἐστι μέσον ἑκάτερον αὐτῶν, καὶ αἱ ΜΝ, ΝΞ δυνάμει εἰσὶν ἀσύμμετροι.

Ἡ ΜΞ ἄρα δύο μέσα δυναμένη ἐστὶ καὶ δύναται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

For let the area $ABCD$ be contained by the rational (straight-line) AB and the sixth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term. So, I say that the square-root of AC is the square-root of (the sum of) two medial (areas).

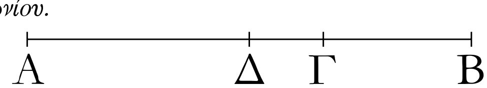
[For] let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of AC, and that MN is incommensurable in square with NO. And since EA is incommensurable in length with AB [Def. 10.10], EA and AB are thus rational (straight-lines which are) commensurable in square only. Thus, AK—that is to say, the sum of the (squares) on MN and NO—is medial [Prop. 10.21]. Again, since ED is incommensurable in length with AB [Def. 10.10], FE is thus also incommensurable (in length) with EK [Prop. 10.13]. Thus, FE and EK are rational (straight-lines which are) commensurable in square only. Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—is medial [Prop. 10.21]. And since AE is incommensurable (in length) with EF, AK is also incommensurable with EL [Props. 6.1, 10.11]. But, AK is the sum of the (squares) on MN and NO, and EL is the (rectangle contained) by MNO. Thus, the sum of the (squares) on MNO is incommensurable with the (rectangle contained) by MNO. And each of them is medial. And MN and NO are incommensurable in square.

Thus, MO is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. And (it is) the square-root of AC. (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a sixth binomial straight-line is the square root of the sum of two medial areas: i.e., a sixth binomial straight-line has a length $\sqrt{k} + \sqrt{k'}$ whose square-root can be written $k^{1/4}\left(\sqrt{[1 + k''/(1 + k''^2)^{1/2}]/2} + \sqrt{[1 - k''/(1 + k''^2)^{1/2}]/2}\right)$, where $k''^2 = (k - k')/k'$. This is the length of the square-root of the sum of two medial areas (see Prop. 10.41).

Λῆμμα.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τετράγωνα μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ἀνίσων περιεχομένου ὀρθογωνίου.

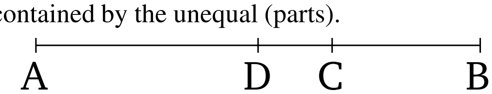


Ἔστω εὐθεῖα ἡ ΑΒ καὶ τετμήσθω εἰς ἄνισα κατὰ τὸ Γ, καὶ ἔστω μείζων ἡ ΑΓ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ.

Τετμήσθω γὰρ ἡ ΑΒ δίχα κατὰ τὸ Δ. ἐπεὶ οὖν εὐθεῖα γραμμὴ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Δ, εἰς δὲ ἄνισα κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν ΑΓ, ΓΒ μετὰ τοῦ ἀπὸ ΓΔ ἴσον ἐστὶ τῷ ἀπὸ ΑΔ· ὥστε τὸ ὑπὸ τῶν ΑΓ, ΓΒ ἔλαττόν ἐστι τοῦ ἀπὸ ΑΔ· τὸ ἄρα δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἔλαττον ἢ διπλάσιόν ἐστι τοῦ ἀπὸ ΑΔ. ἀλλὰ τὰ ἀπὸ τῶν ΑΓ, ΓΒ διπλάσιά [ἐστι] τῶν ἀπὸ τῶν ΑΔ, ΔΓ· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· ὅπερ ἔδει δεῖξαι.

Lemma

If a straight-line is cut unequally then (the sum of) the squares on the unequal (parts) is greater than twice the rectangle contained by the unequal (parts).

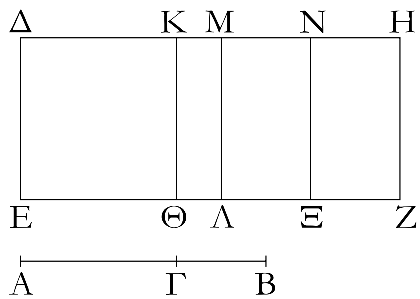


Let AB be a straight-line, and let it be cut unequally at C, and let AC be greater (than CB). I say that (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB.

For let AB be cut in half at D. Therefore, since a straight-line has been cut into equal (parts) at D, and into unequal (parts) at C, the (rectangle contained) by AC and CB, plus the (square) on CD, is thus equal to the (square) on AD [Prop. 2.5]. Hence, the (rectangle contained) by AC and CB is less than the (square) on AD. Thus, twice the (rectangle contained) by AC and CB is less than double the (square) on AD. But, (the sum

ξ′.

Τὸ ἀπὸ τῆς ἐκ δύο ὀνομάτων παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην.



Ἔστω ἐκ δύο ὀνομάτων ἡ ΑΒ διῃρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΕΖΗ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ πρώτη.

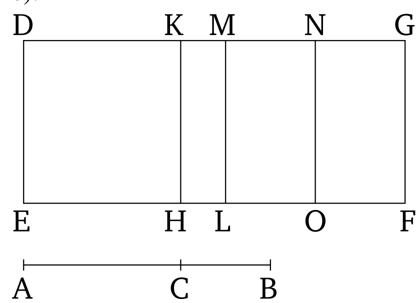
Παραβεβλήσθω γὰρ παρὰ τὴν ΔΕ τῷ μὲν ἀπὸ τῆς ΑΓ ἴσον τὸ ΔΘ, τῷ δὲ ἀπὸ τῆς ΒΓ ἴσον τὸ ΚΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον ἐστὶ τῷ ΜΖ. τετμήσθω ἡ ΜΗ δίχα κατὰ τὸ Ν, καὶ παράλληλος ἤχθω ἡ ΝΞ [ἑκατέρᾳ τῶν ΜΛ, ΗΖ]. ἑκάτερον ἄρα τῶν ΜΞ, ΝΖ ἴσον ἐστὶ τῷ ἅπαξ ὑπὸ τῶν ΑΓΒ. καὶ ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶν ἡ ΑΒ διῃρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΒ ῥητά ἐστι καὶ σύμμετρα ἀλλήλοις· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ. καί ἐστιν ἴσον τῷ ΔΛ· ῥητὸν ἄρα ἐστὶ τὸ ΔΛ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΔΜ καὶ σύμμετρος τῇ ΔΕ μήκει. πάλιν, ἐπεὶ αἱ ΑΓ, ΓΒ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, μέσον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΜΖ. καὶ παρὰ ῥητὴν τὴν ΜΛ παράκειται· ῥητὴ ἄρα καὶ ἡ ΜΗ καὶ ἀσύμμετρος τῇ ΜΛ, τουτέστι τῇ ΔΕ, μήκει. ἔστι δὲ καὶ ἡ ΜΔ ῥητὴ καὶ τῇ ΔΕ μήκει σύμμετρος· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΜ τῇ ΜΗ μήκει. καί εἰσι ῥηταί· αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον δή, ὅτι καὶ πρώτη.

Ἐπεὶ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΓΒ, καὶ τῶν ΔΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΜΞ. ἔστιν ἄρα ὡς τὸ ΔΘ πρὸς τὸ ΜΞ, οὕτως τὸ ΜΞ πρὸς τὸ ΚΛ, τουτέστιν ὡς ἡ ΔΚ πρὸς τὴν ΜΝ, ἡ ΜΝ πρὸς τὴν ΜΚ· τὸ ἄρα ὑπὸ τῶν ΔΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, σύμμετρόν ἐστι καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε καὶ ἡ ΔΚ τῇ ΚΜ σύμμετρός

of) the (squares) on *AC* and *CB* [is] double (the sum of) the (squares) on *AD* and *DC* [Prop. 2.9]. Thus, (the sum of) the (squares) on *AC* and *CB* is greater than twice the (rectangle contained) by *AC* and *CB*. (Which is) the very thing it was required to show.

Proposition 60

The square on a binomial (straight-line) applied to a rational (straight-line) produces as breadth a first binomial (straight-line).†



Let *AB* be a binomial (straight-line), having been divided into its (component) terms at *C*, such that *AC* is the greater term. And let the rational (straight-line) *DE* be laid down. And let the (rectangle) *DEFG*, equal to the (square) on *AB*, be applied to *DE*, producing *DG* as breadth. I say that *DG* is a first binomial (straight-line).

For let *DH*, equal to the (square) on *AC*, and *KL*, equal to the (square) on *BC*, be applied to *DE*. Thus, the remaining twice the (rectangle contained) by *AC* and *CB* is equal to *MF* [Prop. 2.4]. Let *MG* be cut in half at *N*, and let *NO* be drawn parallel [to each of *ML* and *GF*]. *MO* and *NF* are thus each equal to once the (rectangle contained) by *ACB*. And since *AB* is a binomial (straight-line), having been divided into its (component) terms at *C*, *AC* and *CB* are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Thus, the (squares) on *AC* and *CB* are rational, and commensurable with one another. And hence the sum of the (squares) on *AC* and *CB* (is rational) [Prop. 10.15], and is equal to *DL*. Thus, *DL* is rational. And it is applied to the rational (straight-line) *DE*. *DM* is thus rational, and commensurable in length with *DE* [Prop. 10.20]. Again, since *AC* and *CB* are rational (straight-lines which are) commensurable in square only, twice the (rectangle contained) by *AC* and *CB*—that is to say, *MF*—is thus medial [Prop. 10.21]. And it is applied to the rational (straight-line) *ML*. *MG* is thus also rational, and incommensurable in length with *ML*—that is to say, with *DE* [Prop. 10.22]. And *MD* is also rational, and commensurable in length with *DE*. Thus, *DM* is incommensurable in length with *MG* [Prop. 10.13]. And they are rational. *DM* and *MG* are thus rational (straight-lines which are) commensurable in square only. Thus, *DG* is a binomial (straight-line) *ἐστιν. καὶ ἐπεὶ μείζονά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, μεῖζον ἄρα καὶ τὸ ΔΛ τοῦ ΜΖ· ὥστε καὶ ἡ ΔΜ τῆς ΜΗ μείζων ἐστίν. καί ἐστιν ἴσον τὸ ὑπὸ τῶν ΔΚ, ΚΜ τῷ ἀπὸ τῆς ΜΝ, τουτέστι τῷ τετάρτῳ τοῦ ἀπὸ τῆς ΜΗ -.7πτ η, καὶ σύμμετρος ἡ ΔΚ τῇ ΚΜ. ἐὰν δὲ ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρῇ, ἡ μείζων τῆς ἐλάσσονος μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί εἰσι ῥηταὶ αἱ ΔΜ, ΜΗ , καὶ ἡ ΔΜ μεῖζον ὄνομα οὖσα σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΔΕ μήκει.*

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

[Prop. 10.36]. So, we must show that (it is) also a first (binomial straight-line).

Since the (rectangle contained) by *ACB* is the mean proportional to the squares on *AC* and *CB* [Prop. 10.53 lem.], *MO* is thus also the mean proportional to *DH* and *KL*. Thus, as *DH* is to *MO*, so *MO* (is) to *KL*—that is to say, as *DK* (is) to *MN*, (so) *MN* (is) to *MK* [Prop. 6.1]. Thus, the (rectangle contained) by *DK* and *KM* is equal to the (square) on *MN* [Prop. 6.17]. And since the (square) on *AC* is commensurable with the (square) on *CB*, *DH* is also commensurable with *KL*. Hence, *DK* is also commensurable with *KM* [Props. 6.1, 10.11]. And since (the sum of) the squares on *AC* and *CB* is greater than twice the (rectangle contained) by *AC* and *CB* [Prop. 10.59 lem.], *DL* (is) thus also greater than *MF*. Hence, *DM* is also greater than *MG* [Props. 6.1, 5.14]. And the (rectangle contained) by *DK* and *KM* is equal to the (square) on *MN*—that is to say, to one quarter the (square) on *MG*. And *DK* (is) commensurable (in length) with *KM*. And if there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable (in length), then the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable (in length) with the greater [Prop. 10.17]. Thus, the square on *DM* is greater than (the square on) *MG* by the (square) on (some straight-line) commensurable (in length) with (*DM*). And *DM* and *MG* are rational. And *DM*, which is the greater term, is commensurable in length with the (previously) laid down rational (straight-line) *DE*.

Thus, *DG* is a first binomial (straight-line) [Def. 10.5]. (Which is) the very thing it was required to show.

† In other words, the square of a binomial is a first binomial. See Prop. 10.54.

ξα΄.

Τὸ ἀπὸ τῆς ἐκ δύο μέσων πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν.

Ἔστω ἐκ δύο μέσων πρώτη ἡ ΑΒ διῃρημένη εἰς τὰς μέσας κατὰ τὸ Γ, ὧν μείζων ἡ ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ παραβεβλήσθω παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς ΑΒ ἴσον παραλληλόγραμμον τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρὸ τούτου. καὶ ἐπεὶ ἡ ΑΒ ἐκ δύο μέσων ἐστὶ πρώτη διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ὥστε καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ μέσα ἐστίν. μέσον ἄρα ἐστὶ τὸ ΔΛ. καὶ παρὰ ῥητὴν τὴν ΔΕ παραβέβληται· ῥητὴ ἄρα ἐστίν ἡ ΜΔ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. πάλιν, ἐπεὶ ῥητόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ῥητόν ἐστι καὶ τὸ ΜΖ. καὶ παρὰ ῥητὴν τὴν ΜΛ παράκειται· ῥητὴ ἄρα [ἐστὶ]

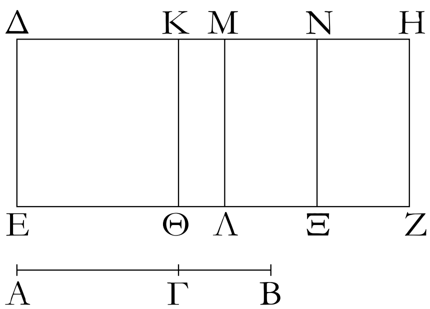
Proposition 61

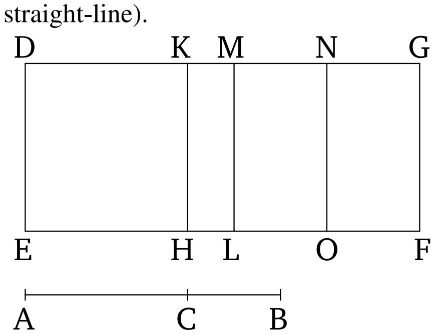
The square on a first bimedial (straight-line) applied to a rational (straight-line) produces as breadth a second binomial (straight-line).†

Let *AB* be a first bimedial (straight-line) having been divided into its (component) medial (straight-lines) at *C*, of which *AC* (is) the greater. And let the rational (straight-line) *DE* be laid down. And let the parallelogram *DF*, equal to the (square) on *AB*, be applied to *DE*, producing *DG* as breadth. I say that *DG* is a second binomial (straight-line).

For let the same construction be made as in the (proposition) before this. And since *AB* is a first bimedial (straight-line), having been divided at *C*, *AC* and *CB* are thus medial (straight-lines) commensurable in square only, and containing a rational (area) [Prop. 10.37]. Hence, the (squares) on *AC* and *CB* are also medial [Prop. 10.21]. Thus, *DL* is me-καὶ ἡ ΜΗ καὶ μήκει σύμμετρος τῇ ΜΛ, τουτέστι τῇ ΔΕ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΜ τῇ ΜΗ μήκει. καί εἰσι ῥηταί· αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον δή, ὅτι καὶ δευτέρα.

dial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) *DE*. *MD* is thus rational, and incommensurable in length with *DE* [Prop. 10.22]. Again, since twice the (rectangle contained) by *AC* and *CB* is rational, *MF* is also rational. And it is applied to the rational (straight-line) *ML*. Thus, *MG* [is] also rational, and commensurable in length with *ML*—that is to say, with *DE* [Prop. 10.20]. *DM* is thus incommensurable in length with *MG* [Prop. 10.13]. And they are rational. *DM* and *MG* are thus rational, and commensurable in square only. *DG* is thus a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).





Ἐπεὶ γὰρ τὰ ἀπὸ τῶν ΑΓ, ΓΒ μείζονά ἐστι τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, μεῖζον ἄρα καὶ τὸ ΔΛ τοῦ ΜΖ· ὥστε καὶ ἡ ΔΜ τῆς ΜΗ . καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, σύμμετρόν ἐστι καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε καὶ ἡ ΔΚ τῇ ΚΜ σύμμετρός ἐστιν. καί ἐστι τὸ ὑπὸ τῶν ΔΚΜ ἴσον τῷ ἀπὸ τῆς ΜΝ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καί ἐστιν ἡ ΜΗ σύμμετρος τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

For since (the sum of) the squares on *AC* and *CB* is greater than twice the (rectangle contained) by *AC* and *CB* [Prop. 10.59], *DL* (is) thus also greater than *MF*. Hence, *DM* (is) also (greater) than *MG* [Prop. 6.1]. And since the (square) on *AC* is commensurable with the (square) on *CB*, *DH* is also commensurable with *KL*. Hence, *DK* is also commensurable (in length) with *KM* [Props. 6.1, 10.11]. And the (rectangle contained) by *DKM* is equal to the (square) on *MN*. Thus, the square on *DM* is greater than (the square on) *MG* by the (square) on (some straight-line) commensurable (in length) with (*DM*) [Prop. 10.17]. And *MG* is commensurable in length with *DE*.

Thus, *DG* is a second binomial (straight-line) [Def. 10.6].

† In other words, the square of a first bimedial is a second binomial. See Prop. 10.55.

ξβ´.

Τὸ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τρίτην.

Ἔστω ἐκ δύο μέσων δευτέρα ἡ ΑΒ διῃρημένη εἰς τὰς μέσας κατὰ τὸ Γ, ὥστε τὸ μεῖζον τμῆμα εἶναι τὸ ΑΓ, ῥητὴ δέ τις ἔστω ἡ ΔΕ, καὶ παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς ΑΒ ἴσον παραλληλόγραμμον παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ τρίτη.

Κατεσκευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ ἐκ δύο μέσων δευτέρα ἐστὶν ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΔΛ· μέσον ἄρα καὶ τὸ ΔΛ. καὶ

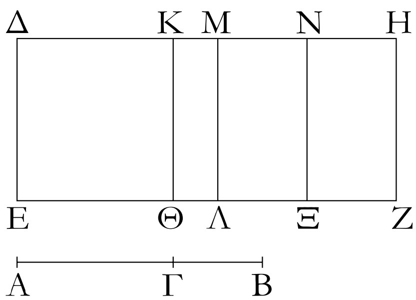
Proposition 62

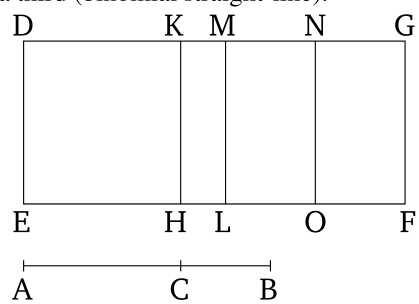
The square on a second bimedial (straight-line) applied to a rational (straight-line) produces as breadth a third binomial (straight-line).†

Let *AB* be a second bimedial (straight-line) having been divided into its (component) medial (straight-lines) at *C*, such that *AC* is the greater segment. And let *DE* be some rational (straight-line). And let the parallelogram *DF*, equal to the (square) on *AB*, be applied to *DE*, producing *DG* as breadth. I say that *DG* is a third binomial (straight-line).

Let the same construction be made as that shown previously. And since *AB* is a second bimedial (straight-line), having been divided at *C*, *AC* and *CB* are thus medial (straight-παράκειται παρὰ ῥητὴν τὴν ΔΕ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΜΔ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΜΗ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΜΛ, τουτέστι τῇ ΔΕ, μήκει· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΜ, ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΓ τῇ ΓΒ μήκει, ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓΒ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓΒ. ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓΒ ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΔΛ τῷ ΜΖ· ὥστε καὶ ἡ ΔΜ τῷ ΜΗ ἀσύμμετρός ἐστιν. καί εἰσι ῥηταί· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον [δή], ὅτι καὶ τρίτη.

lines) commensurable in square only, and containing a medial (area) [Prop. 10.38]. Hence, the sum of the (squares) on *AC* and *CB* is also medial [Props. 10.15, 10.23 corr.]. And it is equal to *DL*. Thus, *DL* (is) also medial. And it is applied to the rational (straight-line) *DE*. *MD* is thus also rational, and incommensurable in length with *DE* [Prop. 10.22]. So, for the same (reasons), *MG* is also rational, and incommensurable in length with *ML*—that is to say, with *DE*. Thus, *DM* and *MG* are each rational, and incommensurable in length with *DE*. And since *AC* is incommensurable in length with *CB*, and as *AC* (is) to *CB*, so the (square) on *AC* (is) to the (rectangle contained) by *ACB* [Prop. 10.21 lem.], the (square) on *AC* (is) also incommensurable with the (rectangle contained) by *ACB* [Prop. 10.11]. And hence the sum of the (squares) on *AC* and *CB* is incommensurable with twice the (rectangle contained) by *ACB*—that is to say, *DL* with *MF* [Props. 10.12, 10.13]. Hence, *DM* is also incommensurable (in length) with *MG* [Props. 6.1, 10.11]. And they are rational. *DG* is thus a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a third (binomial straight-line).





Ὁμοίως δὴ τοῖς προτέροις ἐπιλογιούμεθα, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ , καὶ σύμμετρος ἡ ΔΚ τῇ ΚΜ. καί ἐστι τὸ ὑπὸ τῶν ΔΚΜ ἴσον τῷ ἀπὸ τῆς ΜΝ· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ οὐδετέρα τῶν ΔΜ, ΜΗ σύμμετρός ἐστι τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

So, similarly to the previous (propositions), we can conclude that *DM* is greater than *MG*, and *DK* (is) commensurable (in length) with *KM*. And the (rectangle contained) by *DKM* is equal to the (square) on *MN*. Thus, the square on *DM* is greater than (the square on) *MG* by the (square) on (some straight-line) commensurable (in length) with (*DM*) [Prop. 10.17]. And neither of *DM* and *MG* is commensurable in length with *DE*.

Thus, *DG* is a third binomial (straight-line) [Def. 10.7]. (Which is) the very thing it was required to show.

[†] In other words, the square of a second bimedial is a third binomial. See Prop. 10.56.

ξγ´.

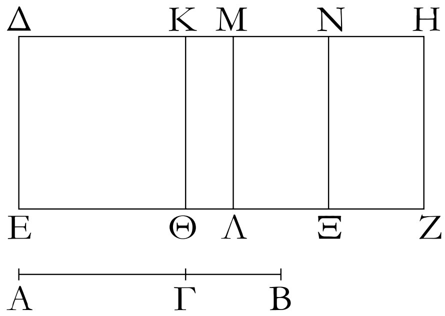
Τὸ ἀπὸ τῆς μείζονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τετάρτην.

Ἔστω μείζων ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ὥστε μείζονα εἶναι τὴν ΑΓ τῆς ΓΒ, ῥητὴ δὲ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ παραλληλόγραμμον πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη.

Proposition 63

The square on a major (straight-line) applied to a rational (straight-line) produces as breadth a fourth binomial (straight-line).[†]

Let *AB* be a major (straight-line) having been divided at *C*, such that *AC* is greater than *CB*, and (let) *DE* (be) a rational (straight-line). And let the parallelogram *DF*, equal to the (square) on *AB*, be applied to *DE*, producing *DG* as breadth. I

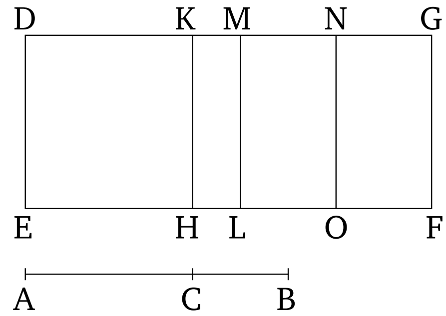


Κατεσκευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ μείζων ἐστὶν ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ᾽ αὐτῶν μέσον. ἐπεὶ οὖν ῥητόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, ῥητὸν ἄρα ἐστὶ τὸ ΔΛ· ῥητὴ ἄρα καὶ ἡ ΔΜ καὶ σύμμετρος τῇ ΔΕ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τουτέστι τὸ ΜΖ, καὶ παρὰ ῥητήν ἐστι τὴν ΜΛ, ῥητὴ ἄρα ἐστὶ καὶ ἡ ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΜ τῇ ΜΗ μήκει. αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον [δή], ὅτι καὶ τετάρτη.

Ὁμοίως δὴ δείξομεν τοῖς πρότερον, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ , καὶ ὅτι τὸ ὑπὸ ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. ἐπεὶ οὖν ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΓΒ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε ἀσύμμετρος καὶ ἡ ΔΚ τῇ ΚΜ ἐστιν. ἐὰν δὲ ὦσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παραλληλόγραμμον παρὰ τὴν μείζονα παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρῇ, ἡ μείζων τῆς ἐλάσσονος μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί εἰσιν αἱ ΔΜ, ΜΗ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΜ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΔΕ.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἔδει δεῖξαι.

say that DG is a fourth binomial (straight-line).



Let the same construction be made as that shown previously. And since AB is a major (straight-line), having been divided at C, AC and CB are incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. Therefore, since the sum of the (squares) on AC and CB is rational, DL is thus rational. Thus, DM (is) also rational, and commensurable in length with DE [Prop. 10.20]. Again, since twice the (rectangle contained) by AC and CB—that is to say, MF—is medial, and is (applied to) the rational (straight-line) ML, MG is thus also rational, and incommensurable in length with DE [Prop. 10.22]. DM is thus also incommensurable in length with MG [Prop. 10.13]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a fourth (binomial straight-line).

So, similarly to the previous (propositions), we can show that DM is greater than MG, and that the (rectangle contained) by DKM is equal to the (square) on MN. Therefore, since the (square) on AC is incommensurable with the (square) on CB, DH is also incommensurable with KL. Hence, DK is also incommensurable with KM [Props. 6.1, 10.11]. And if there are two unequal straight-lines, and a parallelogram equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable (in length), then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) incommensurable in length with the greater [Prop. 10.18]. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM). And DM and MG are rational (straight-lines which are) commensurable in square only. And DM is commensurable (in length) with the (previously) laid down rational (straight-line) DE.

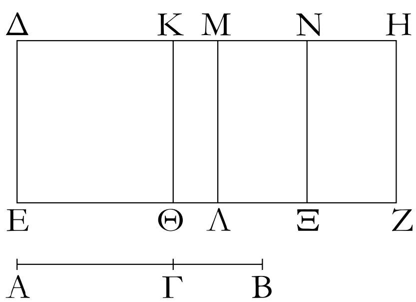
Thus, DG is a fourth binomial (straight-line) [Def. 10.8]. (Which is) the very thing it was required to show.

† In other words, the square of a major is a fourth binomial. See Prop. 10.57.

ξδ΄.

Τὸ ἀπὸ τῆς ῥητὸν καὶ μέσον δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην.

Ἔστω ῥητὸν καὶ μέσον δυναμένη ἡ ΑΒ διῃρημένη εἰς τὰς εὐθείας κατὰ τὸ Γ, ὥστε μείζονα εἶναι τὴν ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη.



Κατεσκευάσθω τὰ αὐτὰ τοῖς πρὸ τούτου. ἐπεὶ οὖν ῥητὸν καὶ μέσον δυναμένη ἐστὶν ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν. ἐπεὶ οὖν μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, μέσον ἄρα ἐστὶ τὸ ΔΛ· ὥστε ῥητή ἐστιν ἡ ΔΜ καὶ μήκει ἀσύμμετρος τῇ ΔΕ. πάλιν, ἐπεὶ ῥητόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓΒ, τουτέστι τὸ ΜΖ, ῥητὴ ἄρα ἡ ΜΗ καὶ σύμμετρος τῇ ΔΕ. ἀσύμμετρος ἄρα ἡ ΔΜ τῇ ΜΗ · αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. λέγω δή, ὅτι καὶ πέμπτη.

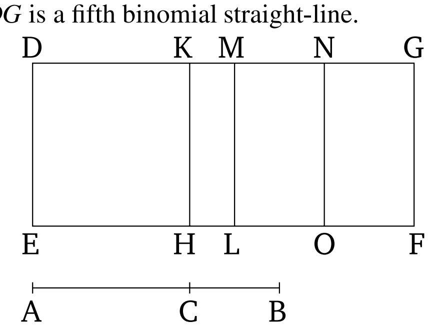
Ὁμοίως γὰρ διεχθήσεται, ὅτι τὸ ὑπὸ τῶν ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, καὶ ἀσύμμετρος ἡ ΔΚ τῇ ΚΜ μήκει· ἡ ΔΜ ἄρα τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰσιν αἱ ΔΜ, ΜΗ [ῥηταὶ] δυνάμει μόνον σύμμετροι, καὶ ἡ ἐλάσσων ἡ ΜΗ σύμμετρος τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

Proposition 64

The square on the square-root of a rational plus a medial (area) applied to a rational (straight-line) produces as breadth a fifth binomial (straight-line).†

Let *AB* be the square-root of a rational plus a medial (area) having been divided into its (component) straight-lines at *C*, such that *AC* is greater. And let the rational (straight-line) *DE* be laid down. And let the (parallelogram) *DF*, equal to the (square) on *AB*, be applied to *DE*, producing *DG* as breadth. I say that *DG* is a fifth binomial straight-line.



Let the same construction be made as in the (propositions) before this. Therefore, since *AB* is the square-root of a rational plus a medial (area), having been divided at *C*, *AC* and *CB* are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. Therefore, since the sum of the (squares) on *AC* and *CB* is medial, *DL* is thus medial. Hence, *DM* is rational and incommensurable in length with *DE* [Prop. 10.22]. Again, since twice the (rectangle contained) by *ACB*—that is to say, *MF*—is rational, *MG* (is) thus rational and commensurable (in length) with *DE* [Prop. 10.20]. *DM* (is) thus incommensurable (in length) with *MG* [Prop. 10.13]. Thus, *DM* and *MG* are rational (straight-lines which are) commensurable in square only. Thus, *DG* is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For, similarly (to the previous propositions), it can be shown that the (rectangle contained) by *DKM* is equal to the (square) on *MN*, and *DK* (is) incommensurable in length with *KM*. Thus, the square on *DM* is greater than (the square on) *MG* by the (square) on (some straight-line) incommensurable (in length) with (*DM*) [Prop. 10.18]. And *DM* and *MG* are [rational] (straight-lines which are) commensurable in square only, and the lesser *MG* is commensurable in length with *DE*.

Thus, *DG* is a fifth binomial (straight-line) [Def. 10.9]. (Which is) the very thing it was required to show.

† In other words, the square of the square-root of a rational plus medial is a fifth binomial. See Prop. 10.58.

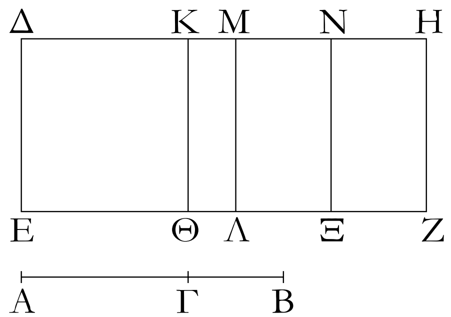
ξε΄.

Τὸ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ῥητὴν παρα-

Proposition 65

The square on the square-root of (the sum of) two medial *βαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην.*

Ἔστω δύο μέσα δυναμένη ἡ ΑΒ διῃρημένη κατὰ τὸ Γ, ῥητὴ δὲ ἔστω ἡ ΔΕ, καὶ παρὰ τὴν ΔΕ τῷ ἀπὸ τῆς ΑΒ ἴσον παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶν ἕκτη.



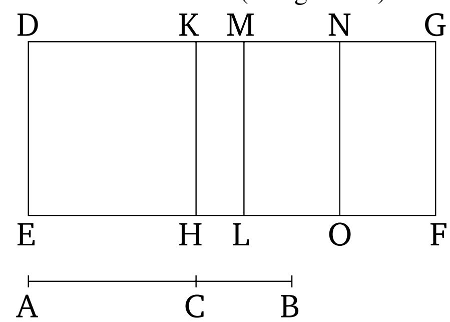
Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ ἡ ΑΒ δύο μέσα δυναμένη ἐστὶ διῃρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων συγκείμενον τῷ ὑπ᾽ αὐτῶν· ὥστε κατὰ τὰ προδεδειγμένα μέσον ἐστὶν ἑκάτερον τῶν ΔΛ, ΜΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρα τῶν ΔΜ, ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρον ἄρα ἐστὶ τὸ ΔΛ τῷ ΜΖ. ἀσύμμετρος ἄρα καὶ ἡ ΔΜ τῇ ΜΗ · αἱ ΔΜ, ΜΗ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. λέγω δή, ὅτι καὶ ἕκτη.

Ὁμοίως δὴ πάλιν δείξομεν, ὅτι τὸ ὑπὸ τῶν ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, καὶ ὅτι ἡ ΔΚ τῇ ΚΜ μήκει ἐστὶν ἀσύμμετρος· καὶ διὰ τὰ αὐτὰ δὴ ἡ ΔΜ τῆς ΜΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καὶ οὐδετέρα τῶν ΔΜ, ΜΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἕκτη· ὅπερ ἔδει δεῖξαι.

(areas) applied to a rational (straight-line) produces as breadth a sixth binomial (straight-line).†

Let AB be the square-root of (the sum of) two medial (areas), having been divided at C. And let DE be a rational (straight-line). And let the (parallelogram) DF, equal to the (square) on AB, be applied to DE, producing DG as breadth. I say that DG is a sixth binomial (straight-line).



For let the same construction be made as in the previous (propositions). And since AB is the square-root of (the sum of) two medial (areas), having been divided at C, AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.41]. Hence, according to what has been previously demonstrated, DL and MF are each medial. And they are applied to the rational (straight-line) DE. Thus, DM and MG are each rational, and incommensurable in length with DE [Prop. 10.22]. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, DL is thus incommensurable with MF. Thus, DM (is) also incommensurable (in length) with MG [Props. 6.1, 10.11]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a sixth (binomial straight-line).

So, similarly (to the previous propositions), we can again show that the (rectangle contained) by DKM is equal to the (square) on MN, and that DK is incommensurable in length with KM. And so, for the same (reasons), the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable in length with (DM) [Prop. 10.18]. And neither of DM and MG is commensurable in length with the (previously) laid down rational (straight-line) DE.

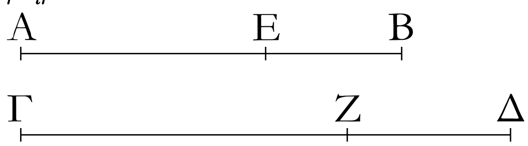
Thus, DG is a sixth binomial (straight-line) [Def. 10.10]. (Which is) the very thing it was required to show.

† In other words, the square of the square-root of two medials is a sixth binomial. See Prop. 10.59.

ξϛ΄.

Ἡ τῇ ἐκ δύο ὀνομάτων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτή.

Ἔστω ἐκ δύο ὀνομάτων ἡ ΑΒ, καὶ τῇ ΑΒ μήκει σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.



Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων ἐστὶν ἡ ΑΒ, διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, καὶ ἔστω μεῖζον ὄνομα τὸ ΑΕ· αἱ ΑΕ, ΕΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. γεγονέτω ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΑΕ πρὸς τὴν ΓΖ· καὶ λοιπὴ ἄρα ἡ ΕΒ πρὸς λοιπὴν τὴν ΖΔ ἐστιν, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ μήκει· σύμμετρος ἄρα ἐστὶ καὶ ἡ μὲν ΑΕ τῇ ΓΖ, ἡ δὲ ΕΒ τῇ ΖΔ. καί εἰσι ῥηταὶ αἱ ΑΕ, ΕΒ· ῥηταὶ ἄρα εἰσὶ καὶ αἱ ΓΖ, ΖΔ. καὶ ἐστιν ὡς ἡ ΑΕ πρὸς ΓΖ, ἡ ΕΒ πρὸς ΖΔ. ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς ΕΒ, ἡ ΓΖ πρὸς ΖΔ. αἱ δὲ ΑΕ, ΕΒ δυνάμει μόνον [εἰσὶ] σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καί εἰσι ῥηταί· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΓΔ. λέγω δή, ὅτι τῇ τάξει ἐστὶν ἡ αὐτὴ τῇ ΑΒ.

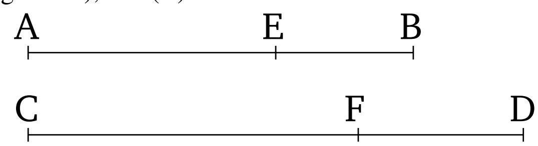
Ἡ γὰρ ΑΕ τῆς ΕΒ μεῖζον δύναται ἤτοι τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. εἰ μὲν ο ὖν ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ ΓΖ σύμμετρος αὐτῇ ἔσται, καὶ διὰ τοῦτο ἑκατέρα τῶν ΑΒ, ΓΔ ἐκ δύο ὀνομάτων ἐστὶ πρώτη, τουτέστι τῇ τάξει ἡ αὐτή. εἰ δὲ ἡ ΕΒ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ ΖΔ σύμμετρός ἐστιν αὐτῇ, καὶ διὰ τοῦτο πάλιν τῇ τάξει ἡ αὐτὴ ἔσται τῇ ΑΒ· ἑκατέρα γὰρ αὐτῶν ἔσται ἐκ δύο ὀνομάτων δευτέρα. εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ, οὐδετέρα τῶν ΓΖ, ΖΔ σύμμετρος αὐτῇ ἔσται, καί ἐστιν ἑκατέρα τρίτη. εἰ δὲ ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τὴς ΖΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν ἡ ΑΕ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητη, καὶ ἡ ΓΖ σύμμετρός ἐστιν αὐτῇ, καὶ ἐστιν ἑκατέρα τετάρτη. εἰ δὲ ἡ ΕΒ, καὶ ἡ ΖΔ, καὶ ἔσται ἑκατέρα πέμπτη. εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, καὶ τῶν ΓΖ, ΖΔ οὐδετέρα σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ, καὶ ἔσται ἑκατέρα ἕκτη.

Ὥστε ἡ τῇ ἐκ δύο ὀνομάτων μήκει σύμμετρος ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

Proposition 66

A (straight-line) commensurable in length with a binomial (straight-line) is itself also binomial, and the same in order.

Let AB be a binomial (straight-line), and let CD be commensurable in length with AB. I say that CD is a binomial (straight-line), and (is) the same in order as AB.



For since AB is a binomial (straight-line), let it be divided into its (component) terms at E, and let AE be the greater term. AE and EB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Let it be contrived that as AB (is) to CD, so AE (is) to CF [Prop. 6.12]. Thus, the remainder EB is also to the remainder FD, as AB (is) to CD [Props. 6.16, 5.19 corr.]. And AB (is) commensurable in length with CD. Thus, AE is also commensurable (in length) with CF, and EB with FD [Prop. 10.11]. And AE and EB are rational. Thus, CF and FD are also rational. And as AE is to CF, (so) EB (is) to FD [Prop. 5.11]. Thus, alternately, as AE is to EB, (so) CF (is) to FD [Prop. 5.16]. And AE and EB [are] commensurable in square only. Thus, CF and FD are also commensurable in square only [Prop. 10.11]. And they are rational. CD is thus a binomial (straight-line) [Prop. 10.36]. So, I say that it is the same in order as AB.

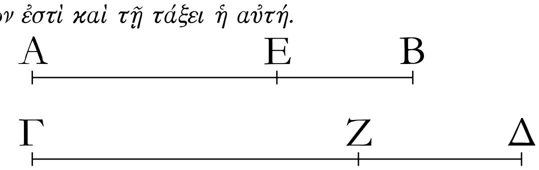
For the square on AE is greater than (the square on) EB by the (square) on (some straight-line) either commensurable or incommensurable (in length) with (AE). Therefore, if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with (some previously) laid down rational (straight-line) then CF will also be commensurable (in length) with it [Prop. 10.12]. And, on account of this, AB and CD are each first binomial (straight-lines) [Def. 10.5]—that is to say, the same in order. And if EB is commensurable (in length) with the (previously) laid down rational (straight-line) then FD is also commensurable (in length) with it [Prop. 10.12], and, again, on account of this, (CD) will be the same in order as AB. For each of them will be second binomial (straight-lines) [Def. 10.6]. And if neither of AE and EB is commensurable (in length) with the (previously) laid down rational (straight-line) then neither of CF and FD will be commensurable (in length) with it [Prop. 10.13], and each (of AB and CD) is a third (binomial straight-line) [Def. 10.7]. And if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) incommensurable (in length) with (AE)

then the square on CF is also greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with the (previously) laid down rational (straight-line) then CF is also commensurable (in length) with it [Prop. 10.12], and each (of AB and CD) is a fourth (binomial straight-line) [Def. 10.8]. And if EB (is commensurable in length with the previously laid down rational straight-line) then FD (is) also (commensurable in length with it), and each (of AB and CD) will be a fifth (binomial straight-line) [Def. 10.9]. And if neither of AE and EB (is commensurable in length with the previously laid down rational straight-line) then also neither of CF and FD is commensurable (in length) with the laid down rational (straight-line), and each (of AB and CD) will be a sixth (binomial straight-line) [Def. 10.10].

Hence, a (straight-line) commensurable in length with a binomial (straight-line) is a binomial (straight-line), and the same in order. (Which is) the very thing it was required to show.

ξζ΄.

Ἡ τῇ ἐκ δύο μέσων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο μέσων ἐστὶ καὶ τῇ τάξει ἡ αὐτή.



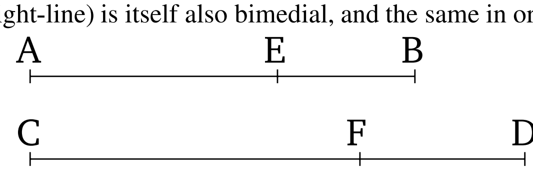
Ἔστω ἐκ δύο μέσων ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἔστω μήκει ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ ἐκ δύο μέσων ἐστὶ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ γὰρ ἐκ δύο μέσων ἐστὶν ἡ ΑΒ, διῃρήσθω εἰς τὰς μέσας κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ γεγονέτω ὡς ἡ ΑΒ πρὸς ΓΔ, ἡ ΑΕ πρὸς ΓΖ· καὶ λοιπὴ ἄρα ἡ ΕΒ πρὸς λοιπὴν τὴν ΖΔ ἐστιν, ὡς ἡ ΑΒ πρὸς ΓΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ μήκει· σύμμετρος ἄρα καὶ ἑκατέρα τῶν ΑΕ, ΕΒ ἑκατέρᾳ τῶν ΓΖ, ΖΔ. μέσαι δὲ αἱ ΑΕ, ΕΒ· μέσαι ἄρα καὶ αἱ ΓΖ, ΖΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΕ πρὸς ΕΒ, ἡ ΓΖ πρὸς ΖΔ, αἱ δὲ ΑΕ, ΕΒ δυνάμει μόνον σύμμετροί εἰσιν, καὶ αἱ ΓΖ, ΖΔ [ἄρα] δυνάμει μόνον σύμμετροί εἰσιν, ἐδείχθησαν δὲ καὶ μέσαι· ἡ ΓΔ ἄρα ἐκ δύο μέσων ἐστίν. λέγω δή, ὅτι καὶ τῇ τάξει ἡ αὐτή ἐστι τῇ ΑΒ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ ΑΕ πρὸς ΕΒ, ἡ ΓΖ πρὸς ΖΔ, καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖΔ· ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ἀπὸ τῆς ΓΖ, οὕτως τὸ ὑπὸ τῶν ΑΕΒ πρὸς τὸ ὑπὸ τῶν ΓΖΔ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΖ· σύμμετρον ἄρα καὶ τὸ ὑπὸ τῶν ΑΕΒ τῷ ὑπὸ τῶν ΓΖΔ. εἴτε οὖν ῥητόν ἐστι τὸ ὑπὸ τῶν ΑΕΒ, καὶ τὸ ὑπὸ τῶν ΓΖΔ ῥητόν ἐστιν [καὶ διὰ τοῦτό ἐστιν ἐκ δύο μέσων πρώτη]. εἴτε μέσον, μέσον, καί ἐστιν ἑκατέρα δευτέρα.

Proposition 67

A (straight-line) commensurable in length with a bimedial (straight-line) is itself also bimedial, and the same in order.



Let AB be a bimedial (straight-line), and let CD be commensurable in length with AB. I say that CD is bimedial, and the same in order as AB.

For since AB is a bimedial (straight-line), let it be divided into its (component) medial (straight-lines) at E. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.37, 10.38]. And let it be contrived that as AB (is) to CD, (so) AE (is) to CF [Prop. 6.12]. And thus as the remainder EB is to the remainder FD, so AB (is) to CD [Props. 5.19 corr., 6.16]. And AB (is) commensurable in length with CD. Thus, AE and EB are also commensurable (in length) with CF and FD, respectively [Prop. 10.11]. And AE and EB (are) medial. Thus, CF and FD (are) also medial [Prop. 10.23]. And since as AE is to EB, (so) CF (is) to FD, and AE and EB are commensurable in square only, CF and FD are [thus] also commensurable in square only [Prop. 10.11]. And they were also shown (to be) medial. Thus, CD is a bimedial (straight-line). So, I say that it is also the same in order as AB.

For since as AE is to EB, (so) CF (is) to FD, thus also as the (square) on AE (is) to the (rectangle contained) by AEB, so the (square) on CF (is) to the (rectangle contained) by CFD [Prop. 10.21 lem.]. Alternately, as the (square) on AE (is) to

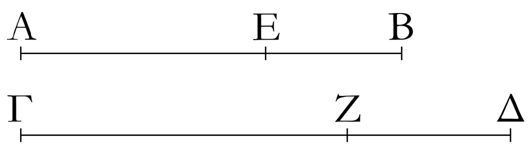
Καὶ διὰ τοῦτο ἔσται ἡ ΓΔ τῇ ΑΒ τῇ τάξει ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

the (square) on CF, so the (rectangle contained) by AEB (is) to the (rectangle contained) by CFD [Prop. 5.16]. And the (square) on AE (is) commensurable with the (square) on CF. Thus, the (rectangle contained) by AEB (is) also commensurable with the (rectangle contained) by CFD [Prop. 10.11]. Therefore, either the (rectangle contained) by AEB is rational, and the (rectangle contained) by CFD is rational [and, on account of this, (AE and CD) are first bimedial (straight-lines)], or (the rectangle contained by AEB is) medial, and (the rectangle contained by CFD is) medial, and (AB and CD) are each second (bimedial straight-lines) [Props. 10.23, 10.37, 10.38].

And, on account of this, CD will be the same in order as AB. (Which is) the very thing it was required to show.

ξη΄.

Ἡ τῇ μείζονι σύμμετρος καὶ αὐτὴ μείζων ἐστίν.

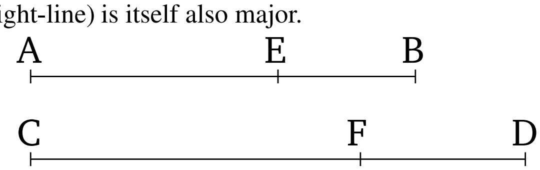


Ἔστω μείζων ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ μείζων ἐστίν.

Διῃρήσθω ἡ ΑΒ κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον· καὶ γεγονέτω τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἥ τε ΑΕ πρὸς τὴν ΓΖ καὶ ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ὡς ἄρα ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ· σύμμετρος ἄρα καὶ ἑκατέρα τῶν ΑΕ, ΕΒ ἑκατέρᾳ τῶν ΓΖ, ΖΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ἐναλλὰξ ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ, καὶ συνθέντι ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΖ· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ἀπὸ τῆς ΔΖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ὡς τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΑΕ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ἀπὸ τῆς ΓΖ. καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὰ ἀπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὰ ἀπὸ τῶν ΓΖ, ΖΔ· καὶ ἐναλλὰξ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΓΔ, οὕτως τὰ ἀπὸ τῶν ΑΕ, ΕΒ πρὸς τὰ ἀπὸ τῶν ΓΖ, ΖΔ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΓΔ· σύμμετρα ἄρα καὶ τὰ ἀπὸ τῶν ΑΕ, ΕΒ τοῖς ἀπὸ τῶν ΓΖ, ΖΔ. καί ἐστι τὰ ἀπὸ τῶν ΑΕ, ΕΒ ἅμα ῥητόν, καὶ τὰ ἀπὸ τῶν ΓΖ, ΖΔ ἅμα ῥητόν ἐστιν. ὁμοίως δὲ καὶ τὸ δὶς ὑπὸ τῶν ΑΕ, ΕΒ σύμμετρόν ἐστι τῷ δὶς ὑπὸ τῶν ΓΖ, ΖΔ. καί ἐστι μέσον τὸ δὶς ὑπὸ τῶν ΑΕ, ΕΒ· μέσον ἄρα καὶ τὸ δὶς ὑπὸ τῶν ΓΖ, ΖΔ. αἱ ΓΖ, ΖΔ ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ἅμα ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον· ὅλη ἄρα ἡ ΓΔ ἄλογός ἐστιν ἡ καλουμένη μείζων.

Proposition 68

A (straight-line) commensurable (in length) with a major (straight-line) is itself also major.



Let AB be a major (straight-line), and let CD be commensurable (in length) with AB. I say that CD is a major (straight-line).

Let AB be divided (into its component terms) at E. AE and EB are thus incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. And let (the) same (things) be contrived as in the previous (propositions). And since as AB is to CD, so AE (is) to CF and EB to FD, thus also as AE (is) to CF, so EB (is) to FD [Prop. 5.11]. And AB (is) commensurable (in length) with CD. Thus, AE and EB (are) also commensurable (in length) with CF and FD, respectively [Prop. 10.11]. And since as AE is to CF, so EB (is) to FD, also, alternately, as AE (is) to EB, so CF (is) to FD [Prop. 5.16], and thus, via composition, as AB is to BE, so CD (is) to DF [Prop. 5.18]. And thus as the (square) on AB (is) to the (square) on BE, so the (square) on CD (is) to the (square) on DF [Prop. 6.20]. So, similarly, we can also show that as the (square) on AB (is) to the (square) on AE, so the (square) on CD (is) to the (square) on CF. And thus as the (square) on AB (is) to (the sum of) the (squares) on AE and EB, so the (square) on CD (is) to (the sum of) the (squares) on CF and FD. And thus, alternately, as the (square) on AB is to the (square) on CD, so (the sum of) the (squares) on AE and EB (is) to (the sum of) the (squares) on CF and FD [Prop. 5.16]. And the (square) on AB (is) commensurable with the (square) on CD. Thus, (the sum of) the (squares) on AE and EB (is) also commensurable with (the sum of) the (squares) on CF and FD [Prop. 10.11]. And the (squares) on AE and EB (added)

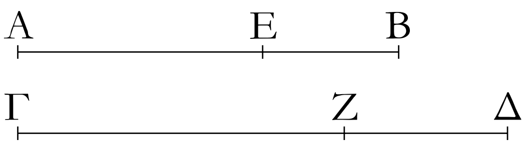
Ἡ ἄρα τῇ μείζονι σύμμετρος μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

together are rational. The (squares) on CF and FD (added) together (are) thus also rational. So, similarly, twice the (rectangle contained) by AE and EB is also commensurable with twice the (rectangle contained) by CF and FD. And twice the (rectangle contained) by AE and EB is medial. Therefore, twice the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.]. CF and FD are thus (straight-lines which are) incommensurable in square [Prop 10.13], simultaneously making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. The whole, CD, is thus that irrational (straight-line) called major [Prop. 10.39].

Thus, a (straight-line) commensurable (in length) with a major (straight-line) is major. (Which is) the very thing it was required to show.

ξθ′.

Ἡ τῇ ῥητὸν καὶ μέσον δυναμένῃ σύμμετρος [καὶ αὐτὴ] ῥητὸν καὶ μέσον δυναμένη ἐστίν.



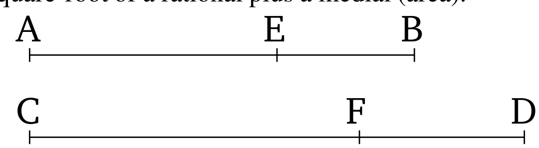
Ἔστω ῥητὸν καὶ μέσον δυναμένη ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἔστω ἡ ΓΔ· δεικτέον, ὅτι καὶ ἡ ΓΔ ῥητὸν καὶ μέσον δυναμένη ἐστίν.

Διῃρήσθω ἡ ΑΒ εἰς τὰς εὐθείας κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν· καὶ τὰ αὐτὰ κατεσκευάσθω τοῖς πρότερον. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι, καὶ σύμμετρον τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ ΑΕ, ΕΒ τῷ ὑπὸ ΓΖ, ΖΔ· ὥστε καὶ τὸ [μὲν] συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων ἐστὶ μέσον, τὸ δ᾽ ὑπὸ τῶν ΓΖ, ΖΔ ῥητόν.

Ῥητὸν ἄρα καὶ μέσον δυναμένη ἐστὶν ἡ ΓΔ· ὅπερ ἔδει δεῖξαι.

Proposition 69

A (straight-line) commensurable (in length) with the square-root of a rational plus a medial (area) is [itself also] the square-root of a rational plus a medial (area).



Let AB be the square-root of a rational plus a medial (area), and let CD be commensurable (in length) with AB. We must show that CD is also the square-root of a rational plus a medial (area).

Let AB be divided into its (component) straight-lines at E. AE and EB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. And let the same construction be made as in the previous (propositions). So, similarly, we can show that CF and FD are also incommensurable in square, and that the sum of the (squares) on AE and EB (is) commensurable with the sum of the (squares) on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. And hence the sum of the squares on CF and FD is medial, and the (rectangle contained) by CF and FD (is) rational.

Thus, CD is the square-root of a rational plus a medial (area) [Prop. 10.40]. (Which is) the very thing it was required to show.

ο′.

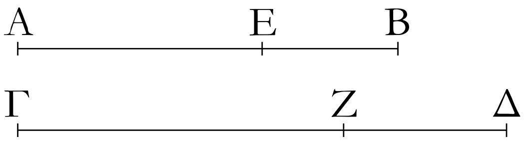
Ἡ τῇ δύο μέσα δυναμένῃ σύμμετρος δύο μέσα δυναμένη ἐστίν.

Ἔστω δύο μέσα δυναμένη ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἡ ΓΔ· δεικτέον, ὅτι καὶ ἡ ΓΔ δύο μέσα δυναμένη ἐστίν.

Proposition 70

A (straight-line) commensurable (in length) with the square-root of (the sum of) two medial (areas) is (itself also) the square-root of (the sum of) two medial (areas).

Let AB be the square-root of (the sum of) two medial (areas), and (let) CD (be) commensurable (in length) with AB.



Ἐπεὶ γὰρ δύο μέσα δυναμένη ἐστὶν ἡ ΑΒ, διῃρήσθω εἰς τὰς εὐθείας κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν [τετραγώνων] μέσον καὶ τὸ ὑπ᾿ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ ὑπὸ τῶν ΑΕ, ΕΒ· καὶ κατεσκευάσθω τὰ αὐτὰ τοῖς πρότερον. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι καὶ σύμμετρον τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων μέσον ἐστὶ καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων τῷ ὑπὸ τῶν ΓΖ, ΖΔ.

Ἡ ἄρα ΓΔ δύο μέσα δυναμένη ἐστίν· ὅπερ ἔδει δεῖξαι.

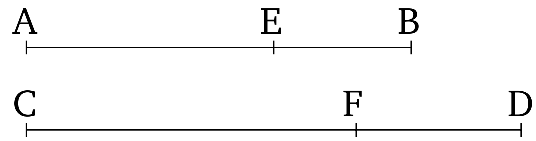
οα΄.

Ῥητοῦ καὶ μέσου συντιθεμένου τέσσαρες ἄλογοι γίγνονται ἤτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη.

Ἔστω ῥητὸν μὲν τὸ ΑΒ, μέσον δὲ τὸ ΓΔ· λέγω, ὅτι ἡ τὸ ΑΔ χωρίον δυναμένη ἤτοι ἐκ δύο ὀνομάτων ἐστὶν ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη.

Τὸ γὰρ ΑΒ τοῦ ΓΔ ἤτοι μεῖζόν ἐστιν ἢ ἔλασσον. ἔστω πρότερον μεῖζον· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ παραβεβλήσθω παρὰ τὴν ΕΖ τῷ ΑΒ ἴσον τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΘ· τῷ δὲ ΔΓ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΘΙ πλάτος ποιοῦν τὴν ΘΚ. καὶ ἐπεὶ ῥητόν ἐστι τὸ ΑΒ καί ἐστιν ἴσον τῷ ΕΗ, ῥητὸν ἄρα καὶ τὸ ΕΗ. καὶ παρὰ [ῥητὴν] τὴν ΕΖ παραβέβληται πλάτος ποιοῦν τὴν ΕΘ· ἡ ΕΘ ἄρα ῥητή ἐστι καὶ σύμμετρος τῇ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ΓΔ καί ἐστιν ἴσον τῷ ΘΙ, μέσον ἄρα ἐστὶ καὶ τὸ ΘΙ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΚ· ῥητὴ ἄρα ἐστὶν ἡ ΘΚ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ μέσον ἐστὶ τὸ ΓΔ, ῥητὸν δὲ τὸ ΑΒ, ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΒ τῷ ΓΔ· ὥστε καὶ τὸ ΕΗ ἀσύμμετρόν ἐστι τῷ ΘΙ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΙ, οὕτως ἐστὶν ἡ ΕΘ πρὸς τὴν ΘΚ· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΕΘ τῇ ΘΚ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΕΘ, ΘΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων

We must show that *CD* is also the square-root of (the sum of) two medial (areas).



For since *AB* is the square-root of (the sum of) two medial (areas), let it be divided into its (component) straight-lines at *E*. Thus, *AE* and *EB* are incommensurable in square, making the sum of the [squares] on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the (squares) on *AE* and *EB* incommensurable with the (rectangle) contained by *AE* and *EB* [Prop. 10.41]. And let the same construction be made as in the previous (propositions). So, similarly, we can show that *CF* and *FD* are also incommensurable in square, and (that) the sum of the (squares) on *AE* and *EB* (is) commensurable with the sum of the (squares) on *CF* and *FD*, and the (rectangle contained) by *AE* and *EB* with the (rectangle contained) by *CF* and *FD*. Hence, the sum of the squares on *CF* and *FD* is also medial, and the (rectangle contained) by *CF* and *FD* (is) medial, and, moreover, the sum of the squares on *CF* and *FD* (is) incommensurable with the (rectangle contained) by *CF* and *FD*.

Thus, *CD* is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. (Which is) the very thing it was required to show.

Proposition 71

When a rational and a medial (area) are added together, four irrational (straight-lines) arise (as the square-roots of the total area)—either a binomial, or a first bimedial, or a major, or the square-root of a rational plus a medial (area).

Let *AB* be a rational (area), and *CD* a medial (area). I say that the square-root of area *AD* is either binomial, or first bimedial, or major, or the square-root of a rational plus a medial (area).

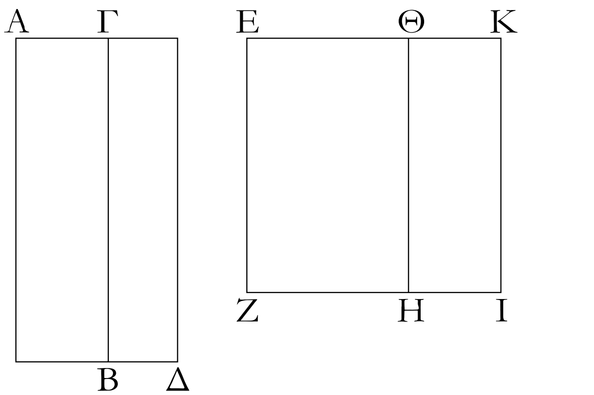
For *AB* is either greater or less than *CD*. Let it, first of all, be greater. And let the rational (straight-line) *EF* be laid down. And let (the rectangle) *EG*, equal to *AB*, be applied to *EF*, producing *EH* as breadth. And let (the recatangle) *HI*, equal to *DC*, be applied to *EF*, producing *HK* as breadth. And since *AB* is rational, and is equal to *EG*, *EG* is thus also rational. And it has been applied to the [rational] (straight-line) *EF*, producing *EH* as breadth. *EH* is thus rational, and commensurable in length with *EF* [Prop. 10.20]. Again, since *CD* is medial, and is equal to *HI*, *HI* is thus also medial. And it is applied to the rational (straight-line) *EF*, producing *HK* as breadth. *HK* is thus rational, and incommensurable in length with *EF* [Prop. 10.22]. And since *CD* is medial, and *AB* rational, *AB* is thus incommensurable with *CD*. Hence, *EG* is *ἐστὶν ἡ ΕΚ διῃρημένη κατὰ τὸ Θ. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΑΒ τοῦ ΓΔ, ἴσον δὲ τὸ μὲν ΑΒ τῷ ΕΗ, τὸ δὲ ΓΔ τῷ ΘΙ, μεῖζον ἄρα καὶ τὸ ΕΗ τοῦ ΘΙ· καὶ ἡ ΕΘ ἄρα μείζων ἐστὶ τῆς ΘΚ. ἤτοι οὖν ἡ ΕΘ τῆς ΘΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ· καί ἐστιν ἡ μείζων ἡ ΘΕ σύμμετρος τῇ ἐκκειμένῃ ῥητῃ τῇ ΕΖ· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ πρώτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πρώτης, ἡ τὸ χωρίον δυναμένη ἐκ δύο ὀνομάτων ἐστίν. ἡ ἄρα τὸ ΕΙ δυναμένη ἐκ δύο ὀνομάτων ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ δυναμένη ἐκ δύο ὀνομάτων ἐστίν. ἀλλὰ δὴ δυνάσθω ἡ ΕΘ τῆς ΘΚ μεῖζον τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ· καί ἐστιν ἡ μείζων ἡ ΕΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων. ἡ ἄρα τὸ ΕΙ χωρίον δυναμένη μείζων ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ δυναμένη μείζων ἐστίν.*

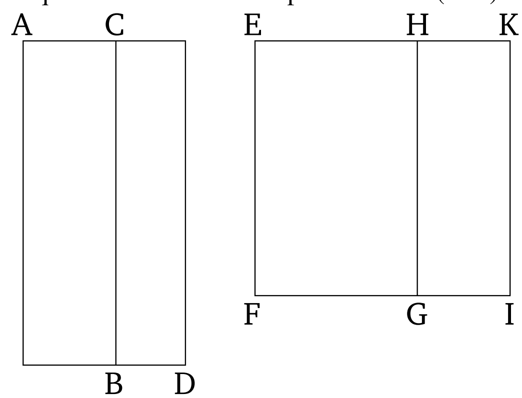
Ἀλλὰ δὴ ἔστω ἔλασσον τὸ ΑΒ τοῦ ΓΔ· καὶ τὸ ΕΗ ἄρα ἔλασσόν ἐστι τοῦ ΘΙ· ὥστε καὶ ἡ ΕΘ ἐλάσσων ἐστὶ τῆς ΘΚ. ἤτοι δὲ ἡ ΘΚ τῆς ΕΘ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· καί ἐστιν ἡ ἐλάσσων ἡ ΕΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ δευτέρα. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη. ἡ ἄρα τὸ ΕΙ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη· ὥστε καὶ ἡ τὸ ΑΔ δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη. ἀλλὰ δὴ ἡ ΘΚ τῆς ΘΕ μεῖζον δυνάσθω τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί ἐστιν ἡ ἐλάσσων ἡ ΕΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ἡ τὸ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν. ἡ ἄρα τὸ ΕΙ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν.

also incommensurable with HI. And as EG (is) to HI, so EH is to HK [Prop. 6.1]. Thus, EH is also incommensurable in length with HK [Prop. 10.11]. And they are both rational. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line), having been divided (into its component terms) at H [Prop. 10.36]. And since AB is greater than CD, and AB (is) equal to EG, and CD to HI, EG (is) thus also greater than HI. Thus, EH is also greater than HK [Prop. 5.14]. Therefore, the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable in length with (EH), or by the (square) on (some straight-line) incommensurable (in length with EH). Let it, first of all, be greater by the (square) on (some straight-line) commensurable (in length with EH). And the greater (of the two components of EK) HE is commensurable (in length) with the (previously) laid down (straight-line) EF. EK is thus a first binomial (straight-line) [Def. 10.5]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is a binomial (straight-line) [Prop. 10.54]. Thus, the square-root of EI is a binomial (straight-line). Hence the square-root of AD is also a binomial (straight-line). And, so, let the square on EH be greater than (the square on) HK by the (square) on (some straight-line) incommensurable (in length) with (EH). And the greater (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a fourth binomial (straight-line) [Def. 10.8]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line) called major [Prop. 10.57]. Thus, the square-root of area EI is a major (straight-line). Hence, the square-root of AD is also major.

And so, let AB be less than CD. Thus, EG is also less than HI. Hence, EH is also less than HK [Props. 6.1, 5.14]. And the square on HK is greater than (the square on) EH either by the (square) on (some straight-line) commensurable (in length) with (HK), or by the (square) on (some straight-line) incommensurable (in length) with (HK). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (HK). And the lesser (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a second binomial (straight-line) [Def. 10.6]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is a first bimedial (straight-line) [Prop. 10.55]. Thus, the square-root of area EI is a first bimedial (straight-line). Hence, the square-root of AD is also a first bimedial (straight-line). And so, let the square on HK be greater than (the square on) HE

by the (square) on (some straight-line) incommensurable (in length) with (HK). And the lesser (of the two components of EK) EH is commensurable (in length) with the (previously) laid down rational (straight-line) EF. Thus, EK is a fifth binomial (straight-line) [Def. 10.9]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the square-root of a rational plus a medial (area) [Prop. 10.58]. Thus, the square-root of area EI is the square-root of a rational plus a medial (area). Hence, the square-root of area AD is also the square-root of a rational plus a medial (area).





Ῥητοῦ ἄρα καὶ μέσου συντιθεμένου τέσσαρες ἄλογοι γίγνονται ἤτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη· ὅπερ ἔδει δεῖξαι.

Thus, when a rational and a medial area are added together, four irrational (straight-lines) arise (as the square-roots of the total area)—either a binomial, or a first bimedial, or a major, or the square-root of a rational plus a medial (area). (Which is) the very thing it was required to show.

ξβ'.

Δύο μέσων ἀσυμμέτρων ἀλλήλοις συντιθεμένων αἱ λοιπαὶ δύο ἄλογοι γίγνονται ἤτοι ἐκ δύο μέσων δευτέρα ἢ [ἡ] δύο μέσα δυναμένη.

Συγκείσθω γὰρ δύο μέσα ἀσύμμετρα ἀλλήλοις τὰ ΑΒ, ΓΔ· λέγω, ὅτι ἡ τὸ ΑΔ χωρίον δυναμένη ἤτοι ἐκ δύο μέσων ἐστὶ δευτέρα ἢ δύο μέσα δυναμένη.

Τὸ γὰρ ΑΒ τοῦ ΓΔ ἤτοι μεῖζόν ἐστιν ἢ ἔλασσον. ἔστω, εἰ τύχον, πρότερον μεῖζον τὸ ΑΒ τοῦ ΓΔ· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τῷ μὲν ΑΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΘ, τῷ δὲ ΓΔ ἴσον τὸ ΘΙ πλάτος ποιοῦν τὴν ΘΚ. καὶ ἐπεὶ μέσον ἐστὶν ἑκάτερον τῶν ΑΒ, ΓΔ, μέσον ἄρα καὶ ἑκάτερον τῶν ΕΗ, ΘΙ. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὰς ΕΘ, ΘΚ· ἑκατέρα ἄρα τῶν ΕΘ, ΘΚ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΑΒ τῷ ΓΔ, καί ἐστιν ἴσον τὸ μὲν ΑΒ τῷ ΕΗ, τὸ δὲ ΓΔ τῷ ΘΙ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΕΗ τῷ ΘΙ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΙ, οὕτως ἐστὶν ἡ ΕΘ πρὸς ΘΚ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΘ τῇ ΘΚ μήκει. αἱ ΕΘ, ΘΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΚ. ἤτοι δὲ ἡ ΕΘ τῆς ΘΚ μεῖζον δύναται τῷ ἀπὸ

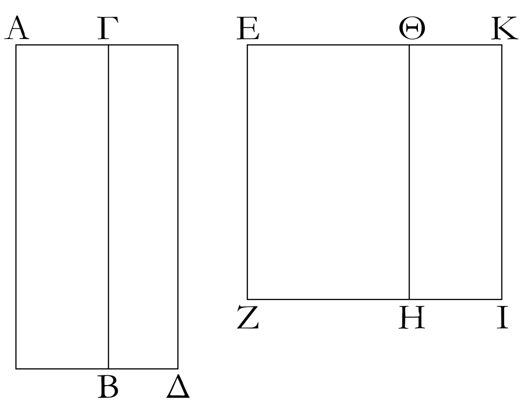
Proposition 72

When two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second bimedial, or the square-root of (the sum of) two medial (areas).

For let the two medial (areas) AB and CD, (which are) incommensurable with one another, be added together. I say that the square-root of area AD is either a second bimedial, or the square-root of (the sum of) two medial (areas).

For AB is either greater than or less than CD. By chance, let AB, first of all, be greater than CD. And let the rational (straight-line) EF be laid down. And let EG, equal to AB, be applied to EF, producing EH as breadth, and HI, equal to CD, producing HK as breadth. And since AB and CD are each medial, EG and HI (are) thus also each medial. And they are applied to the rational straight-line FE, producing EH and HK (respectively) as breadth. Thus, EH and HK are each rational (straight-lines which are) incommensurable in length with EF [Prop. 10.22]. And since AB is incommensurable with CD, and AB is equal to EG, and CD to HI, EG is thus also in-

συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· καὶ οὐδετέρα τῶν ΕΘ, ΘΚ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΕΖ μήκει· ἡ ΕΚ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη. ῥητὴ δὲ ἡ ΕΖ· ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ δευτέρα· ἡ ἄρα τὸ ΕΙ, τουτέστι τὸ ΑΔ, δυναμένη ἐκ δύο μέσων ἐστὶ δευτέρα. ἀλλα δὴ ἡ ΕΘ τῆς ΘΚ μεῖζον δυνάσθω τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει· καὶ ἀσύμμετρός ἐστιν ἑκατέρα τῶν ΕΘ, ΘΚ τῇ ΕΖ μήκει· ἡ ἄρα ΕΚ ἐκ δύο ὀνομάτων ἐστὶν ἕκτη. ἐὰν δὲ χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης, ἡ τὸ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν· ὥστε καὶ ἡ τὸ ΑΔ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

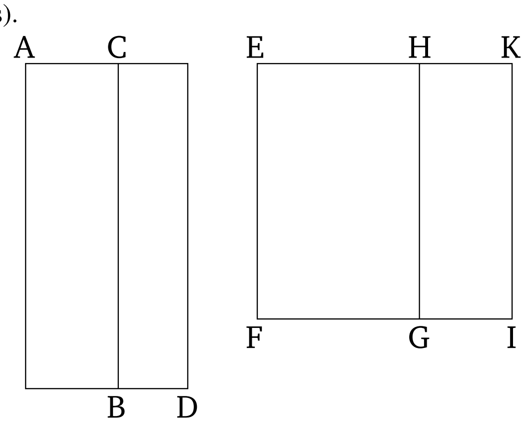


[Ὁμοίως δὴ δείξομεν, ὅτι κἂν ἔλαττον ᾖ τὸ ΑΒ τοῦ ΓΔ, ἡ τὸ ΑΔ χωρίον δυναμένη ἢ ἐκ δύο μέσων δευτέρα ἐστὶν ἤτοι δύο μέσα δυναμένη].

Δύο ἄρα μέσων ἀσυμμέτρων ἀλλήλοις συντιθεμένων αἱ λοιπαὶ δύο ἄλογοι γίγνονται ἤτοι ἐκ δύο μέσων δευτέρα ἢ δύο μέσα δυναμένη.

Ἡ ἐκ δύο ὀνομάτων καὶ αἱ μετ᾽ αὐτὴν ἄλογοι οὔτε τῇ μέσῃ οὔτε ἀλλήλαις εἰσὶν αἱ αὐταί. τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ παρ᾽ ἣν παράκειται μήκει. τὸ δὲ ἀπὸ τῆς

commensurable with HI. And as EG (is) to HI, so EH is to HK [Prop. 6.1]. EH is thus incommensurable in length with HK [Prop. 10.11]. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line) [Prop. 10.36]. And the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable (in length) with (EH), or by the (square) on (some straight-line) incommensurable (in length with EH). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (EH). And neither of EH or HK is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a third binomial (straight-line) [Def. 10.7]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is a second bimedial (straight-line) [Prop. 10.56]. Thus, the square-root of EI—that is to say, of AD—is a second bimedial. And so, let the square on EH be greater than (the square) on HK by the (square) on (some straight-line) incommensurable in length with (EH). And EH and HK are each incommensurable in length with EF. Thus, EK is a sixth binomial (straight-line) [Def. 10.10]. And if an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the square-root of (the sum of) two medial (areas) [Prop. 10.59]. Hence, the square-root of area AD is also the square-root of (the sum of) two medial (areas).



[So, similarly, we can show that, even if AB is less than CD, the square-root of area AD is either a second bimedial or the square-root of (the sum of) two medial (areas).]

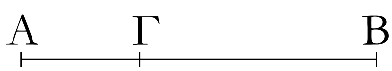
Thus, when two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second bimedial, or the square-root of (the sum of) two medial (areas).

A binomial (straight-line), and the (other) irrational (straight-lines) after it, are neither the same as a medial (straight-line) *ἐκ δύο ὀνομάτων παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην. τὸ δὲ ἀπὸ τῆς ἐκ δύο μέσων πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν. τὸ δὲ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τρίτην. τὸ δὲ ἀπὸ τῆς μείζονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τετάρτην. τὸ δὲ ἀπὸ τῆς ῥητὸν καὶ μέσον δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην. τὸ δὲ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην. τὰ δ᾽ εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ῥητή ἐστιν, ἀλλήλων δέ, ὅτι τῇ τάξει οὐκ εἰσὶν αἱ αὐταί· ὥστε καὶ αὐταὶ αἱ ἄλογοι διαφέρουσιν ἀλλήλων.*

nor (the same) as one another. For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) also incommensurable in length with (the straight-line) to which it is applied [Prop. 10.22]. And the (square) on a binomial (straight-line), applied to a rational (straight-line), produces as breadth a first binomial [Prop. 10.60]. And the (square) on a first bimedial (straight-line), applied to a rational (straight-line), produces as breadth a second binomial [Prop. 10.61]. And the (square) on a second bimedial (straight-line), applied to a rational (straight-line), produces as breadth a third binomial [Prop. 10.62]. And the (square) on a major (straight-line), applied to a rational (straight-line), produces as breadth a fourth binomial [Prop. 10.63]. And the (square) on the square-root of a rational plus a medial (area), applied to a rational (straight-line), produces as breadth a fifth binomial [Prop. 10.64]. And the (square) on the square-root of (the sum of) two medial (areas), applied to a rational (straight-line), produces as breadth a sixth binomial [Prop. 10.65]. And the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational—and from one another, because they are not the same in order. Hence, the (previously mentioned) irrational (straight-lines) themselves also differ from one another.

ογ'.

Ἐὰν ἀπὸ ῥητῆς ῥητὴ ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἀποτομή.

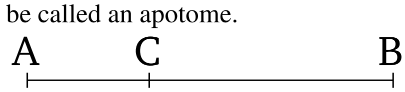


Ἀπὸ γὰρ ῥητῆς τῆς ΑΒ ῥητὴ ἀφῃρήσθω ἡ ΒΓ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐπειδήπερ τὰ ἀπὸ τῶν ΑΒ, ΒΓ ἴσα ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μετὰ τοῦ ἀπὸ ΓΑ, καὶ λοιπῷ ἄρα τῷ ἀπὸ τῆς ΑΓ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΑΒ, ΒΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ· καλείσθω δὲ ἀποτομή. ὅπερ ἔδει δεῖξαι.

Proposition 73

If a rational (straight-line), which is commensurable in square only with the whole, is subtracted from a(nother) rational (straight-line) then the remainder is an irrational (straight-line). Let it be called an apotome.



For let the rational (straight-line) *BC*, which commensurable in square only with the whole, be subtracted from the rational (straight-line) *AB*. I say that the remainder *AC* is that irrational (straight-line) called an apotome.

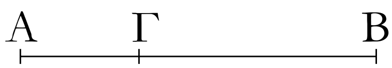
For since *AB* is incommensurable in length with *BC*, and as *AB* is to *BC*, so the (square) on *AB* (is) to the (rectangle contained) by *AB* and *BC* [Prop. 10.21 lem.], the (square) on *AB* is thus incommensurable with the (rectangle contained) by *AB* and *BC* [Prop. 10.11]. But, the (sum of the) squares on *AB* and *BC* is commensurable with the (square) on *AB* [Prop. 10.15], and twice the (rectangle contained) by *AB* and *BC* is commensurable with the (rectangle contained) by *AB* and *BC* [Prop. 10.6]. And, inasmuch as the (sum of the squares) on *AB* and *BC* is equal to twice the (rectangle contained) by *AB* and *BC* plus the (square) on *CA* [Prop. 2.7], the (sum of the squares) on *AB* and *BC* is thus also incommensurable with the remaining (square) on *AC* [Props. 10.13, 10.16]. And the (sum of the squares) on *AB* and *BC* is rational.

AC is thus an irrational (straight-line) [Def. 10.4]. And let it be called an apotome.[†] (Which is) the very thing it was required to show.

[†] See footnote to Prop. 10.36.

οδ΄.

Ἐὰν ἀπὸ μέσης μέση ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

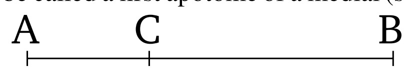


Ἀπὸ γὰρ μέσης τῆς ΑΒ μέση ἀφῃρήσθω ἡ ΒΓ δυνάμει μόνον σύμμετρος οὖσα τῇ ΑΒ, μετὰ δὲ τῆς ΑΒ ῥητὸν ποιοῦσα τὸ ὑπὸ τῶν ΑΒ, ΒΓ· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

Ἐπεὶ γὰρ αἱ ΑΒ, ΒΓ μέσαι εἰσίν, μέσα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ. ῥητὸν δὲ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἀσύμμετρα ἄρα τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ λοιπῷ ἄρα τῷ ἀπὸ τῆς ΑΓ ἀσύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ἐπεὶ κἂν τὸ ὅλον ἑνὶ αὐτῶν ἀσύμμετρον ᾖ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται. ῥητὸν δὲ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

Proposition 74

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a rational (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a first apotome of a medial (straight-line).



For let the medial (straight-line) BC, which is commensurable in square only with AB, and which makes with AB the rational (rectangle contained) by AB and BC, be subtracted from the medial (straight-line) AB [Prop. 10.27]. I say that the remainder AC is an irrational (straight-line). Let it be called the first apotome of a medial (straight-line).

For since AB and BC are medial (straight-lines), the (sum of the squares) on AB and BC is also medial. And twice the (rectangle contained) by AB and BC (is) rational. The (sum of the squares) on AB and BC (is) thus incommensurable with twice the (rectangle contained) by AB and BC. Thus, twice the (rectangle contained) by AB and BC is also incommensurable with the remaining (square) on AC [Prop. 2.7], since if the whole is incommensurable with one of the (constituent magnitudes) then the original magnitudes will also be incommensurable (with one another) [Prop. 10.16]. And twice the (rectangle contained) by AB and BC (is) rational. Thus, the (square) on AC is irrational. Thus, AC is an irrational (straight-line) [Def. 10.4]. Let it be called a first apotome of a medial (straight-line).[†]

[†] See footnote to Prop. 10.37.

οε΄.

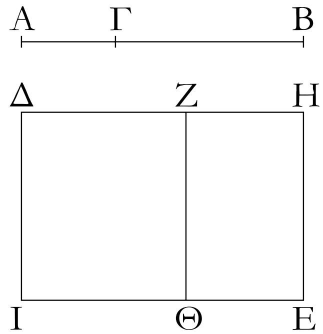
Ἐὰν ἀπὸ μέσης μέση ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα.

Ἀπὸ γὰρ μέσης τῆς ΑΒ μέση ἀφῃρήσθω ἡ ΓΒ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ τῇ ΑΒ, μετὰ δὲ τῆς ὅλης τῆς ΑΒ μέσον περιέχουσα τὸ ὑπὸ τῶν ΑΒ, ΒΓ· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα.

Proposition 75

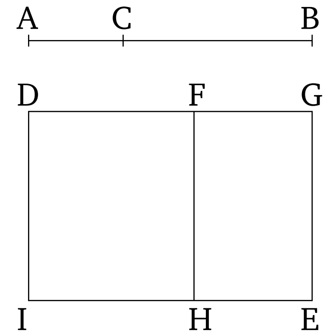
If a medial (straight-line), which is commensurable in square only with the whole, and which contains a medial (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).

For let the medial (straight-line) CB, which is commensurable in square only with the whole, AB, and which contains with the whole, AB, the medial (rectangle contained) by AB and BC, be subtracted from the medial (straight-line) AB [Prop. 10.28]. I say that the remainder AC is an irrational (straight-line). Let it be called a second apotome of a medial



Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΙ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΕ πλάτος ποιοῦν τὴν ΔΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΘ πλάτος ποιοῦν τὴν ΔΖ· λοιπὸν ἄρα τὸ ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καὶ ἐπεὶ μέσα καὶ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, μέσον ἄρα καὶ τὸ ΔΕ. καὶ παρὰ ῥητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΗ· ῥητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ, καὶ τὸ δὶς ἄρα ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΔΘ· καὶ τὸ ΔΘ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΔΙ παραβέβληται πλάτος ποιοῦν τὴν ΔΖ· ῥητὴ ἄρα ἐστὶν ἡ ΔΖ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. καὶ ἐπεὶ αἱ ΑΒ, ΒΓ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΓ μήκει· ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τετράγωνον τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· ἀσύμμετρον ἄρα ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ. ἴσον δὲ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ τὸ ΔΕ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ τὸ ΔΘ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔΕ τῷ ΔΘ. ὡς δὲ τὸ ΔΕ πρὸς τὸ ΔΘ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΗΔ τῇ ΔΖ. καὶ εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΗΔ, ΔΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΖΗ ἄρα ἀποτομή ἐστιν. ῥητὴ δὲ ἡ ΔΙ· τὸ δὲ ὑπὸ ῥητῆς καὶ ἀλόγου περιεχόμενον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. καὶ δύναται τὸ ΖΕ ἡ ΑΓ· ἡ ΑΓ ἄρα ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα. ὅπερ ἔδει δεῖξαι.

(straight-line).



For let the rational (straight-line) *DI* be laid down. And let *DE*, equal to the (sum of the squares) on *AB* and *BC*, be applied to *DI*, producing *DG* as breadth. And let *DH*, equal to twice the (rectangle contained) by *AB* and *BC*, be applied to *DI*, producing *DF* as breadth. The remainder *FE* is thus equal to the (square) on *AC* [Prop. 2.7]. And since the (squares) on *AB* and *BC* are medial and commensurable (with one another), *DE* (is) thus also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) *DI*, producing *DG* as breadth. Thus, *DG* is rational, and incommensurable in length with *DI* [Prop. 10.22]. Again, since the (rectangle contained) by *AB* and *BC* is medial, twice the (rectangle contained) by *AB* and *BC* is thus also medial [Prop. 10.23 corr.]. And it is equal to *DH*. Thus, *DH* is also medial. And it has been applied to the rational (straight-line) *DI*, producing *DF* as breadth. *DF* is thus rational, and incommensurable in length with *DI* [Prop. 10.22]. And since *AB* and *BC* are commensurable in square only, *AB* is thus incommensurable in length with *BC*. Thus, the square on *AB* (is) also incommensurable with the (rectangle contained) by *AB* and *BC* [Props. 10.21 lem., 10.11]. But, the (sum of the squares) on *AB* and *BC* is commensurable with the (square) on *AB* [Prop. 10.15], and twice the (rectangle contained) by *AB* and *BC* is commensurable with the (rectangle contained) by *AB* and *BC* [Prop. 10.6]. Thus, twice the (rectangle contained) by *AB* and *BC* is incommensurable with the (sum of the squares) on *AB* and *BC* [Prop. 10.13]. And *DE* is equal to the (sum of the squares) on *AB* and *BC*, and *DH* to twice the (rectangle contained) by *AB* and *BC*. Thus, *DE* [is] incommensurable with *DH*. And as *DE* (is) to *DH*, so *GD* (is) to *DF* [Prop. 6.1]. Thus, *GD* is incommensurable with *DF* [Prop. 10.11]. And they are both rational (straight-lines). Thus, *GD* and *DF* are rational (straight-lines which are) commensurable in square only. Thus, *FG* is an apotome [Prop. 10.73]. And *DI* (is) rational. And the (area) contained by a rational and an irrational (straight-line) is irrational [Prop. 10.20], and its square-root is irrational. And *AC* is the square-root of *FE*. Thus, *AC* is an irrational (straight-line) [Def. 10.4]. And let it be called the second apotome of a medial (straight-line).† (Which is) the very thing it was required to show.

† See footnote to Prop. 10.38.

ος΄.

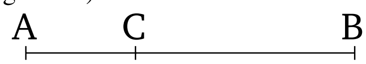
Ἐὰν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ μὲν ἀπ᾿ αὐτῶν ἅμα ῥητόν, τὸ δ᾿ ὑπ᾿ αὐτῶν μέσον, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἐλάσσων.

Ἀπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφῃρήσθω ἡ ΒΓ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ ποιοῦσα τὰ προκείμενα. λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων ῥητόν ἐστιν, τὸ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ ἀναστρέψαντι λοιπῷ τῷ ἀπὸ τῆς ΑΓ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΑΒ, ΒΓ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΑΓ· ἄλογος ἄρα ἡ ΑΓ· καλείσθω δὲ ἐλάσσων. ὅπερ ἔδει δεῖξαι.

Proposition 76

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the (squares) on them (added) together rational, and the (rectangle contained) by them medial, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called a minor (straight-line).



For let the straight-line *BC*, which is incommensurable in square with the whole, and fulfils the (other) prescribed (conditions), be subtracted from the straight-line *AB* [Prop. 10.33]. I say that the remainder *AC* is that irrational (straight-line) called minor.

For since the sum of the squares on *AB* and *BC* is rational, and twice the (rectangle contained) by *AB* and *BC* (is) medial, the (sum of the squares) on *AB* and *BC* is thus incommensurable with twice the (rectangle contained) by *AB* and *BC*. And, via conversion, the (sum of the squares) on *AB* and *BC* is incommensurable with the remaining (square) on *AC* [Props. 2.7, 10.16]. And the (sum of the squares) on *AB* and *BC* (is) rational. The (square) on *AC* (is) thus irrational. Thus, *AC* (is) an irrational (straight-line) [Def. 10.4]. Let it be called a minor (straight-line).† (Which is) the very thing it was required to show.

† See footnote to Prop. 10.39.

οζ΄.

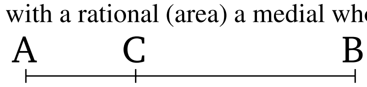
Ἐὰν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾿ αὐτῶν ῥητόν, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

Ἀπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφῃρήσθω ἡ ΒΓ δυνάμει ἀσύμμετος οὖσα τῇ ΑΒ ποιοῦσα τὰ προκείμενα· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ προειρημένη.

Ἐπεὶ γὰρ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων μέσον ἐστίν, τὸ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ῥητόν, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ· καὶ λοιπὸν ἄρα τὸ ἀπὸ τῆς ΑΓ ἀσύμμετρόν ἐστι τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ. καί ἐστι τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ῥητόν· τὸ ἄρα ἀπὸ τῆς ΑΓ ἄλογόν ἐστιν· ἄλογος ἄρα ἐστὶν ἡ ΑΓ·

Proposition 77

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called that which makes with a rational (area) a medial whole.



For let the straight-line *BC*, which is incommensurable in square with *AB*, and fulfils the (other) prescribed (conditions), be subtracted from the straight-line *AB* [Prop. 10.34]. I say that the remainder *AC* is the aforementioned irrational (straight-line).

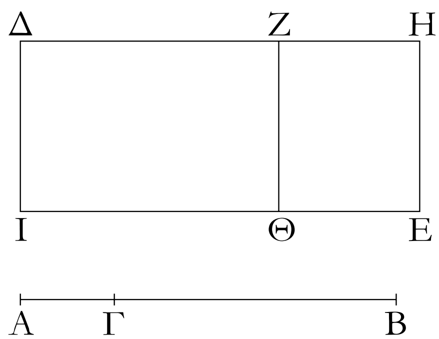
For since the sum of the squares on *AB* and *BC* is medial, and twice the (rectangle contained) by *AB* and *BC* rational, the (sum of the squares) on *AB* and *BC* is thus incommensurable with twice the (rectangle contained) by *AB* and *BC*. Thus, the *καλείσθω δὲ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.*

remaining (square) on *AC* is also incommensurable with twice the (rectangle contained) by *AB* and *BC* [Props. 2.7, 10.16]. And twice the (rectangle contained) by *AB* and *BC* is rational. Thus, the (square) on *AC* is irrational. Thus, *AC* is an irrational (straight-line) [Def. 10.4]. And let it be called that which makes with a rational (area) a medial whole.† (Which is) the very thing it was required to show.

† See footnote to Prop. 10.40.

οη΄.

Ἐὰν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον τό τε δὶς ὑπ᾽ αὐτῶν μέσον καὶ ἔτι τὰ ἀπ᾽ αὐτῶν τετράγωνα ἀσύμμετρα τῷ δὶς ὑπ᾽ αὐτῶν, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

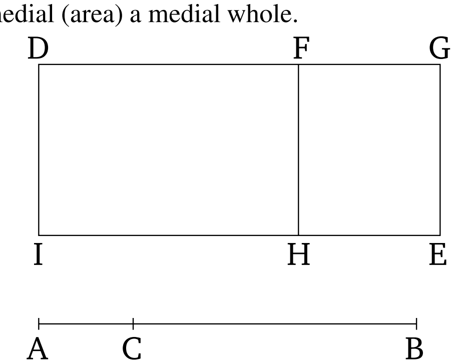


Ἀπὸ γὰρ εὐθείας τῆς ΑΒ εὐθεῖα ἀφῃρήσθω ἡ ΒΓ δυνάμει ἀσύμμετρος οὖσα τῇ ΑΒ ποιοῦσα τὰ προκείμενα· λέγω, ὅτι ἡ λοιπὴ ἡ ΑΓ ἄλογός ἐστιν ἡ καλουμένη ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΙ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΕ πλάτος ποιοῦν τὴν ΔΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἀφῃρήσθω τὸ ΔΘ [πλάτος ποιοῦν τὴν ΔΖ]. λοιπὸν ἄρα τὸ ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ· ὥστε ἡ ΑΓ δύναται τὸ ΖΕ. καὶ ἐπεὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΔΕ, μέσον ἄρα [ἐστὶ] τὸ ΔΕ. καὶ παρὰ ῥητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΗ· ῥητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. πάλιν, ἐπεὶ τὸ δὶς ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστὶ καὶ ἔστιν ἴσον τῷ ΔΘ, τὸ ἄρα ΔΘ μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΖ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΔΖ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τῷ δὶς ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα καὶ τὸ ΔΕ τῷ ΔΘ. ὡς δὲ τὸ ΔΕ πρὸς τὸ ΔΘ, οὕτως ἐστὶ καὶ ἡ ΔΗ πρὸς τὴν ΔΖ· ἀσύμμετρος ἄρα ἡ ΔΗ τῇ ΔΖ. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΗΔ, ΔΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἀποτομὴ ἄρα ἐστὶν ἡ ΖΗ·

Proposition 78

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called that which makes with a medial (area) a medial whole.



For let the straight-line *BC*, which is incommensurable in square *AB*, and fulfils the (other) prescribed (conditions), be subtracted from the (straight-line) *AB* [Prop. 10.35]. I say that the remainder *AC* is the irrational (straight-line) called that which makes with a medial (area) a medial whole.

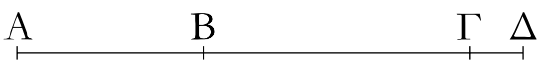
For let the rational (straight-line) *DI* be laid down. And let *DE*, equal to the (sum of the squares) on *AB* and *BC*, be applied to *DI*, producing *DG* as breadth. And let *DH*, equal to twice the (rectangle contained) by *AB* and *BC*, be subtracted (from *DE*) [producing *DF* as breadth]. Thus, the remainder *FE* is equal to the (square) on *AC* [Prop. 2.7]. Hence, *AC* is the square-root of *FE*. And since the sum of the squares on *AB* and *BC* is medial, and is equal to *DE*, *DE* [is] thus medial. And it is applied to the rational (straight-line) *DI*, producing *DG* as breadth. Thus, *DG* is rational, and incommensurable in length with *DI* [Prop 10.22]. Again, since twice the (rectangle contained) by *AB* and *BC* is medial, and is equal to *DH*, *DH* is thus medial. And it is applied to the rational (straight-line) *DI*, producing *DF* as breadth. Thus, *DF* is also rational, and incommensurable in length with *DI* [Prop. 10.22]. And since the (sum of the squares) on *AB* and *BC* is incommensurable *ῥητὴ δὲ ἡ ΖΘ. τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς περιεχόμενον [ὀρθογώνιον] ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν· καὶ δύναται τὸ ΖΕ ἡ ΑΓ· ἡ ΑΓ ἄρα ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.*

with twice the (rectangle contained) by AB and BC, DE (is) also incommensurable with DH. And as DE (is) to DH, so DG also is to DF [Prop. 6.1]. Thus, DG (is) incommensurable (in length) with DF [Prop. 10.11]. And they are both rational. Thus, GD and DF are rational (straight-lines which are) commensurable in square only. Thus, FG is an apotome [Prop. 10.73]. And FH (is) rational. And the [rectangle] contained by a rational (straight-line) and an apotome is irrational [Prop. 10.20], and its square-root is irrational. And AC is the square-root of FE. Thus, AC is irrational. Let it be called that which makes with a medial (area) a medial whole.† (Which is) the very thing it was required to show.

† See footnote to Prop. 10.41.

οθ΄.

Τῇ ἀποτομῇ μία [μόνον] προσαρμόζει εὐθεῖα ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ.



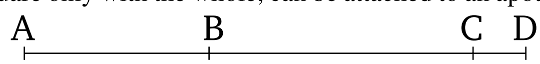
Ἔστω ἀποτομὴ ἡ ΑΒ, προσαρμόζουσα δὲ αὐτῇ ἡ ΒΓ· αἱ ΑΓ, ΓΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεί, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· τῷ γὰρ αὐτῷ τῷ ἀπὸ τῆς ΑΒ ἀμφότερα ὑπερέχει· ἐναλλὰξ ἄρα, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει [καὶ] τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. τὰ δὲ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφότερα. καὶ τὸ δὶς ἄρα ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γὰρ ἀμφότερα, μέσον δὲ μέσου οὐχ ὑπερέχει ῥητῷ. τῇ ἄρα ΑΒ ἑτέρα οὐ προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ.

Μία ἄρα μόνη τῇ ἀποτομῇ προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· ὅπερ ἔδει δεῖξαι.

Proposition 79

[Only] one rational straight-line, which is commensurable in square only with the whole, can be attached to an apotome.†



Let AB be an apotome, with BC (so) attached to it. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.73]. I say that another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area). For both exceed by the same (area)—(namely), the (square) on AB [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB [also] exceeds twice the (rectangle contained) by AC and CB by this (same area). And the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB by a rational (area). For both (are) rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.21], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26]. Thus, another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

Thus, only one rational (straight-line), which is commensurable in square only with the whole, can be attached to an apotome. (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.42, with minus signs instead of plus signs.

π´.

Τῇ μέσης ἀποτομῇ πρώτῃ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.

Ἔστω γὰρ μέσης ἀποτομὴ πρώτη ἡ ΑΒ, καὶ τῇ ΑΒ προσαρμοζέτω ἡ ΒΓ· αἱ ΑΓ, ΓΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν ΑΓ, ΓΒ· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόζει μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω καὶ ἡ ΔΒ· αἱ ἄρα ΑΔ, ΔΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεί, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΔ, ΔΒ, τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· τῷ γὰρ αὐτῷ [πάλιν] ὑπερέχουσι τῷ ἀπὸ τῆς ΑΒ· ἐναλλὰξ ἄρα, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. τὸ δὲ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφότερα. καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ [τετραγώνων] ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα, μέσον δὲ μέσου οὐχ ὑπερέχει ῥητῷ.

Τῇ ἄρα μέσης ἀποτομῇ πρώτῃ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα· ὅπερ ἔδει δεῖξαι.

Proposition 80

Only one medial straight-line, which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line).†

For let *AB* be a first apotome of a medial (straight-line), and let *BC* be (so) attached to *AB*. Thus, *AC* and *CB* are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that contained) by *AC* and *CB* [Prop. 10.74]. I say that a(nother) medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, cannot be attached to *AB*.

For, if possible, let *DB* also be (so) attached to *AB*. Thus, *AD* and *DB* are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that) contained by *AD* and *DB* [Prop. 10.74]. And since by whatever (area) the (sum of the squares) on *AD* and *DB* exceeds twice the (rectangle contained) by *AD* and *DB*, the (sum of the squares) on *AC* and *CB* also exceeds twice the (rectangle contained) by *AC* and *CB* by this (same area). For [again] both exceed by the same (area)—(namely), the (square) on *AB* [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on *AD* and *DB* exceeds the (sum of the squares) on *AC* and *CB*, twice the (rectangle contained) by *AD* and *DB* also exceeds twice the (rectangle contained) by *AC* and *CB* by this (same area). And twice the (rectangle contained) by *AD* and *DB* exceeds twice the (rectangle contained) by *AC* and *CB* by a rational (area). For both (are) rational (areas). Thus, the (sum of the squares) on *AD* and *DB* also exceeds the (sum of the) [squares] on *AC* and *CB* by a rational (area). The very thing is impossible. For both are medial (areas) [Props. 10.15, 10.23 corr.], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26].

Thus, only one medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line). (Which is) the very thing it was required to show.

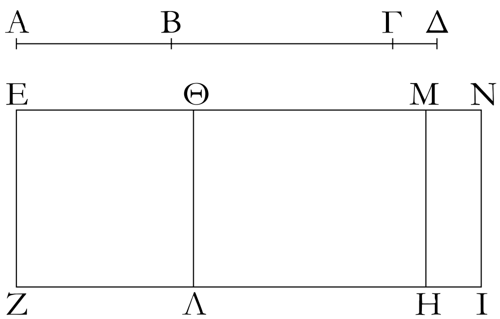
† This proposition is equivalent to Prop. 10.43, with minus signs instead of plus signs.

πα´.

Τῇ μέσης ἀποτομῇ δευτέρᾳ μία μόνον προσαρμόζει εὐθεῖα

Proposition 81

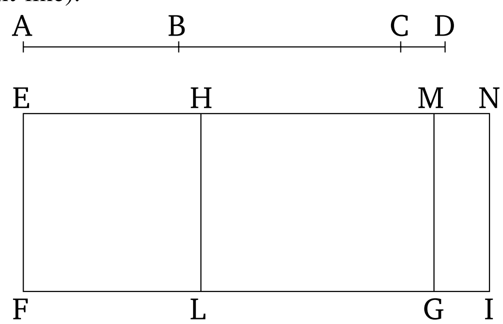
Only one medial straight-line, which is commensurable in μέση δυνάμει μόνον σύμμετρος τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.



Ἔστω μέσης ἀποτομὴ δευτέρα ἡ ΑΒ καὶ τῇ ΑΒ προσαρμόζουσα ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν ΑΓ, ΓΒ· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόσει εὐθεῖα μέση δυνάμει μόνον σύμμετρος ο᾿ὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΜ· τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον ἀφῃρήσθω τὸ ΘΗ πλάτος ποιοῦν τὴν ΘΜ· λοιπὸν ἄρα τὸ ΕΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ· ὥστε ἡ ΑΒ δύναται τὸ ΕΛ. πάλιν δὴ τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΙ πλάτος ποιοῦν τὴν ΕΝ· ἔστι δὲ καὶ τὸ ΕΛ ἴσον τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ· λοιπὸν ἄρα τὸ ΘΙ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΔ, ΔΒ. καὶ ἐπεὶ μέσαι εἰσὶν αἱ ΑΓ, ΓΒ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ. καί ἐστιν ἴσα τῷ ΕΗ· μέσον ἄρα καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΕΜ· ῥητὴ ἄρα ἐστὶν ἡ ΕΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ὑπὸ τῶν ΑΓ, ΓΒ, καὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ μέσον ἐστίν. καί ἐστιν ἴσον τῷ ΘΗ· καὶ τὸ ΘΗ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΜ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΘΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ αἱ ΑΓ, ΓΒ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΓ τῇ ΓΒ μήκει. ὡς δὲ ἡ ΑΓ πρὸς τὴν ΓΒ, οὕτως ἐστὶ τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓ, ΓΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΓ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ, τῷ δὲ ὑπὸ τῶν ΑΓ, ΓΒ σύμμετρόν ἐστι τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ. καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΕΗ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον τὸ ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΗ τῷ ΘΗ. ὡς δὲ τὸ ΕΗ πρὸς τὸ ΘΗ, οὕτως ἐστὶν ἡ ΕΜ πρὸς τὴν ΘΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΜ τῇ ΜΘ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΕΜ, ΜΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΘ, προσαρμόζουσα δὲ αὐτῇ ἡ ΘΜ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΘΝ αὐτῇ προσαρμόζει· τῇ ἄρα ἀποτομῇ ἄλλη καὶ ἄλλη προσαρμόζει εὐθεῖα δυνάμει

square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line).†



Let *AB* be a second apotome of a medial (straight-line), with *BC* (so) attached to *AB*. Thus, *AC* and *CB* are medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by *AC* and *CB* [Prop. 10.75]. I say that a(nother) medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, cannot be attached to *AB*.

For, if possible, let *BD* be (so) attached. Thus, *AD* and *DB* are also medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by *AD* and *DB* [Prop. 10.75]. And let the rational (straight-line) *EF* be laid down. And let *EG*, equal to the (sum of the squares) on *AC* and *CB*, be applied to *EF*, producing *EM* as breadth. And let *HG*, equal to twice the (rectangle contained) by *AC* and *CB*, be subtracted (from *EG*), producing *HM* as breadth. The remainder *EL* is thus equal to the (square) on *AB* [Prop. 2.7]. Hence, *AB* is the square-root of *EL*. So, again, let *EI*, equal to the (sum of the squares) on *AD* and *DB* be applied to *EF*, producing *EN* as breadth. And *EL* is also equal to the square on *AB*. Thus, the remainder *HI* is equal to twice the (rectangle contained) by *AD* and *DB* [Prop. 2.7]. And since *AC* and *CB* are (both) medial (straight-lines), the (sum of the squares) on *AC* and *CB* is also medial. And it is equal to *EG*. Thus, *EG* is also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) *EF*, producing *EM* as breadth. Thus, *EM* is rational, and incommensurable in length with *EF* [Prop. 10.22]. Again, since the (rectangle contained) by *AC* and *CB* is medial, twice the (rectangle contained) by *AC* and *CB* is also medial [Prop. 10.23 corr.]. And it is equal to *HG*. Thus, *HG* is also medial. And it is applied to the rational (straight-line) *EF*, producing *HM* as breadth. Thus, *HM* is also rational, and incommensurable in length with *EF* [Prop. 10.22]. And since *AC* and *CB* are commensurable in square only, *AC* is thus incommensurable in length with *CB*. And as *AC* (is) to *CB*, so the (square) on *AC* is to the (rectangle contained) by *AC* and *CB* [Prop. 10.21 corr.]. Thus, the (square) on *AC* is in-

μόνον σύμμετρος οὖσα τῇ ὅλῃ· ὅπερ ἐστὶν ἀδύνατον.

Τῇ ἄρα μέσης ἀποτομῇ δευτέρᾳ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα· ὅπερ ἔδει δεῖξαι.

commensurable with the (rectangle contained) by AC and CB [Prop. 10.11]. But, the (sum of the squares) on AC and CB is commensurable with the (square) on AC, and twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6]. Thus, the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13]. And EG is equal to the (sum of the squares) on AC and CB. And GH is equal to twice the (rectangle contained) by AC and CB. Thus, EG is incommensurable with HG. And as EG (is) to HG, so EM is to HM [Prop. 6.1]. Thus, EM is incommensurable in length with MH [Prop. 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], and HM (is) attached to it. So, similarly, we can show that HN (is) also (commensurable in square only with EN and is) attached to (EH). Thus, different straight-lines, which are commensurable in square only with the whole, are attached to an apotome. The very thing is impossible [Prop. 10.79].

Thus, only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line). (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.44, with minus signs instead of plus signs.

πβ΄.

Τῇ ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ ποιοῦσα μετὰ τῆς ὅλης τὸ μὲν ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον.

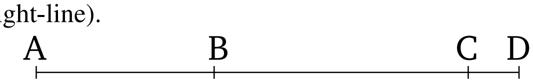
Ἔστω ἡ ἐλάσσων ἡ ΑΒ, καὶ τῇ ΑΒ προσαρμόζουσα ἔστω ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον· λέγω, ὅτι τῇ ΑΒ ἑτέρα εὐθεῖα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. καὶ ἐπεί, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, τὰ δὲ ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα τῶν ἀπὸ τῶν ΑΓ, ΓΒ τετραγώνων ὑπερέχει ῥητῷ· ῥητὰ γάρ ἐστιν ἀμφότερα· καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἄρα τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα.

Τῇ ἄρα ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει

Proposition 82

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the (sum of the) squares on them rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line).



Let AB be a minor (straight-line), and let BC be (so) attached to AB. Thus, AC and CB are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial [Prop. 10.76]. I say that another another straight-line fulfilling the same (conditions) cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also (straight-lines which are) incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.76]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area) [Prop. 2.7]. And the (sum of the) squares *ἀσύμμετρος οὖσα τῇ ὅλῃ καὶ ποιοῦσα τὰ μὲν ἀπ᾿ αὐτῶν τετράγωνα ἅμα ῥητόν, τὸ δὲ δὶς ὑπ᾿ αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.*

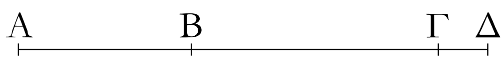
on AD and DB exceeds the (sum of the) squares on AC and CB by a rational (area). For both are rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, only one straight-line, which is incommensurable in square with the whole, and (with the whole) makes the squares on them (added) together rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line). (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.45, with minus signs instead of plus signs.

πγ΄.

Τῇ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσῃ μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾿ αὐτῶν ῥητόν.



Ἔστω ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, καὶ τῇ ΑΒ προσαρμοζέτω ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ· καὶ αἱ ΑΔ, ΔΒ ἄρα εὐθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα. ἐπεὶ οὖν, ᾧ ὑπερέχει τὰ ἀπὸ τῶν ΑΔ, ΔΒ τῶν ἀπὸ τῶν ΑΓ, ΓΒ, τούτῳ ὑπερέχει καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἀκολούθως τοῖς πρὸ αὐτοῦ, τὸ δὲ δὶς ὑπὸ τῶν ΑΔ, ΔΒ τοῦ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ῥητὰ γάρ ἐστιν ἀμφότερα· καὶ τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἄρα τῶν ἀπὸ τῶν ΑΓ, ΓΒ ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφότερα.

Οὐκ ἄρα τῇ ΑΒ ἑτέρα προσαρμόσει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ προειρημένα· μία ἄρα μόνον προσαρμόσει· ὅπερ ἔδει δεῖξαι.

Proposition 83

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, can be attached to that (straight-line) which with a rational (area) makes a medial whole.†

Let AB be a (straight-line) which with a rational (area) makes a medial whole, and let BC be (so) attached to AB. Thus, AC and CB are (straight-lines which are) incommensurable in square, fulfilling the (other) proscribed (conditions) [Prop. 10.77]. I say that another (straight-line) fulfilling the same (conditions) cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also straight-lines (which are) incommensurable in square, fulfilling the (other) prescribed (conditions) [Prop. 10.77]. Therefore, analogously to the (propositions) before this, since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area). And twice the (rectangle contained) by AD and DB exceeds twice the (rectangle contained) by AC and CB by a rational (area). For they are (both) rational (areas). Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the squares) on AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

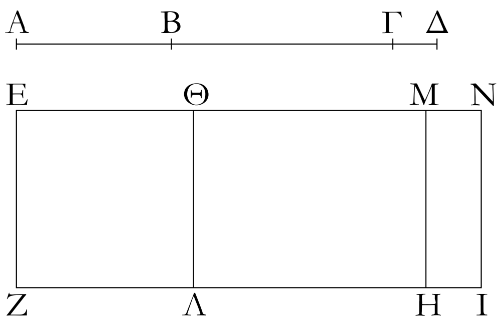
Thus, another straight-line cannot be attached to AB, which is incommensurable in square with the whole, and fulfills the (other) aforementioned (conditions) with the whole. Thus, only one (such straight-line) can be (so) attached. (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.46, with minus signs instead of plus signs.

πδ´.

Τῇ μετὰ μέσου μέσον τὸ ὅλον ποιούσῃ μία μόνη προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον τό τε δὶς ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ᾽ αὐτῶν.

Ἔστω ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, προσαρμόζουσα δὲ αὐτῇ ἡ ΒΓ· αἱ ἄρα ΑΓ, ΓΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. λέγω, ὅτι τῇ ΑΒ ἑτέρα οὐ προσαρμόσει ποιοῦσα προειρημένα.

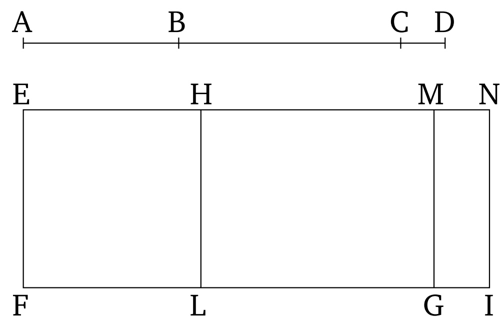


Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ ΒΔ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσυμμέτρους εἶναι ποιούσας τά τε ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα ἅμα μέσον καὶ τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ μέσον καὶ ἔτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἀσύμμετρα τῷ δὶς ὑπὸ τῶν ΑΔ, ΔΒ· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΗ πλάτος ποιοῦν τὴν ΕΜ, τῷ δὲ δὶς ὑπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΘΗ πλάτος ποιοῦν τὴν ΘΜ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ΕΛ· ἡ ἄρα ΑΒ δύναται τὸ ΕΛ. πάλιν τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον παρὰ τὴν ΕΖ παραβεβλήσθω τὸ ΕΙ πλάτος ποιοῦν τὴν ΕΝ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΑΒ ἴσον τῷ ΕΛ· λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΔ, ΔΒ ἴσον [ἐστὶ] τῷ ΘΙ. καὶ ἐπεὶ μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ καί ἐστιν ἴσον τῷ ΕΗ, μέσον ἄρα ἐστὶ καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΕΜ· ῥητὴ ἄρα ἐστὶν ἡ ΕΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ καί ἐστιν ἴσον τῷ ΘΗ, μέσον ἄρα καὶ τὸ ΘΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΘΜ· ῥητὴ ἄρα ἐστὶν ἡ ΘΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῷ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρόν ἐστι καὶ τὸ ΕΗ τῷ ΘΗ· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΕΜ τῇ ΜΘ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΕΜ, ΜΘ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΘ, προσαρμόζουσα δὲ αὐτῇ ἡ ΘΜ. ὁμοίως δὴ

Proposition 84

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the (squares) on them, can be attached to that (straight-line) which with a medial (area) makes a medial whole.†

Let *AB* be a (straight-line) which with a medial (area) makes a medial whole, *BC* being (so) attached to it. Thus, *AC* and *CB* are incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.78]. I say that a(nother) (straight-line) fulfilling the aforementioned (conditions) cannot be attached to *AB*.



For, if possible, let *BD* be (so) attached. Hence, *AD* and *DB* are also (straight-lines which are) incommensurable in square, making the squares on *AD* and *DB* (added) together medial, and twice the (rectangle contained) by *AD* and *DB* medial, and, moreover, the (sum of the squares) on *AD* and *DB* incommensurable with twice the (rectangle contained) by *AD* and *DB* [Prop. 10.78]. And let the rational (straight-line) *EF* be laid down. And let *EG*, equal to the (sum of the squares) on *AC* and *CB*, be applied to *EF*, producing *EM* as breadth. And let *HG*, equal to twice the (rectangle contained) by *AC* and *CB*, be applied to *EF*, producing *HM* as breadth. Thus, the remaining (square) on *AB* is equal to *EL* [Prop. 2.7]. Thus, *AB* is the square-root of *EL*. Again, let *EI*, equal to the (sum of the squares) on *AD* and *DB*, be applied to *EF*, producing *EN* as breadth. And the (square) on *AB* is also equal to *EL*. Thus, the remaining twice the (rectangle contained) by *AD* and *DB* [is] equal to *HI* [Prop. 2.7]. And since the sum of the (squares) on *AC* and *CB* is medial, and is equal to *EG*, *EG* is thus also medial. And it is applied to the rational (straight-line) *EF*, producing *EM* as breadth. *EM* is thus rational, and incommensurable in length with *EF* [Prop. 10.22]. Again, since twice the (rectangle contained) by *AC* and *CB* is medial, and is equal to *HG*, *HG* is thus also medial. And it is applied to the rational (straight-line) *EF*, producing *HM* as breadth.

δείξομεν, ὅτι ἡ ΕΘ πάλιν ἀποτομή ἐστιν, προσαρμόζουσα δὲ αὐτῇ ἡ ΘΝ. τῇ ἄρα ἀποτομῇ ἄλλη καὶ ἄλλη προσαρμόζει ῥητὴ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· ὅπερ ἐδείχθη ἀδύνατον. οὐκ ἄρα τῇ ΑΒ ἑτέρα προσαρμόσει εὐθεῖα.

Τῇ ἄρα ΑΒ μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τά τε ἀπ᾿ αὐτῶν τετράγωνα ἅμα μέσον καὶ τὸ δὶς ὑπ᾿ αὐτῶν μέσον καὶ ἔτι τὰ ἀπ᾿ αὐτῶν τετράγωνα ἀσύμμετρα τῷ δὶς ὑπ᾿ αὐτῶν· ὅπερ ἔδει δεῖξαι.

HM is thus rational, and incommensurable in length with EF [Prop. 10.22]. And since the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is also incommensurable with HG. Thus, EM is also incommensurable in length with MH [Props. 6.1, 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], with HM attached to it. So, similarly, we can show that EH is again an apotome, with HN attached to it. Thus, different rational (straight-lines), which are commensurable in square only with the whole, are attached to an apotome. The very thing was shown (to be) impossible [Prop. 10.79]. Thus, another straight-line cannot be (so) attached to AB.

Thus, only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the squares on them (added) together medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with the (rectangle contained) by them, can be attached to AB. (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.47, with minus signs instead of plus signs.

Ὅροι τρίτοι.

ιαʹ. Ὑποκειμένης ῥητῆς καὶ ἀποτομῆς, ἐὰν μὲν ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει, καὶ ἡ ὅλη σύμμετρος ᾖ τῇ ἐκκειμένῃ ῥητῇ μήκει, καλείσθω ἀποτομὴ πρώτη.

ιβʹ. Ἐὰν δὲ ἡ προσαρμόζουσα σύμμετρος ᾖ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καλείσθω ἀποτομὴ δευτέρα.

ιγʹ. Ἐὰν δὲ μηδετέρα σύμμετρος ᾖ τῇ ἐκκειμένῃ ῥητῇ μήκει, ἡ δὲ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καλείσθω ἀποτομὴ τρίτη.

ιδʹ. Πάλιν, ἐὰν ἡ ὅλη τῆς προσαρμοζούσης μεῖζον δύνηται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει], ἐὰν μὲν ἡ ὅλη σύμμετρος ᾖ τῇ ἐκκειμένῃ ῥητῇ μήκει, καλείσθω ἀποτομὴ τετάρτη.

ιεʹ. Ἐὰν δὲ ἡ προσαρμόζουσα, πέμπτη.

ιϛʹ. Ἐὰν δὲ μηδετέρα, ἕκτη.

Definitions III

11. Given a rational (straight-line) and an apotome, if the square on the whole is greater than the (square on a straight-line) attached (to the apotome) by the (square) on (some straight-line) commensurable in length with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a first apotome.

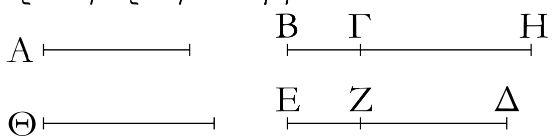
12. And if the attached (straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a second apotome.

13. And if neither of (the whole or the attached straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a third apotome.

14. Again, if the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) incommensurable [in length] with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a fourth apotome.

πε′.

Εὑρεῖν τὴν πρώτην ἀποτομήν.



Ἐκκείσθω ῥητὴ ἡ Α, καὶ τῇ Α μήκει σύμμετρος ἔστω ἡ ΒΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΗ. καὶ ἐκκείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ ΔΕ, ΕΖ, ὧν ἡ ὑπεροχὴ ὁ ΖΔ μὴ ἔστω τετράγωνος· οὐδ᾿ ἄρα ὁ ΕΔ πρὸς τὸν ΔΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΓ τετράγωνον· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΗ τῷ ἀπὸ τῆς ΗΓ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΒΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΓ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΗΓ. καὶ ἐπεὶ ὁ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾿ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ ΗΓ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΒΗ, ΗΓ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα ΒΓ ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ πρώτη.

Ὧι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. καὶ ἐπεί ἐστιν ὡς ὁ ΕΔ πρὸς τὸν ΖΔ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΗΒ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΔΕ πρὸς τὸν ΕΖ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἑκάτερος γὰρ τετράγωνός ἐστιν· καὶ τὸ ἀπὸ τῆς ΗΒ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ΒΗ ἄρα τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ ὅλη ἡ ΒΗ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ Α. ἡ ΒΓ ἄρα ἀποτομή ἐστι πρώτη.

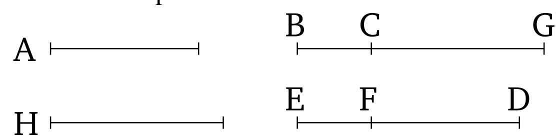
Εὕρηται ἄρα ἡ πρώτη ἀποτομὴ ἡ ΒΓ· ὅπερ ἔδει εὑρεῖν.

15. And if the attached (straight-line is commensurable), a fifth (apotome).

16. And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

Proposition 85

To find a first apotome.



Let the rational (straight-line) *A* be laid down. And let *BG* be commensurable in length with *A*. *BG* is thus also a rational (straight-line). And let two square numbers *DE* and *EF* be laid down, and let their difference *FD* be not square [Prop. 10.28 lem. I]. Thus, *ED* does not have to *DF* the ratio which (some) square number (has) to (some) square number. And let it be contrived that as *ED* (is) to *DF*, so the square on *BG* (is) to the square on *GC* [Prop. 10.6. corr.]. Thus, the (square) on *BG* is commensurable with the (square) on *GC* [Prop. 10.6]. And the (square) on *BG* (is) rational. Thus, the (square) on *GC* (is) also rational. Thus, *GC* is also rational. And since *ED* does not have to *DF* the ratio which (some) square number (has) to (some) square number, the (square) on *BG* thus does not have to the (square) on *GC* the ratio which (some) square number (has) to (some) square number either. Thus, *BG* is incommensurable in length with *GC* [Prop. 10.9]. And they are both rational (straight-lines). Thus, *BG* and *GC* are rational (straight-lines which are) commensurable in square only. Thus, *BC* is an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

Let the (square) on *H* be that (area) by which the (square) on *BG* is greater than the (square) on *GC* [Prop. 10.13 lem.]. And since as *ED* is to *FD*, so the (square) on *BG* (is) to the (square) on *GC*, thus, via conversion, as *DE* is to *EF*, so the (square) on *GB* (is) to the (square) on *H* [Prop. 5.19 corr.]. And *DE* has to *EF* the ratio which (some) square-number (has) to (some) square-number. For each is a square (number). Thus, the (square) on *GB* also has to the (square) on *H* the ratio which (some) square number (has) to (some) square number. Thus, *BG* is commensurable in length with *H* [Prop. 10.9]. And the square on *BG* is greater than (the square on) *GC* by the (square) on *H*. Thus, the square on *BG* is greater than (the square on) *GC* by the (square) on (some straight-line) commensurable in length with (*BG*). And the whole, *BG*, is commensurable in length with the (previously) laid down rational (straight-line) *A*. Thus, *BC* is a first apotome [Def. 10.11].

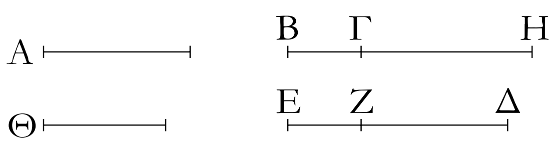
Thus, the first apotome *BC* has been found. (Which is) the very thing it was required to find.

† See footnote to Prop. 10.48.

πϛ´.

Εὑρεῖν τὴν δευτέραν ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α καὶ τῇ Α σύμμετρος μήκει ἡ ΗΓ. ῥητὴ ἄρα ἐστὶν ἡ ΗΓ. καὶ ἐκκείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ ΔΕ, ΕΖ, ὧν ἡ ὑπεροχὴ ὁ ΔΖ μὴ ἔστω τετράγωνος. καὶ πεποιήσθω ὡς ὁ ΖΔ πρὸς τὸν ΔΕ, οὕτως τὸ ἀπὸ τῆς ΓΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΒ τετράγωνον. σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΓΗ τετράγωνον τῷ ἀπὸ τῆς ΗΒ τετραγώνῳ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΓΗ. ῥητὸν ἄρα [ἐστὶ] καὶ τὸ ἀπὸ τῆς ΗΒ· ῥητὴ ἄρα ἐστὶν ἡ ΒΗ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΗΓ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ἀσύμμετρός ἐστιν ἡ ΓΗ τῇ ΗΒ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΓΗ, ΗΒ ἄρα ρηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΒΓ ἄρα ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ δευτέρα.



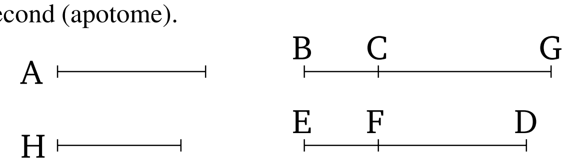
Ὧι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οὖν ἐστιν ὡς τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, οὕτως ὁ ΕΔ ἀριθμὸς πρὸς τὸν ΔΖ ἀριθμόν, ἀναστρέψαντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ, οὕτως ὁ ΔΕ πρὸς τὸν ΕΖ. καί ἐστιν ἑκάτερος τῶν ΔΕ, ΕΖ τετράγωνος· τὸ ἄρα ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ΒΗ ἄρα τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΓΗ τῇ ἐκκειμένῃ ῥητῇ σύμμετρος τῇ Α. ἡ ΒΓ ἄρα ἀποτομὴ ἐστι δευτέτα.

Εὕρηται ἄρα δευτέρα ἀποτομὴ ἡ ΒΓ· ὅπερ ἔδει δεῖξαι.

Proposition 86

To find a second apotome.

Let the rational (straight-line) *A*, and *GC* (which is) commensurable in length with *A*, be laid down. Thus, *GC* is a rational (straight-line). And let the two square numbers *DE* and *EF* be laid down, and let their difference *DF* be not square [Prop. 10.28 lem. I]. And let it be contrived that as *FD* (is) to *DE*, so the square on *CG* (is) to the square on *GB* [Prop. 10.6 corr.]. Thus, the square on *CG* is commensurable with the square on *GB* [Prop. 10.6]. And the (square) on *CG* (is) rational. Thus, the (square) on *GB* [is] also rational. Thus, *BG* is a rational (straight-line). And since the square on *GC* does not have to the (square) on *GB* the ratio which (some) square number (has) to (some) square number, *CG* is incommensurable in length with *GB* [Prop. 10.9]. And they are both rational (straight-lines). Thus, *CG* and *GB* are rational (straight-lines which are) commensurable in square only. Thus, *BC* is an apotome [Prop. 10.73]. So, I say that it is also a second (apotome).



For let the (square) on *H* be that (area) by which the (square) on *BG* is greater than the (square) on *GC* [Prop. 10.1 lem.]. Therefore, since as the (square) on *BG* is to the (square) on *GC*, so the number *ED* (is) to the number *DF*, thus, also, via conversion, as the (square) on *BG* is to the (square) on *H*, so *DE* (is) to *EF* [Prop. 5.19 corr.]. And *DE* and *EF* are each square (numbers). Thus, the (square) on *BG* has to the (square) on *H* the ratio which (some) square number (has) to (some) square number. Thus, *BG* is commensurable in length with *H* [Prop. 10.9]. And the square on *BG* is greater than (the square on) *GC* by the (square) on *H*. Thus, the square on *BG* is greater than (the square on) *GC* by the (square) on (some straight-line) commensurable in length with (*BG*). And the attachment *CG* is commensurable (in length) with the (prevously) laid down rational (straight-line) *A*. Thus, *BC* is a second apotome [Def. 10.12].†

Thus, the second apotome *BC* has been found. (Which is) the very thing it was required to show.

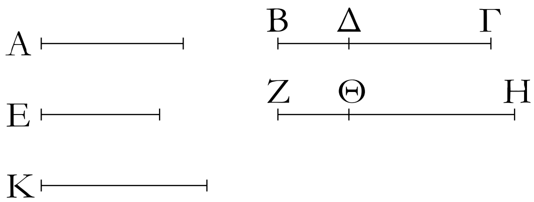
† See footnote to Prop. 10.49.

πζ´.

Εὑρεῖν τὴν τρίτην ἀποτομήν.

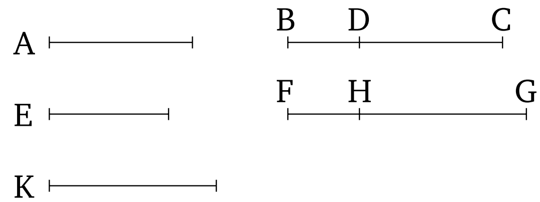
Proposition 87

To find a third apotome.



Ἐκκείσθω ῥητὴ ἡ Α, καὶ ἐκκείσθωσαν τρεῖς ἀριθμοὶ οἱ Ε, ΒΓ, ΓΔ λόγον μὴ ἔχοντες πρὸς ἀλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὁ δὲ ΓΒ πρὸς τὸν ΒΔ λόγον ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ πεποιήσθω ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ τετράγωνον, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τὴς ΗΘ. ἐπεὶ οὖν ἐστιν ὡς ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ τετράγωνον, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Α τετράγωνον τῷ ἀπὸ τῆς ΖΗ τετραγώνῳ. ῥητὸν δὲ τὸ ἀπὸ τῆς Α τετράγωνον. ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΖΗ· ῥητὴ ἄρα ἐστὶν ἡ ΖΗ. καὶ ἐπεὶ ὁ Ε πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ [τετράγωνον] λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ ΖΗ μήκει. πάλιν, ἐπεί ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΘ, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΖΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΘ· ῥητὴ ἄρα ἐστὶν ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΓ πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΖΘ. λέγω δή, ὅτι καὶ τρίτη.

Ἐπεὶ γάρ ἐστιν ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΘΗ, δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΘΗ. ὁ δὲ Ε πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἡ Α τῇ ΗΘ μήκει. οὐδετέρα ἄρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ Α μήκει. ᾧ οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ, ἔστω τὸ ἀπὸ τῆς Κ. ἐπεὶ οὖν ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΒΓ πρὸς τὸν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΒΓ πρὸς τὸν ΒΔ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ τὸ ἀπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. σύμμετρός ἄρα ἐστὶν ἡ ΖΗ τῇ Κ μήκει, καὶ δύναται ἡ ΖΗ τῆς ΗΘ μεῖζον τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ οὐδετέρα



Let the rational (straight-line) A be laid down. And let the three numbers, E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. And let CB have to BD the ratio which (some) square number (has) to (some) square number. And let it be contrived that as E (is) to BC, so the square on A (is) to the square on FG, and as BC (is) to CD, so the square on FG (is) to the (square) on GH [Prop. 10.6 corr.]. Therefore, since as E is to BC, so the square on A (is) to the square on FG, the square on A is thus commensurable with the square on FG [Prop. 10.6]. And the square on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the square on A thus does not have to the [square] on FG the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with FG [Prop. 10.9]. Again, since as BC is to CD, so the square on FG is to the (square) on GH, the square on FG is thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH (is) also rational. Thus, GH is a rational (straight-line). And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straight-lines). FG and GH are thus rational (straight-lines which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since as E is to BC, so the square on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on HG, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on HG [Prop. 5.22]. And E does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. A (is) thus incommensurable in length with GH [Prop. 10.9]. Thus, neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.]. Therefore, since as BC is to CD, so the (square) on FG (is) to the *τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ Α μήκει· ἡ ΖΘ ἄρα ἀποτομή ἐστι τρίτη.*

Εὕρηται ἄρα ἡ τρίτη ἀποτομὴ ἡ ΖΘ· ὅπερ ἔδει δεῖξαι.

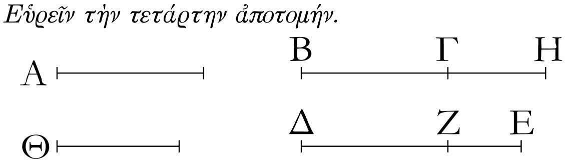
(square) on *GH*, thus, via conversion, as *BC* is to *BD*, so the square on *FG* (is) to the square on *K* [Prop. 5.19 corr.]. And *BC* has to *BD* the ratio which (some) square number (has) to (some) square number. Thus, the (square) on *FG* also has to the (square) on *K* the ratio which (some) square number (has) to (some) square number. *FG* is thus commensurable in length with *K* [Prop. 10.9]. And the square on *FG* is (thus) greater than (the square on) *GH* by the (square) on (some straight-line) commensurable (in length) with (*FG*). And neither of *FG* and *GH* is commensurable in length with the (previously) laid down rational (straight-line) *A*. Thus, *FH* is a third apotome [Def. 10.13].

Thus, the third apotome *FH* has been found. (Which is) very thing it was required to show.

† See footnote to Prop. 10.50.

πη΄.

Εὑρεῖν τὴν τετάρτην ἀποτομήν.

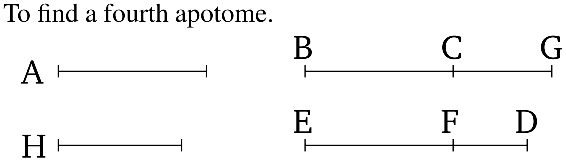


Ἐκκείσθω ῥητὴ ἡ Α καὶ τῇ Α μήκει σύμμετρος ἡ ΒΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΗ. καὶ ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΔΖ, ΖΕ, ὥστε τὸν ΔΕ ὅλον πρὸς ἑκάτερον τῶν ΔΖ, ΕΖ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΓ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΗ τῷ ἀπὸ τῆς ΗΓ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΒΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΓ· ῥητὴ ἄρα ἐστὶν ἡ ΗΓ. καὶ ἐπεὶ ὁ ΔΕ πρὸς τὸν ΕΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ ΗΓ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΒΗ, ΗΓ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΒΓ. [λέγω δή, ὅτι καὶ τετάρτη.]

Ὧι οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οὖν ἐστιν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς ΗΒ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΗΒ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ἄρα ΒΗ τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ ἔστιν ὅλη ἡ ΒΗ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ Α. ἡ ἄρα ΒΓ ἀποτομή ἐστι τετάρτη.

Proposition 88

To find a fourth apotome.



Let the rational (straight-line) *A*, and *BG* (which is) commensurable in length with *A*, be laid down. Thus, *BG* is also a rational (straight-line). And let the two numbers *DF* and *FE* be laid down such that the whole, *DE*, does not have to each of *DF* and *EF* the ratio which (some) square number (has) to (some) square number. And let it be contrived that as *DE* (is) to *EF*, so the square on *BG* (is) to the (square) on *GC* [Prop. 10.6 corr.]. The (square) on *BG* is thus commensurable with the (square) on *GC* [Prop. 10.6]. And the (square) on *BG* (is) rational. Thus, the (square) on *GC* (is) also rational. Thus, *GC* (is) a rational (straight-line). And since *DE* does not have to *EF* the ratio which (some) square number (has) to (some) square number, the (square) on *BG* thus does not have to the (square) on *GC* the ratio which (some) square number (has) to (some) square number either. Thus, *BG* is incommensurable in length with *GC* [Prop. 10.9]. And they are both rational (straight-lines). Thus, *BG* and *GC* are rational (straight-lines which are) commensurable in square only. Thus, *BC* is an apotome [Prop. 10.73]. [So, I say that (it is) also a fourth (apotome).]

Now, let the (square) on *H* be that (area) by which the (square) on *BG* is greater than the (square) on *GC* [Prop. 10.13 lem.]. Therefore, since as *DE* is to *EF*, so the (square) on *BG* (is) to the (square) on *GC*, thus, also, via conversion, as *ED* is to *DF*, so the (square) on *GB* (is) to the (square) on *H* [Prop. 5.19 corr.]. And *ED* does not have to *DF* the ratio which (some) square number (has) to (some)

Εὕρηται ἄρα ἡ τετάρτη ἀποτομή· ὅπερ ἔδει δεῖξαι.

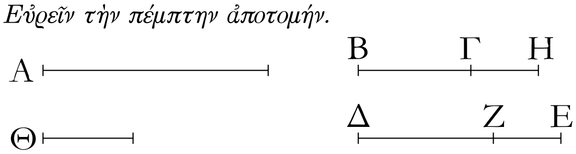
square number. Thus, the (square) on *GB* does not have to the (square) on *H* the ratio which (some) square number (has) to (some) square number either. Thus, *BG* is incommensurable in length with *H* [Prop. 10.9]. And the square on *BG* is greater than (the square on) *GC* by the (square) on *H*. Thus, the square on *BG* is greater than (the square) on *GC* by the (square) on (some straight-line) incommensurable (in length) with (*BG*). And the whole, *BG*, is commensurable in length with the the (previously) laid down rational (straight-line) *A*. Thus, *BC* is a fourth apotome [Def. 10.14].†

Thus, a fourth apotome has been found. (Which is) the very thing it was required to show.

† See footnote to Prop. 10.51.

πθ΄.

Εὑρεῖν τὴν πέμπτην ἀποτομήν.



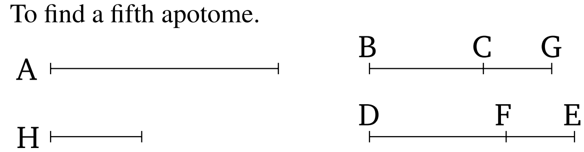
Ἐκκείσθω ῥητὴ ἡ Α, καὶ τῇ Α μήκει σύμμετρος ἔστω ἡ ΓΗ· ῥητὴ ἄρα [ἐστὶν] ἡ ΓΗ. καὶ ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΔΖ, ΖΕ, ὥστε τὸν ΔΕ πρὸς ἑκάτερον τῶν ΔΖ, ΖΕ λόγον πάλιν μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ πεποιήσθω ὡς ὁ ΖΕ πρὸς τὸν ΕΔ, οὕτως τὸ ἀπὸ τῆς ΓΗ πρὸς τὸ ἀπὸ τῆς ΗΒ. ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΒ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΗ. καὶ ἐπεί ἐστιν ὡς ὁ ΔΕ πρὸς τὸν ΕΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, ὁ δὲ ΔΕ πρὸς τὸν ΕΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ ΗΓ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΒΗ, ΗΓ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΒΓ ἄρα ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ πέμπτη.

Ὧι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΗ τοῦ ἀπὸ τῆς ΗΓ, ἔστω τὸ ἀπὸ τῆς Θ. ἐπεὶ οὖν ἐστιν ὡς τὸ ἀπὸ τὴς ΒΗ πρὸς τὸ ἀπὸ τῆς ΗΓ, οὕτως ὁ ΔΕ πρὸς τὸν ΕΖ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΕΔ πρὸς τὸν ΔΖ, οὕτως τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ, ὁ δὲ ΕΔ πρὸς τὸν ΔΖ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΒΗ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΗ τῇ Θ μήκει. καὶ δύναται ἡ ΒΗ τῆς ΗΓ μεῖζον τῷ ἀπὸ τῆς Θ· ἡ ΗΒ ἄρα τῆς ΗΓ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΓΗ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ Α μήκει· ἡ ἄρα ΒΓ ἀποτομή ἐστι πέμπτη.

Εὕρηται ἄρα ἡ πέμπτη ἀποτομὴ ἡ ΒΓ· ὅπερ ἔδει δεῖξαι.

Proposition 89

To find a fifth apotome.



Let the rational (straight-line) *A* be laid down, and let *CG* be commensurable in length with *A*. Thus, *CG* [is] a rational (straight-line). And let the two numbers *DF* and *FE* be laid down such that *DE* again does not have to each of *DF* and *FE* the ratio which (some) square number (has) to (some) square number. And let it be contrived that as *FE* (is) to *ED*, so the (square) on *CG* (is) to the (square) on *GB*. Thus, the (square) on *GB* (is) also rational [Prop. 10.6]. Thus, *BG* is also rational. And since as *DE* is to *EF*, so the (square) on *BG* (is) to the (square) on *GC*. And *DE* does not have to *EF* the ratio which (some) square number (has) to (some) square number. The (square) on *BG* thus does not have to the (square) on *GC* the ratio which (some) square number (has) to (some) square number either. Thus, *BG* is incommensurable in length with *GC* [Prop. 10.9]. And they are both rational (straight-lines). *BG* and *GC* are thus rational (straight-lines which are) commensurable in square only. Thus, *BC* is an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

For, let the (square) on *H* be that (area) by which the (square) on *BG* is greater than the (square) on *GC* [Prop. 10.13 lem.]. Therefore, since as the (square) on *BG* (is) to the (square) on *GC*, so *DE* (is) to *EF*, thus, via conversion, as *ED* is to *DF*, so the (square) on *BG* (is) to the (square) on *H* [Prop. 5.19 corr.]. And *ED* does not have to *DF* the ratio which (some) square number (has) to (some) square number. Thus, the (square) on *BG* does not have to the (square) on *H* the ratio which (some) square number (has) to (some) square number either. Thus, *BG* is incommensurable in length with *H* [Prop. 10.9]. And the square on *BG* is greater than (the square on) GC by the (square) on H. Thus, the square on GB is greater than (the square on) GC by the (square) on (some straight-line) incommensurable in length with (GB). And the attachment CG is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, BC is a fifth apotome [Def. 10.15].†

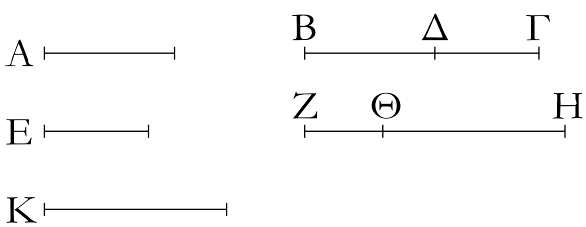
Thus, the fifth apotome BC has been found. (Which is) the very thing it was required to show.

† See footnote to Prop. 10.52.

ϙ´.

Εὑρεῖν τὴν ἕκτην ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ Α καὶ τρεῖς ἀριθμοὶ οἱ Ε, ΒΓ, ΓΔ λόγον μὴ ἔχοντες πρὸς ἀλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἔτι δὲ καὶ ὁ ΓΒ πρὸς τὸν ΒΔ λόγον μὴ ἐχετώ, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ πεποιήσθω ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ.



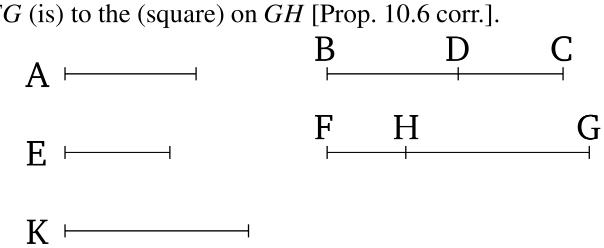
Ἐπεὶ οὖν ἐστιν ὡς ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ, σύμμετρον ἄρα τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς ΖΗ. ῥητὸν δὲ τὸ ἀπὸ τῆς Α· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΖΗ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ Ε πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ ΖΗ μήκει. πάλιν, ἐπεί ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, σύμμετρον ἄρα τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΖΗ· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΗΘ· ῥητὴ ἄρα καὶ ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΓ πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ΖΗ, ΗΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα ΖΘ ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ ἕκτη.

Ἐπεὶ γάρ ἐστιν ὡς μὲν ὁ Ε πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι᾽ ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Ε πρὸς τὸν ΓΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς

Proposition 90

To find a sixth apotome.

Let the rational (straight-line) A, and the three numbers E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. Furthermore, let CB also not have to BD the ratio which (some) square number (has) to (some) square number. And let it be contrived that as E (is) to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.].



Therefore, since as E is to BC, so the (square) on A (is) to the (square) on FG, the (square) on A (is) thus commensurable with the (square) on FG [Prop. 10.6]. And the (square) on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is also a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with FG [Prop. 10.9]. Again, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH (is) also rational. Thus, GH (is) also rational. And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square (number) has to (some) square (number) either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straight-lines). Thus, FG and GH are rational (straight-lines which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς Α πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ Α τῇ ΗΘ μήκει· οὐδετέρα ἄρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ Α ῥητῇ μήκει. ᾧ οὖν μεῖζόν ἐστι τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ, ἔστω τὸ ἀπὸ τῆς Κ. ἐπεὶ οὖν ἐστιν ὡς ὁ ΒΓ πρὸς τὸν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΓΒ πρὸς τὸν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΓΒ πρὸς τὸν ΒΔ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ᾽ ἄρα τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Κ μήκει. καὶ δύναται ἡ ΖΗ τῆς ΗΘ μεῖζον τῷ ἀπὸ τῆς Κ· ἡ ΖΗ ἄρα τῆς ΗΘ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καὶ οὐδετέρα τῶν ΖΗ, ΗΘ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ Α. ἡ ἄρα ΖΘ ἀποτομή ἐστιν ἕκτη.

Εὕρηται ἄρα ἡ ἕκτη ἀποτομὴ ἡ ΖΘ· ὅπερ ἔδει δεῖξαι.

For since as E is to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on GH [Prop. 5.22]. And E does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) GH the ratio which (some) square number (has) to (some) square number either. A is thus incommensurable in length with GH [Prop. 10.9]. Thus, neither of FG and GH is commensurable in length with the rational (straight-line) A. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.]. Therefore, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, thus, via conversion, as CB is to BD, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And CB does not have to BD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. FG is thus incommensurable in length with K [Prop. 10.9]. And the square on FG is greater than (the square on) GH by the (square) on K. Thus, the square on FG is greater than (the square on) GH by the (square) on (some straight-line) incommensurable in length with (FG). And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, FH is a sixth apotome [Def. 10.16].

Thus, the sixth apotome FH has been found. (Which is) the very thing it was required to show.

† See footnote to Prop. 10.53.

ϙα'.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς πρώτης, ἡ τὸ χωρίον δυναμένη ἀπορομή ἐστιν.

Περιεχέσθω γὰρ χωρίον τὸ ΑΒ ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς πρώτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη ἀποτομή ἐστιν.

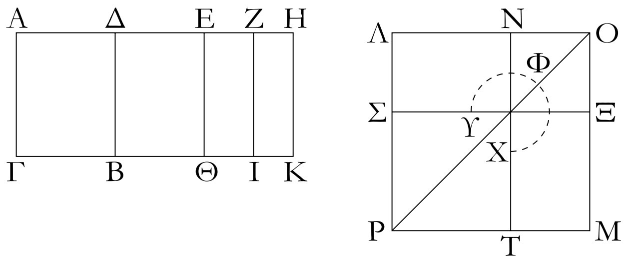
Ἐπεὶ γὰρ ἀποτομή ἐστι πρώτη ἡ ΑΔ, ἔστω αὐτῇ προσαρμόζουσα ἡ ΔΗ· αἱ ΑΗ, ΗΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ὅλη ἡ ΑΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, καὶ ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. τετμήσθω ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· σύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ. καὶ διὰ τῶν Ε, Ζ, Η σημείων τῇ ΑΓ παράλληλοι ἤχθωσαν αἱ ΕΘ, ΖΙ, ΗΚ.

Proposition 91

If an area is contained by a rational (straight-line) and a first apotome then the square-root of the area is an apotome.

For let the area AB be contained by the rational (straight-line) AC and the first apotome AD. I say that the square-root of area AB is an apotome.

For since AD is a first apotome, let DG be its attachment. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And the whole, AG, is commensurable (in length) with the (previously) laid down rational (straight-line) AC, and the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.11]. Thus, if (an area) equal to the fourth part of the (square) on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Let DG be cut in half at E. And let (an area) equal to the (square) on EG be applied to AG, falling short by a square figure. And let it be the (rectangle contained) by

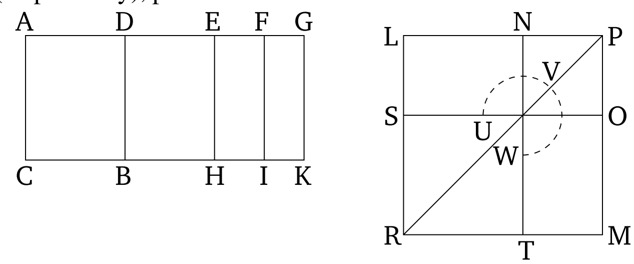


Καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΖ τῇ ΖΗ μήκει, καὶ ἡ ΑΗ ἄρα ἑκατέρᾳ τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ἀλλὰ ἡ ΑΗ σύμμετρός ἐστι τῇ ΑΓ· καὶ ἑκατέρα ἄρα τῶν ΑΖ, ΖΗ σύμμετρός ἐστι τῇ ΑΓ μήκει. καί ἐστι ῥητὴ ἡ ΑΓ· ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΑΖ, ΖΗ· ὥστε καὶ ἑκάτερον τῶν ΑΙ, ΖΚ ῥητόν ἐστιν. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΕΗ μήκει, καὶ ἡ ΔΗ ἄρα ἑκατέρᾳ τῶν ΔΕ, ΕΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν.

Κείσθω δὴ τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον τετράγωνον ἀφῃρήσθω κοινὴν γωνίαν ἔχον αὐτῷ τὴν ὑπὸ ΛΟΜ τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν ἴσον ἐστὶ τὸ ὑπὸ τῶν ΑΖ, ΖΗ περιεχόμενον ὀρθογώνιον τῷ ἀπὸ τῆς ΕΗ τετραγώνῳ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ. ἀλλ᾿ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως τὸ ΑΙ πρὸς τὸ ΕΚ, ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΚΖ· τῶν ἄρα ΑΙ, ΚΖ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ μέσον ἀνάλογον τὸ ΜΝ, ὡς ἐν τοῖς ἔμπροσθεν ἐδείχθη, καί ἐστι τὸ [μὲν] ΑΙ τῷ ΛΜ τετραγώνῳ ἴσον, τὸ δὲ ΚΖ τῷ ΝΞ· καὶ τὸ ΜΝ ἄρα τῷ ΕΚ ἴσον ἐστίν. ἀλλὰ τὸ μὲν ΕΚ τῷ ΔΘ ἐστιν ἴσον, τὸ δὲ ΜΝ τῷ ΛΞ· τὸ ἄρα ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἔστι δὲ καὶ τὸ ΑΚ ἴσον τοῖς ΛΜ, ΝΞ τετραγώνοις· λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΣΤ. τὸ δὲ ΣΤ τὸ ἀπὸ τῆς ΛΝ ἐστι τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΛΝ τετράγωνον ἴσον ἐστὶ τῷ ΑΒ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ. λέγω δή, ὅτι ἡ ΛΝ ἀποτομή ἐστιν.

Ἐπεὶ γὰρ ῥητόν ἐστιν ἑκάτερον τῶν ΑΙ, ΖΚ, καί ἐστιν ἴσον τοῖς ΛΜ, ΝΞ, καὶ ἑκάτερον ἄρα τῶν ΛΜ, ΝΞ ῥητόν ἐστιν, τουτέστι τὸ ἀπὸ ἑκατέρας τῶν ΛΟ, ΟΝ· καὶ ἑκατέρα ἄρα τῶν ΛΟ, ΟΝ ῥητή ἐστιν. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ΔΘ καί ἐστιν ἴσον τῷ ΛΞ, μέσον ἄρα ἐστὶ καὶ τὸ ΛΞ. ἐπεὶ οὖν τὸ μὲν ΛΞ μέσον ἐστίν, τὸ δὲ ΝΞ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ ΛΞ τῷ ΝΞ· ὡς δὲ τὸ ΛΞ πρὸς τὸ ΝΞ, οὕτως ἐστὶν ἡ ΛΟ πρὸς τὴν ΟΝ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΛΟ τῇ ΟΝ μήκει. καὶ εἰσιν ἀμφότεραι ῥηταί· αἱ ΛΟ, ΟΝ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΛΝ. καὶ δύναται τὸ ΑΒ χωρίον· ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη ἀποτομή ἐστιν.

AF and FG. AF is thus commensurable (in length) with FG. And let EH, FI, and GK be drawn through points E, F, and G (respectively), parallel to AC.



And since AF is commensurable in length with FG, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. But AG is commensurable (in length) with AC. Thus, each of AF and FG is also commensurable in length with AC [Prop. 10.12]. And AC is a rational (straight-line). Thus, AF and FG (are) each also rational (straight-lines). Hence, AI and FK are also each rational (areas) [Prop. 10.19]. And since DE is commensurable in length with EG, DG is thus also commensurable in length with each of DE and EG [Prop. 10.15]. And DG (is) rational, and incommensurable in length with AC. DE and EG (are) thus each rational, and incommensurable in length with AC [Prop. 10.13]. Thus, DH and EK are each medial (areas) [Prop. 10.21].

So let the square LM, equal to AI, be laid down. And let the square NO, equal to FK, be subtracted (from LM), having with it the common angle LPM. Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure be drawn. Therefore, since the rectangle contained by AF and FG is equal to the square EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI (is) to EK, and as EG (is) to FG, so EK is to KF [Prop. 6.1]. Thus, EK is the mean proportional to AI and KF [Prop. 5.11]. And MN is also the mean proportional to LM and NO, as shown before [Prop. 10.53 lem.]. And AI is equal to the square LM, and KF to NO. Thus, MN is also equal to EK. But, EK is equal to DH, and MN to LO [Prop. 1.43]. Thus, DK is equal to the gnomon UVW and NO. And AK is also equal to (the sum of) the squares LM and NO. Thus, the remainder AB is equal to ST. And ST is the square on LN. Thus, the square on LN is equal to AB. Thus, LN is the square-root of AB. So, I say that LN is an apotome.

For since AI and FK are each rational (areas), and are equal to LM and NO (respectively), thus LM and NO—that is to say, the (squares) on each of LP and PN (respectively)—are also each rational (areas). Thus, LP and PN are also each rational (straight-lines). Again, since DH is a medial (area), and is equal to LO, LO is thus also a medial (area). Therefore, since LO is medial, and NO rational, LO is thus incommensurable with NO. And as LO (is) to NO, so LP

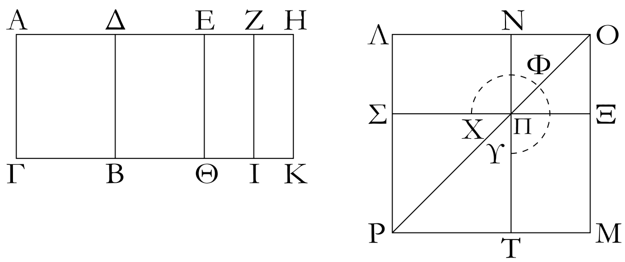
Ἐὰν ἄρα χωρίον περιέχηται ὑπὸ ῥητῆς καὶ τὰ ἑξῆς.

is to *PN* [Prop. 6.1]. *LP* is thus incommensurable in length with *PN* [Prop. 10.11]. And they are both rational (straight-lines). Thus, *LP* and *PN* are rational (straight-lines which are) commensurable in square only. Thus, *LN* is an apotome [Prop. 10.73]. And it is the square-root of area *AB*. Thus, the square-root of area *AB* is an apotome.

Thus, if an area is contained by a rational (straight-line), and so on ....

ρβ'.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς δευτέρας, ἡ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη.



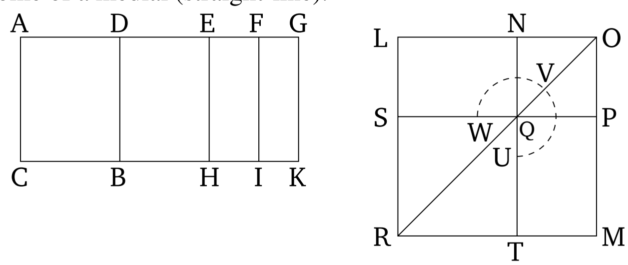
Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς δευτέρας τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προσαρμόζουσα ἡ ΔΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΗΔ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε· καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· σύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει. καὶ ἡ ΑΗ ἄρα ἑκατέρᾳ τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΑΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· καὶ ἑκατέρα ἄρα τῶν ΑΖ, ΖΗ ῥητή ἐστι καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ἑκάτερον ἄρα τῶν ΑΙ, ΖΚ μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΕΗ, καὶ ἡ ΔΗ ἄρα ἑκατέρᾳ τῶν ΔΕ, ΕΗ σύμμετρός ἐστιν. ἀλλ' ἡ ΔΗ σύμμετρός ἐστι τῇ ΑΓ μήκει [ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ σύμμετρος τῇ ΑΓ μήκει]. ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ ῥητόν ἐστιν.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν ὂν τῷ ΛΜ τὴν ὑπὸ τῶν ΛΟΜ· περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὰ ΑΙ, ΖΚ μέσα ἐστὶ καί ἐστιν ἴσα τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, καὶ τὰ ἀπὸ τῶν ΛΟ, ΟΝ [ἄρα] μέσα ἐστίν· καὶ αἱ ΛΟ, ΟΝ ἄρα μέσαι εἰσὶ δυνάμει μόνον

Proposition 92

If an area is contained by a rational (straight-line) and a second apotome then the square-root of the area is a first apotome of a medial (straight-line).



For let the area *AB* be contained by the rational (straight-line) *AC* and the second apotome *AD*. I say that the square-root of area *AB* is the first apotome of a medial (straight-line).

For let *DG* be an attachment to *AD*. Thus, *AG* and *GD* are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment *DG* is commensurable (in length) with the (previously) laid down rational (straight-line) *AC*, and the square on the whole, *AG*, is greater than (the square on) the attachment, *GD*, by the (square) on (some straight-line) commensurable in length with (*AG*) [Def. 10.12]. Therefore, since the square on *AG* is greater than (the square on) *GD* by the (square) on (some straight-line) commensurable (in length) with (*AG*), thus if (an area) equal to the fourth part of the (square) on *GD* is applied to *AG*, falling short by a square figure, then it divides (*AG*) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let *DG* be cut in half at *E*. And let (an area) equal to the (square) on *EG* be applied to *AG*, falling short by a square figure. And let it be the (rectangle contained) by *AF* and *FG*. Thus, *AF* is commensurable in length with *FG*. *AG* is thus also commensurable in length with each of *AF* and *FG* [Prop. 10.15]. And *AG* (is) a rational (straight-line), and incommensurable in length with *AC*. *AF* and *FG* are thus also each rational (straight-lines), and incommensurable in length with *AC* [Prop. 10.13]. Thus, *AI* and *FK* are each medial (areas) [Prop. 10.21]. Again, since *DE* is commensurable (in length) with *EG*, thus *DG* is also commensurable (in length) with each of *DE* and *EG* [Prop. 10.15]. But, *DG* is commensurable in length with *AC* [thus, *DE* and *EG* are also each *σύμμετροι. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ· ἀλλ᾽ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως τὸ ΑΙ πρὸς τὸ ΕΚ· ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως [ἐστὶ] τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ· καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΜΝ ἄρα ἴσον ἐστὶ τῷ ΕΚ. ἀλλὰ τῷ μὲν ΕΚ ἴσον [ἐστὶ] τὸ ΔΘ, τῷ δὲ ΜΝ ἴσον τὸ ΛΞ· ὅλον ἄρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἐπεὶ οὖν ὅλον τὸ ΑΚ ἴσον ἐστὶ τοῖς ΛΜ, ΝΞ, ὧν τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ, λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΤΣ. τὸ δὲ ΤΣ ἐστι τὸ ἀπὸ τῆς ΛΝ· τὸ ἀπὸ τῆς ΛΝ ἄρα ἴσον ἐστὶ τῷ ΑΒ χωρίῳ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον. λέγω [δή], ὅτι ἡ ΛΝ μέσης ἀποτομή ἐστι πρώτη.*

Ἐπεὶ γὰρ ῥητόν ἐστι τὸ ΕΚ καί ἐστιν ἴσον τῷ ΛΞ, ῥητὸν ἄρα ἐστὶ τὸ ΛΞ, τουτέστι τὸ ὑπὸ τῶν ΛΟ, ΟΝ. μέσον δὲ ἐδείχθη τὸ ΝΞ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΛΞ τῷ ΝΞ· ὡς δὲ τὸ ΛΞ πρὸς τὸ ΝΞ, οὕτως ἐστὶν ἡ ΛΟ πρὸς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα ἀσύμμετροί εἰσι μήκει. αἱ ἄρα ΛΟ, ΟΝ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ἡ ΛΝ ἄρα μέσης ἀποτομή ἐστι πρώτη· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι πρώτη· ὅπερ ἔδει δεῖξαι.

rational, and commensurable in length with *AC*]. Thus, *DH* and *EK* are each rational (areas) [Prop. 10.19].

Therefore, let the square *LM*, equal to *AI*, be constructed. And let *NO*, equal to *FK*, which is about the same angle *LPM* as *LM*, be subtracted (from *LM*). Thus, the squares *LM* and *NO* are about the same diagonal [Prop. 6.26]. Let *PR* be their (common) diagonal, and let the (rest of the) figure be drawn. Therefore, since *AI* and *FK* are medial (areas), and are equal to the (squares) on *LP* and *PN* (respectively), [thus] the (squares) on *LP* and *PN* are also medial. Thus, *LP* and *PN* are also medial (straight-lines which are) commensurable in square only.† And since the (rectangle contained) by *AF* and *FG* is equal to the (square) on *EG*, thus as *AF* is to *EG*, so *EG* (is) to *FG* [Prop. 10.17]. But, as *AF* (is) to *EG*, so *AI* (is) to *EK*. And as *EG* (is) to *FG*, so *EK* [is] to *FK* [Prop. 6.1]. Thus, *EK* is the mean proportional to *AI* and *FK* [Prop. 5.11]. And *MN* is also the mean proportional to the squares *LM* and *NO* [Prop. 10.53 lem.]. And *AI* is equal to *LM*, and *FK* to *NO*. Thus, *MN* is also equal to *EK*. But, *DH* [is] equal to *EK*, and *LO* equal to *MN* [Prop. 1.43]. Thus, the whole (of) *DK* is equal to the gnomon *UVW* and *NO*. Therefore, since the whole (of) *AK* is equal to *LM* and *NO*, of which *DK* is equal to the gnomon *UVW* and *NO*, the remainder *AB* is thus equal to *TS*. And *TS* is the (square) on *LN*. Thus, the (square) on *LN* is equal to the area *AB*. *LN* is thus the square-root of area *AB*. [So], I say that *LN* is the first apotome of a medial (straight-line).

For since *EK* is a rational (area), and is equal to *LO*, *LO*—that is to say, the (rectangle contained) by *LP* and *PN*—is thus a rational (area). And *NO* was shown (to be) a medial (area). Thus, *LO* is incommensurable with *NO*. And as *LO* (is) to *NO*, so *LP* is to *PN* [Prop. 6.1]. Thus, *LP* and *PN* are incommensurable in length [Prop. 10.11]. *LP* and *PN* are thus medial (straight-lines which are) commensurable in square only, and which contain a rational (area). Thus, *LN* is the first apotome of a medial (straight-line) [Prop. 10.74]. And it is the square-root of area *AB*.

Thus, the square root of area *AB* is the first apotome of a medial (straight-line). (Which is) the very thing it was required to show.

† There is an error in the argument here. It should just say that *LP* and *PN* are commensurable in square, rather than in square only, since *LP* and *PN* are only shown to be incommensurable in length later on.

ϙγ΄.

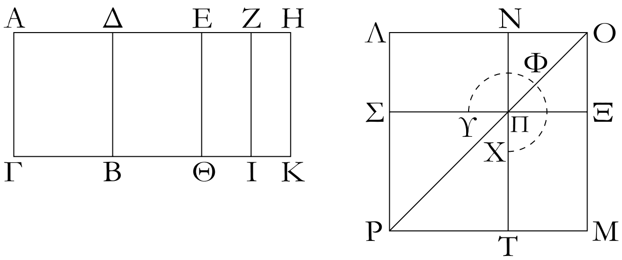
Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς τρίτης, ἡ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα.

Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς τρίτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα.

Proposition 93

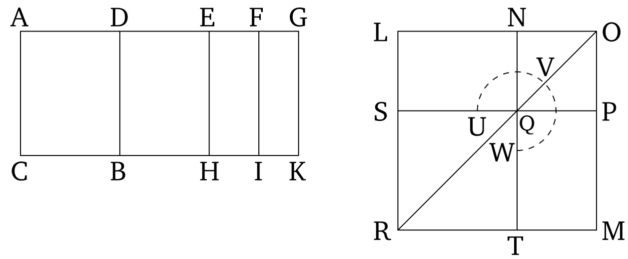
If an area is contained by a rational (straight-line) and a third apotome then the square-root of the area is a second apotome of a medial (straight-line).

For let the area *AB* be contained by the rational (straight-line) *AC* and the third apotome *AD*. I say that the square-root of area *AB* is the second apotome of a medial (straight-line).



Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ΑΗ, ΗΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα τῶν ΑΗ, ΗΔ σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ. καὶ ἤχθωσαν διὰ τῶν Ε, Ζ, Η σημείων τῇ ΑΓ παράλληλοι αἱ ΕΘ, ΖΙ, ΗΚ· σύμμετροι ἄρα εἰσὶν αἱ ΑΖ, ΖΗ· σύμμετρον ἄρα καὶ τὸ ΑΙ τῷ ΖΚ. καὶ ἐπεὶ αἱ ΑΖ, ΖΗ σύμμετροί εἰσι μήκει, καὶ ἡ ΑΗ ἄρα ἑκατέρᾳ τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΑΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ὥστε καὶ αἱ ΑΖ, ΖΗ. ἑκάτερον ἄρα τῶν ΑΙ, ΖΚ μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΕΗ μήκει, καὶ ἡ ΔΗ ἄρα ἑκατέρᾳ τῶν ΔΕ, ΕΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΗΔ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, ΕΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν. καὶ ἐπεὶ αἱ ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶ μήκει ἡ ΑΗ τῇ ΗΔ. ἀλλ᾽ ἡ μὲν ΑΗ τῇ ΑΖ σύμμετρός ἐστι μήκει ἡ δὲ ΔΗ τῇ ΕΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΕΗ μήκει. ὡς δὲ ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΙ τῷ ΕΚ.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν ὂν τῷ ΛΜ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ. ἀλλ᾽ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ· ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΖΚ· καὶ ὡς ἄρα τὸ ΑΙ πρὸς τὸ ΕΚ, οὕτως τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ· καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΕΚ ἄρα ἴσον ἐστὶ τῷ ΜΝ. ἀλλὰ τὸ μὲν ΜΝ ἴσον ἐστὶ τῷ ΛΞ, τὸ δὲ ΕΚ ἴσον [ἐστὶ] τῷ ΔΘ· καὶ ὅλον ἄρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἔστι δὲ καὶ τὸ ΑΚ ἴσον τοῖς ΛΜ, ΝΞ· λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΣΤ, τουτέστι τῷ ἀπὸ τῆς ΛΝ τετραγώνῳ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΛΝ μέσης ἀποτομή ἐστι δευτέρα.



For let DG be an attachment to AD. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of AG and GD is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straight-line) commensurable (in length) with (AG) [Def. 10.13]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG), thus if (an area) equal to the fourth part of the square on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let DG be cut in half at E. And let (an area) equal to the (square) on EG be applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. And let EH, FI, and GK be drawn through points E, F, and G (respectively), parallel to AC. Thus, AF and FG are commensurable (in length). AI (is) thus also commensurable with FK [Props. 6.1, 10.11]. And since AF and FG are commensurable in length, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. And AG (is) rational, and incommensurable in length with AC. Hence, AF and FG (are) also (rational, and incommensurable in length with AC) [Prop. 10.13]. Thus, AI and FK are each medial (areas) [Prop. 10.21]. Again, since DE is commensurable in length with EG, DG is also commensurable in length with each of DE and EG [Prop. 10.15]. And GD (is) rational, and incommensurable in length with AC. Thus, DE and EG (are) each also rational, and incommensurable in length with AC [Prop. 10.13]. DH and EK are thus each medial (areas) [Prop. 10.21]. And since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD. But, AG is commensurable in length with AF, and DG with EG. Thus, AF is incommensurable in length with EG [Prop. 10.13]. And as AF (is) to EG, so AI is to EK [Prop. 6.1]. Thus, AI is incommensurable with EK [Prop. 10.11].

Therefore, let the square LM, equal to AI, be constructed. And let NO, equal to FK, which is about the same angle as LM, be subtracted (from LM). Thus, LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure be drawn. Therefore, since the (rectangle contained) by AF and FG is equal

Ἐπεὶ γὰρ μέσα ἐδείχθη τὰ ΑΙ, ΖΚ καί ἐστιν ἴσα τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, μέσον ἄρα καὶ ἑκάτερον τῶν ἀπὸ τῶν ΛΟ, ΟΝ· μέση ἄρα ἑκατέρα τῶν ΛΟ, ΟΝ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ΑΙ τῷ ΖΚ, σύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ. πάλιν, ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΙ τῷ ΕΚ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΛΜ τῷ ΜΝ, τουτέστι τὸ ἀπὸ τῆς ΛΟ τῷ ὑπὸ τῶν ΛΟ, ΟΝ· ὥστε καὶ ἡ ΛΟ ἀσύμμετρός ἐστι μήκει τῇ ΟΝ· αἱ ΛΟ, ΟΝ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δή, ὅτι καὶ μέσον περιέχουσιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΕΚ καί ἐστιν ἴσον τῷ ὑπὸ τῶν ΛΟ, ΟΝ, μέσον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΛΟ, ΟΝ· ὥστε αἱ ΛΟ, ΟΝ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι. ἡ ΛΝ ἄρα μέσης ἀποτομή ἐστι δευτέρα· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη μέσης ἀποτομή ἐστι δευτέρα· ὅπερ ἔδει δεῖξαι.

to the (square) on EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI is to EK [Prop. 6.1]. And as EG (is) to FG, so EK is to FK [Prop. 6.1]. And thus as AI (is) to EK, so EK (is) to FK [Prop. 5.11]. Thus, EK is the mean proportional to AI and FK. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.]. And AI is equal to LM, and FK to NO. Thus, EK is also equal to MN. But, MN is equal to LO, and EK [is] equal to DH [Prop. 1.43]. And thus the whole of DK is equal to the gnomon UVW and NO. And AK (is) also equal to LM and NO. Thus, the remainder AB is equal to ST—that is to say, to the square on LN. Thus, LN is the square-root of area AB. I say that LN is the second apotome of a medial (straight-line).

For since AI and FK were shown (to be) medial (areas), and are equal to the (squares) on LP and PN (respectively), the (squares) on each of LP and PN (are) thus also medial. Thus, LP and PN (are) each medial (straight-lines). And since AI is commensurable with FK [Props. 6.1, 10.11], the (square) on LP (is) thus also commensurable with the (square) on PN. Again, since AI was shown (to be) incommensurable with EK, LM is thus also incommensurable with MN—that is to say, the (square) on LP with the (rectangle contained) by LP and PN. Hence, LP is also incommensurable in length with PN [Props. 6.1, 10.11]. Thus, LP and PN are medial (straight-lines which are) commensurable in square only. So, I say that they also contain a medial (area).

For since EK was shown (to be) a medial (area), and is equal to the (rectangle contained) by LP and PN, the (rectangle contained) by LP and PN is thus also medial. Hence, LP and PN are medial (straight-lines which are) commensurable in square only, and which contain a medial (area). Thus, LN is the second apotome of a medial (straight-line) [Prop. 10.75]. And it is the square-root of area AB.

Thus, the square-root of area AB is the second apotome of a medial (straight-line). (Which is) the very thing it was required to show.

ρδ΄.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης, ἡ τὸ χωρίον δυναμένη ἐλάσσων ἐστίν.

Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς τετάρτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη ἐλάσσων ἐστίν.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ μήκει, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. ἐπεὶ ο὾ν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τε-

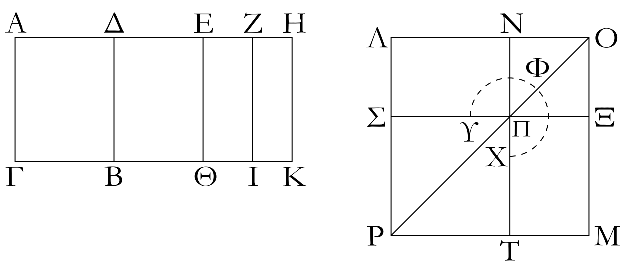
Proposition 94

If an area is contained by a rational (straight-line) and a fourth apotome then the square-root of the area is a minor (straight-line).

For let the area AB be contained by the rational (straight-line) AC and the fourth apotome AD. I say that the square-root of area AB is a minor (straight-line).

For let DG be an attachment to AD. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and AG is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the square on (some straight-line) incom-

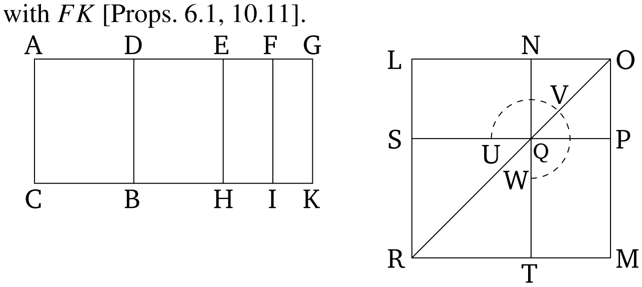
τραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶ μήκει ἡ ΑΖ τῇ ΖΗ. ἤχθωσαν οὖν διὰ τῶν Ε, Ζ, Η παράλληλοι ταῖς ΑΓ, ΒΔ αἱ ΕΘ, ΖΙ, ΗΚ. ἐπεὶ οὖν ῥητή ἐστιν ἡ ΑΗ καὶ σύμμετρος τῇ ΑΓ μήκει, ῥητὸν ἄρα ἐστὶν ὅλον τὸ ΑΚ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΔΗ τῇ ΑΓ μήκει, καί εἰσιν ἀμφότεραι ῥηταί, μέσον ἄρα ἐστὶ τὸ ΔΚ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει, ἀσύμμετρον ἄρα καὶ τὸ ΑΙ τῷ ΖΚ.



Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ τῶν ΛΟΜ τὸ ΝΞ. περὶ τὴν αὐτὴν ἄρα διάμετόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ. ἀλλ᾽ ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ, ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΛΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ, καί ἐστιν ἴσον τὸ μὲν ΑΙ τῷ ΛΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΕΚ ἄρα ἴσον ἐστὶ τῷ ΜΝ. ἀλλὰ τῷ μὲν ΕΚ ἴσον ἐστὶ τὸ ΔΘ, τῷ δὲ ΜΝ ἴσον ἐστὶ τὸ ΛΞ· ὅλον ἄρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ. ἐπεὶ οὖν ὅλον τὸ ΑΚ ἴσον ἐστὶ τοῖς ΛΜ, ΝΞ τετραγώνοις, ὧν τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνώμονι καὶ τῷ ΝΞ τετραγώνῳ, λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΣΤ, τουτέστι τῷ ἀπὸ τῆς ΛΝ τετραγώνῳ· ἡ ΛΝ ἄρα δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΛΝ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ῥητόν ἐστι τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ τετράγωνοις, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ ῥητόν ἐστιν. πάλιν, ἐπεὶ τὸ ΔΚ μέσον ἐστίν, καί ἐστιν ἴσον τὸ ΔΚ τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα δὶς ὑπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΙ τῷ ΖΚ,

mensurable in length with (*AG*) [Def. 10.14]. Therefore, since the square on *AG* is greater than (the square on) *GD* by the (square) on (some straight-line) incommensurable in length with (*AG*), thus if (some area), equal to the fourth part of the (square) on *DG*, is applied to *AG*, falling short by a square figure, then it divides (*AG*) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let *DG* be cut in half at *E*, and let (some area), equal to the (square) on *EG*, be applied to *AG*, falling short by a square figure, and let it be the (rectangle contained) by *AF* and *FG*. Thus, *AF* is incommensurable in length with *FG*. Therefore, let *EH*, *FI*, and *GK* be drawn through *E*, *F*, and *G* (respectively), parallel to *AC* and *BD*. Therefore, since *AG* is rational, and commensurable in length with *AC*, the whole (area) *AK* is thus rational [Prop. 10.19]. Again, since *DG* is incommensurable in length with *AC*, and both are rational (straight-lines), *DK* is thus a medial (area) [Prop. 10.21]. Again, since *AF* is incommensurable in length with *FG*, *AI* (is) thus also incommensurable with *FK* [Props. 6.1, 10.11].



Therefore, let the square *LM*, equal to *AI*, be constructed. And let *NO*, equal to *FK*, (and) about the same angle, *LPM*, be subtracted (from *LM*). Thus, the squares *LM* and *NO* are about the same diagonal [Prop. 6.26]. Let *PR* be their (common) diagonal, and let the (rest of the) figure be drawn. Therefore, since the (rectangle contained) by *AF* and *FG* is equal to the (square) on *EG*, thus, proportionally, as *AF* is to *EG*, so *EG* (is) to *FG* [Prop. 6.17]. But, as *AF* (is) to *EG*, so *AI* is to *EK*, and as *EG* (is) to *FG*, so *EK* is to *FK* [Prop. 6.1]. Thus, *EK* is the mean proportional to *AI* and *FK* [Prop. 5.11]. And *MN* is also the mean proportional to the squares *LM* and *NO* [Prop. 10.13 lem.], and *AI* is equal to *LM*, and *FK* to *NO*. *EK* is thus also equal to *MN*. But, *DH* is equal to *EK*, and *LO* is equal to *MN* [Prop. 1.43]. Thus, the whole of *DK* is equal to the gnomon *UVW* and *NO*. Therefore, since the whole of *AK* is equal to the (sum of the) squares *LM* and *NO*, of which *DK* is equal to the gnomon *UVW* and the square *NO*, the remainder *AB* is thus equal to *ST*—that is to say, to the square on *LN*. Thus, *LN* is the square-root of area *AB*. I say that *LN* is the irrational (straight-line which is) called minor.

For since *AK* is rational, and is equal to the (sum of the) squares *LP* and *PN*, the sum of the (squares) on *LP* and *PN* is thus rational. Again, since *DK* is medial, and *DK* is equal to twice the (rectangle contained) by *LP* and *PN*, thus twice *ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τετράγωνον τῷ ἀπὸ τῆς ΟΝ τετραγώνῳ. αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπ᾽ αὐτῶν μέσον. ἡ ΛΝ ἄρα ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων· καὶ δύναται τὸ ΑΒ χωρίον.*

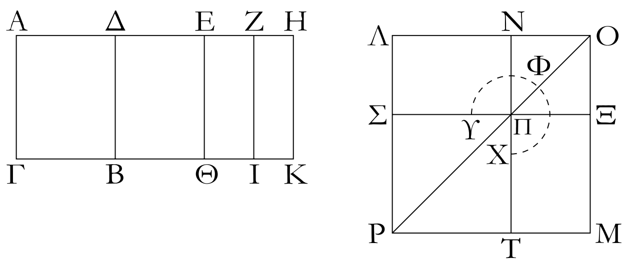
Ἡ ἄρα τὸ ΑΒ χωρίον δυναμένη ἐλάσσων ἐστίν· ὅπερ ἔδει δεῖξαι.

the (rectangle contained) by *LP* and *PN* is medial. And since *AI* was shown (to be) incommensurable with *FK*, the square on *LP* (is) thus also incommensurable with the square on *PN*. Thus, *LP* and *PN* are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. *LN* is thus the irrational (straight-line) called minor [Prop. 10.76]. And it is the square-root of area *AB*.

Thus, the square-root of area *AB* is a minor (straight-line). (Which is) the very thing it was required to show.

ϙε΄.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς πέμπτης, ἡ τὸ χωρίον δυναμένη [ἡ] μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.



Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς πέμπτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη [ἡ] μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

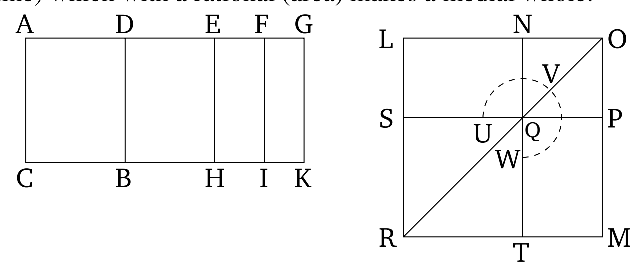
Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προσαρμόζουσα ἡ ΗΔ σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε σημεῖον, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΗ τῇ ΓΑ μήκει, καί εἰσιν ἀμφότεραι ῥηταί, μέσον ἄρα ἐστὶ τὸ ΑΚ. πάλιν, ἐπεὶ ῥητή ἐστιν ἡ ΔΗ καὶ σύμμετρος τῇ ΑΓ μήκει, ῥητόν ἐστι τὸ ΔΚ.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον τετράγωνον ἀφῃρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ ΛΟΜ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ὁμοίως δὴ δείξομεν, ὅτι ἡ ΛΝ δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΛΝ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. πάλιν, ἐπεὶ ῥητόν ἐστι τὸ ΔΚ καί ἐστιν ἴσον τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ, καὶ αὐτὸ ῥητόν ἐστιν. καὶ ἐπεὶ ἀσύμμετρόν

Proposition 95

If an area is contained by a rational (straight-line) and a fifth apotome then the square-root of the area is that (straight-line) which with a rational (area) makes a medial whole.



For let the area *AB* be contained by the rational (straight-line) *AC* and the fifth apotome *AD*. I say that the square-root of area *AB* is that (straight-line) which with a rational (area) makes a medial whole.

For let *DG* be an attachment to *AD*. Thus, *AG* and *DG* are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment *GD* is commensurable in length the the (previously) laid down rational (straight-line) *AC*, and the square on the whole, *AG*, is greater than (the square on) the attachment, *DG*, by the (square) on (some straight-line) incommensurable (in length) with (*AG*) [Def. 10.15]. Thus, if (some area), equal to the fourth part of the (square) on *DG*, is applied to *AG*, falling short by a square figure, then it divides (*AG*) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let *DG* be divided in half at point *E*, and let (some area), equal to the (square) on *EG*, be applied to *AG*, falling short by a square figure, and let it be the (rectangle contained) by *AF* and *FG*. Thus, *AF* is incommensurable in length with *FG*. And since *AG* is incommensurable in length with *CA*, and both are rational (straight-lines), *AK* is thus a medial (area) [Prop. 10.21]. Again, since *DG* is rational, and commensurable in length with *AC*, *DK* is a rational (area) [Prop. 10.19].

Therefore, let the square *LM*, equal to *AI*, be constructed. And let the square *NO*, equal to *FK*, (and) about the same angle, *LPM*, be subtracted (from *NO*). Thus, the squares *LM* and *NO* are about the same diagonal [Prop. 6.26]. Let *PR* be *ἐστι τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾽ αὐτῶν ῥητόν. ἡ λοιπὴ ἄρα ἡ ΛΝ ἄλογός ἐστιν ἡ καλουμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα· καὶ δύναται τὸ ΑΒ χωρίον.*

Ἡ τὸ ΑΒ ἄρα χωρίον δυναμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

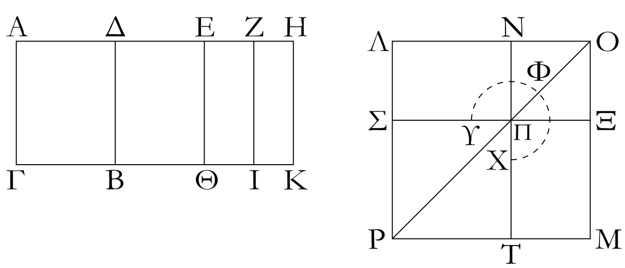
their (common) diagonal, and let (the rest of) the figure be drawn. So, similarly (to the previous propositions), we can show that *LN* is the square-root of area *AB*. I say that *LN* is that (straight-line) which with a rational (area) makes a medial whole.

For since *AK* was shown (to be) a medial (area), and is equal to (the sum of) the squares on *LP* and *PN*, the sum of the (squares) on *LP* and *PN* is thus medial. Again, since *DK* is rational, and is equal to twice the (rectangle contained) by *LP* and *PN*, (the latter) is also rational. And since *AI* is incommensurable with *FK*, the (square) on *LP* is thus also incommensurable with the (square) on *PN*. Thus, *LP* and *PN* are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational. Thus, the remainder *LN* is the irrational (straight-line) called that which with a rational (area) makes a medial whole [Prop. 10.77]. And it is the square-root of area *AB*.

Thus, the square-root of area *AB* is that (straight-line) which with a rational (area) makes a medial whole. (Which is) the very thing it was required to show.

ϟς´.

Ἐὰν χωρίον περιέχηται ὑπὸ ῥητῆς καὶ ἀποτομῆς ἕκτης, ἡ τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

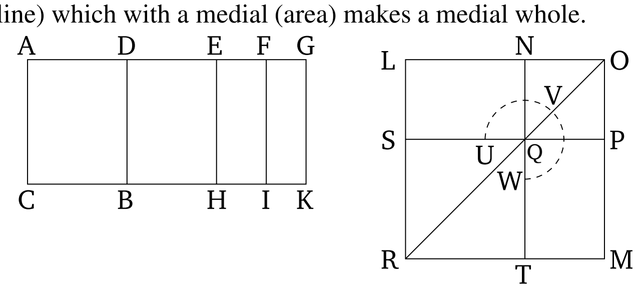


Χωρίον γὰρ τὸ ΑΒ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΓ καὶ ἀποτομῆς ἕκτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΒ χωρίον δυναμένη [ἡ] μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω γὰρ τῇ ΑΔ προσαρμόζουσα ἡ ΔΗ· αἱ ἄρα ΑΗ, ΗΔ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΓ μήκει, ἡ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλεῖπον εἴδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεῖ. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε [σημεῖον], καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλεῖπον εἴδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει. ὡς δὲ ἡ ΑΖ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΖΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΙ τῷ ΖΚ. καὶ ἐπεὶ αἱ ΑΗ, ΑΓ

Proposition 96

If an area is contained by a rational (straight-line) and a sixth apotome then the square-root of the area is that (straight-line) which with a medial (area) makes a medial whole.



For let the area *AB* be contained by the rational (straight-line) *AC* and the sixth apotome *AD*. I say that the square-root of area *AB* is that (straight-line) which with a medial (area) makes a medial whole.

For let *DG* be an attachment to *AD*. Thus, *AG* and *GD* are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of them is commensurable in length with the (previously) laid down rational (straight-line) *AC*, and the square on the whole, *AG*, is greater than (the square on) the attachment, *DG*, by the (square) on (some straight-line) incommensurable in length with (*AG*) [Def. 10.16]. Therefore, since the square on *AG* is greater than (the square on) *GD* by the (square) on (some straight-line) incommensurable in length with (*AG*), thus if (some area), equal to the fourth part of square on *DG*, is applied to *AG*, falling short by a square figure, then it divides (*AG*) into (parts which *ῥηταί εἰσι δυνάμει μόνον σύμμετροι, μέσον ἐστὶ τὸ ΑΚ. πάλιν, ἐπεὶ αἱ ΑΓ, ΔΗ ῥηταί εἰσι καὶ ἀσύμμετροι μήκει, μέσον ἐστὶ καὶ τὸ ΔΚ. ἐπεὶ οὖν αἱ ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΗΔ μήκει. ὡς δὲ ἡ ΑΗ πρὸς τὴν ΗΔ, οὕτως ἐστὶ τὸ ΑΚ πρὸς τὸ ΚΔ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΚ τῷ ΚΔ.*

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφῃρήσθω περὶ τὴν αὐτὴν γωνίαν τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ὁμοίως δὴ τοῖς ἐπάνω δείξομεν, ὅτι ἡ ΛΝ δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΛΝ [ἡ] μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΑΚ καί ἐστιν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. πάλιν, ἐπεὶ μέσον ἐδείχθη τὸ ΔΚ καί ἐστιν ἴσον τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ, καὶ τὸ δὶς ὑπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΚ τῷ ΔΚ, ἀσύμμετρα [ἄρα] ἐστὶ καὶ τὰ ἀπὸ τῶν ΛΟ, ΟΝ τετράγωνα τῷ δὶς ὑπὸ τῶν ΛΟ, ΟΝ. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ δὶς ὑπ᾽ αὐτῶν μέσον ἔτι τε τὰ ἀπ᾽ αὐτῶν τετράγωνα ἀσύμμετρα τῷ δὶς ὑπ᾽ αὐτῶν. ἡ ἄρα ΛΝ ἄλογός ἐστιν ἡ καλουμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

are) incommensurable (in length) [Prop. 10.18]. Therefore, let *DG* be cut in half at [point] *E*. And let (some area), equal to the (square) on *EG*, be applied to *AG*, falling short by a square figure. And let it be the (rectangle contained) by *AF* and *FG*. *AF* is thus incommensurable in length with *FG*. And as *AF* (is) to *FG*, so *AI* is to *FK* [Prop. 6.1]. Thus, *AI* is incommensurable with *FK* [Prop. 10.11]. And since *AG* and *AC* are rational (straight-lines which are) commensurable in square only, *AK* is a medial (area) [Prop. 10.21]. Again, since *AC* and *DG* are rational (straight-lines which are) incommensurable in length, *DK* is also a medial (area) [Prop. 10.21]. Therefore, since *AG* and *GD* are commensurable in square only, *AG* is thus incommensurable in length with *GD*. And as *AG* (is) to *GD*, so *AK* is to *KD* [Prop. 6.1]. Thus, *AK* is incommensurable with *KD* [Prop. 10.11].

Therefore, let the square *LM*, equal to *AI*, be constructed. And let *NO*, equal to *FK*, (and) about the same angle, be subtracted (from *LM*). Thus, the squares *LM* and *NO* are about the same diagonal [Prop. 6.26]. Let *PR* be their (common) diagonal, and let (the rest of) the figure be drawn. So, similarly to the above, we can show that *LN* is the square-root of area *AB*. I say that *LN* is that (straight-line) which with a medial (area) makes a medial whole.

For since *AK* was shown (to be) a medial (area), and is equal to the (sum of the) squares on *LP* and *PN*, the sum of the (squares) on *LP* and *PN* is medial. Again, since *DK* was shown (to be) a medial (area), and is equal to twice the (rectangle contained) by *LP* and *PN*, twice the (rectangle contained) by *LP* and *PN* is also medial. And since *AK* was shown (to be) incommensurable with *DK*, [thus] the (sum of the) squares on *LP* and *PN* is also incommensurable with twice the (rectangle contained) by *LP* and *PN*. And since *AI* is incommensurable with *FK*, the (square) on *LP* (is) thus also incommensurable with the (square) on *PN*. Thus, *LP* and *PN* are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by medial, and, furthermore, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them. Thus, *LN* is the irrational (straight-line) called that which with a medial (area) makes a medial whole [Prop. 10.78]. And it is the square-root of area *AB*.

Thus, the square-root of area (*AB*) is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to show.

ρζ΄.

Τὸ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην.

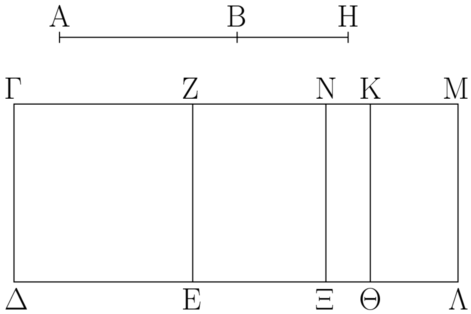
Ἔστω ἀποτομὴ ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ

Proposition 97

The (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth.

Let *AB* be an apotome, and *CD* a rational (straight-line).

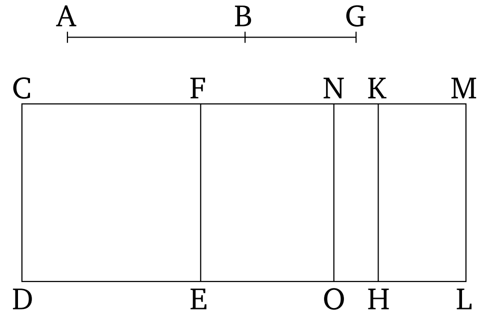
ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι πρώτη.



Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΒΗ τὸ ΚΛ. ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ· λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ ἤχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΛΝ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ ῥητά ἐστιν, καί ἐστι τοῖς ἀπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΔΜ, ῥητὸν ἄρα ἐστὶ τὸ ΔΜ. καὶ παρὰ ῥητὴν τὴν ΓΔ παραβέβληται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ σύμμετρος τῇ ΓΔ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, καὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ, μέσον ἄρα τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὰ μὲν ἀπὸ τῶν ΑΗ, ΗΒ ῥητά ἐστιν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. καὶ τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ τὸ ΖΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΔΜ τῷ ΖΛ. ὡς δὲ τὸ ΔΜ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ. ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΓΖ ἄρα ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ πρώτη.

Ἐπεὶ γὰρ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τὸ ΝΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ᾽ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ· ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. καὶ επεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, σύμμετρόν [ἐστι] καὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ· σύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ, καί ἐστι σύμμετρος ἡ ΓΚ

And let *CE*, equal to the (square) on *AB*, be applied to *CD*, producing *CF* as breadth. I say that *CF* is a first apotome.



For let *BG* be an attachment to *AB*. Thus, *AG* and *GB* are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let *CH*, equal to the (square) on *AG*, and *KL*, (equal) to the (square) on *BG*, be applied to *CD*. Thus, the whole of *CL* is equal to the (sum of the squares) on *AG* and *GB*, of which *CE* is equal to the (square) on *AB*. The remainder *FL* is thus equal to twice the (rectangle contained) by *AG* and *GB* [Prop. 2.7]. Let *FM* be cut in half at point *N*. And let *NO* be drawn through *N*, parallel to *CD*. Thus, *FO* and *LN* are each equal to the (rectangle contained) by *AG* and *GB*. And since the (sum of the squares) on *AG* and *GB* is rational, and *DM* is equal to the (sum of the squares) on *AG* and *GB*, *DM* is thus rational. And it has been applied to the rational (straight-line) *CD*, producing *CM* as breadth. Thus, *CM* is rational, and commensurable in length with *CD* [Prop. 10.20]. Again, since twice the (rectangle contained) by *AG* and *GB* is medial, and *FL* (is) equal to twice the (rectangle contained) by *AG* and *GB*, *FL* (is) thus a medial (area). And it is applied to the rational (straight-line) *CD*, producing *FM* as breadth. *FM* is thus rational, and incommensurable in length with *CD* [Prop. 10.22]. And since the (sum of the squares) on *AG* and *GB* is rational, and twice the (rectangle contained) by *AG* and *GB* medial, the (sum of the squares) on *AG* and *GB* is thus incommensurable with twice the (rectangle contained) by *AG* and *GB*. And *CL* is equal to the (sum of the squares) on *AG* and *GB*, and *FL* to twice the (rectangle contained) by *AG* and *GB*. *DM* is thus incommensurable with *FL*. And as *DM* (is) to *FL*, so *CM* is to *FM* [Prop. 6.1]. *CM* is thus incommensurable in length with *FM* [Prop. 10.11]. And both are rational (straight-lines). Thus, *CM* and *MF* are rational (straight-lines which are) commensurable in square only. *CF* is thus an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

For since the (rectangle contained) by *AG* and *GB* is the mean proportional to the (squares) on *AG* and *GB* [Prop. 10.21 lem.], and *CH* is equal to the (square) on *AG*, and *KL* equal to the (square) on *BG*, and *NL* to the (rectangle contained) by *AG* and *GB*, *NL* is thus also the mean proportional to *CH* and *KL*. Thus, as *CH* is to *NL*, so *NL* (is) to *KL*. But, as *CH* (is) to *NL*, so *CK* is to *NM*, and as *NL* (is) to *KL*, so *NM* *τῇ ΚΜ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ ΓΜ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ μήκει· ἡ ἄρα ΓΖ ἀποτομή ἐστι πρώτη.*

Τὸ ἄρα ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην· ὅπερ ἔδει δεῖξαι.

is to KM [Prop. 6.1]. Thus, the (rectangle contained) by CK and KM is equal to the (square) on NM—that is to say, to the fourth part of the (square) on FM [Prop. 6.17]. And since the (square) on AG is commensurable with the (square) on GB, CH [is] also commensurable with KL. And as CH (is) to KL, so CK (is) to KM [Prop. 6.1]. CK is thus commensurable (in length) with KM [Prop. 10.11]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and CK is commensurable (in length) with KM, the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. And CM is commensurable in length with the (previously) laid down rational (straight-line) CD. Thus, CF is a first apotome [Def. 10.15].

Thus, the (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth. (Which is) the very thing it was required to show.

ρη΄.

Τὸ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν.

Ἔστω μέσης ἀποτομὴ πρώτη ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι δευτέρα.

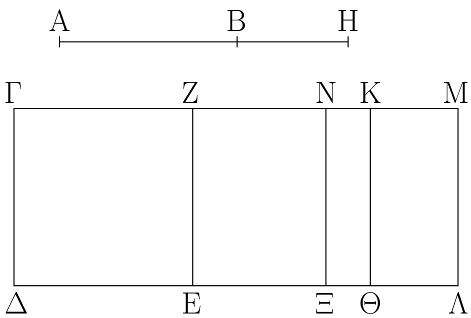
Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ΓΕ, λοιπὸν ἄρα τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τῷ ΖΛ. ῥητὸν δέ [ἐστι] τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ· ῥητὸν ἄρα τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΜ καὶ σύμμετρος τῇ ΓΔ μήκει. ἐπεὶ οὖν τὰ μὲν ἀπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΓΛ, μέσον ἐστίν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΖΛ, ῥητόν ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἡ ΓΜ τῇ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΓΖ ἄρα ἀποτομή ἐστιν. λέγω δή, ὅτι καὶ δευτέρα.

Proposition 98

The (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth.

Let AB be a first apotome of a medial (straight-line), and CD a rational (straight-line). And let CE, equal to the (square) on AB, be applied to CD, producing CF as breadth. I say that CF is a second apotome.

For let BG be an attachment to AB. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a rational (area) [Prop. 10.74]. And let CH, equal to the (square) on AG, be applied to CD, producing CK as breadth, and KL, equal to the (square) on GB, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB. Thus, CL (is) also a medial (area) [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) CD, producing CM as breadth. CM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since CL is equal to the (sum of the squares) on AG and GB, of which the (square) on AB is equal to CE, the remainder, twice the (rectangle contained) by AG and GB, is thus equal to FL [Prop. 2.7]. And twice the (rectangle contained) by AG and GB [is] rational. Thus, FL (is) rational. And it is applied to the rational (straight-line) FE, producing FM as breadth. FM is thus also rational, and commensurable in length with CD [Prop. 10.20]. Therefore, since the (sum of the squares) on AG and GB—that is to say, CL—is medial, and twice the (rectangle contained) by AG and GB—that is to say, FL—(is) rational, CL is thus incommensurable with FL. And as CL (is) to FL, so CM is to FM [Prop. 6.1]. Thus, CM (is) incom-



Τετμήσθω γὰρ ἡ ΖΜ δίχα κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ τετραγώνων μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστιν ἴσον τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τῷ ΝΛ, τὸ δὲ ἀπὸ τῆς ΒΗ τῷ ΚΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ᾽ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΜΚ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΝΜ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ [καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΒΗ, σύμμετρόν ἐστι καὶ τὸ ΓΘ τῷ ΚΛ, τουτέστιν ἡ ΓΚ τῇ ΚΜ]. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΜΖ ἴσον παρὰ τὴν μείζονα τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι δευτέρα.

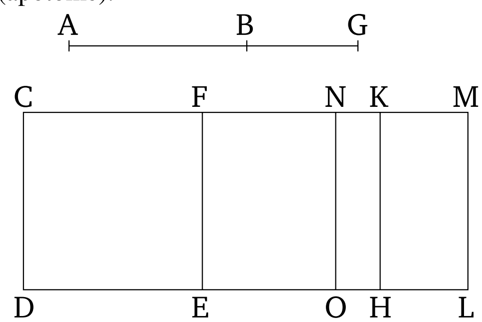
Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν· ὅπερ ἔδει δεῖξαι.

ρθ΄.

Τὸ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην.

Ἔστω μέσης ἀποτομὴ δευτέρα ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ

mensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a second (apotome).



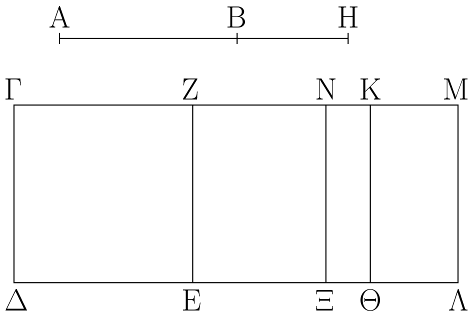
For let FM be cut in half at N. And let NO be drawn through (point) N, parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since the (rectangle contained) by AG and GB is the mean proportional to the squares on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (rectangle contained) by AG and GB to NL, and the (square) on BG to KL, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL [Prop. 5.11]. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to MK [Prop. 6.1]. Thus, as CK (is) to NM, so NM is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on NM [Prop. 6.17]—that is to say, to the fourth part of the (square) on FM [and since the (square) on AG is commensurable with the (square) on BG, CH is also commensurable with KL—that is to say, CK with KM]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to the greater CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. The attachment FM is also commensurable in length with the (previously) laid down rational (straight-line) CD. CF is thus a second apotome [Def. 10.16].

Thus, the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth. (Which is) the very thing it was required to show.

Proposition 99

The (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth.

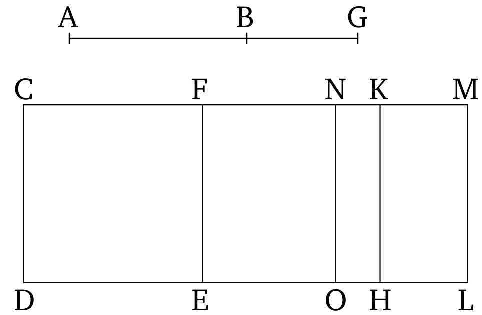
τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι τρίτη.



Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον παρὰ τὴν ΚΘ παραβεβλήσθω τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ [καί ἐστι μέσα τὰ ἀπὸ τῶν ΑΗ, ΗΒ]· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παραβέβληται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΛΖ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ τῇ ΓΔ παράλληλος ἤχθω ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. μέσον δὲ τὸ ὑπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα ἐστὶ καὶ τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα καὶ ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει μόνον εἰσὶ σύμμετροι, ἀσύμμετρος ἄρα [ἐστὶ] μήκει ἡ ΑΗ τῇ ΗΒ· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΗ, ΗΒ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΖΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΖΜ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ. λέγω δή, ὅτι καὶ τρίτη.

Ἐπεὶ γὰρ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, σύμμετρον ἄρα καὶ τὸ ΓΘ τῷ ΚΛ· ὥστε καὶ ἡ ΓΚ τῇ ΚΜ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ᾽ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ,

Let AB be the second apotome of a medial (straight-line), and CD a rational (straight-line). And let CE, equal to the (square) on AB, be applied to CD, producing CF as breadth. I say that CF is a third apotome.



For let BG be an attachment to AB. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a medial (area) [Prop. 10.75]. And let CH, equal to the (square) on AG, be applied to CD, producing CK as breadth. And let KL, equal to the (square) on BG, be applied to KH, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB [and the (sum of the squares) on AG and GB is medial]. CL (is) thus also medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder LF is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM be cut in half at point N. And let NO be drawn parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial. And it is applied to the rational (straight-line) EF, producing FM as breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since AG and GB are commensurable in square only, AG [is] thus incommensurable in length with GB. Thus, the (square) on AG is also incommensurable with the (rectangle contained) by AG and GB [Props. 6.1, 10.11]. But, the (sum of the squares) on AG and GB is commensurable with the (square) on AG, and twice the (rectangle contained) by AG and GB with the (rectangle contained) by AG and GB. The (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.13]. But, CL is equal to the (sum of the squares) on AG and GB, and FL is equal to twice the (rectangle contained) by AG and GB. Thus, CL is incommensurable with FL. And as CL (is) to FL, so CM is to FM [Prop. 6.1]. CM is thus incommensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which *ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ [ἀπὸ τῆς ΜΝ, τουτέστι τῷ] τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ΓΜ ἄρα τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ οὐδετέρα τῶν ΓΜ, ΜΖ σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι τρίτη.*

Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην· ὅπερ ἔδει δεῖξαι.

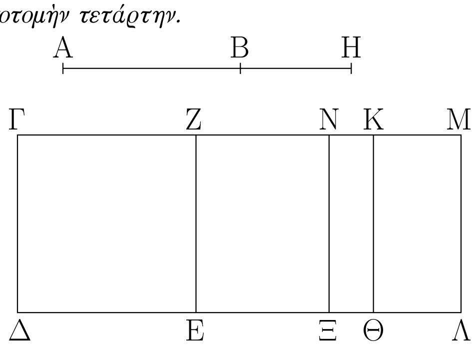
are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since the (square) on AG is commensurable with the (square) on GB, CH (is) thus also commensurable with KL. Hence, CK (is) also (commensurable in length) with KM [Props. 6.1, 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on GB, and NL equal to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM (is) to KM [Prop. 6.1]. Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11]. Thus, the (rectangle contained) by CK and KM is equal to the [(square) on MN—that is to say, to the] fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable (in length) with (CM) [Prop. 10.17]. And neither of CM and MF is commensurable in length with the (previously) laid down rational (straight-line) CD. CF is thus a third apotome [Def. 10.13].

Thus, the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth. (Which is) the very thing it was required to show.

ϱ′.

Τὸ ἀπὸ ἐλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τετάρτην.

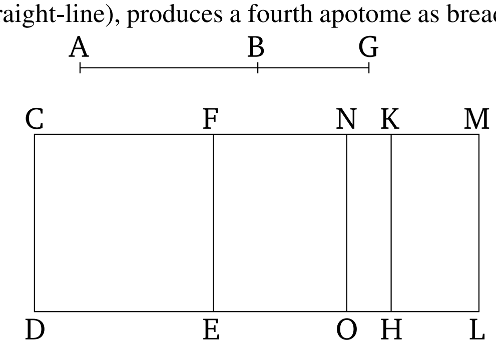


Ἔστω ἐλάσσων ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ ῥητὴν τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι τετάρτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ τετραγώνων ῥητόν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ

Proposition 100

The (square) on a minor (straight-line), applied to a rational (straight-line), produces a fourth apotome as breadth.



Let AB be a minor (straight-line), and CD a rational (straight-line). And let CE, equal to the (square) on AB, be applied to the rational (straight-line) CD, producing CF as breadth. I say that CF is a fourth apotome.

For let BG be an attachment to AB. Thus, AG and GB are incommensurable in square, making the sum of the squares on AG and GB rational, and twice the (rectangle contained) *παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ. καί ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ῥητόν· ῥητὸν ἄρα ἐστὶ καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα καὶ ἡ ΓΜ καὶ σύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ ἤχθω δία τοῦ Ν ὁποτέρᾳ τῶν ΓΔ, ΜΛ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον ἐστὶ καί ἐστιν ἴσον τῷ ΖΛ, καὶ τὸ ΖΛ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ῥητόν ἐστιν, τὸ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον, ἀσύμμετρα [ἄρα] ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. ἴσον δέ [ἐστι] τὸ ΓΛ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ. λέγω [δή], ὅτι καὶ τετάρτη.*

Ἐπεὶ γὰρ αἱ ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ μήκει. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστιν ἴσον τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς ΗΒ τῷ ΚΛ, τὸ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τῷ ΝΛ, τῶν ἄρα ΓΘ, ΚΛ μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ᾽ ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως ἐστίν ἡ ΓΚ πρὸς τὴν ΝΜ, ὡς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετράρτῳ μέρει τοῦ ἀπὸ τῆς ΜΖ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί ἐστιν ὅλη ἡ ΓΜ σύμμετρος μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι τετάρτη.

Τὸ ἄρα ἀπὸ ἐλάσσονος καὶ τὰ ἑξῆς.

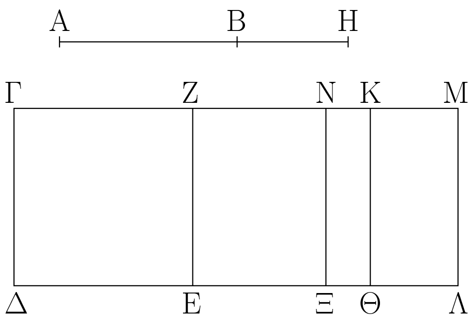
by AG and GB medial [Prop. 10.76]. And let CH, equal to the (square) on AG, be applied to CD, producing CK as breadth, and KL, equal to the (square) on BG, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB. And the sum of the (squares) on AG and GB is rational. CL is thus also rational. And it is applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM (is) also rational, and commensurable in length with CD [Prop. 10.20]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM be cut in half at point N. And let NO be drawn through N, parallel to either of CD or ML. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since twice the (rectangle contained) by AG and GB is medial, and is equal to FL, FL is thus also medial. And it is applied to the rational (straight-line) FE, producing FM as breadth. Thus, FM is rational, and incommensurable in length with CD [Prop. 10.22]. And since the sum of the (squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is [thus] incommensurable with twice the (rectangle contained) by AG and GB. And CL (is) equal to the (sum of the squares) on AG and GB, and FL equal to twice the (rectangle contained) by AG and GB. CL [is] thus incommensurable with FL. And as CL (is) to FL, so CM is to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. [So], I say that (it is) also a fourth (apotome).

For since AG and GB are incommensurable in square, the (square) on AG (is) thus also incommensurable with the (square) on GB. And CH is equal to the (square) on AG, and KL equal to the (square) on GB. Thus, CH is incommensurable with KL. And as CH (is) to KL, so CK is to KM [Prop. 6.1]. CK is thus incommensurable in length with KM [Prop. 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (square) on GB to KL, and the (rectangle contained) by AG and GB to NL, NL is thus the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to KM [Prop. 6.1]. Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on MN—that is to say, to the fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18]. And the whole of CM is commensurable in length with the (previously) laid down rational (straight-line) CD. Thus, CF is a fourth apotome [Def. 10.14].

Thus, the (square) on a minor, and so on . . .

ρα΄.

Τὸ ἀπὸ τῆς μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πέμπτην.

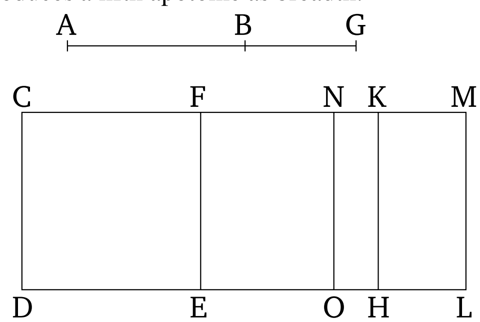


Ἔστω ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι πέμπτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ εὐθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων μέσον, τὸ δὲ δὶς ὑπ᾿ αὐτῶν ῥητόν, καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ. τὸ δὲ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ἅμα μέσον ἐστίν· μέσον ἄρα ἐστὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ. καὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν ὁποτέρᾳ τῶν ΓΔ, ΜΛ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐπεὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ῥητόν ἐστι καί [ἐστιν] ἴσον τῷ ΖΛ, ῥητὸν ἄρα ἐστὶ τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ σύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ μὲν ΓΛ μέσον ἐστίν, τὸ δὲ ΖΛ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ. λέγω δή, ὅτι καὶ πέμπτη.

Proposition 101

The (square) on that (straight-line) which with a rational (area) makes a medial whole, applied to a rational (straight-line), produces a fifth apotome as breadth.



Let AB be that (straight-line) which with a rational (area) makes a medial whole, and CD a rational (straight-line). And let CE, equal to the (square) on AB, be applied to CD, producing CF as breadth. I say that CF is a fifth apotome.

Let BG be an attachment to AB. Thus, the straight-lines AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational [Prop. 10.77]. And let CH, equal to the (square) on AG, be applied to CD, and KL, equal to the (square) on GB. The whole of CL is thus equal to the (sum of the squares) on AG and GB. And the sum of the (squares) on AG and GB together is medial. Thus, CL is medial. And it has been applied to the rational (straight-line) CD, producing CM as breadth. CM is thus rational, and incommensurable (in length) with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM be cut in half at N. And let NO be drawn through N, parallel to either of CD or ML. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since twice the (rectangle contained) by AG and GB is rational, and [is] equal to FL, FL is thus rational. And it is applied to the rational (straight-line) EF, producing FM as breadth. Thus, FM is rational, and commensurable in length with CD [Prop. 10.20]. And since CL is medial, and FL rational, CL is thus incommensurable with FL. And as CL (is) to FL, so CM

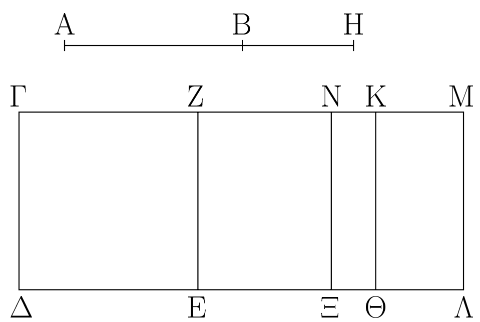
Ὁμοίως γὰρ δείξομεν, ὅτι τὸ ὑπὸ τῶν ΓΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, ἴσον δὲ τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς ΗΒ τῷ ΚΛ, ἀσύμμετρον ἄρα τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἡ ΓΚ τῇ ΚΜ μήκει. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλεῖπον εἴδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστι πέμπτη· ὅπερ ἔδει δεῖξαι.

(is) to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational. Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

For, similarly (to the previous propositions), we can show that the (rectangle contained) by CKM is equal to the (square) on NM—that is to say, to the fourth part of the (square) on FM. And since the (square) on AG is incommensurable with the (square) on GB, and the (square) on AG (is) equal to CH, and the (square) on GB to KL, CH (is) thus incommensurable with KL. And as CH (is) to KL, so CK (is) to KM [Prop. 6.1]. Thus, CK (is) incommensurable in length with KM [Prop. 10.11]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18]. And the attachment FM is commensurable with the (previously) laid down rational (straight-line) CD. Thus, CF is a fifth apotome [Def. 10.15]. (Which is) the very thing it was required to show.

ρβ΄.

Τὸ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν ἕκτην.

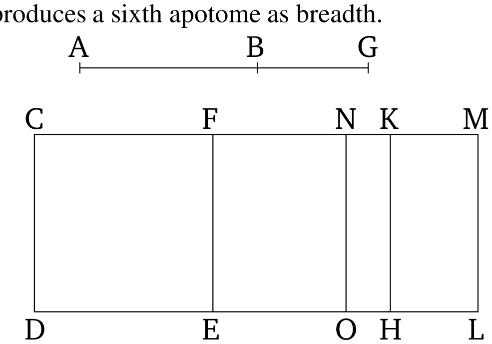


Ἔστω ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστιν ἕκτη.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾿ αὐτῶν τετραγώνων μέσον καὶ τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον καὶ ἀσύμμετρον τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. παραβεβλήσθω οὖν παρὰ τὴν ΓΔ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ τὸ ΚΛ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα [ἐστὶ] καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ

Proposition 102

The (square) on that (straight-line) which with a medial (area) makes a medial whole, applied to a rational (straight-line), produces a sixth apotome as breadth.



Let AB be that (straight-line) which with a medial (area) makes a medial whole, and CD a rational (straight-line). And let CE, equal to the (square) on AB, be applied to CD, producing CF as breadth. I say that CF is a sixth apotome.

For let BG be an attachment to AB. Thus, AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by AG and GB medial, and the (sum of the squares) on AG and GB incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.78]. Therefore, let CH, equal to the (square) on AG, be applied to CD, producing CK as breadth, and KL, equal to the (square) on BG. Thus, the whole of CL is equal *παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. ἐπεὶ οὖν τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ. καί ἐστι τὸ δὶς ὑπὸ τῶν ΑΗ, ΗΒ μέσον· καὶ τὸ ΖΛ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ ἀσύμμετρά ἐστι τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, καί ἐστι τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΓΛ, τῷ δὲ δὶς ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΛ, ἀσύμμετρος ἄρα [ἐστὶ] τὸ ΓΛ τῷ ΖΛ. ὡς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καί εἰσιν ἀμφότεραι ῥηταί. αἱ ΓΜ, ΜΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΓΖ. λέγω δή, ὅτι καὶ ἕκτη.*

Ἐπεὶ γὰρ τὸ ΖΛ ἴσον ἐστὶ τῷ δὶς ὑπὸ τῶν ΑΗ, ΗΒ, τετμήσθω δίχα ἡ ΖΜ κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον ἐστὶ τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον ἐστὶ τὸ ΚΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐστι τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΛ, καὶ τῶν ἄρα ΓΘ, ΚΛ μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. καὶ διὰ τὰ αὐτὰ ἡ ΓΜ τῆς ΜΖ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ΓΖ ἄρα ἀποτομή ἐστιν ἕκτη· ὅπερ ἔδει δεῖξαι.

to the (sum of the squares) on *AG* and *GB*. *CL* [is] thus also medial. And it is applied to the rational (straight-line) *CD*, producing *CM* as breadth. Thus, *CM* is rational, and incommensurable in length with *CD* [Prop. 10.22]. Therefore, since *CL* is equal to the (sum of the squares) on *AG* and *GB*, of which *CE* (is) equal to the (square) on *AB*, the remainder *FL* is thus equal to twice the (rectangle contained) by *AG* and *GB* [Prop. 2.7]. And twice the (rectangle contained) by *AG* and *GB* (is) medial. Thus, *FL* is also medial. And it is applied to the rational (straight-line) *FE*, producing *FM* as breadth. *FM* is thus rational, and incommensurable in length with *CD* [Prop. 10.22]. And since the (sum of the squares) on *AG* and *GB* is incommensurable with twice the (rectangle contained) by *AG* and *GB*, and *CL* equal to the (sum of the squares) on *AG* and *GB*, and *FL* equal to twice the (rectangle contained) by *AG* and *GB*, *CL* [is] thus incommensurable with *FL*. And as *CL* (is) to *FL*, so *CM* is to *MF* [Prop. 6.1]. Thus, *CM* is incommensurable in length with *MF* [Prop. 10.11]. And they are both rational. Thus, *CM* and *MF* are rational (straight-lines which are) commensurable in square only. *CF* is thus an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since *FL* is equal to twice the (rectangle contained) by *AG* and *GB*, let *FM* be cut in half at *N*, and let *NO* be drawn through *N*, parallel to *CD*. Thus, *FO* and *NL* are each equal to the (rectangle contained) by *AG* and *GB*. And since *AG* and *GB* are incommensurable in square, the (square) on *AG* is thus incommensurable with the (square) on *GB*. But, *CH* is equal to the (square) on *AG*, and *KL* is equal to the (square) on *GB*. Thus, *CH* is incommensurable with *KL*. And as *CH* (is) to *KL*, so *CK* is to *KM* [Prop. 6.1]. Thus, *CK* is incommensurable (in length) with *KM* [Prop. 10.11]. And since the (rectangle contained) by *AG* and *GB* is the mean proportional to the (squares) on *AG* and *GB* [Prop. 10.21 lem.], and *CH* is equal to the (square) on *AG*, and *KL* equal to the (square) on *GB*, and *NL* equal to the (rectangle contained) by *AG* and *GB*, *NL* is thus also the mean proportional to *CH* and *KL*. Thus, as *CH* is to *NL*, so *NL* (is) to *KL*. And for the same (reasons as the preceding propositions), the square on *CM* is greater than (the square on) *MF* by the (square) on (some straight-line) incommensurable (in length) with (*CM*) [Prop. 10.18]. And neither of them is commensurable with the (previously) laid down rational (straight-line) *CD*. Thus, *CF* is a sixth apotome [Def. 10.16]. (Which is) the very thing it was required to show.

ργ′.

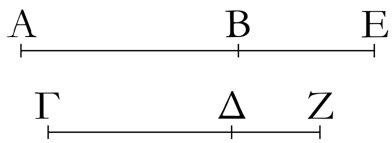
Ἡ τῇ ἀποτομῇ μήκει σύμμετρος ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτή.

Ἔστω ἀποτομὴ ἡ ΑΒ, καὶ τῇ ΑΒ μήκει σύμμετρος ἔστω

Proposition 103

A (straight-line) commensurable in length with an apotome is an apotome, and (is) the same in order.

Let *AB* be an apotome, and let *CD* be commensurable in *ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.*

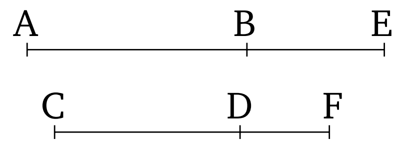


Ἐπεὶ γὰρ ἀποτομή ἐστιν ἡ ΑΒ, ἔστω αὐτῇ προσαρμόζουσα ἡ ΒΕ· αἱ ΑΕ, ΕΒ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ τῆς ΑΒ πρὸς τὴν ΓΔ λόγῳ ὁ αὐτὸς γεγονέτω ὁ τῆς ΒΕ πρὸς τὴν ΔΖ· καὶ ὡς ἓν ἄρα πρὸς ἕν, πάντα [ἐστὶ] πρὸς πάντα· ἔστιν ἄρα καὶ ὡς ὅλη ἡ ΑΕ πρὸς ὅλην τὴν ΓΖ, οὕτως ἡ ΑΒ πρὸς τὴν ΓΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ μήκει· σύμμετρος ἄρα καὶ ἡ ΑΕ μὲν τῇ ΓΖ, ἡ δὲ ΒΕ τῇ ΔΖ. καὶ αἱ ΑΕ, ΕΒ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι [ἀποτομὴ ἄρα ἐστὶν ἡ ΓΔ. λέγω δή, ὅτι καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ].

Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΒΕ πρὸς τὴν ΔΖ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ. ἤτοι δὴ ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. εἰ μὲν οὖν ἡ ΑΕ τῆς ΕΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΓΖ, εἰ δὲ ἡ ΒΕ, καὶ ἡ ΔΖ, εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, καὶ οὐδετέρα τῶν ΓΖ, ΖΔ. εἰ δὲ ἡ ΑΕ [τῆς ΕΒ] μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΓΖ τῆς ΖΔ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΑΕ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΓΖ, εἰ δὲ ἡ ΒΕ, καὶ ἡ ΔΖ, εἰ δὲ οὐδετέρα τῶν ΑΕ, ΕΒ, οὐδετέρα τῶν ΓΖ, ΖΔ.

Ἀποτομὴ ἄρα ἐστὶν ἡ ΓΔ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ· ὅπερ ἔδει δεῖξαι.

length with *AB*. I say that *CD* is also an apotome, and (is) the same in order as *AB*.



For since *AB* is an apotome, let *BE* be an attachment to it. Thus, *AE* and *EB* are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let it be contrived that the (ratio) of *BE* to *DF* is the same as the ratio of *AB* to *CD* [Prop. 6.12]. Thus, also, as one is to one, (so) all [are] to all [Prop. 5.12]. And thus as the whole *AE* is to the whole *CF*, so *AB* (is) to *CD*. And *AB* (is) commensurable in length with *CD*. *AE* (is) thus also commensurable (in length) with *CF*, and *BE* with *DF* [Prop. 10.11]. And *AE* and *BE* are rational (straight-lines which are) commensurable in square only. Thus, *CF* and *FD* are also rational (straight-lines which are) commensurable in square only [Prop. 10.13]. [*CD* is thus an apotome. So, I say that (it is) also the same in order as *AB*.]

Therefore, since as *AE* is to *CF*, so *BE* (is) to *DF*, thus, alternately, as *AE* is to *EB*, so *CF* (is) to *FD* [Prop. 5.16]. So, the square on *AE* is greater than (the square on) *EB* either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (*AE*). Therefore, if the (square) on *AE* is greater than (the square on) *EB* by the (square) on (some straight-line) commensurable (in length) with (*AE*) then the square on *CF* will also be greater than (the square on) *FD* by the (square) on (some straight-line) commensurable (in length) with (*CF*) [Prop. 10.14]. And if *AE* is commensurable in length with a (previously) laid down rational (straight-line) then so (is) *CF* [Prop. 10.12], and if *BE* (is commensurable), so (is) *DF*, and if neither of *AE* or *EB* (are commensurable), neither (are) either of *CF* or *FD* [Prop. 10.13]. And if the (square) on *AE* is greater [than (the square on) *EB*] by the (square) on (some straight-line) incommensurable (in length) with (*AE*) then the (square) on *CF* will also be greater than (the square on) *FD* by the (square) on (some straight-line) incommensurable (in length) with (*CF*) [Prop. 10.14]. And if *AE* is commensurable in length with a (previously) laid down rational (straight-line), so (is) *CF* [Prop. 10.12], and if *BE* (is commensurable), so (is) *DF*, and if neither of *AE* or *EB* (are commensurable), neither (are) either of *CF* or *FD* [Prop. 10.13].

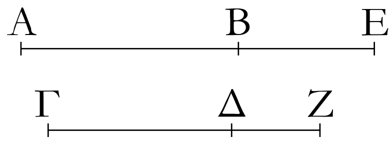
Thus, *CD* is an apotome, and (is) the same in order as *AB* [Defs. 10.11—10.16]. (Which is) the very thing it was required to show.

ρδʹ.

Ἡ τῇ μέσης ἀποτομῇ σύμμετρος μέσης ἀποτομή ἐστι καὶ

Proposition 104

A (straight-line) commensurable (in length) with an apoτῇ τάξει ἡ αὐτή.



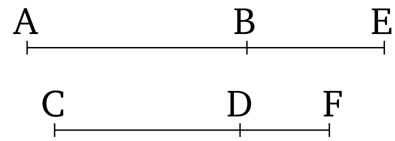
Ἔστω μέσης ἀποτομὴ ἡ ΑΒ, καὶ τῇ ΑΒ μήκει σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μέσης ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ γὰρ μέσης ἀποτομή ἐστιν ἡ ΑΒ, ἔστω αὐτῇ προσαρμόζουσα ἡ ΕΒ. αἱ ΑΕ, ΕΒ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ γεγονέτω ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΒΕ πρὸς τὴν ΔΖ· σύμμετρος ἄρα [ἐστὶ] καὶ ἡ ΑΕ τῇ ΓΖ, ἡ δὲ ΒΕ τῇ ΔΖ. αἱ δὲ ΑΕ, ΕΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· καὶ αἱ ΓΖ, ΖΔ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· μέσης ἄρα ἀποτομή ἐστιν ἡ ΓΔ. λέγω δή, ὅτι καὶ τῇ τάξει ἐστὶν ἡ αὐτὴ τῇ ΑΒ.

Ἐπεὶ [γάρ] ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ [ἀλλ᾽ ὡς μὲν ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, ὡς δὲ ἡ ΓΖ πρὸς τὴν ΖΔ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ], ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ [καὶ ἐναλλὰξ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ἀπὸ τῆς ΓΖ, οὕτως τὸ ὑπὸ τῶν ΑΕ, ΕΒ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ]. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΖ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ. εἴτε ο᾽ὖν ῥητόν ἐστι τὸ ὑπὸ τῶν ΑΕ, ΕΒ, ῥητὸν ἔσται καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ, εἴτε μέσον [ἐστὶ] τὸ ὑπὸ τῶν ΑΕ, ΕΒ, μέσον [ἐστὶ] καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ.

Μέσης ἄρα ἀποτομή ἐστιν ἡ ΓΔ καὶ τῇ τάξει ἡ αὐτὴ τῇ ΑΒ· ὅπερ ἔδει δεῖξαι.

tome of a medial (straight-line) is an apotome of a medial (straight-line), and (is) the same in order.



Let AB be an apotome of a medial (straight-line), and let CD be commensurable in length with AB. I say that CD is also an apotome of a medial (straight-line), and (is) the same in order as AB.

For since AB is an apotome of a medial (straight-line), let EB be an attachment to it. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.74, 10.75]. And let it be contrived that as AB is to CD, so BE (is) to DF [Prop. 6.12]. Thus, AE [is] also commensurable (in length) with CF, and BE with DF [Props. 5.12, 10.11]. And AE and EB are medial (straight-lines which are) commensurable in square only. CF and FD are thus also medial (straight-lines which are) commensurable in square only [Props. 10.23, 10.13]. Thus, CD is an apotome of a medial (straight-line) [Props. 10.74, 10.75]. So, I say that it is also the same in order as AB.

[For] since as AE is to EB, so CF (is) to FD [Props. 5.12, 5.16] [but as AE (is) to EB, so the (square) on AE (is) to the (rectangle contained) by AE and EB, and as CF (is) to FD, so the (square) on CF (is) to the (rectangle contained) by CF and FD], thus as the (square) on AE is to the (rectangle contained) by AE and EB, so the (square) on CF also (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.] [and, alternately, as the (square) on AE (is) to the (square) on CF, so the (rectangle contained) by AE and EB (is) to the (rectangle contained) by CF and FD]. And the (square) on AE (is) commensurable with the (square) on CF. Thus, the (rectangle contained) by AE and EB is also commensurable with the (rectangle contained) by CF and FD [Props. 5.16, 10.11]. Therefore, either the (rectangle contained) by AE and EB is rational, and the (rectangle contained) by CF and FD will also be rational [Def. 10.4], or the (rectangle contained) by AE and EB [is] medial, and the (rectangle contained) by CF and FD [is] also medial [Prop. 10.23 corr.].

Therefore, CD is the apotome of a medial (straight-line), and is the same in order as AB [Props. 10.74, 10.75]. (Which is) the very thing it was required to show.

ρε′.

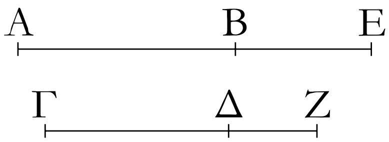
Ἡ τῇ ἐλάσσονι σύμμετρος ἐλάσσων ἐστίν.

Ἔστω γὰρ ἐλάσσων ἡ ΑΒ καὶ τῇ ΑΒ σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ ἐλάσσων ἐστίν.

Proposition 105

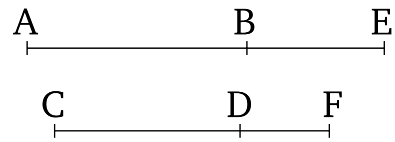
A (straight-line) commensurable (in length) with a minor (straight-line) is a minor (straight-line).

For let AB be a minor (straight-line), and (let) CD (be) commensurable (in length) with AB. I say that CD is also a minor (straight-line).



Γεγονέτω γὰρ τὰ αὐτά· καὶ ἐπεὶ αἱ ΑΕ, ΕΒ δυνάμει εἰσὶν ἀσύμμετροι, καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ, ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ἀπὸ τῆς ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ἀπὸ τῆς ΖΔ. συνθέντι ἄρα ἐστὶν ὡς τὰ ἀπὸ τῶν ΑΕ, ΕΒ πρὸς τὸ ἀπὸ τῆς ΕΒ, οὕτως τὰ ἀπὸ τῶν ΓΖ, ΖΔ πρὸς τὸ ἀπὸ τῆς ΖΔ [καὶ ἐναλλάξ]· σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς ΒΕ τῷ ἀπὸ τῆς ΔΖ· σύμμετρον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων. ῥητὸν δέ ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων· ῥητὸν ἄρα ἐστὶ καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων. πάλιν, ἐπεί ἐστιν ὡς τὸ ἀπὸ τῆς ΑΕ πρὸς τὸ ὑπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΖ πρὸς τὸ ὑπὸ τῶν ΓΖ, ΖΔ, σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τετράγωνον τῷ ἀπὸ τῆς ΓΖ τετραγώνῳ, σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ. μέσον δὲ τὸ ὑπὸ τῶν ΑΕ, ΕΒ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ· αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων ῥητόν, τὸ δ᾽ ὑπ᾽ αὐτῶν μέσον.

Ἐλάσσων ἄρα ἐστὶν ἡ ΓΔ· ὅπερ ἔδει δεῖξαι.



For let the same things be contrived (as in the former proposition). And since *AE* and *EB* are (straight-lines which are) incommensurable in square [Prop. 10.76], *CF* and *FD* are thus also (straight-lines which are) incommensurable in square [Prop. 10.13]. Therefore, since as *AE* is to *EB*, so *CF* (is) to *FD* [Props. 5.12, 5.16], thus also as the (square) on *AE* is to the (square) on *EB*, so the (square) on *CF* (is) to the (square) on *FD* [Prop. 6.22]. Thus, via composition, as the (sum of the squares) on *AE* and *EB* is to the (square) on *EB*, so the (sum of the squares) on *CF* and *FD* (is) to the (square) on *FD* [Prop. 5.18], [also alternately]. And the (square) on *BE* is commensurable with the (square) on *DF* [Prop. 10.104]. The sum of the squares on *AE* and *EB* (is) thus also commensurable with the sum of the squares on *CF* and *FD* [Prop. 5.16, 10.11]. And the sum of the (squares) on *AE* and *EB* is rational [Prop. 10.76]. Thus, the sum of the (squares) on *CF* and *FD* is also rational [Def. 10.4]. Again, since as the (square) on *AE* is to the (rectangle contained) by *AE* and *EB*, so the (square) on *CF* (is) to the (rectangle contained) by *CF* and *FD* [Prop. 10.21 lem.], and the square on *AE* (is) commensurable with the square on *CF*, the (rectangle contained) by *AE* and *EB* is thus also commensurable with the (rectangle contained) by *CF* and *FD*. And the (rectangle contained) by *AE* and *EB* (is) medial [Prop. 10.76]. Thus, the (rectangle contained) by *CF* and *FD* (is) also medial [Prop. 10.23 corr.]. *CF* and *FD* are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Thus, *CD* is a minor (straight-line) [Prop. 10.76]. (Which is) the very thing it was required to show.

ρϛʹ.

Ἡ τῇ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσῃ σύμμετρος μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ καὶ τῇ ΑΒ σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν.

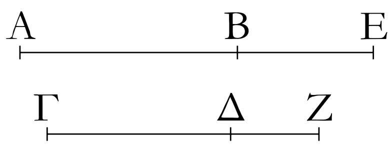
Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΕ· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν. καὶ τὰ αὐτὰ κατεσκευάσθω. ὁμοίως δὴ δείξομεν τοῖς πρότερον, ὅτι αἱ ΓΖ, ΖΔ ἐν τῷ αὐτῷ λόγῳ εἰσὶ ταῖς ΑΕ, ΕΒ, καὶ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων, τὸ δὲ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ

Proposition 106

A (straight-line) commensurable (in length) with a (straight-line) which with a rational (area) makes a medial whole is a (straight-line) which with a rational (area) makes a medial whole.

Let *AB* be a (straight-line) which with a rational (area) makes a medial whole, and (let) *CD* (be) commensurable (in length) with *AB*. I say that *CD* is also a (straight-line) which with a rational (area) makes a medial (whole).

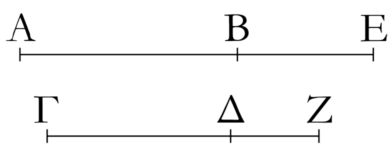
For let *BE* be an attachment to *AB*. Thus, *AE* and *EB* are (straight-lines which are) incommensurable in square, making the sum of the squares on *AE* and *EB* medial, and the (rectangle contained) by them rational [Prop. 10.77]. And let the same construction be made (as in the previous propositions). So, similarly to the previous (propositions), we can show that *μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων μέσον, τὸ δ᾽ ὑπ᾽ αὐτῶν ῥητόν.*



Ἡ ΓΔ ἄρα μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρζ΄.

Ἡ τῇ μετὰ μέσου μέσον τὸ ὅλον ποιούσῃ σύμμετρος καὶ αὐτὴ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

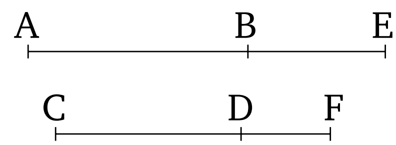


Ἔστω μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ ΑΒ, καὶ τῇ ΑΒ ἔστω σύμμετρος ἡ ΓΔ· λέγω, ὅτι καὶ ἡ ΓΔ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν.

Ἔστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΕ, καὶ τὰ αὐτὰ κατεσκευάσθω· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων τῷ ὑπ᾽ αὐτῶν. καί εἰσιν, ὡς ἐδείχθη, αἱ ΑΕ, ΕΒ σύμμετροι ταῖς ΓΖ, ΖΔ, καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ· καὶ αἱ ΓΖ, ΖΔ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ᾽ ἀὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπ᾽ αὐτῶν [τετραγώνων] τῷ ὑπ᾽ αὐτῶν.

Ἡ ΓΔ ἄρα μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

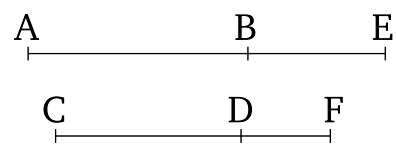
CF and FD are in the same ratio as AE and EB, and the sum of the squares on AE and EB is commensurable with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Hence, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on CF and FD medial, and the (rectangle contained) by them rational.



CD is thus a (straight-line) which with a rational (area) makes a medial whole [Prop. 10.77]. (Which is) the very thing it was required to show.

Proposition 107

A (straight-line) commensurable (in length) with a (straight-line) which with a medial (area) makes a medial whole is itself also a (straight-line) which with a medial (area) makes a medial whole.



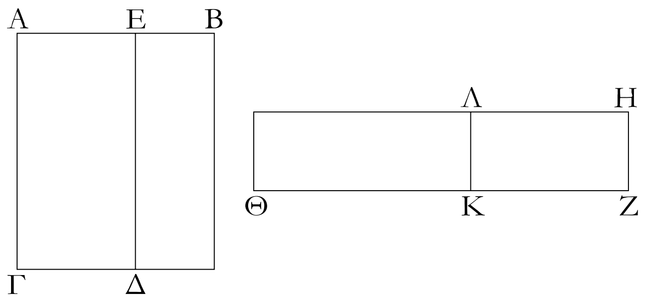
Let AB be a (straight-line) which with a medial (area) makes a medial whole, and let CD be commensurable (in length) with AB. I say that CD is also a (straight-line) which with a medial (area) makes a medial whole.

For let BE be an attachment to AB. And let the same construction be made (as in the previous propositions). Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.78]. And, as was shown (previously), AE and EB are commensurable (in length) with CF and FD (respectively), and the sum of the squares on AE and EB with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Thus, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the [squares] on them incommensurable with the (rectangle contained) by them.

Thus, CD is a (straight-line) which with a medial (area) makes a medial whole [Prop. 10.78]. (Which is) the very thing it was required to show.

ρη´.

Ἀπὸ ῥητοῦ μέσου ἀφαιρουμένου ἡ τὸ λοιπὸν χωρίον δυναμένη μία δύο ἀλόγων γίνεται ἤτοι ἀποτομὴ ἢ ἐλάσσων.



Ἀπὸ γὰρ ῥητοῦ τοῦ ΒΓ μέσον ἀφῃρήσθω τὸ ΒΔ· λέγω, ὅτι ἡ τὸ λοιπὸν δυναμένη τὸ ΕΓ μία δύο ἀλόγων γίνεται ἤτοι ἀποτομὴ ἢ ἐλάσσων.

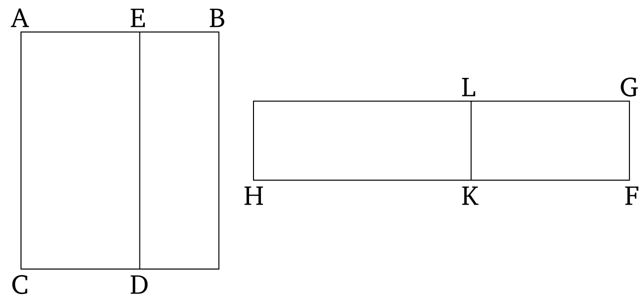
Ἐκκείσθω γὰρ ῥητὴ ἡ ΖΗ, καὶ τῷ μὲν ΒΓ ἴσον παρὰ τὴν ΖΗ παραβεβλήσθω ὀρθογώνιον παραλληλόγραμμον τὸ ΗΘ, τῷ δὲ ΔΒ ἴσον ἀφῃρήσθω τὸ ΗΚ· λοιπὸν ἄρα τὸ ΕΓ ἴσον ἐστὶ τῷ ΛΘ. ἐπεὶ οὖν ῥητὸν μέν ἐστι τὸ ΒΓ, μέσον δὲ τὸ ΒΔ, ἴσον δὲ τὸ μὲν ΒΓ τῷ ΗΘ, τὸ δὲ ΒΔ τῷ ΗΚ, ῥητὸν μὲν ἄρα ἐστὶ τὸ ΗΘ, μέσον δὲ τὸ ΗΚ. καὶ παρὰ ῥητὴν τὴν ΖΗ παράκειται· ῥητὴ μὲν ἄρα ἡ ΖΘ καὶ σύμμετρος τῇ ΖΗ μήκει, ῥητὴ δὲ ἡ ΖΚ καὶ ἀσύμμετρος τῇ ΖΗ μήκει· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΘ τῇ ΖΚ μήκει. αἱ ΖΘ, ΖΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ, προσαρμόζουσα δὲ αὐτῇ ἡ ΚΖ. ἤτοι δὴ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἢ οὔ.

Δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου. καί ἐστιν ὅλη ἡ ΘΖ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ· ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ ΚΘ. τὸ δ᾽ ὑπὸ ῥητῆς καὶ ἀποτομῆς πρώτης περιεχόμενον ἡ δυναμένη ἀποτομή ἐστιν. ἡ ἄρα τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένη ἀποτομή ἐστιν.

Εἰ δὲ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καί ἐστιν ὅλη ἡ ΖΘ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ τετάρτη ἐστὶν ἡ ΚΘ. τὸ δ᾽ ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ἡ δυναμένη ἐλάσσων ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 108

A medial (area) being subtracted from a rational (area), one of two irrational (straight-lines) arise (as) the square-root of the remaining area—either an apotome, or a minor (straight-line).



For let the medial (area) BD be subtracted from the rational (area) BC. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either an apotome, or a minor (straight-line).

For let the rational (straight-line) FG be laid out, and let the right-angled parallelogram GH, equal to BC, be applied to FG, and let GK, equal to DB, be subtracted (from GH). Thus, the remainder EC is equal to LH. Therefore, since BC is a rational (area), and BD a medial (area), and BC (is) equal to GH, and BD to GK, GH is thus a rational (area), and GK a medial (area). And they are applied to the rational (straight-line) FG. Thus, FH (is) rational, and commensurable in length with FG [Prop. 10.20], and FK (is) also rational, and incommensurable in length with FG [Prop. 10.22]. Thus, FH is incommensurable in length with FK [Prop. 10.13]. FH and FK are thus rational (straight-lines which are) commensurable in square only. Thus, KH is an apotome [Prop. 10.73], and KF an attachment to it. So, the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) either commensurable, or not (commensurable), (in length with HF).

First, let the square (on it) be (greater) by the (square) on (some straight-line which is) commensurable (in length with HF). And the whole of HF is commensurable in length with the (previously) laid down rational (straight-line) FG. Thus, KH is a first apotome [Def. 10.1]. And the square-root of an (area) contained by a rational (straight-line) and a first apotome is an apotome [Prop. 10.91]. Thus, the square-root of LH—that is to say, (of) EC—is an apotome.

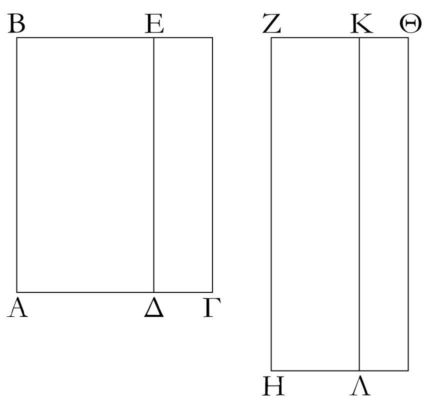
And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) incommensurable (in length) with (HF), and (since) the whole of FH is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fourth apotome [Prop. 10.14]. And the square-root of an (area) contained by a rational (straight-line) and a fourth apotome is a minor (straight-line) [Prop. 10.94]. (Which is) the very thing it was required to show.

ρθ΄.

Ἀπὸ μέσου ῥητοῦ ἀφαιρουμένου ἄλλαι δύο ἄλογοι γίνονται ἤτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

Ἀπὸ γὰρ μέσου τοῦ ΒΓ ῥητὸν ἀφῃρήσθω τὸ ΒΔ. λέγω, ὅτι ἡ τὸ λοιπὸν τὸ ΕΓ δυναμένη μία δύο ἀλόγων γίνεται ἤτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΖΗ, καὶ παραβεβλήσθω ὁμοίως τὰ χωρία. ἔστι δὴ ἀκολούθως ῥητὴ μὲν ἡ ΖΘ καὶ ἀσύμμετρος τῇ ΖΗ μήκει, ῥητὴ δὲ ἡ ΚΖ καὶ σύμμετρος τῇ ΖΗ μήκει· αἱ ΖΘ, ΖΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ, προσαρμόζουσα δὲ ταύτῃ ἡ ΖΚ. ἤτοι δὴ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου.



Εἰ μὲν οὖν ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ δευτέρα ἐστὶν ἡ ΚΘ. ῥητὴ δὲ ἡ ΖΗ· ὥστε ἡ τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένη μέσης ἀποτομὴ πρώτη ἐστίν.

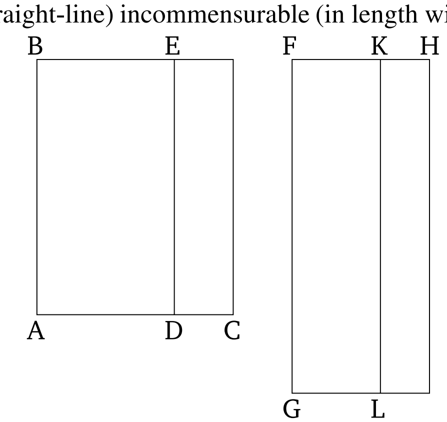
Εἰ δὲ ἡ ΘΖ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου, καί ἐστιν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ πέμπτη ἐστὶν ἡ ΚΘ· ὥστε ἡ τὸ ΕΓ δυναμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

Proposition 109

A rational (area) being subtracted from a medial (area), two other irrational (straight-lines) arise (as the square-root of the remaining area)—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (area) BD be subtracted from the medial (area) BC. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (straight-line) FG be laid down, and let similar areas (to the preceding proposition) have been applied (to it). So, accordingly, FH is rational, and incommensurable in length with FG, and KF (is) also rational, and commensurable in length with FG. Thus, FH and FK are rational (straight-lines which are) commensurable in square only [Prop. 10.13]. KH is thus an apotome [Prop. 10.73], and FK an attachment to it. So, the square on HF is greater than (the square on) FK either by the (square) on (some straight-line) commensurable (in length) with (HF), or by the (square) on (some straight-line) incommensurable (in length with HF).



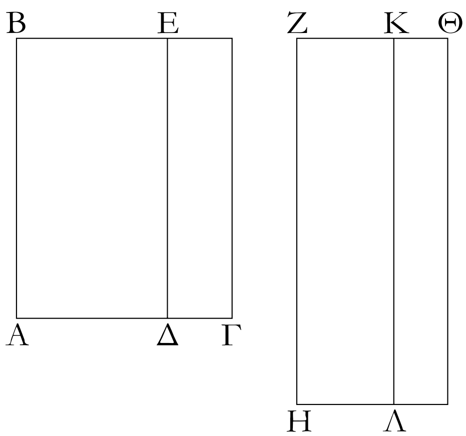
Therefore, if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a second apotome [Def. 10.12]. And FG (is) rational. Hence, the square-root of LH—that is to say, (of) EC—is a first apotome of a medial (straight-line) [Prop. 10.92].

And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) incommensurable (in length with HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fifth apotome [Def. 10.15]. Hence,

ρι΄.

Ἀπὸ μέσου μέσου ἀφαιρουμένου ἀσυμμέτρου τῷ ὅλῳ αἱ λοιπαὶ δύο ἄλογοι γίνονται ἤτοι μέσης ἀποτομὴ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Ἀφῃρήσθω γὰρ ὡς ἐπὶ τῶν προκειμένων καταγραφῶν ἀπὸ μέσου τοῦ ΒΓ μέσον τὸ ΒΔ ἀσύμμετρον τῷ ὅλῳ· λέγω, ὅτι ἡ τὸ ΕΓ δυναμένη μία ἐστὶ δύο ἀλόγων ἤτοι μέσης ἀποτομὴ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.



Ἐπεὶ γὰρ μέσον ἐστὶν ἑκάτερον τῶν ΒΓ, ΒΔ, καὶ ἀσύμμετρον τὸ ΒΓ τῷ ΒΔ, ἔσται ἀκολούθως ῥητὴ ἑκατέρα τῶν ΖΘ, ΖΚ καὶ ἀσύμμετρος τῇ ΖΗ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ΒΓ τῷ ΒΔ, τουτέστι τὸ ΗΘ τῷ ΗΚ, ἀσύμμετρος καὶ ἡ ΘΖ τῇ ΖΚ· αἱ ΖΘ, ΖΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΘ [προσαρμόζουσα δὲ ἡ ΖΚ. ἤτοι δὴ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἢ τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ].

Εἰ μὲν δὴ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ οὐδετέρα τῶν ΖΘ, ΖΚ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ τρίτη ἐστὶν ἡ ΚΘ. ῥητὴ δὲ ἡ ΚΛ, τὸ δ᾽ ὑπὸ ῥητῆς καὶ ἀποτομῆς τρίτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ μέσης ἀποτομὴ δευτέρα· ὥστε ἡ τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένη μέσης ἀποτομή ἐστι δευτερά.

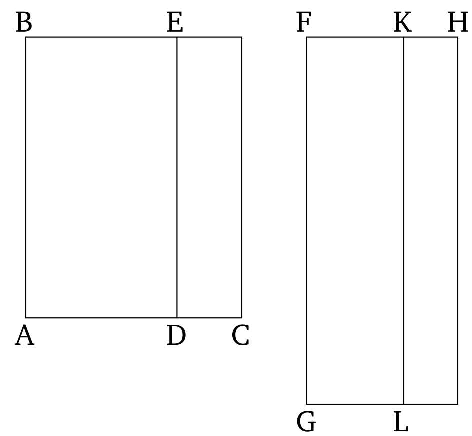
Εἰ δὲ ἡ ΖΘ τῆς ΖΚ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει], καὶ οὐδετέρα τῶν ΘΖ, ΖΚ σύμμετρός ἐστι τῇ ΖΗ μήκει, ἀποτομὴ ἕκτη ἐστὶν ἡ ΚΘ. τὸ δ᾽ ὑπὸ ῥητῆς καὶ ἀποτομῆς ἕκτης ἡ δυναμένη ἐστὶ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. ἡ τὸ ΛΘ ἄρα, τουτέστι τὸ ΕΓ, δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιοῦσά ἐστιν· ὅπερ ἔδει δεῖξαι.

the square-root of EC is that (straight-line) which with a rational (area) makes a medial whole [Prop. 10.95]. (Which is) the very thing it was required to show.

Proposition 110

A medial (area), incommensurable with the whole, being subtracted from a medial (area), the two remaining irrational (straight-lines) arise (as) the (square-root of the area)—either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.

For, as in the previous figures, let the medial (area) BD, incommensurable with the whole, be subtracted from the medial (area) BC. I say that the square-root of EC is one of two irrational (straight-lines)—either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.



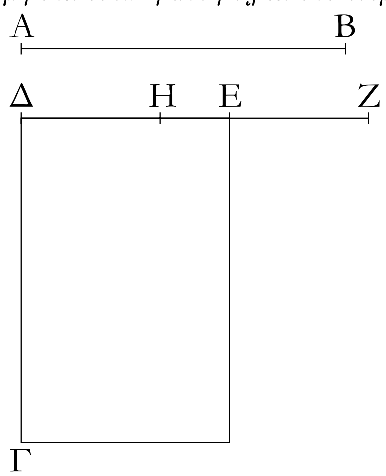
For since BC and BD are each medial (areas), and BC (is) incommensurable with BD, accordingly, FH and FK will each be rational (straight-lines), and incommensurable in length with FG [Prop. 10.22]. And since BC is incommensurable with BD—that is to say, GH with GK—HF (is) also incommensurable (in length) with FK [Props. 6.1, 10.11]. Thus, FH and FK are rational (straight-lines which are) commensurable in square only. KH is thus as apotome [Prop. 10.73], [and FK an attachment (to it). So, the square on FH is greater than (the square on) FK either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (FH).]

So, if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (FH), and (since) neither of FH and FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a third apotome [Def. 10.3]. And KL (is) rational. And the rectangle contained by a rational (straight-line) and a third apotome is irrational, and the square-root of it is that irrational (straight-line) called a second apotome of a medial (straight-line) [Prop. 10.93]. Hence, the square-root of LH—that is to say, (of) EC—is a second apotome of a medial (straight-line).

And if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) incommensurable [in length] with (FH), and (since) neither of HF and FK is commensurable in length with FG, KH is a sixth apotome [Def. 10.16]. And the square-root of the (rectangle contained) by a rational (straight-line) and a sixth apotome is that (straight-line) which with a medial (area) makes a medial whole [Prop. 10.96]. Thus, the square-root of LH—that is to say, (of) EC—is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to show.

ρια´.

Ἡ ἀποτομὴ οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων.

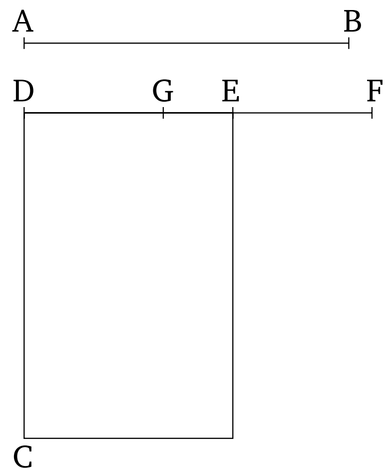


Ἔστω ἀποτομὴ ἡ ΑΒ· λέγω, ὅτι ἡ ΑΒ οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων.

Εἰ γὰρ δυνατόν, ἔστω· καὶ ἐκκείσθω ῥητὴ ἡ ΔΓ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω ὀρθογώνιον τὸ ΓΕ πλάτος ποιοῦν τὴν ΔΕ. ἐπεὶ οὖν ἀποτομή ἐστιν ἡ ΑΒ, ἀποτομὴ πρώτη ἐστὶν ἡ ΔΕ. ἔστω αὐτῇ προσαρμόζουσα ἡ ΕΖ· αἱ ΔΖ, ΖΕ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΖ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΔΖ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΔΓ. πάλιν, ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶν ἡ ΑΒ, ἐκ δύο ἄρα ὀνομάτων πρώτη ἐστὶν ἡ ΔΕ. διῃρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Η, καὶ ἔστω μεῖζον ὄνομα τὸ ΔΗ· αἱ ΔΗ, ΗΕ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΗ τῆς ΗΕ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τὸ μεῖζον ἡ ΔΗ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΔΓ. καὶ ἡ ΔΖ ἄρα τῇ ΔΗ σύμμετρός ἐστι μήκει· καὶ λοιπὴ ἄρα ἡ ΗΖ σύμμετρός ἐστι τῇ ΔΖ μήκει. [ἐπεὶ οὖν σύμμετρός ἐστιν ἡ ΔΖ τῇ ΗΖ, ῥητὴ δέ ἐστιν ἡ ΔΖ, ῥητὴ ἄρα ἐστὶ καὶ ἡ ΗΖ. ἐπεὶ οὖν σύμμετρός ἐστιν ἡ ΔΖ τῇ ΗΖ μήκει] ἀσύμμετρος δὲ ἡ ΔΖ τῇ ΕΖ μήκει. ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΖΗ τῇ ΕΖ μήκει. αἱ ΗΖ, ΖΕ

Proposition 111

An apotome is not the same as a binomial.



Let AB be an apotome. I say that AB is not the same as a binomial.

For, if possible, let it be (the same). And let a rational (straight-line) DC be laid down. And let the rectangle CE, equal to the (square) on AB, be applied to CD, producing DE as breadth. Therefore, since AB is an apotome, DE is a first apotome [Prop. 10.97]. Let EF be an attachment to it. Thus, DF and FE are rational (straight-lines which are) commensurable in square only, and the square on DF is greater than (the square on) FE by the (square) on (some straight-line) commensurable (in length) with (DF), and DF is commensurable in length with the (previously) laid down rational (straight-line) DC [Def. 10.10]. Again, since AB is a binomial, DE is thus a first binomial [Prop. 10.60]. Let (DE) be divided into its (component) terms at G, and let DG be the greater term. Thus, DG and GE are rational (straight-lines which are) commensurable in square only, and the square on DG is greater than (the square on) GE by the (square) on (some straight-line) commensurable (in length) with (DG), and the greater (term) DG is commensurable in length with the (previously) laid down *ἄρα ῥηταί [εἰσι] δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΗ. ἀλλὰ καὶ ῥητή· ὅπερ ἐστὶν ἀδύνατον.*

Ἡ ἄρα ἀποτομὴ οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι.

rational (straight-line) DC [Def. 10.5]. Thus, DF is also commensurable in length with DG [Prop. 10.12]. The remainder GF is thus commensurable in length with DF [Prop. 10.15]. [Therefore, since DF is commensurable with GF, and DF is rational, GF is thus also rational. Therefore, since DF is commensurable in length with GF,] DF (is) incommensurable in length with EF. Thus, FG is also incommensurable in length with EF [Prop. 10.13]. GF and FE [are] thus rational (straight-lines which are) commensurable in square only. Thus, EG is an apotome [Prop. 10.73]. But, (it is) also rational. The very thing is impossible.

Thus, an apotome is not the same as a binomial. (Which is) the very thing it was required to show.

[Πόρισμα.]

Ἡ ἀποτομὴ καὶ αἱ μετʼ αὐτὴν ἄλογοι οὔτε τῇ μέσῃ οὔτε ἀλλήλαις εἰσὶν αἱ αὐταί.

Τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ, παρʼ ἣν παράκειται, μήκει, τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην, τὸ δὲ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν, τὸ δὲ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην, τὸ δὲ ἀπὸ ἐλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τετάρτην, τὸ δὲ ἀπὸ τῆς μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πέμπτην, τὸ δὲ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν ἕκτην. ἐπεὶ οὖν τὰ εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ῥητή ἐστιν, ἀλλήλων δέ, ἐπεὶ τῇ τάξει οὐκ εἰσὶν αἱ αὐταί, δῆλον, ὡς καὶ αὐταὶ αἱ ἄλογοι διαφέρουσιν ἀλλήλων. καὶ ἐπεὶ δέδεικται ἡ ἀποτομὴ οὐκ οὖσα ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων, ποιοῦσι δὲ πλάτη παρὰ ῥητὴν παραβαλλόμεναι αἱ μετὰ τὴν ἀποτομὴν ἀποτομὰς ἀκολούθως ἑκάστη τῇ τάξει τῇ καθʼ αὑτήν, αἱ δὲ μετὰ τὴν ἐκ δύο ὀνομάτων τὰς ἐκ δύο ὀνομάτων καὶ αὐταὶ τῇ τάξει ἀκολούθως, ἕτεραι ἄρα εἰσὶν αἱ μετὰ τὴν ἀποτομὴν καὶ ἕτεραι αἱ μετὰ τὴν ἐκ δύο ὀνομάτων, ὡς εἶναι τῇ τάξει πάσας ἀλόγους $\overline{\iota\gamma}$,

[Corollary]

The apotome and the irrational (straight-lines) after it are neither the same as a medial (straight-line) nor (the same) as one another.

For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) incommensurable in length with the (straight-line) to which (the area) is applied [Prop. 10.22]. And the (square) on an apotome, applied to a rational (straight-line), produces as breadth a first apotome [Prop. 10.97]. And the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a second apotome [Prop. 10.98]. And the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a third apotome [Prop. 10.99]. And (square) on a minor (straight-line), applied to a rational (straight-line), produces as breadth a fourth apotome [Prop. 10.100]. And (square) on that (straight-line) which with a rational (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a fifth apotome [Prop. 10.101]. And (square) on that (straight-line) which with a medial (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a sixth apotome [Prop. 10.102]. Therefore, since the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational, and from one another since they are not the same in order—clearly, the irrational (straight-lines) themselves also differ from one another. And since it has been shown that an apotome is not the same as a binomial [Prop. 10.111], and (that) the (irrational straight-lines) after the apotome, being applied to a rational (straight-line), produce as breadth, each according to its own (order), apotomes, and (that) the (irrational straight-lines) after the binomial themselves also (produce as breadth), according (to their) order, binomials, the (irrational straight-lines) after the apotome are thus different, and the (irrational straight-lines) after the binomial (are also) different, so that there are, *Μέσην,*

Ἐκ δύο ὀνομάτων,

Ἐκ δύο μέσων πρώτην,

Ἐκ δύο μέσων δευτέραν,

Μείζονα,

μὴ Ῥητὸν καὶ μέσον δυναμένην,

Δύο μέσα δυναμένην,

Ἀποτομήν,

Μέσης ἀποτομὴν πρώτην,

Μέσης ἀποτομὴν δευτέραν,

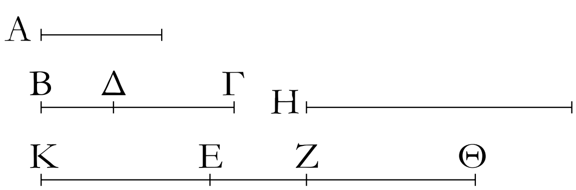
Ἐλάσσονα,

Μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσαν,

Μετὰ μέσου μέσον τὸ ὅλον ποιοῦσαν.

ριβ'.

Τὸ ἀπὸ ῥητῆς παρὰ τὴν ἐκ δύο ὀνομάτων παραβαλλόμενον πλάτος ποιεῖ ἀποτομήν, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι καὶ ἔτι ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ γινομένη ἀποτομὴ τὴν αὐτὴν ἕξει τάξιν τῇ ἐκ δύο ὀνομάτων.



Ἔστω ῥητὴ μὲν ἡ Α, ἐκ δύο ὀνομάτων δὲ ἡ ΒΓ, ἧς μεῖζον ὄνομα ἔστω ἡ ΔΓ, καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω τὸ ὑπὸ τῶν ΒΓ, ΕΖ· λέγω, ὅτι ἡ ΕΖ ἀποτομή ἐστιν, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς ΓΔ, ΔΒ, καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΕΖ τὴν αὐτὴν ἕξει τάξιν τῇ ΒΓ.

Ἔστω γὰρ πάλιν τῷ ἀπὸ τῆς Α ἴσον τὸ ὑπὸ τῶν ΒΔ, Η. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΒΓ, ΕΖ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΔ, Η, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως ἡ Η πρὸς τὴν ΕΖ. μείζων δὲ ἡ ΓΒ τῆς ΒΔ· μείζων ἄρα ἐστὶ καὶ ἡ Η τῆς ΕΖ. ἔστω τῇ Η ἴση ἡ ΕΘ· ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως ἡ ΘΕ πρὸς τὴν ΕΖ· διελόντι ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΒΔ, οὕτως ἡ ΘΖ πρὸς τὴν ΖΕ. γεγονέτω ὡς ἡ ΘΖ πρὸς τὴν ΖΕ, οὕτως ἡ ΖΕ πρὸς τὴν ΚΕ· καὶ ὅλη ἄρα ἡ ΘΚ πρὸς ὅλην τὴν ΚΖ ἐστιν, ὡς ἡ ΖΚ πρὸς ΚΕ· ὡς γὰρ ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα. ὡς δὲ ἡ ΖΚ πρὸς ΚΕ, οὕτως ἐστὶν ἡ ΓΔ πρὸς τὴν ΔΒ· καὶ ὡς ἄρα ἡ ΘΚ πρὸς ΚΖ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΒ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΓΔ τῷ ἀπὸ τῆς ΔΒ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘΚ τῷ ἀπὸ τῆς ΚΖ. καί ἐστιν ὡς τὸ ἀπὸ τῆς ΘΚ πρὸς τὸ ἀπὸ τῆς ΚΖ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΕ, ἐπεὶ αἱ τρεῖς αἱ ΘΚ, ΚΖ, ΚΕ ἀνάλογόν εἰσιν. σύμμετρος

in order, 13 irrational (straight-lines) in all:

Medial,

Binomial,

First bimedial,

Second bimedial,

Major,

Square-root of a rational plus a medial (area),

Square-root of (the sum of) two medial (areas),

Apotome,

First apotome of a medial,

Second apotome of a medial,

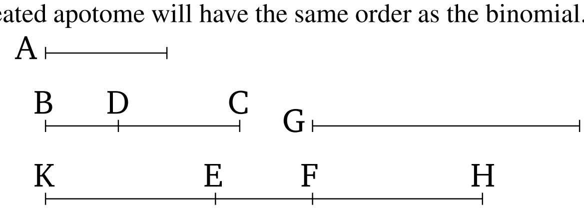
Minor,

That which with a rational (area) produces a medial whole,

That which with a medial (area) produces a medial whole.

Proposition 112[†]

The (square) on a rational (straight-line), applied to a binomial (straight-line), produces as breadth an apotome whose terms are commensurable (in length) with the terms of the binomial, and, furthermore, in the same ratio. Moreover, the created apotome will have the same order as the binomial.



Let A be a rational (straight-line), and BC a binomial (straight-line), of which let DC be the greater term. And let the (rectangle contained) by BC and EF be equal to the (square) on A. I say that EF is an apotome whose terms are commensurable (in length) with CD and DB, and in the same ratio, and, moreover, that EF will have the same order as BC.

For, again, let the (rectangle contained) by BD and G be equal to the (square) on A. Therefore, since the (rectangle contained) by BC and EF is equal to the (rectangle contained) by BD and G, thus as CB is to BD, so G (is) to EF [Prop. 6.16]. And CB (is) greater than BD. Thus, G is also greater than EF [Props. 5.16, 5.14]. Let EH be equal to G. Thus, as CB is to BD, so HE (is) to EF. Thus, via separation, as CD is to BD, so HF (is) to FE [Prop. 5.17]. Let it be contrived that as HF (is) to FE, so FK (is) to KE. And, thus, the whole HK is to the whole KF, as FK (is) to KE. For as one of the leading (proportional magnitudes is) to one of the following, so all of the leading (magnitudes) are to all of the following [Prop. 5.12]. And as FK (is) to KE, so CD is to DB [Prop. 5.11]. And, thus, as HK (is) to KF, so CD is to DB [Prop. 5.11]. And the (square) on CD (is) commensurable with the (square) on *ἄρα ἡ ΘΚ τῇ ΚΕ μήκει. ὥστε καὶ ἡ ΘΕ τῇ ΕΚ σύμμετρός ἐστι μήκει. καὶ ἐπεὶ τὸ ἀπὸ τῆς Α ἴσον ἐστὶ τῷ ὑπὸ τῶν ΕΘ, ΒΔ, ῥητὸν δέ ἐστι τὸ ἀπὸ τῆς Α, ῥητὸν ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΕΘ, ΒΔ. καὶ παρὰ ῥητὴν τὴν ΒΔ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΕΘ καὶ σύμμετρος τῇ ΒΔ μήκει· ὥστε καὶ ἡ σύμμετρος αὐτῇ ἡ ΕΚ ῥητή ἐστι καὶ σύμμετρος τῇ ΒΔ μήκει. ἐπεὶ οὖν ἐστιν ὡς ἡ ΓΔ πρὸς ΔΒ, οὕτως ἡ ΖΚ πρὸς ΚΕ, αἱ δὲ ΓΔ, ΔΒ δυνάμει μόνον εἰσὶ σύμμετροι, καὶ αἱ ΖΚ, ΚΕ δυνάμει μόνον εἰσὶ σύμμετροι. ῥητὴ δέ ἐστιν ἡ ΚΕ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΚ. αἱ ΖΚ, ΚΕ ἄρα ῥηταὶ δυνάμει μόνον εἰσὶ σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΖ.*

Ἤτοι δὲ ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου.

Εἰ μὲν οὖν ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ συμμέτρου [ἑαυτῇ], καὶ ἡ ΖΚ τῆς ΚΕ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΓΔ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΖΚ· εἰ δὲ ἡ ΒΔ, καὶ ἡ ΚΕ· εἰ δὲ οὐδετέρα τῶν ΓΔ, ΔΒ, καὶ οὐδετέρα τῶν ΖΚ, ΚΕ.

Εἰ δὲ ἡ ΓΔ τῆς ΔΒ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΖΚ τῆς ΚΕ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν ἡ ΓΔ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΖΚ· εἰ δὲ ἡ ΒΔ, καὶ ἡ ΚΕ· εἰ δὲ οὐδετέρα τῶν ΓΔ, ΔΒ, καὶ οὐδετέρα τῶν ΖΚ, ΚΕ· ὥστε ἀποτομή ἐστιν ἡ ΖΕ, ἧς τὰ ὀνόματα τὰ ΖΚ, ΚΕ σύμμετρά ἐστι τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι τοῖς ΓΔ, ΔΒ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ τὴν αὐτὴν τάξιν ἔχει τῇ ΒΓ· ὅπερ ἔδει δεῖξαι.

DB [Prop. 10.36]. The (square) on *HK* is thus also commensurable with the (square) on *KF* [Props. 6.22, 10.11]. And as the (square) on *HK* is to the (square) on *KF*, so *HK* (is) to *KE*, since the three (straight-lines) *HK*, *KF*, and *KE* are proportional [Def. 5.9]. *HK* is thus commensurable in length with *KE* [Prop. 10.11]. Hence, *HE* is also commensurable in length with *EK* [Prop. 10.15]. And since the (square) on *A* is equal to the (rectangle contained) by *EH* and *BD*, and the (square) on *A* is rational, the (rectangle contained) by *EH* and *BD* is thus also rational. And it is applied to the rational (straight-line) *BD*. Thus, *EH* is rational, and commensurable in length with *BD* [Prop. 10.20]. And, hence, the (straight-line) commensurable (in length) with it, *EK*, is also rational [Def. 10.3], and commensurable in length with *BD* [Prop. 10.12]. Therefore, since as *CD* is to *DB*, so *FK* (is) to *KE*, and *CD* and *DB* are (straight-lines which are) commensurable in square only, *FK* and *KE* are also commensurable in square only [Prop. 10.11]. And *KE* is rational. Thus, *FK* is also rational. *FK* and *KE* are thus rational (straight-lines which are) commensurable in square only. Thus, *EF* is an apotome [Prop. 10.73].

And the square on *CD* is greater than (the square on) *DB* either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (*CD*).

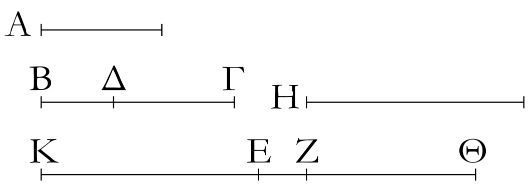
Therefore, if the square on *CD* is greater than (the square on) *DB* by the (square) on (some straight-line) commensurable (in length) with [*CD*] then the square on *FK* will also be greater than (the square on) *KE* by the (square) on (some straight-line) commensurable (in length) with (*FK*) [Prop. 10.14]. And if *CD* is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) *FK* [Props. 10.11, 10.12]. And if *BD* (is commensurable), (so) also (is) *KE* [Prop. 10.12]. And if neither of *CD* or *DB* (is commensurable), neither also (are) either of *FK* or *KE*.

And if the square on *CD* is greater than (the square on) *DB* by the (square) on (some straight-line) incommensurable (in length) with (*CD*) then the square on *FK* will also be greater than (the square on) *KE* by the (square) on (some straight-line) incommensurable (in length) with (*FK*) [Prop. 10.14]. And if *CD* is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) *FK* [Props. 10.11, 10.12]. And if *BD* (is commensurable), (so) also (is) *KE* [Prop. 10.12]. And if neither of *CD* or *DB* (is commensurable), neither also (are) either of *FK* or *KE*. Hence, *FE* is an apotome whose terms, *FK* and *KE*, are commensurable (in length) with the terms, *CD* and *DB*, of the binomial, and in the same ratio. And (*FE*) has the same order as *BC* [Defs. 10.5—10.10]. (Which is) the very thing it was required to show.

† Heiberg considers this proposition, and the succeeding ones, to be relatively early interpolations into the original text.

ριγ'.

Τὸ ἀπὸ ῥητῆς παρὰ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἔτι δὲ ἡ γινομένη ἐκ δύο ὀνομάτων τὴν αὐτὴν τάξιν ἔχει τῇ ἀποτομῇ.



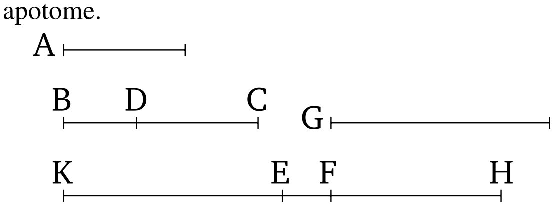
Ἔστω ῥητὴ μὲν ἡ Α, ἀποτομὴ δὲ ἡ ΒΔ, καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω τὸ ὑπὸ τῶν ΒΔ, ΚΘ, ὥστε τὸ ἀπὸ τῆς Α ῥητῆς παρὰ τὴν ΒΔ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ΚΘ· λέγω, ὅτι ἐκ δύο ὀνομάτων ἐστὶν ἡ ΚΘ, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ΒΔ ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΚΘ τὴν αὐτὴν ἔχει τάξιν τῇ ΒΔ.

Ἔστω γὰρ τῇ ΒΔ προσαρμόζουσα ἡ ΔΓ· αἱ ΒΓ, ΓΔ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω καὶ τὸ ὑπὸ τῶν ΒΓ, Η. ῥητὸν δὲ τὸ ἀπὸ τῆς Α· ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν ΒΓ, Η. καὶ παρὰ ῥητὴν τὴν ΒΓ παραβέβληται· ῥητὴ ἄρα ἐστὶν ἡ Η καὶ σύμμετρος τῇ ΒΓ μήκει. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΒΓ, Η ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΔ, ΚΘ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΒ πρὸς ΒΔ, οὕτως ἡ ΚΘ πρὸς Η. μείζων δὲ ἡ ΒΓ τῆς ΒΔ· μείζων ἄρα καὶ ἡ ΚΘ τῆς Η. κείσθω τῇ Η ἴση ἡ ΚΕ· σύμμετρος ἄρα ἐστὶν ἡ ΚΕ τῇ ΒΓ μήκει. καὶ ἐπεί ἐστιν ὡς ἡ ΓΒ πρὸς ΒΔ, οὕτως ἡ ΘΚ πρὸς ΚΕ, ἀναστρέψαντι ἄρα ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν ΓΔ, οὕτως ἡ ΚΘ πρὸς ΘΕ. γεγονέτω ὡς ἡ ΚΘ πρὸς ΘΕ, οὕτως ἡ ΘΖ πρὸς ΖΕ· καὶ λοιπὴ ἄρα ἡ ΚΖ πρὸς ΖΘ ἐστιν, ὡς ἡ ΚΘ πρὸς ΘΕ, τουτέστιν [ὡς] ἡ ΒΓ πρὸς ΓΔ. αἱ δὲ ΒΓ, ΓΔ δυνάμει μόνον [εἰσὶ] σύμμετροι· καὶ αἱ ΚΖ, ΖΘ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι· καὶ ἐπεί ἐστιν ὡς ἡ ΚΘ πρὸς ΘΕ, ἡ ΚΖ πρὸς ΖΘ, ἀλλ' ὡς ἡ ΚΘ πρὸς ΘΕ, ἡ ΘΖ πρὸς ΖΕ, καὶ ὡς ἄρα ἡ ΚΖ πρὸς ΖΘ, ἡ ΘΖ πρὸς ΖΕ· ὥστε καὶ ὡς ἡ πρώτη πρὸς τὴν τρίτην, τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· καὶ ὡς ἄρα ἡ ΚΖ πρὸς ΖΕ, οὕτως τὸ ἀπὸ τῆς ΚΖ πρὸς τὸ ἀπὸ τῆς ΖΘ. σύμμετρον δέ ἐστι τὸ ἀπὸ τῆς ΚΖ τῷ ἀπὸ τῆς ΖΘ· αἱ γὰρ ΚΖ, ΖΘ δυνάμει εἰσὶ σύμμετροι· σύμμετρος ἄρα ἐστὶ καὶ ἡ ΚΖ τῇ ΖΕ μήκει· ὥστε ἡ ΚΖ καὶ τῇ ΚΕ σύμμετρός [ἐστι] μήκει. ῥητὴ δέ ἐστιν ἡ ΚΕ καὶ σύμμετρος τῇ ΒΓ μήκει. ῥητὴ ἄρα καὶ ἡ ΚΖ καὶ σύμμετρος τῇ ΒΓ μήκει. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς ΓΔ, οὕτως ἡ ΚΖ πρὸς ΖΘ, ἐναλλὰξ ὡς ἡ ΒΓ πρὸς ΚΖ, οὕτως ἡ ΔΓ πρὸς ΖΘ. σύμμετρος δὲ ἡ ΒΓ τῇ ΚΖ· σύμμετρος ἄρα καὶ ἡ ΖΘ τῇ ΓΔ μήκει. αἱ ΒΓ, ΓΔ δὲ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· καὶ αἱ ΚΖ, ΖΘ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ὀνομάτων ἐστὶν ἄρα ἡ ΚΘ.

Εἰ μὲν οὖν ἡ ΒΓ τῆς ΓΔ μεῖζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΚΖ τῆς ΖΘ μεῖζον δυνήσεται τῷ ἀπὸ συμμέτρου

Proposition 113

The (square) on a rational (straight-line), applied to an apotome, produces as breadth a binomial whose terms are commensurable with the terms of the apotome, and in the same ratio. Moreover, the created binomial has the same order as the apotome.



Let *A* be a rational (straight-line), and *BD* an apotome. And let the (rectangle contained) by *BD* and *KH* be equal to the (square) on *A*, such that the square on the rational (straight-line) *A*, applied to the apotome *BD*, produces *KH* as breadth. I say that *KH* is a binomial whose terms are commensurable with the terms of *BD*, and in the same ratio, and, moreover, that *KH* has the same order as *BD*.

For let *DC* be an attachment to *BD*. Thus, *BC* and *CD* are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let the (rectangle contained) by *BC* and *G* also be equal to the (square) on *A*. And the (square) on *A* (is) rational. The (rectangle contained) by *BC* and *G* (is) thus also rational. And it has been applied to the rational (straight-line) *BC*. Thus, *G* is rational, and commensurable in length with *BC* [Prop. 10.20]. Therefore, since the (rectangle contained) by *BC* and *G* is equal to the (rectangle contained) by *BD* and *KH*, thus, proportionally, as *CB* is to *BD*, so *KH* (is) to *G* [Prop. 6.16]. And *BC* (is) greater than *BD*. Thus, *KH* (is) also greater than *G* [Prop. 5.16, 5.14]. Let *KE* be made equal to *G*. *KE* is thus commensurable in length with *BC*. And since as *CB* is to *BD*, so *HK* (is) to *KE*, thus, via conversion, as *BC* (is) to *CD*, so *KH* (is) to *HE* [Prop. 5.19 corr.]. Let it be contrived that as *KH* (is) to *HE*, so *HF* (is) to *FE*. And thus the remainder *KF* is to *FH*, as *KH* (is) to *HE*—that is to say, [as] *BC* (is) to *CD* [Prop. 5.19]. And *BC* and *CD* [are] commensurable in square only. *KF* and *FH* are thus also commensurable in square only [Prop. 10.11]. And since as *KH* is to *HE*, (so) *KF* (is) to *FH*, but as *KH* (is) to *HE*, (so) *HF* (is) to *FE*, thus, also as *KF* (is) to *FH*, (so) *HF* (is) to *FE* [Prop. 5.11]. And hence as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9]. And thus as *KF* (is) to *FE*, so the (square) on *KF* (is) to the (square) on *FH*. And the (square) on *KF* is commensurable with the (square) on *FH*. For *KF* and *FH* are commensurable in square. Thus, *KF* is also commensurable in length with *FE* [Prop. 10.11]. Hence, *KF* [is] also commensurable in length with *KE* [Prop. 10.15]. And *KE* is rational, and commensurable in length with *BC*. Thus, *KF* (is) also rational, and commensurable in length with *BC* [Prop. 10.12]. And *ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΒΓ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΚΖ, εἰ δὲ ἡ ΓΔ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΖΘ, εἰ δὲ οὐδετέρα τῶν ΒΓ, ΓΔ, οὐδετέρα τῶν ΚΖ, ΖΘ.*

Εἰ δὲ ἡ ΒΓ τῆς ΓΔ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ΚΖ τῆς ΖΘ μεῖζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ ΒΓ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΚΖ, εἰ δὲ ἡ ΓΔ, καὶ ἡ ΖΘ, εἰ δὲ οὐδετέρα τῶν ΒΓ, ΓΔ, οὐδετέρα τῶν ΚΖ, ΖΘ.

Ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΚΘ, ἧς τὰ ὀνόματα τὰ ΚΖ, ΖΘ σύμμετρά [ἐστι] τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς ΒΓ, ΓΔ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΚΘ τῇ ΒΓ τὴν αὐτὴν ἕξει τάξιν· ὅπερ ἔδει δεῖξαι.

since as BC is to CD, (so) KF (is) to FH, alternately, as BC (is) to KF, so DC (is) to FH [Prop. 5.16]. And BC (is) commensurable (in length) with KF. Thus, FH (is) also commensurable in length with CD [Prop. 10.11]. And BC and CD are rational (straight-lines which are) commensurable in square only. KF and FH are thus also rational (straight-lines which are) commensurable in square only [Def. 10.3, Prop. 10.13]. Thus, KH is a binomial [Prop. 10.36].

Therefore, if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) commensurable (in length) with (BC), then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) commensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

And if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) incommensurable (in length) with (BC) then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) incommensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable, (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

KH is thus a binomial whose terms, KF and FH, [are] commensurable (in length) with the terms, BC and CD, of the apotome, and in the same ratio. Moreover, KH will have the same order as BC [Defs. 10.5—10.10]. (Which is) the very thing it was required to show.

ριδ'.

Ἐὰν χωρίον περιέχηται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ἧς τὰ ὀνόματα σύμμετρά τέ ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ τὸ χωρίον δυναμένη ῥητή ἐστιν.

Περιεχέσθω γὰρ χωρίον τὸ ὑπὸ τῶν ΑΒ, ΓΔ ὑπὸ ἀποτομῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τῆς ΓΔ, ἧς μεῖζον ὄνομα ἔστω τὸ ΓΕ, καὶ ἔστω τὰ ὀνόματα τῆς ἐκ δύο ὀνομάτων τὰ ΓΕ, ΕΔ σύμμετρά τε τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς ΑΖ, ΖΒ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔστω ἡ τὸ ὑπὸ τῶν ΑΒ, ΓΔ δυναμένη ἡ Η· λέγω, ὅτι ῥητή ἐστιν ἡ Η.

Ἐκκείσθω γὰρ ῥητὴ ἡ Θ, καὶ τῷ ἀπὸ τῆς Θ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω πλάτος ποιοῦν τὴν ΚΛ· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΛ, ἧς τὰ ὀνόματα ἔστω τὰ ΚΜ, ΜΛ σύμμετρα τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι τοῖς ΓΕ, ΕΔ καὶ ἐν τῷ αὐτῷ

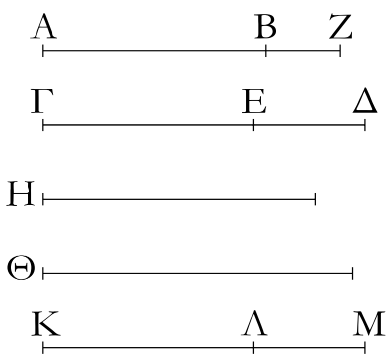
Proposition 114

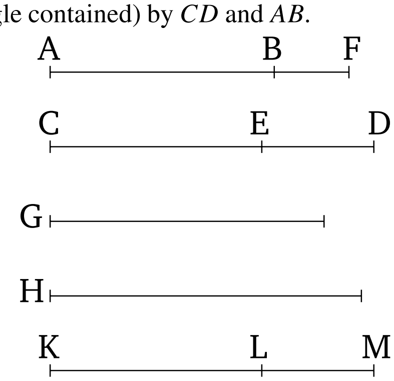
If an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome then the square-root of the area is a rational (straight-line).

For let an area, the (rectangle contained) by AB and CD, be contained by the apotome AB, and the binomial CD, of which let the greater term be CE. And let the terms of the binomial, CE and ED, be commensurable with the terms of the apotome, AF and FB (respectively), and in the same ratio. And let the square-root of the (rectangle contained) by AB and CD be G. I say that G is a rational (straight-line).

For let the rational (straight-line) H be laid down. And let (some rectangle), equal to the (square) on H, be applied to CD, producing KL as breadth. Thus, KL is an apotome, of which *λόγῳ. ἀλλὰ καὶ αἱ ΓΕ, ΕΔ σύμμετροί τέ εἰσι ταῖς ΑΖ, ΖΒ καὶ ἐν τῷ αὐτῷ λόγῳ· ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΖΒ, οὕτως ἡ ΚΜ πρὸς ΜΛ. ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΖ πρὸς τὴν ΚΜ, οὕτως ἡ ΒΖ πρὸς τὴν ΛΜ· καὶ λοιπὴ ἄρα ἡ ΑΒ πρὸς λοιπὴν τὴν ΚΛ ἐστιν ὡς ἡ ΑΖ πρὸς ΚΜ. σύμμετρος δὲ ἡ ΑΖ τῇ ΚΜ· σύμμετρος ἄρα ἐστὶ καὶ ἡ ΑΒ τῇ ΚΛ. καί ἐστιν ὡς ἡ ΑΒ πρὸς ΚΛ, οὕτως τὸ ὑπὸ τῶν ΓΔ, ΑΒ πρὸς τὸ ὑπὸ τῶν ΓΔ, ΚΛ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΓΔ, ΑΒ τῷ ὑπὸ τῶν ΓΔ, ΚΛ. ἴσον δὲ τὸ ὑπὸ τῶν ΓΔ, ΚΛ τῷ ἀπὸ τῆς Θ· σύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΓΔ, ΑΒ τῷ ἀπὸ τῆς Θ. τῷ δὲ ὑπὸ τῶν ΓΔ, ΑΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς Η· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Η τῷ ἀπὸ τῆς Θ. ῥητὸν δὲ τὸ ἀπὸ τῆς Θ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς Η· ῥητὴ ἄρα ἐστὶν ἡ Η. καὶ δύναται τὸ ὑπὸ τῶν ΓΔ, ΑΒ.*

let the terms, KM and ML, be commensurable with the terms of the binomial, CE and ED (respectively), and in the same ratio [Prop. 10.112]. But, CE and ED are also commensurable with AF and FB (respectively), and in the same ratio. Thus, as AF is to FB, so KM (is) to ML. Thus, alternately, as AF is to KM, so BF (is) to LM [Prop. 5.16]. Thus, the remainder AB is also to the remainder KL as AF (is) to KM [Prop. 5.19]. And AF (is) commensurable with KM [Prop. 10.12]. AB is thus also commensurable with KL [Prop. 10.11]. And as AB is to KL, so the (rectangle contained) by CD and AB (is) to the (rectangle contained) by CD and KL [Prop. 6.1]. Thus, the (rectangle contained) by CD and AB is also commensurable with the (rectangle contained) by CD and KL [Prop. 10.11]. And the (rectangle contained) by CD and KL (is) equal to the (square) on H. Thus, the (rectangle contained) by CD and AB is commensurable with the (square) on H. And the (square) on G is equal to the (rectangle contained) by CD and AB. The (square) on G is thus commensurable with the (square) on H. And the (square) on H (is) rational. Thus, the (square) on G is also rational. G is thus rational. And it is the square-root of the (rectangle contained) by CD and AB.





Ἐὰν ἄρα χωρίον περιέχηται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ἧς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ τὸ χωρίον δυναμένη ῥητή ἐστιν.

Thus, if an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome, then the square-root of the area is a rational (straight-line).

Πόρισμα.

Καὶ γέγονεν ἡμῖν καὶ διὰ τούτου φανερόν, ὅτι δυνατόν ἐστι ῥητὸν χωρίον ὑπὸ ἀλόγων εὐθειῶν περιέχεσθαι. ὅπερ ἔδει δεῖξαι.

Corollary

And it has also been made clear to us, through this, that it is possible for a rational area to be contained by irrational straight-lines. (Which is) the very thing it was required to show.

ριεʹ.

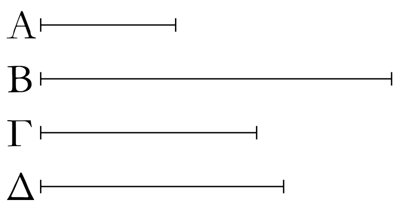
Ἀπὸ μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾷ τῶν πρότερον ἡ αὐτή.

Ἔστω μέση ἡ Α· λέγω, ὅτι ἀπὸ τῆς Α ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾷ τῶν πρότερον ἡ αὐτή.

Proposition 115

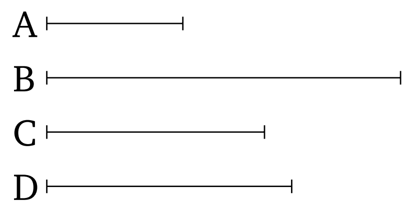
An infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and none of them is the same as any of the preceding (straight-lines).

Let A be a medial (straight-line). I say that an infinite (series) of irrational (straight-lines) can be created from A, and that none of them is the same as any of the preceding (straight-



Ἐκκείσθω ῥητὴ ἡ Β, καὶ τῷ ὑπὸ τῶν Β, Α ἴσον ἔστω τὸ ἀπὸ τῆς Γ· ἄλογος ἄρα ἐστὶν ἡ Γ· τὸ γὰρ ὑπὸ ἀλόγου καὶ ῥητῆς ἄλογόν ἐστιν. καὶ οὐδεμιᾷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπ᾿ οὐδεμιᾶς τῶν πρότερον παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ μέσην. πάλιν δὴ τῷ ὑπὸ τῶν Β, Γ ἴσον ἔστω τὸ ἀπὸ τῆς Δ· ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς Δ. ἄλογος ἄρα ἐστὶν ἡ Δ· καὶ οὐδεμιᾷ τῶν πρότερον ἡ αὐτή· τὸ γὰρ ἀπ᾿ οὐδεμιᾶς τῶν πρότερον παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν Γ. ὁμοίως δὴ τῆς τοιαύτης τάξεως ἐπ᾿ ἄπειρον προβαινούσης φανερόν, ὅτι ἀπὸ τῆς μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾷ τῶν πρότερον ἡ αὐτή· ὅπερ ἔδει δεῖξαι.

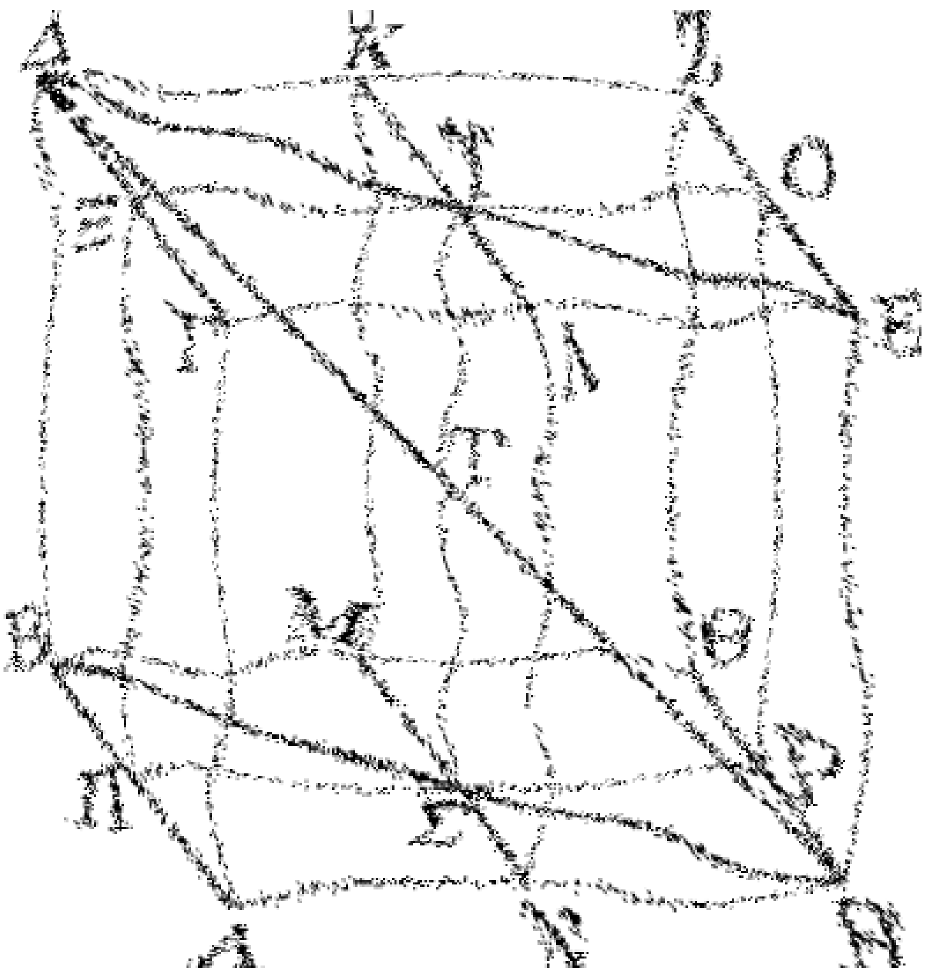
lines).



Let the rational (straight-line) B be laid down. And let the (square) on C be equal to the (rectangle contained) by B and A. Thus, C is irrational [Def. 10.4]. For an (area contained) by an irrational and a rational (straight-line) is irrational [Prop. 10.20]. And (C is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces a medial (straight-line) as breadth. So, again, let the (square) on D be equal to the (rectangle contained) by B and C. Thus, the (square) on D is irrational [Prop. 10.20]. D is thus irrational [Def. 10.4]. And (D is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces C as breadth. So, similarly, this arrangement being advanced to infinity, it is clear that an infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and that none of them is the same as any of the preceding (straight-lines). (Which is) the very thing it was required to show.

ELEMENTS BOOK 11

Elementary Stereometry



Ὅροι.

α΄. Στερεόν ἐστι τὸ μῆκος καὶ πλάτος καὶ βάθος ἔχον.

β΄. Στερεοῦ δὲ πέρας ἐπιφάνεια.

γ΄. Εὐθεῖα πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ [ὑποκειμένῳ] ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας.

δ΄. Ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν.

ε΄. Εὐθείας πρὸς ἐπίπεδον κλίσις ἐστίν, ὅταν ἀπὸ τοῦ μετεώρου πέρατος τῆς εὐθείας ἐπὶ τὸ ἐπίπεδον κάθετος ἀχθῇ, καὶ ἀπὸ τοῦ γενομένου σημείου ἐπὶ τὸ ἐν τῷ ἐπιπέδῳ πέρας τῆς εὐθείας εὐθεῖα ἐπιζευχθῇ, ἡ περιεχομένη γωνία ὑπὸ τῆς ἀχθείσης καὶ τῆς ἐφεστώσης.

ϛ΄. Ἐπιπέδου πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεῖα γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῇ κοινῇ τομῇ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων.

ζ΄. Ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίσθαι λέγεται καὶ ἕτερον πρὸς ἕτερον, ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ὦσιν.

η΄. Παράλληλα ἐπίπεδά ἐστι τὰ ἀσύμπτωτα.

θ΄. Ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τὸ πλῆθος.

ι΄. Ἴσα δὲ καὶ ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

ια΄. Στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐν τῇ αὐτῇ ἐπιφανείᾳ οὐσῶν πρὸς πάσαις ταῖς γραμμαῖς κλίσις. ἄλλως· στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γωνιῶν ἐπιπέδων περιεχομένη μὴ οὐσῶν ἐν τῷ αὐτῷ ἐπιπέδῳ πρὸς ἑνὶ σημείῳ συνισταμένων.

ιβ΄. Πυραμίς ἐστι σχῆμα στερεὸν ἐπιπέδοις περιχόμενον ἀπὸ ἑνὸς ἐπιπέδου πρὸς ἑνὶ σημείῳ συνεστώς.

ιγ΄. Πρίσμα ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον, ὧν δύο τὰ ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστι καὶ παράλληλα, τὰ δὲ λοιπὰ παραλληλόγραμμα.

ιδ΄. Σφαῖρά ἐστιν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

ιε΄. Ἄξων δὲ τῆς σφαίρας ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ ἡμικύκλιον στρέφεται.

ιϛ΄. Κέντρον δὲ τῆς σφαίρας ἐστὶ τὸ αὐτό, ὃ καὶ τοῦ ἡμικυκλίου.

ιζ΄. Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾿ ἑκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

ιη΄. Κῶνός ἐστιν, ὅταν ὀρθογωνίου τριγώνου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ τρίγωνον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα. κἂν μὲν ἡ μένουσα εὐθεῖα ἴση ᾖ τῇ λοιπῇ

Definitions

1. A solid is a (figure) having length and breadth and depth.

2. The extremity of a solid (is) a surface.

3. A straight-line is at right-angles to a plane when it makes right-angles with all of the straight-lines joined to it which are also in the plane.

4. A plane is at right-angles to a(nother) plane when (all of) the straight-lines drawn in one of the planes, at right-angles to the common section of the planes, are at right-angles to the remaining plane.

5. The inclination of a straight-line to a plane is the angle contained by the drawn and standing (straight-lines), when a perpendicular is lead to the plane from the end of the (standing) straight-line raised (out of the plane), and a straight-line is (then) joined from the point (so) generated to the end of the (standing) straight-line (lying) in the plane.

6. The inclination of a plane to a(nother) plane is the acute angle contained by the (straight-lines), (one) in each of the planes, drawn at right-angles to the common segment (of the planes), at the same point.

7. A plane is said to be similarly inclined to a plane, as another to another, when the aforementioned angles of inclination are equal to one another.

8. Parallel planes are those which do not meet (one another).

9. Similar solid figures are those contained by equal numbers of similar planes (which are similarly arranged).

10. But equal and similar solid figures are those contained by similar planes equal in number and in magnitude (which are similarly arranged).

11. A solid angle is the inclination (constituted) by more than two lines joining one another (at the same point), and not being in the same surface, to all of the lines. Otherwise, a solid angle is that contained by more than two plane angles, not being in the same plane, and constructed at one point.

12. A pyramid is a solid figure, contained by planes, (which is) constructed from one plane to one point.

13. A prism is a solid figure, contained by planes, of which the two opposite (planes) are equal, similar, and parallel, and the remaining (planes are) parallelograms.

14. A sphere is the figure enclosed when, the diameter of a semicircle remaining (fixed), the semicircle is carried around, and again established at the same (position) from which it began to be moved.

15. And the axis of the sphere is the fixed straight-line about which the semicircle is turned.

16. And the center of the sphere is the same as that of the semicircle.

17. And the diameter of the sphere is any straight-line

[τῇ] περὶ τὴν ὀρθὴν περιφερομένῃ, ὀρθογώνιος ἔσται ὁ κῶνος, ἐὰν δὲ ἐλάττων, ἀμβλυγώνιος, ἐὰν δὲ μείζων, ὀξυγώνιος.

ιθʹ. Ἄξων δὲ τοῦ κώνου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ τρίγωνον στρέφεται.

κʹ. Βάσις δὲ ὁ κύκλος ὁ ὑπὸ τῆς περιφερομένης εὐθείας γραφόμενος.

καʹ. Κύλινδρός ἐστιν, ὅταν ὀρθογωνίου παραλληλογράμμου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ παραλληλόγραμμον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

κβʹ. Ἄξων δὲ τοῦ κυλίνδρου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ παραλληλόγραμμον στρέφεται.

κγʹ. Βάσεις δὲ οἱ κύκλοι οἱ ὑπὸ τῶν ἀπεναντίον περιαγομένων δύο πλευρῶν γραφόμενοι.

κδʹ. Ὅμοιοι κῶνοι καὶ κύλινδροί εἰσιν, ὧν οἵ τε ἄξονες καὶ αἱ διάμετροι τῶν βάσεων ἀνάλογόν εἰσιν.

κεʹ. Κύβος ἐστὶ σχῆμα στερεὸν ὑπὸ ἓξ τετραγώνων ἴσων περιεχόμενον.

κϛʹ. Ὀκτάεδρόν ἐστὶ σχῆμα στερεὸν ὑπὸ ὀκτὼ τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κζʹ. Εἰκοσάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ εἴκοσι τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κηʹ. Δωδεκάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἴσων καὶ ἰσοπλεύρων καὶ ἰσογωνίων περιεχόμενον.

which is drawn through the center and terminated in both directions by the surface of the sphere.

18. A cone is the figure enclosed when, one of the sides of a right-angled triangle about the right-angle remaining (fixed), the triangle is carried around, and again established at the same (position) from which it began to be moved. And if the fixed straight-line is equal to the remaining (straight-line) about the right-angle, (which is) carried around, then the cone will be right-angled, and if less, obtuse-angled, and if greater, acute-angled.

19. And the axis of the cone is the fixed straight-line about which the triangle is turned.

20. And the base (of the cone is) the circle described by the (remaining) straight-line (about the right-angle which is) carried around (the axis).

21. A cylinder is the figure enclosed when, one of the sides of a right-angled parallelogram about the right-angle remaining (fixed), the parallelogram is carried around, and again established at the same (position) from which it began to be moved.

22. And the axis of the cylinder is the stationary straight-line about which the parallelogram is turned.

23. And the bases (of the cylinder are) the circles described by the two opposite sides (which are) carried around.

24. Similar cones and cylinders are those for which the axes and the diameters of the bases are proportional.

25. A cube is a solid figure contained by six equal squares.

26. An octahedron is a solid figure contained by eight equal and equilateral triangles.

27. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

28. A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

αʹ.

Εὐθείας γραμμῆς μέρος μέν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δέ τι ἐν μετεωροτέρῳ.

Εἰ γὰρ δυνατόν, εὐθείας γραμμῆς τῆς ΑΒΓ μέρος μέν τι τὸ ΑΒ ἔστω ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δέ τι τὸ ΒΓ ἐν μετεωροτέρῳ.

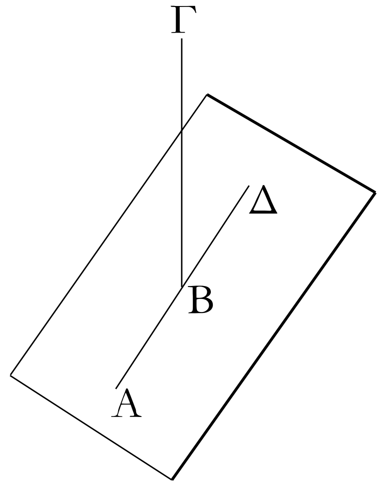
Ἔσται δή τις τῇ ΑΒ συνεχὴς εὐθεῖα ἐπ᾽ εὐθείας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ. ἔστω ἡ ΒΔ· δύο ἄρα εὐθειῶν τῶν ΑΒΓ, ΑΒΔ κοινὸν τμῆμά ἐστιν ἡ ΑΒ· ὅπερ ἐστὶν ἀδύνατον, ἐπειδήπερ ἐὰν κέντρῳ τῷ Β καὶ διαστήματι τῷ ΑΒ κύκλον γράψωμεν, αἱ διάμετροι ἀνίσους ἀπολήψονται τοῦ κύκλου περιφερείας.

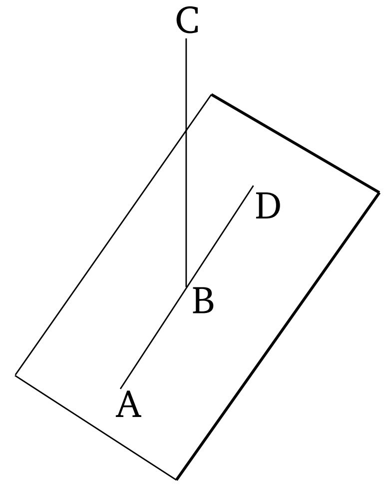
Proposition 1†

Some part of a straight-line cannot be in a reference plane, and some part in a more elevated (plane).

For, if possible, let some part, AB, of the straight-line ABC be in a reference plane, and some part, BC, in a more elevated (plane).

In the reference plane, there will be some straight-line continuous with, and straight-on to, AB.‡ Let it be BD. Thus, AB is a common segment of the two (different) straight-lines ABC and ABD. The very thing is impossible, inasmuch as if we draw a circle with center B and radius AB then the diameters (ABD and ABC) will cut off unequal circumferences of the circle.





Εὐθείας ἄρα γραμμῆς μέρος μέν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν μετεωροτέρῳ· ὅπερ ἔδει δεῖξαι.

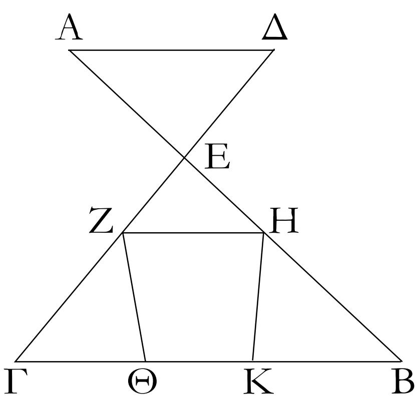
Thus, some part of a straight-line cannot be in a reference plane, and (some part) in a more elevated (plane). (Which is) the very thing it was required to show.

† The proofs of the first three propositions in this book are not at all rigorous. Hence, these three propositions should properly be regarded as additional axioms.

‡ This assumption essentially presupposes the validity of the proposition under discussion.

β′.

Ἐὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.

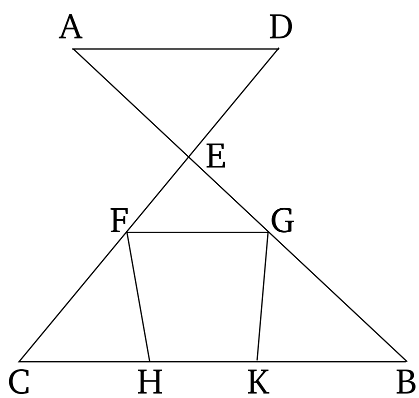


Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον. λέγω, ὅτι αἱ ΑΒ, ΓΔ ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.

Εἰλήφθω γὰρ ἐπὶ τῶν ΕΓ, ΕΒ τυχόντα σημεῖα τὰ Ζ, Η, καὶ ἐπεζεύχθωσαν αἱ ΓΒ, ΖΗ, καὶ διήχθωσαν αἱ ΖΘ, ΗΚ· λέγω πρῶτον, ὅτι τὸ ΕΓΒ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ. εἰ γάρ ἐστι τοῦ ΕΓΒ τριγώνου μέρος ἤτοι τὸ ΖΘΓ ἢ τὸ ΗΒΚ ἐν τῷ ὑποκειμένῳ [ἐπιπέδῳ], τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ μιᾶς τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν αλλῳ. εἰ δὲ τοῦ ΕΓΒ τριγώνου τὸ ΖΓΒΗ μέρος ᾖ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ ἀμφοτέρων τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν

Proposition 2

If two straight-lines cut one another then they are in one plane, and every triangle (formed using segments of both lines) is in one plane.



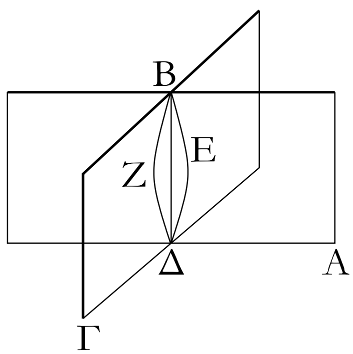
For let the two straight-lines AB and CD have cut one another at point E. I say that AB and CD are in one plane, and that every triangle (formed using segments of both lines) is in one plane.

For let the random points F and G be taken on EC and EB (respectively). And let CB and FG be joined, and let FH and GK be drawn across. I say, first of all, that triangle ECB is in one (reference) plane. For if part of triangle ECB, either FHC or GBK, is in the reference [plane], and the remainder in a different (plane) then a part of one the straight-lines EC and EB will also be in the reference plane, and (a part) in a different (plane). And if the part $FCBG$ of triangle ECB is in *τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν ἄλλῳ· ὅπερ ἄτοπον ἐδείχθη. τὸ ἄρα* $E\Gamma B$ *τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ. ἐν ᾧ δέ ἐστι τὸ* $E\Gamma B$ *τρίγωνον, ἐν τούτῳ καὶ ἑκατέρα τῶν* $E\Gamma$, EB, *ἐν ᾧ δὲ ἑκατέρα τῶν* $E\Gamma$, EB, *ἐν τούτῳ καὶ αἱ* AB, $\Gamma\Delta$. *αἱ* AB, $\Gamma\Delta$ *ἄρα εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.*

the reference plane, and the remainder in a different (plane) then parts of both of the straight-lines EC and EB will also be in the reference plane, and (parts) in a different (plane). The very thing was shown to be absurb [Prop. 11.1]. Thus, triangle ECB is in one plane. And in whichever (plane) triangle ECB is (found), in that (plane) EC and EB (will) each also (be found). And in whichever (plane) EC and EB (are) each (found), in that (plane) AB and CD (will) also (be found) [Prop. 11.1]. Thus, the straight-lines AB and CD are in one plane, and every triangle (formed using segments of both lines) is in one plane. (Which is) the very thing it was required to show.

γ΄.

Ἐὰν δύο ἐπίπεδα τεμνῇ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν.



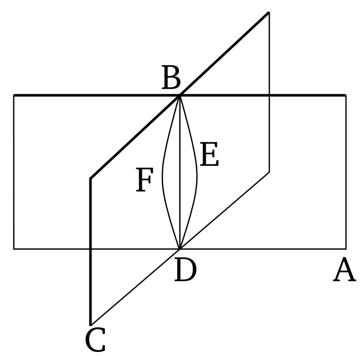
Δύο γὰρ ἐπίπεδα τὰ AB, $B\Gamma$ *τεμνέτω ἄλληλα, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ* ΔB *γραμμή· λέγω, ὅτι ἡ* ΔB *γραμμὴ εὐθεῖά ἐστιν.*

Εἰ γὰρ μή, ἐπεζεύχθω ἀπὸ τοῦ Δ *ἐπὶ τὸ* B *ἐν μὲν τῷ* AB *ἐπιπέδῳ εὐθεῖα ἡ* ΔEB, *ἐν δὲ τῷ* $B\Gamma$ *ἐπιπέδῳ εὐθεῖα ἡ* ΔZB. *ἔσται δὴ δύο εὐθειῶν τῶν* ΔEB, ΔZB *τὰ αὐτὰ πέρατα, καὶ περιέξουσι δηλαδὴ χωρίον· ὅπερ ἄτοπον. οὐκ ἄρα αἱ* ΔEB, ΔZB *εὐθεῖαί εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλη τις ἀπὸ τοῦ* Δ *ἐπὶ τὸ* B *ἐπιζευγνυμένη εὐθεῖα ἔσται πλὴν τῆς* ΔB *κοινῆς τομῆς τῶν* AB, $B\Gamma$ *ἐπιπέδων.*

Ἐὰν ἄρα δύο ἐπίπεδα τέμνῃ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν· ὅπερ ἔδει δεῖξαι.

Proposition 3

If two planes cut one another then their common section is a straight-line.



For let the two planes AB and BC cut one another, and let their common section be the line DB. I say that the line DB is straight.

For, if not, let the straight-line DEB be joined from D to B in the plane AB, and the straight-line DFB in the plane BC. So two straight-lines, DEB and DFB, will have the same ends, and they will clearly enclose an area. The very thing (is) absurd. Thus, DEB and DFB are not straight-lines. So, similarly, we can show than no other straight-line can be joined from D to B except DB, the common section of the planes AB and BC.

Thus, if two planes cut one another then their common section is a straight-line. (Which is) the very thing it was required to show.

δ΄.

Ἐὰν εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι᾿ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Εὐθεῖα γάρ τις ἡ EZ *δύο εὐθείαις ταῖς* AB, $\Gamma\Delta$ *τεμνούσαις ἀλλήλας κατὰ τὸ* E *σημεῖον ἀπὸ τοῦ* E *πρὸς ὀρθὰς ἐφεστάτω· λέγω, ὅτι ἡ* EZ *καὶ τῷ διὰ τῶν* AB, $\Gamma\Delta$ *ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.*

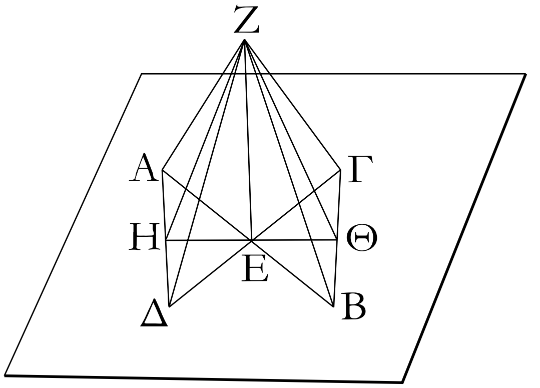
Ἀπειλήφθωσαν γὰρ αἱ AE, EB, ΓE, $E\Delta$ *ἴσαι ἀλλήλαις, καὶ διήχθω τις διὰ τοῦ* E, *ὡς ἔτυχεν, ἡ* $HE\Theta$, *καὶ ἐπε-*

Proposition 4

If a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both).

For let some straight-line EF have (been) set up at right-angles to two straight-lines, AB and CD, cutting one another at point E, at E. I say that EF is also at right-angles to the plane (passing) through AB and CD.

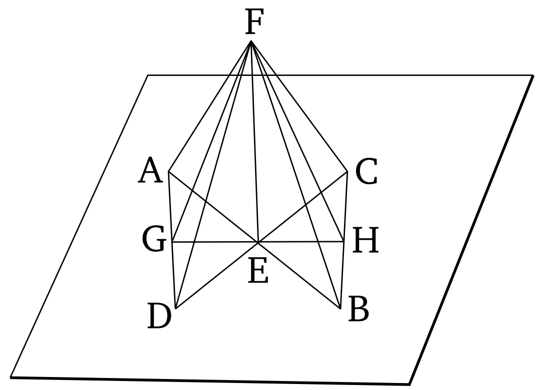
For let AE, EB, CE and ED be cut off from (the two *ζεύχθωσαν αἱ ΑΔ, ΓΒ, καὶ ἔτι ἀπὸ τυχόντος τοῦ Ζ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΗ, ΖΔ, ΖΓ, ΖΘ, ΖΒ.*



Καὶ ἐπεὶ δύο αἱ ΑΕ, ΕΔ δυσὶ ταῖς ΓΕ, ΕΒ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΑΔ βάσει τῇ ΓΒ ἴση ἐστίν, καὶ τὸ ΑΕΔ τρίγωνον τῷ ΓΕΒ τριγώνῳ ἴσον ἔσται· ὥστε καὶ γωνία ἡ ὑπὸ ΔΑΕ γωνίᾳ τῇ ὑπὸ ΕΒΓ ἴση [ἐστίν]. ἔστι δὲ καὶ ἡ ὑπὸ ΑΕΗ γωνία τῇ ὑπὸ ΒΕΘ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΑΗΕ, ΒΕΘ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΑΕ τῇ ΕΒ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση ἄρα ἡ μὲν ΗΕ τῇ ΕΘ, ἡ δὲ ΑΗ τῇ ΒΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΖΕ, βάσις ἄρα ἡ ΖΑ βάσει τῇ ΖΒ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΖΓ τῇ ΖΔ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΓΒ, ἔστι δὲ καὶ ἡ ΖΑ τῇ ΖΒ ἴση, δύο δὴ αἱ ΖΑ, ΑΔ δυσὶ ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΖΔ βάσει τῇ ΖΓ ἐδείχθη ἴση· καὶ γωνία ἄρα ἡ ὑπὸ ΖΑΔ γωνίᾳ τῇ ὑπὸ ΖΒΓ ἴση ἐστίν. καὶ ἐπεὶ πάλιν ἐδείχθη ἡ ΑΗ τῇ ΒΘ ἴση, ἀλλὰ μὴν καὶ ἡ ΖΑ τῇ ΖΒ ἴση, δύο δὴ αἱ ΖΑ, ΑΗ δυσὶ ταῖς ΖΒ, ΒΘ ἴσαι εἰσίν. καὶ γωνία ἡ ὑπὸ ΖΑΗ ἐδείχθη ἴση τῇ ὑπὸ ΖΒΘ· βάσις ἄρα ἡ ΖΗ βάσει τῇ ΖΘ ἐστιν ἴση. καὶ ἐπεὶ πάλιν ἴση ἐδείχθη ἡ ΗΕ τῇ ΕΘ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἱ ΗΕ, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσίν· καὶ βάσις ἡ ΖΗ βάσει τῇ ΖΘ ἴση· γωνία ἄρα ἡ ὑπὸ ΗΕΖ γωνίᾳ τῇ ὑπὸ ΘΕΖ ἴση ἐστίν. ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ ΗΕΖ, ΘΕΖ γωνιῶν. ἡ ΖΕ ἄρα πρὸς τὴν ΗΘ τυχόντως διὰ τοῦ Ε ἀχθεῖσαν ὀρθή ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι ἡ ΖΕ καὶ πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. εὐθεῖα δὲ πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ αὐτῷ ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας· ἡ ΖΕ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ ὑποκείμενον ἐπίπεδόν ἐστι τὸ διὰ τῶν ΑΒ, ΓΔ εὐθειῶν. ἡ ΖΕ ἄρα πρὸς ὀρθάς ἐστι τῷ διὰ τῶν ΑΒ, ΓΔ ἐπιπέδῳ.

Ἐὰν ἄρα εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς

straight-lines so as to be) equal to one another. And let GEH be drawn, at random, through E (in the plane passing through AB and CD). And let AD and CB be joined. And, furthermore, let FA, FG, FD, FC, FH, and FB be joined from the random (point) F (on EF).



For since the two (straight-lines) AE and ED are equal to the two (straight-lines) CE and EB, and they enclose equal angles [Prop. 1.15], the base AD is thus equal to the base CB, and triangle AED will be equal to triangle CEB [Prop. 1.4]. Hence, the angle DAE [is] equal to the angle EBC. And the angle AEG (is) also equal to the angle BEH [Prop. 1.15]. So AGE and BEH are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), those by the equal angles, AE and EB. Thus, they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, GE (is) equal to EH, and AG to BH. And since AE is equal to EB, and FE is common and at right-angles, the base FA is thus equal to the base FB [Prop. 1.4]. So, for the same (reasons), FC is also equal to FD. And since AD is equal to CB, and FA is also equal to FB, the two (straight-lines) FA and AD are equal to the two (straight-lines) FB and BC, respectively. And the base FD was shown (to be) equal to the base FC. Thus, the angle FAD is also equal to the angle FBC [Prop. 1.8]. And, again, since AG was shown (to be) equal to BH, but FA (is) also equal to FB, the two (straight-lines) FA and AG are equal to the two (straight-lines) FB and BH (respectively). And the angle FAG was shown (to be) equal to the angle FBH. Thus, the base FG is equal to the base FH [Prop. 1.4]. And, again, since GE was shown (to be) equal to EH, and EF (is) common, the two (straight-lines) GE and EF are equal to the two (straight-lines) HE and EF (respectively). And the base FG (is) equal to the base FH. Thus, the angle GEF is equal to the angle HEF [Prop. 1.8]. Each of the angles GEF and HEF (are) thus right-angles [Def. 1.10]. Thus, FE is at right-angles to GH, which was drawn at random through E (in the reference plane passing though AB and AC). So, similarly, we can show that FE will make right-angles with all straight-lines joined to it which are in the reference plane.

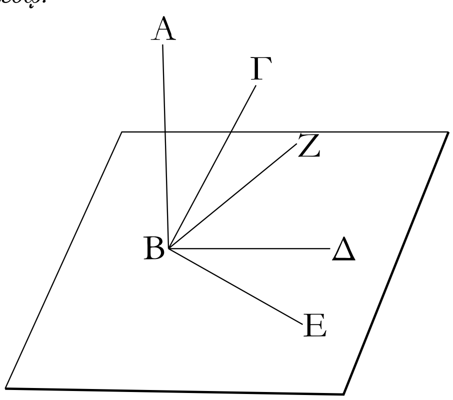
ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι᾽ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

And a straight-line is at right-angles to a plane when it makes right-angles with all straight-lines joined to it which are in the plane [Def. 11.3]. Thus, FE is at right-angles to the reference plane. And the reference plane is that (passing) through the straight-lines AB and CD. Thus, FE is at right-angles to the plane (passing) through AB and CD.

Thus, if a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both). (Which is) the very thing it was required to show.

ε΄.

Ἐὰν εὐθεῖα τρισὶν εὐθείαις ἁπτομέναις ἀλλήλων πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ.



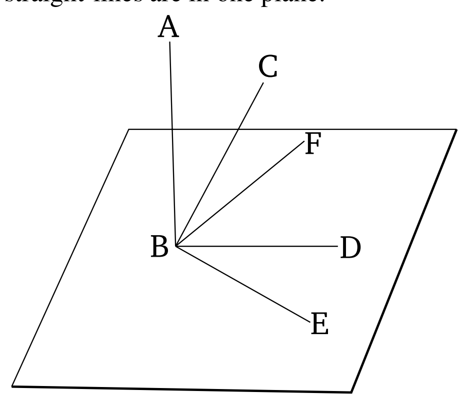
Εὐθεῖα γάρ τις ἡ ΑΒ τρισὶν εὐθείαις ταῖς ΒΓ, ΒΔ, ΒΕ πρὸς ὀρθὰς ἐπὶ τῆς κατὰ τὸ Β ἁφῆς ἐφεστάτω· λέγω, ὅτι αἱ ΒΓ, ΒΔ, ΒΕ ἐν ἑνί εἰσιν ἐπιπέδῳ.

μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, ἔστωσαν αἱ μὲν ΒΔ, ΒΕ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, ἡ δὲ ΒΓ ἐν μετεωροτέρῳ, καὶ ἐκβεβλήσθω τὸ δία τῶν ΑΒ, ΒΓ ἐπίπεδον· κοινὴν δὴ τομὴν ποιήσει ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν ΒΖ. ἐν ἑνὶ ἄρα εἰσὶν ἐπιπέδῳ τῷ διηγμένῳ διὰ τῶν ΑΒ, ΒΓ αἱ τρεῖς εὐθεῖαι αἱ ΑΒ, ΒΓ, ΒΖ. καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΒΔ, ΒΕ, καὶ τῷ διὰ τῶν ΒΔ, ΒΕ ἄρα ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ. τὸ δὲ διὰ τῶν ΒΔ, ΒΕ ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ ΑΒ ἄρα ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὥστε καὶ πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ ΑΒ. ἅπτεται δὲ αὐτῆς ἡ ΒΖ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΑΒΖ γωνία ὀρθή ἐστιν. ὑπόκειται δὲ καὶ ἡ ὑπὸ ΑΒΓ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΑΒΖ γωνία τῇ ὑπὸ ΑΒΓ. καί εἰσιν ἐν ἑνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ΒΓ εὐθεῖα ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΒΓ, ΒΔ, ΒΕ ἐν ἑνί εἰσιν ἐπιπέδῳ.

Ἐὰν ἄρα εὐθεῖα τρισὶν εὐθείαις ἁπτομέναις ἀλλήλων ἐπὶ τῆς ἁφῆς πρὸς ὀρθὰς ἐπισταθῇ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν

Proposition 5

If a straight-line is set up at right-angles to three straight-lines cutting one another, at the common point of section, then the three straight-lines are in one plane.



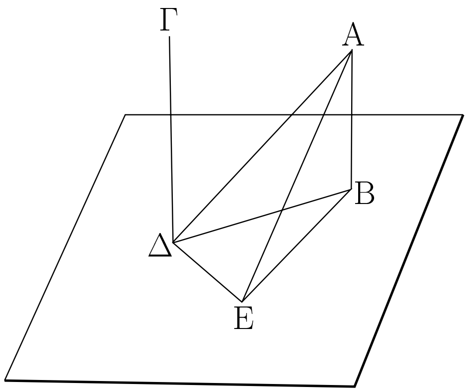
For let some straight-line AB be set up at right-angles to three straight-lines BC, BD, and BE, at the (common) point of section B. I say that BC, BD, and BE are in one plane.

For (if) not, and if possible, let BD and BE be in the reference plane, and BC in a more elevated (plane). And let the plane through AB and BC be produced. So it will make a straight-line as a common section with the reference plane [Def. 11.3]. Let it make BF. Thus, the three straight-lines AB, BC, and BF are in one plane—(namely), that drawn through AB and BC. And since AB is at right-angles to each of BD and BE, AB is thus also at right-angles to the plane (passing) through BD and BE [Prop. 11.4]. And the plane (passing) through BD and BE is the reference plane. Thus, AB is at right-angles to the reference plane. Hence, AB will also make right-angles with all straight-lines joined to it which are also in the reference plane [Def. 11.3]. And BF, which is in the reference plane, is joined to it. Thus, the angle ABF is a right-angle. And ABC was also assumed to be a right-angle. Thus, angle ABF (is) equal to ABC. And they are in one plane. The very thing is impossible. Thus, BC is not in a more elevated plane. Thus, the three straight-lines BC, BD, and BE are in one plane.

ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.

ϛʹ.

Ἐὰν δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι.



Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστωσαν· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Συμβαλλέτωσαν γὰρ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ Β, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ ΒΔ εὐθεῖα, καὶ ἤχθω τῇ ΒΔ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΕ, ΑΕ, ΑΔ.

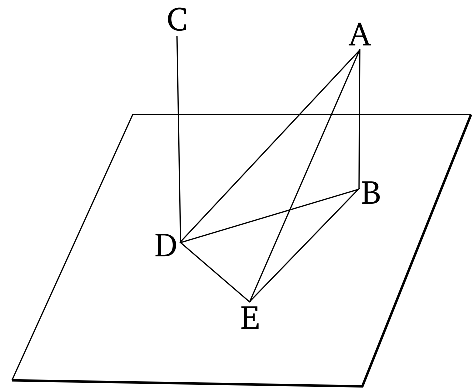
Καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας [ἄρα] τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ τῆς ΑΒ ἑκατέρα τῶν ΒΔ, ΒΕ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΑΒΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΔΒ, ΓΔΕ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, κοινὴ δὲ ἡ ΒΔ, δύο δὴ αἱ ΑΒ, ΒΔ δυσὶ ταῖς ΕΔ, ΔΒ ἴσαι εἰσίν· καὶ γωνίας ὀρθὰς περιέχουσιν· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΕ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, ἀλλὰ καὶ ἡ ΑΔ τῇ ΒΕ, δύο δὴ αἱ ΑΒ, ΒΕ δυσὶ ταῖς ΕΔ, ΔΑ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνιᾷ τῇ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΕ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΔΑ· ἡ ΕΔ ἄρα πρὸς τὴν ΔΑ ὀρθή ἐστιν. ἔστι δὲ καὶ πρὸς ἑκατέραν τῶν ΒΔ, ΔΓ ὀρθή. ἡ ΕΔ ἄρα τρισὶν εὐθείαις ταῖς ΒΔ, ΔΑ, ΔΓ πρὸς ὀρθὰς ἐπὶ τῆς ἁφῆς ἐφέστηκεν· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΒΔ, ΔΑ, ΔΓ ἐν ἑνί εἰσιν ἐπιπέδῳ. ἐν ᾧ δὲ αἱ ΔΒ, ΔΑ, ἐν τούτῳ καὶ ἡ ΑΒ· πᾶν γὰρ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ· αἱ ἄρα ΑΒ, ΒΔ, ΔΓ εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΒΔΓ γωνιῶν· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Ἐὰν ἄρα δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

Thus, if a straight-line is set up at right-angles to three straight-lines cutting one another, at the (common) point of section, then the three straight-lines are in one plane. (Which is) the very thing it was required to show.

Proposition 6

If two straight-lines are at right-angles to the same plane then the straight-lines will be parallel.†



For let the two straight-lines *AB* and *CD* be at right-angles to a reference plane. I say that *AB* is parallel to *CD*.

For let them meet the reference plane at points *B* and *D* (respectively). And let the straight-line *BD* be joined. And let *DE* be drawn at right-angles to *BD* in the reference plane. And let *DE* be made equal to *AB*. And let *BE*, *AE*, and *AD* be joined.

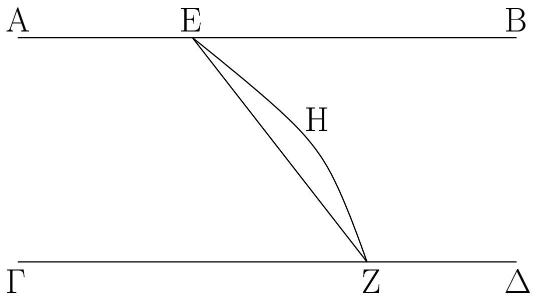
And since *AB* is at right-angles to the reference plane, it will [thus] also make right-angles with all straight-lines joined to it which are in the reference plane [Def. 11.3]. And *BD* and *BE*, which are in the reference plane, are each joined to *AB*. Thus, each of the angles *ABD* and *ABE* are right-angles. So, for the same (reasons), each of the angles *CDB* and *CDE* are also right-angles. And since *AB* is equal to *DE*, and *BD* (is) common, the two (straight-lines) *AB* and *BD* are equal to the two (straight-lines) *ED* and *DB* (respectively). And they contain right-angles. Thus, the base *AD* is equal to the base *BE* [Prop. 1.4]. And since *AB* is equal to *DE*, and *AD* (is) also (equal) to *BE*, the two (straight-lines) *AB* and *BE* are thus equal to the two (straight-lines) *ED* and *DA* (respectively). And their base *AE* (is) common. Thus, angle *ABE* is equal to angle *EDA* [Prop. 1.8]. And *ABE* (is) a right-angle. Thus, *EDA* (is) also a right-angle. *ED* is thus at right-angles to *DA*. And it is also at right-angles to each of *BD* and *DC*. Thus, *ED* is standing at right-angles to the three straight-lines *BD*, *DA*, and *DC* at the (common) point of section. Thus, the three straight-lines *BD*, *DA*, and *DC* are in one plane [Prop. 11.5]. And in which(ever) plane *DB* and *DA* (are found), in that (plane) *AB* (will) also (be found). For every triangle is in one plane [Prop. 11.2]. And each of the angles *ABD* and *BDC* is a right-angle. Thus, *AB* is parallel to *CD* [Prop. 1.28].

Thus, if two straight-lines are at right-angles to the same plane then the straight-lines will be parallel. (Which is) the very thing it was required to show.

† In other words, the two straight-lines lie in the same plane, and never meet when produced in either direction.

ζ΄.

Ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, ληφθῇ δὲ ἐφ᾿ ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.



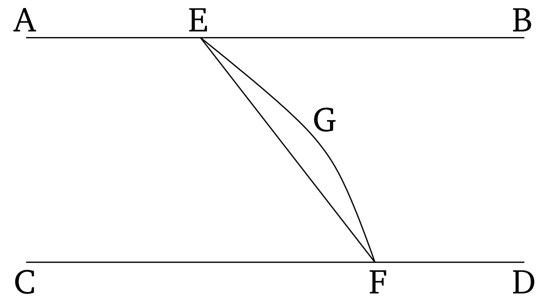
Ἔστωσαν δύο εὐθεῖαι παράλληλοι αἱ ΑΒ, ΓΔ, καὶ εἰλήφθω ἐφ᾿ ἑκατέρας αὐτῶν τυνχόντα σημεῖα τὰ Ε, Ζ· λέγω, ὅτι ἡ ἐπὶ τὰ Ε, Ζ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

μὴ γάρ, ἀλλ᾿ εἰ δυνατόν, ἔστω ἐν μετεωροτέρῳ ὡς ἡ ΕΗΖ, καὶ διήχθω διὰ τῆς ΕΗΖ ἐπίπεδον· τομὴν δὴ ποιήσει ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω ὡς τὴν ΕΖ· δύο ἄρα εὐθεῖαι αἱ ΕΗΖ, ΕΖ χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖαι ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· ἐν τῷ διὰ τῶν ΑΒ, ΓΒ ἄρα παραλλήλων ἐστὶν ἐπιπέδῳ ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖα.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι παράλληλοι, ληφθῇ δὲ ἐφ᾿ ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις· ὅπερ ἔδει δεῖξαι.

Proposition 7

If there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines).



Let *AB* and *CD* be two parallel straight-lines, and let the random points *E* and *F* be taken on each of them (respectively). I say that the straight-line joining points *E* and *F* is in the same (reference) plane as the parallel (straight-lines).

For (if) not, and if possible, let it be in a more elevated (plane), such as *EGF*. And let a plane be drawn through *EGF*. So it will make a straight cutting in the reference plane [Prop. 11.3]. Let it make *EF*. Thus, two straight-lines (with the same end-points), *EGF* and *EF*, will enclose an area. The very thing is impossible. Thus, the straight-line joining *E* to *F* is not in a more elevated plane. The straight-line joining *E* to *F* is thus in the plane through the parallel (straight-lines) *AB* and *CD*.

Thus, if there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines). (Which is) the very thing it was required to show.

η΄.

Ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ ἑτέρα αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Ἔστωσαν δύο εὐθεῖαι παράλληλοι αἱ ΑΒ, ΓΔ, ἡ δὲ ἑτέρα αὐτῶν ἡ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ ἡ λοιπὴ ἡ ΓΔ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

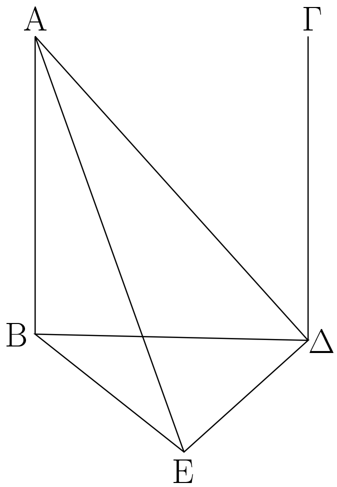
Συμβαλλέτωσαν γὰρ αἱ ΑΒ, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ Β, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ ΒΔ· αἱ ΑΒ, ΓΔ, ΒΔ ἄρα ἐν ἑνί εἰσιν ἐπιπέδῳ. ἤχθω τῇ ΒΑ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΕ, ΑΕ, ΑΔ.

Proposition 8

If two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane.

Let *AB* and *CD* be two parallel straight-lines, and let one of them, *AB*, be at right-angles to a reference plane. I say that the remaining (one), *CD*, will also be at right-angles to the same plane.

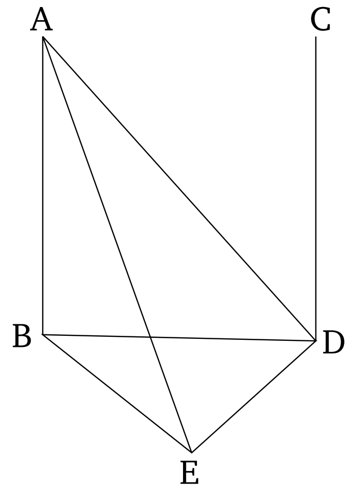
For let *AB* and *CD* meet the reference plane at points *B* and *D* (respectively). And let *BD* be joined. *AB*, *CD*, and *BD* are thus in one plane [Prop. 11.7]. Let *DE* be drawn at right-angles to *BD* in the reference plane, and let *DE* be made equal



Καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν ἡ ΑΒ· ὀρθὴ ἄρα [ἐστὶν] ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΑΒΕ γωνιῶν. καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπέπτωκεν ἡ ΒΔ, αἱ ἄρα ὑπὸ ΑΒΔ, ΓΔΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΔ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΓΔΒ· ἡ ΓΔ ἄρα πρὸς τὴν ΒΔ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, κοινὴ δὲ ἡ ΒΔ, δύο δὴ αἱ ΑΒ, ΒΔ δυσὶ ταῖς ΕΔ, ΔΒ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΒΔ γωνίᾳ τῇ ὑπὸ ΕΔΒ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΕ ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΒ τῇ ΔΕ, ἡ δὲ ΒΕ τῇ ΑΔ, δύο δὴ αἱ ΑΒ, ΒΕ δυσὶ ταῖς ΕΔ, ΔΑ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. καὶ βάσις αὐτῶν κοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνίᾳ τῇ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΕ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΔΑ· ἡ ΕΔ ἄρα πρὸς τὴν ΑΔ ὀρθή ἐστιν. ἔστι δὲ καὶ πρὸς τὴν ΔΒ ὀρθή· ἡ ΕΔ ἄρα καὶ τῷ διὰ τῶν ΒΔ, ΔΑ ἐπιπέδῳ ὀρθή ἐστιν. καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ ΕΔ. ἐν δὲ τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ἐστὶν ἡ ΔΓ, ἐπειδήπερ ἐν τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ἐστὶν αἱ ΑΒ, ΒΔ, ἐν ᾧ δὲ αἱ ΑΒ, ΒΔ, ἐν τούτῳ ἐστὶ καὶ ἡ ΔΓ. ἡ ΕΔ ἄρα τῇ ΔΓ πρὸς ὀρθάς ἐστιν· ὥστε καὶ ἡ ΓΔ τῇ ΔΕ πρὸς ὀρθάς ἐστιν. ἔστι δὲ καὶ ἡ ΓΔ τῇ ΒΔ πρὸς ὀρθάς. ἡ ΓΔ ἄρα δύο εὐθείαις τεμνούσαις ἀλλήλας ταῖς ΔΕ, ΔΒ ἀπὸ τῆς κατὰ τὸ Δ τομῆς πρὸς ὀρθὰς ἐφέστηκεν· ὥστε ἡ ΓΔ καὶ τῷ διὰ τῶν ΔΕ, ΔΒ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΔΕ, ΔΒ ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ ΓΔ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἐὰν ἄρα ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

to *AB*, and let *BE*, *AE*, and *AD* be joined.

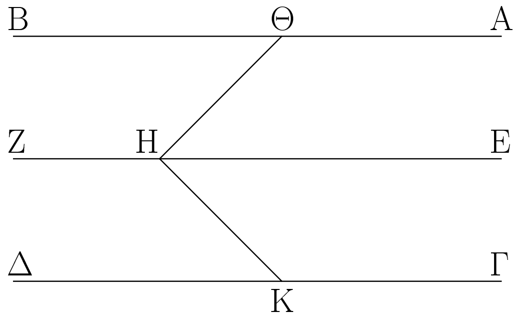


And since *AB* is at right-angles to the reference plane, *AB* is thus also at right-angles to all of the straight-lines joined to it which are in the reference plane [Def. 11.3]. Thus, the angles *ABD* and *ABE* [are] each right-angles. And since the straight-line *BD* has met the parallel (straight-lines) *AB* and *CD*, the (sum of the) angles *ABD* and *CDB* is thus equal to two right-angles [Prop. 1.29]. And *ABD* (is) a right-angle. Thus, *CDB* (is) also a right-angle. *CD* is thus at right-angles to *BD*. And since *AB* is equal to *DE*, and *BD* (is) common, the two (straight-lines) *AB* and *BD* are equal to the two (straight-lines) *ED* and *DB* (respectively). And angle *ABD* (is) equal to angle *EDB*. For each (is) a right-angle. Thus, the base *AD* (is) equal to the base *BE* [Prop. 1.4]. And since *AB* is equal to *DE*, and *BE* to *AD*, the two (sides) *AB*, *BE* are equal to the two (sides) *ED*, *DA*, respectively. And their base *AE* is common. Thus, angle *ABE* is equal to angle *EDA* [Prop. 1.8]. And *ABE* (is) a right-angle. *EDA* (is) thus also a right-angle. Thus, *ED* is at right-angles to *AD*. And it is also at right-angles to *DB*. Thus, *ED* is also at right-angles to the plane through *BD* and *DA* [Prop. 11.4]. And *ED* will thus make right-angles with all of the straight-lines joined to it which are also in the plane through *BDA*. And *DC* is in the plane through *BDA*, inasmuch as *AB* and *BD* are in the plane through *BDA* [Prop. 11.2], and in which(ever plane) *AB* and *BD* (are found), *DC* is also (found). Thus, *ED* is at right-angles to *DC*. Hence, *CD* is also at right-angles to *DE*. And *CD* is also at right-angles to *BD*. Thus, *CD* is standing at right-angles to two straight-lines, *DE* and *DB*, which meet one another, at the (point) of section, *D*. Hence, *CD* is also at right-angles to the plane through *DE* and *DB* [Prop. 11.4]. And the plane through *DE* and *DB* is the reference (plane). *CD* is thus at right-angles to the reference plane.

Thus, if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

θ΄.

Αἱ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι.



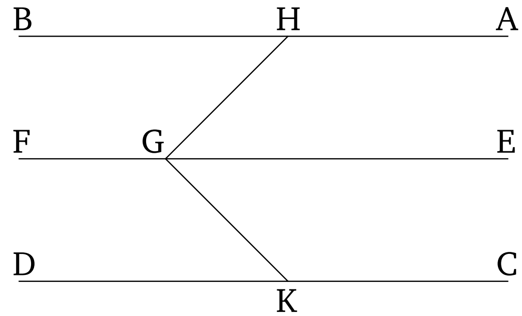
Ἔστω γὰρ ἑκατέρα τῶν ΑΒ, ΓΔ τῇ ΕΖ παράλληλος μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Εἰλήφθω γὰρ ἐπὶ τῆς ΕΖ τυχὸν σημεῖον τὸ Η, καὶ ἀπ᾽ αὐτοῦ τῇ ΕΖ ἐν μὲν τῷ διὰ τῶν ΕΖ, ΑΒ ἐπιπέδῳ πρὸς ὀρθὰς ἤχθω ἡ ΗΘ, ἐν δὲ τῷ διὰ τῶν ΖΕ, ΓΔ τῇ ΕΖ πάλιν πρὸς ὀρθὰς ἤχθω ἡ ΗΚ.

Καὶ ἐπεὶ ἡ ΕΖ πρὸς ἑκατέραν τῶν ΗΘ, ΗΚ ὀρθή ἐστιν, ἡ ΕΖ ἄρα καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν ἡ ΕΖ τῇ ΑΒ παράλληλος· καὶ ἡ ΑΒ ἄρα τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἑκατέρα ἄρα τῶν ΑΒ, ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοί εἰσιν αἱ εὐθεῖαι· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ· ὅπερ ἔδει δεῖξαι.

Proposition 9

(Straight-lines) parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another.



For let AB and CD each be parallel to EF, not being in the same plane as it. I say that AB is parallel to CD.

For let some point G be taken at random on EF. And from it let GH be drawn at right-angles to EF in the plane through EF and AB. And let GK be drawn, again at right-angles to EF, in the plane through FE and CD.

And since EF is at right-angles to each of GH and GK, EF is thus also at right-angles to the plane through GH and GK [Prop. 11.4]. And EF is parallel to AB. Thus, AB is also at right-angles to the plane through HGK [Prop. 11.8]. So, for the same (reasons), CD is also at right-angles to the plane through HGK. Thus, AB and CD are each at right-angles to the plane through HGK. And if two straight-lines are at right–angles to the same plane then the straight-lines are parallel [Prop. 11.6]. Thus, AB is parallel to CD. (Which is) the very thing it was required to show.

ι΄.

Ἐὰν δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΒΓ ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΕ, ΕΖ ἁπτομένας ἀλλήλων ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ.

Ἀπειλήφθωσαν γὰρ αἱ ΒΑ, ΒΓ, ΕΔ, ΕΖ ἴσαι ἀλλήλαις, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΓΖ, ΒΕ, ΑΓ, ΔΖ.

Καὶ ἐπεὶ ἡ ΒΑ τῇ ΕΔ ἴση ἐστὶ καὶ παράλληλος, καὶ ἡ ΑΔ ἄρα τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΖ τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος· ἑκατέρα ἄρα τῶν ΑΔ, ΓΖ τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος. αἱ δὲ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι· παράλληλος ἄρα ἐστὶν ἡ ΑΔ

Proposition 10

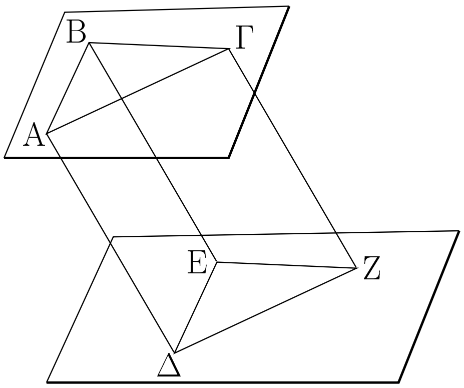
If two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles.

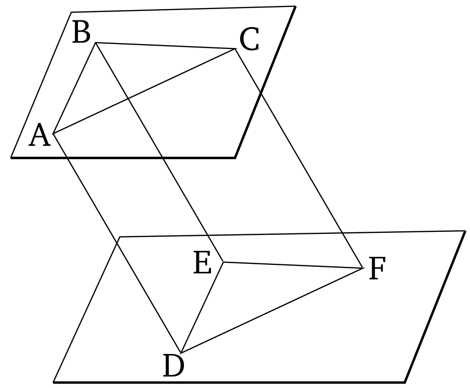
For let the two straight-lines joined to one another, AB and BC, be (respectively) parallel to the two straight-lines joined to one another, DE and EF, (but) not in the same plane. I say that angle ABC is equal to (angle) DEF.

For let BA, BC, ED, and EF be cut off (so as to be, respectively) equal to one another. And let AD, CF, BE, AC, and DF be joined.

And since BA is equal and parallel to ED, AD is thus also equal and parallel to BE [Prop. 1.33]. So, for the same reasons, CF is also equal and parallel to BE. Thus, AD and CF are each equal and parallel to BE. And straight-lines parallel to the same straight-line, and which are not in the same plane as *τῇ ΓΖ καὶ ἴση. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΑΓ, ΔΖ· καὶ ἡ ΑΓ ἄρα τῇ ΔΖ ἴση ἐστὶ καὶ παράλληλος. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ βάσις ἡ ΑΓ βάσει τῇ ΔΖ ἴση, γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση.*

it, are also parallel to one another [Prop. 11.9]. Thus, *AD* is parallel and equal to *CF*. And *AC* and *DF* join them. Thus, *AC* is also equal and parallel to *DF* [Prop. 1.33]. And since the two (straight-lines) *AB* and *BC* are equal to the two (straight-lines) *DE* and *EF* (respectively), and the base *AC* (is) equal to the base *DF*, the angle *ABC* is thus equal to the (angle) *DEF* [Prop. 1.8].





Ἐὰν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν· ὅπερ ἔδει δεῖξαι.

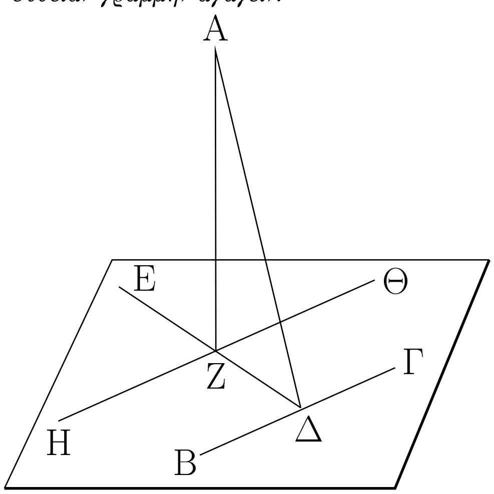
Thus, if two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles. (Which is) the very thing it was required to show.

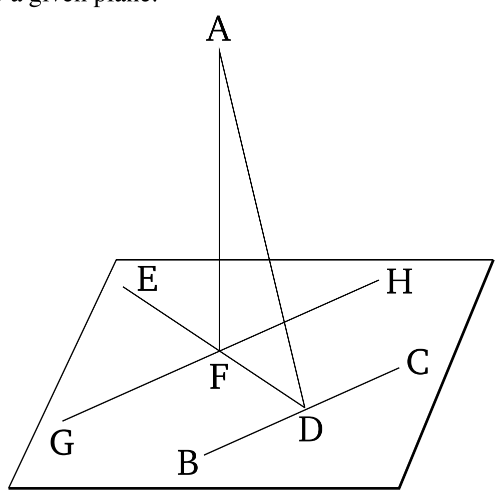
ια´.

Proposition 11

Ἀπὸ τοῦ δοθέντος σημείου μετεώρου ἐπὶ τὸ δοθὲν ἐπίπεδον κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

To draw a perpendicular straight-line from a given raised point to a given plane.





Ἔστω τὸ μὲν δοθὲν σημεῖον μετέωρον τὸ Α, τὸ δὲ δοθὲν ἐπίπεδον τὸ ὑποκείμενον· δεῖ δὴ ἀπὸ τοῦ Α σημείου ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Διήχθω γάρ τις ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖα, ὡς ἔτυχεν, ἡ ΒΓ, καὶ ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ. εἰ μὲν οὖν ἡ ΑΔ κάθετός ἐστι καὶ ἐπὶ τὸ ὑποκείμενον ἐπίπεδον, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ οὔ,

Let *A* be the given raised point, and the given plane the reference (plane). So, it is required to draw a perpendicular straight-line from point *A* to the reference plane.

Let some random straight-line *BC* be drawn across in the reference plane, and let the (straight-line) *AD* be drawn from point *A* perpendicular to *BC* [Prop. 1.12]. If, therefore, *AD* is also perpendicular to the reference plane then that which ἤχθω ἀπὸ τοῦ Δ σημείου τῇ ΒΓ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΔΕ, καὶ ἤχθω ἀπὸ τοῦ Α ἐπὶ τὴν ΔΕ κάθετος ἡ ΑΖ, καὶ διὰ τοῦ Ζ σημείου τῇ ΒΓ παράλληλος ἤχθω ἡ ΗΘ.

Καὶ ἐπεὶ ἡ ΒΓ ἑκατέρᾳ τῶν ΔΑ, ΔΕ πρὸς ὀρθάς ἐστιν, ἡ ΒΓ ἄρα καὶ τῷ διὰ τῶν ΕΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν αὐτῇ παράλληλος ἡ ΗΘ· ἐὰν δὲ ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· καὶ ἡ ΗΘ ἄρα τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΗΘ. ἅπτεται δὲ αὐτῆς ἡ ΑΖ οὖσα ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ· ἡ ΗΘ ἄρα ὀρθή ἐστι πρὸς τὴν ΖΑ· ὥστε καὶ ἡ ΖΑ ὀρθή ἐστι πρὸς τὴν ΘΗ. ἔστι δὲ ἡ ΑΖ καὶ πρὸς τὴν ΔΕ ὀρθή· ἡ ΑΖ ἄρα πρὸς ἑκατέραν τῶν ΗΘ, ΔΕ ὀρθή ἐστιν. ἐὰν δὲ εὐθεῖα δυσὶν εὐθείαις τεμνούσαις ἀλλήλας ἐπὶ τῆς τομῆς πρὸς ὀρθὰς ἐπισταθῇ, καὶ τῷ δι᾽ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ἡ ΖΑ ἄρα τῷ διὰ τῶν ΕΔ, ΗΘ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΕΔ, ΗΘ ἐπίπεδόν ἐστι τὸ ὑποκείμενον· ἡ ΑΖ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἀπὸ τοῦ ἄρα δοθέντος σημείου μετεώρου τοῦ Α ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος εὐθεῖα γραμμὴ ἦκται ἡ ΑΖ· ὅπερ ἔδει ποιῆσαι.

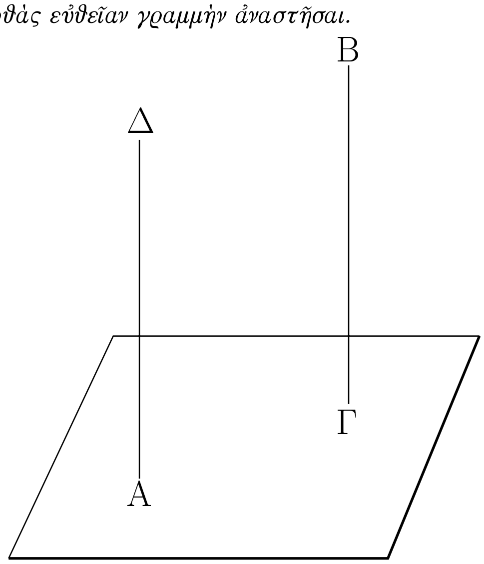
was prescribed will have occurred. And, if not, let *DE* be drawn in the reference plane from point *D* at right-angles to *BC* [Prop. 1.11], and let the (straight-line) *AF* be drawn from *A* perpendicular to *DE* [Prop. 1.12], and let *GH* be drawn through point *F*, parallel to *BC* [Prop. 1.31].

And since *BC* is at right-angles to each of *DA* and *DE*, *BC* is thus also at right-angles to the plane through *EDA* [Prop. 11.4]. And *GH* is parallel to it. And if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (straight-line) will also be at right-angles to the same plane [Prop. 11.8]. Thus, *GH* is also at right-angles to the plane through *ED* and *DA*. And *GH* is thus at right-angles to all of the straight-lines joined to it which are also in the plane through *ED* and *AD* [Def. 11.3]. And *AF*, which is in the plane through *ED* and *DA*, is joined to it. Thus, *GH* is at right-angles to *FA*. Hence, *FA* is also at right-angles to *HG*. And *AF* is also at right-angles to *DE*. Thus, *AF* is at right-angles to each of *GH* and *DE*. And if a straight-line is set up at right-angles to two straight-lines cutting one another, at the point of section, then it will also be at right-angles to the plane through them [Prop. 11.4]. Thus, *FA* is at right-angles to the plane through *ED* and *GH*. And the plane through *ED* and *GH* is the reference (plane). Thus, *AF* is at right-angles to the reference plane.

Thus, the straight-line *AF* has been drawn from the given raised point *A* perpendicular to the reference plane. (Which is) the very thing it was required to do.

ιβ΄.

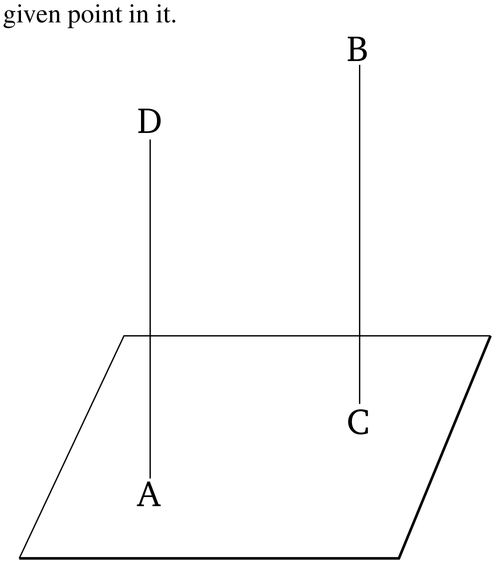
Τῷ δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ δοθέντος σημείου πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.



Ἔστω τὸ μὲν δοθὲν ἐπίπεδον τὸ ὑποκείμενον, τὸ δὲ πρὸς αὐτῷ σημεῖον τὸ Α· δεῖ δὴ ἀπὸ τοῦ Α σημείου τῷ ὑποκειμένῳ

Proposition 12

To set up a straight-line at right-angles to a given plane from a given point in it.



Let the given plane be the reference (plane), and *A* a point in it. So, it is required to set up a straight-line at right-angles *ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.*

Νενοήσθω τι σημεῖον μετέωρον τὸ Β, καὶ ἀπὸ τοῦ Β ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος ἤχθω ἡ ΒΓ, καὶ διὰ τοῦ Α σημείου τῇ ΒΓ παράλληλος ἤχθω ἡ ΑΔ.

Ἐπεὶ οὖν δύο εὐθεῖαι παράλληλοί εἰσιν αἱ ΑΔ, ΓΒ, ἡ δὲ μία αὐτῶν ἡ ΒΓ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, καὶ ἡ λοιπὴ ἄρα ἡ ΑΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Τῷ ἄρα δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ σημείου τοῦ Α πρὸς ὀρθὰς ἀνέσταται ἡ ΑΔ· ὅπερ ἔδει ποιῆσαι.

to the reference plane at point *A*.

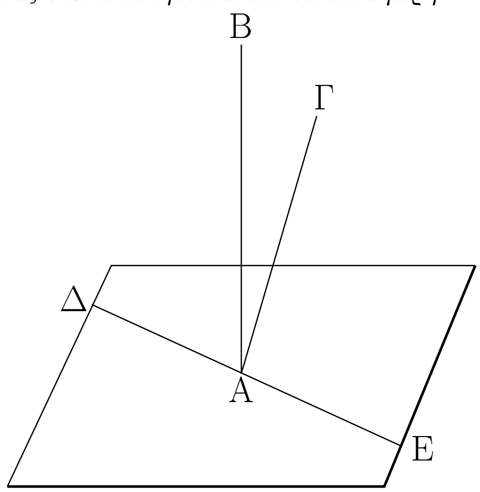
Let some raised point *B* be assumed, and let the perpendicular (straight-line) *BC* be drawn from *B* to the reference plane [Prop. 11.11]. And let *AD* be drawn from point *A* parallel to *BC* [Prop. 1.31].

Therefore, since *AD* and *CB* are two parallel straight-lines, and one of them, *BC*, is at right-angles to the reference plane, the remaining (one) *AD* is thus also at right-angles to the reference plane [Prop. 11.8].

Thus, *AD* has been set up at right-angles to the given plane, from the point in it, *A*. (Which is) the very thing it was required to do.

ιγ′.

Ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς οὐκ ἀναστήσονται ἐπὶ τὰ αὐτὰ μέρη.

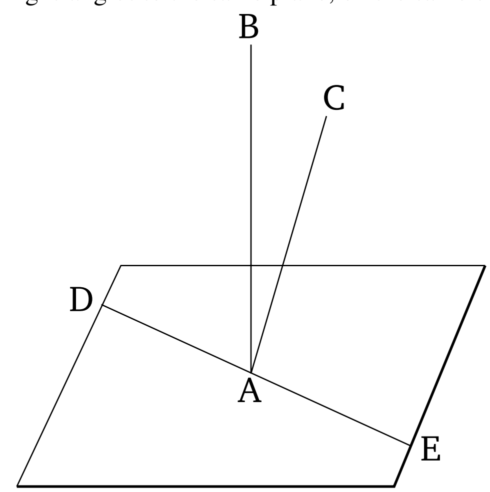


Εἰ γὰρ δυνατόν, ἀπὸ τοῦ αὐτοῦ σημείου τοῦ Α τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι αἱ ΑΒ, ΒΓ πρὸς ὀρθὰς ἀνεστάτωσαν ἐπὶ τὰ αὐτὰ μέρη, καὶ διήχθω τὸ διὰ τῶν ΒΑ, ΑΓ ἐπίπεδον· τομὴν δὴ ποιήσει διὰ τοῦ Α ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν ΔΑΕ· αἱ ἄρα ΑΒ, ΑΓ, ΔΑΕ εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ. καὶ ἐπεὶ ἡ ΓΑ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἡ ΔΑΕ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΓΑΕ γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΕ ὀρθή ἐστιν· ἴση ἄρα ἡ ὑπὸ ΓΑΕ τῇ ὑπὸ ΒΑΕ καί εἰσιν ἐν ἑνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνασταθήσονται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

Proposition 13

Two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side.



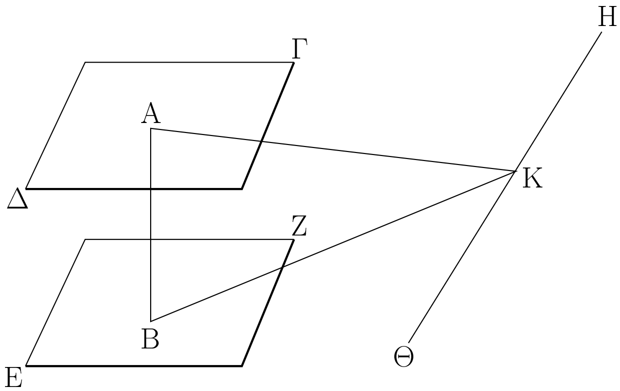
For, if possible, let the two straight-lines *AB* and *AC* be set up at the same point *A* at right-angles to the reference plane, on the same side. And let the plane through *BA* and *AC* be drawn. So it will make a straight cutting (passing) through (point) *A* in the reference plane [Prop. 11.3]. Let it have made *DAE*. Thus, *AB*, *AC*, and *DAE* are straight-lines in one plane. And since *CA* is at right-angles to the reference plane, it will thus also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. And *DAE*, which is in the reference plane, is joined to it. Thus, angle *CAE* is a right-angle. So, for the same (reasons), *BAE* is also a right-angle. Thus, *CAE* (is) equal to *BAE*. And they are in one plane. The very thing is impossible.

Thus, two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side. (Which is) the very thing it was required to show.

ιδ'.

Πρὸς ἃ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλα ἔσται τὰ ἐπίπεδα.

Εὐθεῖα γάρ τις ἡ ΑΒ πρὸς ἑκάτερον τῶν ΓΔ, ΕΖ ἐπιπέδων πρὸς ὀρθὰς ἔστω· λέγω, ὅτι παράλληλά ἐστι τὰ ἐπίπεδα.



Εἰ γὰρ μή, ἐκβαλλόμενα συμπεσοῦνται. συμπιπτέτωσαν· ποιήσουσι δὴ κοινὴν τομὴν εὐθεῖαν. ποιείτωσαν τὴν ΗΘ, καὶ εἰλήφθω ἐπὶ τῆς ΗΘ τυχὸν σημεῖον τὸ Κ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΒΚ.

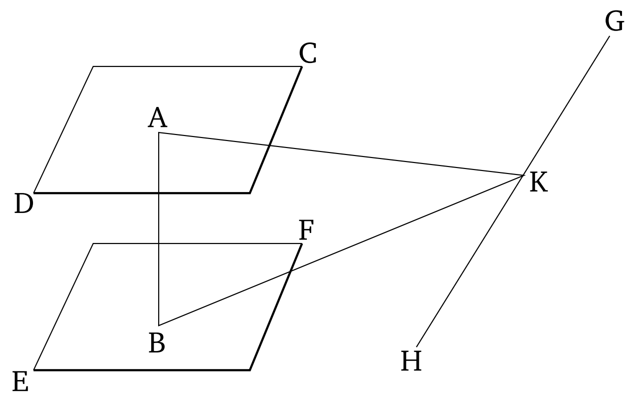
Καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ΕΖ ἐπίπεδον, καὶ πρὸς τὴν ΒΚ ἄρα εὐθεῖαν οὖσαν ἐν τῷ ΕΖ ἐκβληθέντι ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ· ἡ ἄρα ὑπὸ ΑΒΚ γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΚ ὀρθή ἐστιν. τριγώνου δὴ τοῦ ΑΒΚ αἱ δύο γωνίαι αἱ ὑπὸ ΑΒΚ, ΒΑΚ δυσὶν ὀρθαῖς εἴσιν ἴσαι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΓΔ, ΕΖ ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται· παράλληλα ἄρα ἐστὶ τὰ ΓΔ, ΕΖ ἐπίπεδα.

Πρὸς ἃ ἐπίπεδα ἄρα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα· ὅπερ ἔδει δεῖξαι.

Proposition 14

Planes to which the same straight-line is at right-angles will be parallel planes.

For let some straight-line AB be at right-angles to each of the planes CD and EF. I say that the planes are parallel.



For, if not, being produced, they will meet. Let them have met. So they will make a straight-line as a common section [Prop. 11.3]. Let them have made GH. And let some random point K be taken on GH. And let AK and BK be joined.

And since AB is at right-angles to the plane EF, AB is thus also at right-angles to BK, which is a straight-line in the produced plane EF [Def. 11.3]. Thus, angle ABK is a right-angle. So, for the same (reasons), BAK is also a right-angle. So the (sum of the) two angles ABK and BAK in the triangle ABK is equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, planes CD and EF, being produced, will not meet. Planes CD and EF are thus parallel [Def. 11.8].

Thus, planes to which the same straight-line is at right-angles are parallel planes. (Which is) the very thing it was required to show.

ιε'.

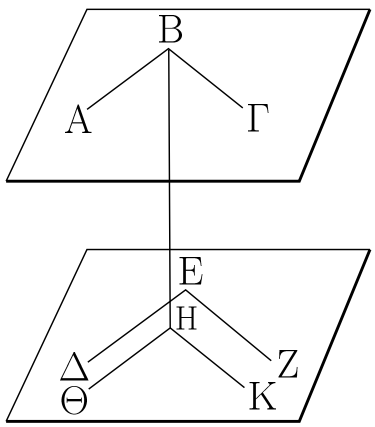
Ἐὰν δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι, παράλληλά ἐστι τὰ δι᾽ αὐτῶν ἐπίπεδα.

Proposition 15

If two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are

Δύο γὰρ εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΑΒ, ΒΓ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΔΕ, ΕΖ ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι· λέγω, ὅτι ἐκβαλλόμενα τὰ διὰ τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ ἐπίπεδα οὐ συμπεσεῖται ἀλλήλοις.

Ἤχθω γὰρ ἀπὸ τοῦ Β σημείου ἐπὶ τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον κάθετος ἡ ΒΗ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η σημεῖον, καὶ διὰ τοῦ Η τῇ μὲν ΕΔ παράλληλος ἤχθω ἡ ΗΘ, τῇ δὲ ΕΖ ἡ ΗΚ.



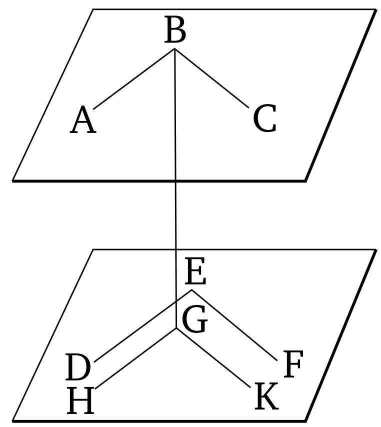
Καὶ ἐπεὶ ἡ ΒΗ ὀρθή ἐστι πρὸς τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἑκατέρα τῶν ΗΘ, ΗΚ οὖσα ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΒΗΘ, ΒΗΚ γωνιῶν. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΒΑ τῇ ΗΘ, αἱ ἄρα ὑπὸ ΗΒΑ, ΒΗΘ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΒΗΘ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΗΒΑ· ἡ ΗΒ ἄρα τῇ ΒΑ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ ἡ ΗΒ καὶ τῇ ΒΓ ἐστι πρὸς ὀρθάς. ἐπεὶ οὖν εὐθεῖα ἡ ΗΒ δυσὶν εὐθείαις ταῖς ΒΑ, ΒΓ τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐφέστηκεν, ἡ ΗΒ ἄρα καὶ τῷ διὰ τῶν ΒΑ, ΒΓ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. [διὰ τὰ αὐτὰ δὴ ἡ ΒΗ καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΗΘ, ΗΚ ἐπίπεδόν ἐστι τὸ διὰ τῶν ΔΕ, ΕΖ· ἡ ΒΗ ἄρα τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ἐστὶ πρὸς ὀρθάς. ἐδείχθη δὲ ἡ ΗΒ καὶ τῷ διὰ τῶν ΑΒ, ΒΓ ἐπιπέδῳ πρὸς ὀρθάς]. πρὸς ἃ δὲ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα· παράλληλον ἄρα ἐστὶ τὸ διὰ τῶν ΑΒ, ΒΓ ἐπίπεδον τῷ διὰ τῶν ΔΕ, ΕΖ.

Ἐὰν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, παράλληλά ἐστι τὰ δι᾽ αὐτῶν ἐπίπεδα· ὅπερ ἔδει δεῖξαι.

parallel (to one another).

For let the two straight-lines joined to one another, *AB* and *BC*, be parallel to the two straight-lines joined to one another, *DE* and *EF* (respectively), not being in the same plane. I say that the planes through *AB*, *BC* and *DE*, *EF* will not meet one another (when) produced.

For let *BG* be drawn from point *B* perpendicular to the plane through *DE* and *EF* [Prop. 11.11], and let it meet the plane at point *G*. And let *GH* be drawn through *G* parallel to *ED*, and *GK* (parallel) to *EF* [Prop. 1.31].



And since *BG* is at right-angles to the plane through *DE* and *EF*, it will thus also make right-angles with all of the straight-lines joined to it, which are also in the plane through *DE* and *EF* [Def. 11.3]. And each of *GH* and *GK*, which are in the plane through *DE* and *EF*, are joined to it. Thus, each of the angles *BGH* and *BGK* are right-angles. And since *BA* is parallel to *GH* [Prop. 11.9], the (sum of the) angles *GBA* and *BGH* is equal to two right-angles [Prop. 1.29]. And *BGH* (is) a right-angle. *GBA* (is) thus also a right-angle. Thus, *GB* is at right-angles to *BA*. So, for the same (reasons), *GB* is also at right-angles to *BC*. Therefore, since the straight-line *GB* has been set up at right-angles to two straight-lines, *BA* and *BC*, cutting one another, *GB* is thus at right-angles to the plane through *BA* and *BC* [Prop. 11.4]. [So, for the same (reasons), *BG* is also at right-angles to the plane through *GH* and *GK*. And the plane through *GH* and *GK* is the (plane) through *DE* and *EF*. And it was also shown that *GB* is at right-angles to the plane through *AB* and *BC*.] And planes to which the same straight-line is at right-angles are parallel planes [Prop. 11.14]. Thus, the plane through *AB* and *BC* is parallel to the (plane) through *DE* and *EF*.

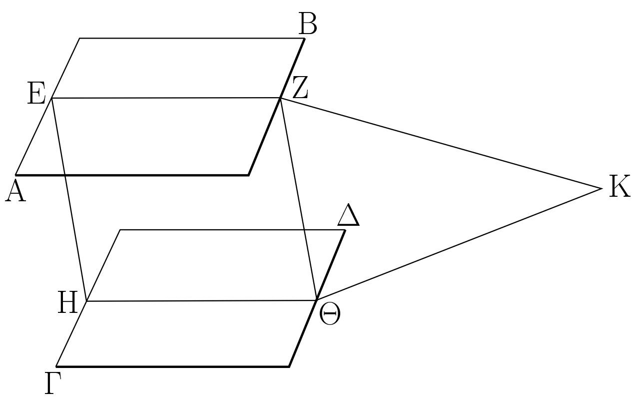
Thus, if two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another). (Which is) the very thing it was required to show.

ιϛ΄.

Ἐὰν δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.

Δύο γὰρ ἐπίπεδα παράλληλα τὰ ΑΒ, ΓΔ ὑπὸ ἐπιπέδου τοῦ ΕΖΗΘ τεμνέσθω, κοιναὶ δὲ αὐτῶν τομαὶ ἔστωσαν αἱ ΕΖ, ΗΘ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΕΖ τῇ ΗΘ.

Εἰ γὰρ μή, ἐκβαλλόμεναι αἱ ΕΖ, ΗΘ ἤτοι ἐπὶ τὰ Ζ, Θ μέρη ἢ ἐπὶ τὰ Ε, Η συμπεσοῦνται. ἐκβεβλήσθωσαν ὡς ἐπὶ τὰ Ζ, Θ μέρη καὶ συμπιπτέτωσαν πρότερον κατὰ τὸ Κ. καὶ ἐπεὶ ἡ ΕΖΚ ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ, καὶ πάντα ἄρα τὰ ἐπὶ τῆς ΕΖΚ σημεῖα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. ἓν δὲ τῶν ἐπὶ τῆς ΕΖΚ εὐθείας σημείων ἐστὶ τὸ Κ· τὸ Κ ἄρα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. διὰ τὰ αὐτὰ δὴ τὸ Κ καὶ ἐν τῷ ΓΔ ἐστιν ἐπιπέδῳ· τὰ ΑΒ, ΓΔ ἄρα ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται. οὐ συμπίπτουσι δὲ διὰ τὸ παράλληλα ὑποκεῖσθαι· οὐκ ἄρα αἱ ΕΖ, ΗΘ εὐθεῖαι ἐκβαλλόμεναι ἐπὶ τὰ Ζ, Θ μέρη συμπεσοῦνται. ὁμοίως δὴ δείξομεν, ὅτι αἱ ΕΖ, ΗΘ εὐθεῖαι οὐδὲ ἐπὶ τὰ Ε, Η μέρη ἐκβαλλόμεναι συμπεσοῦνται. αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΕΖ τῇ ΗΘ.



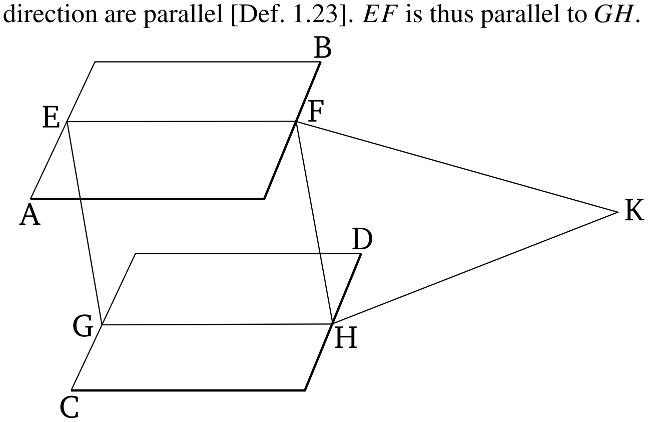
Ἐὰν ἄρα δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν· ὅπερ ἔδει δεῖξαι.

Proposition 16

If two parallel planes are cut by some plane then their common sections are parallel.

For let the two parallel planes AB and CD be cut by the plane $EFGH$. And let EF and GH be their common sections. I say that EF is parallel to GH.

For, if not, being produced, EF and GH will meet either in the direction of F, H, or of E, G. Let them be produced, as in the direction of F, H, and let them, first of all, have met at K. And since EFK is in the plane AB, all of the points on EFK are thus also in the plane AB [Prop. 11.1]. And K is one of the points on EFK. Thus, K is in the plane AB. So, for the same (reasons), K is also in the plane CD. Thus, the planes AB and CD, being produced, will meet. But they do not meet, on account of being (initially) assumed (to be mutually) parallel. Thus, the straight-lines EF and GH, being produced in the direction of F, H, will not meet. So, similarly, we can show that the straight-lines EF and GH, being produced in the direction of E, G, will not meet either. And (straight-lines in one plane which), being produced, do not meet in either direction are parallel [Def. 1.23]. EF is thus parallel to GH.



Thus, if two parallel planes are cut by some plane then their common sections are parallel. (Which is) the very thing it was required to show.

ιζ΄.

Ἐὰν δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπεδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ ὑπὸ παραλλήλων ἐπιπέδων τῶν ΗΘ, ΚΛ, ΜΝ τεμνέσθωσαν κατὰ τὰ Α, Ε, Β, Γ, Ζ, Δ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΔ, ΑΔ, καὶ συμβαλλέτω ἡ ΑΔ τῷ ΚΛ ἐπιπέδῳ κατὰ τὸ Ξ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΞ, ΞΖ.

Καὶ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΚΛ, ΜΝ ὑπὸ ἐπιπέδου τοῦ ΕΒΔΞ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ αἱ ΕΞ,

Proposition 17

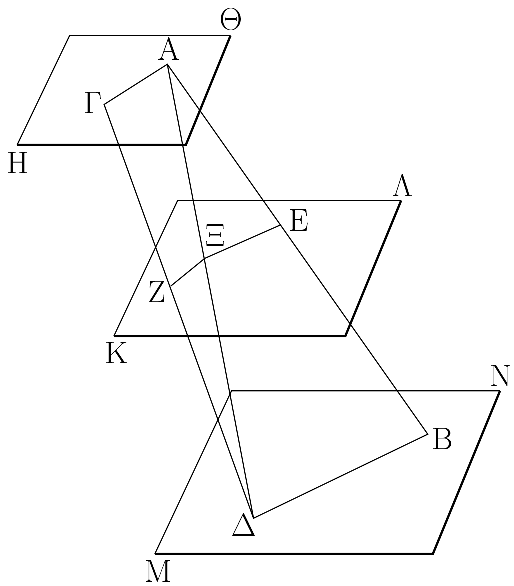
If two straight-lines are cut by parallel planes then they will be cut in the same ratios.

For let the two straight-lines AB and CD be cut by the parallel planes GH, KL, and MN at the points A, E, B, and C, F, D (respectively). I say that as the straight-line AE is to EB, so CF (is) to FD.

For let AC, BD, and AD be joined, and let AD meet the plane KL at point O, and let EO and OF be joined.

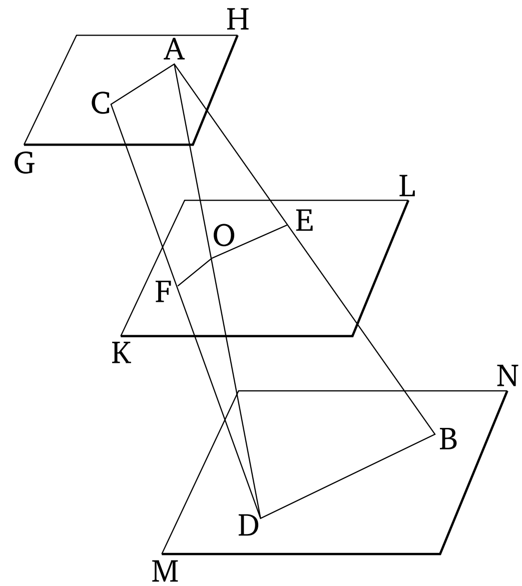
And since two parallel planes KL and MN are cut by the plane $EBDO$, their common sections EO and BD are parallel [Prop. 11.16]. So, for the same (reasons), since two parallel

ΒΔ παράλληλοί εἰσιν. διὰ τὰ αὐτὰ δὴ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΗΘ, ΚΛ ὑπὸ ἐπιπέδου τοῦ ΑΞΖΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ αἱ ΑΓ, ΞΖ παράλληλοί εἰσιν. καὶ ἐπεὶ τριγώνου τοῦ ΑΒΔ παρὰ μίαν τῶν πλευρῶν τὴν ΒΔ εὐθεῖα ἦκται ἡ ΕΞ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΑΞ πρὸς ΞΔ. πάλιν ἐπεὶ τριγώνου τοῦ ΑΔΓ παρὰ μίαν τῶν πλευρῶν τὴν ΑΓ εὐθεῖα ἦκται ἡ ΞΖ, ἀνάλογόν ἐστιν ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΓΖ πρὸς ΖΔ. ἐδείχθη δὲ καὶ ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΑΕ πρὸς ΕΒ· καὶ ὡς ἄρα ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ.



Ἐὰν ἄρα δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπέδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται· ὅπερ ἔδει δεῖξαι.

planes GH and KL are cut by the plane $AOFC$, their common sections AC and OF are parallel [Prop. 11.16]. And since the straight-line EO has been drawn parallel to one of the sides BD of triangle ABD, thus, proportionally, as AE is to EB, so AO (is) to OD [Prop. 6.2]. Again, since the straight-line OF has been drawn parallel to one of the sides AC of triangle ADC, proportionally, as AO is to OD, so CF (is) to FD [Prop. 6.2]. And it was also shown that as AO (is) to OD, so AE (is) to EB. And thus as AE (is) to EB, so CF (is) to FD [Prop. 5.11].



Thus, if two straight-lines are cut by parallel planes then they will be cut in the same ratios. (Which is) the very thing it was required to show.

ιη'.

Ἐὰν εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι᾽ αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Εὐθεῖα γάρ τις ἡ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ πάντα τὰ διὰ τῆς ΑΒ ἐπίπεδα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἐκβεβλήσθω γὰρ διὰ τῆς ΑΒ ἐπίπεδον τὸ ΔΕ, καὶ ἔστω κοινὴ τομὴ τοῦ ΔΕ ἐπιπέδου καὶ τοῦ ὑποκειμένου ἡ ΓΕ, καὶ εἰλήφθω ἐπὶ τῆς ΓΕ τυχὸν σημεῖον τὸ Ζ, καὶ ἀπὸ τοῦ Ζ τῇ ΓΕ πρὸς ὀρθὰς ἤχθω ἐν τῷ ΔΕ ἐπιπέδῳ ἡ ΖΗ.

Καὶ ἐπεὶ ἡ ΑΒ πρὸς τὸ ὑποκείμενον ἐπίπεδον ὀρθή ἐστιν, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ· ὥστε καὶ πρὸς τὴν ΓΕ ὀρθή ἐστιν· ἡ ἄρα ὑπὸ ΑΒΖ γωνία ὀρθή ἐστιν. ἔστι δὲ καὶ ἡ ὑπὸ ΗΖΒ ὀρθή· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΖΗ. ἡ δὲ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· καὶ ἡ ΖΗ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ

Proposition 18

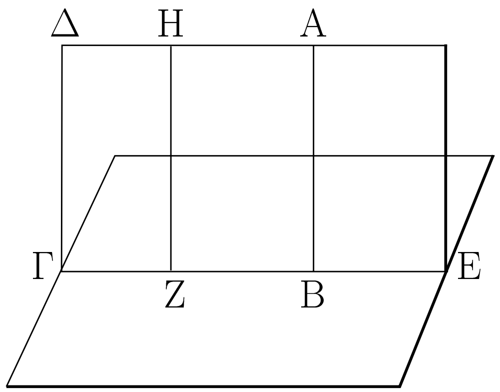
If a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane.

For let some straight-line AB be at right-angles to a reference plane. I say that all of the planes (passing) through AB are also at right-angles to the reference plane.

For let the plane DE be produced through AB. And let CE be the common section of the plane DE and the reference (plane). And let some random point F be taken on CE. And let FG be drawn from F, at right-angles to CE, in the plane DE [Prop. 1.11].

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Hence, it is also at right-angles to CE. Thus, angle ABF is a right-angle. And GFB is also a right-angle. Thus, AB is parallel to FG [Prop. 1.28]. And AB is at right-angles to the refer-

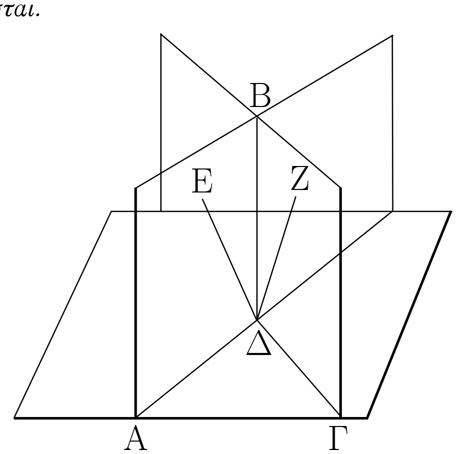
τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν. καὶ τῇ κοινῇ τομῇ τῶν ἐπιπέδων τῇ ΓΕ ἐν ἑνὶ τῶν ἐπιπέδων τῷ ΔΕ πρὸς ὀρθὰς ἀχθεῖσα ἡ ΖΗ ἐδείχθη τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς· τὸ ἄρα ΔΕ ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον. ὁμοίως δὴ δειχθήσεται καὶ πάντα τὰ διὰ τῆς ΑΒ ἐπίπεδα ὀρθὰ τυγχανοντα πρὸς τὸ ὑποκείμενον ἐπίπεδον.



Ἐὰν ἄρα εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι᾽ αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

ιθ΄.

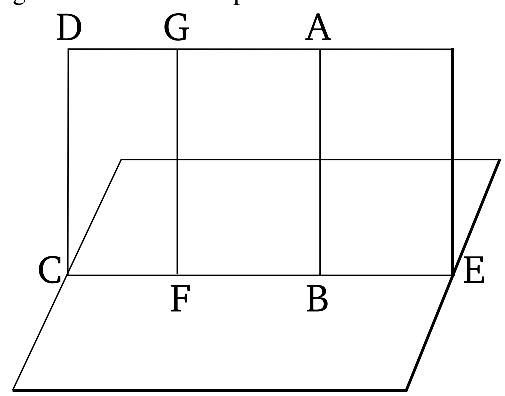
Ἐὰν δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



Δύο γὰρ ἐπίπεδα τὰ ΑΒ, ΒΓ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ ΒΔ· λέγω, ὅτι ἡ ΒΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

μὴ γάρ, καὶ ἤχθωσαν ἀπὸ τοῦ Δ σημείου ἐν μὲν τῷ ΑΒ

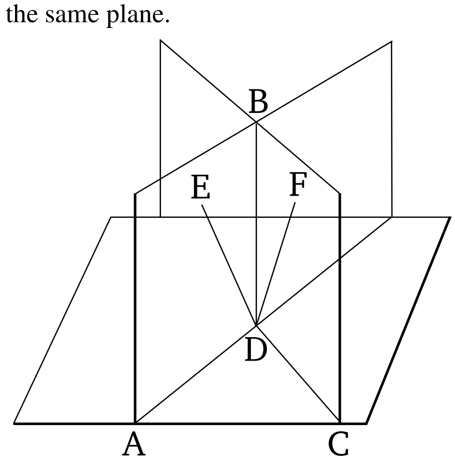
ence plane. Thus, *FG* is also at right-angles to the reference plane [Prop. 11.8]. And a plane is at right-angles to a(nother) plane when the straight-lines drawn at right-angles to the common section of the planes, (and lying) in one of the planes, are at right-angles to the remaining plane [Def. 11.4]. And *FG*, (which was) drawn at right-angles to the common section of the planes, *CE*, in one of the planes, *DE*, was shown to be at right-angles to the reference plane. Thus, plane *DE* is at right-angles to the reference (plane). So, similarly, it can be shown that all of the planes (passing) at random through *AB* (are) at right-angles to the reference plane.



Thus, if a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 19

If two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane.



For let the two planes *AB* and *BC* be at right-angles to a reference plane, and let their common section be *BD*. I say that *BD* is at right-angles to the reference plane.

For (if) not, let *DE* also be drawn from point *D*, in the *ἐπιπέδῳ τῇ ΑΔ εὐθείᾳ πρὸς ὀρθὰς ἡ ΔΕ, ἐν δὲ τῷ ΒΓ ἐπιπέδῳ τῇ ΓΔ πρὸς ὀρθὰς ἡ ΔΖ.*

Καὶ ἐπεὶ τὸ ΑΒ ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ ΑΔ πρὸς ὀρθὰς ἐν τῷ ΑΒ ἐπιπέδῳ ἦκται ἡ ΔΕ, ἡ ΔΕ ἄρα ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΔΖ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ἀπὸ τοῦ αὐτοῦ ἄρα σημείου τοῦ Δ τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνεσταμέναι εἰσὶν ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ ἀπὸ τοῦ Δ σημείου ἀνασταθήσεται πρὸς ὀρθὰς πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Ἐὰν ἄρα δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

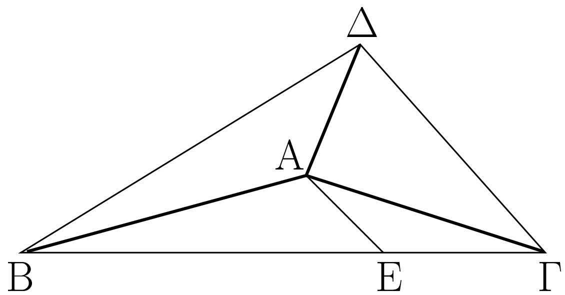
plane *AB*, at right-angles to the straight-line *AD*, and *DF*, in the plane *BC*, at right-angles to *CD*.

And since the plane *AB* is at right-angles to the reference (plane), and *DE* has been drawn at right-angles to their common section *AD*, in the plane *AB*, *DE* is thus at right-angles to the reference plane [Def. 11.4]. So, similarly, we can show that *DF* is also at right-angles to the reference plane. Thus, two (different) straight-lines are set up, at the same point *D*, at right-angles to the reference plane, on the same side. The very thing is impossible [Prop. 11.13]. Thus, no (other straight-line) except the common section *DB* of the planes *AB* and *BC* can be set up at point *D*, at right-angles to the reference plane.

Thus, if two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

κʹ.

Ἐὰν στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.



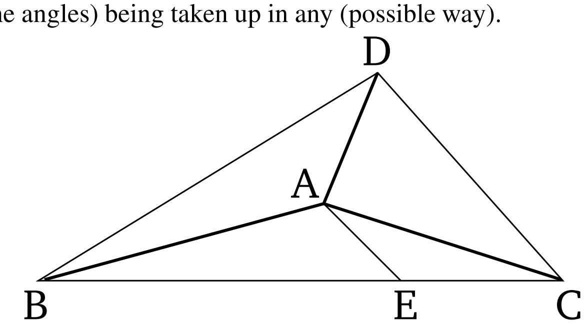
Στερεὰ γὰρ γωνία ἡ πρὸς τῷ Α ὑπὸ τριῶν γωνιῶν ἐπιπέδων τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ περιεχέσθω· λέγω, ὅτι τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνιῶν δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.

Εἰ μὲν οὖν αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν. εἰ δὲ οὔ, ἔστω μείζων ἡ ὑπὸ ΒΑΓ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΔΑΒ γωνίᾳ ἐν τῷ διὰ τῶν ΒΑΓ ἐπιπέδῳ ἴση ἡ ὑπὸ ΒΑΕ, καὶ κείσθω τῇ ΑΔ ἴση ἡ ΑΕ, καὶ διὰ τοῦ Ε σημείου διαχθεῖσα ἡ ΒΕΓ τεμνέτω τὰς ΑΒ, ΑΓ εὐθείας κατὰ τὰ Β, Γ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΔΒ, ΔΓ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΒ, δύο δυσὶν ἴσαι· καὶ γωνία ἡ ὑπὸ ΔΑΒ γωνίᾳ τῇ ὑπὸ ΒΑΕ ἴση· βάσις ἄρα ἡ ΔΒ βάσει τῇ ΒΕ ἐστιν ἴση. καὶ ἐπεὶ δύο αἱ ΒΔ, ΔΓ τῆς ΒΓ μείζονές εἰσιν, ὧν ἡ ΔΒ τῇ ΒΕ ἐδείχθη ἴση, λοιπὴ ἄρα ἡ ΔΓ λοιπῆς τῆς ΕΓ μείζων ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΓ, καὶ βάσις ἡ ΔΓ βάσεως τῆς ΕΓ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνίας τῆς ὑπὸ ΕΑΓ μείζων ἐστίν. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΔΑΒ τῇ ὑπὸ ΒΑΕ ἴση· αἱ

Proposition 20

If a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way).



For let the solid angle *A* be contained by the three plane angles *BAC*, *CAD*, and *DAB*. I say that (the sum of) any two of the angles *BAC*, *CAD*, and *DAB* is greater than the remaining (one), (the angles) being taken up in any (possible way).

For if the angles *BAC*, *CAD*, and *DAB* are equal to one another then (it is) clear that (the sum of) any two is greater than the remaining (one). But, if not, let *BAC* be greater (than *CAD* or *DAB*). And let (angle) *BAE*, equal to the angle *DAB*, be constructed in the plane through *BAC*, on the straight-line *AB*, at the point *A* on it. And let *AE* be made equal to *AD*. And *BEC* being drawn across through point *E*, let it cut the straight-lines *AB* and *AC* at points *B* and *C* (respectively). And let *DB* and *DC* be joined.

And since *DA* is equal to *AE*, and *AB* (is) common, the two (straight-lines *AD* and *AB* are) equal to the two (straight-lines *EA* and *AB*, respectively). And angle *DAB* (is) equal to angle *BAE*. Thus, the base *DB* is equal to the base *BE* [Prop. 1.4]. And since the (sum of the) two (straight-lines) *BD* and *DC* is greater than *BC* [Prop. 1.20], of which *DB* was shown (to be) equal to *BE*, the remainder *DC* is thus greater *ἄρα ὑπὸ ΔΑΒ, ΔΑΓ τῆς ὑπὸ ΒΑΓ μείζονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ λοιπαὶ σύνδυο λαμβανόμεναι τῆς λοιπῆς μείζονές εἰσιν.*

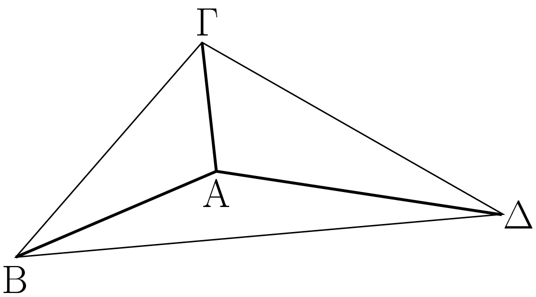
Ἐὰν ἄρα στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

than the remainder *EC*. And since *DA* is equal to *AE*, but *AC* (is) common, and the base *DC* is greater than the base *EC*, the angle *DAC* is thus greater than the angle *EAC* [Prop. 1.25]. And *DAB* was also shown (to be) equal to *BAE*. Thus, (the sum of) *DAB* and *DAC* is greater than *BAC*. So, similarly, we can also show that the remaining (angles), being taken in pairs, are greater than the remaining (one).

Thus, if a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way). (Which is) the very thing it was required to show.

κα΄.

Ἅπασα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἢ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται.



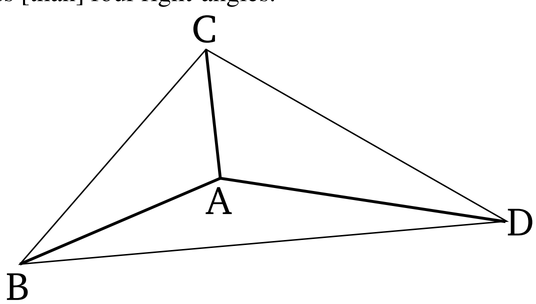
Ἔστω στερεὰ γωνία ἡ πρὸς τῷ Α περιεχομένη ὑπὸ ἐπιπέδων γωνιῶν τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ· λέγω, ὅτι αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Εἰλήφθω γὰρ ἐφ᾽ ἑκάστης τῶν ΑΒ, ΑΓ, ΑΔ τυχόντα σημεῖα τὰ Β, Γ, Δ, καὶ ἐπεζεύχθωσαν αἱ ΒΓ, ΓΔ, ΔΒ. καὶ ἐπεὶ στερεὰ γωνία ἡ πρὸς τῷ Β ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχεται τῶν ὑπὸ ΓΒΑ, ΑΒΔ, ΓΒΔ, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν· αἱ ἄρα ὑπὸ ΓΒΑ, ΑΒΔ τῆς ὑπὸ ΓΒΔ μείζονές εἰσιν. διὰ τὰ αὐτὰ δὴ καὶ αἱ μὲν ὑπὸ ΒΓΑ, ΑΓΔ τῆς ὑπὸ ΒΓΔ μείζονές εἰσιν, αἱ δὲ ὑπὸ ΓΔΑ, ΑΔΒ τῆς ὑπὸ ΓΔΒ μείζονές εἰσιν· αἱ ἓξ ἄρα γωνίαι αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ τριῶν τῶν ὑπὸ ΓΒΔ, ΒΓΑ, ΓΔΒ μείζονές εἰσιν. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ ΓΒΔ, ΒΔΓ, ΒΓΔ δυσὶν ὀρθαῖς ἴσαι εἰσίν· αἱ ἓξ ἄρα αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ δύο ὀρθῶν μείζονές εἰσιν. καὶ ἐπεὶ ἑκάστου τῶν ΑΒΓ, ΑΓΔ, ΑΔΒ τριγώνων αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν, αἱ ἄρα τῶν τριῶν τριγώνων ἐννέα γωνίαι αἱ ὑπὸ ΓΒΑ, ΑΓΒ, ΒΑΓ, ΑΓΔ, ΓΔΑ, ΓΑΔ, ΑΔΒ, ΔΒΑ, ΒΑΔ ἓξ ὀρθαῖς ἴσαι εἰσίν, ὧν αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ, ΔΒΑ ἓξ γωνίαι δύο ὀρθῶν εἰσι μείζονες· λοιπαὶ ἄρα αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τρεῖς [γωνίαι] περιέχουσαι τὴν στερεὰν γωνίαν τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Ἅπασα ἄρα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἢ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται· ὅπερ ἔδει δεῖξαι.

Proposition 21

Any solid angle is contained by plane angles (whose sum is) less [than] four right-angles.[†]



Let the solid angle *A* be contained by the plane angles *BAC*, *CAD*, and *DAB*. I say that (the sum of) *BAC*, *CAD*, and *DAB* is less than four right-angles.

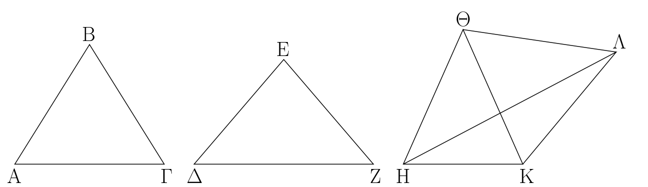
For let the random points *B*, *C*, and *D* be taken on each of (the straight-lines) *AB*, *AC*, and *AD* (respectively). And let *BC*, *CD*, and *DB* be joined. And since the solid angle at *B* is contained by the three plane angles *CBA*, *ABD*, and *CBD*, (the sum of) any two is greater than the remaining (one) [Prop. 11.20]. Thus, (the sum of) *CBA* and *ABD* is greater than *CBD*. So, for the same (reasons), (the sum of) *BCA* and *ACD* is also greater than *BCD*, and (the sum of) *CDA* and *ADB* is greater than *CDB*. Thus, the (sum of the) six angles *CBA*, *ABD*, *BCA*, *ACD*, *CDA*, and *ADB* is greater than the (sum of the) three (angles) *CBD*, *BCD*, and *CDB*. But, the (sum of the) three (angles) *CBD*, *BDC*, and *BCD* is equal to two right-angles [Prop. 1.32]. Thus, the (sum of the) six angles *CBA*, *ABD*, *BCA*, *ACD*, *CDA*, and *ADB* is greater than two right-angles. And since the (sum of the) three angles of each of the triangles *ABC*, *ACD*, and *ADB* is equal to two right-angles, the (sum of the) nine angles *CBA*, *ACB*, *BAC*, *ACD*, *CDA*, *CAD*, *ADB*, *DBA*, and *BAD* of the three triangles is equal to six right-angles, of which the (sum of the) six angles *ABC*, *BCA*, *ACD*, *CDA*, *ADB*, and *DBA* is greater than two right-angles. Thus, the (sum of the) remaining three [angles] *BAC*, *CAD*, and *DAB*, containing the solid angle, is less than four right-angles.

Thus, any solid angle is contained by plane angles (whose sum is) less [than] four right-angles. (Which is) the very thing it was required to show.

† This proposition is only proved for the case of a solid angle contained by three plane angles. However, the generalization to a solid angle contained by more than three plane angles is straightforward.

κβ΄.

Ἐὰν ὦσι τρεῖς γωνίαι ἐπίπεδοι, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, περιέχωσι δὲ αὐτὰς ἴσαι εὐθεῖαι, δυνατόν ἐστιν ἐκ τῶν ἐπιζευγνυουσῶν τὰς ἴσας εὐθείας τρίγωνον συστήσασθαι.

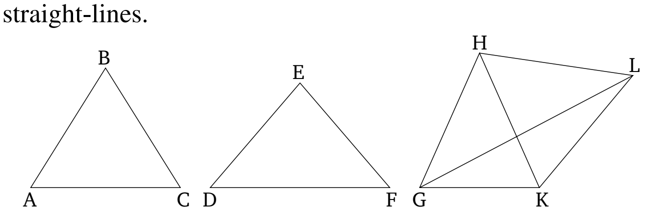


Ἔστωσαν τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, αἱ μὲν ὑπὸ ΑΒΓ, ΔΕΖ τῆς ὑπὸ ΗΘΚ, αἱ δὲ ὑπὸ ΔΕΖ, ΗΘΚ τῆς ὑπὸ ΑΒΓ, καὶ ἔτι αἱ ὑπὸ ΗΘΚ, ΑΒΓ τῆς ὑπὸ ΔΕΖ, καὶ ἔστωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ εὐθεῖαι, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· λέγω, ὅτι δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι, τουτέστιν ὅτι τῶν ΑΓ, ΔΖ, ΗΚ δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν.

Εἰ μὲν οὖν αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι καὶ τῶν ΑΓ, ΔΖ, ΗΚ ἴσων γινομένων δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. εἰ δὲ οὔ, ἔστωσαν ἄνισοι, καὶ συνεστάτω πρὸς τῇ ΘΚ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ τῇ ὑπὸ ΑΒΓ γωνίᾳ ἴση ἡ ὑπὸ ΚΘΛ· καὶ κείσθω μιᾷ τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἴση ἡ ΘΛ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΗΛ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΚΘ, ΘΛ ἴσαι εἰσίν, καὶ γωνία ἡ πρὸς τῷ Β γωνίᾳ τῇ ὑπὸ ΚΘΛ ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῇ ΚΛ ἴση. καὶ ἐπεὶ αἱ ὑπὸ ΑΒΓ, ΗΘΚ τῆς ὑπὸ ΔΕΖ μείζονές εἰσιν, ἴση δὲ ἡ ὑπὸ ΑΒΓ τῇ ὑπὸ ΚΘΛ, ἡ ἄρα ὑπὸ ΗΘΛ τῆς ὑπὸ ΔΕΖ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ΗΘ, ΘΛ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΗΘΛ γωνίας τῆς ὑπὸ ΔΕΖ μείζων, βάσις ἄρα ἡ ΗΛ βάσεως τῆς ΔΖ μείζων ἐστίν. ἀλλὰ αἱ ΗΚ, ΚΛ τῆς ΗΛ μείζονές εἰσιν. πολλῷ ἄρα αἱ ΗΚ, ΚΛ τῆς ΔΖ μείζονές εἰσιν. ἴση δὲ ἡ ΚΛ τῇ ΑΓ· αἱ ΑΓ, ΗΚ ἄρα τῆς λοιπῆς τῆς ΔΖ μείζονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν ΑΓ, ΔΖ τῆς ΗΚ μείζονές εἰσιν, καὶ ἔτι αἱ ΔΖ, ΗΚ τῆς ΑΓ μείζονές εἰσιν. δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι· ὅπερ ἔδει δεῖξαι.

Proposition 22

If there are three plane angles, of which (the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way), and if equal straight-lines contain them, then it is possible to construct a triangle from (the straight-lines created by) joining the (ends of the) equal straight-lines.

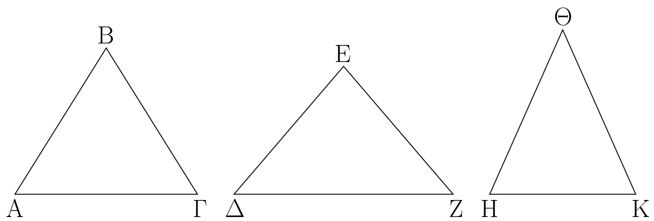


Let ABC, DEF, and GHK be three plane angles, of which the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way)—(that is), ABC and DEF (greater) than GHK, DEF and GHK (greater) than ABC, and, further, GHK and ABC (greater) than DEF. And let AB, BC, DE, EF, GH, and HK be equal straight-lines. And let AC, DF, and GK be joined. I say that that it is possible to construct a triangle out of (straight-lines) equal to AC, DF, and GK—that is to say, that (the sum of) any two of AC, DF, and GK is greater than the remaining (one).

Now, if the angles ABC, DEF, and GHK are equal to one another then (it is) clear that, (with) AC, DF, and GK also becoming equal, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. And if not, let them be unequal, and let KHL, equal to angle ABC, be constructed on the straight-line HK, at the point H on it. And let HL be made equal to one of AB, BC, DE, EF, GH, and HK. And let KL and GL be joined. And since the two (straight-lines) AB and BC are equal to the two (straight-lines) KH and HL (respectively), and the angle at B (is) equal to KHL, the base AC is thus equal to the base KL [Prop. 1.4]. And since (the sum of) ABC and GHK is greater than DEF, and ABC equal to KHL, GHL is thus greater than DEF. And since the two (straight-lines) GH and HL are equal to the two (straight-lines) DE and EF (respectively), and angle GHL (is) greater than DEF, the base GL is thus greater than the base DF [Prop. 1.24]. But, (the sum of) GK and KL is greater than GL [Prop. 1.20]. Thus, (the sum of) GK and KL is much greater than DF. And KL (is) equal to AC. Thus, (the sum of) AC and GK is greater than the remaining (straight-line) DF. So, similarly, we can show that (the sum of) AC and DF is greater than GK, and, further, that (the sum of) DF and GK is greater than AC. Thus, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. (Which is) the very thing it was required to show.

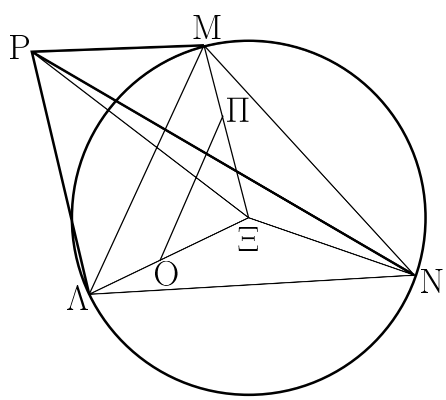
κγ′.

Ἐκ τριῶν γωνιῶν ἐπιπέδων, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, στερεὰν γωνίαν συστήσασθαι· δεῖ δὴ τὰς τρεῖς τεσσάρων ὀρθῶν ἐλάσςονας εἶναι.



Ἔστωσαν αἱ δοθεῖσαι τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, ἔτι δὲ αἱ τρεῖς τεσσάρων ὀρθῶν ἐλάσσονες· δεῖ δὴ ἐκ τῶν ἴσων ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ στερεὰν γωνίαν συστήσασθαι.

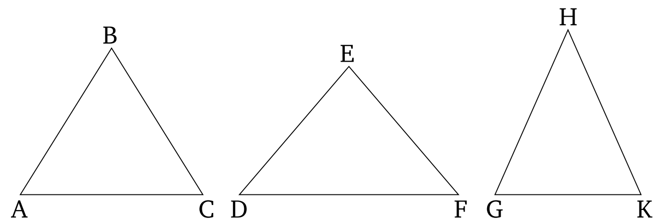
Ἀπειλήφθωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. συνεστάτω τὸ ΛΜΝ, ὥστε ἴσην εἶναι τὴν μὲν ΑΓ τῇ ΛΜ, τὴν δὲ ΔΖ τῇ ΜΝ, καὶ ἔτι τὴν ΗΚ τῇ ΝΛ, καὶ περιγεγράφθω περὶ τὸ ΛΜΝ τρίγωνον κύκλος ὁ ΛΜΝ, καὶ εἰλήφθω αὐτοῦ τὸ κέντρον καὶ ἔστω τὸ Ξ, καὶ ἐπεζεύχθωσαν αἱ ΛΞ, ΜΞ, ΝΞ·



Λέγω, ὅτι ἡ ΑΒ μείζων ἐστὶ τῆς ΛΞ. εἰ γὰρ μή, ἤτοι ἴση ἐστὶν ἡ ΑΒ τῇ ΛΞ ἢ ἐλάττων. ἔστω πρότερον ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΛΞ, ἀλλὰ ἡ μὲν ΑΒ τῇ ΒΓ ἐστιν ἴση, ἡ δὲ ΞΛ τῇ ΞΜ, δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΛΞ, ΞΜ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΑΓ βάσει τῇ ΛΜ

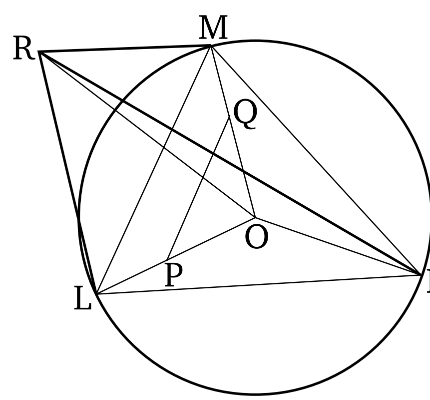
Proposition 23

To construct a solid angle from three (given) plane angles, (the sum of) two of which is greater than the remaining (one, the angles) being taken up in any (possible way). So, it is necessary for the (sum of the) three (angles) to be less than four right-angles [Prop. 11.21].



Let ABC, DEF, and GHK be the three given plane angles, of which let (the sum of) two be greater than the remaining (one, the angles) being taken up in any (possible way), and, further, (let) the (sum of the) three (be) less than four right-angles. So, it is necessary to construct a solid angle from (plane angles) equal to ABC, DEF, and GHK.

Let AB, BC, DE, EF, GH, and HK be cut off (so as to be) equal (to one another). And let AC, DF, and GK be joined. It is, thus, possible to construct a triangle from (straight-lines) equal to AC, DF, and GK [Prop. 11.22]. Let (such a triangle), LMN, have be constructed, such that AC is equal to LM, DF to MN, and, further, GK to NL. And let the circle LMN be circumscribed about triangle LMN [Prop. 4.5]. And let its center be found, and let it be (at) O. And let LO, MO, and NO be joined.



I say that AB is greater than LO. For, if not, AB is either equal to, or less than, LO. Let it, first of all, be equal. And since AB is equal to LO, but AB is equal to BC, and OL to OM, so the two (straight-lines) AB and BC are equal to the two (straight-lines) LO and OM, respectively. And the base *ὑπόκειται ἴση· γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΛΞΜ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΔΕΖ τῇ ὑπὸ ΜΞΝ ἐστιν ἴση, καὶ ἔτι ἡ ὑπὸ ΗΘΚ τῇ ὑπὸ ΝΞΛ· αἱ ἄρα τρεῖς αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι τρισὶ ταῖς ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ εἰσιν ἴσαι. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τέτταρσιν ὀρθαῖς εἰσιν ἴσαι· καὶ αἱ τρεῖς ἄρα αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τέτταρσιν ὀρθαῖς ἴσαι εἰσίν. ὑπόκεινται δὲ καὶ τεσσάρων ὀρθῶν ἐλάσσονες· ὅπερ ἄτοπον. οὐκ ἄρα ἡ ΑΒ τῇ ΛΞ ἴση ἐστίν. λέγω δή, ὅτι οὐδὲ ἐλάττων ἐστὶν ἡ ΑΒ τῆς ΛΞ. εἰ γὰρ δυνατόν, ἔστω· καὶ κείσθω τῇ μὲν ΑΒ ἴση ἡ ΞΟ, τῇ δὲ ΒΓ ἴση ἡ ΞΠ, καὶ ἐπεζεύχθω ἡ ΟΠ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΒΓ, ἴση ἐστὶ καὶ ἡ ΞΟ τῇ ΞΠ· ὥστε καὶ λοιπὴ ἡ ΛΟ τῇ ΠΜ ἐστιν ἴση. παράλληλος ἄρα ἐστὶν ἡ ΛΜ τῇ ΟΠ, καὶ ἰσογώνιον τὸ ΛΜΞ τῷ ΟΠΞ· ἔστιν ἄρα ὡς ἡ ΞΛ πρὸς ΛΜ, οὕτως ἡ ΞΟ πρὸς ΟΠ· ἐναλλὰξ ὡς ἡ ΛΞ πρὸς ΞΟ, οὕτως ἡ ΛΜ πρὸς ΟΠ. μείζων δὲ ἡ ΛΞ τῆς ΞΟ· μείζων ἄρα καὶ ἡ ΛΜ τῆς ΟΠ. ἀλλὰ ἡ ΛΜ κεῖται τῇ ΑΓ ἴση· καὶ ἡ ΑΓ ἄρα τῆς ΟΠ μείζων ἐστίν. ἐπεὶ οὖν δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΟΞ, ΞΠ ἴσαι εἰσίν, καὶ βάσις ἡ ΑΓ βάσεως τῆς ΟΠ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίας τῆς ὑπὸ ΟΞΠ μείζων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ μὲν ὑπὸ ΔΕΖ τῆς ὑπὸ ΜΞΝ μείζων ἐστίν, ἡ δὲ ὑπὸ ΗΘΚ τῆς ὑπὸ ΝΞΛ. αἱ ἄρα τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τριῶν τῶν ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τεσσάρων ὀρθῶν ἐλάσσονες ὑπόκεινται· πολλῷ ἄρα αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν. ἀλλὰ καὶ ἴσαι· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἡ ΑΒ ἐλάσσων ἐστὶ τῆς ΛΞ. ἐδείχθη δέ, ὅτι οὐδὲ ἴση· μείζων ἄρα ἡ ΑΒ τῆς ΛΞ.*

Ἀνεστάτω δὴ ἀπὸ τοῦ Ξ σημείου τῷ τοῦ ΛΜΝ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΞΡ, καὶ ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τετράγωνον τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ τῆς ΞΡ, καὶ ἐπεζεύχθωσαν αἱ ΡΛ, ΡΜ, ΡΝ.

Καὶ ἐπεὶ ἡ ΡΞ ὀρθή ἐστι πρὸς τὸ τοῦ ΛΜΝ κύκλου ἐπίπεδον, καὶ πρὸς ἑκάστην ἄρα τῶν ΛΞ, ΜΞ, ΝΞ ὀρθή ἐστιν ἡ ΡΞ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΛΞ τῇ ΞΜ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΞΡ, βάσις ἄρα ἡ ΡΛ βάσει τῇ ΡΜ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΡΝ ἑκατέρᾳ τῶν ΡΛ, ΡΜ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΡΛ, ΡΜ, ΡΝ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον ὑπόκειται τὸ ἀπὸ τῆς ΞΡ, τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΛΞ, ΞΡ. τοῖς δὲ ἀπὸ τῶν ΛΞ, ΞΡ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΛΡ· ὀρθὴ γὰρ ἡ ὑπὸ ΛΞΡ· τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΡΛ· ἴση ἄρα ἡ ΑΒ τῇ ΡΛ. ἀλλὰ τῇ μὲν ΑΒ ἴση ἐστὶν ἑκάστη τῶν ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, τῇ δὲ ΡΛ ἴση ἑκατέρα τῶν ΡΜ, ΡΝ· ἑκάστη ἄρα τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἑκάστῃ τῶν ΡΛ, ΡΜ, ΡΝ ἴση ἐστίν. καὶ ἐπεὶ δύο αἱ ΛΡ, ΡΜ δυσὶ ταῖς ΑΒ, ΒΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΛΜ βάσει τῇ ΑΓ ὑπόκειται ἴση, γωνία ἄρα ἡ ὑπὸ ΛΡΜ γωνίᾳ τῇ ὑπὸ ΑΒΓ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΜΡΝ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση, ἡ δὲ ὑπὸ ΛΡΝ τῇ ὑπὸ ΗΘΚ.

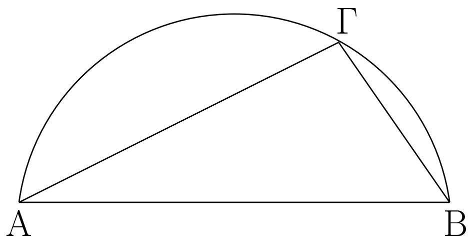
Ἐκ τριῶν ἄρα γωνιῶν ἐπιπέδων τῶν ὑπὸ ΛΡΜ, ΜΡΝ,

AC was assumed (to be) equal to the base *LM*. Thus, angle *ABC* is equal to angle *LOM* [Prop. 1.8]. So, for the same (reasons), *DEF* is also equal to *MON*, and, further, *GHK* to *NOL*. Thus, the three angles *ABC*, *DEF*, and *GHK* are equal to the three angles *LOM*, *MON*, and *NOL*, respectively. But, the (sum of the) three angles *LOM*, *MON*, and *NOL* is equal to four right-angles. Thus, the (sum of the) three angles *ABC*, *DEF*, and *GHK* is also equal to four right-angles. And it was also assumed (to be) less than four right-angles. The very thing (is) absurd. Thus, *AB* is not equal to *LO*. So, I say that *AB* is not less than *LO* either. For, if possible, let it be (less). And let *OP* be made equal to *AB*, and *OQ* equal to *BC*, and let *PQ* be joined. And since *AB* is equal to *BC*, *OP* is also equal to *OQ*. Hence, the remainder *LP* is also equal to (the remainder) *QM*. *LM* is thus parallel to *PQ* [Prop. 6.2], and (triangle) *LMO* (is) equiangular with (triangle) *PQO* [Prop. 1.29]. Thus, as *OL* is to *LM*, so *OP* (is) to *PQ* [Prop. 6.4]. Alternately, as *LO* (is) to *OP*, so *LM* (is) to *PQ* [Prop. 5.16]. And *LO* (is) greater than *OP*. Thus, *LM* (is) also greater than *PQ* [Prop. 5.14]. But *LM* was made equal to *AC*. Thus, *AC* is also greater than *PQ*. Therefore, since the two (straight-lines) *AB* and *BC* are equal to the two (straight-lines) *PO* and *OQ* (respectively), and the base *AC* is greater than the base *PQ*, the angle *ABC* is thus greater than the angle *POQ* [Prop. 1.25]. So, similarly, we can show that *DEF* is also greater than *MON*, and *GHK* than *NOL*. Thus, the (sum of the) three angles *ABC*, *DEF*, and *GHK* is greater than the (sum of the) three angles *LOM*, *MON*, and *NOL*. But, (the sum of) *ABC*, *DEF*, and *GHK* was assumed (to be) less than four right-angles. Thus, (the sum of) *LOM*, *MON*, and *NOL* is much less than four right-angles. But, (it is) also equal (to four right-angles). The very thing is absurd. Thus, *AB* is not less than *LO*. And it was shown (to be) not equal either. Thus, *AB* (is) greater than *LO*.

So let *OR* be set up at point *O* at right-angles to the plane of circle *LMN* [Prop. 11.12]. And let the (square) on *OR* be equal to that (area) by which the square on *AB* is greater than the (square) on *LO* [Prop. 11.23 lem.]. And let *RL*, *RM*, and *RN* be joined.

And since *RO* is at right-angles to the plane of circle *LMN*, *RO* is thus also at right-angles to each of *LO*, *MO*, and *NO*. And since *LO* is equal to *OM*, and *OR* is common and at right-angles, the base *RL* is thus equal to the base *RM* [Prop. 1.4]. So, for the same (reasons), *RN* is also equal to each of *RL* and *RM*. Thus, the three (straight-lines) *RL*, *RM*, and *RN* are equal to one another. And since the (square) on *OR* was assumed to be equal to that (area) by which the (square) on *AB* is greater than the (square) on *LO*, the (square) on *AB* is thus equal to the (sum of the squares) on *LO* and *OR*. And the (square) on *LR* is equal to the (sum of the squares) on *LO* and *OR*. For *LOR* (is) a right-angle [Prop. 1.47]. Thus, the (square) on *AB* is equal to the (square) on *RL*. Thus, *AB* (is) equal to *RL*. But, each

ΛΡΝ, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, στερεὰ γωνία συνέσταται ἡ πρὸς τῷ Ρ περιεχομένη ὑπὸ τῶν ΛΡΜ, ΜΡΝ, ΛΡΝ γωνιῶν· ὅπερ ἔδει ποιῆσαι.



Λῆμμα.

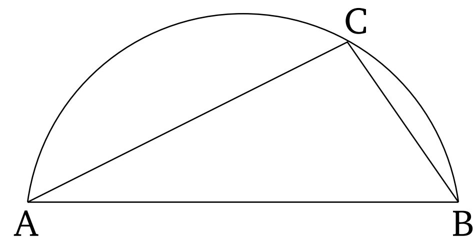
Ὃν δὲ τρόπον, ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον λαβεῖν ἔστι τὸ ἀπὸ τῆς ΞΡ, δείξομεν οὕτως. ἐκκείσθωσαν αἱ ΑΒ, ΛΞ εὐθεῖαι, καὶ ἔστω μείζων ἡ ΑΒ, καὶ γεγράφθω ἐπ᾽ αὐτῆς ἡμικύκλιον τὸ ΑΒΓ, καὶ εἰς τὸ ΑΒΓ ἡμικύκλιον ἐνηρμόσθω τῇ ΛΞ εὐθείᾳ μὴ μείζονι οὔσῃ τῆς ΑΒ διαμέτρου ἴση ἡ ΑΓ, καὶ ἐπεζεύχθω ἡ ΓΒ. ἐπεὶ οὖν ἐν ἡμικυκλίῳ τῷ ΑΓΒ γωνία ἐστὶν ἡ ὑπὸ ΑΓΒ, ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΑΓΒ. τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΓ, ΓΒ. ὥστε τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΑΓ μεῖζόν ἐστι τῷ ἀπὸ τῆς ΓΒ. ἴση δὲ ἡ ΑΓ τῇ ΛΞ. τὸ ἄρα ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ μεῖζόν ἐστι τῷ ἀπὸ τῆς ΓΒ. ἐὰν οὖν τῇ ΒΓ ἴσην τὴν ΞΡ ἀπολάβωμεν, ἔσται τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ μεῖζον τῷ ἀπὸ τῆς ΞΡ· ὅπερ προέκειτο ποιῆσαι.

κδ΄.

Ἐὰν στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

of *BC*, *DE*, *EF*, *GH*, and *HK* is equal to *AB*, and each of *RM* and *RN* equal to *RL*. Thus, each of *AB*, *BC*, *DE*, *EF*, *GH*, and *HK* is equal to each of *RL*, *RM*, and *RN*. And since the two (straight-lines) *LR* and *RM* are equal to the two (straight-lines) *AB* and *BC* (respectively), and the base *LM* was assumed (to be) equal to the base *AC*, the angle *LRM* is thus equal to the angle *ABC* [Prop. 1.8]. So, for the same (reasons), *MRN* is also equal to *DEF*, and *LRN* to *GHK*.

Thus, the solid angle *R*, contained by the angles *LRM*, *MRN*, and *LRN*, has been constructed out of the three plane angles *LRM*, *MRN*, and *LRN*, which are equal to the three given (plane angles) *ABC*, *DEF*, and *GHK* (respectively). (Which is) the very thing it was required to do.

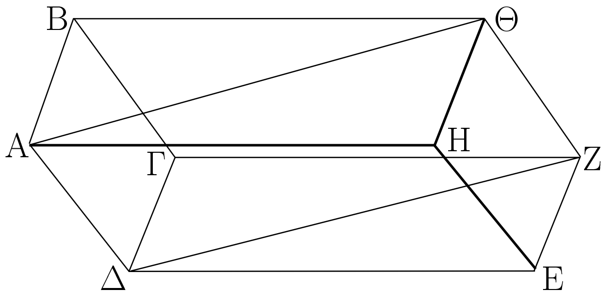


Lemma

And we can demonstrate, thusly, in which manner to take the (square) on *OR* equal to that (area) by which the (square) on *AB* is greater than the (square) on *LO*. Let the straight-lines *AB* and *LO* be set out, and let *AB* be greater, and let the semicircle *ABC* be drawn around it. And let *AC*, equal to the straight-line *LO*, which is not greater than the diameter *AB*, be inserted into the semicircle *ABC* [Prop. 4.1]. And let *CB* be joined. Therefore, since the angle *ACB* is in the semicircle *ACB*, *ACB* is thus a right-angle [Prop. 3.31]. Thus, the (square) on *AB* is equal to the (sum of the) squares on *AC* and *CB* [Prop. 1.47]. Hence, the (square) on *AB* is greater than the (square) on *AC* by the (square) on *CB*. And *AC* (is) equal to *LO*. Thus, the (square) on *AB* is greater than the (square) on *LO* by the (square) on *CB*. Therefore, if we take *OR* equal to *BC* then the (square) on *AB* will be greater than the (square) on *LO* by the (square) on *OR*. (Which is) the very thing it was prescribed to do.

Proposition 24

If a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic.

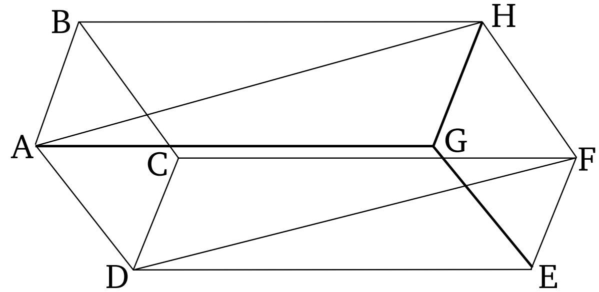


Στερεὸν γὰρ τὸ ΓΔΘΗ ὑπὸ παραλλήλων ἐπιπέδων περιεχέσθω τῶν ΑΓ, ΗΖ, ΑΘ, ΔΖ, ΒΖ, ΑΕ· λέγω, ὅτι τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

Ἐπεὶ γὰρ δύο ἐπίπεδα παράλληλα τὰ ΒΗ, ΓΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΔΓ. πάλιν, ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΒΖ, ΑΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΒΓ τῇ ΑΔ. ἐδείχθη δὲ καὶ ἡ ΑΒ τῇ ΔΓ παράλληλος· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν ΔΖ, ΖΗ, ΗΒ, ΒΖ, ΑΕ παραλληλόγραμμόν ἐστιν.

Ἐπεζεύχθωσαν αἱ ΑΘ, ΔΖ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ μὲν ΑΒ τῇ ΔΓ, ἡ δὲ ΒΘ τῇ ΓΖ, δύο δὴ αἱ ΑΒ, ΒΘ ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΓ, ΓΖ ἁπτομένας ἀλλήλων εἰσὶν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ· ἴσας ἄρα γωνίας περιέξουσιν· ἴση ἄρα ἡ ὑπὸ ΑΒΘ γωνία τῇ ὑπὸ ΔΓΖ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΘ δυσὶ ταῖς ΔΓ, ΓΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΑΒΘ γωνίᾳ τῇ ὑπὸ ΔΓΖ ἐστιν ἴση, βάσις ἄρα ἡ ΑΘ βάσει τῇ ΔΖ ἐστιν ἴση, καὶ τὸ ΑΒΘ τρίγωνον τῷ ΔΓΖ τριγώνῳ ἴσον ἐστίν. καί ἐστι τοῦ μὲν ΑΒΘ διπλάσιον τὸ ΒΗ παραλληλόγραμμον, τοῦ δὲ ΔΓΖ διπλάσιον τὸ ΓΕ παραλληλόγραμμον· ἴσον ἄρα τὸ ΒΗ παραλληλόγραμμον τῷ ΓΕ παραλληλογράμμῳ· ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ μὲν ΑΓ τῷ ΗΖ ἐστιν ἴσον, τὸ δὲ ΑΕ τῷ ΒΖ.

Ἐὰν ἄρα στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν· ὅπερ ἔδει δεῖξαι.



For let the solid (figure) *CDHG* be contained by the parallel planes *AC*, *GF*, and *AH*, *DF*, and *BF*, *AE*. I say that its opposite planes are both equal and parallelogrammic.

For since the two parallel planes *BG* and *CE* are cut by the plane *AC*, their common sections are parallel [Prop. 11.16]. Thus, *AB* is parallel to *DC*. Again, since the two parallel planes *BF* and *AE* are cut by the plane *AC*, their common sections are parallel [Prop. 11.16]. Thus, *BC* is parallel to *AD*. And *AB* was also shown (to be) parallel to *DC*. Thus, *AC* is a parallelogram. So, similarly, we can also show that *DF*, *FG*, *GB*, *BF*, and *AE* are each parallelograms.

Let *AH* and *DF* be joined. And since *AB* is parallel to *DC*, and *BH* to *CF*, so the two (straight-lines) joining one another, *AB* and *BH*, are parallel to the two straight-lines joining one another, *DC* and *CF* (respectively), not (being) in the same plane. Thus, they will contain equal angles [Prop. 11.10]. Thus, angle *ABH* (is) equal to (angle) *DCF*. And since the two (straight-lines) *AB* and *BH* are equal to the two (straight-lines) *DC* and *CF* (respectively) [Prop. 1.34], and angle *ABH* is equal to angle *DCF*, the base *AH* is thus equal to the base *DF*, and triangle *ABH* is equal to triangle *DCF* [Prop. 1.4]. And parallelogram *BG* is double (triangle) *ABH*, and parallelogram *CE* double (triangle) *DCF* [Prop. 1.34]. Thus, parallelogram *BG* (is) equal to parallelogram *CE*. So, similarly, we can show that *AC* is also equal to *GF*, and *AE* to *BF*.

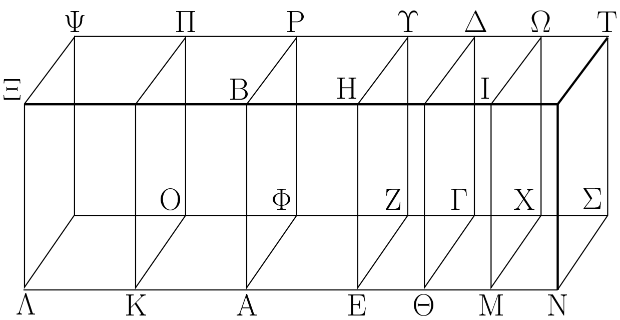
Thus, if a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic. (Which is) the very thing it was required to show.

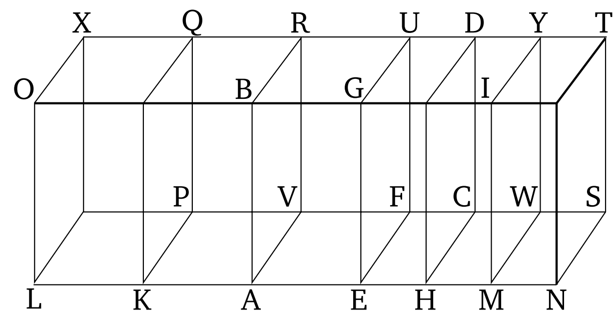
κε΄.

Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ἡ βάσις πρὸς τὴν βάσιν, οὕτως τὸ στερεὸν πρὸς τὸ στερεόν.

Proposition 25

If a parallelipiped solid is cut by a plane which is parallel to the opposite planes (of the parallelipiped) then as the base (is) to the base, so the solid will be to the solid.





Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒΓΔ ἐπιπέδῳ τῷ ΖΗ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΡΑ, ΔΘ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕΖΦ βάσις πρὸς τὴν ΕΘΓΖ βάσιν, οὕτως τὸ ΑΒΖΥ στερεὸν πρὸς τὸ ΕΗΓΔ στερεόν.

Ἐκβεβλήσθω γὰρ ἡ ΑΘ ἐφ᾽ ἑκάτερα τὰ μέρη, καὶ κείσθωσαν τῇ μὲν ΑΕ ἴσαι ὁσαιδηποτοῦν αἱ ΑΚ, ΚΛ, τῇ δὲ ΕΘ ἴσαι ὁσαιδηποτοῦν αἱ ΘΜ, ΜΝ, καὶ συμπεπληρώσθω τὰ ΛΟ, ΚΦ, ΘΧ, ΜΣ παραλληλόγραμμα καὶ τὰ ΛΠ, ΚΡ, ΔΜ, ΜΤ στερεά.

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΛΚ, ΚΑ, ΑΕ εὐθεῖαι ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ μὲν ΛΟ, ΚΦ, ΑΖ παραλληλόγραμμα ἀλλήλοις, τὰ δὲ ΚΞ, ΚΒ, ΑΗ ἀλλήλοις καὶ ἔτι τὰ ΛΨ, ΚΠ, ΑΡ ἀλλήλοις· ἀπεναντίον γάρ. διὰ τὰ αὐτὰ δὴ καὶ τὰ μὲν ΕΓ, ΘΧ, ΜΣ παραλληλόγραμμα ἴσα εἰσὶν ἀλλήλοις, τὰ δὲ ΘΗ, ΘΙ, ΙΝ ἴσα εἰσὶν ἀλλήλοις, καὶ ἔτι τὰ ΔΘ, ΜΩ, ΝΤ· τρία ἄρα ἐπίπεδα τῶν ΛΠ, ΚΡ, ΑΥ στερεῶν τρισὶν ἐπιπέδοις ἐστὶν ἴσα. ἀλλὰ τὰ τρία τρισὶ τοῖς ἀπεναντίον ἐστὶν ἴσα· τὰ ἄρα τρία στερεὰ τὰ ΛΠ, ΚΡ, ΑΥ ἴσα ἀλλήλοις ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ τὰ τρία στερεὰ τὰ ΕΔ, ΔΜ, ΜΤ ἴσα ἀλλήλοις ἐστίν· ὁσαπλασίων ἄρα ἐστὶν ἡ ΛΖ βάσις τῆς ΑΖ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΛΥ στερεὸν τοῦ ΑΥ στερεοῦ. διὰ τὰ αὐτὰ δὴ ὁσαπλασίων ἐστὶν ἡ ΝΖ βάσις τῆς ΖΘ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΝΥ στερεὸν τοῦ ΘΥ στερεοῦ. καὶ εἰ ἴση ἐστὶν ἡ ΛΖ βάσις τῇ ΝΖ βάσει, ἴσον ἐστὶ καὶ τὸ ΛΥ στερεὸν τῷ ΝΥ στερεῷ, καὶ εἰ ὑπερέχει ἡ ΛΖ βάσις τῆς ΝΖ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ στερεοῦ, καὶ εἰ ἐλλείπει, ἐλλείπει. τεσσάρων δὴ ὄντων μεγεθῶν, δύο μὲν βάσεων τῶν ΑΖ, ΖΘ, δύο δὲ στερεῶν τῶν ΑΥ, ΥΘ, εἴληπται ἰσάκις πολλαπλάσια τῆς μὲν ΑΖ βάσεως καὶ τοῦ ΑΥ στερεοῦ ἥ τε ΛΖ βάσις καὶ τὸ ΛΥ στερεόν, τῆς δὲ ΘΖ βάσεως καὶ τοῦ ΘΥ στερεοῦ ἥ τε ΝΖ βάσις καὶ τὸ ΝΥ στερεόν, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ ΛΖ βάσις τῆς ΖΝ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ [στερεοῦ], καὶ εἰ ἴση, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. ἔστιν ἄρα ὡς ἡ ΑΖ βάσις πρὸς τὴν ΖΘ βάσιν, οὕτως τὸ ΑΥ στερεὸν πρὸς τὸ ΥΘ στερεόν· ὅπερ ἔδει δεῖξαι.

For let the parallelipiped solid $ABCD$ be cut by the plane FG which is parallel to the opposite planes RA and DH. I say that as the base $AEFV$ (is) to the base $EHCF$, so the solid $ABFU$ (is) to the solid $EGCD$.

For let AH be produced in each direction. And let any number whatsoever (of lengths), AK and KL, be made equal to AE, and any number whatsoever (of lengths), HM and MN, equal to EH. And let the parallelograms LP, KV, HW, and MS be completed, and the solids LQ, KR, DM, and MT.

And since the straight-lines LK, KA, and AE are equal to one another, the parallelograms LP, KV, and AF are also equal to one another, and KO, KB, and AG (are equal) to one another, and, further, LX, KQ, and AR (are equal) to one another. For (they are) opposite [Prop. 11.24]. So, for the same (reasons), the parallelograms EC, HW, and MS are also equal to one another, and HG, HI, and IN are equal to one another, and, further, DH, MY, and NT (are equal to one another). Thus, three planes of (one of) the solids LQ, KR, and AU are equal to the (corresponding) three planes (of the others). But, the three planes (in one of the soilds) are equal to the three opposite planes [Prop. 11.24]. Thus, the three solids LQ, KR, and AU are equal to one another [Def. 11.10]. So, for the same (reasons), the three solids ED, DM, and MT are also equal to one another. Thus, as many multiples as the base LF is of the base AF, so many multiples is the solid LU also of the the solid AU. So, for the same (reasons), as many multiples as the base NF is of the base FH, so many multiples is the solid NU also of the solid HU. And if the base LF is equal to the base NF then the solid LU is also equal to the solid NU.† And if the base LF exceeds the base NF then the solid LU also exceeds the solid NU. And if (LF) is less than (NF) then (LU) is (also) less than (NU). So, there are four magnitudes, the two bases AF and FH, and the two solids AU and UH, and equal multiples be taken of the base AF and the solid AU— (namely), the base LF and the solid LU—and of the base HF and the solid HU—(namely), the base NF and the solid NU. And it has been shown that if the base LF exceeds the base FN then the solid LU also exceeds the [solid] NU, and if (LF is) equal (to FN) then (LU is) equal (to NU), and if (LF is) less than (FN) then (LU is) less than (NU). Thus, as the base AF is to the base FH, so the solid AU (is) to the solid UH [Def. 5.5].

(Which is) the very thing it was required to show.

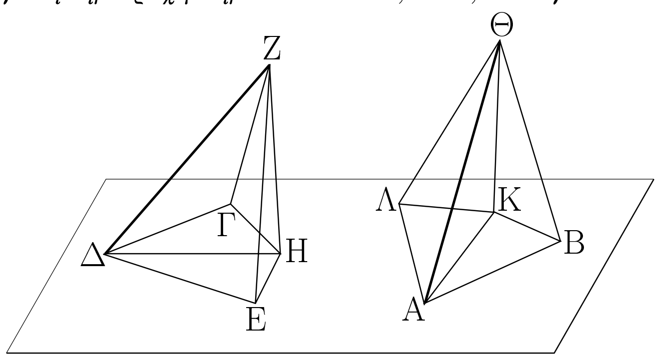
† Here, Euclid assumes that $LF \gtreqless NF$ implies $LU \gtreqless NU$. This is easily demonstrated.

κϛʹ.

Πρὸς τῇ δοθείσῃ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ δοθείσῃ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ πρὸς αὐτῇ δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα στερεὰ γωνία ἡ πρὸς τῷ Δ περιεχομένη ὑπὸ τῶν ὑπὸ ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν ἐπιπέδων· δεῖ δὴ πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

Εἰλήφθω γὰρ ἐπὶ τῆς ΔΖ τυχὸν σημεῖον τὸ Ζ, καὶ ἤχθω ἀπὸ τοῦ Ζ ἐπὶ τὸ διὰ τῶν ΕΔ, ΔΓ ἐπίπεδον κάθετος ἡ ΖΗ, καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΔΗ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ μὲν ὑπὸ ΕΔΓ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΛ, τῇ δὲ ὑπὸ ΕΔΗ ἴση ἡ ὑπὸ ΒΑΚ, καὶ κείσθω τῇ ΔΗ ἴση ἡ ΑΚ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ διὰ τῶν ΒΑΛ ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΚΘ, καὶ κείσθω ἴση τῇ ΗΖ ἡ ΚΘ, καὶ ἐπεζεύχθω ἡ ΘΑ· λέγω, ὅτι ἡ πρὸς τῷ Α στερεὰ γωνία περιεχομένη ὑπὸ τῶν ΒΑΛ, ΒΑΘ, ΘΑΛ γωνιῶν ἴση ἐστὶ τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ τῇ περιεχομένῃ ὑπὸ τῶν ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν.



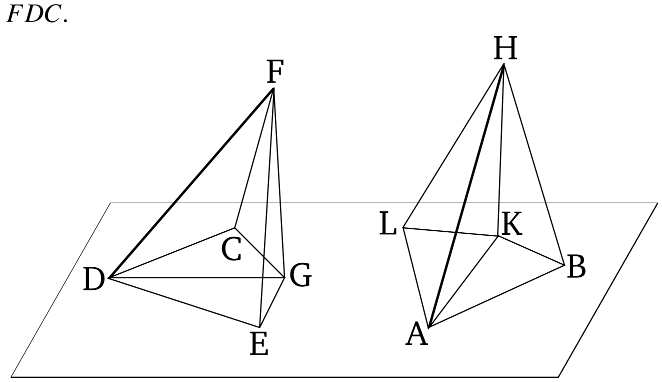
Ἀπειλήφθωσαν γὰρ ἴσαι αἱ ΑΒ, ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΒ, ΚΒ, ΖΕ, ΗΕ. καὶ ἐπεὶ ἡ ΖΗ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΖΗΔ, ΖΗΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΘΚΑ, ΘΚΒ

Proposition 26

To construct a solid angle equal to a given solid angle on a given straight-line, and at a given point on it.

Let *AB* be the given straight-line, and *A* the given point on it, and *D* the given solid angle, contained by the plane angles *EDC*, *EDF*, and *FDC*. So, it is necessary to construct a solid angle equal to the solid angle *D* on the straight-line *AB*, and at the point *A* on it.

For let some random point *F* be taken on *DF*, and let *FG* be drawn from *F* perpendicular to the plane through *ED* and *DC* [Prop. 11.11], and let it meet the plane at *G*, and let *DG* be joined. And let *BAL*, equal to the angle *EDC*, and *BAK*, equal to *EDG*, be constructed on the straight-line *AB* at the point *A* on it [Prop. 1.23]. And let *AK* be made equal to *DG*. And let *KH* be set up at the point *K* at right-angles to the plane through *BAL* [Prop. 11.12]. And let *KH* be made equal to *GF*. And let *HA* be joined. I say that the solid angle at *A*, contained by the (plane) angles *BAL*, *BAH*, and *HAL*, is equal to the solid angle at *D*, contained by the (plane) angles *EDC*, *EDF*, and *FDC*.



For let *AB* and *DE* be cut off (so as to be) equal, and let *HB*, *KB*, *FE*, and *GE* be joined. And since *FG* is at right-angles to the reference plane (*EDC*), it will also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Thus, the angles *FGD* and *γωνιῶν ὀρθή ἐστιν. καὶ ἐπεὶ δύο αἱ ΚΑ, ΑΒ δύο ταῖς ΗΔ, ΔΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΚΒ βάσει τῇ ΗΕ ἴση ἐστίν. ἔστι δὲ καὶ ἡ ΚΘ τῇ ΗΖ ἴση· καὶ γωνίας ὀρθὰς περιέχουσιν· ἴση ἄρα καὶ ἡ ΘΒ τῇ ΖΕ. πάλιν ἐπεὶ δύο αἱ ΑΚ, ΚΘ δυσὶ ταῖς ΔΗ, ΗΖ ἴσαι εἰσίν, καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΑΘ βάσει τῇ ΖΔ ἴση ἐστίν. ἔστι δὲ καὶ ἡ ΑΒ τῇ ΔΕ ἴση· δύο δὴ αἱ ΘΑ, ΑΒ δύο ταῖς ΔΖ, ΔΕ ἴσαι εἰσίν. καὶ βάσις ἡ ΘΒ βάσει τῇ ΖΕ ἴση· γωνία ἄρα ἡ ὑπὸ ΒΑΘ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΘΑΛ τῇ ὑπὸ ΖΔΓ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ὑπὸ ΒΑΛ τῇ ὑπὸ ΕΔΓ ἴση.*

Πρὸς ἄρα τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ στερεᾷ γωνίᾳ τῇ πρὸς τῷ Δ ἴση συνέσταται· ὅπερ ἔδει ποιῆσαι.

FGE are right-angles. So, for the same (reasons), the angles HKA and HKB are also right-angles. And since the two (straight-lines) KA and AB are equal to the two (straight-lines) GD and DE, respectively, and they contain equal angles, the base KB is thus equal to the base GE [Prop. 1.4]. And KH is also equal to GF. And they contain right-angles (with the respective bases). Thus, HB (is) also equal to FE [Prop. 1.4]. Again, since the two (straight-lines) AK and KH are equal to the two (straight-lines) DG and GF (respectively), and they contain right-angles, the base AH is thus equal to the base FD [Prop. 1.4]. And AB (is) also equal to DE. So, the two (straight-lines) HA and AB are equal to the two (straight-lines) DF and DE (respectively). And the base HB (is) equal to the base FE. Thus, the angle BAH is equal to the angle EDF [Prop. 1.8]. So, for the same (reasons), HAL is also equal to FDC. And BAL is also equal to EDC.

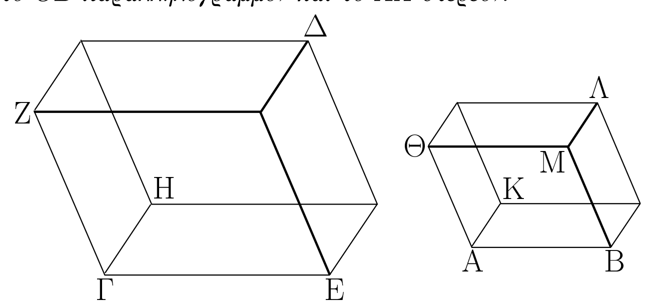
Thus, (a solid angle) has been constructed, equal to the given solid angle at D, on the given straight-line AB, at the given point A on it. (Which is) the very thing it was required to do.

κζ΄.

Ἀπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ ὅμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν στερεὸν παραλληλεπίπεδον τὸ ΓΔ· δεῖ δὴ ἀπὸ τῆς δοθείσης εὐθείας τῆς ΑΒ τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ τῷ ΓΔ ὅμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Συνεστάτω γὰρ πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ πρὸς τῷ Γ στερεᾷ γωνίᾳ ἴση ἡ περιεχομένη ὑπὸ τῶν ΒΑΘ, ΘΑΚ, ΚΑΒ, ὥστε ἴσην εἶναι τὴν μὲν ὑπὸ ΒΑΘ γωνίαν τῇ ὑπὸ ΕΓΖ, τὴν δὲ ὑπὸ ΒΑΚ τῇ ὑπὸ ΕΓΗ, τὴν δὲ ὑπὸ ΚΑΘ τῇ ὑπὸ ΗΓΖ· καὶ γεγονέτω ὡς μὲν ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, ὡς δὲ ἡ ΗΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΚΑ πρὸς τὴν ΑΘ. καὶ δι᾽ ἴσου ἄρα ἐστὶν ὡς ἡ ΕΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΘ. καὶ συμπεπληρώσθω τὸ ΘΒ παραλληλόγραμμον καὶ τὸ ΑΛ στερεόν.



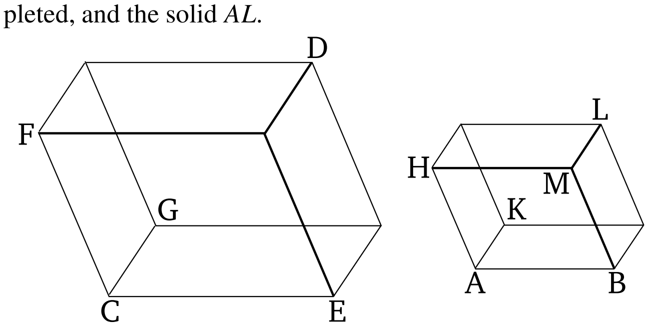
Καὶ ἐπεί ἐστιν ὡς ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΕΓΗ, ΒΑΚ αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ ΗΕ παραλληλόγραμμον

Proposition 27

To describe a parallelepiped solid similar, and similarly laid out, to a given parallelepiped solid on a given straight-line.

Let the given straight-line be AB, and the given parallelepiped solid CD. So, it is necessary to describe a parallelepiped solid similar, and similarly laid out, to the given parallelepiped solid CD on the given straight-line AB.

For, let a (solid angle) contained by the (plane angles) BAH, HAK, and KAB be constructed, equal to solid angle at C, on the straight-line AB at the point A on it [Prop. 11.26], such that angle BAH is equal to ECF, and BAK to ECG, and KAH to GCF. And let it be contrived that as EC (is) to CG, so BA (is) to AK, and as GC (is) to CF, so KA (is) to AH [Prop. 6.12]. And thus, via equality, as EC is to CF, so BA (is) to AH [Prop. 5.22]. And let the parallelogram HB be completed, and the solid AL.



And since as EC is to CG, so BA (is) to AK, and the sides about the equal angles ECG and BAK are (thus) proportional, the parallelogram GE is thus similar to the parallelogram KB.

τῷ ΚΒ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΘ παραλληλόγραμμον τῷ ΗΖ παραλληλογράμμῳ ὅμοιόν ἐστι καὶ ἔτι τὸ ΖΕ τῷ ΘΒ· τρία ἄρα παραλληλόγραμμα τοῦ ΓΔ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΑΛ στερεοῦ ὅμοιά ἐστιν. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια· ὅλον ἄρα τὸ ΓΔ στερεὸν ὅλῳ τῷ ΑΛ στερεῷ ὅμοιόν ἐστιν.

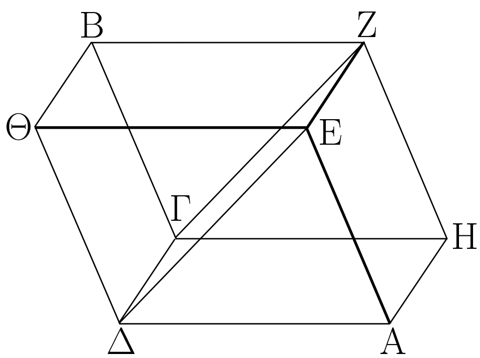
Ἀπὸ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ τῷ ΓΔ ὅμοιόν τε καὶ ὁμοίως κείμενον ἀναγέγραπται τὸ ΑΛ· ὅπερ ἔδει ποιῆσαι.

So, for the same (reasons), the parallelogram KH is also similar to the parallelogram GF, and, further, FE (is similar) to HB. Thus, three of the parallelograms of solid CD are similar to three of the parallelograms of solid AL. But, the (former) three are equal and similar to the three opposite, and the (latter) three are equal and similar to the three opposite. Thus, the whole solid CD is similar to the whole solid AL [Def. 11.9].

Thus, AL, similar, and similarly laid out, to the given parallelepiped solid CD, has been described on the given straight-lines AB. (Which is) the very thing it was required to do.

κη'.

Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ κατὰ τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων, δίχα τμηθήσεται τὸ στερεὸν ὑπὸ τοῦ ἐπιπέδου.

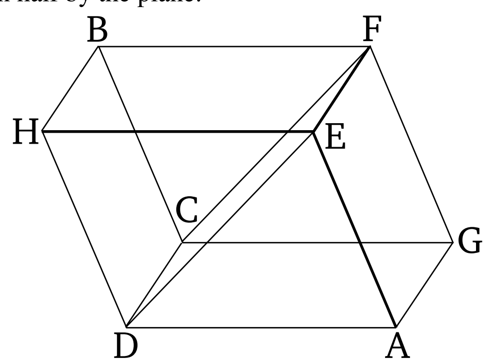


Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒ ἐπιπέδῳ τῷ ΓΔΕΖ τετμήσθω κατὰ τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων τὰς ΓΖ, ΔΕ· λέγω, ὅτι δίχα τμηθήσεται τὸ ΑΒ στερεὸν ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου.

Ἐπεὶ γὰρ ἴσον ἐστὶ τὸ μὲν ΓΗΖ τρίγωνον τῷ ΓΖΒ τριγώνῳ, τὸ δὲ ΑΔΕ τῷ ΔΕΘ, ἔστι δὲ καὶ τὸ μὲν ΓΑ παραλληλόγραμμον τῷ ΕΒ ἴσον· ἀπεναντίον γάρ· τὸ δὲ ΗΕ τῷ ΓΘ, καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΓΗΖ, ΑΔΕ, τριῶν δὲ παραλληλογράμμων τῶν ΗΕ, ΑΓ, ΓΕ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΓΖΒ, ΔΕΘ, τριῶν δὲ παραλληλογράμμων τῶν ΓΘ, ΒΕ, ΓΕ· ὑπὸ γὰρ ἴσων ἐπιπέδων περιέχονται τῷ τε πλήθει καὶ τῷ μεγέθει. ὥστε ὅλον τὸ ΑΒ στερεὸν δίχα τέτμηται ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου· ὅπερ ἔδει δεῖξαι.

Proposition 28

If a parallelepiped solid is cut by a plane (passing) through the diagonals of (a pair of) opposite planes then the solid will be cut in half by the plane.



For let the parallelepiped solid AB be cut by the plane $CDEF$ (passing) through the diagonals of the opposite planes CF and DE.† I say that the solid AB will be cut in half by the plane $CDEF$.

For since triangle CGF is equal to triangle CFB, and ADE (is equal) to DEH [Prop. 1.34], and parallelogram CA is also equal to EB—for (they are) opposite [Prop. 11.24]—and GE (equal) to CH, thus the prism contained by the two triangles CGF and ADE, and the three parallelograms GE, AC, and CE, is also equal to the prism contained by the two triangles CFB and DEH, and the three parallelograms CH, BE, and CE. For they are contained by planes (which are) equal in number and in magnitude [Def. 11.10].‡ Thus, the whole of solid AB is cut in half by the plane $CDEF$. (Which is) the very thing it was required to show.

† Here, it is assumed that the two diagonals lie in the same plane. The proof is easily supplied.

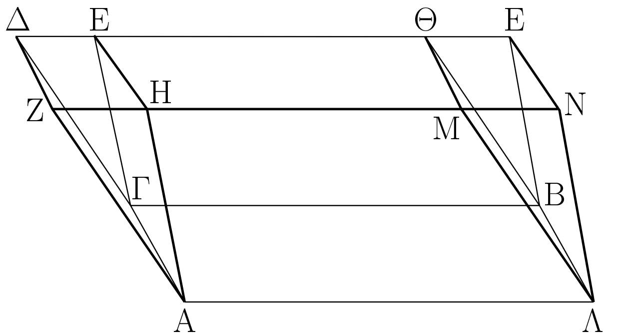
‡ However, strictly speaking, the prisms are not similarly arranged, being mirror images of one another.

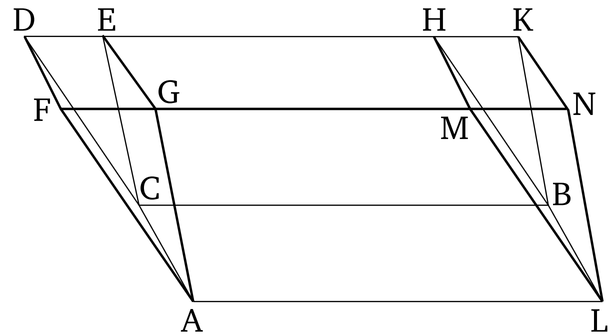
κθ'.

Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.

Proposition 29

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are on the same straight-lines, are equal to one another.





Ἔστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλληλεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΖ, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ ἐπὶ τῶν αὐτῶν εὐθειῶν ἔστωσαν τῶν ΖΝ, ΔΚ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐπεὶ γὰρ παραλληλόγραμμόν ἐστιν ἑκάτερον τῶν ΓΘ, ΓΚ, ἴση ἐστὶν ἡ ΓΒ ἑκατέρᾳ τῶν ΔΘ, ΕΚ· ὥστε καὶ ἡ ΔΘ τῇ ΕΚ ἐστιν ἴση. κοινὴ ἀφῃρήσθω ἡ ΕΘ· λοιπὴ ἄρα ἡ ΔΕ λοιπῇ τῇ ΘΚ ἐστιν ἴση. ὥστε καὶ τὸ μὲν ΔΓΕ τρίγωνον τῷ ΘΒΚ τριγώνῳ ἴσον ἐστίν, τὸ δὲ ΔΗ παραλληλόγραμμον τῷ ΘΝ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΖΗ τρίγωνον τῷ ΜΛΝ τριγώνῳ ἴσον ἐστίν. ἔστι δὲ καὶ τὸ μὲν ΓΖ παραλληλόγραμμον τῷ ΒΜ παραλληλογράμμῳ ἴσον, τὸ δὲ ΓΗ τῷ ΒΝ· ἀπεναντίον γάρ· καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΑΖΗ, ΔΓΕ, τριῶν δὲ παραλληλογράμμων τῶν ΑΔ, ΔΗ, ΓΗ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΜΛΝ, ΘΒΚ, τριῶν δὲ παραλληλογράμμων τῶν ΒΜ, ΘΝ, ΒΝ. κοινὸν προσκείσθω τὸ στερεόν, οὗ βάσις μὲν τὸ ΑΒ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΘΜ· ὅλον ἄρα τὸ ΓΜ στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΓΝ στερεῷ παραλληλεπιπέδῳ ἴσον ἐστίν.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

For let the parallelepiped solids CM and CN be on the same base AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AG, AF, LM, LN, CD, CE, BH, and BK, be on the same straight-lines, FN and DK. I say that solid CM is equal to solid CN.

For since CH and CK are each parallelograms, CB is equal to each of DH and EK [Prop. 1.34]. Hence, DH is also equal to EK. Let EH be subtracted from both. Thus, the remainder DE is equal to the remainder HK. Hence, triangle DCE is also equal to triangle HBK [Props. 1.4, 1.8], and parallelogram DG to parallelogram HN [Prop. 1.36]. So, for the same (reasons), traingle AFG is also equal to triangle MLN. And parallelogram CF is also equal to parallelogram BM, and CG to BN [Prop. 11.24]. For they are opposite. Thus, the prism contained by the two triangles AFG and DCE, and the three parallelograms AD, DG, and CG, is equal to the prism contained by the two triangles MLN and HBK, and the three parallelograms BM, HN, and BN. Let the solid whose base (is) parallelogram AB, and (whose) opposite (face is) $GEHM$, be added to both (prisms). Thus, the whole parallelepiped solid CM is equal to the whole parallelepiped solid CN.

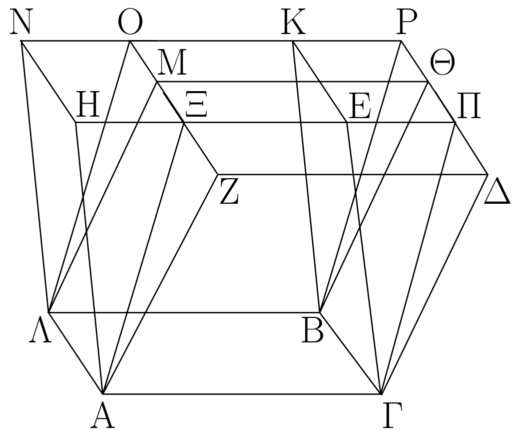
Thus, parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up (are) on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

λ΄.

Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.

Proposition 30

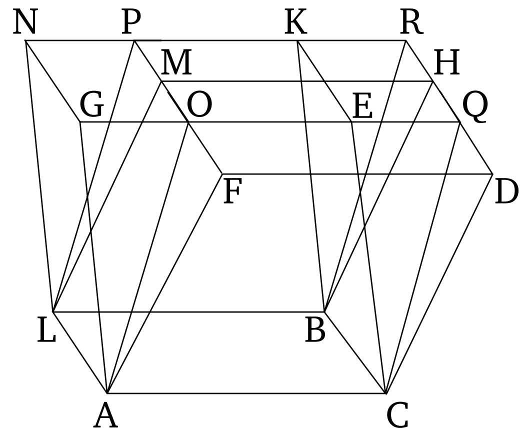
Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another.



Ἔστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλληλεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΗ, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ μὴ ἔστωσαν ἐπὶ τῶν αὐτῶν εὐθειῶν· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐκβεβλήσθωσαν γὰρ αἱ ΝΚ, ΔΘ καὶ συμπιπτέτωσαν ἀλλήλαις κατὰ τὸ Ρ, καὶ ἔτι ἐκβεβλήσθωσαν αἱ ΖΜ, ΗΕ ἐπὶ τὰ Ο, Π, καὶ ἐπεζεύχθωσαν αἱ ΑΞ, ΛΟ, ΓΠ, ΒΡ. ἴσον δή ἐστι τὸ ΓΜ στερεόν, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΖΔΘΜ, τῷ ΓΟ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΑΓΒΛ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΞ, ΛΜ, ΛΟ, ΓΔ, ΓΠ, ΒΘ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΖΟ, ΔΡ. ἀλλὰ τὸ ΓΟ στερεόν, οὗ βάσις μέν ἐστι τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ, ἴσον ἐστὶ τῷ ΓΝ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΚΝ· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς εἰσι τῆς ΑΓΒΛ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΞ, ΓΕ, ΓΠ, ΛΝ, ΛΟ, ΒΚ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΗΠ, ΝΡ. ὥστε καὶ τὸ ΓΜ στερεὸν ἴσον ἐστὶ τῷ ΓΝ στερεῷ.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.



Let the parallelepiped solids CM and CN be on the same base, AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AF, AG, LM, LN, CD, CE, BH, and BK, not be on the same straight-lines. I say that the solid CM is equal to the solid CN.

For let NK and DH be produced, and let them have joined one another at R. And, further, let FM and GE be produced to P and Q (respectively). And let AO, LP, CQ, and BR be joined. So, solid CM, whose base (is) parallelogram $ACBL$, and opposite (face) $FDHM$, is equal to solid CP, whose base (is) parallelogram $ACBL$, and opposite (face) $OQRP$. For they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, AF, AO, LM, LP, CD, CQ, BH, and BR, are on the same straight-lines, FP and DR [Prop. 11.29]. But, solid CP, whose base is parallelogram $ACBL$, and opposite (face) $OQRP$, is equal to solid CN, whose base (is) parallelogram $ACBL$, and opposite (face) $GEKN$. For, again, they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, AG, AO, CE, CQ, LN, LP, BK, and BR, are on the same straight-lines, GQ and NR [Prop. 11.29]. Hence, solid CM is also equal to solid CN.

Thus, parallelepiped solids (which are) on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

λα΄.

Τὰ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν.

Ἔστω ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα τὰ ΑΕ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος. λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ.

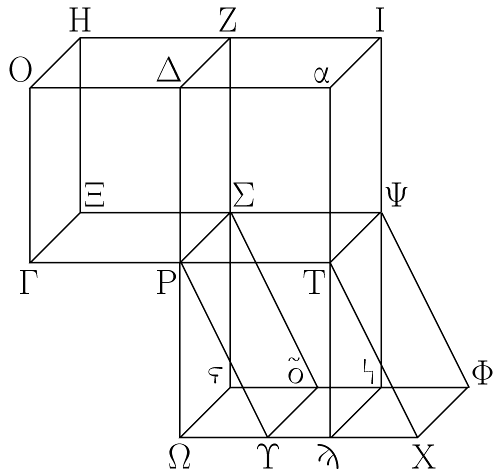
Ἔστωσαν δὴ πρότερον αἱ ἐφεστηκυῖαι αἱ ΘΚ, ΒΕ, ΑΗ, ΛΜ, ΟΠ, ΔΖ, ΓΞ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν, καὶ ἐκβεβλήσθω ἐπ᾽ εὐθείας τῇ ΓΡ εὐθεῖα ἡ ΡΤ, καὶ συνεστάτω

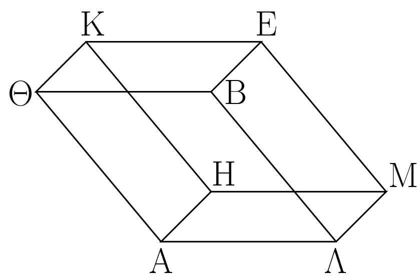
Proposition 31

Parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another.

Let the parallelepiped solids AE and CF be on the equal bases AB and CD (respectively), and (have) the same height. I say that solid AE is equal to solid CF.

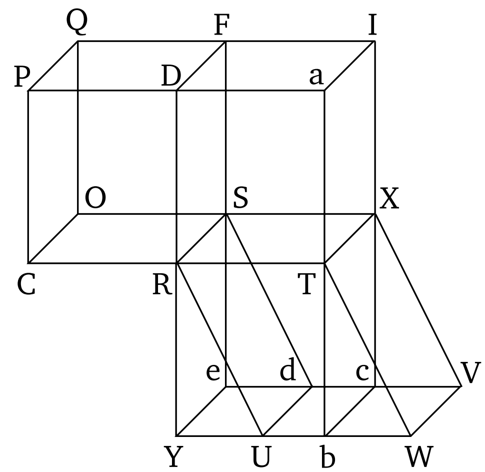
So, let the (straight-lines) standing up, HK, BE, AG, LM, PQ, DF, CO, and RS, first of all, be at right-angles to the bases AB and CD. And let RT be produced in a straight-line with *πρὸς τῇ ΡΤ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ρ τῇ ὑπὸ ΑΛΒ γωνίᾳ ἴση ἡ ὑπὸ ΤΡΥ, καὶ κείσθω τῇ μὲν ΑΛ ἴση ἡ ΡΤ, τῇ δὲ ΛΒ ἴση ἡ ΡΥ, καὶ συμπεπληρώσθω ἥ τε ΡΧ βάσις καὶ τὸ ΨΥ στερεόν.*

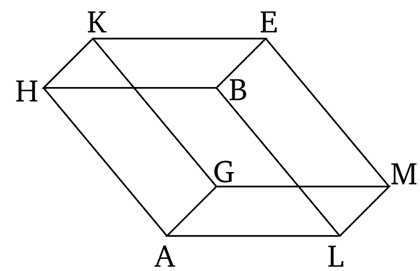




Καὶ ἐπεὶ δύο αἱ ΤΡ, ΡΥ δυσὶ ταῖς ΑΛ, ΛΒ ἴσαι εἰσίν, καὶ γωνίας ἴσας περιέχουσιν, ἴσον ἄρα καὶ ὅμοιον τὸ ΡΧ παραλληλόγραμμον τῷ ΘΛ παραλληλογράμμῳ. καὶ ἐπεὶ πάλιν ἴση μὲν ἡ ΑΛ τῇ ΡΤ, ἡ δὲ ΛΜ τῇ ΡΣ, καὶ γωνίας ὀρθὰς περιέχουσιν, ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ ΡΨ παραλληλόγραμμον τῷ ΑΜ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΛΕ τῷ ΣΥ ἴσον τέ ἐστι καὶ ὅμοιον· τρία ἄρα παραλληλόγραμμα τοῦ ΑΕ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΨΥ στερεοῦ ἴσα τέ ἐστι καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον· ὅλον ἄρα τὸ ΑΕ στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΨΥ στερεῷ παραλληλεπιπέδῳ ἴσον ἐστίν. διήχθωσαν αἱ ΔΡ, ΧΥ καὶ συμπιπτέτωσαν ἀλλήλαις κατὰ τὸ Ω, καὶ διὰ τοῦ Τ τῇ ΔΩ παράλληλος ἤχθω ἡ αΤϡ, καὶ ἐκβεβλήσθω ἡ ΟΔ κατὰ τὸ α, καὶ συμπεπληρώσθω τὰ ΩΨ, ΡΙ στερεά. ἴσον δή ἐστι τὸ ΨΩ στερεόν, οὗ βάσις μέν ἐστι τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ Ωϙ, τῷ ΨΥ στερεῷ, οὗ βάσις μὲν τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΥΦ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΨ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΡΩ, ΡΥ, Τϡ, ΤΧ, Σϛ, Σο̃, Ψϙ, ΨΦ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΩΧ, ϛΦ. ἀλλὰ τὸ ΨΥ στερεὸν τῷ ΑΕ ἐστιν ἴσον· καὶ τὸ ΨΩ ἄρα στερεὸν τῷ ΑΕ στερεῷ ἐστιν

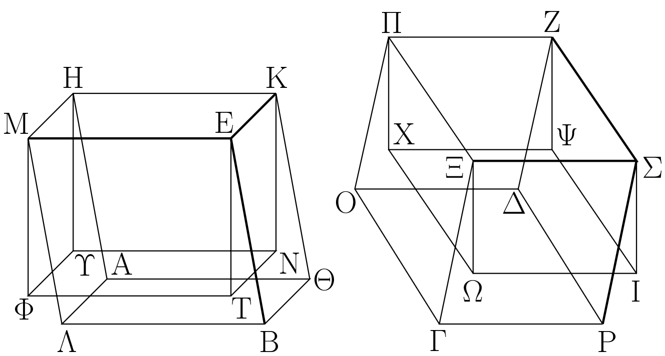
CR. And let (angle) *TRU*, equal to angle *ALB*, be constructed on the straight-line *RT*, at the point *R* on it [Prop. 1.23]. And let *RT* be made equal to *AL*, and *RU* to *LB*. And let the base *RW*, and the solid *XU*, be completed.





And since the two (straight-lines) *TR* and *RU* are equal to the two (straight-lines) *AL* and *LB* (respectively), and they contain equal angles, parallelogram *RW* is thus equal and similar to parallelogram *HL* [Prop. 6.14]. And, again, since *AL* is equal to *RT*, and *LM* to *RS*, and they contain right-angles, parallelogram *RX* is thus equal and similar to parallelogram *AM* [Prop. 6.14]. So, for the same (reasons), *LE* is also equal and similar to *SU*. Thus, three parallelograms of solid *AE* are equal and similar to three parallelograms of solid *XU*. But, the three (faces of the former solid) are equal and similar to the three opposite (faces), and the three (faces of the latter solid) to the three opposite (faces) [Prop. 11.24]. Thus, the whole parallelepiped solid *AE* is equal to the whole parallelepiped solid *XU* [Def. 11.10]. Let *DR* and *WU* be drawn across, and let them have met one another at *Y*. And let *aTb* be drawn through *T* parallel to *DY*. And let *PD* be produced to *a*. And let the solids *YX* and *RI* be completed. So, solid *XY*, whose base is parallelogram *RX*, and opposite (face) *Yc*, is equal to solid *XU*, whose base (is) parallelogram *RX*, and opposite (face) *UV*. For they are on the same base *RX*, and (have) the same height, and the (ends of the straight-lines) standing up in them, *RY*, *RU*, *Tb*, *TW*, *Se*, *Sd*, *Xc* and *XV*, are on the same straight-

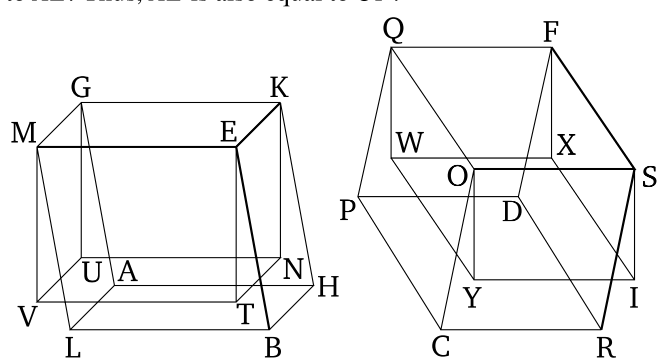
ἴσον. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΡΥΧΤ παραλληλόγραμμον τῷ ΩΤ παραλληλογράμμῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΤ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΡΤ, ΩΧ· ἀλλὰ τὸ ΡΥΧΤ τῷ ΓΔ ἐστιν ἴσον, ἐπεὶ καὶ τῷ ΑΒ, καὶ τὸ ΩΤ ἄρα παραλληλόγραμμον τῷ ΓΔ ἐστιν ἴσον. ἄλλο δὲ τὸ ΔΤ· ἔστιν ἄρα ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΙ ἐπιπέδῳ τῷ ΡΖ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ βάσιν, οὕτως τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν. διὰ τὰ αὐτὰ δή, ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΩΙ ἐπιπέδῳ τῷ ΡΨ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΩΤ βάσις πρὸς τὴν ΤΛ βάσιν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἀλλ᾽ ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ· καὶ ὡς ἄρα τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἑκάτερον ἄρα τῶν ΓΖ, ΩΨ στερεῶν πρὸς τὸ ΡΙ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΓΖ στερεὸν τῷ ΩΨ στερεῷ. ἀλλὰ τὸ ΩΨ τῷ ΑΕ ἐδείχθη ἴσον· καὶ τὸ ΑΕ ἄρα τῷ ΓΖ ἐστιν ἴσον.



μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΑΗ, ΘΚ, ΒΕ, ΛΜ, ΓΞ, ΟΠ, ΔΖ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν· λέγω πάλιν, ὅτι ἴσον τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ. ἤχθωσαν γὰρ ἀπὸ τῶν Κ, Ε, Η, Μ, Π, Ζ, Ξ, Σ σημείων ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετοι αἱ ΚΝ, ΕΤ, ΗΥ, ΜΦ, ΠΧ, ΖΨ, ΞΩ, ΣΙ, καὶ συμβαλλέτωσαν τῷ ἐπιπέδῳ κατὰ τὰ Ν, Τ, Υ, Φ, Χ, Ψ, Ω, Ι σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΝΤ, ΝΥ, ΥΦ, ΤΦ, ΧΨ, ΧΩ, ΩΙ, ΙΨ. ἴσον δή ἐστι τὸ ΚΦ στερεὸν τῷ ΠΙ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΚΜ, ΠΣ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι πρὸς ὀρθάς εἰσι ταῖς βάσεσιν. ἀλλὰ τὸ μὲν ΚΦ στερεὸν τῷ ΑΕ στερεῷ ἐστιν ἴσον, τὸ δὲ ΠΙ τῷ ΓΖ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσιν ἐπὶ τῶν αὐτῶν εὐθειῶν. καὶ τὸ ΑΕ ἄρα στερεὸν τῷ ΓΖ στερεῷ ἐστιν ἴσον.

Τὰ ἄρα ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

lines, YW and eV [Prop. 11.29]. But, solid XU is equal to AE. Thus, solid XY is also equal to solid AE. And since parallelogram $RUWT$ is equal to parallelogram YT. For they are on the same base RT, and between the same parallels RT and YW [Prop. 1.35]. But, $RUWT$ is equal to CD, since (it is) also (equal) to AB. Parallelogram YT is thus also equal to CD. And DT is another (parallelogram). Thus, as base CD is to DT, so YT (is) to DT [Prop. 5.7]. And since the parallelepiped solid CI has been cut by the plane RF, which is parallel to the opposite planes (of CI), as base CD is to base DT, so solid CF (is) to solid RI [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid YI has been cut by the plane RX, which is parallel to the opposite planes (of YI), as base YT is to base TD, so solid YX (is) to solid RI [Prop. 11.25]. But, as base CD (is) to DT, so YT (is) to DT. And, thus, as solid CF (is) to solid RI, so solid YX (is) to solid RI. Thus, solids CF and YX each have the same ratio to RI [Prop. 5.11]. Thus, solid CF is equal to solid YX [Prop. 5.9]. But, YX was show (to be) equal to AE. Thus, AE is also equal to CF.



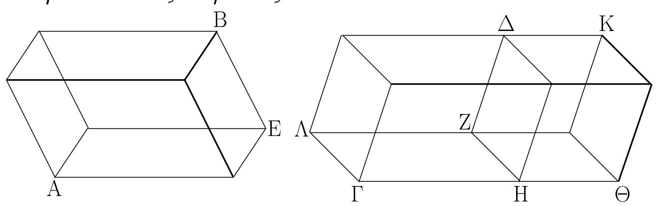
And so let the (straight-lines) standing up, AG, HK, BE, LM, CO, PQ, DF, and RS, not be at right-angles to the bases AB and CD. Again, I say that solid AE (is) equal to solid CF. For let KN, ET, GU, MV, QW, FX, OY, and SI be drawn from points K, E, G, M, Q, F, O, and S (respectively) perpendicular to the reference plane (i.e., the plane of the bases AB and CD), and let them have met the plane at points N, T, U, V, W, X, Y, and I (respectively). And let NT, NU, UV, TV, WX, WY, YI, and IX be joined. So solid KV is equal to solid QI. For they are on the equal bases KM and QS, and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid KV is equal to solid AE, and QI to CF. For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid AE is also equal to solid CF.

Thus, parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.

λβʹ.

Proposition 32

Τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.



Ἔστω ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ· λέγω, ὅτι τὰ ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις, τουτέστιν ὅτι ἐστὶν ὡς ἡ ΑΕ βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεόν.

Παραβεβλήσθω γὰρ παρὰ τὴν ΖΗ τῷ ΑΕ ἴσον τὸ ΖΘ, καὶ ἀπὸ βάσεως μὲν τῆς ΖΘ, ὕψους δὲ τοῦ αὐτοῦ τῷ ΓΔ στερεὸν παραλληλεπίπεδον συμπεπληρώσθω τὸ ΗΚ. ἴσον δή ἐστι τὸ ΑΒ στερεὸν τῷ ΗΚ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΑΕ, ΖΘ καὶ ὑπὸ τὸ αὐτο ὕψος. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΚ ἐπιπέδῳ τῷ ΔΗ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ἡ ΓΖ βάσις πρὸς τὴν ΖΘ βάσιν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΔΘ στερεόν. ἴση δὲ ἡ μὲν ΖΘ βάσις τῇ ΑΕ βάσει, τὸ δὲ ΗΚ στερεὸν τῷ ΑΒ στερεῷ· ἔστιν ἄρα καὶ ὡς ἡ ΑΕ βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεόν.

Τὰ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

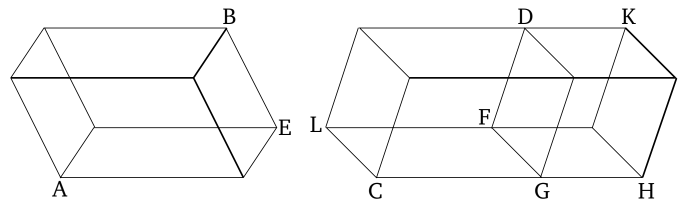
λγ΄.

Τὰ ὅμοια στερεὰ παραλληλεπίπεδα πρὸς ἄλληλα ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἔστω ὅμοια στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ, ὁμόλογος δὲ ἔστω ἡ ΑΕ τῇ ΓΖ· λέγω, ὅτι τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεὸν τριπλασίονα λόγον ἔχει, ἤπερ ἡ ΑΕ πρὸς τὴν ΓΖ.

Ἐκβεβλήσθωσαν γὰρ ἐπ᾽ εὐθείας ταῖς ΑΕ, ΗΕ, ΘΕ αἱ ΕΚ, ΕΛ, ΕΜ, καὶ κείσθω τῇ μὲν ΓΖ ἴση ἡ ΕΚ, τῇ δὲ ΖΝ ἴση ἡ ΕΛ, καὶ ἔτι τῇ ΖΡ ἴση ἡ ΕΜ, καὶ συμπεπληρώσθω τὸ ΚΛ παραλληλόγραμμον καὶ τὸ ΚΟ στερεόν.

Parallelepiped solids which (have) the same height are to one another as their bases.



Let *AB* and *CD* be parallelepiped solids (having) the same height. I say that the parallelepiped solids *AB* and *CD* are to one another as their bases. That is to say, as base *AE* is to base *CF*, so solid *AB* (is) to solid *CD*.

For let *FH*, equal to *AE*, be applied to *FG* (in the angle *FGH* equal to angle *LCG*) [Prop. 1.45]. And let the parallelepiped solid *GK*, (having) the same height as *CD*, be completed on the base *FH*. So solid *AB* is equal to solid *GK*. For they are on the equal bases *AE* and *FH*, and (have) the same height [Prop. 11.31]. And since the parallelepiped solid *CK* has been cut by the plane *DG*, which is parallel to the opposite planes (of *CK*), thus as the base *CF* is to the base *FH*, so the solid *CD* (is) to the solid *DH* [Prop. 11.25]. And base *FH* (is) equal to base *AE*, and solid *GK* to solid *AB*. And thus as base *AE* is to base *CF*, so solid *AB* (is) to solid *CD*.

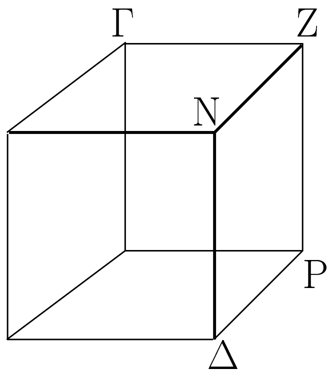
Thus, parallelepiped solids which (have) the same height are to one another as their bases. (Which is) the very thing it was required to show.

Proposition 33

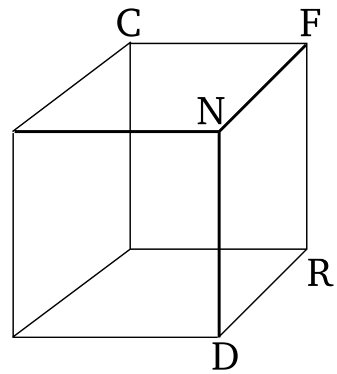
Similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides.

Let *AB* and *CD* be similar parallelepiped solids, and let *AE* correspond to *CF*. I say that solid *AB* has to solid *CD* the cubed ratio that *AE* (has) to *CF*.

For let *EK*, *EL*, and *EM* be produced in a straight-line with *AE*, *GE*, and *HE* (respectively). And let *EK* be made equal to *CF*, and *EL* equal to *FN*, and, further, *EM* equal to *FR*. And let the parallelogram *KL* be completed, and the solid *KP*.

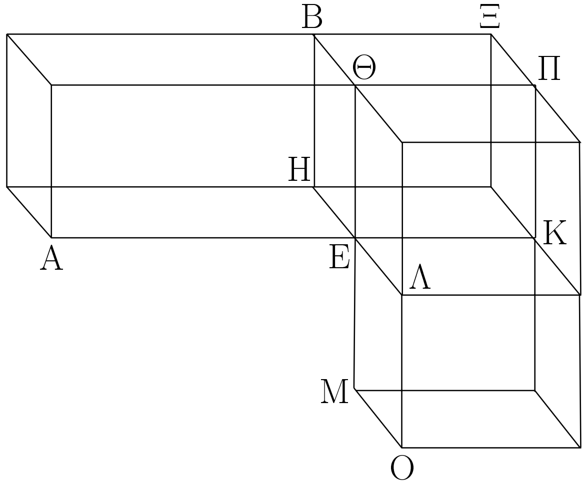


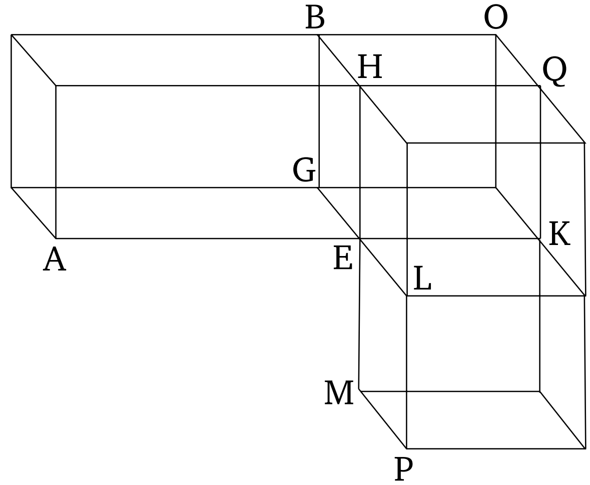
Καὶ ἐπεὶ δύο αἱ ΚΕ, ΕΛ δυσὶ ταῖς ΓΖ, ΖΝ ἴσαι εἰσίν, ἀλλὰ καὶ γωνία ἡ ὑπὸ ΚΕΛ γωνίᾳ τῇ ὑπὸ ΓΖΝ ἐστιν ἴση, ἐπειδήπερ καὶ ἡ ὑπὸ ΑΕΗ τῇ ὑπὸ ΓΖΝ ἐστιν ἴση διὰ τὴν ὁμοιότητα τῶν ΑΒ, ΓΔ στερεῶν, ἴσον ἄρα ἐστὶ [καὶ ὅμοιον] τὸ ΚΛ παραλληλόγραμμον τῷ ΓΝ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΜ παραλληλόγραμμον ἴσον ἐστὶ καὶ ὅμοιον τῷ ΓΡ [παραλληλογράμμῳ] καὶ ἔτι τὸ ΕΟ τῷ ΔΖ· τρία ἄρα παραλληλόγραμμα τοῦ ΚΟ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΓΔ στερεοῦ ἴσα ἐστὶ καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια· ὅλον ἄρα τὸ ΚΟ στερεὸν ὅλῳ τῷ ΓΔ στερεῷ ἴσον ἐστὶ καὶ ὅμοιον. συμπεπληρώσθω τὸ ΗΚ παραλληλόγραμμον, καὶ ἀπὸ βάσεων μὲν τῶν ΗΚ, ΚΛ παραλληλόγραμμων, ὕψους δὲ τοῦ αὐτοῦ τῷ ΑΒ στερεὰ συμπεπληρώσθω τὰ ΕΞ, ΛΠ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΑΒ, ΓΔ στερεῶν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΝ, καὶ ἡ ΕΘ πρὸς τὴν ΖΡ, ἴση δὲ ἡ μὲν ΓΖ τῇ ΕΚ, ἡ δὲ ΖΝ τῇ ΕΛ, ἡ δὲ ΖΡ τῇ ΕΜ, ἔστιν ἄρα ὡς ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΛ καὶ ἡ ΘΕ πρὸς τὴν ΕΜ. ἀλλ᾿ ὡς μὲν ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως τὸ ΑΗ [παραλληλόγραμμον] πρὸς τὸ ΗΚ παραλληλόγραμμον, ὡς δὲ ἡ ΗΕ πρὸς τὴν ΕΛ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ, ὡς δὲ ἡ ΘΕ πρὸς ΕΜ, οὕτως τὸ ΠΕ πρὸς τὸ ΚΜ· καὶ ὡς ἄρα τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ ΗΚ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ καὶ τὸ ΠΕ πρὸς τὸ ΚΜ. ἀλλ᾿ ὡς μὲν τὸ ΑΗ πρὸς τὸ ΗΚ, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ στερεόν, ὡς δὲ τὸ ΗΚ πρὸς τὸ ΚΛ, οὕτως τὸ ΞΕ στερεὸν πρὸς τὸ ΠΛ στερεόν, ὡς δὲ τὸ ΠΕ πρὸς τὸ ΚΜ, οὕτως τὸ ΠΛ στερεὸν πρὸς τὸ ΚΟ στερεόν· καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ, οὕτως τὸ ΕΞ πρὸς τὸ ΠΛ καὶ τὸ ΠΛ πρὸς τὸ ΚΟ. ἐὰν δὲ τέσσαρα μεγέθη κατὰ τὸ συνεχὲς ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχει ἤπερ πρὸς τὸ δεύτερον· τὸ ΑΒ ἄρα στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἤπερ τὸ ΑΒ πρὸς τὸ ΕΞ. ἀλλ᾿ ὡς τὸ ΑΒ πρὸς τὸ ΕΞ, οὕτως τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ ΗΚ καὶ ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΚ· ὥστε καὶ τὸ ΑΒ στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΕ πρὸς τὴν ΕΚ. ἴσον δὲ τὸ [μὲν] ΚΟ στερεὸν τῷ ΓΔ στερεῷ, ἡ δὲ ΕΚ εὐθεῖα τῇ ΓΖ· καὶ τὸ ΑΒ ἄρα στερεὸν πρὸς τὸ ΓΔ στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος αὐτοῦ πλευρὰ ἡ ΑΕ



And since the two (straight-lines) *KE* and *EL* are equal to the two (straight-lines) *CF* and *FN*, but angle *KEL* is also equal to angle *CFN*, inasmuch as *AEG* is also equal to *CFN*, on account of the similarity of the solids *AB* and *CD*, parallelogram *KL* is thus equal [and similar] to parallelogram *CN*. So, for the same (reasons), parallelogram *KM* is also equal and similar to [parallelogram] *CR*, and, further, *EP* to *DF*. Thus, three parallelograms of solid *KP* are equal and similar to three parallelograms of solid *CD*. But the three (former parallelograms) are equal and similar to the three opposite (parallelograms), and the three (latter parallelograms) are equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the whole of solid *KP* is equal and similar to the whole of solid *CD* [Def. 11.10]. Let parallelogram *GK* be completed. And let the the solids *EO* and *LQ*, with bases the parallelograms *GK* and *KL* (respectively), and with the same height as *AB*, be completed. And since, on account of the similarity of solids *AB* and *CD*, as *AE* is to *CF*, so *EG* (is) to *FN*, and *EH* to *FR* [Defs. 6.1, 11.9], and *CF* (is) equal to *EK*, and *FN* to *EL*, and *FR* to *EM*, thus as *AE* is to *EK*, so *GE* (is) to *EL*, and *HE* to *EM*. But, as *AE* (is) to *EK*, so [parallelogram] *AG* (is) to parallelogram *GK*, and as *GE* (is) to *EL*, so *GK* (is) to *KL*, and as *HE* (is) to *EM*, so *QE* (is) to *KM* [Prop. 6.1]. And thus as parallelogram *AG* (is) to *GK*, so *GK* (is) to *KL*, and *QE* (is) to *KM*. But, as *AG* (is) to *GK*, so solid *AB* (is) to solid *EO*, and as *GK* (is) to *KL*, so solid *OE* (is) to solid *QL*, and as *QE* (is) to *KM*, so solid *QL* (is) to solid *KP* [Prop. 11.32]. And, thus, as solid *AB* is to *EO*, so *EO* (is) to *QL*, and *QL* to *KP*. And if four magnitudes are continuously proportional then the first has to the fourth the cubed ratio that (it has) to the second [Def. 5.10]. Thus, solid *AB* has to *KP* the cubed ratio which *AB* (has) to *EO*. But, as *AB* (is) to *EO*, so parallelogram *AG* (is) to *GK*, and the straight-line *AE* to *EK* [Prop. 6.1]. Hence, solid *AB* also has to *KP* the cubed ratio that *AE* (has) to *EK*. And solid *KP* (is) equal to solid *CD*, and straight-line *EK* to *CF*. Thus, solid *AB* also has to solid *CD* the cubed ratio which its corresponding side *AE* (has) to the corresponding side *CF*.

πρὸς τὴν ὁμόλογον πλευρὰν τὴν ΓΖ.





Τὰ ἄρα ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· ὅπερ ἔδει δεῖξαι.

Thus, similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides. (Which is) the very thing it was required to show.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, ἔσται ὡς ἡ πρώτη πρὸς τὴν τετάρτην, οὕτω τὸ ἀπὸ τῆς πρώτης στερεὸν παραλληλεπίπεδον πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἐπείπερ καὶ ἡ πρώτη πρὸς τὴν τετάρτην τριπλασίονα λόγον ἔχει ἤπερ πρὸς τὴν δευτέραν.

Corollary

So, (it is) clear, from this, that if four straight-lines are (continuously) proportional then as the first is to the fourth, so the parallelepiped solid on the first will be to the similar, and similarly described, parallelepiped solid on the second, since the first also has to the fourth the cubed ratio that (it has) to the second.

λδ´.

Τῶν ἴσων στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα.

Ἔστω ἴσα στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ· λέγω, ὅτι τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἔστωσαν γὰρ πρότερον αἱ ἐφεστηκυῖαι αἱ ΑΗ, ΕΖ, ΛΒ, ΘΚ, ΓΜ, ΝΞ, ΟΔ, ΠΡ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν· λέγω, ὅτι ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΑΗ.

Εἰ μὲν οὖν ἴση ἐστὶν ἡ ΕΘ βάσις τῇ ΝΠ βάσει, ἔστι δὲ καὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ ἴσον, ἔσται καὶ ἡ ΓΜ τῇ ΑΗ ἴση. τὰ γὰρ ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις. καὶ ἔσται ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ, οὕτως ἡ ΓΜ πρὸς τὴν ΑΗ, καὶ φανερόν, ὅτι τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

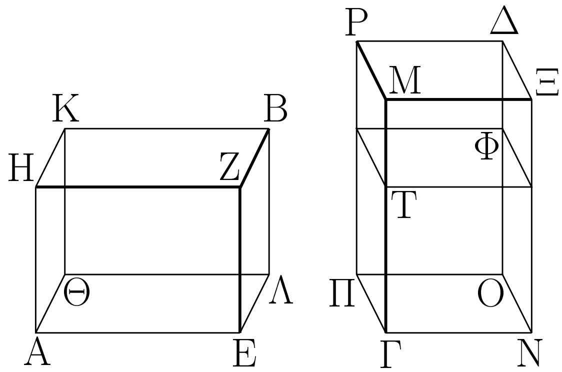
Proposition 34†

The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let *AB* and *CD* be equal parallelepiped solids. I say that the bases of the parallelepiped solids *AB* and *CD* are reciprocally proportional to their heights, and (so) as base *EH* is to base *NQ*, so the height of solid *CD* (is) to the height of solid *AB*.

For, first of all, let the (straight-lines) standing up, *AG*, *EF*, *LB*, *HK*, *CM*, *NO*, *PD*, and *QR*, be at right-angles to their bases. I say that as base *EH* is to base *NQ*, so *CM* (is) to *AG*.

Therefore, if base *EH* is equal to base *NQ*, and solid *AB* is also equal to solid *CD*, *CM* will also be equal to *AG*. For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base *EH* (is) to *NQ*, so *CM* will be to *AG*. And (so it is) clear that the bases of the parallelepiped solids *AB* and *CD* are reciprocally proportional to their heights.

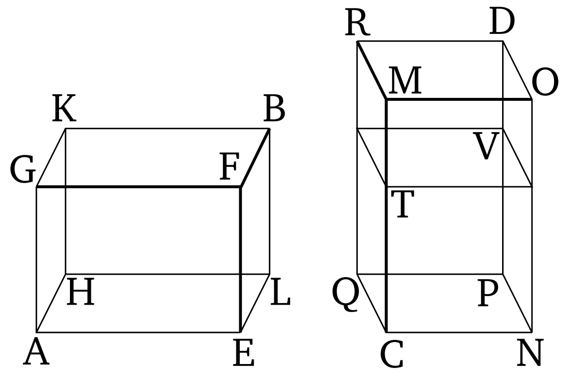


μὴ ἔστω δὴ ἴση ἡ ΕΘ βάσις τῇ ΝΠ βάσει, ἀλλ᾽ ἔστω μείζων ἡ ΕΘ. ἔστι δὲ καὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ ἴσον· μείζων ἄρα ἐστὶ καὶ ἡ ΓΜ τῆς ΑΗ. κείσθω οὖν τῇ ΑΗ ἴση ἡ ΓΤ, καὶ συμπεπληρώσθω ἀπὸ βάσεως μὲν τῆς ΝΠ, ὕψους δὲ τοῦ ΓΤ, στερεὸν παραλληλεπίπεδον τὸ ΦΓ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἔξωθεν δὲ τὸ ΓΦ, τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. ἀλλ᾽ ὡς μὲν τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν· ἰσοϋψῆ γὰρ τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως ἡ ΜΠ βάσις πρὸς τὴν ΤΠ βάσιν καὶ ἡ ΓΜ πρὸς τὴν ΓΤ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΓΤ. ἴση δὲ ἡ ΓΤ τῇ ΑΗ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΑΗ. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Ἔστωσαν [γὰρ] πάλιν αἱ ἐφεστηκυῖαι πρὸς ὀρθὰς ταῖς βάσεσιν. καὶ εἰ μὲν ἴση ἐστὶν ἡ ΕΘ βάσις τῇ ΝΠ βάσει, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴσον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τῷ τοῦ ΑΒ στερεοῦ ὕψει. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

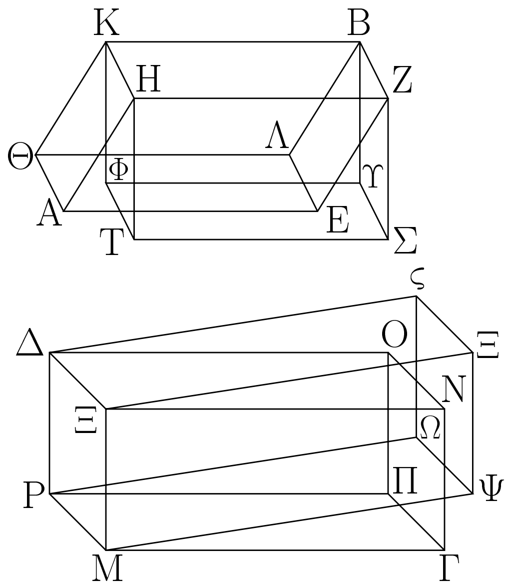
μὴ ἔστω δὴ ἡ ΕΘ βάσις τῇ ΝΠ [βάσει] ἴση, ἀλλ᾽ ἔστω μείζων ἡ ΕΘ· μεῖζον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τοῦ τοῦ ΑΒ στερεοῦ ὕψους, τουτέστιν ἡ ΓΜ τῆς ΑΗ. κείσθω τῇ ΑΗ ἴση πάλιν ἡ ΓΤ, καὶ συμπεπληρώσθω ὁμοίως τὸ ΓΦ στερεόν. ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΑΗ, ἴση δὲ ἡ ΑΗ τῇ ΓΤ, ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΓΤ. ἀλλ᾽ ὡς μὲν ἡ ΕΘ [βάσις] πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἰσοϋψῆ γάρ ἐστι τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ ἡ ΓΜ πρὸς τὴν ΓΤ, οὕτως ἥ τε ΜΠ βάσις πρὸς τὴν ΠΤ



So let base EH not be equal to base NQ, but let EH be greater. And solid AB is also equal to solid CD. Thus, CM is also greater than AG. Therefore, let CT be made equal to AG. And let the parallelepiped solid VC be completed on the base NQ, with height CT. And since solid AB is equal to solid CD, and CV (is) extrinsic (to them), and equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7], thus as solid AB is to solid CV, so solid CD (is) to solid CV. But, as solid AB (is) to solid CV, so base EH (is) to base NQ. For the solids AB and CV (are) of equal height [Prop. 11.32]. And as solid CD (is) to solid CV, so base MQ (is) to base TQ [Prop. 11.25], and CM to CT [Prop. 6.1]. And, thus, as base EH is to base NQ, so MC (is) to AG. And CT (is) equal to AG. And thus as base EH (is) to base NQ, so MC (is) to AG. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepipid solids AB and CD be reciprocally proportional to their heights, and let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD. [For] let the (straight-lines) standing up again be at right-angles to the bases. And if base EH is equal to base NQ, and as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, the height of solid CD is thus also equal to the height of solid AB. And parallelepiped solids on equal bases, and also with the same height, are equal to one another [Prop. 11.31]. Thus, solid AB is equal to solid CD.

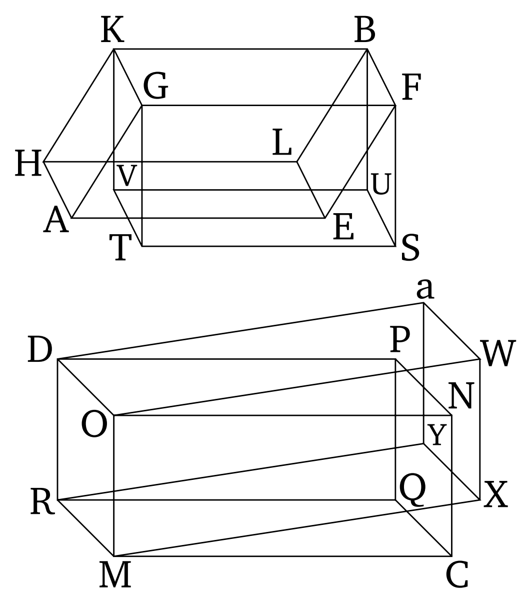
So, let base EH not be equal to [base] NQ, but let EH be greater. Thus, the height of solid CD is also greater than the height of solid AB, that is to say CM (greater) than AG. Let CT again be made equal to AG, and let the solid CV be similarly completed. Since as base EH is to base NQ, so MC (is) to AG, and AG (is) equal to CT, thus as base EH (is) to base NQ, so CM (is) to CT. But, as [base] EH (is) to base NQ, so solid AB (is) to solid CV. For solids AB and CV are of equal heights [Prop. 11.32]. And as CM (is) to CT, so (is) base MQ to base QT [Prop. 6.1], and solid CD to solid CV [Prop. 11.25]. And thus as solid AB (is) to solid CV, so solid CD (is) to solid CV. Thus, AB and CD each have the same *βάσιν καὶ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἑκάτερον ἄρα τῶν ΑΒ, ΓΔ πρὸς τὸ ΓΦ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.*



μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΖΕ, ΒΛ, ΗΑ, ΚΘ, ΞΝ, ΔΟ, ΜΓ, ΡΠ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν, καὶ ἤχθωσαν ἀπὸ τῶν Ζ, Η, Β, Κ, Ξ, Μ, Ρ, Δ σημείων ἐπὶ τὰ διὰ τῶν ΕΘ, ΝΠ ἐπίπεδα κάθετοι καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω, ς, καὶ συμπεπληρώσθω τὰ ΖΦ, ΞΩ στερεά· λέγω, ὅτι καὶ οὕτως ἴσων ὄντων τῶν ΑΒ, ΓΔ στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσιν πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἀλλὰ τὸ μὲν

ratio to CV. Thus, solid AB is equal to solid CD [Prop. 5.9].



So, let the (straight-lines) standing up, FE, BL, GA, KH, ON, DP, MC, and RQ, not be at right-angles to their bases. And let perpendiculars be drawn to the planes through EH and NQ from points F, G, B, K, O, M, R, and D, and let them have joined the planes at (points) S, T, U, V, W, X, Y, and a (respectively). And let the solids FV and OY be completed. In this case, also, I say that the solids AB and CD being equal, their bases are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is)

ΑΒ τῷ ΒΤ ἐστιν ἴσον· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος· τὸ δὲ ΓΔ στερεὸν τῷ ΔΨ ἐστιν ἴσον· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΞ καὶ ὑπὸ τὸ αὐτὸ ὕψος· καὶ τὸ ΒΤ ἄρα στερεὸν τῷ ΔΨ στερεῷ ἴσον ἐστίν. ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. ἴση δὲ ἡ μὲν ΖΚ βάσις τῇ ΕΘ βάσει, ἡ δὲ ΞΡ βάσις τῇ ΝΠ βάσει· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τὰ δ᾽ αὐτὰ ὕψη ἐστὶ τῶν ΔΨ, ΒΤ στερεῶν καὶ τῶν ΔΓ, ΒΑ· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΔΓ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴση δὲ ἡ μὲν ΕΘ βάσις τῇ ΖΚ βάσει, ἡ δὲ ΝΠ τῇ ΞΡ, ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τὰ δ᾽ αὐτὰ ὕψη ἐστὶ τῶν ΑΒ, ΓΔ στερεῶν καὶ τῶν ΒΤ, ΔΨ· ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τῶν ΒΤ, ΔΨ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἴσον ἄρα ἐστὶ τὸ ΒΤ στερεὸν τῷ ΔΨ στερεῷ. ἀλλὰ τὸ μὲν ΒΤ τῷ ΒΑ ἴσον ἐστίν· ἐπί τε γὰρ τῆς αὐτῆς βάσεως [εἰσι] τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος. τὸ δὲ ΔΨ στερεὸν τῷ ΔΓ στερεῷ ἴσον ἐστίν. καὶ τὸ ΑΒ ἄρα στερεὸν τῷ ΓΔ στερεῷ ἐστιν ἴσον· ὅπερ ἔδει δεῖξαι.

to the height of solid *AB*.

Since solid *AB* is equal to solid *CD*, but *AB* is equal to *BT*. For they are on the same base *FK*, and (have) the same height [Props. 11.29, 11.30]. And solid *CD* is equal is equal to *DX*. For, again, they are on the same base *RO*, and (have) the same height [Props. 11.29, 11.30]. Solid *BT* is thus also equal to solid *DX*. Thus, as base *FK* (is) to base *OR*, so the height of solid *DX* (is) to the height of solid *BT* (see first part of proposition). And base *FK* (is) equal to base *EH*, and base *OR* to *NQ*. Thus, as base *EH* is to base *NQ*, so the height of solid *DX* (is) to the height of solid *BT*. And solids *DX*, *BT* are the same height as (solids) *DC*, *BA* (respectively). Thus, as base *EH* is to base *NQ*, so the height of solid *DC* (is) to the height of solid *AB*. Thus, the bases of the parallelepiped solids *AB* and *CD* are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids *AB* and *CD* be reciprocally proportional to their heights, and (so) let base *EH* be to base *NQ*, as the height of solid *CD* (is) to the height of solid *AB*. I say that solid *AB* is equal to solid *CD*.

For, with the same construction (as before), since as base *EH* is to base *NQ*, so the height of solid *CD* (is) to the height of solid *AB*, and base *EH* (is) equal to base *FK*, and *NQ* to *OR*, thus as base *FK* is to base *OR*, so the height of solid *CD* (is) to the height of solid *AB*. And solids *AB*, *CD* are the same height as (solids) *BT*, *DX* (respectively). Thus, as base *FK* is to base *OR*, so the height of solid *DX* (is) to the height of solid *BT*. Thus, the bases of the parallelepiped solids *BT* and *DX* are reciprocally proportional to their heights. Thus, solid *BT* is equal to solid *DX* (see first part of proposition). But, *BT* is equal to *BA*. For [they are] on the same base *FK*, and (have) the same height [Props. 11.29, 11.30]. And solid *DX* is equal to solid *DC* [Props. 11.29, 11.30]. Thus, solid *AB* is also equal to solid *CD*. (Which is) the very thing it was required to show.

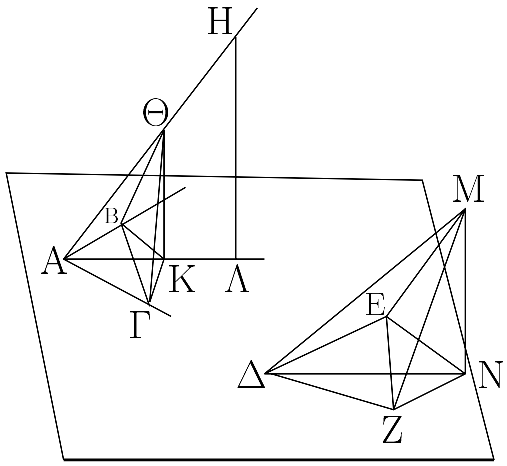
† This proposition assumes that (a) if two parallelepipeds are equal, and have equal bases, then their heights are equal, and (b) if the bases of two equal parallelepipeds are unequal, then that solid which has the lesser base has the greater height.

λε΄.

Ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπὶ δὲ τῶν κορυφῶν αὐτῶν μετέωροι εὐθεῖαι ἐπισταθῶσιν ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, ἐπὶ δὲ τῶν μετεώρων ληφθῇ τυχόντα σημεῖα, καὶ ἀπ᾽ αὐτῶν ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, κάθετοι ἀχθῶσιν, ἀπὸ δὲ τῶν γενομένων σημείων ἐν τοῖς ἐπιπέδοις ἐπὶ τὰς ἐξ ἀρχῆς γωνίας ἐπιζευχθῶσιν εὐθεῖαι, ἴσας γωνίας περιέξουσι μετὰ τῶν μετεώρων.

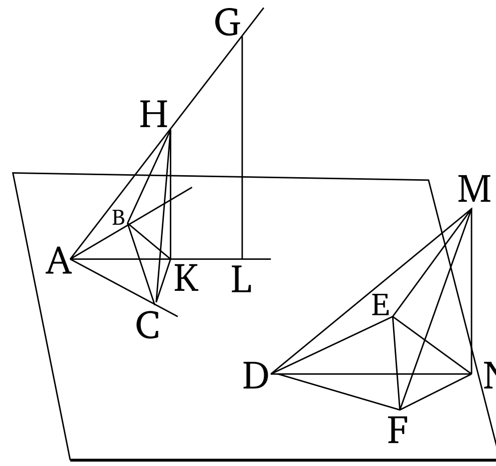
Proposition 35

If there are two equal plane angles, and raised straight-lines are stood on the apexes of them, containing equal angles respectively with the original straight-lines (forming the angles), and random points are taken on the raised (straight-lines), and perpendiculars are drawn from them to the planes in which the original angles are, and straight-lines are joined from the points created in the planes to the (vertices of the) original angles, then they will enclose equal angles with the raised (straight-lines).



Ἔστωσαν δύο γωνίαι εὐθύγραμμοι ἴσαι αἱ ὑπὸ ΒΑΓ, ΕΔΖ, ἀπὸ δὲ τῶν Α, Δ σημείων μετέωροι εὐθεῖαι ἐφεστάτωσαν αἱ ΑΗ, ΔΜ ἴσας γωνίας περιέχουσιν μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ ΜΔΕ τῇ ὑπὸ ΗΑΒ, τὴν δὲ ὑπὸ ΜΔΖ τῇ ὑπὸ ΗΑΓ, καὶ εἰλήφθω ἐπὶ τῶν ΑΗ, ΔΜ τυχόντα σημεῖα τὰ Η, Μ, καὶ ἤχθωσαν ἀπὸ τῶν Η, Μ σημείων ἐπὶ τὰ διὰ τῶν ΒΑΓ, ΕΔΖ ἐπίπεδα κάθετοι αἱ ΗΛ, ΜΝ, καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Λ, Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΑ, ΝΔ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΗΑΛ γωνία τῇ ὑπὸ ΜΔΝ γωνίᾳ.

Κείσθω τῇ ΔΜ ἴση ἡ ΑΘ, καὶ ἤχθω διὰ τοῦ Θ σημείου τῇ ΗΛ παράλληλος ἡ ΘΚ. ἡ δὲ ΗΛ κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον· καὶ ἡ ΘΚ ἄρα κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον. ἤχθωσαν ἀπὸ τῶν Κ, Ν σημείων ἐπὶ τὰς ΑΓ, ΔΖ, ΑΒ, ΔΕ εὐθείας κάθετοι αἱ ΚΓ, ΝΖ, ΚΒ, ΝΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΓ, ΓΒ, ΜΖ, ΖΕ. ἐπεὶ τὸ ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΚΓ, ΓΑ, καὶ τὸ ἀπὸ τῆς ΘΑ ἄρα ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΓ, ΓΑ. τοῖς δὲ ἀπὸ τῶν ΘΚ, ΚΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΘΓ· τὸ ἄρα ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΓ, ΓΑ. ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΘΓΑ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΔΖΜ γωνία ὀρθή ἐστιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΘ γωνία τῇ ὑπὸ ΔΖΜ. ἔστι δὲ καὶ ἡ ὑπὸ ΘΑΓ τῇ ὑπὸ ΜΔΖ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΜΔΖ, ΘΑΓ δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΘΑ τῇ ΜΔ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει ἑκατέραν ἑκαρέρᾳ. ἴση ἄρα ἐστὶν ἡ ΑΓ τῇ ΔΖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΑΒ τῇ ΔΕ ἐστιν ἴση. ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΓ τῇ ΔΖ, ἡ δὲ ΑΒ τῇ ΔΕ, δύο δὴ αἱ ΓΑ, ΑΒ δυσὶ ταῖς ΖΔ, ΔΕ ἴσαι εἰσίν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΓΑΒ γωνίᾳ τῇ ὑπὸ ΖΔΕ ἐστιν ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστὶ καὶ τὸ τρίγωνον τῷ τριγώνῳ καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΔΖΕ. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΓΚ ὀρθῇ τῇ ὑπὸ ΔΖΝ ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΓΚ λοιπῇ τῇ ὑπὸ ΕΖΝ ἐστιν



Let *BAC* and *EDF* be two equal rectilinear angles. And let the raised straight-lines *AG* and *DM* be stood on points *A* and *D*, containing equal angles respectively with the original straight-lines. (That is) *MDE* (equal) to *GAB*, and *MDF* (to) *GAC*. And let the random points *G* and *M* be taken on *AG* and *DM* (respectively). And let the *GL* and *MN* be drawn from points *G* and *M* perpendicular to the planes through *BAC* and *EDF* (respectively). And let them have joined the planes at points *L* and *N* (respectively). And let *LA* and *ND* be joined. I say that angle *GAL* is equal to angle *MDN*.

Let *AH* be made equal to *DM*. And let *HK* be drawn through point *H* parallel to *GL*. And *GL* is perpendicular to the plane through *BAC*. Thus, *HK* is also perpendicular to the plane through *BAC* [Prop. 11.8]. And let *KC*, *NF*, *KB*, and *NE* be drawn from points *K* and *N* perpendicular to the straight-lines *AC*, *DF*, *AB*, and *DE*. And let *HC*, *CB*, *MF*, and *FE* be joined. Since the (square) on *HA* is equal to the (sum of the squares) on *HK* and *KA* [Prop. 1.47], and the (sum of the squares) on *KC* and *CA* is equal to the (square) on *KA* [Prop. 1.47], thus the (square) on *HA* is equal to the (sum of the squares) on *HK*, *KC*, and *CA*. And the (square) on *HC* is equal to the (sum of the squares) on *HK* and *KC* [Prop. 1.47]. Thus, the (square) on *HA* is equal to the (sum of the squares) on *HC* and *CA*. Thus, angle *HCA* is a right-angle [Prop. 1.48]. So, for the same (reasons), angle *DFM* is also a right-angle. Thus, angle *ACH* is equal to (angle) *DFM*. And *HAC* is also equal to *MDF*. So, *MDF* and *HAC* are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that subtending one of the equal angles —(that is), *HA* (equal) to *MD*. Thus, they will also have the remaining sides equal to the remaining sides, respectively [Prop. 1.26]. Thus, *AC* is equal to *DF*. So, similarly, we can show that *AB* is also equal to *DE*. Therefore, since *AC* is equal to *DF*, and *AB* to *DE*, so the two (straight-lines) *CA* and *AB* are equal to the two (straight-lines) *FD* and *DE* (respectively). But, angle *CAB* is also equal to angle *FDE*. Thus, *ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΒΚ τῇ ὑπὸ ΖΕΝ ἐστιν ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΒΓΚ, ΕΖΝ [τὰς] δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΒΓ τῇ ΕΖ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση ἄρα ἐστὶν ἡ ΓΚ τῇ ΖΝ. ἔστι δὲ καὶ ἡ ΑΓ τῇ ΔΖ ἴση· δύο δὴ αἱ ΑΓ, ΓΚ δυσὶ ταῖς ΔΖ, ΖΝ ἴσαι εἰσίν· καὶ ὀρθὰς γωνίας περιέχουσιν. βάσις ἄρα ἡ ΑΚ βάσει τῇ ΔΝ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΘ τῇ ΔΜ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΘ τῷ ἀπὸ τῆς ΔΜ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΚ, ΚΘ· ὀρθὴ γὰρ ἡ ὑπὸ ΑΚΘ· τῷ δὲ ἀπὸ τῆς ΔΜ ἴσα τὰ ἀπὸ τῶν ΔΝ, ΝΜ· ὀρθὴ γὰρ ἡ ὑπὸ ΔΝΜ· τὰ ἄρα ἀπὸ τῶν ΑΚ, ΚΘ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΔΝ, ΝΜ, ὧν τὸ ἀπὸ τῆς ΑΚ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΝ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΚΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ· ἴση ἄρα ἡ ΘΚ τῇ ΜΝ. καὶ ἐπεὶ δύο αἱ ΘΑ, ΑΚ δυσὶ ταῖς ΜΔ, ΔΝ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ ΘΚ βάσει τῇ ΜΝ ἐδείχθη ἴση, γωνία ἄρα ἡ ὑπὸ ΘΑΚ γωνίᾳ τῇ ὑπὸ ΜΔΝ ἐστιν ἴση.*

Ἐὰν ἄρα ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι καὶ τὰ ἑξῆς τῆς προτάσεως [ὅπερ ἔδει δεῖξαι].

base BC is equal to base EF, and triangle (ACB) to triangle (DFE), and the remaining angles to the remaining angles (respectively) [Prop. 1.4]. Thus, angle ACB (is) equal to DFE. And the right-angle ACK is also equal to the right-angle DFN. Thus, the remainder BCK is equal to the remainder EFN. So, for the same (reasons), CBK is also equal to FEN. So, BCK and EFN are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that by the equal angles—(that is), BC (equal) to EF. Thus, they will also have the remaining sides equal to the remaining sides (respectively) [Prop. 1.26]. Thus, CK is equal to FN. And AC (is) also equal to DF. So, the two (straight-lines) AC and CK are equal to the two (straight-lines) DF and FN (respectively). And they enclose right-angles. Thus, base AK is equal to base DN [Prop. 1.4]. And since AH is equal to DM, the (square) on AH is also equal to the (square) on DM. But, the the (sum of the squares) on AK and KH is equal to the (square) on AH. For angle AKH (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on DN and NM (is) equal to the square on DM. For angle DNM (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AK and KH is equal to the (sum of the squares) on DN and NM, of which the (square) on AK is equal to the (square) on DN. Thus, the remaining (square) on KH is equal to the (square) on NM. Thus, HK (is) equal to MN. And since the two (straight-lines) HA and AK are equal to the two (straight-lines) MD and DN, respectively, and base HK was shown (to be) equal to base MN, angle HAK is thus equal to angle MDN [Prop. 1.8].

Thus, if there are two equal plane angles, and so on of the proposition. [(Which is) the very thing it was required to show].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπισταθῶσι δὲ ἐπ᾿ αὐτῶν μετέωροι εὐθεῖαι ἴσαι ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἀπ᾿ αὐτῶν κάθετοι ἀγόμεναι ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, ἴσαι ἀλλήλαις εἰσίν. ὅπερ ἔδει δεῖξαι.

Corollary

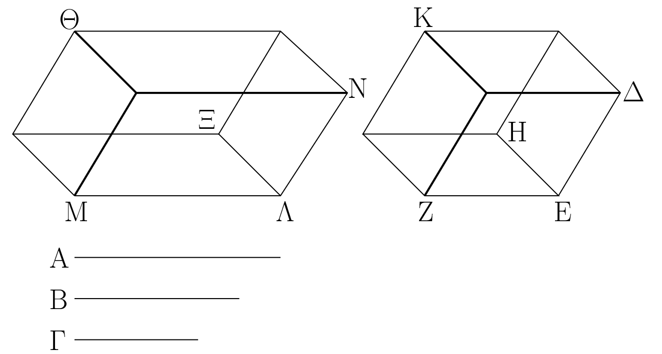
So, it is clear, from this, that if there are two equal plane angles, and equal raised straight-lines are stood on them (at their apexes), containing equal angles respectively with the original straight-lines (forming the angles), then the perpendiculars drawn from (the raised ends of) them to the planes in which the original angles lie are equal to one another. (Which is) the very thing it was required to show.

λϛ΄.

Ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ἐκ τῶν τριῶν στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης στερεῷ παραλληλεπιπέδῳ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

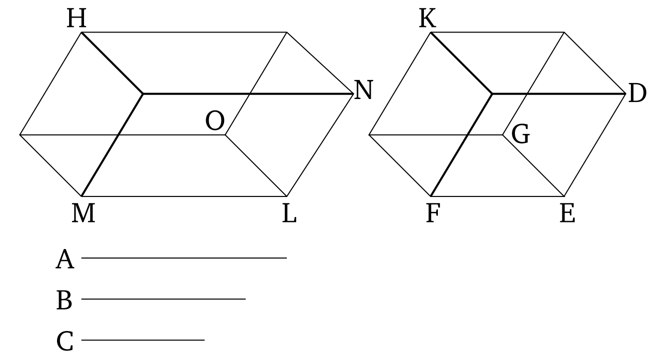
Proposition 36

If three straight-lines are (continuously) proportional then the parallelepiped solid (formed) from the three (straight-lines) is equal to the equilateral parallelepiped solid on the middle (straight-line which is) equiangular to the aforementioned (parallelepiped solid).



Ἔστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ· λέγω, ὅτι τὸ ἐκ τῶν Α, Β, Γ στερεὸν ἴσον ἐστὶ τῷ ἀπὸ τῆς Β στερεῷ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

Ἐκκείσθω στερεὰ γωνία ἡ πρὸς τῷ Ε περιεχομένη ὑπὸ τῶν ὑπὸ ΔΕΗ, ΗΕΖ, ΖΕΔ, καὶ κείσθω τῇ μὲν Β ἴση ἑκάστη τῶν ΔΕ, ΗΕ, ΕΖ, καὶ συμπεπληρώσθω τὸ ΕΚ στερεὸν παραλληλεπίπεδον, τῇ δὲ Α ἴση ἡ ΛΜ, καὶ συνεστάτω πρὸς τῇ ΛΜ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Λ τῇ πρὸς τῷ Ε στερεᾷ γωνίᾳ ἴση στερεὰ γωνία ἡ περειχομένη ὑπὸ τῶν ΝΛΞ, ΞΛΜ, ΜΛΝ, καὶ κείσθω τῇ μὲν Β ἴση ἡ ΛΞ, τῇ δὲ Γ ἴση ἡ ΛΝ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ, ἴση δὲ ἡ μὲν Α τῇ ΛΜ, ἡ δὲ Β ἑκατέρᾳ τῶν ΛΞ, ΕΔ, ἡ δὲ Γ τῇ ΛΝ, ἔστιν ἄρα ὡς ἡ ΛΜ πρὸς τὴν ΕΖ, οὕτως ἡ ΔΕ πρὸς τὴν ΛΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΝΛΜ, ΔΕΖ αἱ πλευραὶ ἀντιπεπόνθασιν· ἴσον ἄρα ἐστὶ τὸ ΜΝ παραλληλόγραμμον τῷ ΔΖ παραλληλογραμάμμῳ. καὶ ἐπεὶ δύο γωνίαι ἐπίπεδοι εὐθύγραμμοι ἴσαι εἰσὶν αἱ ὑπὸ ΔΕΖ, ΝΛΜ, καὶ ἐπ᾽ αὐτῶν μετέωροι εὐθεῖαι ἐφεστᾶσιν αἱ ΛΞ, ΕΗ ἴσαι τε ἀλλήλαις καὶ ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἄρα ἀπὸ τῶν Η, Ξ σημείων κάθετοι ἀγόμεναι ἐπὶ τὰ διὰ τῶν ΝΛΜ, ΔΕΖ ἐπίπεδα ἴσαι ἀλλήλαις εἰσίν· ὥστε τὰ ΛΘ, ΕΚ στερεὰ ὑπὸ τὸ αὐτὸ ὕψος ἐστίν. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΘΛ στερεὸν τῷ ΕΚ στερεῷ. καὶ ἔστι τὸ μὲν ΛΘ τὸ ἐκ τῶν Α, Β, Γ στερεόν, τὸ δὲ ΕΚ τὸ ἀπὸ τῆς Β στερεόν· τὸ ἄρα ἐκ τῶν Α, Β, Γ στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β στερεῷ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ· ὅπερ ἔδει δεῖξαι.



Let *A*, *B*, and *C* be three (continuously) proportional straight-lines, (such that) as *A* (is) to *B*, so *B* (is) to *C*. I say that the (parallelepiped) solid (formed) from *A*, *B*, and *C* is equal to the equilateral solid on *B* (which is) equiangular with the aforementioned (solid).

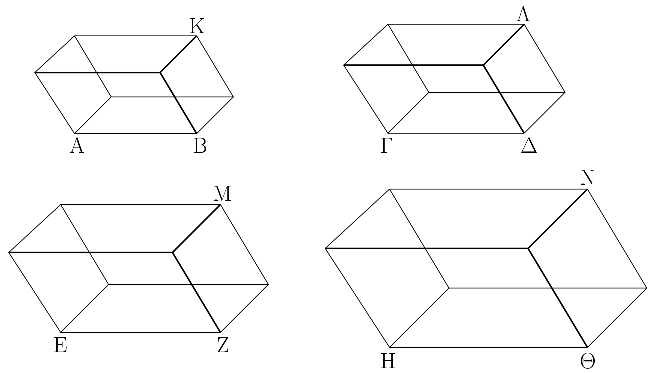
Let the solid angle at *E*, contained by *DEG*, *GEF*, and *FED*, be set out. And let *DE*, *GE*, and *EF* each be made equal to *B*. And let the parallelepiped solid *EK* be completed. And (let) *LM* (be made) equal to *A*. And let the solid angle contained by *NLO*, *OLM*, and *MLN* be constructed on the straight-line *LM*, and at the point *L* on it, (so as to be) equal to the solid angle *E* [Prop. 11.23]. And let *LO* be made equal to *B*, and *LN* equal to *C*. And since as *A* (is) to *B*, so *B* (is) to *C*, and *A* (is) equal to *LM*, and *B* to each of *LO* and *ED*, and *C* to *LN*, thus as *LM* (is) to *EF*, so *DE* (is) to *LN*. And (so) the sides around the equal angles *NLM* and *DEF* are reciprocally proportional. Thus, parallelogram *MN* is equal to parallelogram *DF* [Prop. 6.14]. And since the two plane rectilinear angles *DEF* and *NLM* are equal, and the raised straight-lines stood on them (at their apexes), *LO* and *EG*, are equal to one another, and contain equal angles respectively with the original straight-lines (forming the angles), the perpendiculars drawn from points *G* and *O* to the planes through *NLM* and *DEF* (respectively) are thus equal to one another [Prop. 11.35 corr.]. Thus, the solids *LH* and *EK* (have) the same height. And parallelepiped solids on equal bases, and with the same height, are equal to one another [Prop. 11.31]. Thus, solid *HL* is equal to solid *EK*. And *LH* is the solid (formed) from *A*, *B*, and *C*, and *EK* the solid on *B*. Thus, the parallelepiped solid (formed) from *A*, *B*, and *C* is equal to the equilateral solid on *B* (which is) equiangular with the aforementioned (solid). (Which is) the very thing it was required to show.

λζ'.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ᾽ αὐτῶν στερεὰ παραλληλεπίπεδα ὅμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ἔσται· καὶ ἐὰν τὰ ἀπ᾽ αὐτῶν στερεὰ παραλληλεπίπεδα ὅμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ᾖ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται.

Proposition 37[†]

If four straight-lines are proportional then the similar, and similarly described, parallelepiped solids on them will also be proportional. And if the similar, and similarly described, parallelepiped solids on them are proportional then the straight-lines themselves will be proportional.



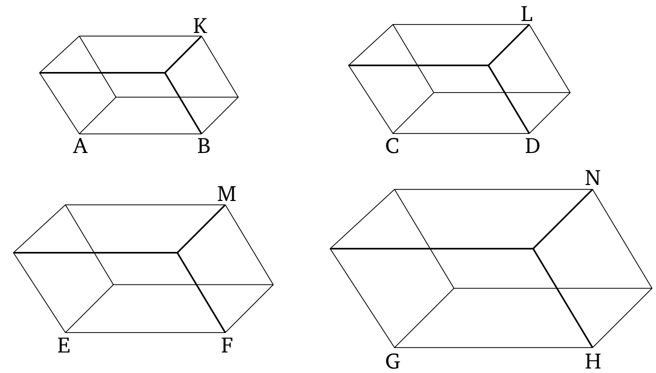
Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ ΑΒ, ΓΔ, ΕΖ, ΗΘ, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, καὶ ἀναγεγράφθωσαν ἀπὸ τῶν ΑΒ, ΓΔ, ΕΖ, ΗΘ ὅμοιά τε καὶ ὁμοίως κείμενα στερεὰ παραλληλεπίπεδα τὰ ΚΑ, ΛΓ, ΜΕ, ΝΗ· λέγω, ὅτι ἐστὶν ὡς τὸ ΚΑ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ.

Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ ΚΑ στερεὸν παραλληλεπίπεδον τῷ ΛΓ, τὸ ΚΑ ἄρα πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ πρὸς τὴν ΓΔ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΜΕ πρὸς τὸ ΝΗ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΗΘ. καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ. καὶ ὡς ἄρα τὸ ΑΚ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ.

Ἀλλὰ δὴ ἔστω ὡς τὸ ΑΚ στερεὸν πρὸς τὸ ΛΓ στερεόν, οὕτως τὸ ΜΕ στερεὸν πρὸς τὸ ΝΗ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒ εὐθεῖα πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐπεὶ γὰρ πάλιν τὸ ΚΑ πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ πρὸς τὴν ΓΔ, ἔχει δὲ καὶ τὸ ΜΕ πρὸς τὸ ΝΗ τριπλασίονα λόγον ἤπερ ἡ ΕΖ πρὸς τὴν ΗΘ, καί ἐστιν ὡς τὸ ΚΑ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ, καὶ ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ὦσι καὶ τὰ ἑξῆς τῆς προτάσεως· ὅπερ ἔδει δεῖξαι.



Let *AB*, *CD*, *EF*, and *GH*, be four proportional straight-lines, (such that) as *AB* (is) to *CD*, so *EF* (is) to *GH*. And let the similar, and similarly laid out, parallelepiped solids *KA*, *LC*, *ME* and *NG* be described on *AB*, *CD*, *EF*, and *GH* (respectively). I say that as *KA* is to *LC*, so *ME* (is) to *NG*.

For since the parallelepiped solid *KA* is similar to *LC*, *KA* thus has to *LC* the cubed ratio that *AB* (has) to *CD* [Prop. 11.33]. So, for the same (reasons), *ME* also has to *NG* the cubed ratio that *EF* (has) to *GH* [Prop. 11.33]. And since as *AB* is to *CD*, so *EF* (is) to *GH*, thus, also, as *AK* (is) to *LC*, so *ME* (is) to *NG*.

And so let solid *AK* be to solid *LC*, as solid *ME* (is) to *NG*. I say that as straight-line *AB* is to *CD*, so *EF* (is) to *GH*.

For, again, since *KA* has to *LC* the cubed ratio that *AB* (has) to *CD* [Prop. 11.33], and *ME* also has to *NG* the cubed ratio that *EF* (has) to *GH* [Prop. 11.33], and as *KA* is to *LC*, so *ME* (is) to *NG*, thus, also, as *AB* (is) to *CD*, so *EF* (is) to *GH*.

Thus, if four straight-lines are proportional, and so on of the proposition. (Which is) the very thing it was required to show.

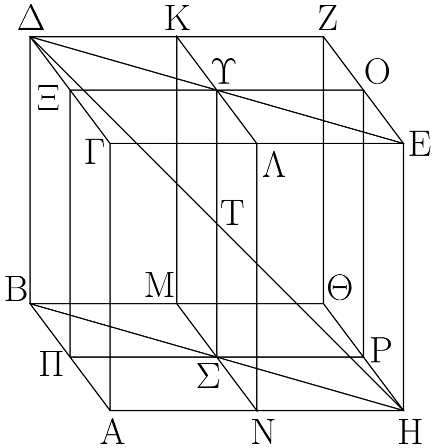
† This proposition assumes that if two ratios are equal then the cube of the former is also equal to the cube of the latter, and vice versa.

λη΄.

Ἐὰν κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας.

Proposition 38

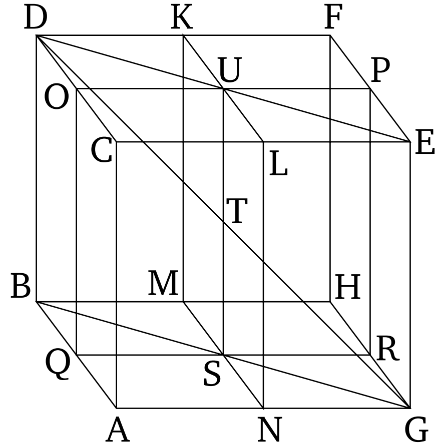
If the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half.



Κύβου γὰρ τοῦ ΑΖ τῶν ἀπεναντίον ἐπιπέδων τῶν ΓΖ, ΑΘ αἱ πλευραὶ δίχα τετμήσθωσαν κατὰ τὰ Κ, Λ, Μ, Ν, Ξ, Π, Ο, Ρ σημεῖα, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβεβλήσθω τὰ ΚΝ, ΞΡ, κοινὴ δὲ τομὴ τῶν ἐπιπέδων ἔστω ἡ ΥΣ, τοῦ δὲ ΑΖ κύβου διαγώνιος ἡ ΔΗ. λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ΥΤ τῇ ΤΣ, ἡ δὲ ΔΤ τῇ ΤΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΔΥ, ΥΕ, ΒΣ, ΣΗ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΔΞ τῇ ΟΕ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΔΞΥ, ΥΟΕ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΞ τῇ ΟΕ, ἡ δὲ ΞΥ τῇ ΥΟ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΔΥ τῇ ΥΕ ἐστιν ἴση, καὶ τὸ ΔΞΥ τρίγωνον τῷ ΟΥΕ τριγώνῳ ἐστὶν ἴσον καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι· ἴση ἄρα ἡ ὑπὸ ΞΥΔ γωνία τῇ ὑπὸ ΟΥΕ γωνίᾳ. διὰ δὴ τοῦτο εὐθεῖά ἐστιν ἡ ΔΥΕ. διὰ τὰ αὐτὰ δὴ καὶ ΒΣΗ εὐθεῖά ἐστιν, καὶ ἴση ἡ ΒΣ τῇ ΣΗ. καὶ ἐπεὶ ἡ ΓΑ τῇ ΔΒ ἴση ἐστὶ καὶ παράλληλος, ἀλλὰ ἡ ΓΑ καὶ τῇ ΕΗ ἴση τέ ἐστι καὶ παράλληλος, καὶ ἡ ΔΒ ἄρα τῇ ΕΗ ἴση τέ ἐστι καὶ παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΔΕ, ΒΗ· παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΗ. ἴση ἄρα ἡ μὲν ὑπὸ ΕΔΤ γωνία τῇ ὑπὸ ΒΗΤ· ἐναλλὰξ γάρ· ἡ δὲ ὑπὸ ΔΤΥ τῇ ὑπὸ ΗΤΣ. δύο δὴ τρίγωνά ἐστι τὰ ΔΤΥ, ΗΤΣ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΔΥ τῇ ΗΣ· ἡμίσειαι γάρ εἰσι τῶν ΔΕ, ΒΗ· καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει. ἴση ἄρα ἡ μὲν ΔΤ τῇ ΤΗ, ἡ δὲ ΥΤ τῇ ΤΣ.

Ἐὰν ἄρα κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας· ὅπερ ἔδει δεῖξαι.



For let the opposite planes CF and AH of the cube AF be cut in half at the points K, L, M, N, O, Q, P, and R. And let the planes KN and OR be produced through the pieces. And let US be the common section of the planes, and DG the diameter of cube AF. I say that UT is equal to TS, and DT to TG.

For let DU, UE, BS, and SG be joined. And since DO is parallel to PE, the alternate angles DOU and UPE are equal to one another [Prop. 1.29]. And since DO is equal to PE, and OU to UP, and they contain equal angles, base DU is thus equal to base UE, and triangle DOU is equal to triangle PUE, and the remaining angles (are) equal to the remaining angles [Prop. 1.4]. Thus, angle OUD (is) equal to angle PUE. So, for this (reason), DUE is a straight-line [Prop. 1.14]. So, for the same (reason), BSG is also a straight-line, and BS equal to SG. And since CA is equal and parallel to DB, but CA is also equal and parallel to EG, DB is thus also equal and parallel to EG [Prop. 11.9]. And the straight-lines DE and BG join them. DE is thus parallel to BG [Prop. 1.33]. Thus, angle EDT (is) equal to BGT. For (they are) alternate [Prop. 1.29]. And (angle) DTU (is equal) to GTS [Prop. 1.15]. So, DTU and GTS are two triangles having two angles equal to two angles, and one side equal to one side—(namely), that subtended by one of the equal angles—(that is), DU (equal) to GS. For they are halves of DE and BG (respectively). (Thus), they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, DT (is) equal to TG, and UT to TS.

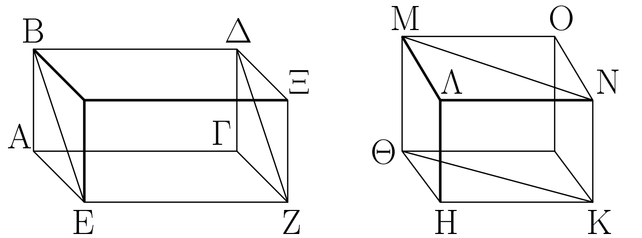
Thus, if the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half. (Which is) the very thing it was required to show.

λθʹ.

Ἐὰν ᾖ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα.

Proposition 39

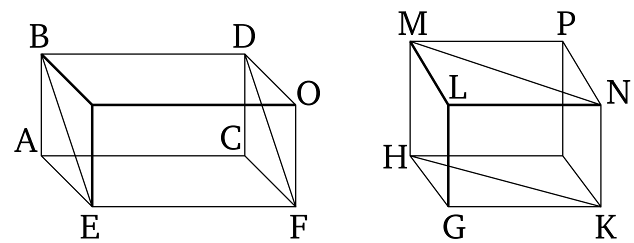
If there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms will be equal.



Ἔστω δύο πρίσματα ἰσοϋψῆ τὰ ΑΒΓΔΕΖ, ΗΘΚΛΜΝ, καὶ τὸ μὲν ἐχέτω βάσιν τὸ ΑΖ παραλληλόγραμμον, τὸ δὲ τὸ ΗΘΚ τρίγωνον, διπλάσιον δὲ ἔστω τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Συμπεπληρώσθω γὰρ τὰ ΑΞ, ΗΟ στερεά. ἐπεὶ διπλάσιόν ἐστι τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου, ἔστι δὲ καὶ τὸ ΘΚ παραλληλόγραμμον διπλάσιον τοῦ ΗΘΚ τριγώνου, ἴσον ἄρα ἐστὶ τὸ ΑΖ παραλληλόγραμμον τῷ ΘΚ παραλληλογράμμῳ. τὰ δὲ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΞ στερεὸν τῷ ΗΟ στερεῷ. καί ἐστι τοῦ μὲν ΑΞ στερεοῦ ἥμισυ τὸ ΑΒΓΔΕΖ πρίσμα, τοῦ δὲ ΗΟ στερεοῦ ἥμισυ τὸ ΗΘΚΛΜΝ πρίσμα· ἴσον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Ἐὰν ἄρα ᾖ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔστὶ τὰ πρίσματα· ὅπερ ἔδει δεῖξαι.



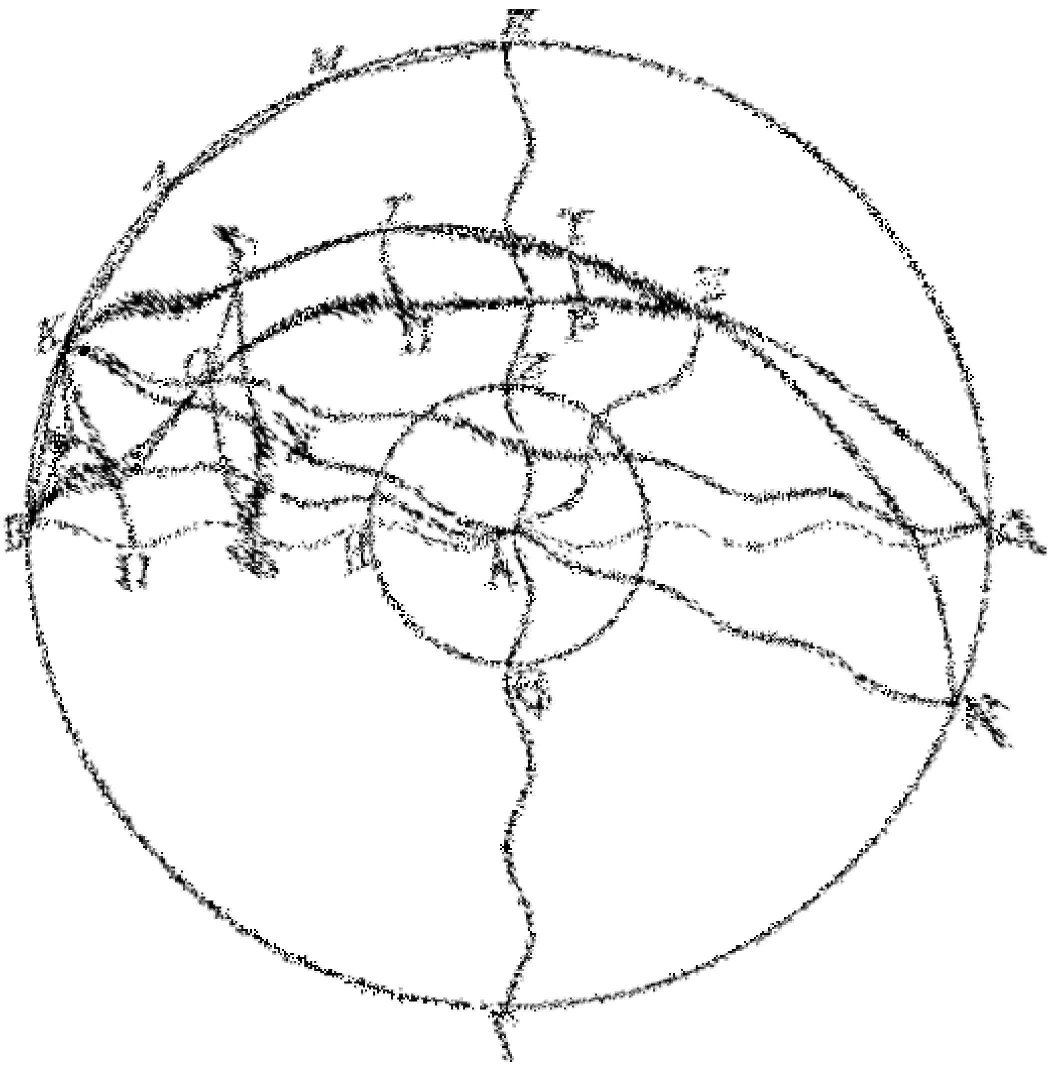
Let $ABCDEF$ and $GHKLMN$ be two equal height prisms, and let the former have the parallelogram AF, and the latter the triangle GHK, as a base. And let parallelogram AF be twice triangle GHK. I say that prism $ABCDEF$ is equal to prism $GHKLMN$.

For let the solids AO and GP be completed. Since parallelogram AF is double triangle GHK, and parallelogram HK is also double triangle GHK [Prop. 1.34], parallelogram AF is thus equal to parallelogram HK. And parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another [Prop. 11.31]. Thus, solid AO is equal to solid GP. And prism $ABCDEF$ is half of solid AO, and prism $GHKLMN$ half of solid GP [Prop. 11.28]. Prism $ABCDEF$ is thus equal to prism $GHKLMN$.

Thus, if there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms are equal. (Which is) the very thing it was required to show.

ELEMENTS BOOK 12

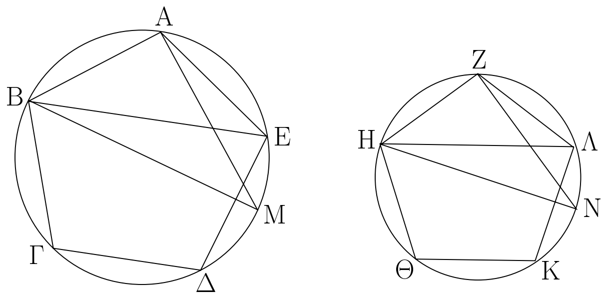
Proportional Stereometry[†]



[†]The novel feature of this book is the use of the so-called method of exhaustion (see Prop. 10.1), a precursor to integration which is generally attributed to Eudoxus of Cnidus.

α'.

Τὰ ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἄλληλά ἐστιν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.



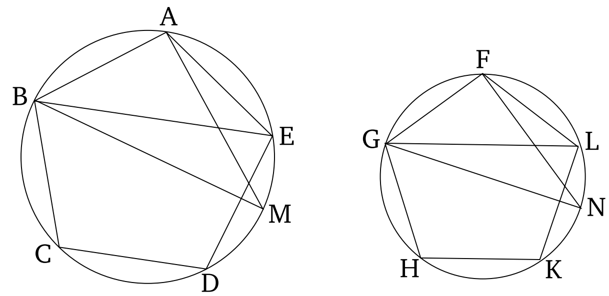
Ἔστωσαν κύκλοι οἱ ΑΒΓ, ΖΗΘ, καὶ ἐν αὐτοῖς ὅμοια πολύγωνα ἔστω τὰ ΑΒΓΔΕ, ΖΗΘΚΛ, διάμετροι δὲ τῶν κύκλων ἔστωσαν ΒΜ, ΗΝ· λέγω, ὅτι ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΑΜ, ΗΛ, ΖΝ. καὶ ἐπεὶ ὅμοιον τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνῳ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΒΑΕ γωνία τῇ ὑπὸ ΗΖΛ, καί ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΕ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΛ. δύο δὴ τρίγωνά ἐστι τὰ ΒΑΕ, ΗΖΛ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα τὴν ὑπὸ ΒΑΕ τῇ ὑπὸ ΗΖΛ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΗΛ τριγώνῳ. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῇ ὑπὸ ΖΛΗ. ἀλλ᾽ ἡ μὲν ὑπὸ ΑΕΒ τῇ ὑπὸ ΑΜΒ ἐστιν ἴση· ἐπὶ γὰρ τῆς αὐτῆς περιφερείας βεβήκασιν· ἡ δὲ ὑπὸ ΖΛΗ τῇ ὑπὸ ΖΝΗ· καὶ ἡ ὑπὸ ΑΜΒ ἄρα τῇ ὑπὸ ΖΝΗ ἐστιν ἴση. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΒΑΜ ὀρθῇ τῇ ὑπὸ ΗΖΝ ἴση· καὶ ἡ λοιπὴ ἄρα τῇ λοιπῇ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΜ τρίγωνον τῷ ΖΗΝ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΜ πρὸς τὴν ΗΝ, οὕτως ἡ ΒΑ πρὸς τὴν ΗΖ. ἀλλὰ τοῦ μὲν τῆς ΒΜ πρὸς τὴν ΗΝ λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς ΒΜ τετραγώνου πρὸς τὸ ἀπὸ τῆς ΗΝ τετραγώνου, τοῦ δὲ τῆς ΒΑ πρὸς τὴν ΗΖ διπλασίων ἐστὶν ὁ τοῦ ΑΒΓΔΕ πολυγώνου πρὸς τὸ ΖΗΘΚΛ πολύγωνον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Τὰ ἄρα ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἄλληλά ἐστιν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

Proposition 1

Similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles).



Let ABC and FGH be circles, and let $ABCDE$ and $FGHKL$ be similar polygons (inscribed) in them (respectively), and let BM and GN be the diameters of the circles (respectively). I say that as the square on BM is to the square on GN, so polygon $ABCDE$ (is) to polygon $FGHKL$.

For let BE, AM, GL, and FN be joined. And since polygon $ABCDE$ (is) similar to polygon $FGHKL$, angle BAE is also equal to (angle) GFL, and as BA is to AE, so GF (is) to FL [Def. 6.1]. So, BAE and GFL are two triangles having one angle equal to one angle, (namely), BAE (equal) to GFL, and the sides around the equal angles proportional. Triangle ABE is thus equiangular with triangle FGL [Prop. 6.6]. Thus, angle AEB is equal to (angle) FLG. But, AEB is equal to AMB, and FLG to FNG, for they stand on the same circumference [Prop. 3.27]. Thus, AMB is also equal to FNG. And the right-angle BAM is also equal to the right-angle GFN [Prop. 3.31]. Thus, the remaining (angle) is also equal to the remaining (angle) [Prop. 1.32]. Thus, triangle ABM is equiangular with triangle FGN. Thus, proportionally, as BM is to GN, so BA (is) to GF [Prop. 6.4]. But, the (ratio) of the square on BM to the square on GN is the square of the ratio of BM to GN, and the (ratio) of polygon $ABCDE$ to polygon $FGHKL$ is the square of the (ratio) of BA to GF [Prop. 6.20]. And, thus, as the square on BM (is) to the square on GN, so polygon $ABCDE$ (is) to polygon $FGHKL$.

Thus, similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles). (Which is) the very thing it was required to show.

β'.

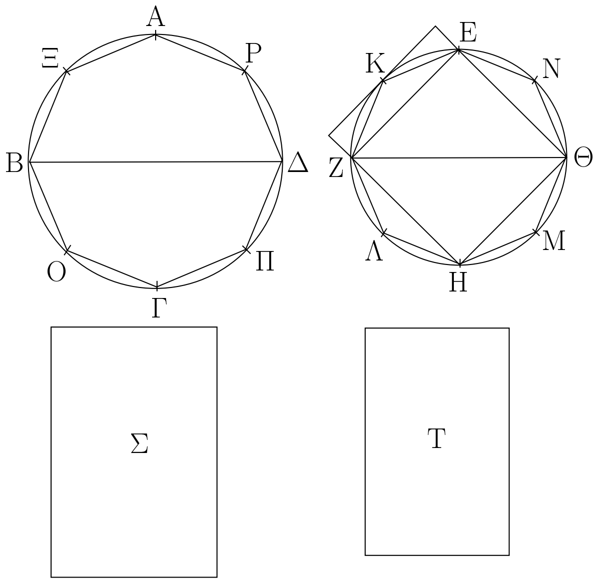
Οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.

Ἔστωσαν κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ, διάμετροι δὲ αὐτῶν [ἔστωσαν] αἱ ΒΔ, ΖΘ· λέγω, ὅτι ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ τετράγωνον.

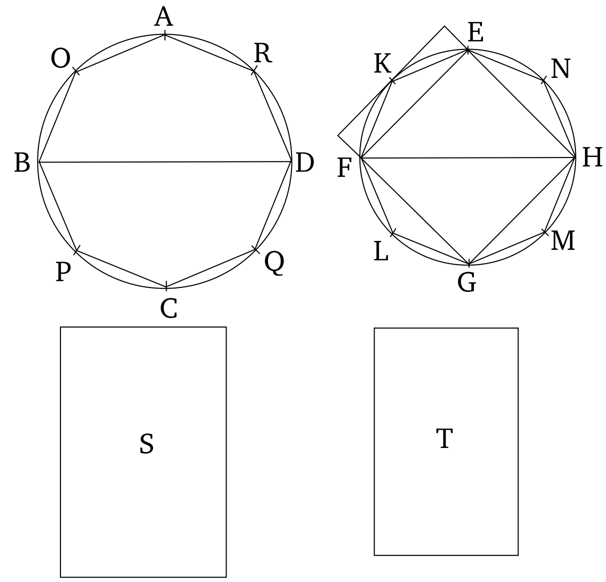
Proposition 2

Circles are to one another as the squares on (their) diameters.

Let $ABCD$ and $EFGH$ be circles, and [let] BD and FH [be] their diameters. I say that as circle $ABCD$ is to circle $EFGH$, so the square on BD (is) to the square on FH.



Εἰ γὰρ μή ἐστιν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ, οὕτως τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, ἔσται ὡς τὸ ἀπὸ τῆς ΒΔ πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος ἤτοι πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ κύκλου χωρίον ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Σ. καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ. τὸ δὴ ἐγγεγραμμένον τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΕΖΗΘ κύκλου, ἐπειδήπερ ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων ἐφαπτομένας [εὐθείας] τοῦ κύκλου ἀγάγωμεν, τοῦ περιγραφομένου περὶ τὸν κύκλον τετραγώνου ἥμισύ ἐστι τὸ ΕΖΗΘ τετράγωνον, τοῦ δὲ περιγραφέντος τετραγώνου ἐλάττων ἐστὶν ὁ κύκλος· ὥστε τὸ ΕΖΗΘ ἐγγεγραμμένομῆν τετράγωνον μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ ΕΖΗΘ κύκλου. τετμήσθωσαν δίχα αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι κατὰ τὰ Κ, Λ, Μ, Ν σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ· καὶ ἕκαστον ἄρα τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ καθ᾿ ἑαυτὸ τμήματος τοῦ κύκλου, ἐπειδήπερ ἐὰν διὰ τῶν Κ, Λ, Μ, Ν σημείων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν καὶ ἀναπληρώσωμεν τὰ ἐπὶ τῶν ΕΖ, ΖΗ, ΗΘ, ΘΕ εὐθειῶν παραλληλόγραμμα, ἕκαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων ἥμισυ ἔσται τοῦ καθ᾿ ἑαυτὸ παραλληλογράμμου, ἀλλὰ τὸ καθ᾿ ἑαυτὸ τμῆμα ἔλαττόν ἐστι τοῦ παραλληλογράμμου· ὥστε ἕκαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ καθ᾿ ἑαυτὸ τμήματος τοῦ κύκλου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κύκλου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου. ἐδείχθη γὰρ ἐν τῷ πρώτῳ θεωρήματι τοῦ δεκάτου βιβλίου, ὅτι δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους. λελείφθω οὖν,



For if the circle $ABCD$ is not to the (circle) $EFGH$, as the square on BD (is) to the (square) on FH, then as the (square) on BD (is) to the (square) on FH, so circle $ABCD$ will be to some area either less than, or greater than, circle $EFGH$. Let it, first of all, be (in that ratio) to (some) lesser (area), S. And let the square $EFGH$ be inscribed in circle $EFGH$ [Prop. 4.6]. So the inscribed square is greater than half of circle $EFGH$, inasmuch as if we draw tangents to the circle through the points E, F, G, and H, then square $EFGH$ is half of the square circumscribed about the circle [Prop. 1.47], and the circle is less than the circumscribed square. Hence, the inscribed square $EFGH$ is greater than half of circle $EFGH$. Let the circumferences EF, FG, GH, and HE be cut in half at points K, L, M, and N (respectively), and let EK, KF, FL, LG, GM, MH, HN, and NE be joined. And, thus, each of the triangles EKF, FLG, GMH, and HNE is greater than half of the segment of the circle about it, inasmuch as if we draw tangents to the circle through points K, L, M, and N, and complete the parallelograms on the straight-lines EF, FG, GH, and HE, then each of the triangles EKF, FLG, GMH, and HNE will be half of the parallelogram about it, but the segment about it is less than the parallelogram. Hence, each of the triangles EKF, FLG, GMH, and HNE is greater than half of the segment of the circle about it. So, by cutting the circumferences remaining behind in half, and joining straight-lines, and doing this continually, we will (eventually) leave behind some segments of the circle whose (sum) will be less than the excess by which circle $EFGH$ exceeds the area S. For we showed in the first theorem of the tenth book that if two unequal magnitudes are laid out, and if (a part) greater than a half is subtracted from the greater, and (if from) the remainder (a part) greater than a half (is subtracted), and this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid καὶ ἔστω τὰ ἐπὶ τῶν ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ τμήματα τοῦ ΕΖΗΘ κύκλου ἐλάττονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου. λοιπὸν ἄρα τὸ ΕΚΖΛΗΜΘΝ πολύγωνον μεῖζόν ἐστι τοῦ Σ χωρίου. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΚΖΛΗΜΘΝ πολυγώνῳ ὅμοιον πολύγωνον τὸ ΑΞΒΟΓΠΔΡ· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ τετράγωνον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον. ἀλλὰ καὶ ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον· καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον· ἐναλλὰξ ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸ ἐν αὐτῷ πολύγωνον, οὕτως τὸ Σ χωρίον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον. μείζων δὲ ὁ ΑΒΓΔ κύκλος τοῦ ἐν αὐτῷ πολυγώνου· μεῖζον ἄρα καὶ τὸ Σ χωρίον τοῦ ΕΚΖΛΗΜΘΝ πολυγώνου. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ὡς τὸ ἀπὸ ΖΘ πρὸς τὸ ἀπὸ ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλασσόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Λέγω δή, ὅτι οὐδὲ ὡς τὸ ἀπὸ τῆς ΒΔ πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς μεῖζόν τι τοῦ ΕΖΗΘ κύκλου χωρίον.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Σ. ἀνάπαλιν ἄρα [ἐστὶν] ὡς τὸ ἀπὸ τῆς ΖΘ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΔΒ, οὕτως τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον. ἀλλ᾽ ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΖΘ πρὸς τὸ ἀπὸ τῆς ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλασσόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς μεῖζόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον.

Οἱ ἄρα κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

out magnitude [Prop. 10.1]. Therefore, let the (segments) be left, and let the (sum of the) segments of the circle $EFGH$ on EK, KF, FL, LG, GM, MH, HN, and NE be less than the excess by which circle $EFGH$ exceeds area S. Thus, the remaining polygon $EKFLGMHN$ is greater than area S. And let the polygon $AOBPCQDR$, similar to the polygon $EKFLGMHN$, be inscribed in circle $ABCD$. Thus, as the square on BD is to the square on FH, so polygon $AOBPCQDR$ (is) to polygon $EKFLGMHN$ [Prop. 12.1]. But, also, as the square on BD (is) to the square on FH, so circle $ABCD$ (is) to area S. And, thus, as circle $ABCD$ (is) to area S, so polygon $AOBPGQDR$ (is) to polygon $EKFLGMHN$ [Prop. 5.11]. Thus, alternately, as circle $ABCD$ (is) to the polygon (inscribed) within it, so area S (is) to polygon $EKFLGMHN$ [Prop. 5.16]. And circle $ABCD$ (is) greater than the polygon (inscribed) within it. Thus, area S is also greater than polygon $EKFLGMHN$. But, (it is) also less. The very thing is impossible. Thus, the square on BD is not to the (square) on FH, as circle $ABCD$ (is) to some area less than circle $EFGH$. So, similarly, we can show that the (square) on FH (is) not to the (square) on BD as circle $EFGH$ (is) to some area less than circle $ABCD$ either.

So, I say that neither (is) the (square) on BD to the (square) on FH, as circle $ABCD$ (is) to some area greater than circle $EFGH$.

For, if possible, let it be (in that ratio) to (some) greater (area), S. Thus, inversely, as the square on FH [is] to the (square) on DB, so area S (is) to circle $ABCD$ [Prop. 5.7 corr.]. But, as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$ (see lemma). And, thus, as the (square) on FH (is) to the (square) on BD, so circle $EFGH$ (is) to some area less than circle $ABCD$ [Prop. 5.11]. The very thing was shown (to be) impossible. Thus, as the square on BD is to the (square) on FH, so circle $ABCD$ (is) not to some area greater than circle $EFGH$. And it was shown that neither (is it in that ratio) to (some) lesser (area). Thus, as the square on BD is to the (square) on FH, so circle $ABCD$ (is) to circle $EFGH$.

Thus, circles are to one another as the squares on (their) diameters. (Which is) the very thing it was required to show.

Λῆμμα.

Λέγω δή, ὅτι τοῦ Σ χωρίου μείζονος ὄντος τοῦ ΕΖΗΘ κύκλου ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Γεγονέτω γὰρ ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον. λέγω, ὅτι ἔλαττόν ἐστι τὸ Τ χωρίον τοῦ ΑΒΓΔ κύκλου. ἐπεὶ γάρ ἐστιν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον, ἐναλλάξ ἐστιν ὡς τὸ Σ χωρίον πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Τ χωρίον. μεῖζον

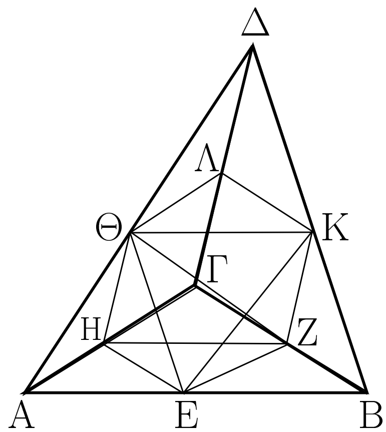
Lemma

So, I say that, area S being greater than circle $EFGH$, as area S is to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$.

For let it be contrived that as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to area T. I say that area T is less than circle $ABCD$. For since as area S is to circle $ABCD$, so circle $EFGH$ (is) to area T, alternately, as area S is to circle $EFGH$, so circle $ABCD$ (is) to area T [Prop. 5.16]. And area S (is) greater than circle $EFGH$. Thus, circle $ABCD$ (is) also greater than area *δὲ τὸ Σ χωρίον τοῦ ΕΖΗΘ κύκλου· μείζων ἄρα καὶ ὁ ΑΒΓΔ κύκλος τοῦ Τ χωρίου. ὥστε ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· ὅπερ ἔδει δεῖξαι.*

γ΄.

Πᾶσα πυραμὶς τρίγωνον ἔχουσα βάσιν διαιρεῖται εἰς δύο πυραμίδας ἴσας τε καὶ ὁμοίας ἀλλήλαις καὶ [ὁμοίας] τῇ ὅλῃ τριγώνους ἐχούσας βάσεις καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος.



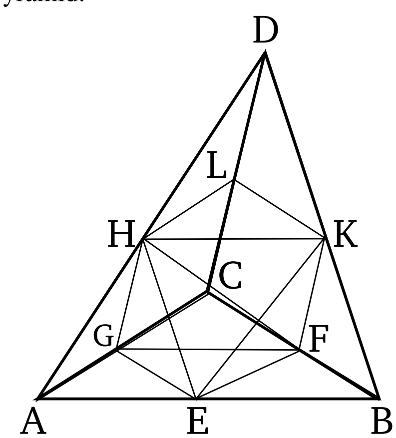
Ἔστω πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· λέγω, ὅτι ἡ ΑΒΓΔ πυραμὶς διαιρεῖται εἰς δύο πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἐχούσας καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος.

Τετμήσθωσαν γὰρ αἱ ΑΒ, ΒΓ, ΓΑ, ΑΔ, ΔΒ, ΔΓ δίχα κατὰ τὰ Ε, Ζ, Η, Θ, Κ, Λ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΘΕ, ΕΗ, ΗΘ, ΘΚ, ΚΛ, ΛΘ, ΚΖ, ΖΗ. ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΕ τῇ ΕΒ, ἡ δὲ ΑΘ τῇ ΔΘ, παράλληλος ἄρα ἐστὶν ἡ ΕΘ τῇ ΔΒ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΚ τῇ ΑΒ παράλληλός ἐστιν. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΕΒΚ· ἴση ἄρα ἐστὶν ἡ ΘΚ τῇ ΕΒ. ἀλλὰ ἡ ΕΒ τῇ ΕΑ ἐστιν ἴση· καὶ ἡ ΑΕ ἄρα τῇ ΘΚ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΑΘ τῇ ΘΔ ἴση· δύο δὴ αἱ ΕΑ, ΑΘ δυσὶ ταῖς ΚΘ, ΘΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΕΑΘ γωνίᾳ τῇ ὑπὸ ΚΘΔ ἴση· βάσις ἄρα ἡ ΕΘ βάσει τῇ ΚΔ ἐστιν ἴση. ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ ΑΕΘ τρίγωνον τῷ ΘΚΔ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΘΗ τρίγωνον τῷ ΘΛΔ τριγώνῳ ἴσον τέ ἐστι καὶ ὅμοιον. καὶ ἐπεὶ δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΕΘ, ΘΗ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΚΔ, ΔΛ εἰσιν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι, ἴσας γωνίας περιέξουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΕΘΗ γωνία τῇ ὑπὸ ΚΔΛ γωνίᾳ. καὶ ἐπεὶ δύο εὐθεῖαι αἱ ΕΘ, ΘΗ δυσὶ ταῖς ΚΔ, ΔΛ ἴσαι εἰσὶν ἑκατέρα εκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΕΘΗ γωνίᾳ τῇ ὑπὸ ΚΔΛ ἐστιν ἴση, βάσις ἄρα ἡ ΕΗ βάσει τῇ ΚΛ [ἐστιν] ἴση· ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ ΕΘΗ τρίγωνον τῷ ΚΔΛ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ

T [Prop. 5.14]. Hence, as area *S* is to circle *ABCD*, so circle *EFGH* (is) to some area less than circle *ABCD*. (Which is) the very thing it was required to show.

Proposition 3

Any pyramid having a triangular base is divided into two pyramids having triangular bases (which are) equal, similar to one another, and [similar] to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.



Let there be a pyramid whose base is triangle *ABC*, and (whose) apex (is) point *D*. I say that pyramid *ABCD* is divided into two pyramids having triangular bases (which are) equal to one another, and similar to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.

For let *AB*, *BC*, *CA*, *AD*, *DB*, and *DC* be cut in half at points *E*, *F*, *G*, *H*, *K*, and *L* (respectively). And let *HE*, *EG*, *GH*, *HK*, *KL*, *LH*, *KF*, and *FG* be joined. Since *AE* is equal to *EB*, and *AH* to *DH*, *EH* is thus parallel to *DB* [Prop. 6.2]. So, for the same (reasons), *HK* is also parallel to *AB*. Thus, *HEBK* is a parallelogram. Thus, *HK* is equal to *EB* [Prop. 1.34]. But, *EB* is equal to *EA*. Thus, *AE* is also equal to *HK*. And *AH* is also equal to *HD*. So the two (straight-lines) *EA* and *AH* are equal to the two (straight-lines) *KH* and *HD*, respectively. And angle *EAH* (is) equal to angle *KHD* [Prop. 1.29]. Thus, base *EH* is equal to base *KD* [Prop. 1.4]. Thus, triangle *AEH* is equal and similar to triangle *HKD* [Prop. 1.4]. So, for the same (reasons), triangle *AHG* is also equal and similar to triangle *HLD*. And since *EH* and *HG* are two straight-lines joining one another (which are respectively) parallel to two straight-lines joining one another, *KD* and *DL*, not being in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle *EHG* is equal to angle *KDL*. And since the two straight-lines *EH* and *HG* are equal to the two straight-lines *KD* and *DL*, respectively, and

ΑΕΗ τρίγωνον τῷ ΘΚΛ τριγώνῳ ἴσον τε καὶ ὅμοιόν ἐστιν. ἡ ἄρα πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον, ἴση καὶ ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. καὶ ἐπεὶ τριγώνου τοῦ ΑΔΒ παρὰ μίαν τῶν πλευρῶν τὴν ΑΒ ἦκται ἡ ΘΚ, ἰσογώνιόν ἐστι τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνῳ, καὶ τὰς πλευρὰς ἀνάλογον ἔχουσιν· ὅμοιον ἄρα ἐστὶ τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΔΒΓ τρίγωνον τῷ ΔΚΛ τριγώνῳ ὅμοιόν ἐστιν, τὸ δὲ ΑΔΓ τῷ ΔΛΘ. καὶ ἐπεὶ δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΒΑ, ΑΓ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΚΘ, ΘΛ εἴσιν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΚΘΛ. καί ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΚΘ πρὸς τὴν ΘΛ· ὅμοιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΘΚΛ τριγώνῳ. καὶ πυραμὶς ἄρα, ἧς βάσις μέν ἐστι τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. ἀλλὰ πυραμίς, ἧς βάσις μέν [ἐστι] τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ὁμοία ἐδείχθη πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον. ἑκατέρα ἄρα τῶν ΑΕΗΘ, ΘΚΛΔ πυραμίδων ὁμοία ἐστὶ τῇ ὅλῃ τῇ ΑΒΓΔ πυραμίδι.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΖ τῇ ΖΓ, διπλάσιόν ἐστι τὸ ΕΒΖΗ παραλληλόγραμμον τοῦ ΗΖΓ τριγώνου. καὶ ἐπεί, ἐὰν ᾖ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἐστὶ τὰ πρίσματα, ἴσον ἄρα ἐστὶ τὸ πρίσμα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΒΚΖ, ΕΘΗ, τριῶν δὲ παραλληλογράμμων τῶν ΕΒΖΗ, ΕΒΚΘ, ΘΚΖΗ τῷ πρισματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΗΖΓ, ΘΚΛ, τριῶν δὲ παραλληλογράμμων τῶν ΚΖΓΛ, ΛΓΗΘ, ΘΚΖΗ. καὶ φανερόν, ὅτι ἑκάτερον τῶν πρισμάτων, οὗ τε βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, καὶ οὗ βάσις τὸ ΗΖΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΘΚΛ τρίγωνον, μεῖζόν ἐστιν ἑκατέρας τῶν πυραμίδων, ὧν βάσεις μὲν τὰ ΑΕΗ, ΘΚΛ τρίγωνα, κορυφαὶ δὲ τὰ Θ, Δ σημεῖα, ἐπειδήπερ [καὶ] ἐὰν ἐπιζεύξωμεν τὰς ΕΖ, ΕΚ εὐθείας, τὸ μὲν πρίσμα, οὗ βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, μεῖζόν ἐστι τῆς πυραμίδος, ἧς βάσις τὸ ΕΒΖ τρίγωνον, κορυφὴ δὲ τὸ Κ σημεῖον. ἀλλ᾽ ἡ πυραμίς, ἧς βάσις τὸ ΕΒΖ τρίγωνον, κορυφὴ δὲ τὸ Κ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον· ὑπὸ γὰρ ἴσων καὶ ὁμοίων ἐπιπέδων περιέχονται. ὥστε καὶ τὸ πρίσμα, οὗ βάσις μὲν τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, μεῖζόν ἐστι πυραμίδος, ἧς βάσις μὲν τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον. ἴσον δὲ τὸ μὲν πρίσμα, οὗ βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, τῷ πρίσματι, οὗ βάσις μὲν τὸ ΗΖΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΘΚΛ τρίγωνον· ἡ δὲ πυραμίς, ἧς βάσις τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. τὰ

angle *EHG* is equal to angle *KDL*, base *EG* [is] thus equal to base *KL* [Prop. 1.4]. Thus, triangle *EHG* is equal and similar to triangle *KDL*. So, for the same (reasons), triangle *AEG* is also equal and similar to triangle *HKL*. Thus, the pyramid whose base is triangle *AEG*, and apex the point *H*, is equal and similar to the pyramid whose base is triangle *HKL*, and apex the point *D* [Def. 11.10]. And since *HK* has been drawn parallel to one of the sides, *AB*, of triangle *ADB*, triangle *ADB* is equiangular to triangle *DHK* [Prop. 1.29], and they have proportional sides. Thus, triangle *ADB* is similar to triangle *DHK* [Def. 6.1]. So, for the same (reasons), triangle *DBC* is also similar to triangle *DKL*, and *ADC* to *DLH*. And since two straight-lines joining one another, *BA* and *AC*, are parallel to two straight-lines joining one another, *KH* and *HL*, not in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle *BAC* is equal to (angle) *KHL*. And as *BA* is to *AC*, so *KH* (is) to *HL*. Thus, triangle *ABC* is similar to triangle *HKL* [Prop. 6.6]. And, thus, the pyramid whose base is triangle *ABC*, and apex the point *D*, is similar to the pyramid whose base is triangle *HKL*, and apex the point *D* [Def. 11.9]. But, the pyramid whose base [is] triangle *HKL*, and apex the point *D*, was shown (to be) similar to the pyramid whose base is triangle *AEG*, and apex the point *H*. Thus, each of the pyramids *AEGH* and *HKLD* is similar to the whole pyramid *ABCD*.

And since *BF* is equal to *FC*, parallelogram *EBFG* is double triangle *GFC* [Prop. 1.41]. And since, if two prisms (have) equal heights, and the former has a parallelogram as a base, and the latter a triangle, and the parallelogram (is) double the triangle, then the prisms are equal [Prop. 11.39], the prism contained by the two triangles *BKF* and *EHG*, and the three parallelograms *EBFG*, *EBKH*, and *HKFG*, is thus equal to the prism contained by the two triangles *GFC* and *HKL*, and the three parallelograms *KFCL*, *LCGH*, and *HKFG*. And (it is) clear that each of the prisms whose base (is) parallelogram *EBFG*, and opposite (side) straight-line *HK*, and whose base (is) triangle *GFC*, and opposite (plane) triangle *HKL*, is greater than each of the pyramids whose bases are triangles *AEG* and *HKL*, and apexes the points *H* and *D* (respectively), inasmuch as, if we [also] join the straight-lines *EF* and *EK* then the prism whose base (is) parallelogram *EBFG*, and opposite (side) straight-line *HK*, is greater than the pyramid whose base (is) triangle *EBF*, and apex the point *K*. But the pyramid whose base (is) triangle *EBF*, and apex the point *K*, is equal to the pyramid whose base is triangle *AEG*, and apex point *H*. For they are contained by equal and similar planes. And, hence, the prism whose base (is) parallelogram *EBFG*, and opposite (side) straight-line *HK*, is greater than the pyramid whose base (is) triangle *AEG*, and apex the point *H*. And the prism whose base is parallelogram *EBFG*, and opposite (side) straight-line *HK*, (is) equal to the prism whose base (is) triangle *GFC*, and opposite (plane) triangle *HKL*. And the *ἄρα εἰρημένα δύο πρίσματα μείζονά ἐστι τῶν εἰρημένων δύο πυραμίδων, ὧν βάσεις μὲν τὰ ΑΕΗ, ΘΚΛ τρίγωνα, κορυφαὶ δὲ τὰ Θ, Δ σημεῖα.*

Ἡ ἄρα ὅλη πυραμίς, ἧς βάσις τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, διῄρηται εἴς τε δύο πυραμίδας ἴσας ἀλλήλαις [καὶ ὁμοίας τῇ ὅλῃ] καὶ εἰς δύο πρίσματα ἴσα, καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος· ὅπερ ἔδει δεῖξαι.

pyramid whose base (is) triangle *AEG*, and apex the point *H*, is equal to the pyramid whose base (is) triangle *HKL*, and apex the point *D*. Thus, the (sum of the) aforementioned two prisms is greater than the (sum of the) aforementioned two pyramids, whose bases (are) triangles *AEG* and *HKL*, and apexes the points *H* and *D* (respectively).

Thus, the whole pyramid, whose base (is) triangle *ABC*, and apex the point *D*, has been divided into two pyramids (which are) equal to one another [and similar to the whole], and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid. (Which is) the very thing it was required to show.

δ΄.

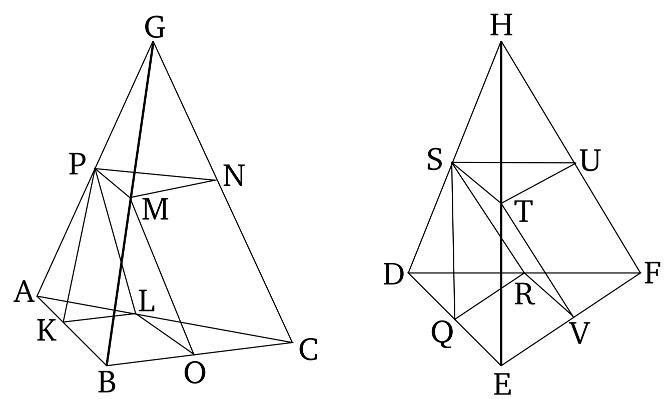
Ἐὰν ὦσι δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις, διαιρεθῇ δὲ ἑκατέρα αὐτῶν εἴς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα, ἔσται ὡς ἡ τῆς μιᾶς πυραμίδος βάσις πρὸς τὴν τῆς ἑτέρας πυραμίδος βάσιν, οὕτως τὰ ἐν τῇ μιᾷ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ἑτάρᾳ πυραμίδι πρίσματα πάντα ἰσοπληθῆ.

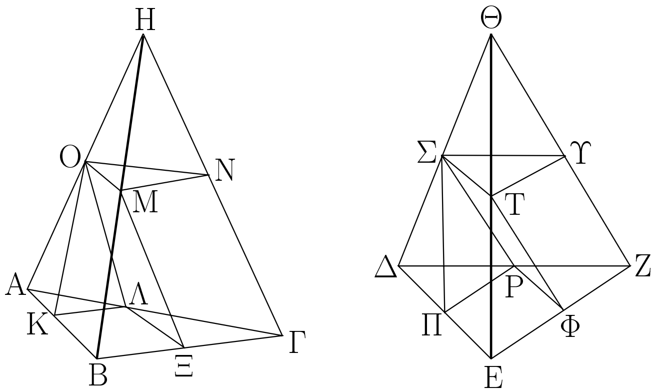
Ἔστωσαν δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις τὰς ΑΒΓ, ΔΕΖ, κορυφὰς δὲ τὰ Η, Θ σημεῖα, καὶ διῃρήσθω ἑκατέρα αὐτῶν εἴς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ΑΒΓΗ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα ἰσοπληθῆ.

Proposition 4

If there are two pyramids with the same height, having triangular bases, and each of them is divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms then as the base of one pyramid (is) to the base of the other pyramid, so (the sum of) all the prisms in one pyramid will be to (the sum of all) the equal number of prisms in the other pyramid.

Let there be two pyramids with the same height, having the triangular bases *ABC* and *DEF*, (with) apexes the points *G* and *H* (respectively). And let each of them be divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms [Prop. 12.3]. I say that as base *ABC* is to base *DEF*, so (the sum of) all the prisms in pyramid *ABCG* (is) to (the sum of) all the equal number of prisms in pyramid *DEFH*.





Ἐπεὶ γὰρ ἴση ἐστὶν ἡ μὲν ΒΞ τῇ ΞΓ, ἡ δὲ ΑΛ τῇ ΛΓ, παράλληλος ἄρα ἐστὶν ἡ ΛΞ τῇ ΑΒ καὶ ὅμοιον τὸ ΑΒΓ τρίγωνον τῷ ΛΞΓ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΔΕΖ τρίγωνον τῷ ΡΦΖ τριγώνῳ ὅμοιόν ἐστιν. καὶ ἐπεὶ διπλασίων ἐστὶν ἡ μὲν ΒΓ τῆς ΓΞ, ἡ δὲ ΕΖ τῆς ΖΦ, ἔστιν ἄρα ὡς ἡ ΒΓ πρὸς τὴν ΓΞ, οὕτως ἡ ΕΖ πρὸς τὴν ΖΦ. καὶ ἀναγέγραπται ἀπὸ μὲν τῶν ΒΓ, ΓΞ ὅμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ ΑΒΓ, ΛΞΓ, ἀπὸ δὲ τῶν ΕΖ, ΖΦ ὅμοιά τε καὶ ὁμοίως κείμενα [εὐθύγραμμα] τὰ ΔΕΖ, ΡΦΖ· ἔστιν ἄρα ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΛΞΓ τρίγωνον, οὕτως τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον· ἐναλλὰξ ἄρα ἐστὶν ὡς τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ [τρίγωνον], οὕτως τὸ ΛΞΓ [τρίγωνον] πρὸς τὸ ΡΦΖ τρίγωνον. ἀλλ᾽ ὡς τὸ ΛΞΓ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις μέν [ἐστι] τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ· καὶ ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις μὲν τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ. ὡς δὲ τὰ εἰρημένα πρίσματα πρὸς ἄλληλα, οὕτως τὸ πρίσμα, οὗ βάσις μὲν τὸ ΚΒΞΛ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΟΜ εὐθεῖα, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΠΕΦΡ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΣΤ εὐθεῖα. καὶ τὰ δύο ἄρα πρίσματα, οὗ τε βάσις μὲν τὸ ΚΒΞΛ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΟΜ, καὶ οὗ βάσις μὲν τὸ ΛΞΓ, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὰ πρίσματα, οὗ τε βάσις μὲν τὸ ΠΕΦΡ, ἀπεναντίον δὲ ἡ ΣΤ εὐθεῖα, καὶ οὗ βάσις μὲν τὸ ΡΦΖ τρίγωνον, ἀπεναντίον δὲ τὸ ΣΤΥ. καὶ ὡς ἄρα ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ εἰρημένα δύο πρίσματα πρὸς τὰ εἰρημένα δύο πρίσματα.

Καὶ ὁμοίως, ἐὰν διαιρεθῶσιν αἱ ΟΜΝΗ, ΣΤΥΘ πυραμίδες εἴς τε δύο πρίσματα καὶ δύο πυραμίδας, ἔσται ὡς ἡ ΟΜΝ βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως τὰ ἐν τῇ ΟΜΝΗ πυραμίδι δύο πρίσματα πρὸς τὰ ἐν τῇ ΣΤΥΘ πυραμίδι δύο πρίσματα. ἀλλ᾽ ὡς ἡ ΟΜΝ βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν· ἴσον γὰρ ἑκάτερον τῶν ΟΜΝ, ΣΤΥ τριγώνων ἑκατέρῳ τῶν ΛΞΓ, ΡΦΖ. καὶ ὡς ἄρα ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ τέσσαρα πρίσματα πρὸς τὰ τέσσαρα πρίσματα. ὁμοίως δὲ κἂν τὰς ὑπο-

For since *BO* is equal to *OC*, and *AL* to *LC*, *LO* is thus parallel to *AB*, and triangle *ABC* similar to triangle *LOC* [Prop. 12.3]. So, for the same (reasons), triangle *DEF* is also similar to triangle *RVF*. And since *BC* is double *CO*, and *EF* (double) *FV*, thus as *BC* (is) to *CO*, so *EF* (is) to *FV*. And the similar, and similarly laid out, rectilinear (figures) *ABC* and *LOC* be described on *BC* and *CO* (respectively), and the similar, and similarly laid out, [rectilinear] (figures) *DEF* and *RVF* on *EF* and *FV* (respectively). Thus, as triangle *ABC* is to triangle *LOC*, so triangle *DEF* (is) to triangle *RVF* [Prop. 6.22]. Thus, alternately, as triangle *ABC* is to [triangle] *DEF*, so [triangle] *LOC* (is) to triangle *RVF* [Prop. 5.16]. But, as triangle *LOC* (is) to triangle *RVF*, so the prism whose base [is] triangle *LOC*, and opposite (plane) *PMN*, (is) to the prism whose base (is) triangle *RVF*, and opposite (plane) *STU* (see lemma). And, thus, as triangle *ABC* (is) to triangle *DEF*, so the prism whose base (is) triangle *LOC*, and opposite (plane) *PMN*, (is) to the prism whose base (is) triangle *RVF*, and opposite (plane) *STU*. And as the aforementioned prisms (are) to one another, so the prism whose base (is) parallelogram *KBOL*, and opposite (side) straight-line *PM*, (is) to the prism whose base (is) parallelogram *QEVR*, and opposite (side) straight-line *ST* [Props. 11.39, 12.3]. Thus, also, (is) the (sum of the) two prisms—that whose base (is) parallelogram *KBOL*, and opposite (side) *PM*, and that whose base (is) *LOC*, and opposite (plane) *PMN*—to (the sum of) the (two) prisms—that whose base (is) *QEVR*, and opposite (side) straight-line *ST*, and that whose base (is) triangle *RVF*, and opposite (plane) *STU* [Prop. 5.12]. And, thus, as base *ABC* (is) to base *DEF*, so the (sum of the first) aforementioned two prisms (is) to the (sum of the second) aforementioned two prisms.

And, similarly, if pyramids *PMNG* and *STUH* are divided into two prisms, and two pyramids, as base *PMN* (is) to base *STU*, so (the sum of) the two prisms in pyramid *PMNG* will be to (the sum of) the two prisms in pyramid *STUH*. But, as base *PMN* (is) to base *STU*, so base *ABC* (is) to base *DEF*. For the triangles *PMN* and *STU* (are) equal to *LOC* and *RVF*, respectively. And, thus, as base *ABC* (is) to base *DEF*, so (the sum of) the four prisms (is) to (the sum of) the four prisms [Prop. 5.12]. So, similarly, even if we divide the pyramids left *λειπομένας πυραμίδας διέλωμεν εἴς τε δύο πυραμίδας καὶ εἰς δύο πρίσματα, ἔσται ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ΑΒΓΗ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα πάντα ἰσοπληθῆ· ὅπερ ἔδει δεῖξαι.*

behind into two pyramids and into two prisms, as base ABC (is) to base DEF, so (the sum of) all the prisms in pyramid $ABCG$ will be to (the sum of) all the equal number of prisms in pyramid $DEFH$. (Which is) the very thing it was required to show.

Λῆμμα.

Ὅτι δέ ἐστιν ὡς τὸ ΛΞΓ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις τὸ ΛΞΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΟΜΝ, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΡΦΖ [τρίγωνον], ἀπεναντίον δὲ τὸ ΣΤΥ, οὕτω δεικτέον.

Ἐπὶ γὰρ τῆς αὐτῆς καταγραφῆς νενοήσθωσαν ἀπὸ τῶν Η, Θ κάθετοι ἐπὶ τὰ ΑΒΓ, ΔΕΖ ἐπίπεδα, ἴσαι δηλαδὴ τυγχάνουσαι διὰ τὸ ἰσοϋψεῖς ὑποκεῖσθαι τὰς πυραμίδας. καὶ ἐπεὶ δύο εὐθεῖαι ἥ τε ΗΓ καὶ ἡ ἀπὸ τοῦ Η κάθετος ὑπὸ παραλλήλων ἐπιπέδων τῶν ΑΒΓ, ΟΜΝ τέμνονται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται. καὶ τέτμηται ἡ ΗΓ δίχα ὑπὸ τοῦ ΟΜΝ ἐπιπέδου κατὰ τὸ Ν· καὶ ἡ ἀπὸ τοῦ Η ἄρα κάθετος ἐπὶ τὸ ΑΒΓ ἐπίπεδον δίχα τμηθήσεται ὑπὸ τοῦ ΟΜΝ ἐπιπέδου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἀπὸ τοῦ Θ κάθετος ἐπὶ τὸ ΔΕΖ ἐπίπεδον δίχα τμηθήσεται ὑπὸ τοῦ ΣΤΥ ἐπιπέδου. καί εἰσιν ἴσαι αἱ ἀπὸ τῶν Η, Θ κάθετοι ἐπὶ τὰ ΑΒΓ, ΔΕΖ ἐπίπεδα· ἴσαι ἄρα καὶ αἱ ἀπὸ τῶν ΟΜΝ, ΣΤΥ τριγώνων ἐπὶ τὰ ΑΒΓ, ΔΕΖ κάθετοι. ἰσοϋψῆ ἄρα [ἐστὶ] τὰ πρίσματα, ὧν βάσεις μέν εἰσι τὰ ΛΞΓ, ΡΦΖ τρίγωνα, ἀπεναντίον δὲ τὰ ΟΜΝ, ΣΤΥ. ὥστε καὶ τὰ στερεὰ παραλληλεπίπεδα τὰ ἀπὸ τῶν εἰρημένων πρισμάτων ἀναγραφόμενα ἰσοϋψῆ καὶ πρὸς ἄλληλά [εἰσιν] ὡς αἱ βάσεις· καὶ τὰ ἡμίση ἄρα ἐστὶν ὡς ἡ ΛΞΓ βάσις πρὸς τὴν ΡΦΖ βάσιν, οὕτως τὰ εἰρημένα πρίσματα πρὸς ἄλληλα· ὅπερ ἔδει δεῖξαι.

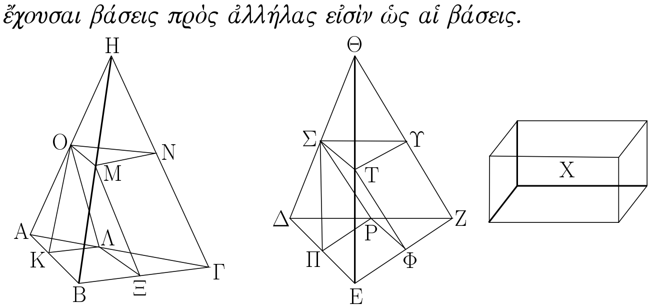
Lemma

And one may show, as follows, that as triangle LOC is to triangle RVF, so the prism whose base (is) triangle LOC, and opposite (plane) PMN, (is) to the prism whose base (is) [triangle] RVF, and opposite (plane) STU.

For, in the same figure, let perpendiculars be conceived (drawn) from (points) G and H to the planes ABC and DEF (respectively). These clearly turn out to be equal, on account of the pyramids being assumed (to be) of equal height. And since two straight-lines, GC and the perpendicular from G, are cut by the parallel planes ABC and PMN they will be cut in the same ratios [Prop. 11.17]. And GC was cut in half by the plane PMN at N. Thus, the perpendicular from G to the plane ABC will also be cut in half by the plane PMN. So, for the same (reasons), the perpendicular from H to the plane DEF will also be cut in half by the plane STU. And the perpendiculars from G and H to the planes ABC and DEF (respectively) are equal. Thus, the perpendiculars from the triangles PMN and STU to ABC and DEF (respectively, are) also equal. Thus, the prisms whose bases are triangles LOC and RVF, and opposite (sides) PMN and STU (respectively), [are] of equal height. And, hence, the parallelepiped solids described on the aforementioned prisms [are] of equal height and (are) to one another as their bases [Prop. 11.32]. Likewise, the halves (of the solids) [Prop. 11.28]. Thus, as base LOC is to base RVF, so the aforementioned prisms (are) to one another. (Which is) the very thing it was required to show.

ε'.

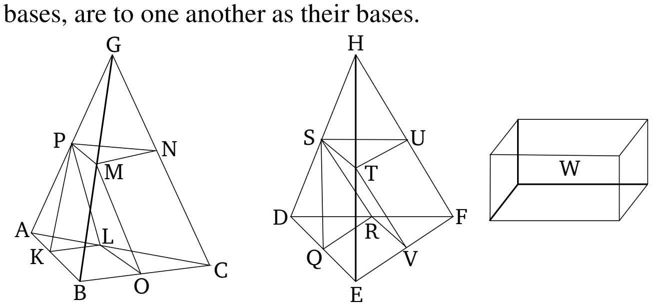
Αἱ ὑπὸ τὸ αὐτὸ ὕψος οὖσαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.



Ἔστωσαν ὑπὸ τὸ αὐτὸ ὕψος πυραμίδες, ὧν βάσεις μὲν τὰ ΑΒΓ, ΔΕΖ τρίγωνα, κορυφαὶ δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα.

Proposition 5

Pyramids which are of the same height, and have triangular bases, are to one another as their bases.



Let there be pyramids of the same height whose bases (are) the triangles ABC and DEF, and apexes the points G and H (respectively). I say that as base ABC is to base DEF, so pyramid $ABCG$ (is) to pyramid $DEFH$.

Εἰ γὰρ μή ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα, ἔσται ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς ἤτοι πρὸς ἔλασσόν τι τῆς ΔΕΖΘ πυραμίδος στερεὸν ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Χ, καὶ διῃρήσθω ἡ ΔΕΖΘ πυραμὶς εἴς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· τὰ δὴ δύο πρίσαμτα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος. καὶ πάλιν αἱ ἐκ τῆς διαιρέσεως γινόμεναι πυραμίδες ὁμοίως διῃρήσθωσαν, καὶ τοῦτο ἀεὶ γινέσθω, ἕως οὗ λειφθῶσί τινες πυραμίδες ἀπὸ τῆς ΔΕΖΘ πυραμίδος, αἵ εἰσιν ἐλάττονες τῆς ὑπεροχῆς, ᾗ ὑπερέχει ἡ ΔΕΖΘ πυραμὶς τοῦ Χ στερεοῦ. λελείφθωσαν καὶ ἔστωσαν λόγου ἕνεκεν αἱ ΔΠΡΣ, ΣΤΥΘ· λοιπὰ ἄρα τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα μείζονά ἐστι τοῦ Χ στερεοῦ. διῃρήσθω καὶ ἡ ΑΒΓΗ πυραμὶς ὁμοίως καὶ ἰσοπληθῶς τῇ ΔΕΖΘ πυραμίδι· ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ΑΒΓΗ πυραμίδι πρίσματα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα, ἀλλὰ καὶ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὸ Χ στερεόν· καὶ ὡς ἄρα ἡ ΑΒΓΗ πυραμὶς πρὸς τὸ Χ στερεόν, οὕτως τὰ ἐν τῇ ΑΒΓΗ πυραμίδι πρίσματα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα· ἐναλλὰξ ἄρα ὡς ἡ ΑΒΓΗ πυραμὶς πρὸς τὰ ἐν αὐτῇ πρίσματα, οὕτως τὸ Χ στερεὸν πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα. μείζων δὲ ἡ ΑΒΓΗ πυραμὶς τῶν ἐν αὐτῇ πρισμάτων· μεῖζον ἄρα καὶ τὸ Χ στερεὸν τῶν ἐν τῇ ΔΕΖΘ πυραμίδι πρισμάτων. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς ἔλασσόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν. ὁμοίως δὴ δειχθήσεται, ὅτι οὐδὲ ὡς ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλαττόν τι τῆς ΑΒΓΗ πυραμίδος στερεόν.

Λέγω δή, ὅτι οὐκ ἔστιν οὐδὲ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς μεῖζόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Χ· ἀνάπαλιν ἄρα ἐστὶν ὡς ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως τὸ Χ στερεὸν πρὸς τὴν ΑΒΓΗ πυραμίδα. ὡς δὲ τὸ Χ στερεὸν πρὸς τὴν ΑΒΓΗ πυραμίδα, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλασσόν τι τῆς ΑΒΓΗ πυραμίδος, ὡς ἔμπροσθεν ἐδείχθη· καὶ ὡς ἄρα ἡ ΔΕΖ βάσις πρὸς τὴν ΑΒΓ βάσιν, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλασσόν τι τῆς ΑΒΓΗ πυραμίδος· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς μεῖζόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον. ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα· ὅπερ ἔδει δεῖξαι.

For if base ABC is not to base DEF, as pyramid $ABCG$ (is) to pyramid $DEFH$, then base ABC will be to base DEF, as pyramid $ABCG$ (is) to some solid either less than, or greater than, pyramid $DEFH$. Let it, first of all, be (in this ratio) to (some) lesser (solid), W. And let pyramid $DEFH$ be divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms. So, the (sum of the) two prisms is greater than half of the whole pyramid [Prop. 12.3]. And, again, let the pyramids generated by the division be similarly divided, and let this be done continually until some pyramids are left from pyramid $DEFH$ which (when added together) are less than the excess by which pyramid $DEFH$ exceeds the solid W [Prop. 10.1]. Let them be left, and, for the sake of argument, let them be $DQRS$ and $STUH$. Thus, the (sum of the) remaining prisms within pyramid $DEFH$ is greater than solid W. Let pyramid $ABCG$ also be divided similarly, and a similar number of times, as pyramid $DEFH$. Thus, as base ABC is to base DEF, so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 12.4]. But, also, as base ABC (is) to base DEF, so pyramid $ABCG$ (is) to solid W. And, thus, as pyramid $ABCG$ (is) to solid W, so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.11]. Thus, alternately, as pyramid $ABCG$ (is) to the (sum of the) prisms within it, so solid W (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.16]. And pyramid $ABCG$ (is) greater than the (sum of the) prisms within it. Thus, solid W (is) also greater than the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.14]. But, (it is) also less. This very thing is impossible. Thus, as base ABC is to base DEF, so pyramid $ABCG$ (is) not to some solid less than pyramid $DEFH$. So, similarly, we can show that base DEF is not to base ABC, as pyramid $DEFH$ (is) to some solid less than pyramid $ABCG$ either.

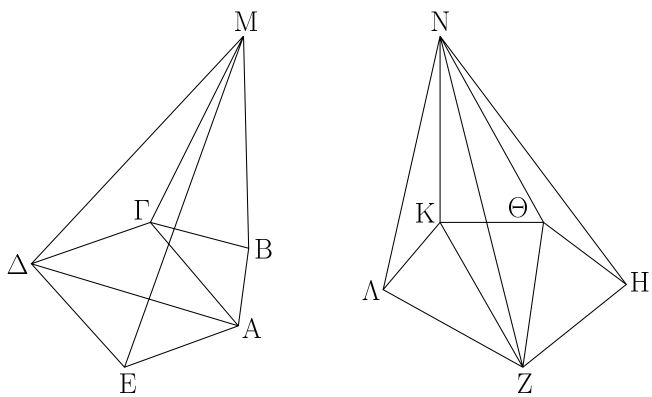
So, I say that neither is base ABC to base DEF, as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$.

For, if possible, let it be (in this ratio) to some greater (solid), W. Thus, inversely, as base DEF (is) to base ABC, so solid W (is) to pyramid $ABCG$ [Prop. 5.7. corr.]. And as solid W (is) to pyramid $ABCG$, so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$, as shown before [Prop. 12.2 lem.]. And, thus, as base DEF (is) to base ABC, so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$ [Prop. 5.11]. The very thing was shown (to be) absurd. Thus, base ABC is not to base DEF, as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$. And, it was shown that neither (is it in this ratio) to a lesser (solid). Thus, as base ABC is to base DEF, so pyramid $ABCG$ (is) to pyramid $DEFH$. (Which is) the very thing it was required to show.

ϛʹ.

Proposition 6

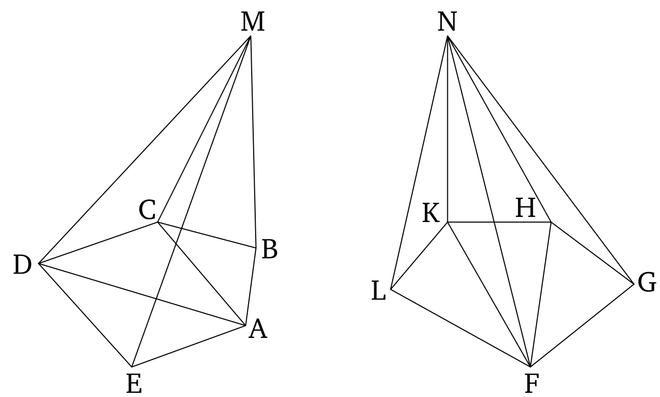
Αἱ ὑπὸ τὸ αὐτὸ ὕψος οὖσαι πυραμίδες καὶ πολυγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.



Ἔστωσαν ὑπὸ τὸ αὐτὸ ὕψος πυραμίδες, ὧν [αἱ] βάσεις μὲν τὰ ΑΒΓΔΕ, ΖΗΘΚΛ πολύγωνα, κορυφαὶ δὲ τὰ Μ, Ν σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΚΛΝ πυραμίδα.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΑΔ, ΖΘ, ΖΚ. ἐπεὶ οὖν δύο πυραμίδες εἰσὶν αἱ ΑΒΓΜ, ΑΓΔΜ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις· ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΑΓΔ βάσιν, οὕτως ἡ ΑΒΓΜ πυραμὶς πρὸς τὴν ΑΓΔΜ πυραμίδα. καὶ συνθέντι ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΑΓΔ βάσιν, οὕτως ἡ ΑΒΓΔΜ πυραμὶς πρὸς τὴν ΑΓΔΜ πυραμίδα. ἀλλὰ καὶ ὡς ἡ ΑΓΔ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΓΔΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΒΓΔΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. καὶ συνθέντι πάλιν, ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ὡς ἡ ΖΗΘΚΛ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως καὶ ἡ ΖΗΘΚΛΝ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. καὶ ἐπεὶ δύο πυραμίδες εἰσὶν αἱ ΑΔΕΜ, ΖΗΘΝ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, ἔστιν ἄρα ὡς ἡ ΑΔΕ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως ἡ ΑΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. ἀλλ' ὡς ἡ ΑΔΕ βάσις πρὸς τὴν ΑΒΓΔΕ βάσιν, οὕτως ἦν ἡ ΑΔΕΜ πυραμὶς πρὸς τὴν ΑΒΓΔΕΜ πυραμίδα. καὶ δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. ἀλλὰ μὴν καὶ ὡς ἡ ΖΗΘ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἦν καὶ ἡ ΖΗΘΝ πυραμὶς πρὸς τὴν ΖΗΘΚΛΝ πυραμίδα, καὶ δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘΚΛ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΚΛΝ πυραμίδα· ὅπερ ἔδει δεῖξαι.

Pyramids which are of the same height, and have polygonal bases, are to one another as their bases.



Let there be pyramids of the same height whose bases (are) the polygons $ABCDE$ and $FGHKL$, and apexes the points M and N (respectively). I say that as base $ABCDE$ is to base $FGHKL$, so pyramid $ABCDEM$ (is) to pyramid $FGHKLN$.

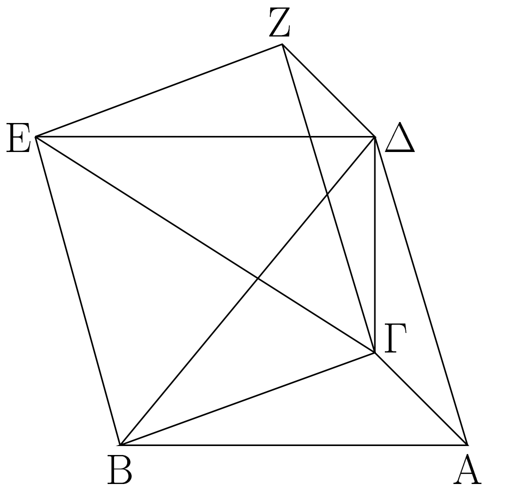
For let AC, AD, FH, and FK be joined. Therefore, since $ABCM$ and $ACDM$ are two pyramids having triangular bases and equal height, they are to one another as their bases [Prop. 12.5]. Thus, as base ABC is to base ACD, so pyramid $ABCM$ (is) to pyramid $ACDM$. And, via composition, as base $ABCD$ (is) to base ACD, so pyramid $ABCDM$ (is) to pyramid $ACDM$ [Prop. 5.18]. But, as base ACD (is) to base ADE, so pyramid $ACDM$ (is) also to pyramid $ADEM$ [Prop. 12.5]. Thus, via equality, as base $ABCD$ (is) to base ADE, so pyramid $ABCDM$ (is) to pyramid $ADEM$ [Prop. 5.22]. And, again, via composition, as base $ABCDE$ (is) to base ADE, so pyramid $ABCDEM$ (is) to pyramid $ADEM$ [Prop. 5.18]. So, similarly, it can also be shown that as base $FGHKL$ (is) to base FGH, so pyramid $FGHKLN$ (is) also to pyramid $FGHN$. And since $ADEM$ and $FGHN$ are two pyramids having triangular bases and equal height, thus as base ADE (is) to base FGH, so pyramid $ADEM$ (is) to pyramid $FGHN$ [Prop. 12.5]. But, as base ADE (is) to base $ABCDE$, so pyramid $ADEM$ (was) to pyramid $ABCDEM$. Thus, via equality, as base $ABCDE$ (is) to base FGH, so pyramid $ABCDEM$ (is) also to pyramid $FGHN$ [Prop. 5.22]. But, furthermore, as base FGH (is) to base $FGHKL$, so pyramid $FGHN$ was also to pyramid $FGHKLN$. Thus, via equality, as base $ABCDE$ (is) to base $FGHKL$, so pyramid $ABCDEM$ (is) also to pyramid $FGHKLN$ [Prop. 5.22]. (Which is) the very thing it was required to show.

ζ'.

Πᾶν πρίσμα τρίγωνον ἔχον βάσιν διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἐχούσας.

Proposition 7

Any prism having a triangular base is divided into three pyramids having triangular bases (which are) equal to one an-

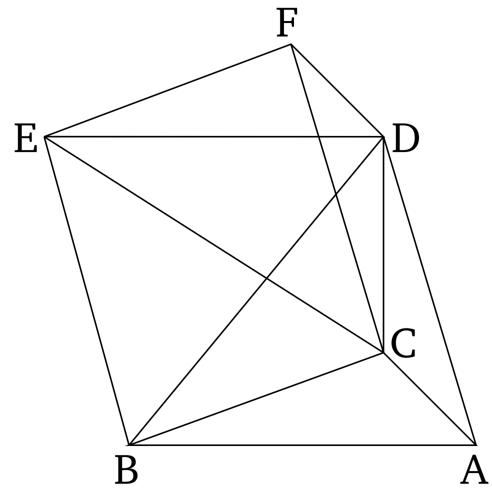


Ἔστω πρίσμα, οὗ βάσις μὲν τὸ ΑΒΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ· λέγω, ὅτι τὸ ΑΒΓΔΕΖ πρίσμα διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους ἐχούσας βάσεις.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΔ, ΕΓ, ΓΔ. ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΑΒΕΔ, διάμετρος δὲ αὐτὸῦ ἐστιν ἡ ΒΔ, ἴσον ἄρα ἐστὶ τὸ ΑΒΔ τρίγωνον τῷ ΕΒΔ τριγώνῳ· καὶ ἡ πυραμὶς ἄρα, ἧς βάσις μὲν τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΔΕΒ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον. ἀλλὰ ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΔΕΒ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἡ αὐτή ἐστι πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΕΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχεται. καὶ πυραμὶς ἄρα, ἧς βάσις μέν ἐστι τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΕΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. πάλιν, ἐπεὶ παραλληλόγραμμόν ἐστι τὸ ΖΓΒΕ, διάμετρος δέ ἐστιν αὐτοῦ ἡ ΓΕ, ἴσον ἐστὶ τὸ ΓΕΖ τρίγωνον τῷ ΓΒΕ τριγώνῳ. καὶ πυραμὶς ἄρα, ἧς βάσις μέν ἐστι τὸ ΒΓΕ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΕΓΖ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. ἡ δὲ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΒΓΕ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐδείχθη πυραμίδι, ἧς βάσις μέν ἐστι τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον· καὶ πυραμὶς ἄρα, ἧς βάσις μέν ἐστι τὸ ΓΕΖ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐστὶ πυραμίδι, ἧς βάσις μὲν [ἐστι] τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον· διῄρηται ἄρα τὸ ΑΒΓΔΕΖ πρίσμα εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους ἐχούσας βάσεις.

Καὶ ἐπεὶ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἡ αὐτή ἐστι πυραμίδι, ἧς βάσις τὸ ΓΑΒ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχονται· ἡ δὲ πυραμίς, ἧς βάσις τὸ ΑΒΔ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, τρίτον ἐδείχθη τοῦ πρίσματος, οὗ βάσις τὸ ΑΒΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ, καὶ ἡ πυραμὶς ἄρα, ἧς βάσις τὸ ΑΒΓ τρίγωνον, κο-

other.



Let there be a prism whose base (is) triangle ABC, and opposite (plane) DEF. I say that prism $ABCDEF$ is divided into three pyramids having triangular bases (which are) equal to one another.

For let BD, EC, and CD be joined. Since $ABED$ is a parallelogram, and BD is its diagonal, triangle ABD is thus equal to triangle EBD [Prop. 1.34]. And, thus, the pyramid whose base (is) triangle ABD, and apex the point C, is equal to the pyramid whose base is triangle DEB, and apex the point C [Prop. 12.5]. But, the pyramid whose base is triangle DEB, and apex the point C, is the same as the pyramid whose base is triangle EBC, and apex the point D. For they are contained by the same planes. And, thus, the pyramid whose base is ABD, and apex the point C, is equal to the pyramid whose base is EBC and apex the point D. Again, since $FCBE$ is a parallelogram, and CE is its diagonal, triangle CEF is equal to triangle CBE [Prop. 1.34]. And, thus, the pyramid whose base is triangle BCE, and apex the point D, is equal to the pyramid whose base is triangle ECF, and apex the point D [Prop. 12.5]. And the pyramid whose base is triangle BCE, and apex the point D, was shown (to be) equal to the pyramid whose base is triangle ABD, and apex the point C. Thus, the pyramid whose base is triangle CEF, and apex the point D, is also equal to the pyramid whose base [is] triangle ABD, and apex the point C. Thus, the prism $ABCDEF$ has been divided into three pyramids having triangular bases (which are) equal to one another.

And since the pyramid whose base is triangle ABD, and apex the point C, is the same as the pyramid whose base is triangle CAB, and apex the point D. For they are contained by the same planes. And the pyramid whose base (is) triangle ABD, and apex the point C, was shown (to be) a third of the prism whose base is triangle ABC, and opposite (plane) DEF, thus the pyramid whose base is triangle ABC, and apex the point D, is also a third of the pyramid having the same base, ρυφὴ δὲ τὸ Δ σημεῖον, τρίτον ἐστὶ τοῦ πρίσματος τοῦ ἔχοντος βάσις τὴν αὐτὴν τὸ ΑΒΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ.

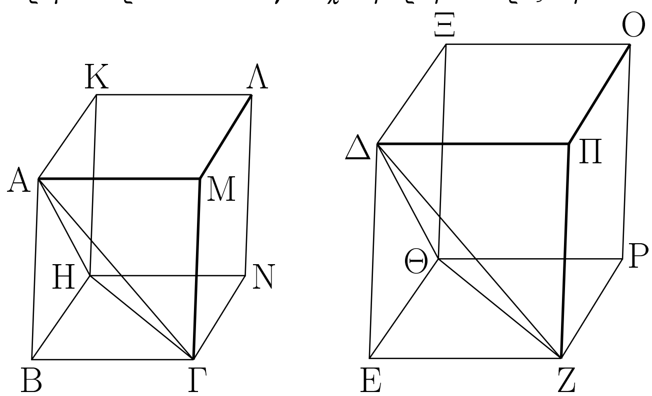
Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι πᾶσα πυραμὶς τρίτον μέρος ἐστὶ τοῦ πρίσματος τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῇ καὶ ὕψος ἴσον· ὅπερ ἔδει δεῖξαι.

η'.

Αἱ ὅμοιαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἔστωσαν ὅμοιαι καὶ ὁμοίως κείμεναι πυραμίδες, ὧν βάσεις μέν εἰσι τὰ ΑΒΓ, ΔΕΖ τρίγωνα, κορυφαὶ δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.



Συμπεπληρώσθω γὰρ τὰ ΒΗΜΛ, ΕΘΠΟ στερεὰ παραλληλεπίπεδα. καὶ ἐπεὶ ὁμοία ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι, ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ γωνίᾳ, ἡ δὲ ὑπὸ ΗΒΓ τῇ ὑπὸ ΘΕΖ, ἡ δὲ ὑπὸ ΑΒΗ τῇ ὑπὸ ΔΕΘ, καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρὸς τὴν ΕΖ, καὶ ἡ ΒΗ πρὸς τὴν ΕΘ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρὸς τὴν ΕΖ, καὶ περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ ΒΜ παραλληλόγραμμον τῷ ΕΠ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΒΝ τῷ ΕΡ ὅμοιόν ἐστι, τὸ δὲ ΒΚ τῷ ΕΞ· τὰ τρία ἄρα τὰ ΜΒ, ΒΚ, ΒΝ τρισὶ τοῖς ΕΠ, ΕΞ, ΕΡ ὅμοιά ἐστιν. ἀλλὰ τὰ μὲν τρία τὰ ΜΒ, ΒΚ, ΒΝ τρισὶ τοῖς ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστιν, τὰ δὲ τρία τὰ ΕΠ, ΕΞ, ΕΡ τρισὶ τοῖς ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστιν. τὰ ΒΗΜΛ, ΕΘΠΟ ἄρα στερεὰ ὑπὸ ὁμοίων ἐπιπέδων ἴσων τὸ πλῆθος περιέχεται. ὅμοιον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν τῷ ΕΘΠΟ στερεῷ. τὰ δὲ ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. τὸ ΒΗΜΛ ἄρα στερεὸν πρὸς τὸ ΕΘΠΟ στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος πλευρὰ ἡ ΒΓ πρὸς τὴν ὁμόλογον πλευρὰν τὴν ΕΖ. ὡς δὲ τὸ ΒΗΜΛ στερεὸν πρὸς τὸ ΕΘΠΟ στερεόν, οὕτως ἡ ΑΒΓΗ πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα, ἐπειδήπερ ἡ πυραμὶς ἕκτον μέρος ἐστὶ τοῦ στερεοῦ διὰ τὸ καὶ τὸ πρίσμα ἥμισυ ὂν τοῦ στερεοῦ παραλληλεπιπέδου

triangle *ABC*, and opposite (plane) *DEF*.

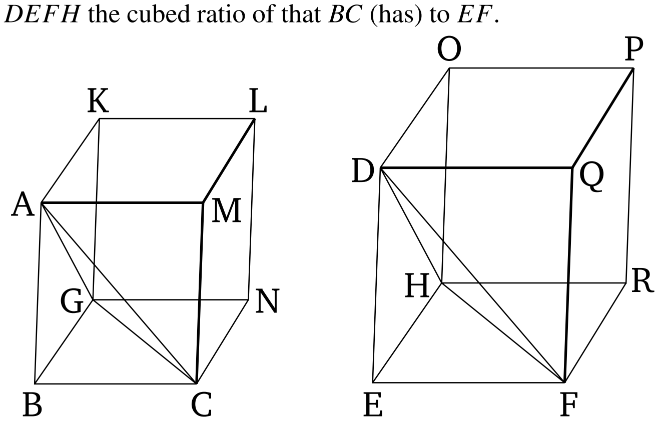
Corollary

And, from this, (it is) clear that any pyramid is the third part of the prism having the same base as it, and an equal height. (Which is) the very thing it was required to show.

Proposition 8

Similar pyramids which also have triangular bases are in the cubed ratio of their corresponding sides.

Let there be similar, and similarly laid out, pyramids whose bases are triangles *ABC* and *DEF*, and apexes the points *G* and *H* (respectively). I say that pyramid *ABCG* has to pyramid *DEFH* the cubed ratio of that *BC* (has) to *EF*.



For let the parallelepiped solids *BGML* and *EHQP* be completed. And since pyramid *ABCG* is similar to pyramid *DEFH*, angle *ABC* is thus equal to angle *DEF*, and *GBC* to *HEF*, and *ABG* to *DEH*. And as *AB* is to *DE*, so *BC* (is) to *EF*, and *BG* to *EH* [Def. 11.9]. And since as *AB* is to *DE*, so *BC* (is) to *EF*, and (so) the sides around equal angles are proportional, parallelogram *BM* is thus similar to paralleleogram *EQ*. So, for the same (reasons), *BN* is also similar to *ER*, and *BK* to *EO*. Thus, the three (parallelograms) *MB*, *BK*, and *BN* are similar to the three (parallelograms) *EQ*, *EO*, *ER* (respectively). But, the three (parallelograms) *MB*, *BK*, and *BN* are (both) equal and similar to the three opposite (parallelograms), and the three (parallelograms) *EQ*, *EO*, and *ER* are (both) equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the solids *BGML* and *EHQP* are contained by equal numbers of similar (and similarly laid out) planes. Thus, solid *BGML* is similar to solid *EHQP* [Def. 11.9]. And similar parallelepiped solids are in the cubed ratio of corresponding sides [Prop. 11.33]. Thus, solid *BGML* has to solid *EHQP* the cubed ratio that the corresponding side *BC* (has) to the corresponding side *EF*. And as solid *BGML* (is) to solid *EHQP*, so pyramid *ABCG* (is) to pyramid *DEFH*, inasmuch as the pyramid is the sixth part *τριπλάσιον εἶναι τῆς πυραμίδος. καὶ ἡ ΑΒΓΗ ἄρα πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.*

of the solid, on account of the prism, being half of the parallelepiped solid [Prop. 11.28], also being three times the pyramid [Prop. 12.7]. Thus, pyramid $ABCG$ also has to pyramid $DEFH$ the cubed ratio that BC (has) to EF. (Which is) the very thing it was required to show.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι καὶ αἱ πολυγώνους ἔχουσαι βάσεις ὅμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. διαιρεθεισῶν γὰρ αὐτῶν εἰς τὰς ἐν αὐταῖς πυραμίδας τριγώνους βάσεις ἐχούσας τῷ καὶ τὰ ὅμοια πολύγωνα τῶν βάσεων εἰς ὅμοια τρίγωνα διαιρεῖσθαι καὶ ἴσα τῷ πλήθει καὶ ὁμόλογα τοῖς ὅλοις ἔσται ὡς [ἡ] ἐν τῇ ἑτέρᾳ μία πυραμὶς τρίγωνον ἔχουσα βάσιν πρὸς τὴν ἐν τῇ ἑτέρᾳ μίαν πυραμίδα τρίγωνον ἔχουσαν βάσιν, οὕτως καὶ ἅπασαι αἱ ἐν τῇ ἑτέρᾳ πυραμίδι πυραμίδες τριγώνους ἔχουσαι βάσεις πρὸς τὰς ἐν τῇ ἑτέρᾳ πυραμίδι πυραμίδας τριγώνους βάσεις ἐχούσας, τουτέστιν αὐτὴ ἡ πολύγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν πολύγωνον βάσιν ἔχουσαν πυραμίδα. ἡ δὲ τρίγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν τρίγωνον βάσιν ἔχουσαν ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγον πλευρῶν· καὶ ἡ πολύγωνον ἄρα βάσιν ἔχουσα πρὸς τὴν ὁμοίαν βάσιν ἔχουσαν τριπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

Corollary

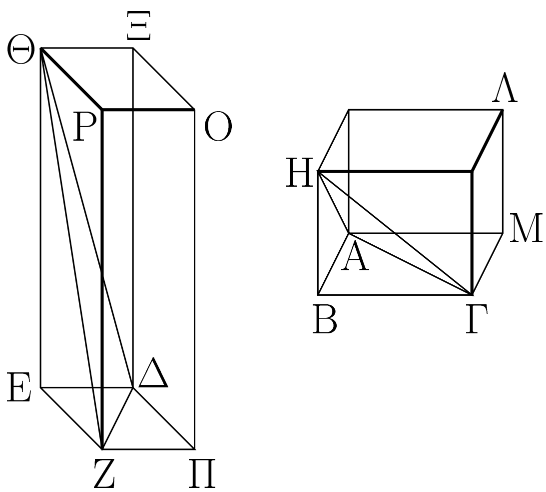
So, from this, (it is) also clear that similar pyramids having polygonal bases (are) to one another as the cubed ratio of their corresponding sides. For, dividing them into the pyramids (contained) within them which have triangular bases, with the similar polygons of the bases also being divided into similar triangles (which are) both equal in number, and corresponding, to the wholes [Prop. 6.20]. As one pyramid having a triangular base in the former (pyramid having a polygonal base is) to one pyramid having a triangular base in the latter (pyramid having a polygonal base), so (the sum of) all the pyramids having triangular bases in the former pyramid will also be to (the sum of) all the pyramids having triangular bases in the latter pyramid [Prop. 5.12]—that is to say, the (former) pyramid itself having a polygonal base to the (latter) pyramid having a polygonal base. And a pyramid having a triangular base is to a (pyramid) having a triangular base in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, a (pyramid) having a polygonal base also has to to a (pyramid) having a similar base the cubed ratio of a (corresponding) side to a (corresponding) side.

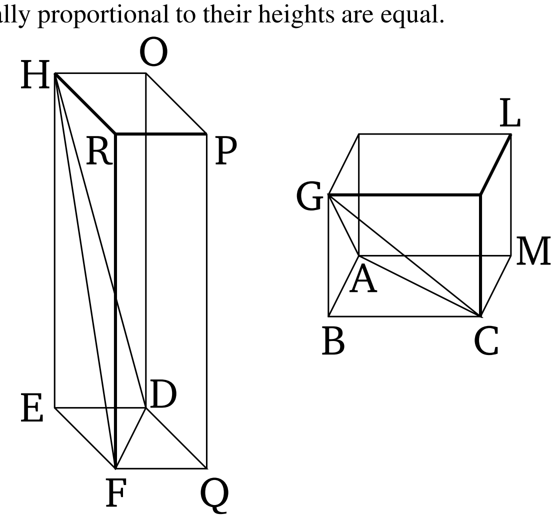
θ'.

Τῶν ἴσων πυραμίδων καὶ τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν πυραμίδων τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσαι εἰσὶν ἐκεῖναι.

Proposition 9

The bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids which have triangular bases whose bases are reciprocally proportional to their heights are equal.





Ἔστωσαν γὰρ ἴσαι πυραμίδες τριγώνους βάσεις ἔχουσαι

For let there be (two) equal pyramids having the triangular *τὰς ΑΒΓ, ΔΕΖ, κορυφὰς δὲ τὰ Η, Θ σημεῖα· λέγω, ὅτι τῶν ΑΒΓΗ, ΔΕΖΘ πυραμίδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος.*

Συμπεπληρώσθω γὰρ τὰ ΒΗΜΛ, ΕΘΠΟ στερεὰ παραλληλεπίπεδα. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι, καί ἐστι τῆς μὲν ΑΒΓΗ πυραμίδος ἑξαπλάσιον τὸ ΒΗΜΛ στερεόν, τῆς δὲ ΔΕΖΘ πυραμίδος ἑξαπλάσιον τὸ ΕΘΠΟ στερεόν, ἴσον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν τῷ ΕΘΠΟ στερεῷ. τῶν δὲ ἴσων στερεῶν παραλληλεπιπώδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἔστιν ἄρα ὡς ἡ ΒΜ βάσις πρὸς τὴν ΕΠ βάσιν, οὕτως τὸ τοῦ ΕΘΠΟ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΗΜΛ στερεοῦ ὕψος. ἀλλ᾽ ὡς ἡ ΒΜ βάσις πρὸς τὴν ΕΠ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον. καὶ ὡς ἄρα τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον, οὕτως τὸ τοῦ ΕΘΠΟ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΗΜΛ στερεοῦ ὕψος. ἀλλὰ τὸ μὲν τοῦ ΕΘΠΟ στερεοῦ ὕψος τὸ αὐτό ἐστι τῷ τῆς ΔΕΖΘ πυραμίδος ὕψει, τὸ δὲ τοῦ ΒΗΜΛ στερεοῦ ὕψος τὸ αὐτό ἐστι τῷ τῆς ΑΒΓΗ πυραμίδος ὕψει· ἔστιν ἄρα ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος. τῶν ΑΒΓΗ, ΔΕΖΘ ἄρα πυραμίδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Ἀλλὰ δὴ τῶν ΑΒΓΗ, ΔΕΖΘ πυραμίδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος· λέγω, ὅτι ἴση ἐστὶν ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος, ἀλλ᾽ ὡς ἡ ΑΒΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ ΒΜ παραλληλόγραμμον πρὸς τὸ ΕΠ παραλληλόγραμμον, καὶ ὡς ἄρα τὸ ΒΜ παραλληλόγραμμον πρὸς τὸ ΕΠ παραλληλόγραμμον, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ΑΒΓΗ πυραμίδος ὕψος. ἀλλὰ τὸ [μὲν] τῆς ΔΕΖΘ πυραμίδος ὕψος τὸ αὐτό ἐστι τῷ τοῦ ΕΘΠΟ παραλληλεπιπέδου ὕψει, τὸ δὲ τῆς ΑΒΓΗ πυραμίδος ὕψος τὸ αὐτό ἐστι τῷ τοῦ ΒΗΜΛ παραλληλεπιπέδου ὕψει· ἔστιν ἄρα ὡς ἡ ΒΜ βάσις πρὸς τὴν ΕΠ βάσιν, οὕτως τὸ τοῦ ΕΘΠΟ παραλληλεπιπέδου ὕψος πρὸς τὸ τοῦ ΒΗΜΛ παραλληλεπιπέδου ὕψος. ὧν δὲ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα· ἴσον ἄρα ἐστὶ τὸ ΒΗΜΛ στερεὸν παραλληλεπίπεδον τῷ ΕΘΠΟ στερεῷ παραλληλεπιπέδῳ. καί ἐστι τοῦ μὲν ΒΗΜΛ ἕκτον μέρος ἡ ΑΒΓΗ πυραμίς, τοῦ δὲ ΕΘΠΟ παραλληλεπιπέδου ἕκτον μέρος ἡ ΔΕΖΘ πυραμίς· ἴση ἄρα ἡ ΑΒΓΗ πυραμὶς τῇ ΔΕΖΘ πυραμίδι.

Τῶν ἄρα ἴσων πυραμίδων καὶ τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν πυραμίδων τριγώνους βάσεις ἐχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψε-

bases ABC and DEF, and apexes the points G and H (respectively). I say that the bases of the pyramids $ABCG$ and $DEFH$ are reciprocally proportional to their heights, and (so) that as base ABC is to base DEF, so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$.

For let the parallelepiped solids $BGML$ and $EHQP$ be completed. And since pyramid $ABCG$ is equal to pyramid $DEFH$, and solid $BGML$ is six times pyramid $ABCG$ (see previous proposition), and solid $EHQP$ (is) six times pyramid $DEFH$, solid $BGML$ is thus equal to solid $EHQP$. And the bases of equal parallelepiped solids are reciprocally proportional to their heights [Prop. 11.34]. Thus, as base BM is to base EQ, so the height of solid $EHQP$ (is) to the height of solid $BGML$. But, as base BM (is) to base EQ, so triangle ABC (is) to triangle DEF [Prop. 1.34]. And, thus, as triangle ABC (is) to triangle DEF, so the height of solid $EHQP$ (is) to the height of solid $BGML$ [Prop. 5.11]. But, the height of solid $EHQP$ is the same as the height of pyramid $DEFH$, and the height of solid $BGML$ is the same as the height of pyramid $ABCG$. Thus, as base ABC is to base DEF, so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$. Thus, the bases of pyramids $ABCG$ and $DEFH$ are reciprocally proportional to their heights.

And so, let the bases of pyramids $ABCG$ and $DEFH$ be reciprocally proportional to their heights, and (thus) let base ABC be to base DEF, as the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$. I say that pyramid $ABCG$ is equal to pyramid $DEFH$.

For, with the same construction, since as base ABC is to base DEF, so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$, but as base ABC (is) to base DEF, so parallelogram BM (is) to parallelogram EQ [Prop. 1.34], thus as parallelogram BM (is) to parallelogram EQ, so the height of pyramid $DEFH$ (is) also to the height of pyramid $ABCG$ [Prop. 5.11]. But, the height of pyramid $DEFH$ is the same as the height of parallelepiped $EHQP$, and the height of pyramid $ABCG$ is the same as the height of parallelepiped $BGML$. Thus, as base BM is to base EQ, so the height of parallelepiped $EHQP$ (is) to the height of parallelepiped $BGML$. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal [Prop. 11.34]. Thus, the parallelepiped solid $BGML$ is equal to the parallelepiped solid $EHQP$. And pyramid $ABCG$ is a sixth part of $BGML$, and pyramid $DEFH$ a sixth part of parallelepiped $EHQP$. Thus, pyramid $ABCG$ is equal to pyramid $DEFH$.

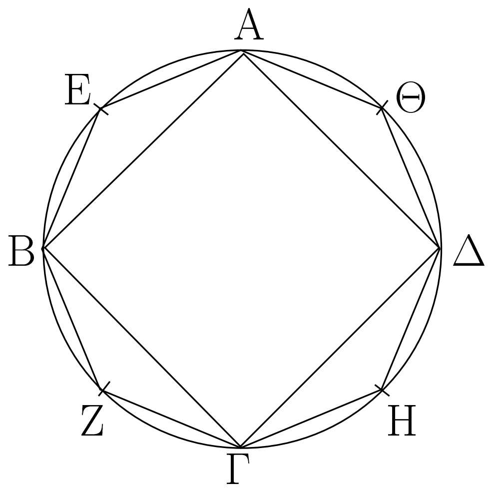
Thus, the bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids having triangular bases whose bases are reciprocally proportional to their heights are equal. (Which is) the very thing it was required to show.

σιν, ἴσαι εἰσὶν ἐκεῖναι· ὅπερ ἔδει δεῖξαι.

ι'.

Πᾶς κῶνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον.

Ἐχέτω γὰρ κῶνος κυλίνδρῳ βάσιν τε τὴν αὐτὴν τὸν ΑΒΓΔ κύκλον καὶ ὕψος ἴσον· λέγω, ὅτι ὁ κῶνος τοῦ κυλίνδρου τρίτον ἐστὶ μέρος, τουτέστιν ὅτι ὁ κύλινδρος τοῦ κώνου τριπλασίων ἐστίν.

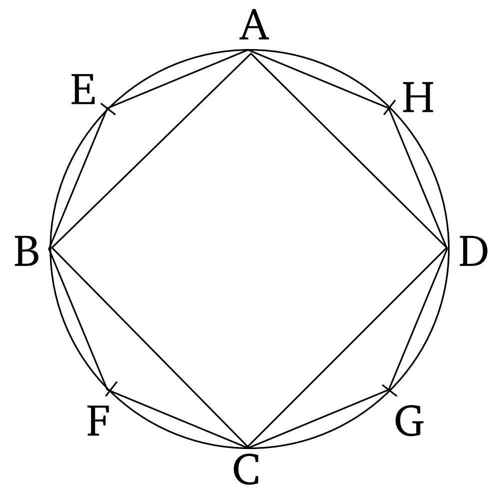


Εἰ γὰρ μή ἐστιν ὁ κύλινδρος τοῦ κώνου τριπλασίων, ἔσται ὁ κύλινδρος τοῦ κώνου ἤτοι μείζων ἢ τριπλασίων ἢ ἐλάσσων ἢ τριπλασίων. ἔστω πρότερον μείζων ἢ τριπλασίων, καὶ ἐγγεγράφθω εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον τὸ ΑΒΓΔ· τὸ δὴ ΑΒΓΔ τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου πρίσμα ἰσοϋψὲς τῷ κυλίνδρῳ. τὸ δὴ ἀνιστάμενον πρίσμα μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ κυλίνδου, ἐπειδήπερ κἂν περὶ τὸν ΑΒΓΔ κύκλον τετράγωνον περιγράψωμεν, τὸ ἐγγεγραμμένον εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον ἥμισύ ἐστι τοῦ περιγεγραμμένου· καί ἐστι τὰ ἀπ᾽ αὐτῶν ἀνιστάμενα στερεὰ παραλληλεπίπεδα πρίσματα ἰσοϋψῆ· τὰ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· καὶ τὸ ἐπὶ τοῦ ΑΒΓΔ ἄρα τετραγώνου ἀνασταθὲν πρίσμα ἥμισύ ἐστι τοῦ ἀνασταθέντος πρίσματος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου· καί ἐστιν ὁ κύλινδρος ἐλάττων τοῦ πρίσματος τοῦ ἀνατραθέντος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου· τὸ ἄρα πρίσμα τὸ ἀνασταθὲν ἀπὸ τοῦ ΑΒΓΔ τετραγώνου ἰσοϋψὲς τῷ κυλίνδρῳ μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ κυλίνδρου. τετμήσθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ περιφέρειαι δίχα κατὰ τὰ Ε, Ζ, Η, Θ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μειζόν ἐστιν ἢ τὸ ἥμισυ τοῦ καθ᾽ ἑαυτὸ τηήματος τοῦ ΑΒΓΔ κύκλου, ὡς ἔμπροσθεν ἐδείκνυμεν. ἀνεστάτω ἐφ᾽ ἑκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πρίσματα ἰσοϋψῆ τῷ κυλίνδρῳ·

Proposition 10

Every cone is the third part of the cylinder which has the same base as it, and an equal height.

For let there be a cone (with) the same base as a cylinder, (namely) the circle *ABCD*, and an equal height. I say that the cone is the third part of the cylinder—that is to say, that the cylinder is three times the cone.



For if the cylinder is not three times the cone then the cylinder will be either more than three times, or less than three times, (the cone). Let it, first of all, be more than three times (the cone). And let the square *ABCD* be inscribed in circle *ABCD* [Prop. 4.6]. So, square *ABCD* is more than half of circle *ABCD* [Prop. 12.2]. And let a prism of equal height to the cylinder be set up on square *ABCD*. So, the prism set up is more than half of the cylinder, inasmuch as if we also circumscribe a square around circle *ABCD* [Prop. 4.7] then the square inscribed in circle *ABCD* is half of the circumscribed (square). And the solids set up on them are parallelepiped prisms of equal height. And parallelepiped solids having the same height are to one another as their bases [Prop. 11.32]. And, thus, the prism set up on square *ABCD* is half of the prism set up on the square circumscribed about circle *ABCD*. And the cylinder is less than the prism set up on the square circumscribed about circle *ABCD*. Thus, the prism set up on square *ABCD* of the same height as the cylinder is more than half of the cylinder. Let the circumferences *AB*, *BC*, *CD*, and *DA* be cut in half at points *E*, *F*, *G*, and *H*. And let *AE*, *EB*, *BF*, *FC*, *CG*, *GD*, *DH*, and *HA* be joined. And thus each of the triangles *AEB*, *BFC*, *CGD*, and *DHA* is more than half of the segment of circle *ABCD* about it, as was shown previously [Prop. 12.2]. Let prisms of equal height to the cylinder be set up on each of the triangles *AEB*, *BFC*, *CGD*, and *DHA*. And each of the prisms set up is greater than the half *καὶ ἕκαστον ἄρα τῶν ἀνασταθέντων πρισμάτων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος τοῦ καθ᾽ ἑαυτὸ τμήματος τοῦ κυλίνδρου, ἐπειδήπερ ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων παραλλήλους ταῖς ΑΒ, ΒΓ, ΓΔ, ΔΑ ἀγάγωμεν, καὶ συμπληρώσωμεν τὰ ἐπὶ τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ παραλληλόγραμμα, καὶ ἀπ᾽ αὐτῶν ἀναστήσωμεν στερεὰ παραλληλεπίπεδα ἰσοϋψῆ τῷ κυλίνδρῳ, ἑκάσου τῶν ἀνασταθέντων ἡμίση ἐστὶ τὰ πρίσματα τὰ ἐπὶ τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων· καί ἐστι τὰ τοῦ κυλίνδρου τμήματα ἐλάττονα τῶν ἀνασταθέντων στερεῶν παραλληλεπιπέδων· ὥστε καὶ τὰ ἐπὶ τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῶν καθ᾽ ἑαυτὰ τοῦ κυλίνδρου τμημάτων. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ᾽ ἑκάσου τῶν τριγώνων πρίσματα ἰσοϋψῆ τῷ κυλίνδρῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κυλίνδρου, ἃ ἔσται ἐλάττονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ κύλινδρος τοῦ τριπλασίου τοῦ κώνου. λελείφθω, καὶ ἔστω τὰ ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· λοιπὸν ἄρα τὸ πρίσμα, οὗ βάσις μὲν τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, μεῖζόν ἐστιν ἢ τριπλάσιον τοῦ κώνου. ἀλλὰ τὸ πρίσμα, οὗ βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, τριπλάσιόν ἐστι τῆς πυραμίδος, ἧς βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ· καὶ ἡ πυραμὶς ἄρα, ἧς βάσις μέν [ἐστι] τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μείζων ἐστὶ τοῦ κώνου τοῦ βάσιν ἔχοντες τὸν ΑΒΓΔ κύκλον. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ᾽ αὐτοῦ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὁ κύλινδρος τοῦ κώνου μείζων ἢ τριπλάσιος.*

Λέγω δή, ὅτι οὐδὲ ἐλάττων ἐστὶν ἢ τριπλάσιος ὁ κύλινδρος τοῦ κώνου.

Εἰ γὰρ δυνατόν, ἔστω ἐλάττων ἢ τριπλάσιος ὁ κύλινδρος τοῦ κώνου· ἀνάπαλιν ἄρα ὁ κῶνος τοῦ κυλίνδρου μεῖζων ἐστὶν ἢ τρίτον μέρος. ἐγγεγράφθω δὴ εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον τὸ ΑΒΓΔ· τὸ ΑΒΓΔ ἄρα τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ κώνου, ἐπειδήπερ, ὡς ἔμπροσθεν ἐδείκνυμεν, ὅτι ἐὰν περὶ τὸν κύκλον τετράγωνον περιγράψωμεν, ἔσται τὸ ΑΒΓΔ τετράγωνον ἥμισυ τοῦ περὶ τὸν κύκλον περιγεγραμμένου τετραγώνου· καὶ ἐὰν ἀπὸ τῶν τετραγώνων στερεὰ παραλληλεπίπεδα ἀναστήσωμεν ἰσοϋψῆ τῷ κώνῳ, ἃ καὶ καλεῖται πρίσματα, ἔσται τὸ ἀνασταθὲν ἀπὸ τοῦ ΑΒΓΔ τετραγώνου ἥμισυ τοῦ ἀνασταθέντος ἀπὸ τοῦ περὶ τὸν κύκλον περιγραφέντος τετραγώνου· πρὸς ἄλληλα γάρ εἰσιν ὡς αἱ βάσεις. ὥστε καὶ τὰ τρίτα· καὶ πυραμὶς ἄρα, ἧς βάσις τὸ ΑΒΓΔ τετράγωνον, ἥμισύ ἐστι τῆς πυραμίδος τῆς ἀνασταθείσης ἀπὸ τοῦ περὶ τὸν κύκλον περιγραφέντος τετραγώνου. καί ἐστι μείζων ἡ πυραμὶς ἡ ἀνασταθεῖσα ἀπὸ τοῦ περὶ τὸν κύκλον τετραγώνου τοῦ κώνου· ἐμπεριέχει γὰρ αὐτόν. ἡ ἄρα πυραμίς, ἧς βάσις τὸ ΑΒΓΔ τετράγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ,

part of the segment of the cylinder about it—inasmuch as if we draw (straight-lines) parallel to AB, BC, CD, and DA through points E, F, G, and H (respectively), and complete the parallelograms on AB, BC, CD, and DA, and set up parallelepiped solids of equal height to the cylinder on them, then the prisms on triangles AEB, BFC, CGD, and DHA are each half of the set up (parallelepipeds). And the segments of the cylinder are less than the set up parallelepiped solids. Hence, the prisms on triangles AEB, BFC, CGD, and DHA are also greater than half of the segments of the cylinder about them. So (if) the remaining circumferences are cut in half, and straight-lines are joined, and prisms of equal height to the cylinder are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cylinder whose (sum) is less than the excess by which the cylinder exceeds three times the cone [Prop. 10.1]. Let them be left, and let them be AE, EB, BF, FC, CG, GD, DH, and HA. Thus, the remaining prism whose base (is) polygon $AEBFCGDH$, and height the same as the cylinder, is greater than three times the cone. But, the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder, is three times the pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone [Prop. 12.7 corr.]. And thus the pyramid whose base [is] polygon $AEBFCGDH$, and apex the same as the cone, is greater than the cone having (as) base circle $ABCD$. But (it is) also less. For it is encompassed by it. The very thing (is) impossible. Thus, the cylinder is not more than three times the cone.

So, I say that neither (is) the cylinder less than three times the cone.

For, if possible, let the cylinder be less than three times the cone. Thus, inversely, the cone is greater than the third part of the cylinder. So, let the square $ABCD$ be inscribed in circle $ABCD$ [Prop. 4.6]. Thus, square $ABCD$ is greater than half of circle $ABCD$. And let a pyramid having the same apex as the cone be set up on square $ABCD$. Thus, the pyramid set up is greater than the half part of the cone, inasmuch as we showed previously that if we circumscribe a square about the circle [Prop. 4.7] then the square $ABCD$ will be half of the square circumscribed about the circle [Prop. 12.2]. And if we set up on the squares parallelepiped solids—which are also called prisms—of the same height as the cone, then the (prism) set up on square $ABCD$ will be half of the (prism) set up on the square circumscribed about the circle. For they are to one another as their bases [Prop. 11.32]. Hence, (the same) also (goes for) the thirds. Thus, the pyramid whose base is square $ABCD$ is half of the pyramid set up on the square circumscribed about the circle [Prop. 12.7 corr.]. And the pyramid set up on the square circumscribed about the circle is greater than the cone. For it encompasses it. Thus, the pyramid whose base is square $ABCD$, and apex the same as the cone, is greater than half of *μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ κώνου. τετμήσθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ περιφέρειαι δίχα κατὰ τὰ Ε, Ζ, Η, Θ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος του καθ᾽ ἑαυτὸ τμήματος τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτωσαν ἐφ᾽ ἑκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πυραμίδες τὴν αὐτὴν κορυφὴν ἔχουσαι τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων κατὰ τὸν αὐτὸν τρόπον μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ καθ᾽ ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ᾽ ἑκάστου τῶν τριγώνων πυραμίδα τὴν αὐτὴν κορυφὴν ἔχουσαν τῷ κώνῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάττονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ κῶνος τοῦ τρίτου μέρους τοῦ κυλίνδρου. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· λοιπὴ ἄρα ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μείζων ἐστὶν ἢ τρίτον μέρος τοῦ κυλίνδρου. ἀλλ᾽ ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αυτὴ τῷ κώνῳ, τρίτον ἐστὶ μέρος τοῦ πρίσματος, οὗ βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ· τὸ ἄρα πρίσμα, οὗ βάσις μέν ἐστι τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, μεῖζόν ἐστι τοῦ κυλίνδρου, οὗ βάσις ἐστὶν ὁ ΑΒΓΔ κύκλος. ἀλλὰ καὶ ἔλαττον· ἐμπεριέχεται γὰρ ὑπ᾽ αὐτοῦ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κύλινδρος τοῦ κώνου ἐλάττων ἐστὶν ἢ τριπλάσιος. ἐδείχθη δέ, ὅτι οὐδὲ μείζων ἢ τριπλάσιος· τριπλάσιος ἄρα ὁ κύλινδρος τοῦ κώνου· ὥστε ὁ κῶνος τρίτον ἐστὶ μέρος τοῦ κυλίνδρου.*

Πᾶς ἄρα κῶνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον· ὅπερ ἔδει δεῖξαι.

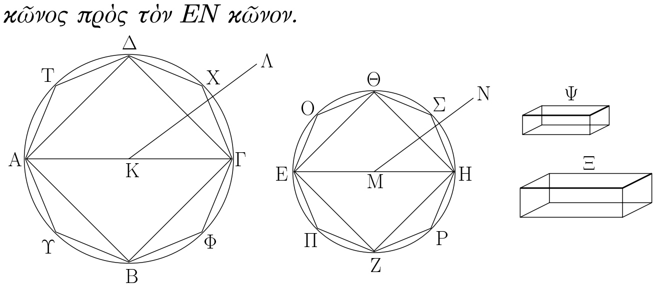
the cone. Let the circumferences AB, BC, CD, and DA be cut in half at points E, F, G, and H (respectively). And let AE, EB, BF, FC, CG, GD, DH, and HA be joined. And, thus, each of the triangles AEB, BFC, CGD, and DHA is greater than the half part of the segment of circle $ABCD$ about it [Prop. 12.2]. And let pyramids having the same apex as the cone be set up on each of the triangles AEB, BFC, CGD, and DHA. And, thus, in the same way, each of the pyramids set up is more than the half part of the segment of the cone about it. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which the cone exceeds the third part of the cylinder [Prop. 10.1]. Let them be left, and let them be the (segments) on AE, EB, BF, FC, CG, GD, DH, and HA. Thus, the remaining pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone, is greater than the third part of the cylinder. But, the pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone, is the third part of the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder [Prop. 12.7 corr.]. Thus, the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder, is greater than the cylinder whose base is circle $ABCD$. But, (it is) also less. For it is encompassed by it. The very thing is impossible. Thus, the cylinder is not less than three times the cone. And it was shown that neither (is it) greater than three times (the cone). Thus, the cylinder (is) three times the cone. Hence, the cone is the third part of the cylinder.

Thus, every cone is the third part of the cylinder which has the same base as it, and an equal height. (Which is) the very thing it was required to show.

ια΄.

Οἱ ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις.

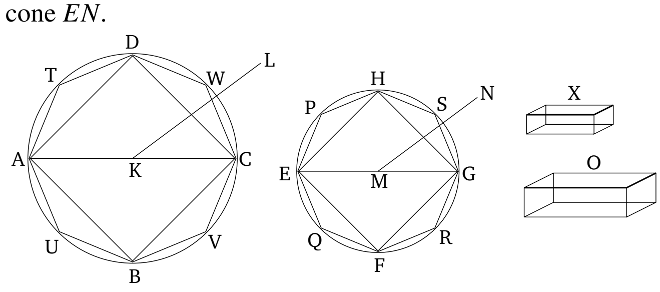
Ἔστωσαν ὑπὸ τὸ αὐτὸ ὕψος κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν [εἰσιν] οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, ἄξονες δὲ οἱ ΚΛ, ΜΝ, διάμετροι δὲ τῶν βάσεων αἱ ΑΓ, ΕΗ· λέγω, ὅτι ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.



Proposition 11

Cones and cylinders having the same height are to one another as their bases.

Let there be cones and cylinders of the same height whose bases [are] the circles $ABCD$ and $EFGH$, axes KL and MN, and diameters of the bases AC and EG (respectively). I say that as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN.



Εἰ γὰρ μή, ἔσται ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος ἤτοι πρὸς ἔλασσόν τι τοῦ ΕΝ κώνου στερεὸν ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Ξ, καὶ ᾧ ἔλασσόν ἐστι τὸ Ξ στερεὸν τοῦ ΕΝ κώνου, ἐκείνῳ ἴσον ἔστω τὸ Ψ στερεόν· ὁ ΕΝ κῶνος ἄρα ἴσος ἐστὶ τοῖς Ξ, Ψ στερεοῖς. ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ ἄρα τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ κύκλου. ἀνεστάτω ἀπὸ τοῦ ΕΖΗΘ τετραγώνου πυραμὶς ἰσοϋψὴς τῷ κώνῳ· ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ κώνου, ἐπειδήπερ ἐὰν περιγράψωμεν περὶ τὸν κύκλον τετράγωνον, καὶ ἀπ᾽ αὐτοῦ ἀναστήσωμεν πυραμίδα ἰσοϋψῆ τῷ κώνῳ, ἡ ἐγγραφεῖσα πυραμὶς ἥμισύ ἐστι τῆς περιγραφείσης· πρὸς ἀλλήλας γάρ εἰσιν ὡς αἱ βάσεις· ἐλάττων δὲ ὁ κῶνος τῆς περιγραφείσης πυραμίδος. τετμήσθωσαν αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι δίχα κατὰ τὰ Ο, Π, Ρ, Σ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΘΟ, ΟΕ, ΕΠ, ΠΖ, ΖΡ, ΡΗ, ΗΣ, ΣΘ. ἕκαστον ἄρα τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΖΘ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ καθ᾽ ἑαυτὸ τμήματος τοῦ κύκλου. ἀνεστάτω ἐφ᾽ ἑκάστου τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ τριγώνων πυραμὶς ἰσοϋψὴς τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ καθ᾽ ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐπὶ ἑκάστου τῶν τριγώνων πυραμίδας ἰσοϋψεῖς τῷ κώνῳ καὶ ἀεὶ τοῦτο ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάσσονα τοῦ Ψ στερεοῦ. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΘΟΕ, ΕΠΖ, ΖΡΗ, ΗΣΘ· λοιπὴ ἄρα ἡ πυραμίς, ἧς βάσις τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κώνῳ, μείζων ἐστὶ τοῦ Ξ στερεοῦ. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΘΟΕΠΖΡΗΣ πολυγώνῳ ὅμοιόν τε καὶ ὁμοίως κείμενον πολύγωνον τὸ ΔΤΑΥΒΦΓΧ, καὶ ἀνεστάτω ἐπ᾽ αὐτοῦ πυραμὶς ἰσοϋψὴς τῷ ΑΛ κώνῳ. ἐπεὶ οὖν ἐστιν ὡς τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρὸς τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, ὡς δὲ τὸ ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρὸς τὸ ΘΟΕΠΖΡΗΣ πολύγωνον. ὡς δὲ ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸ Ξ στερεόν, ὡς δὲ τὸ ΔΤΑΥΒΦΓΧ πολύγωνον πρὸς τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, οὕτως ἡ πυραμίς, ἧς βάσις μὲν τὸ ΔΤΑΥΒΦΓΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν πυραμίδα, ἧς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον. καὶ ὡς ἄρα ὁ ΑΛ κῶνος πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμίς, ἧς βάσις μὲν τὸ ΔΤΑΥΒΦΓΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν πυραμίδα, ἧς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ἐναλλὰξ ἄρα ἐστὶν ὡς ὁ ΑΛ κῶνος πρὸς τὴν ἐν αὐτῷ πυραμίδα, οὕτως τὸ Ξ στερεὸν πρὸς τὴν ἐν τῷ ΕΝ κώνῳ πυραμίδα. μείζων δὲ ὁ ΑΛ κῶνος τῆς ἐν αὐτῷ πυραμίδος· μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς ἐν τῷ ΕΝ κώνῳ πυραμίδος. ἀλλὰ καὶ ἔλασσον· ὅπερ ἄτοπον.

For if not, then as circle $ABCD$ (is) to circle $EFGH$, so cone AL will be to some solid either less than, or greater than, cone EN. Let it, first of all, be (in this ratio) to (some) lesser (solid), O. And let solid X be equal to that (magnitude) by which solid O is less than cone EN. Thus, cone EN is equal to (the sum of) solids O and X. Let the square $EFGH$ be inscribed in circle $EFGH$ [Prop. 4.6]. Thus, the square is greater than half of the circle [Prop. 12.2]. Let a pyramid of the same height as the cone be set up on square $EFGH$. Thus, the pyramid set up is greater than half of the cone, inasmuch as, if we circumscribe a square about the circle [Prop. 4.7], and set up on it a pyramid of the same height as the cone, then the inscribed pyramid is half of the circumscribed pyramid. For they are to one another as their bases [Prop. 12.6]. And the cone (is) less than the circumscribed pyramid. Let the circumferences EF, FG, GH, and HE be cut in half at points P, Q, R, and S. And let HP, PE, EQ, QF, FR, RG, GS, and SH be joined. Thus, each of the triangles HPE, EQF, FRG, and GSH is greater than half of the segment of the circle about it [Prop. 12.2]. Let pyramids of the same height as the cone be set up on each of the triangles HPE, EQF, FRG, and GSH. And, thus, each of the pyramids set up is greater than half of the segment of the cone about it [Prop. 12.10]. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids of equal height to the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone (the sum of) which is less than solid X [Prop. 10.1]. Let them be left, and let them be the (segments) on HPE, EQF, FRG, and GSH. Thus, the remaining pyramid whose base is polygon $HPEQFRGS$, and height the same as the cone, is greater than solid O [Prop. 6.18]. And let the polygon $DTAUBVCW$, similar, and similarly laid out, to polygon $HPEQFRGS$, be inscribed in circle $ABCD$. And on it let a pyramid of the same height as cone AL be set up. Therefore, since as the (square) on AC is to the (square) on EG, so polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$ [Prop. 12.1], and as the (square) on AC (is) to the (square) on EG, so circle $ABCD$ (is) to circle $EFGH$ [Prop. 12.2], thus as circle $ABCD$ (is) to circle $EFGH$, so polygon $DTAUBVCW$ also (is) to polygon $HPEQFRGS$. And as circle $ABCD$ (is) to circle $EFGH$, so cone AL (is) to solid O. And as polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$, so the pyramid whose base is polygon $DTAUBVCW$, and apex the point L, (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 12.6]. And, thus, as cone AL (is) to solid O, so the pyramid whose base is $DTAUBVCW$, and apex the point L, (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 5.11]. Thus, alternately, as cone AL is to the pyramid within it, so solid O (is) to the pyramid within cone EN [Prop. 5.16]. But, cone AL (is) greater than the pyra-

οὐκ ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς ἔλασσόν τι τοῦ ΕΝ κώνου στερεόν. ὁμοίως δὲ δείξομεν, ὅτι οὐδέ ἐστιν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν.

Λέγω δή, ὅτι οὐδέ ἐστιν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΝ κώνου στερεόν.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Ξ· ἀνάπαλιν ἄρα ἐστὶν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως τὸ Ξ στερεὸν πρὸς τὸν ΑΛ κῶνον. ἀλλʼ ὡς τὸ Ξ στερεὸν πρὸς τὸν ΑΛ κῶνον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν· καὶ ὡς ἄρα ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΝ κώνου στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.

Ἀλλʼ ὡς ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλασίων γὰρ ἑκάτερος ἑκατέρου. καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως οἱ ἐπʼ αὐτῶν ἰσοϋψεῖς.

Οἱ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

mid within it. Thus, solid O (is) also greater than the pyramid within cone EN [Prop. 5.14]. But, (it is) also less. The very thing (is) absurd. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid less than cone EN. So, similarly, we can show that neither is circle $EFGH$ to circle $ABCD$, as cone EN (is) to some solid less than cone AL.

So, I say that neither is circle $ABCD$ to circle $EFGH$, as cone AL (is) to some solid greater than cone EN.

For, if possible, let it be (in this ratio) to (some) greater (solid), O. Thus, inversely, as circle $EFGH$ is to circle $ABCD$, so solid O (is) to cone AL [Prop. 5.7 corr.]. But, as solid O (is) to cone AL, so cone EN (is) to some solid less than cone AL [Prop. 12.2 lem.]. And, thus, as circle $EFGH$ (is) to circle $ABCD$, so cone EN (is) to some solid less than cone AL. The very thing was shown (to be) impossible. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid greater than cone EN. And, it was shown that neither (is it in this ratio) to (some) lesser (solid). Thus, as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN.

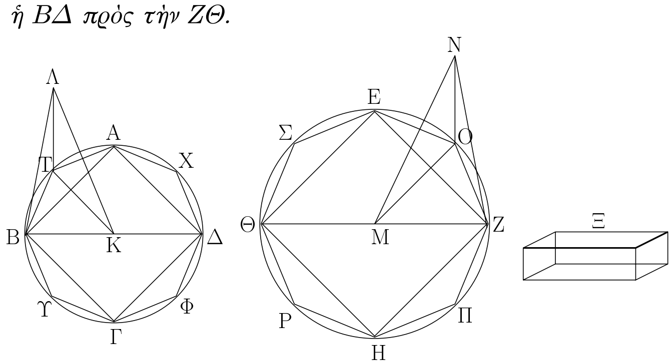
But, as the cone (is) to the cone, (so) the cylinder (is) to the cylinder. For each (is) three times each [Prop. 12.10]. Thus, circle $ABCD$ (is) also to circle $EFGH$, as (the ratio of the cylinders) on them (having) the same height.

Thus, cones and cylinders having the same height are to one another as their bases. (Which is) the very thing it was required to show.

ιβʹ.

Οἱ ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων.

Ἔστωσαν ὅμοιοι κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ τῶν βάσεων αἱ ΒΔ, ΖΘ, ἄξονες δὲ τῶν κώνων καὶ κυλίνδρων οἱ ΚΛ, ΜΝ· λέγω, ὅτι ὁ κῶνος, οὗ βάσις μέν [ἐστιν] ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸν κῶνον, οὗ βάσις μέν [ἐστιν] ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

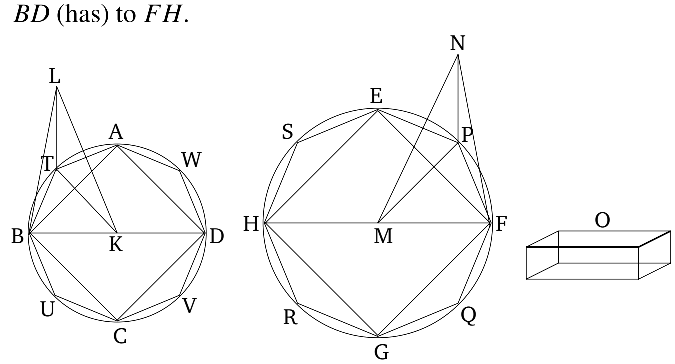


Εἰ γὰρ μὴ ἔχει ὁ ΑΒΓΔΛ κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον τριπλασίονα λόγον ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ, ἕξει ὁ ΑΒΓΔΛ κῶνος ἢ πρὸς ἔλασσόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τρι-

Proposition 12

Similar cones and cylinders are to one another in the cubed ratio of the diameters of their bases.

Let there be similar cones and cylinders of which the bases (are) the circles $ABCD$ and $EFGH$, the diameters of the bases (are) BD and FH, and the axes of the cones and cylinders (are) KL and MN (respectively). I say that the cone whose base [is] circle $ABCD$, and apex the point L, has to the cone whose base [is] circle $EFGH$, and apex the point N, the cubed ratio that BD (has) to FH.



For if cone $ABCDL$ does not have to cone $EFGHN$ the cubed ratio that BD (has) to FH then cone $ABCDL$ will have the cubed ratio to some solid either less than, or greater than, *πλασίονα λόγον ἢ πρὸς μεῖζον. ἐχέτω πρότερον πρὸς ἔλασσον τὸ Ξ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ ἄρα ΕΖΗΘ τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐπὶ τοῦ ΕΖΗΘ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ κώνου. τετμήσθωσαν δὴ αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι δίχα κατὰ τὰ Ο, Π, Ρ, Σ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ. καὶ ἕκαστον ἄρα τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος τοῦ καθ᾽ ἑαυτὸ τμήματος τοῦ ΕΖΗΘ κύκλου. καὶ ἀνεστάτω ἐφ᾽ ἑκάστου τῶν ΕΟΖ, ΖΠΗ, ΗΡΘ, ΘΣΕ τριγώνων πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ καθ᾽ ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ᾽ ἑκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἐχούσας τῷ κώνῳ καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘΝ κῶνος τοῦ Ξ στερεοῦ. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΕΟ, ΟΖ, ΖΠ, ΠΗ, ΗΡ, ΡΘ, ΘΣ, ΣΕ· λοιπὴ ἄρα ἡ πυραμίς, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, μείζων ἐστὶ τοῦ Ξ στερεοῦ. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΟΖΠΗΡΘΣ πολυγώνῳ ὅμοιόν τε καὶ ὁμοίως κείμενον πολύγωνον τὸ ΑΤΒΥΓΦΔΧ, καὶ ἀνεστάτω ἐπὶ τοῦ ΑΤΒΥΓΦΔΧ πολυγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κώνῳ, καὶ τῶν μὲν περιεχόντων τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ἓν τρίγωνον ἔστω τὸ ΛΒΤ, τῶν δὲ περειχόντων τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, ἓν τρίγωνον ἔστω τὸ ΝΖΟ, καὶ ἐπεζεύχθωσαν αἱ ΚΤ, ΜΟ. καὶ ἐπεὶ ὅμοιός ἐστιν ὁ ΑΒΓΔΛ κῶνος τῷ ΕΖΗΘΝ κώνῳ, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ὁ ΚΛ ἄξων πρὸς τὸν ΜΝ ἄξονα. ὡς δὲ ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ἡ ΒΚ πρὸς τὴν ΖΜ· καὶ ὡς ἄρα ἡ ΒΚ πρὸς τὴν ΖΜ, οὕτως ἡ ΚΛ πρὸς τὴν ΜΝ. καὶ ἐναλλὰξ ὡς ἡ ΒΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΛ, ΖΜΝ αἱ πλευραὶ ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ ΒΚΛ τρίγωνον τῷ ΖΜΝ τριγώνῳ. πάλιν, ἐπεί ἐστιν ὡς ἡ ΒΚ πρὸς τὴν ΚΤ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΟ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΤ, ΖΜΟ, ἐπειδήπερ, ὃ μέρος ἐστὶν ἡ ὑπὸ ΒΚΤ γωνία τῶν πρὸς τῷ Κ κέντρῳ τεσσάρων ὀρθῶν, τὸ αὐτὸ μέρος ἐστὶ καὶ ἡ ὑπὸ ΖΜΟ γωνία τῶν πρὸς τῷ Μ κέντρῳ τεσσάρων ὀρθῶν· ἐπεὶ οὖν περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστι τὸ ΒΚΤ τρίγωνον τῷ ΖΜΟ τριγώνῳ. πάλιν, ἐπεὶ ἐδείχθη ὡς ἡ ΒΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ, ἴση δὲ ἡ μὲν ΒΚ τῇ ΚΤ, ἡ δὲ ΖΜ τῇ ΟΜ, ἔστιν ἄρα ὡς ἡ ΤΚ πρὸς τὴν ΚΛ, οὕτως ἡ ΟΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΤΚΛ, ΟΜΝ· ὀρθαὶ γάρ· αἱ πλευραὶ ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ ΛΚΤ τρίγωνον τῷ ΝΜΟ τριγώνῳ. καὶ*

cone $EFGHN$. Let it, first of all, have (such a ratio) to (some) lesser (solid), O. And let the square $EFGH$ be inscribed in circle $EFGH$ [Prop. 4.6]. Thus, square $EFGH$ is greater than half of circle $EFGH$ [Prop. 12.2]. And let a pyramid having the same apex as the cone be set up on square $EFGH$. Thus, the pyramid set up is greater than the half part of the cone [Prop. 12.10]. So, let the circumferences EF, FG, GH, and HE be cut in half at points P, Q, R, and S (respectively). And let EP, PF, FQ, QG, GR, RH, HS, and SE be joined. And, thus, each of the triangles EPF, FQG, GRH, and HSE is greater than the half part of the segment of circle $EFGH$ about it [Prop. 12.2]. And let a pyramid having the same apex as the cone be set up on each of the triangles EPF, FQG, GRH, and HSE. And thus each of the pyramids set up is greater than the half part of the segment of the cone about it [Prop. 12.10]. So, (if) the the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which cone $EFGHN$ exceeds solid O [Prop. 10.1]. Let them be left, and let them be the (segments) on EP, PF, FQ, QG, GR, RH, HS, and SE. Thus, the remaining pyramid whose base is polygon $EPFQGRHS$, and apex the point N, is greater than solid O. And let the polygon $ATBUCVDW$, similar, and similarly laid out, to polygon $EPFQGRHS$, be inscribed in circle $ABCD$ [Prop. 6.18]. And let a pyramid having the same apex as the cone be set up on polygon $ATBUCVDW$. And let LBT be one of the triangles containing the pyramid whose base is polygon $ATBUCVDW$, and apex the point L. And let NFP be one of the triangles containing the pyramid whose base is triangle $EPFQGRHS$, and apex the point N. And let KT and MP be joined. And since cone $ABCDL$ is similar to cone $EFGHN$, thus as BD is to FH, so axis KL (is) to axis MN [Def. 11.24]. And as BD (is) to FH, so BK (is) to FM. And, thus, as BK (is) to FM, so KL (is) to MN. And, alternately, as BK (is) to KL, so FM (is) to MN [Prop. 5.16]. And the sides around the equal angles BKL and FMN are proportional. Thus, triangle BKL is similar to triangle FMN [Prop. 6.6]. Again, since as BK (is) to KT, so FM (is) to MP, and (they are) about the equal angles BKT and FMP, inasmuch as whatever part angle BKT is of the four right-angles at the center K, angle FMP is also the same part of the four right-angles at the center M. Therefore, since the sides about equal angles are proportional, triangle BKT is thus similar to traingle FMP [Prop. 6.6]. Again, since it was shown that as BK (is) to KL, so FM (is) to MN, and BK (is) equal to KT, and FM to PM, thus as TK (is) to KL, so PM (is) to MN. And the sides about the equal angles TKL and PMN—for (they are both) right-angles—are proportional. Thus, triangle LKT (is) similar to triangle NMP [Prop. 6.6]. And since, on account of the similarity of triangles LKB and *ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΛΚΒ, ΝΜΖ τριγώνων ἐστὶν ὡς ἡ ΛΒ πρὸς τὴν ΒΚ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΒΚΤ, ΖΜΟ τριγώνων ἐστὶν ὡς ἡ ΚΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΜΖ πρὸς τὴν ΖΟ, δι᾿ ἴσου ἄρα ὡς ἡ ΛΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΟ. πάλιν, ἐπεὶ διὰ τὴν ομοιότητα τῶν ΛΤΚ, ΝΟΜ τριγώνων ἐστὶν ὡς ἡ ΛΤ πρὸς τὴν ΤΚ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΤΚΒ, ΟΜΖ τριγώνων ἐστὶν ὡς ἡ ΚΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΜΟ πρὸς τὴν ΟΖ, δι᾿ ἴσου ἄρα ὡς ἡ ΛΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΖ. ἐδείχθη δὲ καὶ ὡς ἡ ΤΒ πρὸς τὴν ΒΛ, οὕτως ἡ ΟΖ πρὸς τὴν ΖΝ. δι᾿ ἴσου ἄρα ὡς ἡ ΤΛ πρὸς τὴν ΛΒ, οὕτως ἡ ΟΝ πρὸς τὴν ΝΖ. τῶν ΛΤΒ, ΝΟΖ ἄρα τριγώνων ἀνάλογόν εἰσιν αἱ πλευραί· ἰσογώνια ἄρα ἐστὶ τὰ ΛΤΒ, ΝΟΖ τρίγωνα· ὥστε καὶ ὅμοια. καὶ πυραμὶς ἄρα, ἧς βάσις μὲν τὸ ΒΚΤ τρίγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ὁμοία ἐστὶ πυραμίδι, ἧς βάσις μὲν τὸ ΖΜΟ τρίγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ὑπὸ γὰρ ὁμοίων ἐπιπέδων περιέχονται ἴσων τὸ πλῆθος. αἱ δὲ ὅμοιαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ἡ ἄρα ΒΚΤΛ πυραμὶς πρὸς τὴν ΖΜΟΝ πυραμίδα τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΚ πρὸς τὴν ΖΜ. ὁμοίως δὴ ἐπιζευγνύντες ἀπὸ τῶν Α, Χ, Δ, Φ, Γ, Υ ἐπὶ τὸ Κ εὐθείας καὶ ἀπὸ τῶν Ε, Σ, Θ, Ρ, Η, Π ἐπὶ τὸ Μ καὶ ἀνιστάντες ἐφ᾿ ἑκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἐχούσας τοῖς κώνοις δείξομεν, ὅτι καὶ ἑκάστη τῶν ὁμοταγῶν πυραμίδων πρὸς ἑκάστην ὁμοταγῆ πυραμίδα τριπλασίονα λόγον ἕξει ἤπερ ἡ ΒΚ ὁμόλογος πλευρὰ πρὸς τὴν ΖΜ ὁμόλογον πλευράν, τουτέστιν ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ἔστιν ἄρα καὶ ὡς ἡ ΒΚΤΛ πυραμὶς πρὸς τὴν ΖΜΟΝ πυραμίδα, οὕτως ἡ ὅλη πυραμίς, ἧς βάσις τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν ὅλην πυραμίδα, ἧς βάσις μὲν τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ὥστε καὶ πυραμίς, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ, κορυφὴ δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἧς βάσις [μὲν] τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ὑπόκειται δὲ καὶ ὁ κῶνος, οὗ βάσις [μὲν] ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸ Ξ στερεὸν τριπλασίονα λόγον ἔχων ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ· ἔστιν ἄρα ὡς ὁ κῶνος, οὗ βάσις μέν ἐστιν ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ, πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμίς, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ [πολύγωνον], κορυφὴ δὲ τὸ Λ, πρὸς τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν· ἐναλλὰξ ἄρα, ὡς ὁ κῶνος, οὗ βάσις μὲν ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ, πρὸς τὴν ἐν αὐτῷ πυραμίδα, ἧς βάσις μὲν τὸ ΑΤΒΥΓΦΔΧ πολύγωνον, κορυφὴ δὲ τὸ Λ, οὕτως τὸ Ξ [στερεὸν] πρὸς τὴν πυραμίδα, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν. μείζων δὲ ὁ εἰρημένος κῶνος τῆς ἐν αὐτῷ πυραμίδος· ἐμπεριέχει γὰρ αὐτήν. μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς πυραμίδος, ἧς βάσις μέν ἐστι τὸ ΕΟΖΠΗΡΘΣ πολύγωνον, κορυφὴ δὲ τὸ Ν. ἀλλὰ*

NMF, as LB (is) to BK, so NF (is) to FM, and, on account of the similarity of triangles BKT and FMP, as KB (is) to BT, so MF (is) to FP [Def. 6.1], thus, via equality, as LB (is) to BT, so NF (is) to FP [Prop. 5.22]. Again, since, on account of the similarity of triangles LTK and NPM, as LT (is) to TK, so NP (is) to PM, and, on account of the similarity of triangles TKB and PMF, as KT (is) to TB, so MP (is) to PF, thus, via equality, as LT (is) to TB, so NP (is) to PF [Prop. 5.22]. And it was shown that as TB (is) to BL, so PF (is) to FN. Thus, via equality, as TL (is) to LB, so PN (is) to NF [Prop. 5.22]. Thus, the sides of triangles LTB and NPF are proportional. Thus, triangles LTB and NPF are equiangular [Prop. 6.5]. And, hence, (they are) similar [Def. 6.1]. And, thus, the pyramid whose base is triangle BKT, and apex the point L, is similar to the pyramid whose base is triangle FMP, and apex the point N. For they are contained by equal numbers of similar planes [Def. 11.9]. And similar pyramids which also have triangular bases are in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, pyramid $BKTL$ has to pyramid $FMPN$ the cubed ratio that BK (has) to FM. So, similarly, joining straight-lines from (points) A, W, D, V, C, and U to (center) K, and from (points) E, S, H, R, G, and Q to (center) M, and setting up pyramids having the same apexes as the cones on each of the triangles (so formed), we can also show that each of the pyramids (on base $ABCD$ taken) in order will have to each of the pyramids (on base $EFGH$ taken) in order the cubed ratio that the corresponding side BK (has) to the corresponding side FM—that is to say, that BD (has) to FH. And (for two sets of proportional magnitudes) as one of the leading (magnitudes is) to one of the following, so (the sum of) all of the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. And, thus, as pyramid $BKTL$ (is) to pyramid $FMPN$, so the whole pyramid whose base is polygon $ATBUCVDW$, and apex the point L, (is) to the whole pyramid whose base is polygon $EPFQGRHS$, and apex the point N. And, hence, the pyramid whose base is polygon $ATBUCVDW$, and apex the point L, has to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N, the cubed ratio that BD (has) to FH. And it was also assumed that the cone whose base is circle $ABCD$, and apex the point L, has to solid O the cubed ratio that BD (has) to FH. Thus, as the cone whose base is circle $ABCD$, and apex the point L, is to solid O, so the pyramid whose base (is) [polygon] $ATBUCVDW$, and apex the point L, (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N. Thus, alternately, as the cone whose base (is) circle $ABCD$, and apex the point L, (is) to the pyramid within it whose base (is) the polygon $ATBUCVDW$, and apex the point L, so the [solid] O (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N [Prop. 5.16]. And the aforementioned cone (is) greater than the pyramid within it. For it encompasses it.

καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κῶνος, οὗ βάσις ὁ ΑΒΓΔ κύκλος, κορυφὴ δὲ τὸ Λ [σημεῖον], πρὸς ἔλαττόν τι τοῦ κώνου στερεόν, οὗ βάσις μὲν ὁ ΕΖΗΘ κύκλος, κορυφὴ δὲ τὸ Ν σημεῖον, τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ.

Λέγω δή, ὅτι οὐδὲ ὁ ΑΒΓΔΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μεῖζον τὸ Ξ. ἀνάπαλιν ἄρα τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ. ὡς δὲ τὸ Ξ στερεὸν πρὸς τὸν ΑΒΓΔΛ κῶνον, οὕτως ὁ ΕΖΗΘΝ κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεόν. καὶ ὁ ΕΖΗΘΝ ἄρα κῶνος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔΛ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΖΘ πρὸς τὴν ΒΔ· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ὁ ΑΒΓΔΛ κῶνος πρὸς μεῖζόν τι τοῦ ΕΖΗΘΝ κώνου στερεὸν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλαττον. ὁ ΑΒΓΔΛ ἄρα κῶνος πρὸς τὸν ΕΖΗΘΝ κῶνον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Ὡς δὲ ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλάσιος γὰρ ὁ κύλινδρος τοῦ κώνου ὁ ἐπὶ τῆς αὐτῆς βάσεως τῷ κώνῳ καὶ ἰσοϋψὴς αὐτῷ. καὶ ὁ κύλινδρος ἄρα πρὸς τὸν κύλινδρον τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΔ πρὸς τὴν ΖΘ.

Οἱ ἄρα ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων· ὅπερ ἔδει δεῖξαι.

Thus, solid *O* (is) also greater than the pyramid whose base is polygon *EPFQGRHS*, and apex the point *N*. But, (it is) also less. The very thing is impossible. Thus, the cone whose base (is) circle *ABCD*, and apex the [point] *L*, does not have to some solid less than the cone whose base (is) circle *EFGH*, and apex the point *N*, the cubed ratio that *BD* (has) to *EH*. So, similarly, we can show that neither does cone *EFGHN* have to some solid less than cone *ABCDL* the cubed ratio that *FH* (has) to *BD*.

So, I say that neither does cone *ABCDL* have to some solid greater than cone *EFGHN* the cubed ratio that *BD* (has) to *FH*.

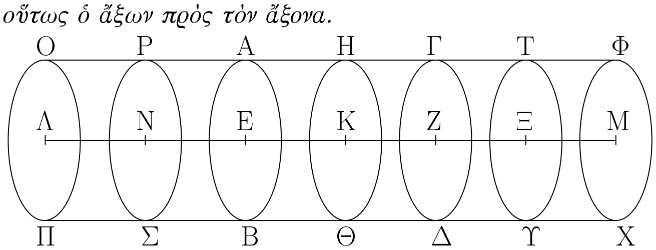
For, if possible, let it have (such a ratio) to a greater (solid), *O*. Thus, inversely, solid *O* has to cone *ABCDL* the cubed ratio that *FH* (has) to *BD* [Prop. 5.7 corr.]. And as solid *O* (is) to cone *ABCDL*, so cone *EFGHN* (is) to some solid less than cone *ABCDL* [12.2 lem.]. Thus, cone *EFGHN* also has to some solid less than cone *ABCDL* the cubed ratio that *FH* (has) to *BD*. The very thing was shown (to be) impossible. Thus, cone *ABCDL* does not have to some solid greater than cone *EFGHN* the cubed ratio than *BD* (has) to *FH*. And it was shown that neither (does it have such a ratio) to a lesser (solid). Thus, cone *ABCDL* has to cone *EFGHN* the cubed ratio that *BD* (has) to *FG*.

And as the cone (is) to the cone, so the cylinder (is) to the cylinder. For a cylinder is three times a cone on the same base as the cone, and of the same height as it [Prop. 12.10]. Thus, the cylinder also has to the cylinder the cubed ratio that *BD* (has) to *FH*.

Thus, similar cones and cylinders are in the cubed ratio of the diameters of their bases. (Which is) the very thing it was required to show.

ιγ'.

Ἐὰν κύλινδρος ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ὁ κύλινδρος πρὸς τὸν κύλινδρον, οὕτως ὁ ἄξων πρὸς τὸν ἄξονα.

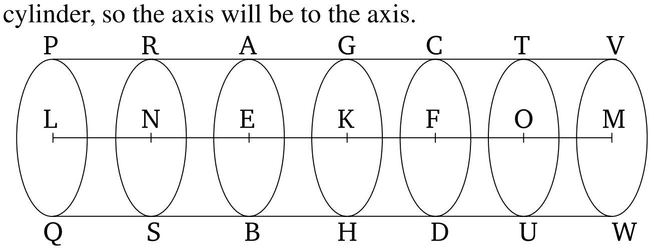


Κύλινδρος γὰρ ὁ ΑΔ ἐπιπέδῳ τῷ ΗΘ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΑΒ, ΓΔ, καὶ συμβαλλέτω τῷ ἄξονι τὸ ΗΘ ἐπίπεδον κατὰ τὸ Κ σημεῖον· λέγω, ὅτι ἐστὶν ὡς ὁ ΒΗ κύλινδρος πρὸς τὸν ΗΔ κύλινδρον, οὕτως ὁ ΕΚ ἄξων πρὸς τὸν ΚΖ ἄξονα.

Ἐκβεβλήσθω γὰρ ὁ ΕΖ ἄξων ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ

Proposition 13

If a cylinder is cut by a plane which is parallel to the opposite planes (of the cylinder) then as the cylinder (is) to the cylinder, so the axis will be to the axis.

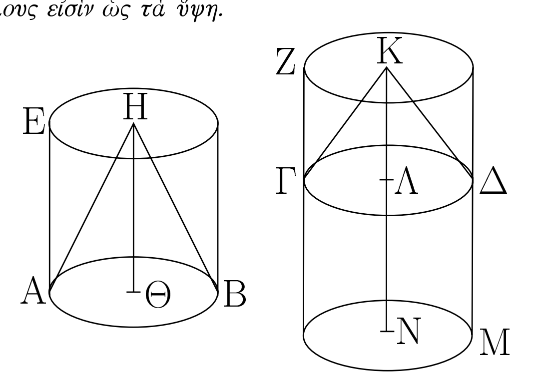


For let the cylinder *AD* be cut by the plane *GH* which is parallel to the opposite planes (of the cylinder), *AB* and *CD*. And let the plane *GH* have met the axis at point *K*. I say that as cylinder *BG* is to cylinder *GD*, so axis *EK* (is) to axis *KF*.

For let axis *EF* be produced in each direction to points *L* and *M*. And let any number whatsoever (of lengths), *EN* and τὰ Λ, Μ σημεῖα, καὶ ἐκκείσθωσαν τῷ ΕΚ ἄξονι ἴσοι ὁσοιδηποτοῦν οἱ ΕΝ, ΝΛ, τῷ δὲ ΖΚ ἴσοι ὁσοιδηποτοῦν οἱ ΖΞ, ΞΜ, καὶ νοείσθω ὁ ἐπὶ τοῦ ΛΜ ἄξονος κύλινδρος ὁ ΟΧ, οὗ βάσεις οἱ ΟΠ, ΦΧ κύκλοι. καὶ ἐκβεβλήσθω διὰ τῶν Ν, Ξ σημείων ἐπίπεδα παράλληλα τοῖς ΑΒ, ΓΔ καὶ ταῖς βάσεσι τοῦ ΟΧ κυλίνδρου καὶ ποιείτωσαν τοὺς ΡΣ, ΤΥ κύκλους περὶ τὰ Ν, Ξ κέντρα. καὶ ἐπεὶ οἱ ΛΝ, ΝΕ, ΕΚ ἄξονες ἴσοι εἰσὶν ἀλλήλοις, οἱ ἄρα ΠΡ, ΡΒ, ΒΗ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσιν αἱ βάσεις· ἴσοι ἄρα καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἀλλήλοις. ἐπεὶ οὖν οἱ ΛΝ, ΝΕ, ΕΚ ἄξονες ἴσοι εἰσὶν ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΠΡ, ΡΒ, ΒΗ κύλινδροι ἴσοι ἀλλήλοις, καί ἐστιν ἴσον τὸ πλῆθος τῷ πλήθει, ὁσαπλασίων ἄρα ὁ ΚΛ ἄξων τοῦ ΕΚ ἄξονος, τοσαυταπλασίων ἔσται καὶ ὁ ΠΗ κύλινδρος τοῦ ΗΒ κυλίνδρου. διὰ τὰ αὐτὰ δὴ καὶ ὁσαπλασίων ἐστὶν ὁ ΜΚ ἄξων τοῦ ΚΖ ἄξονος, τοσαυταπλασίων ἐστὶ καὶ ὁ ΧΗ κύλινδρος τοῦ ΗΔ κυλίνδρου. καὶ εἰ μὲν ἴσος ἐστὶν ὁ ΚΛ ἄξων τῷ ΚΜ ἄξονι, ἴσος ἔσται καὶ ὁ ΠΗ κύλινδρος τῷ ΗΧ κυλίνδρῳ, εἰ δὲ μείζων ὁ ἄξων τοῦ ἄξονος, μείζων καὶ ὁ κύλινδρος τοῦ κυλίνδρου, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὴ μεγεθῶν ὄντων, ἀξόνων μὲν τῶν ΕΚ, ΚΖ, κυλίνδρων δὲ τῶν ΒΗ, ΗΔ, εἴληπται ἰσάκις πολλαπλάσια, τοῦ μὲν ΕΚ ἄξονος καὶ τοῦ ΒΗ κυλίνδρου ὅ τε ΛΚ ἄξων καὶ ὁ ΠΗ κύλινδρος, τοῦ δὲ ΚΖ ἄξονες καὶ τοῦ ΗΔ κυλίνδρου ὅ τε ΚΜ ἄξων καὶ ὁ ΗΧ κύλινδρος, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ὁ ΚΛ ἄξων τοῦ ΚΜ ἄξονος, ὑπερέχει καὶ ὁ ΠΗ κύλινδρος τοῦ ΗΧ κυλίνδρου, καὶ εἰ ἴσος, ἴσος, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα ὡς ὁ ΕΚ ἄξων πρὸς τὸν ΚΖ ἄξονα, οὕτως ὁ ΒΗ κύλινδρος πρὸς τὸν ΗΔ κύλινδρον· ὅπερ ἔδει δεῖξαι.

ιδ'.

Οἱ ἐπὶ ἴσων βάσεων ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ὕψη.

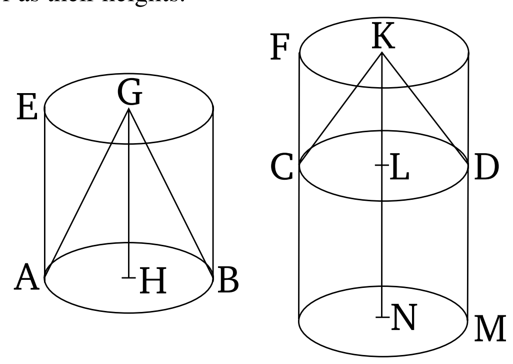


Ἔστωσαν γὰρ ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ κύκλων κύλινδροι οἱ ΕΒ, ΖΔ· λέγω, ὅτι ἐστὶν ὡς ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ

NL, equal to axis EK, be set out (on the axis EL), and any number whatsoever (of lengths), FO and OM, equal to (axis) FK, (on the axis KM). And let the cylinder PW, whose bases (are) the circles PQ and VW, be conceived on axis LM. And let planes parallel to AB, CD, and the bases of cylinder PW, be produced through points N and O, and let them have made the circles RS and TU around the centers N and O (respectively). And since axes LN, NE, and EK are equal to one another, the cylinders QR, RB, and BG are to one another as their bases [Prop. 12.11]. But the bases are equal. Thus, the cylinders QR, RB, and BG (are) also equal to one another. Therefore, since the axes LN, NE, and EK are equal to one another, and the cylinders QR, RB, and BG are also equal to one another, and the number (of the former) is equal to the number (of the latter), thus as many multiples as axis KL is of axis EK, so many multiples is cylinder QG also of cylinder GB. And so, for the same (reasons), as many multiples as axis MK is of axis KF, so many multiples is cylinder WG also of cylinder GD. And if axis KL is equal to axis KM then cylinder QG will also be equal to cylinder GW, and if the axis (is) greater than the axis then the cylinder (will also be) greater than the cylinder, and if (the axis is) less then (the cylinder will also be) less. So, there are four magnitudes—the axes EK and KF, and the cylinders BG and GD—and equal multiples be taken of axis EK and cylinder BG—(namely), axis LK and cylinder QG—and of axis KF and cylinder GD—(namely), axis KM and cylinder GW. And it has been shown that if axis KL exceeds axis KM then cylinder QG also exceeds cylinder GW, and if (the axes are) equal then (the cylinders are) equal, and if (KL is) less then (QG is) less. Thus, as axis EK is to axis KF, so cylinder BG (is) to cylinder GD [Def. 5.5]. (Which is) the very thing it was required to show.

Proposition 14

Cones and cylinders which are on equal bases are to one another as their heights.



For let EB and FD be cylinders on equal bases, (namely) the circles AB and CD (respectively). I say that as cylinder EB is to cylinder FD, so axis GH (is) to axis KL.

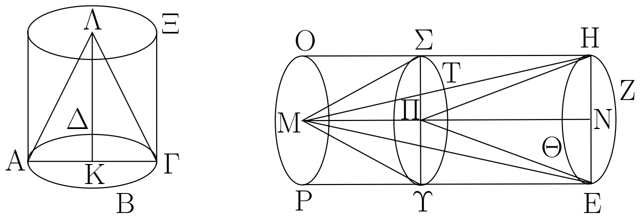
ἄξονα.

Ἐκβεβλήσθω γὰρ ὁ ΚΛ ἄξων ἐπὶ τὸ Ν σημεῖον, καὶ κείσθω τῷ ΗΘ ἄξονι ἴσος ὁ ΛΝ, καὶ περὶ ἄξονα τὸν ΛΝ κύλινδρος νενοήσθω ὁ ΓΜ. ἐπεὶ οὖν οἱ ΕΒ, ΓΜ κύλινδροι ὑπὸ τὸ αὐτὸ ὕψος εἰσίν, πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δέ εἰσιν αἱ βάσεις ἀλλήλαις· ἴσοι ἄρα εἰσὶ καὶ οἱ ΕΒ, ΓΜ κύλινδροι. καὶ ἐπεὶ κύλινδρος ὁ ΖΜ ἐπιπέδῳ τέτμηται τῷ ΓΔ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ὁ ΓΜ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΛΝ ἄξων πρὸς τὸν ΚΛ ἄξονα. ἴσος δέ ἐστιν ὁ μὲν ΓΜ κύλινδρος τῷ ΕΒ κυλίνδρῳ, ὁ δὲ ΛΝ ἄξων τῷ ΗΘ ἄξονι· ἔστιν ἄρα ὡς ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα. ὡς δὲ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον, οὕτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον. καὶ ὡς ἄρα ὁ ΗΘ ἄξων πρὸς τὸν ΚΛ ἄξονα, οὕτως ὁ ΑΒΗ κῶνος πρὸς τὸν ΓΔΚ κῶνον καὶ ὁ ΕΒ κύλινδρος πρὸς τὸν ΖΔ κύλινδρον· ὅπερ ἔδει δεῖξαι.

For let the axis KL be produced to point N. And let LN be made equal to axis GH. And let the cylinder CM be conceived about axis LN. Therefore, since cylinders EB and CM have the same height they are to one another as their bases [Prop. 12.11]. And the bases are equal to one another. Thus, cylinders EB and CM are also equal to one another. And since cylinder FM has been cut by the plane CD, which is parallel to its opposite planes, thus as cylinder CM is to cylinder FD, so axis LN (is) to axis KL [Prop. 12.13]. And cylinder CM is equal to cylinder EB, and axis LN to axis GH. Thus, as cylinder EB is to cylinder FD, so axis GH (is) to axis KL. And as cylinder EB (is) to cylinder FD, so cone ABG (is) to cone CDK [Prop. 12.10]. Thus, also, as axis GH (is) to axis KL, so cone ABG (is) to cone CDK, and cylinder EB to cylinder FD. (Which is) the very thing it was required to show.

ιεʹ.

Τῶν ἴσων κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν κώνων καὶ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσοι εἰσὶν ἐκεῖνοι.

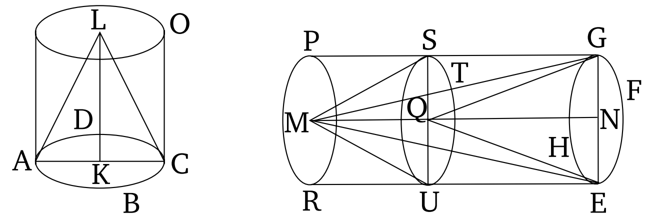


Ἔστωσαν ἴσοι κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ αὐτῶν αἱ ΑΓ, ΕΗ, ἄξονες δὲ οἱ ΚΛ, ΜΝ, οἵτινες καὶ ὕψη εἰσὶ τῶν κώνων ἢ κυλίνδρων, καὶ συμπεπληρώσθωσαν οἱ ΑΞ, ΕΟ κύλινδροι. λέγω, ὅτι τῶν ΑΞ, ΕΟ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος.

Τὸ γὰρ ΛΚ ὕψος τῷ ΜΝ ὕψει ἤτοι ἴσον ἐστὶν ἢ οὔ. ἔστω πρότερον ἴσον. ἔστι δὲ καὶ ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ ἴσος. οἱ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ἴση ἄρα καὶ ἡ ΑΒΓΔ βάσις τῇ ΕΖΗΘ βάσει. ὥστε καὶ ἀντιπέπονθεν, ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος. ἀλλὰ δὴ μὴ ἔστω τὸ ΛΚ ὕψος τῷ ΜΝ ἴσον, ἀλλ᾽ ἔστω μεῖζον τὸ ΜΝ, καὶ ἀφῃρήσθω ἀπὸ τοῦ ΜΝ ὕψους τῷ ΚΛ ἴσον τὸ ΠΝ, καὶ διὰ τοῦ Π σημείου τετμήσθω ὁ ΕΟ κύλινδρος ἐπιπέδῳ τῷ ΤΥΣ παραλλήλῳ τοῖς τῶν ΕΖΗΘ, ΡΟ κύκλων ἐπιπέδοις, καὶ ἀπὸ βάσεως μὲν τοῦ ΕΖΗΘ κύκλου, ὕψους δὲ τοῦ ΝΠ κύλινδρος νενοήσθω ὁ ΕΣ. καὶ ἐπεὶ ἴσος ἐστὶν ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ, ἔστιν ἄρα ὡς ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ

Proposition 15

The bases of equal cones and cylinders are reciprocally proportional to their heights. And, those cones and cylinders whose bases (are) reciprocally proportional to their heights are equal.



Let there be equal cones and cylinders whose bases are the circles $ABCD$ and $EFGH$, and the diameters of (the bases) AC and EG, and (whose) axes (are) KL and MN, which are also the heights of the cones and cylinders (respectively). And let the cylinders AO and EP be completed. I say that the bases of cylinders AO and EP are reciprocally proportional to their heights, and (so) as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL.

For height LK is either equal to height MN, or not. Let it, first of all, be equal. And cylinder AO is also equal to cylinder EP. And cones and cylinders having the same height are to one another as their bases [Prop. 12.11]. Thus, base $ABCD$ (is) also equal to base $EFGH$. And, hence, reciprocally, as base $ABCD$ (is) to base $EFGH$, so height MN (is) to height KL. And so, let height LK not be equal to MN, but let MN be greater. And let QN, equal to KL, be cut off from height MN. And let the cylinder EP be cut, through point Q, by the plane TUS (which is) parallel to the planes of the circles $EFGH$ and RP. And let cylinder ES be conceived, with base the circle $EFGH$, and height NQ. And since cylinder AO is *κυλίνδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον. ἀλλ᾽ ὡς μὲν ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσὶν οἱ ΑΞ, ΕΣ κύλινδροι· ὡς δὲ ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος· ὁ γὰρ ΕΟ κύλινδρος ἐπιπέδῳ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις. ἔστιν ἄρα καὶ ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος. ἴσον δὲ τὸ ΠΝ ὕψος τῷ ΚΛ ὕψει· ἔστιν ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος. τῶν ἄρα ΑΞ, ΕΟ κυλίνδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.*

Ἀλλὰ δὴ τῶν ΑΞ, ΕΟ κυλίνδρων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος· λέγω, ὅτι ἴσος ἐστὶν ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ἐπεί ἐστιν ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΚΛ ὕψος, ἴσον δὲ τὸ ΚΛ ὕψος τῷ ΠΝ ὕψει, ἔσται ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ ὕψος. ἀλλ᾽ ὡς μὲν ἡ ΑΒΓΔ βάσις πρὸς τὴν ΕΖΗΘ βάσιν, οὕτως ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσίν· ὡς δὲ τὸ ΜΝ ὕψος πρὸς τὸ ΠΝ [ὕψος], οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον· ἔστιν ἄρα ὡς ὁ ΑΞ κύλινδρος πρὸς τὸν ΕΣ κύλινδρον, οὕτως ὁ ΕΟ κύλινδρος πρὸς τὸν ΕΣ. ἴσος ἄρα ὁ ΑΞ κύλινδρος τῷ ΕΟ κυλίνδρῳ. ὡσαύτως δὲ καὶ ἐπὶ τῶν κώνων· ὅπερ ἔδει δεῖξαι.

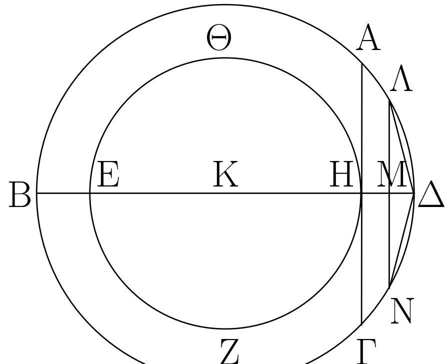
equal to cylinder *EP*, thus, as cylinder *AO* (is) to cylinder *ES*, so cylinder *EP* (is) to cylinder *ES* [Prop. 5.7]. But, as cylinder *AO* (is) to cylinder *ES*, so base *ABCD* (is) to base *EFGH*. For cylinders *AO* and *ES* (have) the same height [Prop. 12.11]. And as cylinder *EP* (is) to (cylinder) *ES*, so height *MN* (is) to height *QN*. For cylinder *EP* has been cut by a plane which is parallel to its opposite planes [Prop. 12.13]. And, thus, as base *ABCD* is to base *EFGH*, so height *MN* (is) to height *QN* [Prop. 5.11]. And height *QN* (is) equal to height *KL*. Thus, as base *ABCD* is to base *EFGH*, so height *MN* (is) to height *KL*. Thus, the bases of cylinders *AO* and *EP* are reciprocally proportional to their heights.

And, so, let the bases of cylinders *AO* and *EP* be reciprocally proportional to their heights, and (thus) let base *ABCD* be to base *EFGH*, as height *MN* (is) to height *KL*. I say that cylinder *AO* is equal to cylinder *EP*.

For, with the same construction, since as base *ABCD* is to base *EFGH*, so height *MN* (is) to height *KL*, and height *KL* (is) equal to height *QN*, thus, as base *ABCD* (is) to base *EFGH*, so height *MN* will be to height *QN*. But, as base *ABCD* (is) to base *EFGH*, so cylinder *AO* (is) to cylinder *ES*. For they are the same height [Prop. 12.11]. And as height *MN* (is) to [height] *QN*, so cylinder *EP* (is) to cylinder *ES* [Prop. 12.13]. Thus, as cylinder *AO* is to cylinder *ES*, so cylinder *EP* (is) to (cylinder) *ES* [Prop. 5.11]. Thus, cylinder *AO* (is) equal to cylinder *EP* [Prop. 5.9]. In the same manner, (the proposition can) also (be demonstrated) for the cones. (Which is) the very thing it was required to show.

ιϛʹ.

Δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων εἰς τὸν μείζονα κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ἐλάσσονος κύκλου.

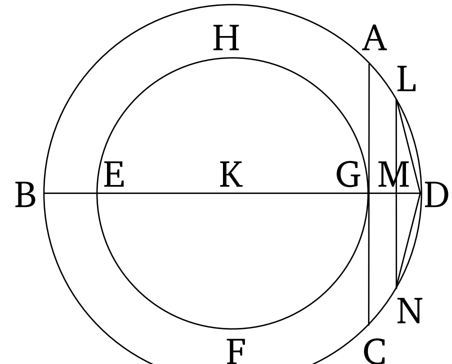


Ἔστωσαν οἱ δοθέντες δύο κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ περὶ τὸ αὐτὸ κέντρον τὸ Κ· δεῖ δὴ εἰς τὸν μείζονα κύκλον τὸν ΑΒΓΔ πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ΕΖΗΘ κύκλου.

Ἤχθω γὰρ διὰ τοῦ Κ κέντρου εὐθεῖα ἡ ΒΚΔ, καὶ ἀπὸ τοῦ

Proposition 16

There being two circles about the same center, to inscribe an equilateral and even-sided polygon in the greater circle, not touching the lesser circle.



Let *ABCD* and *EFGH* be the given two circles, about the same center, *K*. So, it is necessary to inscribe an equilateral and even-sided polygon in the greater circle *ABCD*, not touching circle *EFGH*.

Let the straight-line *BKD* be drawn through the center *K*.

Η σημείου τῇ ΒΔ εὐθείᾳ πρὸς ὀρθὰς ἤχθω ἡ ΗΑ καὶ διήχθω ἐπὶ τὸ Γ· ἡ ΑΓ ἄρα ἐφάπτεται τοῦ ΕΖΗΘ κύκλου. τέμνοντες δὴ τὴν ΒΑΔ περιφέρειαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομεν περιφέρειαν ἐλάσσονα τῆς ΑΔ. λελείφθω, καὶ ἔστω ἡ ΛΔ, καὶ ἀπὸ τοῦ Λ ἐπὶ τὴν ΒΔ κάθετος ἤχθω ἡ ΛΜ καὶ διήχθω ἐπὶ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΔ, ΔΝ· ἴση ἄρα ἐστὶν ἡ ΛΔ τῇ ΔΝ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΛΝ τῇ ΑΓ, ἡ δὲ ΑΓ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου, ἡ ΛΝ ἄρα οὐκ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου· πολλῷ ἄρα αἱ ΛΔ, ΔΝ οὐκ ἐφάπτονται τοῦ ΕΖΗΘ κύκλου. ἐὰν δὴ τῇ ΛΔ εὐθείᾳ ἴσας κατὰ τὸ συνεχὲς ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ κύκλον, ἐγγραφήσεται εἰς τὸν ΑΒΓΔ κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΕΖΗΘ· ὅπερ ἔδει ποιῆσαι.

And let *GA* be drawn, at right-angles to the straight-line *BD*, through point *G*, and let it be drawn through to *C*. Thus, *AC* touches circle *EFGH* [Prop. 3.16 corr.]. So, (by) cutting circumference *BAD* in half, and the half of it in half, and doing this continually, we will (eventually) leave a circumference less than *AD* [Prop. 10.1]. Let it be left, and let it be *LD*. And let *LM* be drawn, from *L*, perpendicular to *BD*, and let it be drawn through to *N*. And let *LD* and *DN* be joined. Thus, *LD* is equal to *DN* [Props. 3.3, 1.4]. And since *LN* is parallel to *AC* [Prop. 1.28], and *AC* touches circle *EFGH*, *LN* thus does not touch circle *EFGH*. Thus, even more so, *LD* and *DN* do not touch circle *EFGH*. And if we continuously insert (straight-lines) equal to straight-line *LD* into circle *ABCD* [Prop. 4.1] then an equilateral and even-sided polygon, not touching the lesser circle *EFGH*, will be inscribed in circle *ABCD*.† (Which is) the very thing it was required to do.

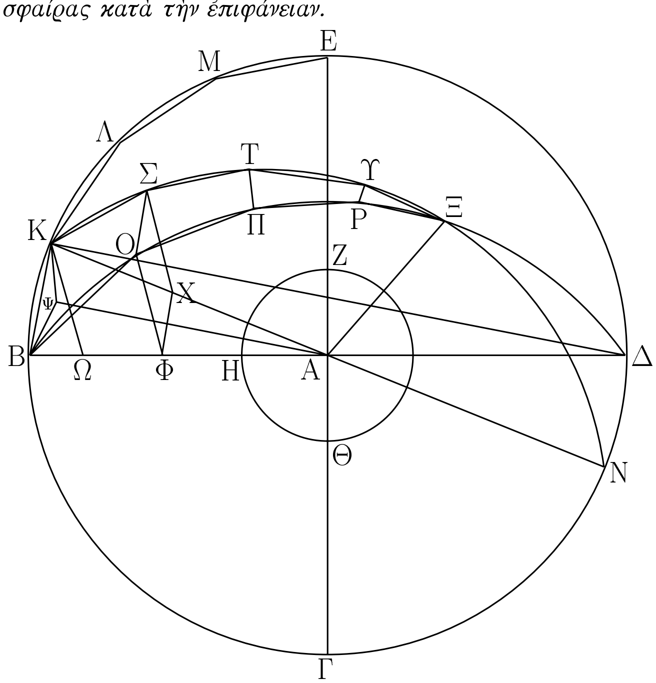
† Note that the chord of the polygon, *LN*, does not touch the inner circle either.

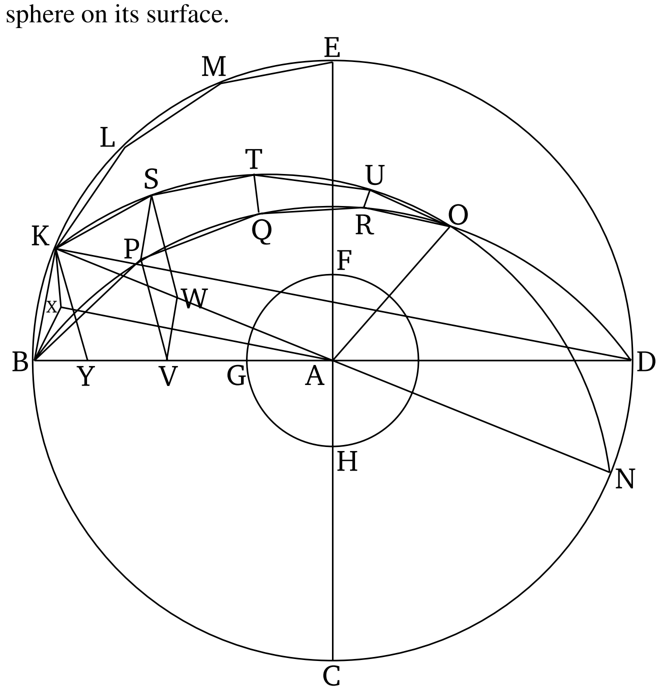
ιζ'.

Δύο σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Proposition 17

There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.





Νενοήσθωσαν δύο σφαῖραι περὶ τὸ αὐτὸ κέντρον τὸ Α· δεῖ δὴ εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Τετμήσθωσαν αἱ σφαῖραι ἐπιπέδῳ τινὶ διὰ τοῦ κέντρου· ἔσονται δὴ αἱ τομαὶ κύκλοι, ἐπειδήπερ μενούσης τῆς διαμέτρου καὶ περιφερομένου τοῦ ἡμικυκλίου ἐγίγνετο ἡ σφαῖρα· ὥστε καὶ καθ' οἵας ἂν θέσεως ἐπινοήσωμεν τὸ ἡμικύκλιον,

Let two spheres be conceived about the same center, *A*. So, it is necessary to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Let the spheres be cut by some plane through the center. So, the sections will be circles, inasmuch as a sphere is generated by the diameter remaining behind, and a semi-circle being carried around [Def. 11.14]. And, hence, whatever position we *τὸ δι᾽ αὐτοῦ ἐκβαλλόμενον ἐπίπεδον ποιήσει ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας κύκλον. καὶ φανερόν, ὅτι καὶ μέγιστον, ἐπειδήπερ ἡ διάμετρος τῆς σφαίρας, ἥτις ἐστὶ καὶ τοῦ ἡμικυκλίου διάμετρος δηλαδὴ καὶ τοῦ κύκλου, μείζων ἐστὶ πασῶν τῶν εἰς τὸν κύκλον ἢ τὴν σφαῖραν διαγομένων [εὐθειῶν]. ἔστω οὖν ἐν μὲν τῇ μείζονι σφαίρᾳ κύκλος ὁ ΒΓΔΕ, ἐν δὲ τῇ ἐλάσσονι σφαίρᾳ κύκλος ὁ ΖΗΘ, καὶ ἤχθωσαν αὐτῶν δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΒΔ, ΓΕ, καὶ δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων τῶν ΒΓΔΕ, ΖΗΘ εἰς τὸν μείζονα κύκλον τὸν ΒΓΔΕ πολύγωνον ἰσόπλευρον καὶ ἀρτιόπλευρον ἐγγεγράφθω μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΖΗΘ, οὗ πλευραὶ ἔστωσαν ἐν τῷ ΒΕ τεταρτημορίῳ αἱ ΒΚ, ΚΛ, ΛΜ, ΜΕ, καὶ ἐπιζευχθεῖσα ἡ ΚΑ διήχθω ἐπὶ τὸ Ν, καὶ ἀνεστάτω ἀπὸ τοῦ Α σημείου τῷ τοῦ ΒΓΔΕ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΑΞ καὶ συμβαλλέτω τῇ ἐπιφανείᾳ τῆς σφαίρας κατὰ τὸ Ξ, καὶ διὰ τῆς ΑΞ καὶ ἑκατέρας τῶν ΒΔ, ΚΝ ἐπίπεδα ἐκβεβλήσθω· ποιήσουσι δὴ διὰ τὰ εἰρημένα ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας μεγίστους κύκλους. ποιείτωσαν, ὧν ἡμικύκλια ἔστω ἐπὶ τῶν ΒΔ, ΚΝ διαμέτρων τὰ ΒΞΔ, ΚΞΝ. καὶ ἐπεὶ ἡ ΞΑ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, καὶ πάντα ἄρα τὰ διὰ τῆς ΞΑ ἐπίπεδά ἐστιν ὀρθὰ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον· ὥστε καὶ τὰ ΒΞΔ, ΚΞΝ ἡμικύκλια ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΒΕΔ, ΒΞΔ, ΚΞΝ ἡμικύκλια· ἐπὶ γὰρ ἴσων εἰσὶ διαμέτρων τῶν ΒΔ, ΚΝ· ἴσα ἐστὶ καὶ τὰ ΒΕ, ΒΞ, ΚΞ τεταρτημόρια ἀλλήλοις. ὅσαι ἄρα εἰσὶν ἐν τῷ ΒΕ τεταρτημορίῳ πλευραὶ τοῦ πολυγώνου, τοσαῦταί εἰσι καὶ ἐν τοῖς ΒΞ, ΚΞ τεταρτημορίοις ἴσαι ταῖς ΒΚ, ΚΛ, ΛΜ, ΜΕ εὐθείαις. ἐγγεγράφθωσαν καὶ ἔστωσαν αἱ ΒΟ, ΟΠ, ΠΡ, ΡΞ, ΚΣ, ΣΤ, ΤΥ, ΥΞ, καὶ ἐπεζεύχθωσαν αἱ ΣΟ, ΤΠ, ΥΡ, καὶ ἀπὸ τῶν Ο, Σ ἐπὶ τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον κάθετοι ἤχθωσαν· πεσοῦνται δὴ ἐπὶ τὰς κοινὰς τομὰς τῶν ἐπιπέδων τὰς ΒΔ, ΚΝ, ἐπειδήπερ καὶ τὰ τῶν ΒΞΔ, ΚΞΝ ἐπίπεδα ὀρθά ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. πιπτέτωσαν, καὶ ἔστωσαν αἱ ΟΦ, ΣΧ, καὶ ἐπεζεύχθω ἡ ΧΦ. καὶ ἐπεὶ ἐν ἴσοις ἡμικυκλίοις τοῖς ΒΞΔ, ΚΞΝ ἴσαι ἀπειλημμέναι εἰσὶν αἱ ΒΟ, ΚΣ, καὶ κάθετοι ἠγμέναι εἰσὶν αἱ ΟΦ, ΣΧ, ἴση [ἄρα] ἐστὶν ἡ μὲν ΟΦ τῇ ΣΧ, ἡ δὲ ΒΦ τῇ ΚΧ. ἔστι δὲ καὶ ὅλη ἡ ΒΑ ὅλῃ τῇ ΚΑ ἴση· καὶ λοιπὴ ἄρα ἡ ΦΑ λοιπῇ τῇ ΧΑ ἐστιν ἴση· ἔστιν ἄρα ὡς ἡ ΒΦ πρὸς τὴν ΦΑ, οὕτως ἡ ΚΧ πρὸς τὴν ΧΑ· παράλληλος ἄρα ἐστὶν ἡ ΧΦ τῇ ΚΒ. καὶ ἐπεὶ ἑκατέρα τῶν ΟΦ, ΣΧ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, παράλληλος ἄρα ἐστὶν ἡ ΟΦ τῇ ΣΧ. ἐδείχθη δὲ αὐτῇ καὶ ἴση· καὶ αἱ ΧΦ, ΣΟ ἄρα ἴσαι εἰσὶ καὶ παράλληλοι. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΧΦ τῇ ΣΟ, ἀλλὰ ἡ ΧΦ τῇ ΚΒ ἐστι παράλληλος, καὶ ἡ ΣΟ ἄρα τῇ ΚΒ ἐστι παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΒΟ, ΚΣ· τὸ ΚΒΟΣ ἄρα τετράπλευρον ἐν ἑνί ἐστιν ἐπιπέδῳ, ἐπειδήπερ, ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, καὶ ἐφ᾽ ἑκατέρας αὐτῶν ληφθῇ τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις. διὰ τὰ αὐτὰ δὴ καὶ ἑκάτερον τῶν ΣΟΠΤ, ΤΠΡΥ τετραπλεύρων ἐν ἑνί ἐστιν*

conceive (of for) the semi-circle, the plane produced through it will make a circle on the surface of the sphere. And (it is) clear that (it is) also a great (circle), inasmuch as the diameter of the sphere, which is also manifestly the diameter of the semi-circle and the circle, is greater than all of the (other) [straight-lines] drawn across in the circle or the sphere [Prop. 3.15]. Therefore, let $BCDE$ be the circle in the greater sphere, and FGH the circle in the lesser sphere. And let two diameters of them be drawn at right-angles to one another, (namely), BD and CE. And there being two circles about the same center—(namely), $BCDE$ and FGH—let an equilateral and even-sided polygon be inscribed in the greater circle, $BCDE$, not touching the lesser circle, FGH [Prop. 12.16], of which let the sides in the quadrant BE be BK, KL, LM, and ME. And, KA being joined, let it be drawn across to N. And let AO be set up at point A, at right-angles to the plane of circle $BCDE$. And let it meet the surface of the (greater) sphere at O. And let planes be produced through AO and each of BD and KN. So, according to the aforementioned (discussion), they will make great circles on the surface of the (greater) sphere. Let them make (great circles), of which let BOD and KON be semi-circles on the diameters BD and KN (respectively). And since OA is at right-angles to the plane of circle $BCDE$, all of the planes through OA are thus also at right-angles to the plane of circle $BCDE$ [Prop. 11.18]. And, hence, the semi-circles BOD and KON are also at right-angles to the plane of circle $BCDE$. And since semi-circles BED, BOD, and KON are equal—for (they are) on the equal diameters BD and KN [Def. 3.1]—the quadrants BE, BO, and KO are also equal to one another. Thus, as many sides of the polygon as are in quadrant BE, so many are also in quadrants BO and KO equal to the straight-lines BK, KL, LM, and ME. Let them be inscribed, and let them be BP, PQ, QR, RO, KS, ST, TU, and UO. And let SP, TQ, and UR be joined. And let perpendiculars be drawn from P and S to the plane of circle $BCDE$ [Prop. 11.11]. So, they will fall on the common sections of the planes BD and KN (with $BCDE$), inasmuch as the planes of BOD and KON are also at right-angles to the plane of circle $BCDE$ [Def. 11.4]. Let them have fallen, and let them be PV and SW. And let WV be joined. And since BP and KS are equal (circumferences) having been cut off in the equal semi-circles BOD and KON [Def. 3.28], and PV and SW are perpendiculars having been drawn (from them), PV is [thus] equal to SW, and BV to KW [Props. 3.27, 1.26]. And the whole of BA is also equal to the whole of KA. And, thus, as BV is to VA, so KW (is) to WA. WV is thus parallel to KB [Prop. 6.2]. And since PV and SW are each at right-angles to the plane of circle $BCDE$, PV is thus parallel to SW [Prop. 11.6]. And it was also shown (to be) equal to it. And, thus, WV and SP are equal and parallel [Prop. 1.33]. And since WV is parallel to SP, but WV is parallel to KB, SP is thus also parallel to KB [Prop. 11.1]. And BP and KS join ἐπιπέδῳ. ἔστι δὲ καὶ τὸ ΥΡΞ τρίγωνον ἐν ἑνὶ ἐπιπέδῳ. ἐὰν δὴ νοήσωμεν ἀπὸ τῶν Ο, Σ, Π, Τ, Ρ, Υ σημείων ἐπὶ τὸ Α ἐπιζευγνυμένας εὐθείας, συσταθήσεταί τι σχῆμα στερεὸν πολύεδρον ματαξὺ τῶν ΒΞ, ΚΞ περιφερειῶν ἐκ πυραμίδων συγκείμενον, ὧν βάσεις μὲν τὰ ΚΒΟΣ, ΣΟΠΤ, ΤΠΡΥ τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον, κορυφὴ δὲ τὸ Α σημεῖον. ἐὰν δὲ καὶ ἐπὶ ἑκάστης τῶν ΚΛ, ΛΜ, ΜΕ πλευρῶν καθάπερ ἐπὶ τῆς ΒΚ τὰ αὐτὰ κατασκευάσωμεν καὶ ἔτι τῶν λοιπῶν τριῶν τεταρτημορίων, συσταθήσεταί τι σχῆμα πολύεδρον ἐγγεγραμμένον εἰς τὴν σφαῖραν πυραμίσι περιεχόμενον, ὧν βάσιες [μὲν] τὰ εἰρημένα τετράπλευρα καὶ τὸ ΥΡΞ τρίγωνον καὶ τὰ ὁμοταγῆ αὐτοῖς, κορυφὴ δὲ τὸ Α σημεῖον.

Λέγω ὅτι τὸ εἰρημένον πολύεδρον οὐκ ἐφάψεται τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν, ἐφ᾽ ἧς ἐστιν ὁ ΖΗΘ κύκλος.

Ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον κάθετος ἡ ΑΨ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Ψ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΨΒ, ΨΚ. καὶ ἐπεὶ ἡ ΑΨ ὀρθή ἐστι πρὸς τὸ τοῦ ΚΒΟΣ τετραπλεύρου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ τοῦ τετραπλεύρου ἐπιπέδῳ ὀρθή ἐστιν. ἡ ΑΨ ἄρα ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΒΨ, ΨΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΑΚ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΑΚ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΑΒ ἴσα τὰ ἀπὸ τῶν ΑΨ, ΨΒ· ὀρθὴ γὰρ ἡ πρὸς τῷ Ψ· τῷ δὲ ἀπὸ τῆς ΑΚ ἴσα τὰ ἀπὸ τῶν ΑΨ, ΨΚ. τὰ ἄρα ἀπὸ τῶν ΑΨ, ΨΒ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΑΨ, ΨΚ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΑΨ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΒΨ λοιπῷ τῷ ἀπὸ τῆς ΨΚ ἴσον ἐστίν· ἴση ἄρα ἡ ΒΨ τῇ ΨΚ. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ ἀπὸ τοῦ Ψ ἐπὶ τὰ Ο, Σ ἐπιζευγνύμεναι εὐθεῖαι ἴσαι εἰσὶν ἑκατέρᾳ τῶν ΒΨ, ΨΚ. ὁ ἄρα κέντρῳ τῷ Ψ καὶ διαστήματι ἑνὶ τῶν ΨΒ, ΨΚ γραφόμενος κύκλος ἥξει καὶ διὰ τῶν Ο, Σ, καὶ ἔσται ἐν κύκλῳ τὸ ΚΒΟΣ τετράπλευρον.

Καὶ ἐπεὶ μείζων ἐστὶν ἡ ΚΒ τῆς ΧΦ, ἴση δὲ ἡ ΧΦ τῇ ΣΟ, μείζων ἄρα ἡ ΚΒ τῆς ΣΟ. ἴση δὲ ἡ ΚΒ ἑκατέρᾳ τῶν ΚΣ, ΒΟ· καὶ ἑκατέρα ἄρα τῶν ΚΣ, ΒΟ τῆς ΣΟ μείζων ἐστίν. καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ ΚΒΟΣ, καὶ ἴσαι αἱ ΚΒ, ΒΟ, ΚΣ, καὶ ἐλάττων ἡ ΟΣ, καὶ ἐκ τοῦ κέντρου τοῦ κύκλου ἐστὶν ἡ ΒΨ, τὸ ἄρα ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μεῖζόν ἐστιν ἢ διπλάσιον. ἤχθω ἀπὸ τοῦ Κ ἐπὶ τὴν ΒΦ κάθετος ἡ ΚΩ. καὶ ἐπεὶ ἡ ΒΔ τῆς ΔΩ ἐλάττων ἐστὶν ἢ διπλῆ, καί ἐστιν ὡς ἡ ΒΔ πρὸς τὴν ΔΩ, οὕτως τὸ ὑπὸ τῶν ΔΒ, ΒΩ πρὸς τὸ ὑπὸ [τῶν] ΔΩ, ΩΒ, ἀναγραφομένου ἀπὸ τῆς ΒΩ τετραγώνου καὶ συμπληρουμένου τοῦ ἐπὶ τῆς ΩΔ παραλληλογράμμου καὶ τὸ ὑπὸ ΔΒ, ΒΩ ἄρα τοῦ ὑπὸ ΔΩ, ΩΒ ἔλαττόν ἐστιν ἢ διπλάσιον. καί ἐστι τῆς ΚΔ ἐπιζευγνυμένης τὸ μὲν ὑπὸ ΔΒ, ΒΩ ἴσον τῷ ἀπὸ τῆς ΒΚ, τὸ δὲ ὑπὸ τῶν ΔΩ, ΩΒ ἴσον τῷ ἀπὸ τῆς ΚΩ· τὸ ἄρα ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΚΩ ἔλασσόν ἐστιν ἢ διπλάσιον. ἀλλὰ τὸ ἀπὸ τῆς ΚΒ τοῦ ἀπὸ τῆς ΒΨ μεῖζόν ἐστιν ἢ διπλάσιον· μεῖζον ἄρα τὸ ἀπὸ τῆς ΚΩ τοῦ ἀπὸ τῆς ΒΨ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΑ τῇ ΚΑ, ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΑ τῷ ἀπὸ τῆς ΑΚ. καί ἐστι τῷ μὲν ἀπὸ τῆς ΒΑ ἴσα τὰ ἀπὸ

them. Thus, the quadrilateral *KBPS* is in one plane, inasmuch as if there are two parallel straight-lines, and a random point is taken on each of them, then the straight-line joining the points is in the same plane as the parallel (straight-lines) [Prop. 11.7]. So, for the same (reasons), each of the quadrilaterals *SPQT* and *TQRU* is also in one plane. And triangle *URO* is also in one plane [Prop. 11.2]. So, if we conceive straight-lines joining points *P*, *S*, *Q*, *T*, *R*, and *U* to *A* then some solid polyhedral figure will be constructed between the circumferences *BO* and *KO*, being composed of pyramids whose bases (are) the quadrilaterals *KBPS*, *SPQT*, *TQRU*, and the triangle *URO*, and apex the point *A*. And if we also make the same construction on each of the sides *KL*, *LM*, and *ME*, just as on *BK*, and, further, (repeat the construction) in the remaining three quadrants, then some polyhedral figure which has been inscribed in the sphere will be constructed, being contained by pyramids whose bases (are) the aforementioned quadrilaterals, and triangle *URO*, and the (quadrilaterals and triangles) similarly arranged to them, and apex the point *A*.

So, I say that the aforementioned polyhedron will not touch the lesser sphere on the surface on which the circle *FGH* is (situated).

Let the perpendicular (straight-line) *AX* be drawn from point *A* to the plane *KBPS*, and let it meet the plane at point *X* [Prop. 11.11]. And let *XB* and *XK* be joined. And since *AX* is at right-angles to the plane of quadrilateral *KBPS*, it is thus also at right-angles to all of the straight-lines joined to it which are also in the plane of the quadrilateral [Def. 11.3]. Thus, *AX* is at right-angles to each of *BX* and *XK*. And since *AB* is equal to *AK*, the (square) on *AB* is also equal to the (square) on *AK*. And the (sum of the squares) on *AX* and *XB* is equal to the (square) on *AB*. For the angle at *X* (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on *AX* and *XK* is equal to the (square) on *AK* [Prop. 1.47]. Thus, the (sum of the squares) on *AX* and *XB* is equal to the (sum of the squares) on *AX* and *XK*. Let the (square) on *AX* be subtracted from both. Thus, the remaining (square) on *BX* is equal to the remaining (square) on *XK*. Thus, *BX* (is) equal to *XK*. So, similarly, we can show that the straight-lines joined from *X* to *P* and *S* are equal to each of *BX* and *XK*. Thus, a circle drawn (in the plane of the quadrilateral) with center *X*, and radius one of *XB* or *XK*, will also pass through *P* and *S*, and the quadrilateral *KBPS* will be inside the circle.

And since *KB* is greater than *WV*, and *WV* (is) equal to *SP*, *KB* (is) thus greater than *SP*. And *KB* (is) equal to each of *KS* and *BP*. Thus, *KS* and *BP* are each greater than *SP*. And since quadrilateral *KBPS* is in a circle, and *KB*, *BP*, and *KS* are equal (to one another), and *PS* (is) less (than them), and *BX* is the radius of the circle, the (square) on *KB* is thus greater than double the (square) on *BX*.† Let the perpendicular *KY* be drawn from *K* to *BV*.‡ And since *BD* is less than dou-

τῶν ΒΨ, ΨΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα τὰ ἀπὸ τῶν ΚΩ, ΩΑ· τὰ ἄρα ἀπὸ τῶν ΒΨ, ΨΑ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΚΩ, ΩΑ, ὧν τὸ ἀπὸ τῆς ΚΩ μεῖζον τοῦ ἀπὸ τῆς ΒΨ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΩΑ ἔλασσόν ἐστι τοῦ ἀπὸ τῆς ΨΑ. μείζων ἄρα ἡ ΑΨ τῆς ΑΩ· πολλῷ ἄρα ἡ ΑΨ μείζων ἐστὶ τῆς ΑΗ. καί ἐστιν ἡ μὲν ΑΨ ἐπὶ μίαν τοῦ πολυέδρου βάσιν, ἡ δὲ ΑΗ ἐπὶ τὴν τῆς ἐλάσσονος σφαίρας ἐπιφάνειαν· ὥστε τὸ πολύεδρον οὐ ψαύσει τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Δύο ἄρα σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγέγραπται μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν· ὅπερ ἔδει ποιῆσαι.

ble *DY*, and as *BD* is to *DY*, so the (rectangle contained) by *DB* and *BY* (is) to the (rectangle contained) by *DY* and *YB*—a square being described on *BY*, and a (rectangular) parallelogram (with short side equal to *BY*) completed on *YD*—the (rectangle contained) by *DB* and *BY* is thus also less than double the (rectangle contained) by *DY* and *YB*. And, *KD* being joined, the (rectangle contained) by *DB* and *BY* is equal to the (square) on *BK*, and the (rectangle contained) by *DY* and *YB* equal to the (square) on *KY* [Props. 3.31, 6.8 corr.]. Thus, the (square) on *KB* is less than double the (square) on *KY*. But, the (square) on *KB* is greater than double the (square) on *BX*. Thus, the (square) on *KY* (is) greater than the (square) on *BX*. And since *BA* is equal to *KA*, the (square) on *BA* is equal to the (square) on *AK*. And the (sum of the squares) on *BX* and *XA* is equal to the (square) on *BA*, and the (sum of the squares) on *KY* and *YA* (is) equal to the (square) on *KA* [Prop. 1.47]. Thus, the (sum of the squares) on *BX* and *XA* is equal to the (sum of the squares) on *KY* and *YA*, of which the (square) on *KY* (is) greater than the (square) on *BX*. Thus, the remaining (square) on *YA* is less than the (square) on *XA*. Thus, *AX* (is) greater than *AY*. Thus, *AX* is much greater than *AG*.§ And *AX* is (a perpendicular) on one of the bases of the polyhedron, and *AG* (is a perpendicular) on the surface of the lesser sphere. Hence, the polyhedron will not touch the lesser sphere on its surface.

Thus, there being two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere on its surface. (Which is) the very thing it was required to do.

† Since *KB*, *BP*, and *KS* are greater than the sides of an inscribed square, which are each of length $\sqrt{2}\,BX$.

‡ Note that points *Y* and *V* are actually identical.

§ This conclusion depends on the fact that the chord of the polygon in proposition 12.16 does not touch the inner circle.

Πόρισμα.

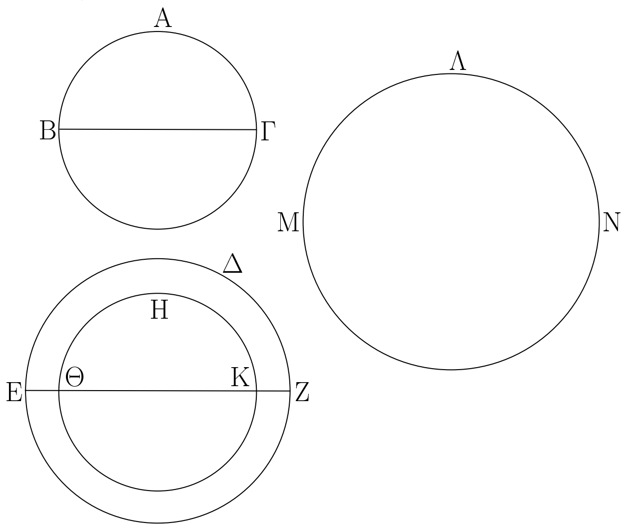
Ἐὰν δὲ καὶ εἰς ἑτέραν σφαῖραν τῷ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεῷ πολυέδρῳ ὅμοιον στερεὸν πολύεδρον ἐγγραφῇ, τὸ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ἑτέρᾳ σφαίρᾳ στερεὸν πολύεδρον τριπλασίονα λόγον ἔχει, ἤπερ ἡ τῆς ΒΓΔΕ σφαίρας διάμετρος πρὸς τὴν τῆς ἑτέρας σφαίρας διάμετρον. διαιρεθέντων γὰρ τῶν στερεῶν εἰς τὰς ὁμοιοπληθεῖς καὶ ὁμοιοταγεῖς πυραμίδας ἔσονται αἱ πυραμίδες ὅμοιαι. αἱ δὲ ὅμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν· ἡ ἄρα πυραμίς, ἧς βάσις μέν ἐστι τὸ ΚΒΟΣ τετράπλευρον, κορυφὴ δὲ τὸ Α σημεῖον, πρὸς τὴν ἐν τῇ ἑτέρᾳ σφαίρᾳ ὁμοιοταγῆ πυραμίδα τριπλασίονα λόγον ἔχει, ἤπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἤπερ ἡ ΑΒ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περὶ κέντρον τὸ Α πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας. ὁμοίως καὶ ἑκάστη πυραμὶς τῶν ἐν τῇ περὶ κέντρον τὸ Α

Corollary

And, also, if a similar polyhedral solid to that in sphere *BCDE* is inscribed in another sphere then the polyhedral solid in sphere *BCDE* has to the polyhedral solid in the other sphere the cubed ratio that the diameter of sphere *BCDE* has to the diameter of the other sphere. For if the solids are divided into similarly numbered, and similarly situated, pyramids, then the pyramids will be similar. And similar pyramids are in the cubed ratio of corresponding sides [Prop. 12.8 corr.]. Thus, the pyramid whose base is quadrilateral *KBPS*, and apex the point *A*, will have to the similarly situated pyramid in the other sphere the cubed ratio that a corresponding side (has) to a corresponding side. That is to say, that of radius *AB* of the sphere about center *A* to the radius of the other sphere. And, similarly, each pyramid in the sphere about center *A* will have to each similarly situated pyramid in the other sphere the cubed *σφαίρᾳ πρὸς ἑκάστην ὁμοταγῆ πυραμίδα τῶν ἐν τῇ ἑτέρᾳ σφαίρᾳ τριπλασίονα λόγον ἕξει, ἤπερ ἡ AB πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας. καὶ ὡς ἓν τῶν ἡγουμένων πρὸς ἓν τῶν ἑπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἑπόμενα· ὥστε ὅλον τὸ ἐν τῇ περὶ κέντρον τὸ A σφαίρᾳ στερεὸν πολύεδρον πρὸς ὅλον τὸ ἐν τῇ ἑτέρᾳ [σφαίρᾳ] στερεὸν πολύεδρον τριπλασίονα λόγον ἕξει, ἤπερ ἡ AB πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἑτέρας σφαίρας, τουτέστιν ἤπερ ἡ $B\Delta$ διάμετρος πρὸς τὴν τῆς ἑτέρας σφαίρας διάμετρον· ὅπερ ἔδει δεῖξαι.*

ιηʹ.

Αἱ σφαῖραι πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἰδίων διαμέτρων.



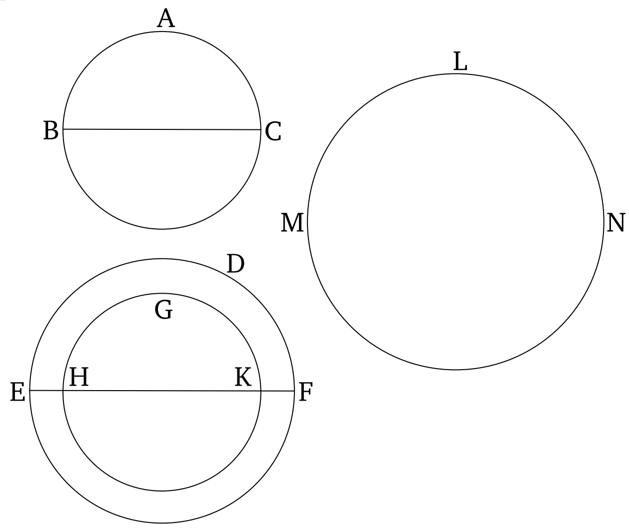
Νενοήσθωσαν σφαῖραι αἱ $AB\Gamma$, ΔEZ, διάμετροι δὲ αὐτῶν αἱ $B\Gamma$, EZ· λέγω, ὅτι ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν ΔEZ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ $B\Gamma$ πρὸς τὴν EZ.

Εἰ γὰρ μὴ ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν ΔEZ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ $B\Gamma$ πρὸς τὴν EZ, ἕξει ἄρα ἡ $AB\Gamma$ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΔEZ σφαίρας τριπλασίονα λόγον ἢ πρὸς μείζονα ἤπερ ἡ $B\Gamma$ πρὸς τὴν EZ. ἐχέτω πρότερον πρὸς ἐλάσσονα τὴν $H\Theta K$, καὶ νενοήσθω ἡ ΔEZ τῇ $H\Theta K$ περὶ τὸ αὐτὸ κέντρον, καὶ ἐγγεγράφθω εἰς τὴν μείζονα σφαῖραν τὴν ΔEZ στερεὸν πολύεδρον μὴ ψαῦον τῆς ἐλάσσονος σφαίρας τῆς $H\Theta K$ κατὰ τὴν ἐπιφάνειαν, ἐγγεγράφθω δὲ καὶ εἰς τὴν $AB\Gamma$ σφαῖραν τῷ ἐν τῇ ΔEZ σφαίρᾳ στερεῷ πολυέδρῳ ὅμοιον στερεὸν πολύεδρον· τὸ ἄρα ἐν τῇ $AB\Gamma$ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ΔEZ στερεὸν πολύεδρον τριπλασίονα λόγον ἔχει ἤπερ ἡ $B\Gamma$ πρὸς τὴν EZ. ἔχει δὲ καὶ ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν $H\Theta K$ σφαῖραν τριπλασίονα λόγον ἤπερ ἡ $B\Gamma$ πρὸς τὴν EZ· ἔστιν ἄρα ὡς ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν $H\Theta K$ σφαῖραν, οὕτως τὸ ἐν τῇ $AB\Gamma$ σφαίρᾳ στερεὸν πολύεδρον πρὸς τὸ ἐν τῇ ΔEZ σφαίρᾳ στερεὸν πολύεδρον· ἐναλλὰξ [ἄρα] ὡς ἡ $AB\Gamma$ σφαῖρα πρὸς τὸ ἐν αὐτῇ πολύεδρον, οὕτως ἡ $H\Theta K$ σφαῖρα πρὸς

ratio that AB (has) to the radius of the other sphere. And as one of the leading (magnitudes is) to one of the following (in two sets of proportional magnitudes), so (the sum of) all the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. Hence, the whole polyhedral solid in the sphere about center A will have to the whole polyhedral solid in the other [sphere] the cubed ratio that (radius) AB (has) to the radius of the other sphere. That is to say, that diameter BD (has) to the diameter of the other sphere. (Which is) the very thing it was required to show.

Proposition 18

Spheres are to one another in the cubed ratio of their respective diameters.



Let the spheres ABC and DEF be conceived, and (let) their diameters (be) BC and EF (respectively). I say that sphere ABC has to sphere DEF the cubed ratio that BC (has) to EF.

For if sphere ABC does not have to sphere DEF the cubed ratio that BC (has) to EF then sphere ABC will have to some (sphere) either less than, or greater than, sphere DEF the cubed ratio that BC (has) to EF. Let it, first of all, have (such a ratio) to a lesser (sphere), GHK. And let DEF be conceived about the same center as GHK. And let a polyhedral solid be inscribed in the greater sphere DEF, not touching the lesser sphere GHK on its surface [Prop. 12.17]. And let a polyhedral solid, similar to the polyhedral solid in sphere DEF, have also been inscribed in sphere ABC. Thus, the polyhedral solid in sphere ABC has to the polyhedral solid in sphere DEF the cubed ratio that BC (has) to EF [Prop. 12.17 corr.]. And sphere ABC also has to sphere GHK the cubed ratio that BC (has) to EF. Thus, as sphere ABC is to sphere GHK, so the polyhedral solid in sphere ABC (is) to the polyhedral solid is sphere DEF. [Thus], alternately, as sphere ABC (is) to the polygon within it, so sphere GHK (is) to the polyhedral solid *τὸ ἐν τῇ ΔΕΖ σφαίρᾳ στερεὸν πολύεδρον. μείζων δὲ ἡ ΑΒΓ σφαῖρα τοῦ ἐν αὐτῇ πολυέδρου· μείζων ἄρα καὶ ἡ ΗΘΚ σφαῖρα τοῦ ἐν τῇ ΔΕΖ σφαίρᾳ πολυέδρου. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ᾿ αὐτοῦ. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς ἐλάσσονα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ διάμετρος πρὸς τὴν ΕΖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἡ ΔΕΖ σφαῖρα πρὸς ἐλάσσονα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ.*

Λέγω δή, ὅτι οὐδὲ ἡ ΑΒΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μείζονα τὴν ΛΜΝ· ἀνάπαλιν ἄρα ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ διάμετρος πρὸς τὴν ΒΓ διάμετρον. ὡς δὲ ἡ ΛΜΝ σφαῖρα πρὸς τὴν ΑΒΓ σφαῖραν, οὕτως ἡ ΔΕΖ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας, ἐπειδήπερ μείζων ἐστὶν ἡ ΛΜΝ τῆς ΔΕΖ, ὡς ἔμπροσθεν ἐδείχθη. καὶ ἡ ΔΕΖ ἄρα σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΑΒΓ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΒΓ· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἡ ΑΒΓ σφαῖρα πρὸς μείζονά τινα τῆς ΔΕΖ σφαίρας τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἐλάσσονα. ἡ ἄρα ΑΒΓ σφαῖρα πρὸς τὴν ΔΕΖ σφαῖραν τριπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

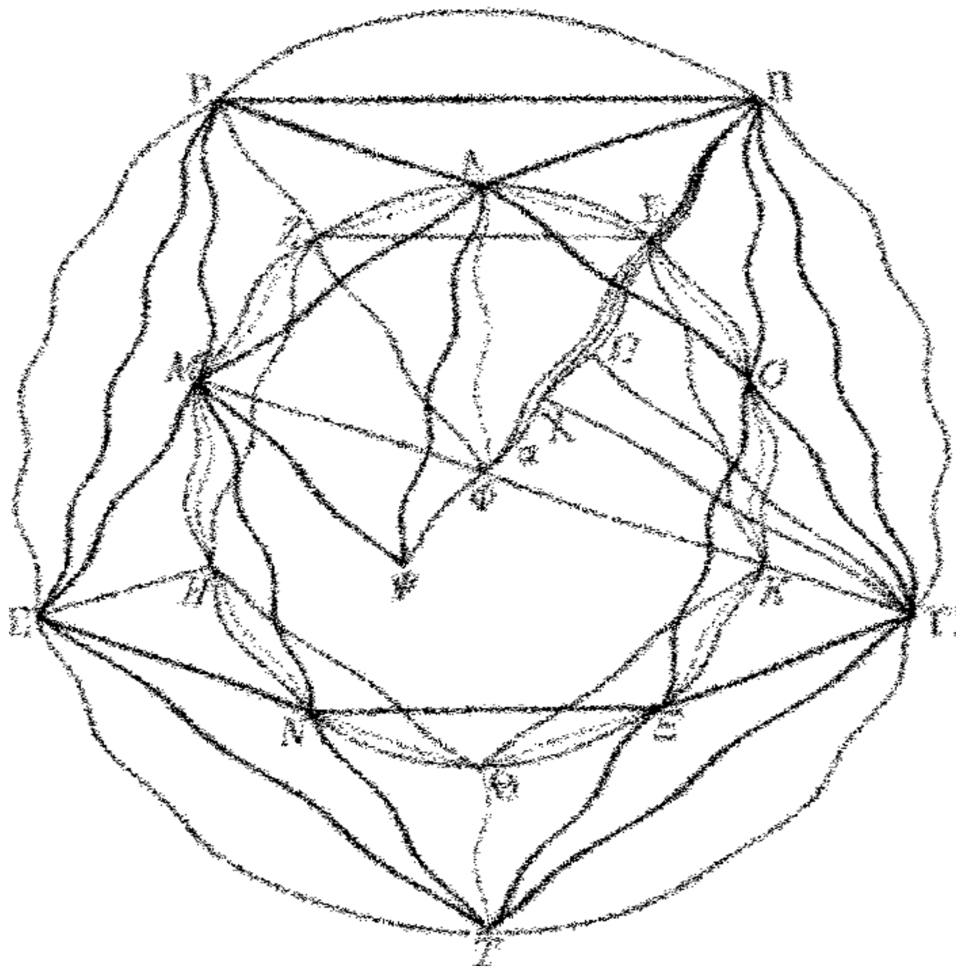
within sphere *DEF* [Prop. 5.16]. And sphere *ABC* (is) greater than the polyhedron within it. Thus, sphere *GHK* (is) also greater than the polyhedron within sphere *DEF* [Prop. 5.14]. But, (it is) also less. For it is encompassed by it. Thus, sphere *ABC* does not have to (a sphere) less than sphere *DEF* the cubed ratio that diameter *BC* (has) to *EF*. So, similarly, we can show that sphere *DEF* does not have to (a sphere) less than sphere *ABC* the cubed ratio that *EF* (has) to *BC* either.

So, I say that sphere *ABC* does not have to some (sphere) greater than sphere *DEF* the cubed ratio that *BC* (has) to *EF* either.

For, if possible, let it have (the cubed ratio) to a greater (sphere), *LMN*. Thus, inversely, sphere *LMN* (has) to sphere *ABC* the cubed ratio that diameter *EF* (has) to diameter *BC* [Prop. 5.7 corr.]. And as sphere *LMN* (is) to sphere *ABC*, so sphere *DEF* (is) to some (sphere) less than sphere *ABC*, inasmuch as *LMN* is greater than *DEF*, as was shown before [Prop. 12.2 lem.]. And, thus, sphere *DEF* has to some (sphere) less than sphere *ABC* the cubed ratio that *EF* (has) to *BC*. The very thing was shown (to be) impossible. Thus, sphere *ABC* does not have to some (sphere) greater than sphere *DEF* the cubed ratio that *BC* (has) to *EF*. And it was shown that neither (does it have such a ratio) to a lesser (sphere). Thus, sphere *ABC* has to sphere *DEF* the cubed ratio that *BC* (has) to *EF*. (Which is) the very thing it was required to show.

ELEMENTS BOOK 13

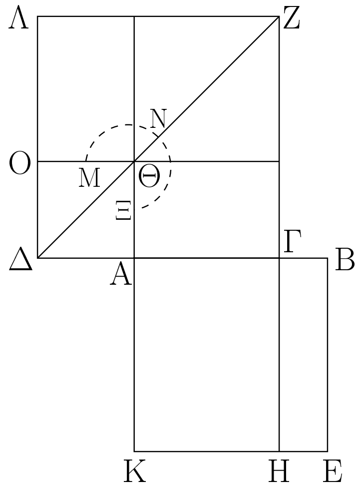
The Platonic Solids†



†The five regular solids—the cube, tetrahedron (i.e., pyramid), octahedron, icosahedron, and dodecahedron—were problably discovered by the school of Pythagoras. They are generally termed "Platonic" solids because they feature prominently in Plato's famous dialogue *Timaeus*. Many of the theorems contained in this book—particularly those which pertain to the last two solids—are ascribed to Theaetetus of Athens.

αʹ.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου.



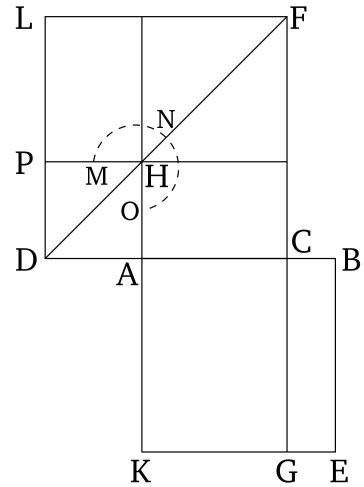
Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ, καὶ ἐκβεβλήσθω ἐπ᾿ εὐθείας τῇ ΓΑ εὐθεῖα ἡ ΑΔ, καὶ κείσθω τῆς ΑΒ ἡμίσεια ἡ ΑΔ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΓΔ τοῦ ἀπὸ τῆς ΔΑ.

Ἀναγεγράφθωσαν γὰρ ἀπὸ τῶν ΑΒ, ΔΓ τετράγωνα τὰ ΑΕ, ΔΖ, καὶ καταγεγράφθω ἐν τῷ ΔΖ τὸ σχῆμα, καὶ διήχθω ἡ ΖΓ ἐπὶ τὸ Η. καὶ ἐπεὶ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΖΘ· ἴσον ἄρα τὸ ΓΕ τῷ ΖΘ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΑ τῆς ΑΔ, ἴση δὲ ἡ μὲν ΒΑ τῇ ΚΑ, ἡ δὲ ΑΔ τῇ ΑΘ, διπλῆ ἄρα καὶ ἡ ΚΑ τῆς ΑΘ. ὡς δὲ ἡ ΚΑ πρὸς τὴν ΑΘ, οὕτως τὸ ΓΚ πρὸς τὸ ΓΘ· διπλάσιον ἄρα τὸ ΓΚ τοῦ ΓΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΓ διπλάσια τοῦ ΓΘ. ἴσον ἄρα τὸ ΚΓ τοῖς ΛΘ, ΘΓ. ἐδείχθη δὲ καὶ τὸ ΓΕ τῷ ΘΖ ἴσον· ὅλον ἄρα τὸ ΑΕ τετράγωνον ἴσον ἐστὶ τῷ ΜΝΞ γνώμονι. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΒΑ τῆς ΑΔ, τετραπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ, τουτέστι τὸ ΑΕ τοῦ ΔΘ. ἴσον δὲ τὸ ΑΕ τῷ ΜΝΞ γνώμονι· καὶ ὁ ΜΝΞ ἄρα γνώμων τετραπλάσιός ἐστι τοῦ ΑΟ· ὅλον ἄρα τὸ ΔΖ πενταπλάσιόν ἐστι τοῦ ΑΟ. καί ἐστι τὸ μὲν ΔΖ τὸ ἀπὸ τῆς ΔΓ, τὸ δὲ ΑΟ τὸ ἀπὸ τῆς ΔΑ· τὸ ἄρα ἀπὸ τῆς ΓΔ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΑ.

Ἐὰν ἄρα εὐθεῖα ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου· ὅπερ ἔδει δεῖξαι.

Proposition 1

If a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half.



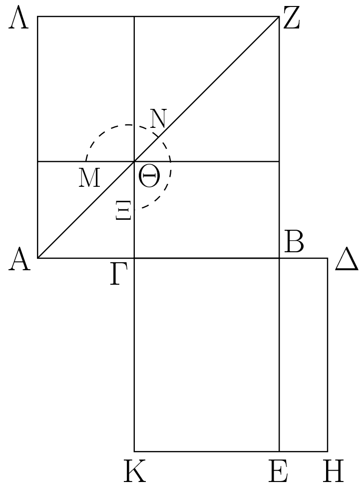
For let the straight-line *AB* be cut in extreme and mean ratio at point *C*, and let *AC* be the greater piece. And let the straight-line *AD* be produced in a straight-line with *CA*. And let *AD* be made (equal to) half of *AB*. I say that the (square) on *CD* is five times the (square) on *DA*.

For let the squares *AE* and *DF* be described on *AB* and *DC* (respectively). And let the figure in *DF* be drawn. And let *FC* be drawn across to *G*. And since *AB* has been cut in extreme and mean ratio at *C*, the (rectangle contained) by *ABC* is thus equal to the (square) on *AC* [Def. 6.3, Prop. 6.17]. And *CE* is the (rectangle contained) by *ABC*, and *FH* the (square) on *AC*. Thus, *CE* (is) equal to *FH*. And since *BA* is double *AD*, and *BA* (is) equal to *KA*, and *AD* to *AH*, *KA* (is) thus also double *AH*. And as *KA* (is) to *AH*, so *CK* (is) to *CH* [Prop. 6.1]. Thus, *CK* (is) double *CH*. And *LH* plus *HC* is also double *CH* [Prop. 1.43]. Thus, *KC* (is) equal to *LH* plus *HC*. And *CE* was also shown (to be) equal to *HF*. Thus, the whole square *AE* is equal to the gnomon *MNO*. And since *BA* is double *AD*, the (square) on *BA* is four times the (square) on *AD*—that is to say, *AE* (is four times) *DH*. And *AE* (is) equal to gnomon *MNO*. And, thus, gnomon *MNO* is also four times *AP*. Thus, the whole of *DF* is five times *AP*. And *DF* is the (square) on *DC*, and *AP* the (square) on *DA*. Thus, the (square) on *CD* is five times the (square) on *DA*.

Thus, if a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half. (Which is) the very thing it was required to show.

β′.

Ἐὰν εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα τὸ λοιπὸν μέρος ἐστὶ τῆς ἐξ ἀρχῆς εὐθείας.



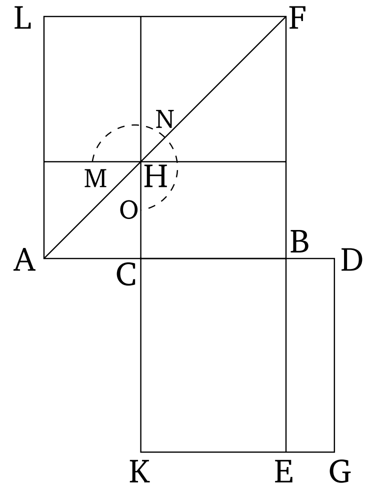
Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ τμήματος ἑαυτῆς τοῦ ΑΓ πενταπλάσιον δυνάσθω, τῆς δὲ ΑΓ διπλῆ ἔστω ἡ ΓΔ. λέγω, ὅτι τῆς ΓΔ ἄκρον καὶ μέσον λόγον τεμνομένος τὸ μεῖζον τμῆμά ἐστιν ἡ ΓΒ.

Ἀναγεγράφθω γὰρ ἀφ᾽ ἑκατέρας τῶν ΑΒ, ΓΔ τετράγωνα τὰ ΑΖ, ΓΗ, καὶ καταγεγράφθω ἐν τῷ ΑΖ τὸ σχῆμα, καὶ διήχθω ἡ ΒΕ. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΓ, πενταπλάσιόν ἐστι τὸ ΑΖ τοῦ ΑΘ. τετραπλάσιος ἄρα ὁ ΜΝΞ γνώμων τοῦ ΑΘ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΔΓ τῆς ΓΑ, τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ ΔΓ τοῦ ἀπὸ ΓΑ, τουτέστι τὸ ΓΗ τοῦ ΑΘ. ἐδείχθη δὲ καὶ ὁ ΜΝΞ γνώμων τετραπλάσιος τοῦ ΑΘ· ἴσος ἄρα ὁ ΜΝΞ γνώμων τῷ ΓΗ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΔΓ τῆς ΓΑ, ἴση δὲ ἡ μὲν ΔΓ τῇ ΓΚ, ἡ δὲ ΑΓ τῇ ΓΘ, [διπλῆ ἄρα καὶ ἡ ΚΓ τῆς ΓΘ], διπλάσιον ἄρα καὶ τὸ ΚΒ τοῦ ΒΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΒ τοῦ ΘΒ διπλάσια· ἴσον ἄρα τὸ ΚΒ τοῖς ΛΘ, ΘΒ. ἐδείχθη δὲ καὶ ὅλος ὁ ΜΝΞ γνώμων ὅλῳ τῷ ΓΗ ἴσος· καὶ λοιπὸν ἄρα τὸ ΘΖ τῷ ΒΗ ἐστιν ἴσον. καί ἐστι τὸ μὲν ΒΗ τὸ ὑπὸ τῶν ΓΔΒ· ἴση γὰρ ἡ ΓΔ τῇ ΔΗ· τὸ δὲ ΘΖ τὸ ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΓΔΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ. ἔστιν ἄρα ὡς ἡ ΔΓ πρὸς τὴν ΓΒ, οὕτως ἡ ΓΒ πρὸς τὴν ΒΔ. μείζων δὲ ἡ ΔΓ τῆς ΓΒ· μείζων ἄρα καὶ ἡ ΓΒ τῆς ΒΔ. τῆς ΓΔ ἄρα εὐθείας ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΓΒ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα τὸ λοιπὸν μέρος ἐστὶ τῆς ἐξ ἀρχῆς εὐθείας· ὅπερ ἔδει δεῖξαι.

Proposition 2

If the square on a straight-line is five times the (square) on a piece of it, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line.



For let the square on the straight-line AB be five times the (square) on the piece of it, AC. And let CD be double AC. I say that if CD is cut in extreme and mean ratio then the greater piece is CB.

For let the squares AF and CG be described on each of AB and CD (respectively). And let the figure in AF be drawn. And let BE be drawn across. And since the (square) on BA is five times the (square) on AC, AF is five times AH. Thus, gnomon MNO (is) four times AH. And since DC is double CA, the (square) on DC is thus four times the (square) on CA—that is to say, CG (is four times) AH. And the gnomon MNO was also shown (to be) four times AH. Thus, gnomon MNO (is) equal to CG. And since DC is double CA, and DC (is) equal to CK, and AC to CH, [KC (is) thus also double CH], (and) KB (is) also double BH [Prop. 6.1]. And LH plus HB is also double HB [Prop. 1.43]. Thus, KB (is) equal to LH plus HB. And the whole gnomon MNO was also shown (to be) equal to the whole of CG. Thus, the remainder HF is also equal to (the remainder) BG. And BG is the (rectangle contained) by CDB. For CD (is) equal to DG. And HF (is) the square on CB. Thus, the (rectangle contained) by CDB is equal to the (square) on CB. Thus, as DC is to CB, so CB (is) to BD [Prop. 6.17]. And DC (is) greater than CB (see lemma). Thus, CB (is) also greater than BD [Prop. 5.14]. Thus, if the straight-line CD is cut in extreme and mean ratio then the greater piece is CB.

Thus, if the square on a straight-line is five times the (square) on a piece of itself, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line. (Which is) the very thing it was required to show.

Λῆμμα.

Ὅτι δὲ ἡ διπλῆ τῆς ΑΓ μείζων ἐστὶ τῆς ΒΓ, οὕτως δεικτέον.

Εἰ γὰρ μή, ἔστω, εἰ δυνατόν, ἡ ΒΓ διπλῆ τῆς ΓΑ. τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΑ· πενταπλάσια ἄρα τὰ ἀπὸ τῶν ΒΓ, ΓΑ τοῦ ἀπὸ τῆς ΓΑ. ὑπόκειται δὲ καὶ τὸ ἀπὸ τῆς ΒΑ πενταπλάσιον τοῦ ἀπὸ τῆς ΓΑ· τὸ ἄρα ἀπὸ τῆς ΒΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΓ, ΓΑ· ὅπερ ἀδύνατον. οὐκ ἄρα ἡ ΓΒ διπλασία ἐστὶ τῆς ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἡ ἐλάττων τῆς ΓΒ διπλασίων ἐστὶ τῆς ΓΑ· πολλῷ γὰρ [μεῖζον] τὸ ἄτοπον.

Ἡ ἄρα τῆς ΑΓ διπλῆ μείζων ἐστὶ τῆς ΓΒ· ὅπερ ἔδει δεῖξαι.

Lemma

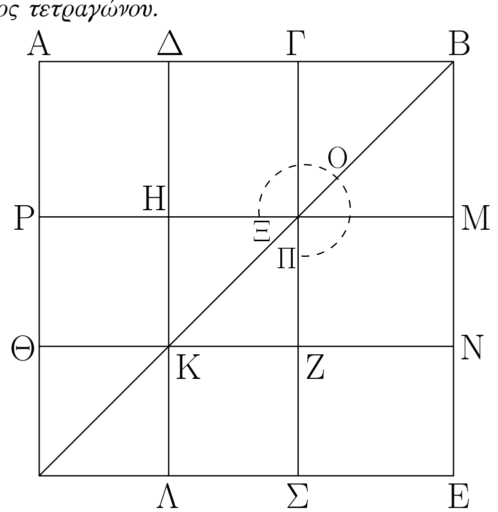
And it can be shown that double *AC* (*i.e.*, *DC*) is greater than *BC*, as follows.

For if (double *AC* is) not (greater than *BC*), if possible, let *BC* be double *CA*. Thus, the (square) on *BC* (is) four times the (square) on *CA*. Thus, the (sum of) the (squares) on *BC* and *CA* (is) five times the (square) on *CA*. And the (square) on *BA* was assumed (to be) five times the (square) on *CA*. Thus, the (square) on *BA* is equal to the (sum of) the (squares) on *BC* and *CA*. The very thing (is) impossible [Prop. 2.4]. Thus, *CB* is not double *AC*. So, similarly, we can show that a (straight-line) less than *CB* is not double *AC* either. For (in this case) the absurdity is much [greater].

Thus, double *AC* is greater than *CB*. (Which is) the very thing it was required to show.

γ'.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ ἔλασσον τμῆμα προσλαβὸν τὴν ἡμίσειαν τοῦ μείζονος τμήματος πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος τετραγώνου.

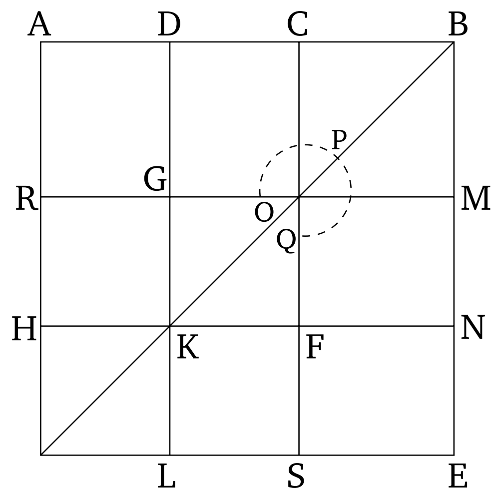


Εὐθεῖα γάρ τις ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ, καὶ τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Δ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΔ τοῦ ἀπὸ τῆς ΔΓ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΕ, καὶ καταγεγράφθω διπλοῦν τὸ σχῆμα. ἐπεὶ διπλῆ ἐστιν ἡ ΑΓ τῆς ΔΓ, τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΓ τοῦ ἀπὸ τῆς ΔΓ, τουτέστι τὸ ΡΣ τοῦ ΖΗ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ, καί ἐστι τὸ ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ ἄρα ΓΕ ἴσον ἐστὶ τῷ ΡΣ. τετραπλάσιον δὲ τὸ ΡΣ τοῦ ΖΗ· τετραπλάσιον ἄρα καὶ τὸ ΓΕ τοῦ ΖΗ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΓ, ἴση ἐστὶ καὶ ἡ ΘΚ τῇ ΚΖ. ὥστε καὶ τὸ ΗΖ τετράγωνον ἴσον ἐστὶ τῷ ΘΛ τετραγώνῳ. ἴση ἄρα ἡ ΗΚ τῇ ΚΛ, τουτέστιν ἡ

Proposition 3

If a straight-line is cut in extreme and mean ratio then the square on the lesser piece added to half of the greater piece is five times the square on half of the greater piece.



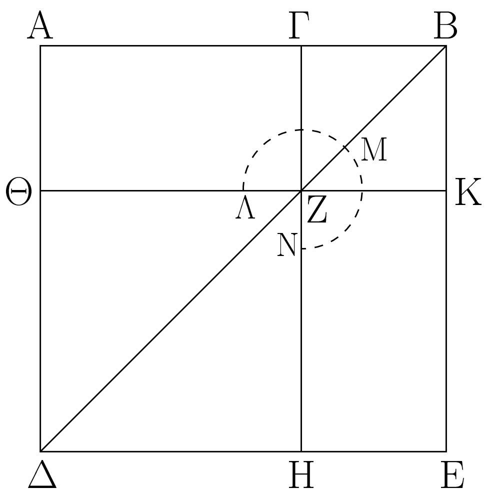
For let some straight-line *AB* be cut in extreme and mean ratio at point *C*. And let *AC* be the greater piece. And let *AC* be cut in half at *D*. I say that the (square) on *BD* is five times the (square) on *DC*.

For let the square *AE* be described on *AB*. And let the figure be drawn double. Since *AC* is double *DC*, the (square) on *AC* (is) thus four times the (square) on *DC*—that is to say, *RS* (is four times) *FG*. And since the (rectangle contained) by *ABC* is equal to the (square) on *AC* [Def. 6.3, Prop. 6.17], and *CE* is the (rectangle contained) by *ABC*, *CE* is thus equal to *RS*. And *RS* (is) four times *FG*. Thus, *CE* (is) also four times *FG*. Again, since *AD* is equal to *DC*, *HK* is also equal to *KF*. Hence, square *GF* is also equal to square *HL*. Thus, *ΜΝ τῇ ΝΕ· ὥστε καὶ τὸ ΜΖ τῷ ΖΕ ἐστιν ἴσον. ἀλλὰ τὸ ΜΖ τῷ ΓΗ ἐστιν ἴσον· καὶ τὸ ΓΗ ἄρα τῷ ΖΕ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΓΝ· ὁ ἄρα ΞΟΠ γνώμων ἴσος ἐστὶ τῷ ΓΕ. ἀλλὰ τὸ ΓΕ τετραπλάσιον ἐδείχθη τοῦ ΗΖ· καὶ ὁ ΞΟΠ ἄρα γνώμων τετραπλάσιός ἐστι τοῦ ΖΗ τετραγώνου. ὁ ΞΟΠ ἄρα γνώμων καὶ τὸ ΖΗ τετράγωνον πενταπλάσιός ἐστι τοῦ ΖΗ. ἀλλὰ ὁ ΞΟΠ γνώμων καὶ τὸ ΖΗ τετράγωνόν ἐστι τὸ ΔΝ. καί ἐστι τὸ μὲν ΔΝ τὸ ἀπὸ τῆς ΔΒ, τὸ δὲ ΗΖ τὸ ἀπὸ τῆς ΔΓ. τὸ ἄρα ἀπὸ τῆς ΔΒ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.*

GK (is) equal to *KL*—that is to say, *MN* to *NE*. Hence, *MF* is also equal to *FE*. But, *MF* is equal to *CG*. Thus, *CG* is also equal to *FE*. Let *CN* be added to both. Thus, gnomon *OPQ* is equal to *CE*. But, *CE* was shown (to be) equal to four times *GF*. Thus, gnomon *OPQ* is also four times square *FG*. Thus, gnomon *OPQ* plus square *FG* is five times *FG*. But, gnomon *OPQ* plus square *FG* is (square) *DN*. And *DN* is the (square) on *DB*, and *GF* the (square) on *DC*. Thus, the (square) on *DB* is five times the (square) on *DC*. (Which is) the very thing it was required to show.

δʹ.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ ἀπὸ τῆς ὅλης καὶ τοῦ ἐλάσσονος τμήματος, τὰ συναμφότερα τετράγωνα, τριπλάσιά ἐστι τοῦ ἀπὸ τοῦ μείζονος τμήματος τετραγώνου.

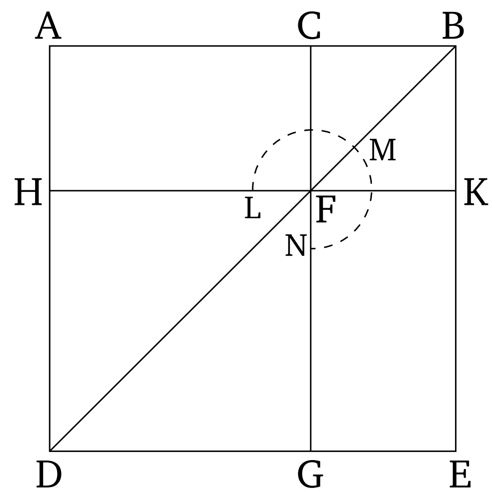


Ἔστω εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μεῖζον τμῆμα τὸ ΑΓ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΓ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΑΚ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΘΗ· ἴσον ἄρα ἐστὶ τὸ ΑΚ τῷ ΘΗ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΖ τῷ ΖΕ, κοινὸν προσκείσθω τὸ ΓΚ· ὅλον ἄρα τὸ ΑΚ ὅλῳ τῷ ΓΕ ἐστιν ἴσον· τὰ ἄρα ΑΚ, ΓΕ τοῦ ΑΚ ἐστι διπλάσια. ἀλλὰ τὰ ΑΚ, ΓΕ ὁ ΛΜΝ γνώμων ἐστὶ καὶ τὸ ΓΚ τετράγωνον· ὁ ἄρα ΛΜΝ γνώμων καὶ τὸ ΓΚ τετράγωνον διπλάσιά ἐστι τοῦ ΑΚ. ἀλλὰ μὴν καὶ τὸ ΑΚ τῷ ΘΗ ἐδείχθη ἴσον· ὁ ἄρα ΛΜΝ γνώμων καὶ [τὸ ΓΚ τετράγωνον διπλάσιά ἐστι τοῦ ΘΗ· ὥστε ὁ ΛΜΝ γνώμων καὶ] τὰ ΓΚ, ΘΗ τετράγωνα τριπλάσιά ἐστι τοῦ ΘΗ τετραγώνου. καί ἐστιν ὁ [μὲν] ΛΜΝ γνώμων καὶ τὰ ΓΚ, ΘΗ τετράγωνα ὅλον τὸ ΑΕ καὶ τὸ ΓΚ, ἅπερ ἐστὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα, τὸ δὲ ΗΘ τὸ ἀπὸ τῆς ΑΓ τετράγωνον.

Proposition 4

If a straight-line is cut in extreme and mean ratio then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece.



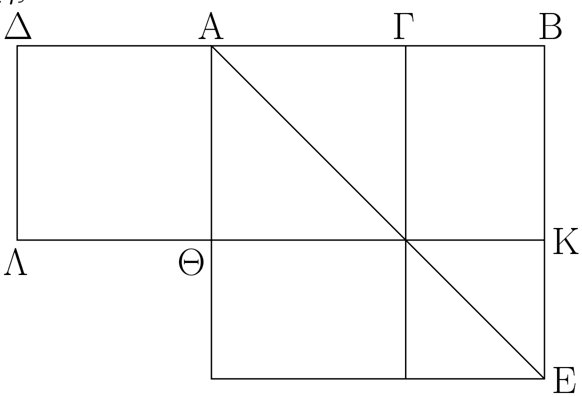
Let *AB* be a straight-line, and let it be cut in extreme and mean ratio at *C*, and let *AC* be the greater piece. I say that the (sum of the squares) on *AB* and *BC* is three times the (square) on *CA*.

For let the square *ADEB* be described on *AB*, and let the (remainder of the) figure be drawn. Therefore, since *AB* has been cut in extreme and mean ratio at *C*, and *AC* is the greater piece, the (rectangle contained) by *ABC* is thus equal to the (square) on *AC* [Def. 6.3, Prop. 6.17]. And *AK* is the (rectangle contained) by *ABC*, and *HG* the (square) on *AC*. Thus, *AK* is equal to *HG*. And since *AF* is equal to *FE* [Prop. 1.43], let *CK* be added to both. Thus, the whole of *AK* is equal to the whole of *CE*. Thus, *AK* plus *CE* is double *AK*. But, *AK* plus *CE* is the gnomon *LMN* plus the square *CK*. Thus, gnomon *LMN* plus square *CK* is double *AK*. But, indeed, *AK* was also shown (to be) equal to *HG*. Thus, gnomon *LMN* plus [square *CK* is double *HG*. Hence, gnomon *LMN* plus] the squares *CK* and *HG* is three times the square *HG*. And gnomon *LMN* plus the squares *CK* and *HG* is the whole of *AE* plus *CK*—which *τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΑΓ τετραγώνου· ὅπερ ἔδει δεῖξαι.*

are the squares on AB and BC (respectively)—and GH (is) the square on AC. Thus, the (sum of the) squares on AB and BC is three times the square on AC. (Which is) the very thing it was required to show.

ε'.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, καὶ προστεθῇ αὐτῇ ἴση τῷ μείζονι τμήματι, ἡ ὅλη εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ἐξ ἀρχῆς εὐθεῖα.



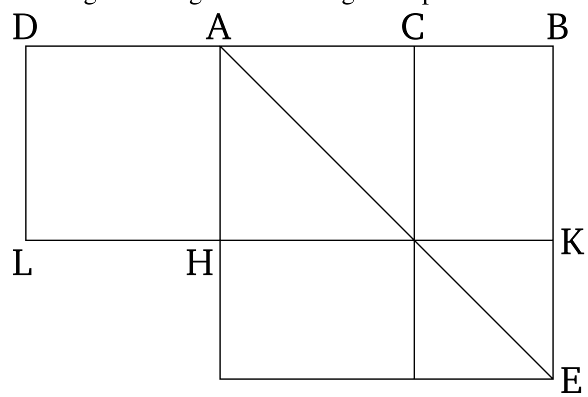
Εὐθεῖα γὰρ γραμμὴ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μεῖζον τμῆμα ἡ ΑΓ, καὶ τῇ ΑΓ ἴση [κείσθω] ἡ ΑΔ. λέγω, ὅτι ἡ ΔΒ εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Α, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ἐξ ἀρχῆς εὐθεῖα ἡ ΑΒ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΕ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ ΑΓ. καί ἐστι τὸ μὲν ὑπὸ ΑΒΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΓΘ· ἴσον ἄρα τὸ ΓΕ τῷ ΘΓ. ἀλλὰ τῷ μὲν ΓΕ ἴσον ἐστὶ τὸ ΘΕ, τῷ δὲ ΘΓ ἴσον τὸ ΔΘ· καὶ τὸ ΔΘ ἄρα ἴσον ἐστὶ τῷ ΘΕ [κοινὸν προσκείσθω τὸ ΘΒ]. ὅλον ἄρα τὸ ΔΚ ὅλῳ τῷ ΑΕ ἐστιν ἴσον. καί ἐστι τὸ μὲν ΔΚ τὸ ὑπὸ τῶν ΒΔ, ΔΑ· ἴση γὰρ ἡ ΑΔ τῇ ΔΛ· τὸ δὲ ΑΕ τὸ ἀπὸ τῆς ΑΒ· τὸ ἄρα ὑπὸ τῶν ΒΔΑ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ. ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΔ. μείζων δὲ ἡ ΔΒ τῆς ΒΑ· μείζων ἄρα καὶ ἡ ΒΑ τῆς ΑΔ.

Ἡ ἄρα ΔΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Α, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΒ· ὅπερ ἔδει δεῖξαι.

Proposition 5

If a straight-line is cut in extreme and mean ratio, and a (straight-line) equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.



For let the straight-line AB be cut in extreme and mean ratio at point C. And let AC be the greater piece. And let AD be [made] equal to AC. I say that the straight-line DB has been cut in extreme and mean ratio at A, and that the original straight-line AB is the greater piece.

For let the square AE be described on AB, and let the (remainder of the) figure be drawn. And since AB has been cut in extreme and mean ratio at C, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC, and CH the (square) on AC. But, HE is equal to CE [Prop. 1.43], and DH equal to HC. Thus, DH is also equal to HE. [Let HB be added to both.] Thus, the whole of DK is equal to the whole of AE. And DK is the (rectangle contained) by BD and DA. For AD (is) equal to DL. And AE (is) the (square) on AB. Thus, the (rectangle contained) by BDA is equal to the (square) on AB. Thus, as DB (is) to BA, so BA (is) to AD [Prop. 6.17]. And DB (is) greater than BA. Thus, BA (is) also greater than AD [Prop. 5.14].

Thus, DB has been cut in extreme and mean ratio at A, and the greater piece is AB. (Which is) the very thing it was required to show.

ϛ'.

Ἐὰν εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Proposition 6

If a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

Ἔστω εὐθεῖα ῥητὴ ἡ ΑΒ καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μεῖζον τμῆμα ἡ ΑΓ· λέγω, ὅτι ἑκατέρα τῶν ΑΓ, ΓΒ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΒΑ, καὶ κείσθω τῆς ΒΑ ἡμίσεια ἡ ΑΔ. ἐπεὶ οὖν εὐθεῖα ἡ ΑΒ τέτμηται ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ τῷ μείζονι τμήματι τῷ ΑΓ πρόσκειται ἡ ΑΔ ἡμίσεια οὖσα τῆς ΑΒ, τὸ ἄρα ἀπὸ ΓΔ τοῦ ἀπὸ ΔΑ πενταπλάσιόν ἐστιν. τὸ ἄρα ἀπὸ ΓΔ πρὸς τὸ ἀπὸ ΔΑ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· σύμμετρον ἄρα τὸ ἀπὸ ΓΔ τῷ ἀπὸ ΔΑ. ῥητὸν δὲ τὸ ἀπὸ ΔΑ· ῥητὴ γάρ [ἐστιν] ἡ ΔΑ ἡμίσεια οὖσα τῆς ΑΒ ῥητῆς οὔσης· ῥητὸν ἄρα καὶ τὸ ἀπὸ ΓΔ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΓΔ. καὶ ἐπεὶ τὸ ἀπὸ ΓΔ πρὸς τὸ ἀπὸ ΔΑ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ἀσύμμετρος ἄρα μήκει ἡ ΓΔ τῇ ΔΑ· αἱ ΓΔ, ΔΑ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΑΓ. πάλιν, ἐπεὶ ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΑΓ, τὸ ἄρα ὑπὸ ΑΒ, ΒΓ τῷ ἀπὸ ΑΓ ἴσον ἐστίν. τὸ ἄρα ἀπὸ τῆς ΑΓ ἀποτομῆς παρὰ τὴν ΑΒ ῥητὴν παραβληθὲν πλάτος ποιεῖ τὴν ΒΓ. τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην· ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ ΓΒ. ἐδείχθη δὲ καὶ ἡ ΓΑ ἀποτομή.

Ἐὰν ἄρα εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή· ὅπερ ἔδει δεῖξαι.

Let *AB* be a rational straight-line cut in extreme and mean ratio at *C*, and let *AC* be the greater piece. I say that *AC* and *CB* is each that irrational (straight-line) called an apotome.

For let *BA* be produced, and let *AD* be made (equal) to half of *BA*. Therefore, since the straight-line *AB* has been cut in extreme and mean ratio at *C*, and *AD*, which is half of *AB*, has been added to the greater piece *AC*, the (square) on *CD* is thus five times the (square) on *DA* [Prop. 13.1]. Thus, the (square) on *CD* has to the (square) on *DA* the ratio which a number (has) to a number. The (square) on *CD* (is) thus commensurable with the (square) on *DA* [Prop. 10.6]. And the (square) on *DA* (is) rational. For *DA* [is] rational, being half of *AB*, which is rational. Thus, the (square) on *CD* (is) also rational [Def. 10.4]. Thus, *CD* is also rational. And since the (square) on *CD* does not have to the (square) on *DA* the ratio which a square number (has) to a square number, *CD* (is) thus incommensurable in length with *DA* [Prop. 10.9]. Thus, *CD* and *DA* are rational (straight-lines which are) commensurable in square only. Thus, *AC* is an apotome [Prop. 10.73]. Again, since *AB* has been cut in extreme and mean ratio, and *AC* is the greater piece, the (rectangle contained) by *AB* and *BC* is thus equal to the (square) on *AC* [Def. 6.3, Prop. 6.17]. Thus, the (square) on the apotome *AC*, applied to the rational (straight-line) *AB*, makes *BC* as width. And the (square) on an apotome, applied to a rational (straight-line), makes a first apotome as width [Prop. 10.97]. Thus, *CB* is a first apotome. And *CA* was also shown (to be) an apotome.

Thus, if a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

ζʹ.

Ἐὰν πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ἤτοι αἱ κατὰ τὸ ἑξῆς ἢ αἱ μὴ κατὰ τὸ ἑξῆς ἴσαι ὦσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον.

Πενταγώνου γὰρ ἰσοπλεύρου τοῦ ΑΒΓΔΕ αἱ τρεῖς γωνίαι πρότερον αἱ κατὰ τὸ ἑξῆς αἱ πρὸς τοῖς Α, Β, Γ ἴσαι ἀλλήλαις ἔστωσαν· λέγω, ὅτι ἰσογώνιόν ἐστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΕ, ΖΔ. καὶ ἐπεὶ δύο αἱ ΓΒ, ΒΑ δυσὶ ταῖς ΒΑ, ΑΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΓΒΑ γωνίᾳ τῇ ὑπὸ ΒΑΕ ἐστιν ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῇ ΒΕ ἐστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΕ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ᾽ ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΒΓΑ τῇ ὑπὸ ΒΕΑ, ἡ δὲ ὑπὸ ΑΒΕ τῇ ὑπὸ ΓΑΒ· ὥστε καὶ πλευρὰ ἡ ΑΖ πλευρᾷ τῇ ΒΖ ἐστιν ἴση. ἐδείχθη δὲ καὶ ὅλη ἡ ΑΓ ὅλῃ τῇ ΒΕ ἴση· καὶ λοιπὴ ἄρα ἡ ΖΓ λοιπῇ τῇ ΖΕ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ΓΔ τῇ ΔΕ ἴση. δύο δὴ αἱ ΖΓ, ΓΔ δυσὶ ταῖς ΖΕ,

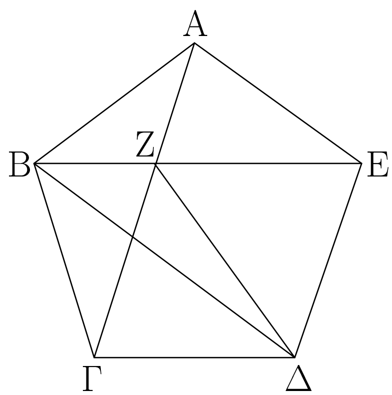
Proposition 7

If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.

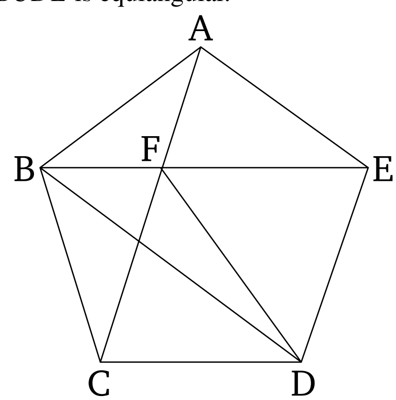
For let three angles of the equilateral pentagon *ABCDE*—first of all, the consecutive (angles) at *A*, *B*, and *C*—-be equal to one another. I say that pentagon *ABCDE* is equiangular.

For let *AC*, *BE*, and *FD* be joined. And since the two (straight-lines) *CB* and *BA* are equal to the two (straight-lines) *BA* and *AE*, respectively, and angle *CBA* is equal to angle *BAE*, base *AC* is thus equal to base *BE*, and triangle *ABC* equal to triangle *ABE*, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4], (that is), *BCA* (equal) to *BEA*, and *ABE* to *CAB*. And hence side *AF* is also equal to side *BF* [Prop. 1.6]. And the whole of *AC* was also shown (to be) equal to the whole of *BE*. Thus, the remainder *FC* is also equal to the remainder

ΕΔ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΖΔ· γωνία ἄρα ἡ ὑπὸ ΖΓΔ γωνίᾳ τῇ ὑπὸ ΖΕΔ ἐστιν ἴση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΓΑ τῇ ὑπὸ ΑΕΒ ἴση· καὶ ὅλη ἄρα ἡ ὑπὸ ΒΓΔ ὅλῃ τῇ ὑπὸ ΑΕΔ ἴση. ἀλλ' ἡ ὑπὸ ΒΓΔ ἴση ὑπόκειται ταῖς πρὸς τοῖς Α, Β γωνίαις· καὶ ἡ ὑπὸ ΑΕΔ ἄρα ταῖς πρὸς τοῖς Α, Β γωνίαις ἴση ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΓΔΕ γωνία ἴση ἐστὶ ταῖς πρὸς τοῖς Α, Β, Γ γωνίαις· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον.



FE. And CD is also equal to DE. So, the two (straight-lines) FC and CD are equal to the two (straight-lines) FE and ED (respectively). And FD is their common base. Thus, angle FCD is equal to angle FED [Prop. 1.8]. And BCA was also shown (to be) equal to AEB. And thus the whole of BCD (is) equal to the whole of AED. But, (angle) BCD was assumed (to be) equal to the angles at A and B. Thus, (angle) AED is also equal to the angles at A and B. So, similarly, we can show that angle CDE is also equal to the angles at A, B, C. Thus, pentagon $ABCDE$ is equiangular.



Ἀλλὰ δὴ μὴ ἔστωσαν ἴσαι αἱ κατὰ τὸ ἑξῆς γωνίαι, ἀλλ' ἔστωσαν ἴσαι αἱ πρὸς τοῖς Α, Γ, Δ σημείοις· λέγω, ὅτι καὶ οὕτως ἰσογώνιόν ἐστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθω γὰρ ἡ ΒΔ. καὶ ἐπεὶ δύο αἱ ΒΑ, ΑΕ δυσὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῇ ΒΔ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῇ ὑπὸ ΓΔΒ. ἔστι δὲ καὶ ἡ ὑπὸ ΒΕΔ γωνία τῇ ὑπὸ ΒΔΕ ἴση, ἐπεὶ καὶ πλευρὰ ἡ ΒΕ πλευρᾷ τῇ ΒΔ ἐστιν ἴση. καὶ ὅλη ἄρα ἡ ὑπὸ ΑΕΔ γωνία ὅλῃ τῇ ὑπὸ ΓΔΕ ἐστιν ἴση. ἀλλὰ ἡ ὑπὸ ΓΔΕ ταῖς πρὸς τοῖς Α, Γ γωνίαις ὑπόκειται ἴση· καὶ ἡ ὑπὸ ΑΕΔ ἄρα γωνία ταῖς πρὸς τοῖς Α, Γ ἴση ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΒΓ ἴση ἐστὶ ταῖς πρὸς τοῖς Α, Γ, Δ γωνίαις. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον· ὅπερ ἔδει δεῖξαι.

And so let consecutive angles not be equal, but let the (angles) at points A, C, and D be equal. I say that pentagon $ABCDE$ is also equiangular in this case.

For let BD be joined. And since the two (straight-lines) BA and AE are equal to the (straight-lines) BC and CD, and they contain equal angles, base BE is thus equal to base BD, and triangle ABE is equal to triangle BCD, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle AEB is equal to (angle) CDB. And angle BED is also equal to (angle) BDE, since side BE is also equal to side BD [Prop. 1.5]. Thus, the whole angle AED is also equal to the whole (angle) CDE. But, (angle) CDE was assumed (to be) equal to the angles at A and C. Thus, angle AED is also equal to the (angles) at A and C. So, for the same (reasons), (angle) ABC is also equal to the angles at A, C, and D. Thus, pentagon $ABCDE$ is equiangular. (Which is) the very thing it was required to show.

η'.

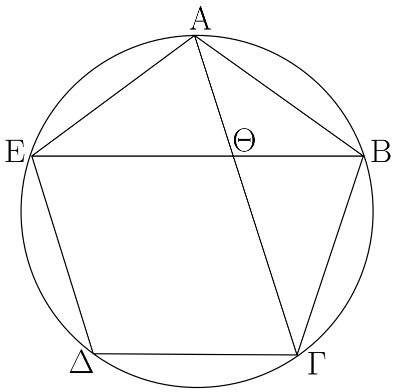
Ἐὰν πενταγώνου ἰσοπλεύρου καὶ ἰσογωνίου τὰς κατὰ τὸ ἑξῆς δύο γωνίας ὑποτείνωσιν εὐθεῖαι, ἄκρον καὶ μέσον λόγον τέμνουσιν ἀλλήλας, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.

Πενταγώνου γὰρ ἰσοπλεύρου καὶ ἰσογωνίου τοῦ ΑΒΓΔΕ δύο γωνίας τὰς κατὰ τὸ ἑξῆς τὰς πρὸς τοῖς Α, Β ὑποτεινέτωσαν εὐθεῖαι αἱ ΑΓ, ΒΕ τέμνουσαι ἀλλήλας κατὰ τὸ Θ σημεῖον· λέγω, ὅτι ἑκατέρα αὐτῶν ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ σημεῖον, καὶ τὰ μείζονα αὐτῶν τμήματα

Proposition 8

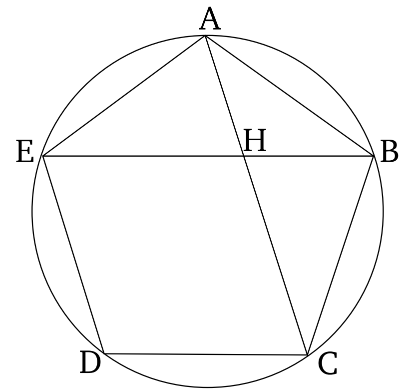
If straight-lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio, and their greater pieces are equal to the sides of the pentagon.

For let the two straight-lines, AC and BE, cutting one another at point H, have subtended two consecutive angles, at A and B (respectively), of the equilateral and equiangular pentagon $ABCDE$. I say that each of them has been cut in extreme and mean ratio at point H, and that their greater pieces *ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.*



Περιγεγράφθω γὰρ περὶ τὸ ΑΒΓΔΕ πεντάγωνον κύκλος ὁ ΑΒΓΔΕ. καὶ ἐπεὶ δύο εὐθεῖαι αἱ ΕΑ, ΑΒ δυσὶ ταῖς ΑΒ, ΒΓ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῇ ΑΓ ἴση ἐστίν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΒΓ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὑφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΑΒΕ· διπλῆ ἄρα ἡ ὑπὸ ΑΘΕ τῆς ὑπὸ ΒΑΘ. ἔστι δὲ καὶ ἡ ὑπὸ ΕΑΓ τῆς ὑπὸ ΒΑΓ διπλῆ, ἐπειδήπερ καὶ περιφέρεια ἡ ΕΔΓ περιφερείας τῆς ΓΒ ἐστι διπλῆ· ἴση ἄρα ἡ ὑπὸ ΘΑΕ γωνία τῇ ὑπὸ ΑΘΕ· ὥστε καὶ ἡ ΘΕ εὐθεῖα τῇ ΕΑ, τουτέστι τῇ ΑΒ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΑ εὐθεῖα τῇ ΑΕ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΑΒΕ τῇ ὑπὸ ΑΕΒ. ἀλλὰ ἡ ὑπὸ ΑΒΕ τῇ ὑπὸ ΒΑΘ ἐδείχθη ἴση· καὶ ἡ ὑπὸ ΒΕΑ ἄρα τῇ ὑπὸ ΒΑΘ ἐστιν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΒΕ καὶ τοῦ ΑΒΘ ἐστιν ἡ ὑπὸ ΑΒΕ· λοιπὴ ἄρα ἡ ὑπὸ ΒΑΕ γωνία λοιπῇ τῇ ὑπὸ ΑΘΒ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΒΘ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΕΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΑΒ πρὸς τὴν ΒΘ. ἴση δὲ ἡ ΒΑ τῇ ΕΘ· ὡς ἄρα ἡ ΒΕ πρὸς τὴν ΕΘ, οὕτως ἡ ΕΘ πρὸς τὴν ΘΒ. μείζων δὲ ἡ ΒΕ τῆς ΕΘ· μείζων ἄρα καὶ ἡ ΕΘ τῆς ΘΒ. ἡ ΒΕ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ, καὶ τὸ μεῖζον τμῆμα τὸ ΘΕ ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΑΓ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ, καὶ τὸ μεῖζον αὐτῆς τμῆμα ἡ ΓΘ ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ· ὅπερ ἔδει δεῖξαι.

are equal to the sides of the pentagon.



For let the circle *ABCDE* be circumscribed about pentagon *ABCDE* [Prop. 4.14]. And since the two straight-lines *EA* and *AB* are equal to the two (straight-lines) *AB* and *BC* (respectively), and they contain equal angles, the base *BE* is thus equal to the base *AC*, and triangle *ABE* is equal to triangle *ABC*, and the remaining angles will be equal to the remaining angles, respectively, which the equal sides subtend [Prop. 1.4]. Thus, angle *BAC* is equal to (angle) *ABE*. Thus, (angle) *AHE* (is) double (angle) *BAH* [Prop. 1.32]. And *EAC* is also double *BAC*, inasmuch as circumference *EDC* is also double circumference *CB* [Props. 3.28, 6.33]. Thus, angle *HAE* (is) equal to (angle) *AHE*. Hence, straight-line *HE* is also equal to (straight-line) *EA*—that is to say, to (straight-line) *AB* [Prop. 1.6]. And since straight-line *BA* is equal to *AE*, angle *ABE* is also equal to *AEB* [Prop. 1.5]. But, *ABE* was shown (to be) equal to *BAH*. Thus, *BEA* is also equal to *BAH*. And (angle) *ABE* is common to the two triangles *ABE* and *ABH*. Thus, the remaining angle *BAE* is equal to the remaining (angle) *AHB* [Prop. 1.32]. Thus, triangle *ABE* is equiangular to triangle *ABH*. Thus, proportionally, as *EB* is to *BA*, so *AB* (is) to *BH* [Prop. 6.4]. And *BA* (is) equal to *EH*. Thus, as *BE* (is) to *EH*, so *EH* (is) to *HB*. And *BE* (is) greater than *EH*. *EH* (is) thus also greater than *HB* [Prop. 5.14]. Thus, *BE* has been cut in extreme and mean ratio at *H*, and the greater piece *HE* is equal to the side of the pentagon. So, similarly, we can show that *AC* has also been cut in extreme and mean ratio at *H*, and that its greater piece *CH* is equal to the side of the pentagon. (Which is) the very thing it was required to show.

θ'.

Ἐὰν ἡ τοῦ ἑξαγώνου πλευρὰ καὶ ἡ τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων συντεθῶσιν, ἡ ὅλη εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ τοῦ ἑξαγώνου πλευρά.

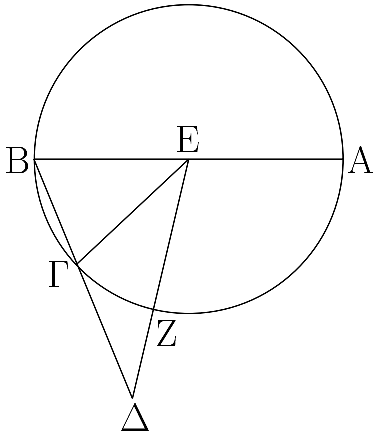
Ἔστω κύκλος ὁ ΑΒΓ, καὶ τῶν εἰς τὸν ΑΒΓ κύκλον

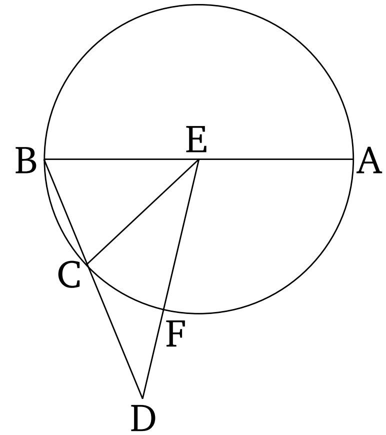
Proposition 9

If the side of a hexagon and of a decagon inscribed in the same circle are added together then the whole straight-line has been cut in extreme and mean ratio (at the junction point), and its greater piece is the side of the hexagon.†

Let *ABC* be a circle. And of the figures inscribed in circle *ABC*, let *BC* be the side of a decagon, and *CD* (the side) of *ἐγγραφομένων σχημάτων, δεκαγώνου μὲν ἔστω πλευρὰ ἡ ΒΓ, ἑξαγώνου δὲ ἡ ΓΔ, καὶ ἔστωσαν ἐπ᾽ εὐθείας· λέγω, ὅτι ἡ ὅλη εὐθεῖα ἡ ΒΔ ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΓΔ.*

a hexagon. And let them be (laid down) straight-on (to one another). I say that the whole straight-line *BD* has been cut in extreme and mean ratio (at *C*), and that *CD* is its greater piece.





Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ε σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΒ, ΕΓ, ΕΔ, καὶ διήχθω ἡ ΒΕ ἐπὶ τὸ Α. ἐπεὶ δεκαγώνου ἰσοπλεύρου πλευρά ἐστιν ἡ ΒΓ, πενταπλασίων ἄρα ἡ ΑΓΒ περιφέρεια τῆς ΒΓ περιφερείας· τετραπλασίων ἄρα ἡ ΑΓ περιφέρεια τῆς ΓΒ. ὡς δὲ ἡ ΑΓ περιφέρεια πρὸς τὴν ΓΒ, οὕτως ἡ ὑπὸ ΑΕΓ γωνία πρὸς τὴν ὑπὸ ΓΕΒ· τετραπλασίων ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΓΕΒ. καὶ ἐπεὶ ἴση ἡ ὑπὸ ΕΒΓ γωνία τῇ ὑπὸ ΕΓΒ, ἡ ἄρα ὑπὸ ΑΕΓ γωνία διπλασία ἐστὶ τῆς ὑπὸ ΕΓΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΓ εὐθεῖα τῇ ΓΔ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ τοῦ ἑξαγώνου πλευρᾷ τοῦ εἰς τὸν ΑΒΓ κύκλον [ἐγγραφομένου]· ἴση ἐστὶ καὶ ἡ ὑπὸ ΓΕΔ γωνία τῇ ὑπὸ ΓΔΕ γωνίᾳ· διπλασία ἄρα ἡ ὑπὸ ΕΓΒ γωνία τῆς ὑπὸ ΕΔΓ. ἀλλὰ τῆς ὑπὸ ΕΓΒ διπλασία ἐδείχθη ἡ ὑπὸ ΑΕΓ· τετραπλασία ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΕΔΓ. ἐδείχθη δὲ καὶ τῆς ὑπὸ ΒΕΓ τετραπλασία ἡ ὑπὸ ΑΕΓ· ἴση ἄρα ἡ ὑπὸ ΕΔΓ τῇ ὑπὸ ΒΕΓ. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ΒΕΓ καὶ τοῦ ΒΕΔ, ἡ ὑπὸ ΕΒΔ γωνία· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΕΔ τῇ ὑπὸ ΕΓΒ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΕΒΔ τρίγωνον τῷ ΕΒΓ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΕΒ πρὸς τὴν ΒΓ. ἴση δὲ ἡ ΕΒ τῇ ΓΔ. ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΒ. μείζων δὲ ἡ ΒΔ τῆς ΔΓ· μείζων ἄρα καὶ ἡ ΔΓ τῆς ΓΒ. ἡ ΒΔ ἄρα εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται [κατὰ τὸ Γ], καὶ τὸ μεῖζον τμῆμα αὐτῆς ἐστιν ἡ ΔΓ· ὅπερ ἔδει δεῖξαι.

For let the center of the circle, point *E*, be found [Prop. 3.1], and let *EB*, *EC*, and *ED* be joined, and let *BE* be drawn across to *A*. Since *BC* is a side on an equilateral decagon, circumference *ACB* (is) thus five times circumference *BC*. Thus, circumference *AC* (is) four times *CB*. And as circumference *AC* (is) to *CB*, so angle *AEC* (is) to *CEB* [Prop. 6.33]. Thus, (angle) *AEC* (is) four times *CEB*. And since angle *EBC* (is) equal to *ECB* [Prop. 1.5], angle *AEC* is thus double *ECB* [Prop. 1.32]. And since straight-line *EC* is equal to *CD*—for each of them is equal to the side of the hexagon [inscribed] in circle *ABC* [Prop. 4.15 corr.]— angle *CED* is also equal to angle *CDE* [Prop. 1.5]. Thus, angle *ECB* (is) double *EDC* [Prop. 1.32]. But, *AEC* was shown (to be) double *ECB*. Thus, *AEC* (is) four times *EDC*. And *AEC* was also shown (to be) four times *BEC*. Thus, *EDC* (is) equal to *BEC*. And angle *EBD* (is) common to the two triangles *BEC* and *BED*. Thus, the remaining (angle) *BED* is equal to the (remaining angle) *ECB* [Prop. 1.32]. Thus, triangle *EBD* is equiangular to triangle *EBC*. Thus, proportionally, as *DB* is to *BE*, so *EB* (is) to *BC* [Prop. 6.4]. And *EB* (is) equal to *CD*. Thus, as *BD* is to *DC*, so *DC* (is) to *CB*. And *BD* (is) greater than *DC*. Thus, *DC* (is) also greater than *CB* [Prop. 5.14]. Thus, the straight-line *BD* has been cut in extreme and mean ratio [at *C*], and *DC* is its greater piece. (Which is), the very thing it was required to show.

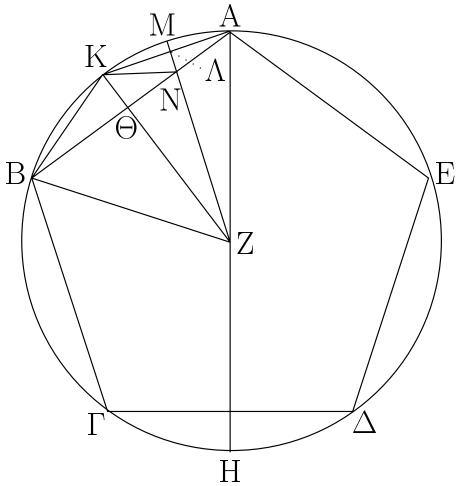
† If the circle is of unit radius then the side of the hexagon is 1, whereas the side of the decagon is $(1/2)(\sqrt{5}-1)$.

ι'.

Proposition 10

Ἐὰν εἰς κύκλον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων.

If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon

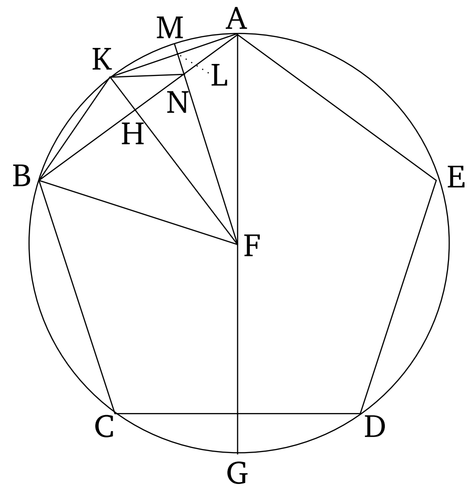


Ἔστω κύκλος ὁ ΑΒΓΔΕ, καὶ εἰς τὸ ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓΔΕ. λέγω, ὅτι ἡ τοῦ ΑΒΓΔΕ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου πλευρὰν τῶν εἰς τὸν ΑΒΓΔΕ κύκλον ἐγγραφομένων.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ζ σημεῖον, καὶ ἐπιζευχθεῖσα ἡ ΑΖ διήχθω ἐπὶ τὸ Η σημεῖον, καὶ ἐπεζεύχθω ἡ ΖΒ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΖΘ, καὶ διήχθω ἐπὶ τὸ Κ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΚΒ, καὶ πάλιν ἀπὸ τοῦ Ζ ἐπὶ τὴν ΑΚ κάθετος ἤχθω ἡ ΖΛ, καὶ διήχθω ἐπὶ τὸ Μ, καὶ ἐπεζεύχθω ἡ ΚΝ.

Ἐπεὶ ἴση ἐστὶν ἡ ΑΒΓΗ περιφέρεια τῇ ΑΕΔΗ περιφερείᾳ, ὧν ἡ ΑΒΓ τῇ ΑΕΔ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΓΗ περιφέρεια λοιπῇ τῇ ΗΔ ἐστιν ἴση. πενταγώνου δὲ ἡ ΓΔ· δεκαγώνου ἄρα ἡ ΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΖΑ τῇ ΖΒ, καὶ κάθετος ἡ ΖΘ, ἴση ἄρα καὶ ἡ ὑπὸ ΑΖΚ γωνία τῇ ὑπὸ ΚΖΒ. ὥστε καὶ περιφέρεια ἡ ΑΚ τῇ ΚΒ ἐστιν ἴση· διπλῆ ἄρα ἡ ΑΒ περιφέρεια τῆς ΒΚ περιφερείας· δεκαγώνου ἄρα πλευρά ἐστιν ἡ ΑΚ εὐθεῖα. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΑΚ τῆς ΚΜ ἐστι διπλῆ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ περιφέρεια τῆς ΒΚ περιφερείας, ἴση δὲ ἡ ΓΔ περιφέρεια τῇ ΑΒ περιφερείᾳ, διπλῆ ἄρα καὶ ἡ ΓΔ περιφέρεια τῆς ΒΚ περιφερείας. ἔστι δὲ ἡ ΓΔ περιφέρεια καὶ τῆς ΓΗ διπλῆ· ἴση ἄρα ἡ ΓΗ περιφέρεια τῇ ΒΚ περιφερείᾳ. ἀλλὰ ἡ ΒΚ τῆς ΚΜ ἐστι διπλῆ, ἐπεὶ καὶ ἡ ΚΑ· καὶ ἡ ΓΗ ἄρα τῆς ΚΜ ἐστι διπλῆ. ἀλλὰ μὴν καὶ ἡ ΓΒ περιφέρεια τῆς ΒΚ περιφερείας ἐστὶ διπλῆ· ἴση γὰρ ἡ ΓΒ περιφέρεια τῇ ΒΑ. καὶ ὅλη ἄρα ἡ ΗΒ περιφέρεια τῆς ΒΜ ἐστι διπλῆ· ὥστε καὶ γωνία ἡ ὑπὸ ΗΖΒ γωνίας τῆς ὑπὸ ΒΖΜ [ἐστι] διπλῆ. ἔστι δὲ ἡ ὑπὸ ΗΖΒ καὶ τῆς ὑπὸ ΖΑΒ διπλῆ· ἴση γὰρ ἡ ὑπὸ ΖΑΒ τῇ ὑπὸ ΑΒΖ. καὶ ἡ ὑπὸ ΒΖΝ ἄρα τῇ ὑπὸ ΖΑΒ ἐστιν ἴση. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ΑΒΖ καὶ τοῦ ΒΖΝ, ἡ ὑπὸ ΑΒΖ γωνία· λοιπὴ ἄρα ἡ ὑπὸ ΑΖΒ λοιπῇ τῇ ὑπὸ ΒΝΖ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΖ τρίγωνον τῷ ΒΖΝ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΒ εὐθεῖα πρὸς

inscribed in the same circle.†



Let $ABCDE$ be a circle. And let the equilateral pentagon $ABCDE$ be inscribed in circle $ABCDE$. I say that the square on the side of pentagon $ABCDE$ is the (sum of the squares) on the sides of the hexagon and of the decagon inscribed in circle $ABCDE$.

For let the center of the circle, point F, be found [Prop. 3.1]. And, AF being joined, let it be drawn across to point G. And let FB be joined. And let FH be drawn from F perpendicular to AB. And let it be drawn across to K. And let AK and KB be joined. And, again, let FL have been drawn from F perpendicular to AK. And let it be drawn across to M. And let KN be joined.

Since circumference $ABCG$ is equal to circumference $AEDG$, of which ABC is equal to AED, the remaining circumference CG is thus equal to the remaining (circumference) GD. And CD (is the side) of the pentagon. CG (is) thus (the side) of the decagon. And since FA is equal to FB, and FH is perpendicular (to AB), angle AFK (is) thus also equal to KFB [Props. 1.5, 1.26]. Hence, circumference AK is also equal to KB [Prop. 3.26]. Thus, circumference AB (is) double circumference BK. Thus, straight-line AK is the side of the decagon. So, for the same (reasons, circumference) AK is also double KM. And since circumference AB is double circumference BK, and circumference CD (is) equal to circumference AB, circumference CD (is) thus also double circumference BK. And circumference CD is also double CG. Thus, circumference CG (is) equal to circumference BK. But, BK is double KM, since KA (is) also (double KM). Thus, (circumference) CG is also double KM. But, indeed, circumference CB is also double circumference BK. For circumference CB (is) equal to BA. Thus, the whole circumference GB is also double BM. Hence, angle GFB [is] also double angle BFM [Prop. 6.33]. And GFB (is) also double FAB. For FAB (is) equal to ABF. Thus, BFN is also equal to FAB. And angle ABF (is) common *τὴν ΒΖ, οὕτως ἡ ΖΒ πρὸς τὴν ΒΝ· τὸ ἄρα ὑπὸ τῶν ΑΒΝ ἴσον ἐστὶ τῷ ἀπὸ ΒΖ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΛ τῇ ΛΚ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΛΝ, βάσις ἄρα ἡ ΚΝ βάσει τῇ ΑΝ ἐστιν ἴση· καὶ γωνία ἄρα ἡ ὑπὸ ΛΚΝ γωνίᾳ τῇ ὑπὸ ΛΑΝ ἐστιν ἴση. ἀλλὰ ἡ ὑπὸ ΛΑΝ τῇ ὑπὸ ΚΒΝ ἐστιν ἴση· καὶ ἡ ὑπὸ ΛΚΝ ἄρα τῇ ὑπὸ ΚΒΝ ἐστιν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ΑΚΒ καὶ τοῦ ΑΚΝ ἡ πρὸς τῷ Α. λοιπὴ ἄρα ἡ ὑπὸ ΑΚΒ λοιπῇ τῇ ὑπὸ ΚΝΑ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΚΒΑ τρίγωνον τῷ ΚΝΑ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΑ εὐθεῖα πρὸς τὴν ΑΚ, οὕτως ἡ ΚΑ πρὸς τὴν ΑΝ· τὸ ἄρα ὑπὸ τῶν ΒΑΝ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΚ. ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ΑΒΝ ἴσον τῷ ἀπὸ τῆς ΒΖ· τὸ ἄρα ὑπὸ τῶν ΑΒΝ μετὰ τοῦ ὑπὸ ΒΑΝ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΒΑ, ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΖ μετὰ τοῦ ἀπὸ τῆς ΑΚ. καί ἐστιν ἡ μὲν ΒΑ πενταγώνου πλευρά, ἡ δὲ ΒΖ ἑξαγώνου, ἡ δὲ ΑΚ δεκαγώνου.*

Ἡ ἄρα τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων· ὅπερ ἔδει δεῖξαι.

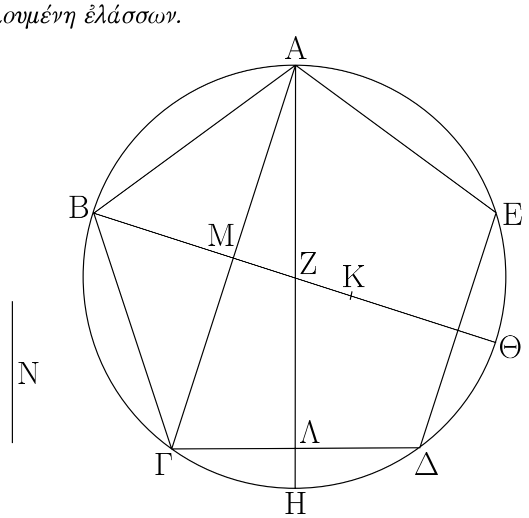
to the two triangles ABF and BFN. Thus, the remaining (angle) AFB is equal to the remaining (angle) BNF [Prop. 1.32]. Thus, triangle ABF is equiangular to triangle BFN. Thus, proportionally, as straight-line AB (is) to BF, so FB (is) to BN [Prop. 6.4]. Thus, the (rectangle contained) by ABN is equal to the (square) on BF [Prop. 6.17]. Again, since AL is equal to LK, and LN is common and at right-angles (to KA), base KN is thus equal to base AN [Prop. 1.4]. And, thus, angle LKN is equal to angle LAN. But, LAN is equal to KBN [Props. 3.29, 1.5]. Thus, LKN is also equal to KBN. And the (angle) at A (is) common to the two triangles AKB and AKN. Thus, the remaining (angle) AKB is equal to the remaining (angle) KNA [Prop. 1.32]. Thus, triangle KBA is equiangular to triangle KNA. Thus, proportionally, as straight-line BA is to AK, so KA (is) to AN [Prop. 6.4]. Thus, the (rectangle contained) by BAN is equal to the (square) on AK [Prop. 6.17]. And the (rectangle contained) by ABN was also shown (to be) equal to the (square) on BF. Thus, the (rectangle contained) by ABN plus the (rectangle contained) by BAN, which is the (square) on BA [Prop. 2.2], is equal to the (square) on BF plus the (square) on AK. And BA is the side of the pentagon, and BF (the side) of the hexagon [Prop. 4.15 corr.], and AK (the side) of the decagon.

Thus, the square on the side of the pentagon (inscribed in a circle) is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.

† If the circle is of unit radius then the side of the pentagon is (1/2) $\sqrt{10-2\sqrt{5}}$.

ιαʹ.

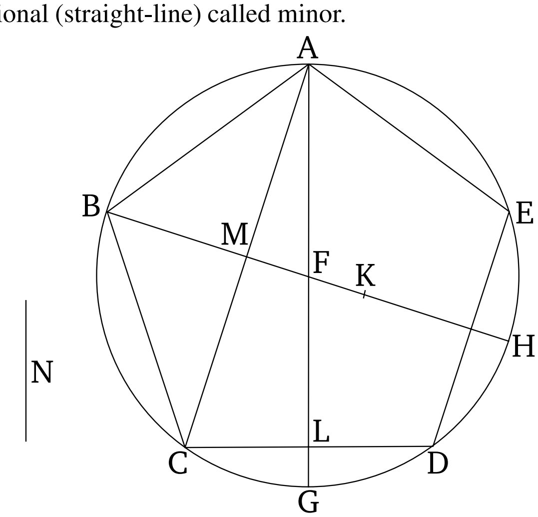
Ἐὰν εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.



Εἰς γὰρ κύκλον τὸν ΑΒΓΔΕ ῥητὴν ἔχοντα τὴν δίαμετρον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓΔΕ· λέγω, ὅτι ἡ

Proposition 11

If an equilateral pentagon is inscribed in a circle which has a rational diameter then the side of the pentagon is that irrational (straight-line) called minor.



For let the equilateral pentagon $ABCDE$ be inscribed in the circle $ABCDE$ which has a rational diameter. I say that *τοῦ [ΑΒΓΔΕ] πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.*

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ζ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ καὶ διήχθωσαν ἐπὶ τὰ Η, Θ σημεῖα, καὶ ἐπεζεύχθω ἡ ΑΓ, καὶ κείσθω τῆς ΑΖ τέταρτον μέρος ἡ ΖΚ. ῥητὴ δὲ ἡ ΑΖ· ῥητὴ ἄρα καὶ ἡ ΖΚ. ἔστι δὲ καὶ ἡ ΒΖ ῥητή· ὅλη ἄρα ἡ ΒΚ ῥητή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓΗ περιφέρεια τῇ ΑΔΗ περιφερείᾳ, ὧν ἡ ΑΒΓ τῇ ΑΕΔ ἐστιν ἴση, λοιπὴ ἄρα ἡ ΓΗ λοιπῇ τῇ ΗΔ ἐστιν ἴση. καὶ ἐὰν ἐπιζεύξωμεν τὴν ΑΔ, συνάγονται ὀρθαὶ αἱ πρὸς τῷ Λ γωνίαι, καὶ διπλῆ ἡ ΓΔ τῆς ΓΛ. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τῷ Μ ὀρθαί εἰσιν, καὶ διπλῆ ἡ ΑΓ τῆς ΓΜ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΑΛΓ γωνία τῇ ὑπὸ ΑΜΖ, κοινὴ δὲ τῶν δύο τριγώνων τοῦ τε ΑΓΛ καὶ τοῦ ΑΜΖ ἡ ὑπὸ ΛΑΓ, λοιπὴ ἄρα ἡ ὑπὸ ΑΓΛ λοιπῇ τῇ ὑπὸ ΜΖΑ ἐστιν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΓΛ τρίγωνον τῷ ΑΜΖ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΛΓ πρὸς ΓΑ, οὕτως ἡ ΜΖ πρὸς ΖΑ· καὶ τῶν ἡγουμένων τὰ διπλάσια· ὡς ἄρα ἡ τῆς ΛΓ διπλῆ πρὸς τὴν ΓΑ, οὕτως ἡ τῆς ΜΖ διπλῆ πρὸς τὴν ΖΑ. ὡς δὲ ἡ τῆς ΜΖ διπλῆ πρὸς τὴν ΖΑ, οὕτως ἡ ΜΖ πρὸς τὴν ἡμίσειαν τῆς ΖΑ· καὶ ὡς ἄρα ἡ τῆς ΛΓ διπλῆ πρὸς τὴν ΓΑ, οὕτως ἡ ΜΖ πρὸς τὴν ἡμίσειαν τῆς ΖΑ· καὶ τῶν ἑπομένων τὰ ἡμίσεα· ὡς ἄρα ἡ τῆς ΛΓ διπλῆ πρὸς τὴν ἡμίσειαν τῆς ΓΑ, οὕτως ἡ ΜΖ πρὸς τὸ τέταρτον τῆς ΖΑ. καί ἐστι τῆς μὲν ΛΓ διπλῆ ἡ ΔΓ, τῆς δὲ ΓΑ ἡμίσεια ἡ ΓΜ, τῆς δὲ ΖΑ τέτατρον μέρος ἡ ΖΚ· ἔστιν ἄρα ὡς ἡ ΔΓ πρὸς τὴν ΓΜ, οὕτως ἡ ΜΖ πρὸς τὴν ΖΚ. συνθέντι καὶ ὡς συναμφότερος ἡ ΔΓΜ πρὸς τὴν ΓΜ, οὕτως ἡ ΜΚ πρὸς ΚΖ· καὶ ὡς ἄρα τὸ ἀπὸ συναμφοτέρου τῆς ΔΓΜ πρὸς τὸ ἀπὸ ΓΜ, οὕτως τὸ ἀπὸ ΜΚ πρὸς τὸ ἀπὸ ΚΖ. καὶ ἐπεὶ τῆς ὑπὸ δύο πλευρὰς τοῦ πενταγώνου ὑποτεινούσης, οἷον τῆς ΑΓ, ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμα ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ, τουτέστι τῇ ΔΓ, τὸ δὲ μεῖζον τμῆμα προσλαβὸν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τῆς ὅλης, καί ἐστιν ὅλης τῆς ΑΓ ἡμίσεια ἡ ΓΜ, τὸ ἄρα ἀπὸ τῆς ΔΓΜ ὡς μιᾶς πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΜ. ὡς δὲ τὸ ἀπὸ τῆς ΔΓΜ ὡς μιᾶς πρὸς τὸ ἀπὸ τῆς ΓΜ, οὕτως ἐδείχθη τὸ ἀπὸ τῆς ΜΚ πρὸς τὸ ἀπὸ τῆς ΚΖ· πενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΜΚ τοῦ ἀπὸ τῆς ΚΖ. ῥητὸν δὲ τὸ ἀπὸ τῆς ΚΖ· ῥητὴ γὰρ ἡ διάμετρος· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ΜΚ· ῥητὴ ἄρα ἐστὶν ἡ ΜΚ [δυνάμει μόνον]. καὶ ἐπεὶ τετραπλασία ἐστὶν ἡ ΒΖ τῆς ΖΚ, πενταπλασία ἄρα ἐστὶν ἡ ΒΚ τῆς ΚΖ· εἰκοσιπενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΖ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΜΚ τοῦ ἀπὸ τῆς ΚΖ· πενταπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ· τὸ ἄρα ἀπὸ τῆς ΒΚ πρὸς τὸ ἀπὸ ΚΜ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΚ τῇ ΚΜ μήκει. καί ἐστι ῥητὴ ἑκατέρα αὐτῶν. αἱ ΒΚ, ΚΜ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ ἀπὸ ῥητῆς ῥητὴ ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν ἀποτομή· ἀποτομὴ ἄρα ἐστὶν ἡ ΜΒ, προσαρμόζουσα δὲ αὐτῇ ἡ ΜΚ. λέγω δή, ὅτι καὶ τετάρτη. ᾧ

the side of pentagon [$ABCDE$] is that irrational (straight-line) called minor.

For let the center of the circle, point F, be found [Prop. 3.1]. And let AF and FB be joined. And let them be drawn across to points G and H (respectively). And let AC be joined. And let FK made (equal) to the fourth part of AF. And AF (is) rational. FK (is) thus also rational. And BF is also rational. Thus, the whole of BK is rational. And since circumference ACG is equal to circumference ADG, of which ABC is equal to AED, the remainder CG is thus equal to the remainder GD. And if we join AD then the angles at L are inferred (to be) right-angles, and CD (is inferred to be) double CL [Prop. 1.4]. So, for the same (reasons), the (angles) at M are also right-angles, and AC (is) double CM. Therefore, since angle ALC (is) equal to AMF, and (angle) LAC (is) common to the two triangles ACL and AMF, the remaining (angle) ACL is thus equal to the remaining (angle) MFA [Prop. 1.32]. Thus, triangle ACL is equiangular to triangle AMF. Thus, proportionally, as LC (is) to CA, so MF (is) to FA [Prop. 6.4]. And (we can take) the doubles of the leading (magnitudes). Thus, as double LC (is) to CA, so double MF (is) to FA. And as double MF (is) to FA, so MF (is) to half of FA. And, thus, as double LC (is) to CA, so MF (is) to half of FA. And (we can take) the halves of the following (magnitudes). Thus, as double LC (is) to half of CA, so MF (is) to the fourth of FA. And DC is double LC, and CM half of CA, and FK the fourth part of FA. Thus, as DC is to CM, so MF (is) to FK. Via composition, as the sum of DCM (*i.e.*, DC and CM) (is) to CM, so MK (is) to KF [Prop. 5.18]. And, thus, as the (square) on the sum of DCM (is) to the (square) on CM, so the (square) on MK (is) to the (square) on KF. And since the greater piece of a (straight-line) subtending two sides of a pentagon, such as AC, (which is) cut in extreme and mean ratio is equal to the side of the pentagon [Prop. 13.8]—that is to say, to DC—-and the square on the greater piece added to half of the whole is five times the (square) on half of the whole [Prop. 13.1], and CM (is) half of the whole, AC, thus the (square) on DCM, (taken) as one, is five times the (square) on CM. And the (square) on DCM, (taken) as one, (is) to the (square) on CM, so the (square) on MK was shown (to be) to the (square) on KF. Thus, the (square) on MK (is) five times the (square) on KF. And the square on KF (is) rational. For the diameter (is) rational. Thus, the (square) on MK (is) also rational. Thus, MK is rational [in square only]. And since BF is four times FK, BK is thus five times KF. Thus, the (square) on BK (is) twenty-five times the (square) on KF. And the (square) on MK (is) five times the square on KF. Thus, the (square) on BK (is) five times the (square) on KM. Thus, the (square) on BK does not have to the (square) on KM the ratio which a square number (has) to a square number. Thus, BK is incommensurable in length with KM [Prop. 10.9]. And each of them is a rational *δὴ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ, ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ τῆς Ν· ἡ ΒΚ ἄρα τῆς ΚΜ μεῖζον δύναται τῇ Ν. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΚΖ τῇ ΖΒ, καὶ συνθέντι σύμμετρός ἐστι ἡ ΚΒ τῇ ΖΒ. ἀλλὰ ἡ ΒΖ τῇ ΒΘ σύμμετρός ἐστιν· καὶ ἡ ΒΚ ἄρα τῇ ΒΘ σύμμετρός ἐστιν. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΚ τοῦ ἀπὸ τῆς ΚΜ, τὸ ἄρα ἀπὸ τῆς ΒΚ πρὸς τὸ ἀπὸ τῆς ΚΜ λόγον ἔχει, ὃν $\overline{ε}$ πρὸς ἕν. ἀναστρέψαντι ἄρα τὸ ἀπὸ τῆς ΒΚ πρὸς τὸ ἀπὸ τῆς Ν λόγον ἔχει, ὃν $\overline{ε}$ πρὸς $\overline{δ}$, οὐχ ὃν τετράγωνος πρὸς τετράγωνον· ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΚ τῇ Ν· ἡ ΒΚ ἄρα τῆς ΚΜ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. ἐπεὶ οὖν ὅλη ἡ ΒΚ τῆς προσαρμοζούσης τῆς ΚΜ μεῖζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ὅλη ἡ ΒΚ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΒΘ, ἀποτομὴ ἄρα τετάρτη ἐστὶν ἡ ΜΒ. τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ ἐλάττων. δύναται δὲ τὸ ὑπὸ τῶν ΘΒΜ ἡ ΑΒ διὰ τὸ ἐπιζευγνυμένης τῆς ΑΘ ἰσογώνιον γίνεσθαι τὸ ΑΒΘ τρίγωνον τῷ ΑΒΜ τριγώνῳ καὶ εἶναι ὡς τὴν ΘΒ πρὸς τὴν ΒΑ, οὕτως τὴν ΑΒ πρὸς τὴν ΒΜ.*

Ἡ ἄρα ΑΒ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων· ὅπερ ἔδει δεῖξαι.

(straight-line). Thus, BK and KM are rational (straight-lines which are) commensurable in square only. And if from a rational (straight-line) a rational (straight-line) is subtracted, which is commensurable in square only with the whole, then the remainder is that irrational (straight-line called) an apotome [Prop. 10.73]. Thus, MB is an apotome, and MK its attachment. So, I say that (it is) also a fourth (apotome). So, let the (square) on N be (made) equal to that (magnitude) by which the (square) on BK is greater than the (square) on KM. Thus, the square on BK is greater than the (square) on KM by the (square) on N. And since KF is commensurable (in length) with FB then, via composition, KB is also commensurable (in length) with FB [Prop. 10.15]. But, BF is commensurable (in length) with BH. Thus, BK is also commensurable (in length) with BH [Prop. 10.12]. And since the (square) on BK is five times the (square) on KM, the (square) on BK thus has to the (square) on KM the ratio which 5 (has) to one. Thus, via conversion, the (square) on BK has to the (square) on N the ratio which 5 (has) to 4 [Prop. 5.19 corr.], which is not (that) of a square (number) to a square (number). BK is thus incommensurable (in length) with N [Prop. 10.9]. Thus, the square on BK is greater than the (square) on KM by the (square) on (some straight-line which is) incommensurable (in length) with (BK). Therefore, since the square on the whole, BK, is greater than the (square) on the attachment, KM, by the (square) on (some straight-line which is) incommensurable (in length) with (BK), and the whole, BK, is commensurable (in length) with the (previously) laid down rational (straight-line) BH, MB is thus a fourth apotome [Def. 10.14]. And the rectangle contained by a rational (straight-line) and a fourth apotome is irrational, and its square-root is that irrational (straight-line) called minor [Prop. 10.94]. And the square on AB is the rectangle contained by HBM, on account of joining AH, (so that) triangle ABH becomes equiangular with triangle ABM [Prop. 6.8], and (proportionally) as HB is to BA, so AB (is) to BM.

Thus, the side AB of the pentagon is that irrational (straight-line) called minor.† (Which is) the very thing it was required to show.

† If the circle has unit radius then the side of the pentagon is $(1/2)\sqrt{10-2\sqrt{5}}$. However, this length can be written in the "minor" form (see Prop. 10.94) $(\rho/\sqrt{2})\sqrt{1+k/\sqrt{1+k^2}} - (\rho/\sqrt{2})\sqrt{1-k/\sqrt{1+k^2}}$, with $\rho = \sqrt{5/2}$ and $k = 2$.

ιβʹ.

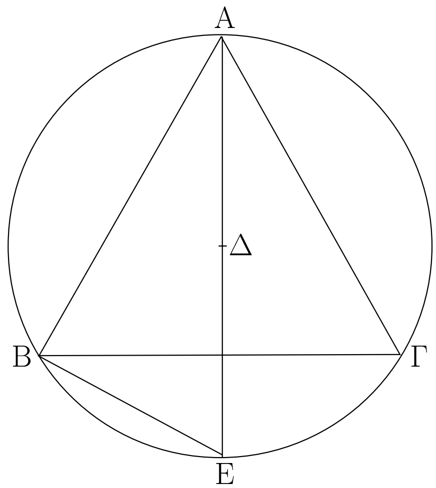
Ἐὰν εἰς κύκλον τρίγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ τριγώνου πλευρὰ δυνάμει τριπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου.

Ἔστω κύκλος ὁ ΑΒΓ, καὶ εἰς αὐτὸν τρίγωνον ἰσόπλευρον ἐγγεγράφθω τὸ ΑΒΓ· λέγω, ὅτι τοῦ ΑΒΓ τριγώνου μία πλευρὰ δυνάμει τριπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ ΑΒΓ

Proposition 12

If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the (square) on the radius of the circle.

Let there be a circle ABC, and let the equilateral triangle ABC have been inscribed in it [Prop. 4.2]. I say that the square on one side of triangle ABC is three times the (square) on the *κύκλου.*

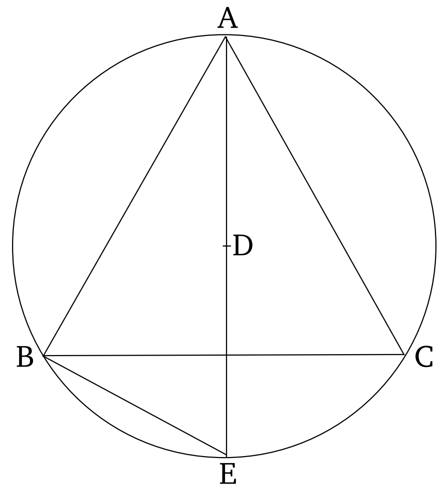


Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓ κύκλου τὸ Δ, καὶ ἐπιζευχθεῖσα ἡ ΑΔ διήχθω ἐπὶ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΒΕ.

Καὶ ἐπεὶ ἰσόπλευρόν ἐστι τὸ ΑΒΓ τρίγωνον, ἡ ΒΕΓ ἄρα περιφέρεια τρίτον μέρος ἐστὶ τῆς τοῦ ΑΒΓ κύκλου περιφερείας. ἡ ἄρα ΒΕ περιφέρεια ἕκτον ἐστὶ μέρος τῆς τοῦ κύκλου περιφερείας· ἑξαγώνου ἄρα ἐστὶν ἡ ΒΕ εὐθεῖα· ἴση ἄρα ἐστὶ τῇ ἐκ τοῦ κέντρου τῇ ΔΕ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΕ τῆς ΔΕ, τετραπλάσιον ἐστι τὸ ἀπὸ τῆς ΑΕ τοῦ ἀπὸ τῆς ΕΔ, τουτέστι τοῦ ἀπὸ τῆς ΒΕ. ἴσον δὲ τὸ ἀπὸ τῆς ΑΕ τοῖς ἀπὸ τῶν ΑΒ, ΒΕ· τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΕ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΒΕ. διελόντι ἄρα τὸ ἀπὸ τῆς ΑΒ τριπλάσιόν ἐστι τοῦ ἀπὸ ΒΕ. ἴση δὲ ἡ ΒΕ τῇ ΔΕ· τὸ ἄρα ἀπὸ τῆς ΑΒ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΕ.

Ἡ ἄρα τοῦ τριγώνου πλευρὰ δυνάμει τριπλασία ἐστὶ τῆς ἐκ τοῦ κέντρου [τοῦ κύκλου]· ὅπερ ἔδει δεῖξαι.

radius of circle *ABC*.



For let the center, *D*, of circle *ABC* be found [Prop. 3.1]. And *AD* (being) joined, let it be drawn across to *E*. And let *BE* be joined.

And since triangle *ABC* is equilateral, circumference *BEC* is thus the third part of the circumference of circle *ABC*. Thus, circumference *BE* is the sixth part of the circumference of the circle. Thus, straight-line *BE* is (the side) of a hexagon. Thus, it is equal to the radius *DE* [Prop. 4.15 corr.]. And since *AE* is double *DE*, the (square) on *AE* is four times the (square) on *ED*—that is to say, of the (square) on *BE*. And the (square) on *AE* (is) equal to the (sum of the squares) on *AB* and *BE* [Props. 3.31, 1.47]. Thus, the (sum of the squares) on *AB* and *BE* is four times the (square) on *BE*. Thus, via separation, the (square) on *AB* is three times the (square) on *BE*. And *BE* (is) equal to *DE*. Thus, the (square) on *AB* is three times the (square) on *DE*.

Thus, the square on the side of the triangle is three times the (square) on the radius [of the circle]. (Which is) the very thing it was required to show.

ιγʹ.

Πυραμίδα συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος.

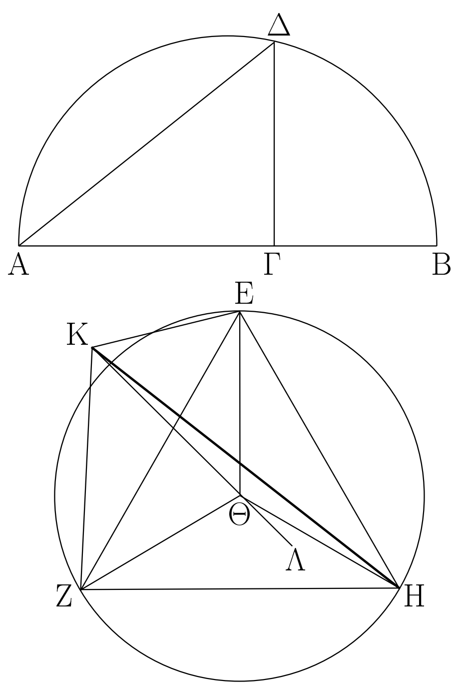
Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας δίαμετρος ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ σημεῖον, ὥστε διπλασίαν εἶναι τὴν ΑΓ τῆς ΓΒ· καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ σημείου τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΑ· καὶ ἐκκείσθω κύκλος ὁ ΕΖΗ ἴσην ἔχων τὴν ἐκ τοῦ κέντρου τῇ ΔΓ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗ κύκλον τρίγωνον ἰσόπλευρον τὸ ΕΖΗ· καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Θ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΘ, ΘΖ, ΘΗ· καὶ ἀνεστάτω ἀπὸ τοῦ Θ σημείου τῷ τοῦ ΕΖΗ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΘΚ, καὶ ἀφῃρήσθω ἀπὸ τῆς ΘΚ τῇ ΑΓ εὐθείᾳ

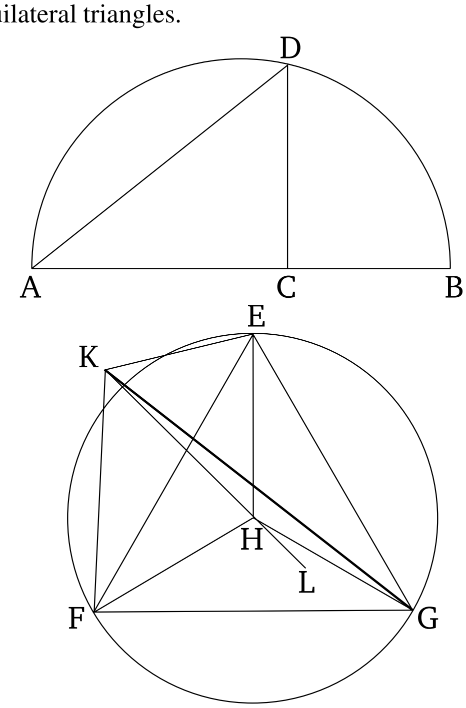
Proposition 13

To construct a (regular) pyramid (*i.e.*, a tetrahedron), and to enclose (it) in a given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

Let the diameter *AB* of the given sphere be laid out, and let it be cut at point *C* such that *AC* is double *CB* [Prop. 6.10]. And let the semi-circle *ADB* be drawn on *AB*. And let *CD* be drawn from point *C* at right-angles to *AB*. And let *DA* be joined. And let the circle *EFG* be laid down having a radius equal to *DC*, and let the equilateral triangle *EFG* be inscribed in circle *EFG* [Prop. 4.2]. And let the center of the circle, point *H*, be found [Prop. 3.1]. And let *EH*, *HF*, and *HG* be joined. And let *HK* be set up, at point *H*, at right-angles to *ἴση ἡ ΘΚ, καὶ ἐπεζεύχθωσαν αἱ ΚΕ, ΚΖ, ΚΗ. καὶ ἐπεὶ ἡ ΚΘ ὀρθή ἐστι πρὸς τὸ τοῦ ΕΖΗ κύκλου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ τοῦ ΕΖΗ κύκλου ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἑκάστη τῶν ΘΕ, ΘΖ, ΘΗ· ἡ ΘΚ ἄρα πρὸς ἑκάστη τῶν ΘΕ, ΘΖ, ΘΗ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΓ τῇ ΘΚ, ἡ δὲ ΓΔ τῇ ΘΕ, καὶ ὀρθὰς γωνίας περιέχουσιν, βάσις ἄρα ἡ ΔΑ βάσει τῇ ΚΕ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ΚΖ, ΚΗ τῇ ΔΑ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΚΕ, ΚΖ, ΚΗ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, τριπλῆ ἄρα ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ, ὡς ἑξῆς δειχθήσεται. τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΔΓ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΖΕ τοῦ ἀπὸ τῆς ΕΘ τριπλάσιον, καί ἐστιν ἴση ἡ ΔΓ τῇ ΕΘ· ἴση ἄρα καὶ ἡ ΔΑ τῇ ΕΖ. ἀλλὰ ἡ ΔΑ ἑκάστη τῶν ΚΕ, ΚΖ, ΚΗ ἐδείχθη ἴση· καὶ ἑκάστη ἄρα τῶν ΕΖ, ΖΗ, ΗΕ ἑκάστη τῶν ΚΕ, ΚΖ, ΚΗ ἐστιν ἴση· ἰσόπλευρα ἄρα ἐστὶ τὰ τέσσαρα τρίγωνα τὰ ΕΖΗ, ΚΕΖ, ΚΖΗ, ΚΕΗ. πυραμὶς ἄρα συνέσταται ἐκ τεσσάρων τριγώνων ἰσοπλεύρων, ἧς βάσις μέν ἐστι τὸ ΕΖΗ τρίγωνον, κορυφὴ δὲ τὸ Κ σημεῖον.*

the plane of circle EFG [Prop. 11.12]. And let HK, equal to the straight-line AC, be cut off from HK. And let KE, KF, and KG be joined. And since KH is at right-angles to the plane of circle EFG, it will thus also make right-angles with all of the straight-lines joining it (which are) also in the plane of circle EFG [Def. 11.3]. And HE, HF, and HG each join it. Thus, HK is at right-angles to each of HE, HF, and HG. And since AC is equal to HK, and CD to HE, and they contain right-angles, the base DA is thus equal to the base KE [Prop. 1.4]. So, for the same (reasons), KF and KG is each equal to DA. Thus, the three (straight-lines) KE, KF, and KG are equal to one another. And since AC is double CB, AB (is) thus triple BC. And as AB (is) to BC, so the (square) on AD (is) to the (square) on DC, as will be shown later [see lemma]. Thus, the (square) on AD (is) three times the (square) on DC. And the (square) on FE is also three times the (square) on EH [Prop. 13.12], and DC is equal to EH. Thus, DA (is) also equal to EF. But, DA was shown (to be) equal to each of KE, KF, and KG. Thus, EF, FG, and GE are equal to KE, KF, and KG, respectively. Thus, the four triangles EFG, KEF, KFG, and KEG are equilateral. Thus, a pyramid, whose base is triangle EFG, and apex the point K, has been constructed from four equilateral triangles.





Δεῖ δὴ αὐτὴν καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐκβεβλήσθω γὰρ ἐπ᾽ εὐθείας τῇ ΚΘ εὐθεῖα ἡ ΘΛ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΘΛ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΓ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΔ πρὸς τὴν ΓΒ, ἴση δὲ ἡ μὲν ΑΓ τῇ ΚΘ, ἡ

So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For let the straight-line HL be produced in a straight-line with KH, and let HL be made equal to CB. And since as AC (is) to CD, so CD (is) to CB [Prop. 6.8 corr.], and AC (is) equal *δὲ ΓΔ τῇ ΘΕ, ἡ δὲ ΓΒ τῇ ΘΛ, ἔστιν ἄρα ὡς ἡ ΚΘ πρὸς τὴν ΘΕ, οὕτως ἡ ΕΘ πρὸς τὴν ΘΛ· τὸ ἄρα ὑπὸ τῶν ΚΘ, ΘΛ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΚΘΕ, ΕΘΛ γωνιῶν· τὸ ἄρα ἐπὶ τῆς ΚΛ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε [ἐπειδήπερ ἐὰν ἐπιζεύξωμεν τὴν ΕΛ, ὀρθὴ γίνεται ἡ ὑπὸ ΛΕΚ γωνία διὰ τὸ ἰσογώνιον γίνεσθαι τὸ ΕΛΚ τρίγωνον ἑκατέρῳ τῶν ΕΛΘ, ΕΘΚ τριγώνων]. ἐὰν δὴ μενούσης τῆς ΚΛ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η σημείων ἐπιζευγνυμένων τῶν ΖΛ, ΛΗ καὶ ὀρθῶν ὁμοίως γινομένων τῶν πρὸς τοῖς Ζ, Η γωνιῶν· καὶ ἔσται ἡ πυραμὶς σφαίρᾳ περιειλημμένη τῇ δοθείσῃ. ἡ γὰρ ΚΛ τῆς σφαίρας διάμετρος ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμετρῳ τῇ ΑΒ, ἐπειδήπερ τῇ μὲν ΑΓ ἴση κεῖται ἡ ΚΘ, τῇ δὲ ΓΒ ἡ ΘΛ.*

Λέγω δή, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐπεὶ γὰρ διπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ· ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΓ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΔ [ἐπειδήπερ ἐπιζευγνμένης τῆς ΔΒ ἐστιν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ διὰ τὴν ὁμοιότητα τῶν ΔΑΒ, ΔΑΓ τριγώνων, καὶ εἶναι ὡς τὴν πρώτην πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς τρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας]. ἡμιόλιον ἄρα καὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ. καί ἐστιν ἡ μὲν ΒΑ ἡ τῆς δοθείσης σφαίρας διάμετρος, ἡ δὲ ΑΔ ἴση τῇ πλευρᾷ τῆς πυραμίδος.

Ἡ ἄρα τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος· ὅπερ ἔδει δεῖξαι.

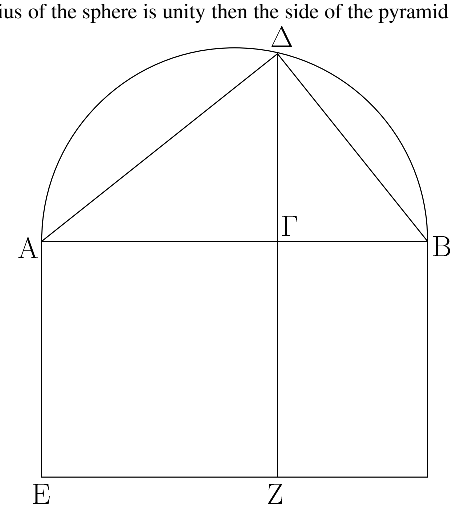
to KH, and CD to HE, and CB to HL, thus as KH is to HE, so EH (is) to HL. Thus, the (rectangle contained) by KH and HL is equal to the (square) on EH [Prop. 6.17]. And each of the angles KHE and EHL is a right-angle. Thus, the semi-circle drawn on KL will also pass through E [inasmuch as if we join EL then the angle LEK becomes a right-angle, on account of triangle ELK becoming equiangular to each of the triangles ELH and EHK [Props. 6.8, 3.31]]. So, if KL remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points F and G, (because) if FL and LG are joined, the angles at F and G will similarly become right-angles. And the pyramid will be enclosed by the given sphere. For the diameter, KL, of the sphere is equal to the diameter, AB, of the given sphere—inasmuch as KH was made equal to AC, and HL to CB.

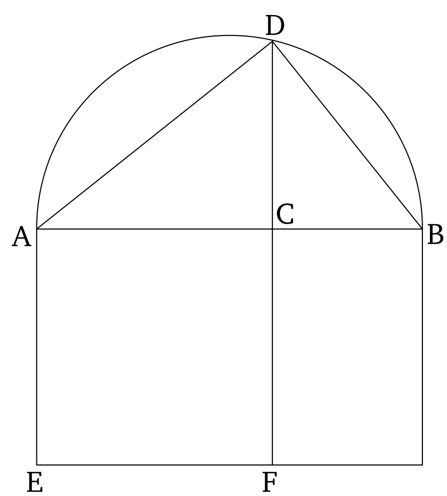
So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since AC is double CB, AB is thus triple BC. Thus, via conversion, BA is one and a half times AC. And as BA (is) to AC, so the (square) on BA (is) to the (square) on AD [inasmuch as if DB is joined then as BA is to AD, so DA (is) to AC, on account of the similarity of triangles DAB and DAC. And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on BA (is) also one and a half times the (square) on AD. And BA is the diameter of the given sphere, and AD (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.† (Which is) the very thing it was required to show.

† If the radius of the sphere is unity then the side of the pyramid (*i.e.*, tetrahedron) is $\sqrt{8/3}$.





Λῆμμα.

Δεικτέον, ὅτι ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ

Lemma

It must be shown that as AB is to BC, so the (square) on *τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ.*

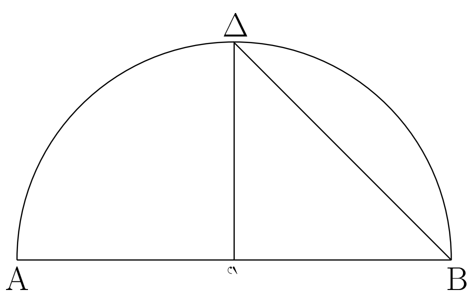
Ἐκκείσθω γὰρ ἡ τοῦ ἡμικυκλίου καταγραφή, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΓ τετράγωνον τὸ ΕΓ, καὶ συμπεπληρώσθω τὸ ΖΒ παραλληλόγραμμον. ἐπεὶ οὖν διὰ τὸ ἰσογώνιον εἶναι τὸ ΔΑΒ τρίγωνον τῷ ΔΑΓ τριγώνῳ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ, τὸ ἄρα ὑπὸ τῶν ΒΑ, ΑΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΔ. καὶ ἐπεί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΕΒ πρὸς τὸ ΒΖ, καί ἐστι τὸ μὲν ΕΒ τὸ ὑπὸ τῶν ΒΑ, ΑΓ· ἴση γὰρ ἡ ΕΑ τῇ ΑΓ· τὸ δὲ ΒΖ τὸ ὑπὸ τῶν ΑΓ, ΓΒ, ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ὑπὸ τῶν ΒΑ, ΑΓ πρὸ τὸ ὑπὸ τῶν ΑΓ, ΓΒ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΒΑ, ΑΓ ἴσον τῷ ἀπὸ τῆς ΑΔ, τὸ δὲ ὑπὸ τῶν ΑΓΒ ἴσον τῷ ἀπὸ τῆς ΔΓ· ἡ γὰρ ΔΓ κάθετος τῶν τῆς βάσεως τμημάτων τῶν ΑΓ, ΓΒ μέση ἀνάλογόν ἐστι διὰ τὸ ὀρθὴν εἶναι τὴν ὑπὸ ΑΔΒ. ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.

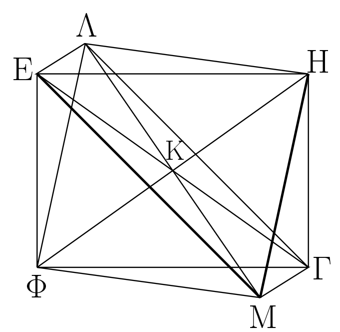
AD (is) to the (square) on *DC*.

For, let the figure of the semi-circle be set out, and let *DB* have been joined. And let the square *EC* be described on *AC*. And let the parallelogram *FB* be completed. Therefore, since, on account of triangle *DAB* being equiangular to triangle *DAC* [Props. 6.8, 6.4], (proportionally) as *BA* is to *AD*, so *DA* (is) to *AC*, the (rectangle contained) by *BA* and *AC* is thus equal to the (square) on *AD* [Prop. 6.17]. And since as *AB* is to *BC*, so *EB* (is) to *BF* [Prop. 6.1]. And *EB* is the (rectangle contained) by *BA* and *AC*—for *EA* (is) equal to *AC*. And *BF* the (rectangle contained) by *AC* and *CB*. Thus, as *AB* (is) to *BC*, so the (rectangle contained) by *BA* and *AC* (is) to the (rectangle contained) by *AC* and *CB*. And the (rectangle contained) by *BA* and *AC* is equal to the (square) on *AD*, and the (rectangle contained) by *ACB* (is) equal to the (square) on *DC*. For the perpendicular *DC* is the mean proportional to the pieces of the base, *AC* and *CB*, on account of *ADB* being a right-angle [Prop. 6.8 corr.]. Thus, as *AB* (is) to *BC*, so the (square) on *AD* (is) to the (square) on *DC*. (Which is) the very thing it was required to show.

ιδʹ.

Ὀκτάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ πρότερα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασία ἐστὶ τῆς πλευρᾶς τοῦ ὀκταέδρου.

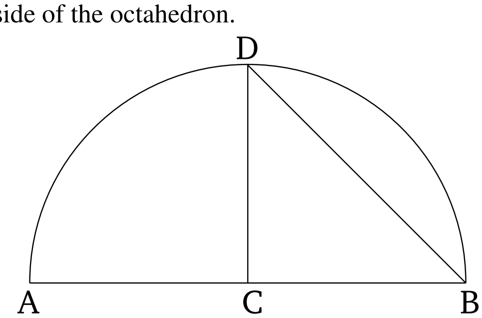


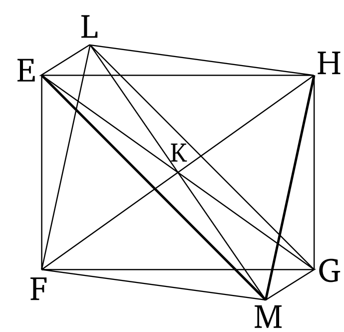


Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Γ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω τετράγωνον τὸ

Proposition 14

To construct an octahedron, and to enclose (it) in a (given) sphere, like in the preceding (proposition), and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.





Let the diameter *AB* of the given sphere be laid out, and let it have been cut in half at *C*. And let the semi-circle *ADB* be drawn on *AB*. And let *CD* be drawn from *C* at right-angles to *AB*. And let *DB* be joined. And let the square *EFGH*, having

ΕΖΗΘ ἴσην ἔχον ἑκάστην τῶν πλευρῶν τῇ ΔΒ, καὶ ἐπεζεύχθωσαν αἱ ΘΖ, ΕΗ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖα ἡ ΚΛ καὶ διήχθω ἐπὶ τὰ ἕτερα μέρη τοῦ ἐπιπέδου ὡς ἡ ΚΜ, καὶ ἀφῃρήσθω ἀφ' ἑκατέρας τῶν ΚΛ, ΚΜ μιᾷ τῶν ΕΚ, ΖΚ, ΗΚ, ΘΚ ἴση ἑκατέρα τῶν ΚΛ, ΚΜ, καὶ ἐπεζεύχθωσαν αἱ ΛΕ, ΛΖ, ΛΗ, ΛΘ, ΜΕ, ΜΖ, ΜΗ, ΜΘ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΕ τῇ ΚΘ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΕΚΘ γωνία, τὸ ἄρα ἀπὸ τῆς ΘΕ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΛΚ τῇ ΚΕ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΛΚΕ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΛ διπλάσιόν ἐστι τοῦ ἀπὸ ΕΚ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΘΕ διπλάσιον τοῦ ἀπὸ τῆς ΕΚ· τὸ ἄρα ἀπὸ τῆς ΛΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ· ἴση ἄρα ἐστὶν ἡ ΛΕ τῇ ΕΘ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΛΘ τῇ ΘΕ ἐστιν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΛΕΘ τρίγωνον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ τοῦ ΕΖΗΘ τετραγώνου πλευραί, κορυφαὶ δὲ τὰ Λ, Μ σημεῖα, ἰσόπλευρόν ἐστιν· ὀκτάεδρον ἄρα συνέσταται ὑπὸ ὀκτὼ τριγώνων ἰσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς.

Ἐπεὶ γὰρ αἱ τρεῖς αἱ ΛΚ, ΚΜ, ΚΕ ἴσαι ἀλλήλαις εἰσίν, τὸ ἄρα ἐπὶ τῆς ΛΜ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε. καὶ διὰ τὰ αὐτά, ἐὰν μενούσης τῆς ΛΜ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η, Θ σημείων, καὶ ἔσται σφαίρᾳ περιειλημμένον τὸ ὀκτάεδρον. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΛΚ τῇ ΚΜ, κοινὴ δὲ ἡ ΚΕ, καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΛΕ βάσει τῇ ΕΜ ἐστιν ἴση. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΛΕΜ γωνία· ἐν ἡμικυκλίῳ γάρ· τὸ ἄρα ἀπὸ τῆς ΛΜ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΛΕ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, διπλασία ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΛΜ διπλάσιον τοῦ ἀπὸ τῆς ΛΕ. καί ἐστιν ἴσον τὸ ἀπὸ τῆς ΔΒ τῷ ἀπὸ τῆς ΛΕ· ἴση γὰρ κεῖται ἡ ΕΘ τῇ ΔΒ. ἴσον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τῷ ἀπὸ τῆς ΛΜ· ἴση ἄρα ἡ ΑΒ τῇ ΛΜ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΛΜ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Περιείληπται ἄρα τὸ ὀκτάεδρον τῇ δοθείσῃ σφαίρᾳ. καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς· ὅπερ ἔδει δεῖξαι.

each of its sides equal to *DB*, be laid out. And let *HF* and *EG* be joined. And let the straight-line *KL* be set up, at point *K*, at right-angles to the plane of square *EFGH* [Prop. 11.12]. And let it be drawn across on the other side of the plane, like *KM*. And let *KL* and *KM*, equal to one of *EK*, *FK*, *GK*, and *HK*, be cut off from *KL* and *KM*, respectively. And let *LE*, *LF*, *LG*, *LH*, *ME*, *MF*, *MG*, and *MH* be joined.

And since *KE* is equal to *KH*, and angle *EKH* is a right-angle, the (square) on the *HE* is thus double the (square) on *EK* [Prop. 1.47]. Again, since *LK* is equal to *KE*, and angle *LKE* is a right-angle, the (square) on *EL* is thus double the (square) on *EK* [Prop. 1.47]. And the (square) on *HE* was also shown (to be) double the (square) on *EK*. Thus, the (square) on *LE* is equal to the (square) on *EH*. Thus, *LE* is equal to *EH*. So, for the same (reasons), *LH* is also equal to *HE*. Triangle *LEH* is thus equilateral. So, similarly, we can show that each of the remaining triangles, whose bases are the sides of the square *EFGH*, and apexes the points *L* and *M*, are equilateral. Thus, an octahedron contained by eight equilateral triangles has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

For since the three (straight-lines) *LK*, *KM*, and *KE* are equal to one another, the semi-circle drawn on *LM* will thus also pass through *E*. And, for the same (reasons), if *LM* remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through points *F*, *G*, and *H*, and the octahedron will be enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since *LK* is equal to *KM*, and *KE* (is) common, and they contain right-angles, the base *LE* is thus equal to the base *EM* [Prop. 1.4]. And since angle *LEM* is a right-angle—for (it is) in a semi-circle [Prop. 3.31]—the (square) on *LM* is thus double the (square) on *LE* [Prop. 1.47]. Again, since *AC* is equal to *CB*, *AB* is double *BC*. And as *AB* (is) to *BC*, so the (square) on *AB* (is) to the (square) on *BD* [Prop. 6.8, Def. 5.9]. Thus, the (square) on *AB* is double the (square) on *BD*. And the (square) on *LM* was also shown (to be) double the (square) on *LE*. And the (square) on *DB* is equal to the (square) on *LE*. For *EH* was made equal to *DB*. Thus, the (square) on *AB* (is) also equal to the (square) on *LM*. Thus, *AB* (is) equal to *LM*. And *AB* is the diameter of the given sphere. Thus, *LM* is equal to the diameter of the given sphere.

Thus, the octahedron has been enclosed by the given sphere, and it has been simultaneously proved that the square on the diameter of the sphere is double the (square) on the side of the octahedron.† (Which is) the very thing it was required to show.

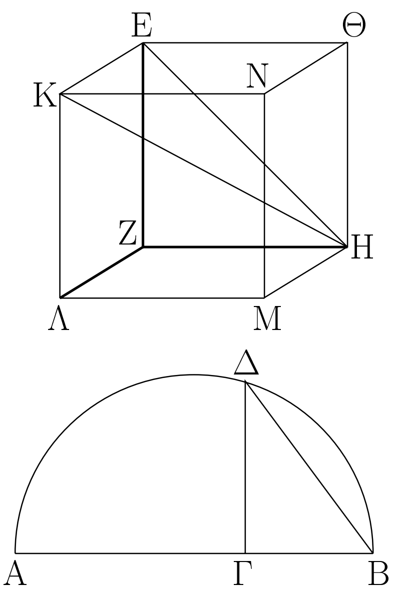
† If the radius of the sphere is unity then the side of octahedron is $\sqrt{2}$.

ιε'.

Κύβον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὴν πυραμίδα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατὰ τὸ Γ ὥστε διπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω τετράγωνον τὸ ΕΖΗΘ ἴσην ἔχον τὴν πλευρὰν τῇ ΔΒ, καὶ ἀπὸ τῶν Ε, Ζ, Η, Θ τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἤχθωσαν αἱ ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἀφῃρήσθω ἀπὸ ἑκάστης τῶν ΕΚ, ΖΛ, ΗΜ, ΘΝ μιᾷ τῶν ΕΖ, ΖΗ, ΗΘ, ΘΕ ἴση ἑκάστη τῶν ΕΚ, ΖΛ, ΗΜ, ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΛΜ, ΜΝ, ΝΚ· κύβος ἄρα συνέσταται ὁ ΖΝ ὑπὸ ἓξ τετραγώνων ἴσων περιεχόμενος.

Δεῖ δὴ αὐτὸν καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασία ἐστὶ τῆς πλευρᾶς τοῦ κύβου.



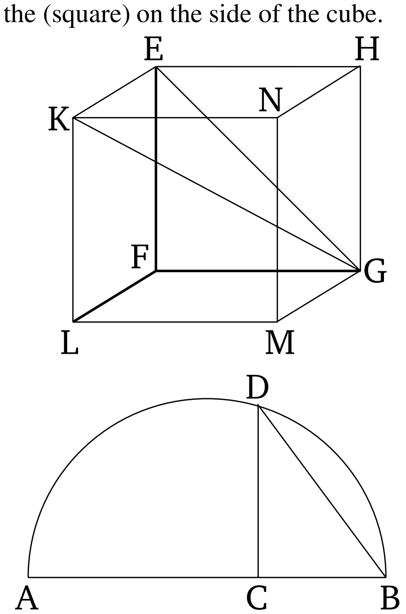
Ἐπεζεύχθωσαν γὰρ αἱ ΚΗ, ΕΗ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ ὑπὸ ΚΕΗ γωνία διὰ τὸ καὶ τὴν ΚΕ ὀρθὴν εἶναι πρὸς τὸ ΕΗ ἐπίπεδον δηλαδὴ καὶ πρὸς τὴν ΕΗ εὐθεῖαν, τὸ ἄρα ἐπὶ τῆς ΚΗ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε σημείου. πάλιν, ἐπεὶ ἡ ΗΖ ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΖΛ, ΖΕ, καὶ πρὸς τὸ ΖΚ ἄρα ἐπίπεδον ὀρθή ἐστιν ἡ ΗΖ· ὥστε καὶ ἐὰν ἐπιζεύξωμεν τὴν ΖΚ, ἡ ΗΖ ὀρθὴ ἔσται καὶ πρὸς τὴν ΖΚ· καὶ δία τοῦτο πάλιν τὸ ἐπὶ τῆς ΗΚ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ζ. ὁμοίως καὶ δία τῶν λοιπῶν τοῦ κύβου σημείων ἥξει. ἐὰν δὴ μενούσης τῆς ΚΗ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἔσται σφαίρᾳ

Proposition 15

To construct a cube, and to enclose (it) in a sphere, like in the (case of the) pyramid, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

Let the diameter AB of the given sphere be laid out, and let it be cut at C such that AC is double CB. And let the semi-circle ADB be drawn on AB. And let CD be drawn from C at right-angles to AB. And let DB be joined. And let the square $EFGH$, having (its) side equal to DB, be laid out. And let EK, FL, GM, and HN be drawn from (points) E, F, G, and H, (respectively), at right-angles to the plane of square $EFGH$. And let EK, FL, GM, and HN, equal to one of EF, FG, GH, and HE, be cut off from EK, FL, GM, and HN, respectively. And let KL, LM, MN, and NK be joined. Thus, a cube contained by six equal squares has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.



For let KG and EG be joined. And since angle KEG is a right-angle—on account of KE also being at right-angles to the plane EG, and manifestly also to the straight-line EG [Def. 11.3]—the semi-circle drawn on KG will thus also pass through point E. Again, since GF is at right-angles to each of FL and FE, GF is thus also at right-angles to the plane FK [Prop. 11.4]. Hence, if we also join FK then GF will also be at right-angles to FK. And, again, on account of this, the semi-circle drawn on GK will also pass through point F. Similarly, it will also pass through the remaining (angular) points of the cube. So, if KG remains (fixed), and the semi-circle is car-

περιειλημμένος ὁ κύβος. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΗΖ τῇ ΖΕ, καί ἐστιν ὀρθὴ ἡ πρὸς τῷ Ζ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΖ. ἴση δὲ ἡ ΕΖ τῇ ΕΚ· τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ· ὥστε τὰ ἀπὸ τῶν ΗΕ, ΕΚ, τουτέστι τὸ ἀπὸ τῆς ΗΚ, τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. καὶ ἐπεὶ τριπλασίων ἐστὶν ἡ ΑΒ τῆς ΒΓ, ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ, τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΗΚ τοῦ ἀπὸ τῆς ΚΕ τριπλάσιον. καὶ κεῖται ἴση ἡ ΚΕ τῇ ΔΒ· ἴση ἄρα καὶ ἡ ΚΗ τῇ ΑΒ. καί ἐστιν ἡ ΑΒ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΚΗ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Τῇ δοθείσῃ ἄρα σφαίρᾳ περιείληπται ὁ κύβος· καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς τοῦ κύβου πλευρᾶς· ὅπερ ἔδει δεῖξαι.

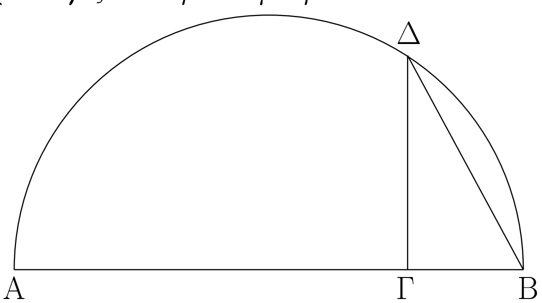
ried around, and again established at the same (position) from which it began to be moved, then the cube will be enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since GF is equal to FE, and the angle at F is a right-angle, the (square) on EG is thus double the (square) on EF [Prop. 1.47]. And EF (is) equal to EK. Thus, the (square) on EG is double the (square) on EK. Hence, the (sum of the squares) on GE and EK—that is to say, the (square) on GK [Prop. 1.47]—is three times the (square) on EK. And since AB is three times BC, and as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9], the (square) on AB (is) thus three times the (square) on BD. And the (square) on GK was also shown (to be) three times the (square) on KE. And KE was made equal to DB. Thus, KG (is) also equal to AB. And AB is the radius of the given sphere. Thus, KG is also equal to the diameter of the given sphere.

Thus, the cube has been enclosed by the given sphere. And it has simultaneously been shown that the square on the diameter of the sphere is three times the (square) on the side of the cube.† (Which is) the very thing it was required to show.

† If the radius of the sphere is unity then the side of the cube is $\sqrt{4/3}$.

ις'.

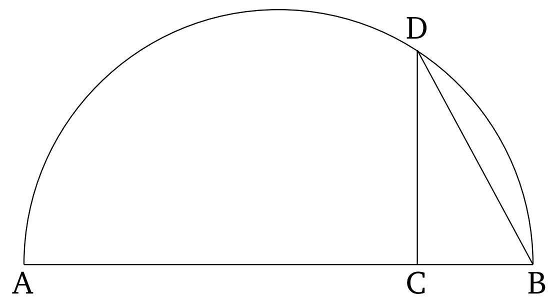
Εἰκοσάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατὰ τὸ Γ ὥστε τετραπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἤχθω ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ορθὰς γωνίας εὐθεῖα γραμμὴ ἡ ΓΔ, καὶ ἐπεζεύχθω ἡ ΔΒ, καὶ ἐκκείσθω κύκλος ὁ ΕΖΗΘΚ, οὗ ἡ ἐν τοῦ κέντρου ἴση ἔστω τῇ ΔΒ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘΚ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ ΕΖΗΘΚ, καὶ τετμήσθωσαν αἱ ΕΖ, ΖΗ, ΗΘ, ΘΚ, ΚΕ περιφέρειαι δίχα κατὰ τὸ Λ, Μ, Ν, Ξ, Ο σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΛΜ, ΜΝ, ΝΞ, ΞΟ, ΟΛ, ΕΟ. ἰσόπλευρον ἄρα ἐστὶ καὶ τὸ ΛΜΝΞΟ πεντάγωνον, καὶ δεκαγώνου ἡ ΕΟ εὐθεῖα. καὶ ἀνεστάτωσαν ἀπὸ τῶν Ε, Ζ, Η, Θ, Κ σημείων τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς γωνίας εὐθεῖαι αἱ ΕΠ, ΖΡ, ΗΣ, ΘΤ, ΚΥ ἴσαι οὖσαι τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ

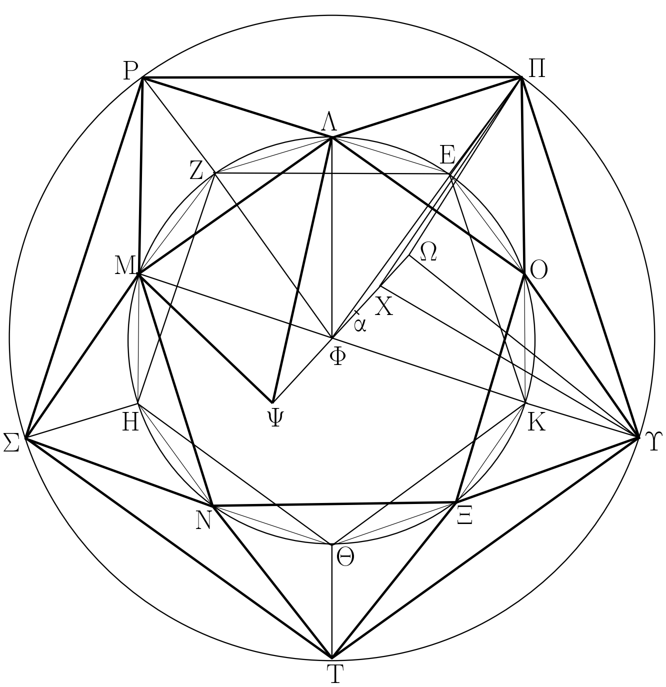
Proposition 16

To construct an icosahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the icosahedron is that irrational (straight-line) called minor.



Let the diameter AB of the given sphere be laid out, and let it be cut at C such that AC is four times CB [Prop. 6.10]. And let the semi-circle ADB be drawn on AB. And let the straight-line CD be drawn from C at right-angles to AB. And let DB be joined. And let the circle $EFGHK$ be set down, and let its radius be equal to DB. And let the equilateral and equiangular pentagon $EFGHK$ be inscribed in circle $EFGHK$ [Prop. 4.11]. And let the circumferences EF, FG, GH, HK, and KE be cut in half at points L, M, N, O, and P (respectively). And let LM, MN, NO, OP, PL, and EP be joined. Thus, pentagon $LMNOP$ is also equilateral, and EP (is) the side of the decagon (inscribed in the circle). And let the straight-lines EQ, FR, GS, HT, and KU, which are equal to the radius of circle $EFGHK$, have been set up at right-angles *κύκλου, καὶ ἐπεζεύχθωσαν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ, ΥΠ, ΠΛ, ΛΡ, ΡΜ, ΜΣ, ΣΝ, ΝΤ, ΤΞ, ΞΥ, ΥΟ, ΟΠ.*

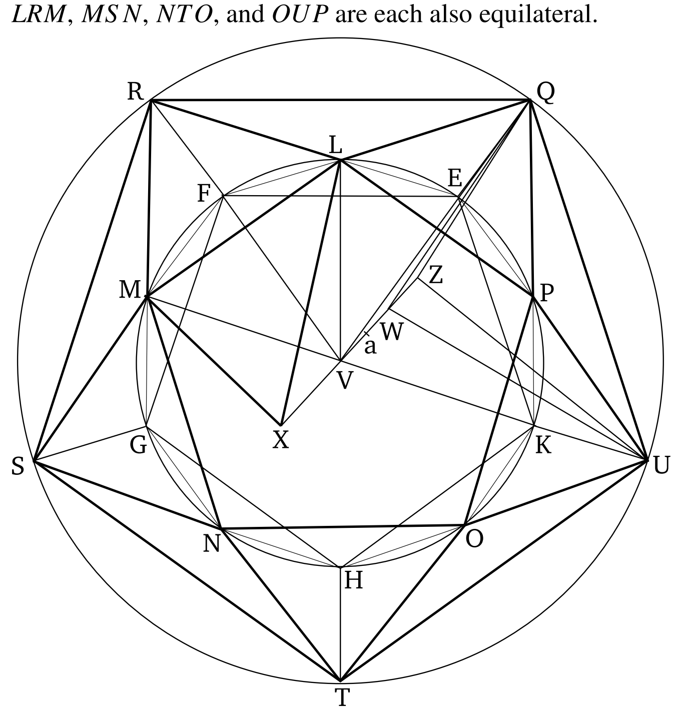
Καὶ ἐπεὶ ἑκατέρα τῶν ΕΠ, ΚΥ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστὶν ἡ ΕΠ τῇ ΚΥ. ἔστι δὲ αὐτῇ καὶ ἴση· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπιζευγνύουσαι ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι ἴσαι τε καὶ παράλληλοί εἰσιν. ἡ ΠΥ ἄρα τῇ ΕΚ ἴση τε καὶ παράλληλός ἐστιν. πενταγώνου δὲ ἰσοπλεύρου ἡ ΕΚ· πενταγώνου ἄρα ἰσοπλεύρου καὶ ἡ ΠΥ τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγραφομένου. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ΠΡ, ΡΣ, ΣΤ, ΤΥ πενταγώνου ἐστὶν ἰσοπλεύρου τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγραφομένου· ἰσόπλευρον ἄρα τὸ ΠΡΣΤΥ πεντάγωνον. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΕ, δεκαγώνου δὲ ἡ ΕΟ, καί ἐστιν ὀρθὴ ἡ ὑπὸ ΠΕΟ, πενταγώνου ἄρα ἐστὶν ἡ ΠΟ· ἡ γὰρ τοῦ πενταγώνου πλευρὰ δύναται τήν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΟΥ πενταγώνου ἐστὶ πλευρά. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΟΥ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΠΛΡ, ΡΜΣ, ΣΝΤ, ΤΞΥ ἰσόπλευρόν ἐστιν. καὶ ἐπεὶ πενταγώνου ἐδείχθη ἑκατέρα τῶν ΠΛ, ΠΟ, ἔστι δὲ καὶ ἡ ΛΟ πενταγώνου, ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΛΟ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΛΡΜ, ΜΣΝ, ΝΤΞ, ΞΥΟ τριγώνων ἰσόπλευρόν ἐστιν.



Εἰλήφθω τὸ κέντρον τοῦ ΕΖΗΘΚ κύκλου τὸ Φ σημεῖον· καὶ ἀπὸ τοῦ Φ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω

to the plane of the circle, at points E, F, G, H, and K (respectively). And let QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, and PQ be joined.

And since EQ and KU are each at right-angles to the same plane, EQ is thus parallel to KU [Prop. 11.6]. And it is also equal to it. And straight-lines joining equal and parallel (straight-lines) on the same side are (themselves) equal and parallel [Prop. 1.33]. Thus, QU is equal and parallel to EK. And EK (is the side) of an equilateral pentagon (inscribed in circle $EFGHK$). Thus, QU (is) also the side of an equilateral pentagon inscribed in circle $EFGHK$. So, for the same (reasons), QR, RS, ST, and TU are also the sides of an equilateral pentagon inscribed in circle $EFGHK$. Pentagon $QRSTU$ (is) thus equilateral. And side QE is (the side) of a hexagon (inscribed in circle $EFGHK$), and EP (the side) of a decagon, and (angle) QEP is a right-angle, thus QP is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), PU is also the side of a pentagon. And QU is also (the side) of a pentagon. Thus, triangle QPU is equilateral. So, for the same (reasons), (triangles) QLR, RMS, SNT, and TOU are each also equilateral. And since QL and QP were each shown (to be the sides) of a pentagon, and LP is also (the side) of a pentagon, triangle QLP is thus equilateral. So, for the same (reasons), triangles LRM, MSN, NTO, and OUP are each also equilateral.



Let the center, point V, of circle $EFGHK$ be found [Prop. 3.1]. And let VZ be set up, at (point) V, at right-angles ἡ ΦΩ, καὶ ἐκβεβλήσθω ἐπὶ τὰ ἕτερα μέρη ὡς ἡ ΦΨ, καὶ ἀφῃρήσθω ἑξαγώνου μὲν ἡ ΦΧ, δεκαγώνου δὲ ἑκατέρα τῶν ΦΨ, ΧΩ, καὶ ἐπεζεύχθωσαν αἱ ΠΩ, ΠΧ, ΥΩ, ΕΦ, ΛΦ, ΛΨ, ΨΜ.

Καὶ ἐπεὶ ἑκατέρα τῶν ΦΧ, ΠΕ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστὶν ἡ ΦΧ τῇ ΠΕ. εἰσὶ δὲ καὶ ἴσαι· καὶ αἱ ΕΦ, ΠΧ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν. ἑξαγώνου δὲ ἡ ΕΦ· ἑξαγώνου ἄρα καὶ ἡ ΠΧ. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΧ, δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΠΧΩ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΠΩ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΥΩ πενταγώνου ἐστίν, ἐπειδήπερ, ἐὰν ἐπιζεύξωμεν τὰς ΦΚ, ΧΥ, ἴσαι καὶ ἀπεναντίον ἔσονται, καί ἐστιν ἡ ΦΚ ἐκ τοῦ κέντρου οὖσα ἑξαγώνου. ἑξαγώνου ἄρα καὶ ἡ ΧΥ. δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθὴ ἡ ὑπὸ ΥΧΩ· πενταγώνου ἄρα ἡ ΥΩ. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΥΩ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ εὐθεῖαι, κορυφὴ δὲ τὸ Ω σημεῖον, ἰσόπλευρόν ἐστιν. πάλιν, ἐπεὶ ἑξαγώνου μὲν ἡ ΦΛ, δεκαγώνου δὲ ἡ ΦΨ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΛΦΨ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΛΨ. διὰ τὰ αὐτὰ δὴ ἐὰν ἐπιζεύξωμεν τὴν ΜΦ οὖσαν ἑξαγώνου, συνάγεται καὶ ἡ ΜΨ πενταγώνου. ἔστι δὲ καὶ ἡ ΛΜ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΛΜΨ τρίγωνον. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μέν εἰσιν αἱ ΜΝ, ΝΞ, ΞΟ, ΟΛ, κορυφὴ δὲ τὸ Ψ σημεῖον, ἰσόπλευρόν ἐστιν. συνέσταται ἄρα εἰκοσάεδρον ὑπὸ εἴκοσι τριγώνων ἰσοπλεύρων περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ἑξαγώνου ἐστὶν ἡ ΦΧ, δεκαγώνου δὲ ἡ ΧΩ, ἡ ΦΩ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΦΧ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ. ἴση δὲ ἡ μὲν ΦΧ τῇ ΦΕ, ἡ δὲ ΧΩ τῇ ΦΨ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΕ, οὕτως ἡ ΕΦ πρὸς τὴν ΦΨ. καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΩΦΕ, ΕΦΨ γωνίαι· ἐὰν ἄρα ἐπιζεύξωμεν τὴν ΕΩ εὐθεῖαν, ὀρθὴ ἔσται ἡ ὑπὸ ΨΕΩ γωνία διὰ τὴν ὁμοιότητα τῶν ΨΕΩ, ΦΕΩ τριγώνων. διὰ τὰ αὐτὰ δὴ ἐπεί ἐστιν ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ, ἴση δὲ ἡ μὲν ΩΦ τῇ ΨΧ, ἡ δὲ ΦΧ τῇ ΧΠ, ἔστιν ἄρα ὡς ἡ ΨΧ πρὸς τὴν ΧΠ, οὕτως ἡ ΠΧ πρὸς τὴν ΧΩ. καὶ διὰ τοῦτο πάλιν ἐὰν ἐπιζεύξωμεν τὴν ΠΨ, ὀρθὴ ἔσται ἡ πρὸς τῷ Π γωνία· τὸ ἄρα ἐπὶ τῆς ΨΩ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Π. καὶ ἐὰν μενούσης τῆς ΨΩ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τοῦ Π καὶ τῶν λοιπῶν σημείων τοῦ εἰκοσαέδρου, καὶ ἔσται σφαίρᾳ περιειλημμένον τὸ εἰκοσάεδρον. λέγω δή, ὅτι καὶ τῇ δοθείσῃ. τετμήσθω γὰρ ἡ ΦΧ δίχα κατὰ τὸ α. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΦΩ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ ἔλασσον αὐτῆς τμῆμά ἐστιν ἡ ΩΧ, ἡ ἄρα ΩΧ προσλαβοῦσα τὴν ἡμίσειαν τοῦ μείζονος τμήματος τὴν Χα πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας

to the plane of the circle. And let it be produced on the other side (of the circle), like VX. And let VW be cut off (from XZ so as to be equal to the side) of a hexagon, and each of VX and WZ (so as to be equal to the side) of a decagon. And let QZ, QW, UZ, EV, LV, LX, and XM be joined.

And since VW and QE are each at right-angles to the plane of the circle, VW is thus parallel to QE [Prop. 11.6]. And they are also equal. EV and QW are thus equal and parallel (to one another) [Prop. 1.33]. And EV (is the side) of a hexagon. Thus, QW (is) also (the side) of a hexagon. And since QW is (the side) of a hexagon, and WZ (the side) of a decagon, and angle QWZ is a right-angle [Def. 11.3, Prop. 1.29], QZ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), UZ is also (the side) of a pentagon—inasmuch as, if we join VK and WU then they will be equal and opposite. And VK, being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, WU (is) also the side of a hexagon. And WZ (is the side) of a decagon, and (angle) UWZ (is) a right-angle. Thus, UZ (is the side) of a pentagon [Prop. 13.10]. And QU is also (the side) of a pentagon. Triangle QUZ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines QR, RS, ST, and TU, and apexes the point Z, are also equilateral. Again, since VL (is the side) of a hexagon, and VX (the side) of a decagon, and angle LVX is a right-angle, LX is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join MV, which is (the side) of a hexagon, MX is also inferred (to be the side) of a pentagon. And LM is also (the side) of a pentagon. Thus, triangle LMX is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) MN, NO, OP, and PL, and apexes the point X, are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since VW is (the side) of a hexagon, and WZ (the side) of a decagon, VZ has thus been cut in extreme and mean ratio at W, and VW is its greater piece [Prop. 13.9]. Thus, as ZV is to VW, so VW (is) to WZ. And VW (is) equal to VE, and WZ to VX. Thus, as ZV is to VE, so EV (is) to VX. And angles ZVE and EVX are right-angles. Thus, if we join straight-line EZ then angle XEZ will be a right-angle, on account of the similarity of triangles XEZ and VEZ. [Prop. 6.8]. So, for the same (reasons), since as ZV is to VW, so VW (is) to WZ, and ZV (is) equal to XW, and VW to WQ, thus as XW is to WQ, so QW (is) to WZ. And, again, on account of this, if we join QX then the angle at Q will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on XZ will also pass through Q [Prop. 3.31]. And if XZ remains fixed, and the semi-circle is carried around, and again established at the same (position) from which it beτοῦ μείζονος τμήματος· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς Ωa τοῦ ἀπὸ τῆς aX. καί ἐστι τῆς μὲν Ωa διπλῆ ἡ ΩΨ, τὴς δὲ aX διπλῆ ἡ ΦΧ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΩΨ τοῦ ἀπὸ τῆς ΧΦ. καὶ ἐπεὶ τετραπλῆ ἐστιν ἡ ΑΓ τῆς ΓΒ, πενταπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΩΨ πενταπλάσιον τοῦ ἀπὸ τῆς ΦΧ. καί ἐστιν ἴση ἡ ΔΒ τῇ ΦΧ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ἴση ἄρα καὶ ἡ ΑΒ τῇ ΨΩ. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΨΩ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ· τῇ ἄρα δοθείσῃ σφαίρᾳ περιείληπται τὸ εἰκοσάεδρον.

Λέγω δή, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἐπεὶ γὰρ ῥητή ἐστιν ἡ τῆς σφαίρας διάμετρος, καί ἐστι δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, ῥητὴ ἄρα ἐστὶ καὶ ἡ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ὥστε καὶ ἡ διάμετρος αὐτοῦ ῥητή ἐστιν. ἐὰν δὲ εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῃ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἡ δὲ τοῦ ΕΖΗΘΚ πενταγώνου πλευρὰ ἡ τοῦ εἰκοσαέδρου ἐστίν. ἡ ἄρα τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

gan to be moved, then it will also pass through (point) Q, and (through) the remaining (angular) points of the icosahedron. And the icosahedron will be enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let VW be cut in half at a. And since the straight-line VZ has been cut in extreme and mean ratio at W, and ZW is its lesser piece, then the square on ZW added to half of the greater piece, Wa, is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on Za is five times the (square) on aW. And ZX is double Za, and VW double aW. Thus, the (square) on ZX is five times the (square) on WV. And since AC is four times CB, AB is thus five times BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is five times the (square) on BD. And the (square) on ZX was also shown (to be) five times the (square) on VW. And DB is equal to VW. For each of them is equal to the radius of circle $EFGHK$. Thus, AB (is) also equal to XZ. And AB is the diameter of the given sphere. Thus, XZ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle $EFGHK$, the radius of circle $EFGHK$ is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon $EFGHK$ is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται, καὶ ὅτι ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. ὅπερ ἔδει δεῖξαι.

Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle.[†]

[†] If the radius of the sphere is unity then the radius of the circle is $2/\sqrt{5}$, and the sides of the hexagon, decagon, and pentagon/icosahedron are $2/\sqrt{5}$, $1-1/\sqrt{5}$, and $(1/\sqrt{5})\sqrt{10-2\sqrt{5}}$, respectively.

ιζ'.

Δωδεκάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ᾗ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

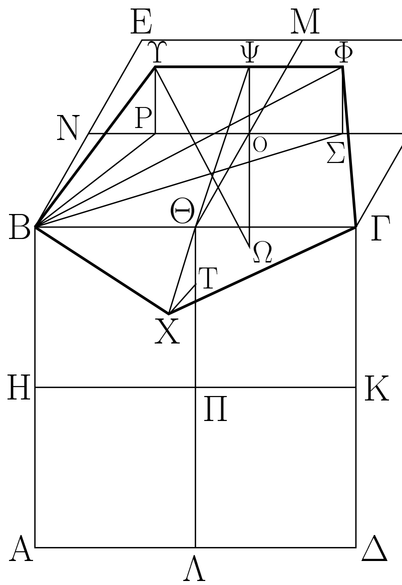
Ἐκκείσθωσαν τοῦ προειρημένου κύβου δύο ἐπίπεδα πρὸς ὀρθὰς ἀλλήλοις τὰ ΑΒΓΔ, ΓΒΕΖ, καὶ τετμήσθω ἑκάστη τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ, ΕΖ, ΕΒ, ΖΓ πλευρῶν δίχα κατὰ τὰ Η, Θ, Κ, Λ, Μ, Ν, Ξ, καὶ ἐπεζεύχθωσαν αἱ ΗΚ, ΘΛ, ΜΘ, ΝΞ, καὶ

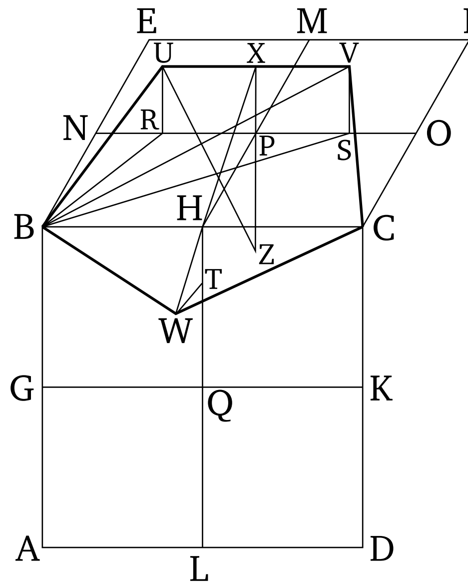
Proposition 17

To construct a dodecahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

Let two planes of the aforementioned cube [Prop. 13.15], $ABCD$ and $CBEF$, (which are) at right-angles to one another, be laid out. And let the sides AB, BC, CD, DA, EF, EB, and τετμήσθω ἑκάστη τῶν ΝΟ, ΟΞ, ΘΠ ἄκρον καὶ μέσον λόγον κατὰ τὰ Ρ, Σ, Τ σημεῖα, καὶ ἔστω αὐτῶν μείζονα τμήματα τὰ ΡΟ, ΟΣ, ΤΠ, καὶ ἀνεστάτωσαν ἀπὸ τῶν Ρ, Σ, Τ σημείων τοῖς τοῦ κύβου ἐπιπέδοις πρὸς ὀρθὰς ἐπὶ τὰ ἐκτὸς μέρη τοῦ κύβου αἱ ΡΥ, ΣΦ, ΤΧ, καὶ κείσθωσαν ἴσαι ταῖς ΡΟ, ΟΣ, ΤΠ, καὶ ἐπεζεύχθωσαν αἱ ΥΒ, ΒΧ, ΧΓ, ΓΦ, ΦΥ.

FC have each been cut in half at points G, H, K, L, M, N, and O (respectively). And let GK, HL, MH, and NO be joined. And let NP, PO, and HQ have each been cut in extreme and mean ratio at points R, S, and T (respectively). And let their greater pieces be RP, PS, and TQ (respectively). And let RU, SV, and TW be set up on the exterior side of the cube, at points R, S, and T (respectively), at right-angles to the planes of the cube. And let them be made equal to RP, PS, and TQ. And let UB, BW, WC, CV, and VU be joined.





Λέγω, ὅτι τὸ ΥΒΧΓΦ πεντάγωνον ἰσόπλευρόν τε καὶ ἐν ἑνὶ ἐπιπέδῳ καὶ ἔτι ἰσογώνιόν ἐστιν. ἐπεζεύχθωσαν γὰρ αἱ ΡΒ, ΣΒ, ΦΒ. καὶ ἐπεὶ εὐθεῖα ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΟ, τὰ ἄρα ἀπὸ τῶν ΟΝ, ΝΡ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΟ. ἴση δὲ ἡ μὲν ΟΝ τῇ ΝΒ, ἡ δὲ ΟΡ τῇ ΡΥ· τὰ ἄρα ἀπὸ τῶν ΒΝ, ΝΡ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΝ, ΝΡ τὸ ἀπὸ τῆς ΒΡ ἐστιν ἴσον· τὸ ἄρα ἀπὸ τῆς ΒΡ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΡΥ· ὥστε τὰ ἀπὸ τῶν ΒΡ, ΡΥ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΡΥ. τοῖς δὲ ἀπὸ τῶν ΒΡ, ΡΥ ἴσον ἐστι τὸ ἀπὸ τῆς ΒΥ· τὸ ἄρα ἄπὸ τῆς ΒΥ τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΥΡ· διπλῆ ἄρα ἐστὶν ἡ ΒΥ τῆς ΡΥ. ἔστι δὲ καὶ ἡ ΦΥ τῆς ΥΡ διπλῆ, ἐπειδήπερ καὶ ἡ ΣΡ τῆς ΟΡ, τουτέστι τῆς ΡΥ, ἐστι διπλῆ· ἴση ἄρα ἡ ΒΥ τῇ ΥΦ. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΒΧ, ΧΓ, ΓΦ ἑκατέρᾳ τῶν ΒΥ, ΥΦ ἐστιν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. λέγω δή, ὅτι καὶ ἐν ἑνί ἐστιν ἐπιπέδῳ. ἤχθω γὰρ ἀπὸ τοῦ Ο ἑκατέρᾳ τῶν ΡΥ, ΣΦ παράλληλος ἐπὶ τὰ ἐκτὸς τοῦ κύβου μέρη ἡ ΟΨ, καὶ ἐπεζεύχθωσαν αἱ ΨΘ, ΘΧ· λέγω, ὅτι ἡ ΨΘΧ εὐθεῖά ἐστιν. ἐπεὶ γὰρ ἡ ΘΠ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Τ, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΠΤ, ἔστιν ἄρα ὡς ἡ ΘΠ πρὸς τὴν ΠΤ, οὕτως ἡ ΠΤ πρὸς τὴν ΤΘ. ἴση δὲ ἡ μὲν ΘΠ τῇ ΘΟ, ἡ δὲ ΠΤ ἑκατέρᾳ τῶν ΤΧ, ΟΨ· ἔστιν ἄρα

I say that the pentagon UBWCV is equilateral, and in one plane, and, further, equiangular. For let RB, SB, and VB be joined. And since the straight-line NP has been cut in extreme and mean ratio at R, and RP is the greater piece, the (sum of the squares) on PN and NR is thus three times the (square) on RP [Prop. 13.4]. And PN (is) equal to NB, and PR to RU. Thus, the (sum of the squares) on BN and NR is three times the (square) on RU. And the (square) on BR is equal to the (sum of the squares) on BN and NR [Prop. 1.47]. Thus, the (square) on BR is three times the (square) on RU. Hence, the (sum of the squares) on BR and RU is four times the (square) on RU. And the (square) on BU is equal to the (sum of the squares) on BR and RU [Prop. 1.47]. Thus, the (square) on BU is four times the (square) on UR. Thus, BU is double RU. And VU is also double UR, inasmuch as SR is also double PR—that is to say, RU. Thus, BU (is) equal to UV. So, similarly, it can be shown that each of BW, WC, CV is equal to each of BU and UV. Thus, pentagon BUVCW is equilateral. So, I say that it is also in one plane. For let PX be drawn from P, parallel to each of RU and SV, on the exterior side of the cube. And let XH and HW be joined. I say that XHW is a straight-line. For since HQ has been cut in extreme and mean ratio at T, and QT is its greater piece, thus as HQ is to QT, so QT (is) to TH.

ὡς ἡ ΘΟ πρὸς τὴν ΟΨ, οὕτως ἡ ΧΤ πρὸς τὴν ΤΘ. καί ἐστι παράλληλος ἡ μὲν ΘΟ τῇ ΤΧ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἡ δὲ ΤΘ τῇ ΟΨ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΖ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν, ὡς τὰ ΨΟΘ, ΘΤΧ, τὰς δύο πλευρὰς ταῖς δυσὶν ἀνάλογον ἔχοντα, ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ εὐθεῖαι ἐπ᾽ εὐθείας ἔσονται· ἐπ᾽ εὐθείας ἄρα ἐστὶν ἡ ΨΘ τῇ ΘΧ. πᾶσα δὲ εὐθεῖα ἐν ἑνί ἐστιν ἐπιπέδῳ· ἐν ἑνὶ ἄρα ἐπιπέδῳ ἐστὶ τὸ ΥΒΧΓΦ πεντάγωνον.

Λέγω δή, ὅτι καὶ ἰσογώνιόν ἐστιν.

Ἐπεὶ γὰρ εὐθεῖα γραμμὴ ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΟΡ [ἔστιν ἄρα ὡς συναμφότερος ἡ ΝΟ, ΟΡ πρὸς τὴν ΟΝ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΡ], ἴση δὲ ἡ ΟΡ τῇ ΟΣ [ἔστιν ἄρα ὡς ἡ ΣΝ πρὸς τὴν ΝΟ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΣ], ἡ ΝΣ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον τμῆμά ἐστιν ἡ ΝΟ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΟ τῇ ΝΒ, ἡ δὲ ΟΣ τῇ ΣΦ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΦ τετράγωνα τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΒ· ὥστε τὰ ἀπὸ τῶν ΦΣ, ΣΝ, ΝΒ τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΒ. τοῖς δὲ ἀπὸ τῶν ΣΝ, ΝΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΣΒ· τὰ ἄρα ἀπὸ τῶν ΒΣ, ΣΦ, τουτέστι τὸ ἀπὸ τῆς ΒΦ [ὀρθὴ γὰρ ἡ ὑπὸ ΦΣΒ γωνία], τετραπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΒ· διπλῆ ἄρα ἐστὶν ἡ ΦΒ τῆς ΒΝ. ἔστι δὲ καὶ ἡ ΒΓ τῆς ΒΝ διπλῆ· ἴση ἄρα ἐστὶν ἡ ΒΦ τῇ ΒΓ. καὶ ἐπεὶ δύο αἱ ΒΥ, ΥΦ δυσὶ ταῖς ΒΧ, ΧΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΒΦ βάσει τῇ ΒΓ ἴση, γωνία ἄρα ἡ ὑπὸ ΒΥΦ γωνίᾳ τῇ ὑπὸ ΒΧΓ ἐστιν ἴση. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΥΦΓ γωνία ἴση ἐστὶ τῇ ὑπὸ ΒΧΓ· αἱ ἄρα ὑπὸ ΒΧΓ, ΒΥΦ, ΥΦΓ τρεῖς γωνίαι ἴσαι ἀλλήλαις εἰσίν. ἐὰν δὲ πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ἴσαι ἀλλήλαις ὦσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον· ἰσογώνιον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τὸ ἄρα ΒΥΦΓΧ πεντάγωνον ἰσόπλευρόν ἐστι καὶ ἰσογώνιον, καί ἐστιν ἐπὶ μιᾶς τοῦ κύβου πλευρᾶς τῆς ΒΓ. ἐὰν ἄρα ἐφ᾽ ἑκάστης τῶν τοῦ κύβου δώδεκα πλευρῶν τὰ αὐτὰ κατασκευάσωμεν, συσταθήσεταί τι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων περιεχόμενον, ὃ καλεῖται δωδεκάεδρον.

Δεῖ δὴ αὐτὸ καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΨΟ, καὶ ἔστω ἡ ΨΩ· συμβάλλει ἄρα ἡ ΟΩ τῇ τοῦ κύβου διαμέτρῳ, καὶ δίχα τέμνουσιν ἀλλήλας· τοῦτο γὰρ δέδεικται ἐν τῷ παρατελεύτῳ θεωρήματι τοῦ ἑνδεκάτου βιβλίου. τεμνέτωσαν κατὰ τὸ Ω· τὸ Ω ἄρα κέντρον ἐστὶ τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον, καὶ ἡ ΩΟ ἡμίσεια τῆς πλευρᾶς τοῦ κύβου. ἐπεζεύχθω δὴ ἡ ΥΩ. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΝΣ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΝΟ, τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ.

And HQ (is) equal to HP, and QT to each of TW and PX. Thus, as HP is to PX, so WT (is) to TH. And HP is parallel to TW. For of each of them is at right-angles to the plane BD [Prop. 11.6]. And TH (is parallel) to PX. For each of them is at right-angles to the plane BF [Prop. 11.6]. And if two triangles, like XPH and HTW, having two sides proportional to two sides, are placed together at a single angle such that their corresponding sides are also parallel then the remaining sides will be straight-on (to one another) [Prop. 6.32]. Thus, XH is straight-on to HW. And every straight-line is in one plane [Prop. 11.1]. Thus, pentagon $UBWCV$ is in one plane.

So, I say that it is also equiangular.

For since the straight-line NP has been cut in extreme and mean ratio at R, and PR is the greater piece [thus as the sum of NP and PR is to PN, so NP (is) to PR], and PR (is) equal to PS [thus as SN is to NP, so NP (is) to PS], NS has thus also been cut in extreme and mean ratio at P, and NP is the greater piece [Prop. 13.5]. Thus, the (sum of the squares) on NS and SP is three times the (square) on NP [Prop. 13.4]. And NP (is) equal to NB, and PS to SV. Thus, the (sum of the) squares on NS and SV is three times the (square) on NB. Hence, the (sum of the squares) on VS, SN, and NB is four times the (square) on NB. And the (square) on SB is equal to the (sum of the squares) on SN and NB [Prop. 1.47]. Thus, the (sum of the squares) on BS and SV—that is to say, the (square) on BV [for angle VSB (is) a right-angle]—is four times the (square) on NB [Def. 11.3, Prop. 1.47]. Thus, VB is double BN. And BC (is) also double BN. Thus, BV is equal to BC. And since the two (straight-lines) BU and UV are equal to the two (straight-lines) BW and WC (respectively), and the base BV (is) equal to the base BC, angle BUV is thus equal to angle BWC [Prop. 1.8]. So, similarly, we can show that angle UVC is equal to angle BWC. Thus, the three angles BWC, BUV, and UVC are equal to one another. And if three angles of an equilateral pentagon are equal to one another then the pentagon is equiangular [Prop. 13.7]. Thus, pentagon $BUVCW$ is equiangular. And it was also shown (to be) equilateral. Thus, pentagon $BUVCW$ is equilateral and equiangular, and it is on one of the sides, BC, of the cube. Thus, if we make the same construction on each of the twelve sides of the cube then some solid figure contained by twelve equilateral and equiangular pentagons will be constructed, which is called a dodecahedron.

So, it is necessary to enclose it in the given sphere, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

For let XP be produced, and let (the produced straight-line) be XZ. Thus, PZ meets the diameter of the cube, and they cut one another in half. For, this has been proved in the penultimate theorem of the eleventh book [Prop. 11.38]. Let them cut (one another) at Z. Thus, Z is the center of the sphere enclosing the cube, and ZP (is) half the side of the cube. So, let UZ

ἴση δὲ ἡ μὲν ΝΣ τῇ ΨΩ, ἐπειδήπερ καὶ ἡ μὲν ΝΟ τῇ ΟΩ ἐστιν ἴση, ἡ δὲ ΨΟ τῇ ΟΣ. ἀλλὰ μὴν καὶ ἡ ΟΣ τῇ ΨΥ, ἐπεὶ καὶ τῇ ΡΟ· τὰ ἄρα ἀπὸ τῶν ΩΨ, ΨΥ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. τοῖς δὲ ἀπὸ τῶν ΩΨ, ΨΥ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΥΩ· τὸ ἄρα ἀπὸ τῆς ΥΩ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἔστι δὲ καὶ ἡ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον δυνάμει τριπλασίων τῆς ἡμισείας τῆς τοῦ κύβου πλευρᾶς· προδέδεικται γὰρ κύβον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων ἐστὶ τῆς πλευρᾶς τοῦ κύβου. εἰ δὲ ὅλη τῆς ὅλης, καὶ [ἡ] ἡμίσεια τῆς ἡμισείας· καί ἐστιν ἡ ΝΟ ἡμίσεια τῆς τοῦ κύβου πλευρᾶς· ἡ ἄρα ΥΩ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον. καί ἐστι τὸ Ω κέντρον τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον· τὸ Υ ἄρα σημεῖον πρὸς τῇ ἐπιφανείᾳ ἐστι τῆς σφαίρας. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκάστη τῶν λοιπῶν γωνιῶν τοῦ δωδεκαέδρου πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας· περιείληπται ἄρα τὸ δωδεκαέδρον τῇ δοθείσῃ σφαίρᾳ.

Λέγω δή, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ τῆς ΝΟ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ὁ ΡΟ, τῆς δὲ ΟΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΟΣ, ὅλης ἄρα τῆς ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΡΣ. [οἷον ἐπεί ἐστιν ὡς ἡ ΝΟ πρὸς τὴν ΟΡ, ἡ ΟΡ πρὸς τὴν ΡΝ, καὶ τὰ διπλάσια· τὰ γὰρ μέρη τοῖς ἰσάκις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον· ὡς ἄρα ἡ ΝΞ πρὸς τὴν ΡΣ, οὕτως ἡ ΡΣ πρὸς συναμφότερον τὴν ΝΡ, ΣΞ. μείζων δὲ ἡ ΝΞ τῆς ΡΣ· μείζων ἄρα καὶ ἡ ΡΣ συναμφοτέρου τῆς ΝΡ, ΣΞ· ἡ ΝΞ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μεῖζον αὐτῆς τμῆμά ἐστιν ἡ ΡΣ.] ἴση δὲ ἡ ΡΣ τῇ ΥΦ· τῆς ἄρα ΝΞ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΥΦ. καὶ ἐπεὶ ῥητή ἐστιν τῆς σφαίρας διάμετρος καί ἐστι δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς, ῥητὴ ἄρα ἐστὶν ἡ ΝΞ πλευρὰ οὖσα τοῦ κύβου. ἐὰν δὲ ῥητὴ γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἑκάτερον τῶν τμημάτων ἄλογός ἐστιν ἀποτομή.

Ἡ ΥΦ ἄρα πλευρὰ οὖσα τοῦ δωδεκαέδρου ἄλογός ἐστιν ἀποτομή.

be joined. And since the straight-line NS has been cut in extreme and mean ratio at P, and its greater piece is NP, the (sum of the squares) on NS and SP is thus three times the (square) on NP [Prop. 13.4]. And NS (is) equal to XZ, inasmuch as NP is also equal to PZ, and XP to PS. But, indeed, PS (is) also (equal) to XU, since (it is) also (equal) to RP. Thus, the (sum of the squares) on ZX and XU is three times the (square) on NP. And the (square) on UZ is equal to the (sum of the squares) on ZX and XU [Prop. 1.47]. Thus, the (square) on UZ is three times the (square) on NP. And the square on the radius of the sphere enclosing the cube is also three times the (square) on half the side of the cube. For it has previously been demonstrated (how to) construct the cube, and to enclose (it) in a sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube [Prop. 13.15]. And if the (square on the) whole (is three times) the (square on the) whole, then the (square on the) half (is) also (three times) the (square on the) half. And NP is half of the side of the cube. Thus, UZ is equal to the radius of the sphere enclosing the cube. And Z is the center of the sphere enclosing the cube. Thus, point U is on the surface of the sphere. So, similarly, we can show that each of the remaining angles of the dodecahedron is also on the surface of the sphere. Thus, the dodecahedron has been enclosed by the given sphere.

So, I say that the side of the dodecahedron is that irrational straight-line called an apotome.

For since RP is the greater piece of NP, which has been cut in extreme and mean ratio, and PS is the greater piece of PO, which has been cut in extreme and mean ratio, RS is thus the greater piece of the whole of NO, which has been cut in extreme and mean ratio. [Thus, since as NP is to PR, (so) PR (is) to RN, and (the same is also true) of the doubles. For parts have the same ratio as similar multiples (taken in corresponding order) [Prop. 5.15]. Thus, as NO (is) to RS, so RS (is) to the sum of NR and SO. And NO (is) greater than RS. Thus, RS (is) also greater than the sum of NR and SO [Prop. 5.14]. Thus, NO has been cut in extreme and mean ratio, and RS is its greater piece.] And RS (is) equal to UV. Thus, UV is the greater piece of NO, which has been cut in extreme and mean ratio. And since the diameter of the sphere is rational, and the square on it is three times the (square) on the side of the cube, NO, which is the side of the cube, is thus rational. And if a rational (straight)-line is cut in extreme and mean ratio then each of the pieces is the irrational (straight-line called) an apotome.

Thus, UV, which is the side of the dodecahedron, is the irrational (straight-line called) an apotome [Prop. 13.6].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι τῆς τοῦ κύβου πλευρᾶς ἄκρον

Corollary

So, (it is) clear, from this, that the side of the dodecahe-

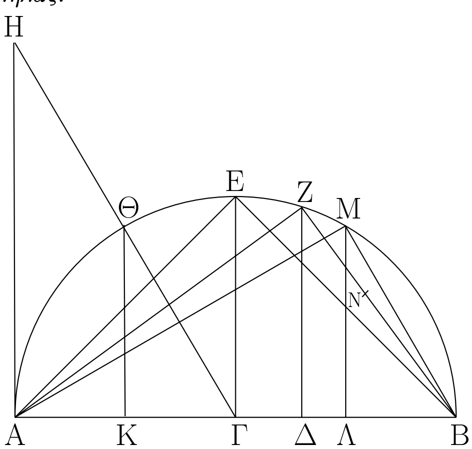
καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ τοῦ δωδεκαέδρου πλευρά. ὅπερ ἔδει δεῖξαι.

dron is the greater piece of the side of the cube, when it is cut in extreme and mean ratio.[†] (Which is) the very thing it was required to show.

[†] If the radius of the circumscribed sphere is unity then the side of the cube is $\sqrt{4/3}$, and the side of the dodecahedron is $(1/3)\,(\sqrt{15}-\sqrt{3})$.

ιηʹ.

Τὰς πλευρὰς τῶν πέντε σχημάτων ἐκθέσθαι καὶ συγκρῖναι πρὸς ἀλλήλας.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ ὥστε ἴσην εἶναι τὴν ΑΓ τῇ ΓΒ, κατὰ δὲ τὸ Δ ὥστε διπλασίονα εἶναι τὴν ΑΔ τῆς ΔΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΕΒ, καὶ ἀπὸ τῶν Γ, Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθωσαν αἱ ΓΕ, ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ, ΕΒ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΔ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ· ἰσογώνιον γάρ ἐστι τὸ ΑΖΒ τρίγωνον τῷ ΑΖΔ τριγώνῳ· ἡμιόλιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία τῆς πλευρᾶς τῆς πυραμίδος. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΑΖ ἄρα ἴση ἐστὶ τῇ πλευρᾷ τῆς πυραμίδος.

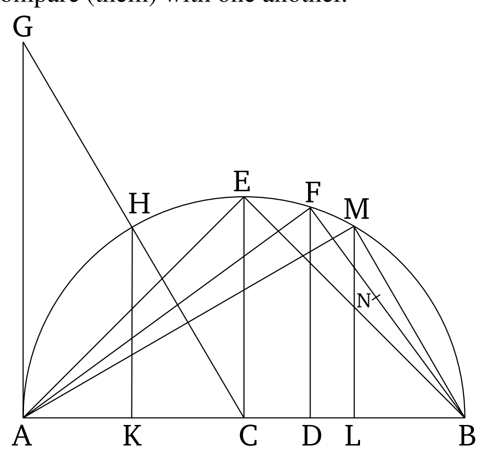
Πάλιν, ἐπεὶ διπλασίων ἐστὶν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· τριπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΒΖ ἄρα τοῦ κύβου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, διπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΕ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων τῆς τοῦ ὀκταέδρου πλευρᾶς. καὶ ἔστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΒΕ ἄρα τοῦ ὀκταέδρου ἐστὶ πλευρά.

Ἤχθω δὴ ἀπὸ τοῦ Α σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς ἡ ΑΗ, καὶ κείσθω ἡ ΑΗ ἴση τῇ ΑΒ, καὶ ἐπεζεύχθω ἡ ΗΓ,

Proposition 18

To set out the sides of the five (aforementioned) figures, and to compare (them) with one another.[†]



Let the diameter, AB, of the given sphere be laid out. And let it be cut at C, such that AC is equal to CB, and at D, such that AD is double DB. And let the semi-circle AEB be drawn on AB. And let CE and DF be drawn from C and D (respectively), at right-angles to AB. And let AF, FB, and EB be joined. And since AD is double DB, AB is thus triple BD. Thus, via conversion, BA is one and a half times AD. And as BA (is) to AD, so the (square) on BA (is) to the (square) on AF [Def. 5.9]. For triangle AFB is equiangular to triangle AFD [Prop. 6.8]. Thus, the (square) on BA is one and a half times the (square) on AF. And the square on the diameter of the sphere is also one and a half times the (square) on the side of the pyramid [Prop. 13.13]. And AB is the diameter of the sphere. Thus, AF is equal to the side of the pyramid.

Again, since AD is double DB, AB is thus triple BD. And as AB (is) to BD, so the (square) on AB (is) to the (square) on BF [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is three times the (square) on BF. And the square on the diameter of the sphere is also three times the (square) on the side of the cube [Prop. 13.15]. And AB is the diameter of the sphere. Thus, BF is the side of the cube.

And since AC is equal to CB, AB is thus double BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BE [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BE. And the square on the diameter of the sphere is also double the (square) on the side of the octagon [Prop. 13.14]. And AB is the diameter of the given sphere. Thus, BE is the side of the octagon.

καὶ ἀπὸ τοῦ Θ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΗΑ τῆς ΑΓ· ἴση γὰρ ἡ ΗΑ τῇ ΑΒ· ὡς δὲ ἡ ΗΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΓ, διπλῆ ἄρα καὶ ἡ ΘΚ τῆς ΚΓ. τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΘΚ τοῦ ἀπὸ τῆς ΚΓ· τὰ ἄρα ἀπὸ τῶν ΘΚ, ΚΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΘΓ, πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΚΓ. ἴση δὲ ἡ ΘΓ τῇ ΓΒ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ τῆς ΓΒ, ὧν ἡ ΑΔ τῆς ΔΒ ἐστι διπλῆ, λοιπὴ ἄρα ἡ ΒΔ λοιπῆς τῆς ΔΓ ἐστι διπλῆ. τριπλῆ ἄρα ἡ ΒΓ τῆς ΓΔ· ἐνναπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΔ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΓΚ τοῦ ἀπὸ τῆς ΓΔ. μείζων ἄρα ἐστὶν ἡ ΓΚ τῆς ΓΔ. κείσθω τῇ ΓΚ ἴση ἡ ΓΛ, καὶ ἀπὸ τοῦ Λ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΛΜ, καὶ ἐπεζεύχθω ἡ ΜΒ. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ, καί ἐστι τῆς μὲν ΒΓ διπλῆ ἡ ΑΒ, τῆς δὲ ΓΚ διπλῆ ἡ ΚΛ, πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΚΛ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΚΛ ἄρα ἐκ τοῦ κέντρου ἐστὶ τοῦ κύκλου, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται· ἡ ΚΛ ἄρα ἑξαγώνου ἐστὶ πλευρὰ τοῦ εἰρημένου κύκλου. καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν εἰρημένον κύκλον ἐγγραφομένων, καί ἐστιν ἡ μὲν ΑΒ ἡ τῆς σφαίρας διάμετρος, ἡ δὲ ΚΛ ἑξαγώνου πλευρά, καὶ ἴση ἡ ΑΚ τῇ ΛΒ, ἑκατέρα ἄρα τῶν ΑΚ, ΛΒ δεκαγώνου ἐστὶ πλευρὰ τοῦ ἐγγραφομένου εἰς τὸν κύκλον, ἀφ᾽ οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐπεὶ δεκαγώνου μὲν ἡ ΛΒ, ἑξαγώνου δὲ ἡ ΜΛ· ἴση γάρ ἐστι τῇ ΚΛ, ἐπεὶ καὶ τῇ ΘΚ· ἴσον γὰρ ἀπέχουσιν ἀπὸ τοῦ κέντρου· καί ἐστιν ἑκατέρα τῶν ΘΚ, ΚΛ διπλασίων τῆς ΚΓ· πενταγώνου ἄρα ἐστὶν ἡ ΜΒ. ἡ δὲ τοῦ πενταγώνου ἐστὶν ἡ τοῦ εἰκοσαέδρου· εἰκοσαέδρου ἄρα ἐστὶν ἡ ΜΒ.

Καὶ ἐπεὶ ἡ ΖΒ κύβου ἐστὶ πλευρά, τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Ν, καὶ ἔστω μεῖζον τμῆμα τὸ ΝΒ· ἡ ΝΒ ἄρα δωδεκαέδρου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος ἐδείχθη τῆς μὲν ΑΖ πλευρᾶς τῆς πυραμίδος δυνάμει ἡμιολία, τῆς δὲ τοῦ ὀκταέδρου τῆς ΒΕ δυνάμει διπλασίων, τῆς δὲ τοῦ κύβου τῆς ΖΒ δυνάμει τριπλασίων, οἵων ἄρα ἡ τῆς σφαίρας διάμετρος δυνάμει ἕξ, τοιούτων ἡ μὲν τῆς πυραμίδος τεσσάρων, ἡ δὲ τοῦ ὀκταέδρου τριῶν, ἡ δὲ τοῦ κύβου δύο. ἡ μὲν ἄρα τῆς πυραμίδος πλευρὰ τῆς μὲν τοῦ ὀκταέδρου πλευρᾶς δυνάμει ἐστὶν ἐπίτριτος, τῆς δὲ τοῦ κύβου δυνάμει διπλῆ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ἡμιολία. αἱ μὲν οὖν εἰρημέναι τῶν τριῶν σχημάτων πλευραί, λέγω δὴ πυραμίδος καὶ ὀκταέδρου καὶ κύβου, πρὸς ἀλλήλας εἰσὶν ἐν λόγοις ῥητοῖς. αἱ δὲ λοιπαὶ δύο, λέγω δὴ ἥ τε τοῦ εἰκοσαέδρου καὶ ἡ τοῦ δωδεκαέδρου, οὔτε πρὸς ἀλλήλας οὔτε πρὸς τὰς προειρημένας εἰσὶν ἐν λόγοις ῥητοῖς· ἄλογοι γάρ εἰσιν, ἡ μὲν ἐλάττων, ἡ δὲ ἀποτομή.

Ὅτι μείζων ἐστὶν ἡ τοῦ εἰκοσαέδρου πλευρὰ ἡ ΜΒ τῆς

So let AG be drawn from point A at right-angles to the straight-line AB. And let AG be made equal to AB. And let GC be joined. And let HK be drawn from H, perpendicular to AB. And since GA is double AC. For GA (is) equal to AB. And as GA (is) to AC, so HK (is) to KC [Prop. 6.4]. HK (is) thus also double KC. Thus, the (square) on HK is four times the (square) on KC. Thus, the (sum of the squares) on HK and KC, which is the (square) on HC [Prop. 1.47], is five times the (square) on KC. And HC (is) equal to CB. Thus, the (square) on BC (is) five times the (square) on CK. And since AB is double CB, of which AD is double DB, the remainder BD is thus double the remainder DC. BC (is) thus triple CD. The (square) on BC (is) thus nine times the (square) on CD. And the (square) on BC (is) five times the (square) on CK. Thus, the (square) on CK (is) greater than the (square) on CD. CK is thus greater than CD. Let CL be made equal to CK. And let LM be drawn from L at right-angles to AB. And let MB be joined. And since the (square) on BC is five times the (square) on CK, and AB is double BC, and KL double CK, the (square) on AB is thus five times the (square) on KL. And the square on the diameter of the sphere is also five times the (square) on the radius of the circle from which the icosahedron has been described [Prop. 13.16 corr.]. And AB is the diameter of the sphere. Thus, KL is the radius of the circle from which the icosahedron has been described. Thus, KL is (the side) of the hexagon (inscribed) in the aforementioned circle [Prop. 4.15 corr.]. And since the diameter of the sphere is composed of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the aforementioned circle, and AB is the diameter of the sphere, and KL the side of the hexagon, and AK (is) equal to LB, thus AK and LB are each sides of the decagon inscribed in the circle from which the icosahedron has been described. And since LB is (the side) of the decagon. And ML (is the side) of the hexagon—for (it is) equal to KL, since (it is) also (equal) to HK, for they are equally far from the center. And HK and KL are each double KC. MB is thus (the side) of the pentagon (inscribed in the circle) [Props. 13.10, 1.47]. And (the side) of the pentagon is (the side) of the icosahedron [Prop. 13.16]. Thus, MB is (the side) of the icosahedron.

And since FB is the side of the cube, let it be cut in extreme and mean ratio at N, and let NB be the greater piece. Thus, NB is the side of the dodecahedron [Prop. 13.17 corr.].

And since the (square) on the diameter of the sphere was shown (to be) one and a half times the square on the side, AF, of the pyramid, and twice the square on (the side), BE, of the octagon, and three times the square on (the side), FB, of the cube, thus, of whatever (parts) the (square) on the diameter of the sphere (makes) six, of such (parts) the (square) on (the side) of the pyramid (makes) four, and (the square) on (the side) of the octagon three, and (the square) on (the side) *τοῦ δωδεκαέδρου τῆς ΝΒ, δείξομεν οὕτως.*

Ἐπεὶ γὰρ ἰσογώνιόν ἐστι τὸ ΖΔΒ τρίγωνον τῷ ΖΑΒ τριγώνῳ, ἀνάλογόν ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΖ, οὕτως ἡ ΒΖ πρὸς τὴν ΒΑ. καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΑ, οὕτως τὸ ἀπὸ τῆς ΔΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· ἀνάπαλιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΒ πρὸς τὸ ἀπὸ τῆς ΒΔ. τριπλῆ δὲ ἡ ΑΒ τῆς ΒΔ· τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΖΒ τοῦ ἀπὸ τῆς ΒΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΔΒ τετραπλάσιον· διπλῆ γὰρ ἡ ΑΔ τῆς ΔΒ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΖΒ· μείζων ἄρα ἡ ΑΔ τῆς ΖΒ· πολλῷ ἄρα ἡ ΑΛ τῆς ΖΒ μείζων ἐστίν. καὶ τῆς μὲν ΑΛ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΚΛ, ἐπειδήπερ ἡ μὲν ΛΚ ἑξαγώνου ἐστίν, ἡ δὲ ΚΑ δεκαγώνου· τῆς δὲ ΖΒ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΝΒ· μείζων ἄρα ἡ ΚΛ τῆς ΝΒ. ἴση δὲ ἡ ΚΛ τῇ ΛΜ· μείζων ἄρα ἡ ΛΜ τῆς ΝΒ [τῆς δὲ ΛΜ μείζων ἐστὶν ἡ ΜΒ]. πολλῷ ἄρα ἡ ΜΒ πλευρὰ οὖσα τοῦ εἰκοσαέδρου μείζων ἐστὶ τῆς ΝΒ πλευρᾶς οὔσης τοῦ δωδεκαέδρου· ὅπερ ἔδει δεῖξαι.

of the cube two. Thus, the (square) on the side of the pyramid is one and a third times the square on the side of the octagon, and double the square on (the side) of the cube. And the (square) on (the side) of the octahedron is one and a half times the square on (the side) of the cube. Therefore, the aforementioned sides of the three figures—I mean, of the pyramid, and of the octahedron, and of the cube—are in rational ratios to one another. And (the sides of) the remaining two (figures)—I mean, of the icosahedron, and of the dodecahedron—are neither in rational ratios to one another, nor to the (sides) of the aforementioned (three figures). For they are irrational (straight-lines): (namely), a minor [Prop. 13.16], and an apotome [Prop. 13.17].

(And), we can show that the side, MB, of the icosahedron is greater that the (side), NB, or the dodecahedron, as follows.

For, since triangle FDB is equiangular to triangle FAB [Prop. 6.8], proportionally, as DB is to BF, so BF (is) to BA [Prop. 6.4]. And since three straight-lines are (continually) proportional, as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9, Prop. 6.20 corr.]. Thus, as DB is to BA, so the (square) on DB (is) to the (square) on BF. Thus, inversely, as AB (is) to BD, so the (square) on FB (is) to the (square) on BD. And AB (is) triple BD. Thus, the (square) on FB (is) three times the (square) on BD. And the (square) on AD is also four times the (square) on DB. For AD (is) double DB. Thus, the (square) on AD (is) greater than the (square) on FB. Thus, AD (is) greater than FB. Thus, AL is much greater than FB. And KL is the greater piece of AL, which is cut in extreme and mean ratio—inasmuch as LK is (the side) of the hexagon, and KA (the side) of the decagon [Prop. 13.9]. And NB is the greater piece of FB, which is cut in extreme and mean ratio. Thus, KL (is) greater than NB. And KL (is) equal to LM. Thus, LM (is) greater than NB [and MB is greater than LM]. Thus, MB, which is (the side) of the icosahedron, is much greater than NB, which is (the side) of the dodecahedron. (Which is) the very thing it was required to show.

† If the radius of the given sphere is unity then the sides of the pyramid (*i.e.*, tetrahedron), octahedron, cube, icosahedron, and dodecahedron, respectively, satisfy the following inequality: $\sqrt{8/3} > \sqrt{2} > \sqrt{4/3} > (1/\sqrt{5})\sqrt{10 - 2\sqrt{5}} > (1/3)(\sqrt{15} - \sqrt{3})$.

Λέγω δή, ὅτι παρὰ τὰ εἰρημένα πέντε σχήματα οὐ συσταθήσεται ἕτερον σχῆμα περιεχόμενον ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων ἴσων ἀλλήλοις.

Ὑπὸ μὲν γὰρ δύο τριγώνων ἢ ὅλως ἐπιπέδων στερεὰ γωνία οὐ συνίσταται. ὑπὸ δὲ τριῶν τριγώνων ἡ τῆς πυραμίδος, ὑπὸ δὲ τεσσάρων ἡ τοῦ ὀκταέδρου, ὑπὸ δὲ πέντε ἡ τοῦ εἰκοσαέδρου· ὑπὸ δὲ ἓξ τριγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων πρὸς ἑνὶ σημείῳ συνισταμένων οὐκ ἔσται στερεὰ γωνία· οὔσης γὰρ τῆς τοῦ ἰσοπλεύρου τριγώνου γωνίας διμοίρου ὀρθῆς ἔσον-

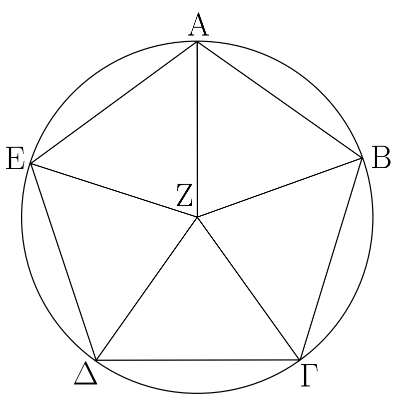
So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.

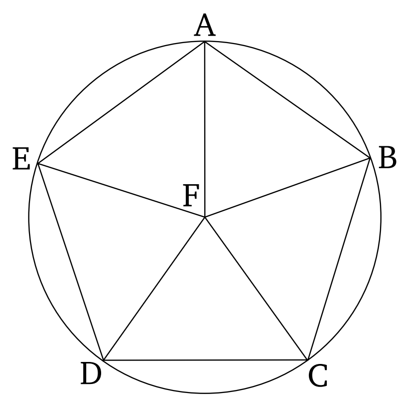
For a solid angle cannot be constructed from two triangles, or indeed (two) planes (of any sort) [Def. 11.11]. And (the solid angle) of the pyramid (is constructed) from three (equiangular) triangles, and (that) of the octahedron from four (triangles), and (that) of the icosahedron from (five) triangles. And a solid angle cannot be (made) from six equilateral and *ται αἱ ἓξ τέσσαρσιν ὀρθαῖς ἴσαι· ὅπερ ἀδύνατον· ἅπασα γὰρ στερεὰ γωνία ὑπὸ ἐλασσόνων ἢ τεσσάρων ὀρθῶν περιέχεται. διὰ τὰ αὐτὰ δὴ οὐδὲ ὑπὸ πλειόνων ἢ ἓξ γωνιῶν ἐπιπέδων στερεὰ γωνία συνίσταται. ὑπὸ δὲ τετραγώνων τριῶν ἡ τοῦ κύβου γωνία περιέχεται· ὑπὸ δὲ τεσσάρων ἀδύνατον· ἔσονται γὰρ πάλιν τέσσαρες ὀρθαί. ὑπὸ δὲ πενταγώνων ἰσοπλεύρων καὶ ἰσογωνίων, ὑπὸ μὲν τριῶν ἡ τοῦ δωδεκαέδρου· ὑπὸ δὲ τεσσάρων ἀδύνατον· οὔσης γὰρ τῆς τοῦ πενταγώνου ἰσοπλεύρου γωνίας ὀρθῆς καὶ πέμπτου, ἔσονται αἱ τέσσαρες γωνίαι τεσσάρων ὀρθῶν μείζους· ὅπερ ἀδύνατον. οὐδὲ μὴν ὑπὸ πολυγώνων ἑτέρων σχημάτων περισχεθήσεται στερεὰ γωνία διὰ τὸ αὐτὸ ἄτοπον.*

Οὐκ ἄρα παρὰ τὰ εἰρημένα πέντε σχήματα ἕτερον σχῆμα στερεὸν συσταθήσεται ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων περιεχόμενον· ὅπερ ἔδει δεῖξαι.

equiangular triangles set up together at one point. For, since the angles of a equilateral triangle are (each) two-thirds of a right-angle, the (sum of the) six (plane) angles (containing the solid angle) will be four right-angles. The very thing (is) impossible. For every solid angle is contained by (plane angles whose sum is) less than four right-angles [Prop. 11.21]. So, for the same (reasons), a solid angle cannot be constructed from more than six plane angles (equal to two-thirds of a right-angle) either. And the (solid) angle of a cube is contained by three squares. And (a solid angle contained) by four (squares is) impossible. For, again, the (sum of the plane angles containing the solid angle) will be four right-angles. And (the solid angle) of a dodecahedron (is contained) by three equilateral and equiangular pentagons. And (a solid angle contained) by four (equiangular pentagons is) impossible. For, the angle of an equilateral pentagon being one and one-fifth of right-angle, four (such) angles will be greater (in sum) than four right-angles. The very thing (is) impossible. And, on account of the same absurdity, a solid angle cannot be constructed from any other (equiangular) polygonal figures either.

Thus, beside the five aforementioned figures, no other solid figure can be constructed (which is) contained by equilateral and equiangular (planes). (Which is) the very thing it was required to show.





Λῆμμα.

Ὅτι δὲ ἡ τοῦ ἰσοπλεύρου καὶ ἰσογωνίου πενταγώνου γωνία ὀρθή ἐστι καὶ πέμπτου, οὕτω δεικτέον.

Ἔστω γὰρ πεντάγωνον ἰσόπλευρον καὶ ἰσογώνιον τὸ ΑΒΓΔΕ, καὶ περιγεγράφθω περὶ αὐτὸ κύκλος ὁ ΑΒΓΔΕ, καὶ εἰλήφθω αὐτοῦ τὸ κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ. δίχα ἄρα τέμνουσι τὰς πρὸς τοῖς Α, Β, Γ, Δ, Ε τοῦ πενταγώνου γωνίας. καὶ ἐπεὶ αἱ πρὸς τῷ Ζ πέντε γωνίαι τέσσαρσιν ὀρθαῖς ἴσαι εἰσὶ καί εἰσιν ἴσαι, μία ἄρα αὐτῶν, ὡς ἡ ὑπὸ ΑΖΒ, μιᾶς ὀρθῆς ἐστι παρὰ πέμπτον· λοιπαὶ ἄρα αἱ ὑπὸ ΖΑΒ, ΑΒΖ μιᾶς εἰσιν ὀρθῆς καὶ πέμπτου. ἴση δὲ ἡ ὑπὸ ΖΑΒ τῇ ὑπὸ ΖΒΓ· καὶ ὅλη ἄρα ἡ ὑπὸ ΑΒΓ τοῦ πενταγώνου γωνία μιᾶς ἐστιν ὀρθῆς καὶ πέμπτου· ὅπερ ἔδει δεῖξαι.

Lemma

It can be shown that the angle of an equilateral and equiangular pentagon is one and one-fifth of a right-angle, as follows.

For let $ABCDE$ be an equilateral and equiangular pentagon, and let the circle $ABCDE$ be circumscribed about it [Prop. 4.14]. And let its center, F, be found [Prop. 3.1]. And let FA, FB, FC, FD, and FE be joined. Thus, they cut the angles of the pentagon in half at (points) A, B, C, D, and E [Prop. 1.4]. And since the five angles at F are equal (in sum) to four right-angles, and are also equal (to one another), (any) one of them, like AFB, is thus one less a fifth of a right-angle. Thus, the (sum of the) remaining (angles in triangle ABF), FAB and ABF, is one plus a fifth of a right-angle [Prop. 1.32].

And FAB (is) equal to FBC. Thus, the whole angle, ABC, of the pentagon is also one and one-fifth of a right-angle. (Which is) the very thing it was required to show.

GREEK-ENGLISH LEXICON

ABBREVIATIONS: *act* - active; *adj* - adjective; *adv* - adverb; *conj* - conjunction; *fut* - future; *gen* - genitive; *imperat* - imperative; *impf* - imperfect; *ind* - indeclinable; *indic* - indicative; *intr* - intransitive; *mid* - middle; *neut* - neuter; *no* - noun; *par* - particle; *part* - participle; *pass* - passive; *perf* - perfect; *pre* - preposition; *pres* - present; *pro* - pronoun; *sg* - singular; *tr* - transitive; *vb* - verb.

ἄγω, *ἄξω, ἤγαγον, -῏ηχα, ῏ηγμαι, ἤχθην* : *vb*, lead, draw (a line).

ἀδύνατος -ον : *adj*, impossible.

ἀεί : *adv*, always, for ever.

αἱρέω, αἱρήσω, ε[ἷ]λον, ᾕρηκα, ᾕρημαι, ᾑρέθην : *vb*, grasp.

ἀιτέω, αἰτήσω, ᾔτησα, ᾔτηκα, ᾔτημαι, ᾐτήθη : *vb*, postulate.

αἴτημα -ατος, τό : *no*, postulate.

ἀκόλουθος -ον : *adj*, analogous, consequent on, in conformity with.

ἄκρος -α -ον : *adj*, outermost, end, extreme.

ἀλλά : *conj*, but, otherwise.

ἄλογος -ον : *adj*, irrational.

ἅμα : *adv*, at once, at the same time, together.

ἀμβλυγώνιος -ον : *adj*, obtuse-angled; *τὸ ἀμβλυγώνιον*, *no*, obtuse angle.

ἀμβλύς -εῖα -ύ : *adj*, obtuse.

ἀμφότερος -α -ον : *pro*, both.

ἀναγράφω : *vb*, describe (a figure); see *γράφω.*

ἀναλογία, ἡ : *no*, proportion, (geometric) progression.

ἀνάλογος -ον : *adj*, proportional.

ἀνάπαλιν : *adv*, inverse(ly).

αναπληρόω : *vb*, fill up.

ἀναστρέφω : *vb*, turn upside down, convert (ratio); see *στρέφω.*

ἀναστροφή, ἡ : *no*, turning upside down, conversion (of ratio).

ἀνθυφαιρέω : *vb*, take away in turn; see *αἱρέω.*

ἀνίστημι : *vb*, set up; see *ἵστημι.*

ἄνισος -ον : *adj*, unequal, uneven.

ἀντιπάσχω : *vb*, be reciprocally proportional; see *πάσχω.*

ἄξων -ονος, ὁ : *vb*, axis.

ἅπαξ : *adv*, once.

ἅπας, ἅπασα, ἅπαν : *adj*, quite all, the whole.

ἄπειρος -ον : *adj*, infinite.

ἀπεναντίον : *ind*, opposite.

ἀπέχω : *vb*, be far from, be away from; see *ἔχω.*

ἀπλατής -ές : *adj*, without breadth.

ἀπόδειξις -εως, ἡ : *no*, proof.

ἀποκαθίστημι : *vb*, re-establish, restore; see *ἵστημι.*

ἀπολαμβάνω : *vb*, take from, subtract from, cut off from; see *λαμβάνω.*

ἀποτέμνω : *vb*, cut off, subtend.

ἀπότμημα -ατος, τό : *no*, piece cut off, segment.

ἀποτομή, ἡ : *vb*, piece cut off, apotome.

ἅπτω, ἅψω, ῞ηψα, —, ῞ημμαι, — : *vb*, touch, join, meet.

ἀπώτερος -α -ον : *adj*, further off.

ἄρα : *par*, thus, as it seems (inferential).

ἀριθμός, ὁ : *no*, number.

ἀρτιάκις : *adv*, an even number of times.

αρτιόπλευρος -ον : *adj*, having a even number of sides.

ἄρχω, ἄρξω, ῏ηρξα, ῏ηρχα, ῏ηργμαι, ῏ηρχθην : *vb*, rule; *mid.*, begin.

ἀσύμμετρος -ον : *adj*, incommensurable.

ἀσύμπτωτος -ον : *adj*, not touching, not meeting.

ἄρτιος -α -ον : *adj*, even, perfect.

ἄτμητος -ον : *adj*, uncut.

ἀτόπος -ον : *adj*, absurd, paradoxical.

αὐτόθεν : *adv*, immediately, obviously.

ἀφαίρεω : *vb*, take from, subtract from, cut off from; see *αἱρέω.*

ἀφή, ἡ : *no*, point of contact.

βάθος -εος, τό : *no*, depth, height.

βαίνω, -βήσομαι, -έβην, βέβηκα, —, — : *vb*, walk; *perf*, stand (of angle).

βάλλω, βαλῶ, ἔβαλον, βέβληκα, βέβλημαι, ἐβλήθην : *vb*, throw.

βάσις -εως, ἡ : *no*, base (of a triangle).

γάρ : *conj*, for (explanatory).

γί[γ]νομαι, γενήσομαι, ἐγενόμην, γέγονα, γεγένημαι, — : *vb*, happen, become.

γνώμων -ονος, ἡ : *no*, gnomon.

γραμμή, ἡ : *no*, line.

γράφω, γράψω, ἔγρα[ψ/φ]α, γέγραφα, γέγραμμαι, ἐραψάμην : *vb*, draw (a figure).

γωνία, ἡ : *no*, angle.

δεῖ : *vb*, be necessary; *δεῖ*, it is necessary; *ἔδει*, it was necassary; *δέον*, being necessary.

δείκνυμι, δείξω, ἔδειξα, δέδειχα, δέδειγμαι, ἐδείχθην : *vb*, show, demonstrate.

δεικτέον : *ind*, one must show.

δεῖξις -εως, ἡ : *no*, proof.

δεκαγώνος -ον : *adj*, ten-sided; *τὸ δεκαγώνον*, *no*, decagon.

δέχομαι, δέξομαι, ἐδεξάμην, —, δέδεγμαι, ἐδέχθην : *vb*, receive, accept.

δή : *conj*, so (explanatory).

δηλαδή : *ind*, quite clear, manifest.

δῆλος -η -ον : *adj*, clear.

δηλονότι : *adv*, manifestly.

διάγω : *vb*, carry over, draw through, draw across; see *ἄγω.*

διαγώνιος -ον : *adj*, diagonal.

διαλείπω : *vb*, leave an interval between.

διάμετρος -ον : *adj*, diametrical; *ἡ διάμετρος*, *no*, diameter, diagonal.

διαίρεσις -εως, ἡ : *no*, division, separation.

διαιρέω : *vb*, divide (in two); *διαρεθέντος -η -ον*, *adj*, separated (ratio); see *αἱρέω.*

διάστημα -ατος, τό : *no*, radius.

διαφέρω : *vb*, differ; see *φέρω.*

δίδωμι, δώσω, ἔδωκα, δέδωκα, δέδομαι, ἐδόθην : *vb*, give.

διμοίρος -ον : *adj*, two-thirds.

διπλασιάζω : *vb*, double.

διπλάσιος -α -ον : *adj*, double, twofold.

διπλασίων -ον : *adj*, double, twofold.

διπλοῦς -ῆ -οῦν : *adj*, double.

δίς : *adv*, twice.

δίχα : *adv*, in two, in half.

διχορομία, ἡ : *no*, point of bisection.

δυάς -άδος, ἡ : *no*, the number two, dyad.

δύναμαι : *vb*, be able, be capable, generate, square, be when squared; *δυναμένη, ἡ, no*, square-root (of area)—*i.e.*, straight-line whose square is equal to a given area.

δύναμις -εως, ἡ : *no*, power (usually 2nd power when used in mathematical sence, hence), square.

δυνατός -ή -όν : *adj*, possible.

δωδεκάεδρος -ον : *adj*, twelve-sided.

ἑαυτοῦ -ῆς -οῦ : *adj*, of him/her/it/self, his/her/its/own.

ἐγγίων -ον : *adj*, nearer, nearest.

ἐγγράφω : *vb*, inscribe; see *γράφω.*

εἶδος -εος, τό : *no*, figure, form, shape.

εἰκοσάεδρος -ον : *adj*, twenty-sided.

εἴρω/λέγω, ἐρῶ/ερέω, εἶπον, εἴρηκα, εἴρημαι, ἐρρήθην : *vb*, say, speak; *per pass part*, *ειρημένος -η -ον, adj*, said, aforementioned.

εἴτε ... εἴτε : *ind*, either ... or.

ἕκαστος -η -ον : *pro*, each, every one.

ἑκατέρος -α -ον : *pro*, each (of two).

ἐκβάλλω : *vb*, produce (a line); see *βάλλω.*

ἐκθέω : *vb*, set out.

ἔκκειμαι : *vb*, be set out, be taken; see *κεῖμαι.*

ἐκτίθημι : *vb*, set out; see *τίθημι.*

ἐκτός : *pre + gen*, outside, external.

ἐλάσσων/ἐλάττων -ον : *adj*, less, lesser.

ἐλάχιστος -η -ον : *adj*, least.

ἐλλείπω : *vb*, be less than, fall short of.

ἐμπίπτω : *vb*, meet (of lines), fall on; see *πίπτω.*

ἔμπροσθεν : *adv*, in front.

ἐναλλάξ : *adv*, alternate(ly).

ἐναρμόζω : *vb*, insert; *perf indic pass 3rd sg*, *ἐνήρμοσται.*

ἐνδέχομαι : *vb*, admit, allow.

ἕνεκεν : *ind*, on account of, for the sake of.

ἐνναπλάσιος -α -ον : *adj*, nine-fold, nine-times.

ἔννοια, ἡ : *no*, notion.

ενπεριέχω : *vb*, encompass; see *ἔχω.*

ἐνπίπτω : see *ἐμπίπτω.*

ἐντός : *pre + gen*, inside, interior, within, internal.

ἑξάγωνος -ον : *adj*, hexagonal; *τὸ ἑξάγωνον, no*, hexagon.

ἑξαπλάσιος -α -ον : *adj*, sixfold.

ἑξῆς : *adv*, in order, successively, consecutively.

ἔξωθεν : *adv*, outside, extrinsic.

ἐπάνω : *adv*, above.

ἐπαφή, ἡ : *no*, point of contact.

ἐπεί : *conj*, since (causal).

ἐπειδήπερ : *ind*, inasmuch as, seeing that.

ἐπιζεύγνῦμι, ἐπιζεύξω, ἐπέζευξα, —, ἐπέζευγμαι, ἐπέζεύχθην : *vb*, join (by a line).

ἐπιλογίζομαι : *vb*, conclude.

ἐπινοέω : *vb*, think of, contrive.

ἐπιπέδος -ον : *adj*, level, flat, plane; *τὸ ἐπιπέδον, no*, plane.

ἐπισκέπτομαι : *vb*, investigate.

ἐπίσκεψις -εως, ἡ : *no*, inspection, investigation.

ἐπιτάσσω : *vb*, put upon, enjoin; *τὸ ἐπιταχθέν, no*, the (thing) prescribed; see *τάσσω.*

ἐπίτριτος -ον : *adj*, one and a third times.

ἐπιφάνεια, ἡ : *no*, surface.

ἕπομαι : *vb*, follow.

ἔρχομαι, ἐλεύσομαι, ἦλθον, ἐλήλυθα, —, — : *vb*, come, go.

ἔσχατος -η -ον : *adj*, outermost, uttermost, last.

ἑτερόμηκης -ες : *adj*, oblong; *τὸ ἑτερόμηκες, no*, rectangle.

ἕτερος -α -ον : *adj*, other (of two).

ἔτι : *par*, yet, still, besides.

εὐθύγραμμος -ον : *adj*, rectilinear; *τὸ εὐθύγραμμον, no*, rectilinear figure.

εὐθύς -εῖα -ύ : *adj*, straight; *ἡ εὐθεῖα, no*, straight-line; *ἐπ᾽ εὐθεῖας*, in a straight-line, straight-on.

εὑρίσκω, εὑρήσκω, ηὗρον, εὕρεκα, εὕρημαι, εὑρέθην : *vb*, find.

ἐφάπτω : *vb*, bind to; *mid*, touch; *ἡ ἐφαπτομένη, no*, tangent; see *ἅπτω.*

ἐφαρμόζω, ἐφαρμόσω, ἐφήρμοσα, ἐφήμοκα, ἐφήμοσμαι, ἐφήμόσθην : *vb*, coincide; *pass*, be applied.

ἐφεξῆς : *adv*, in order, adjacent.

ἐφίστημι : *vb*, set, stand, place upon; see *ἵστημι.*

ἔχω, ἕξω, ἔσχον, ἔσχηκα, -έσχημαι, — : *vb*, have.

ἡγέομαι, ἡγήσομαι, ἡγησάμην, ἡγημαι, —, ἡγήθην : *vb*, lead.

ἤδη : *ind*, already, now.

ἥκω, ἥξω, —, —, —, — : *vb*, have come, be present.

ἡμικύκλιον, τό : *no*, semi-circle.

ἡμιόλιος -α -ον : *adj*, containing one and a half, one and a half times.

ἥμισυς -εια -υ : *adj*, half.

ἤπερ = ἤ + περ : *conj*, than, than indeed.

ἤτοι ... ἤ : *par*, surely, either ... or; in fact, either ... or.

θέσις -εως, ἡ : *no*, placing, setting, position.

θεωρημα -ατος, τό : *no*, theorem.

ἴδιος -α -ον : *adj*, one's own.

ἰσάκις : *adv*, the same number of times; *ἰσάκις πολλαπλάσια*, the same multiples, equal multiples.

ἰσογώνιος -ον : *adj*, equiangular.

ἰσόπλευρος -ον : *adj*, equilateral.

ἰσοπληθής -ές : *adj*, equal in number.

ἴσος -η -ον : *adj*, equal; *ἐξ ἴσου*, equally, evenly.

ἰσοσκελής -ές : *adj*, isosceles.

ἵστημι, στήσω, ἔστησα, —, —, ἐστάθην : *vb tr*, stand (something).

ἵστημι, στήσω, ἔστην, ἕστηκα, ἕσταμαι, ἐσταθην : *vb intr*, stand up (oneself); Note: perfect *I have stood up* can be taken to mean present *I am standing*.

ἰσοϋψής -ές : *adj*, of equal height.

καθάπερ : *ind*, according as, just as.

κάθετος -ον : *adj*, perpendicular.

καθόλου : *adv*, on the whole, in general.

καλέω : *vb*, call.

κἀκεινος = καὶ ἐκεῖνος .

κἄν = καὶ ἄν : *ind*, even if, and if.

καταγραφή, ἡ : *no*, diagram, figure.

καταγράφω : *vb*, describe/draw, inscribe (a figure); see *γράφω*.

κατακολουθέω : *vb*, follow after.

καταλείπω : *vb*, leave behind; see *λείπω*; *τὰ καταλειπόμενα*, *no*, remainder.

κατάλληλος -ον : *adj*, in succession, in corresponding order.

καταμετρέω : *vb*, measure (exactly).

καταντάω : *vb*, come to, arrive at.

κατασκευάζω : *vb*, furnish, construct.

κεῖμαι, κεῖσομαι, —, —, —, — : *vb*, have been placed, lie, be made; see *τίθημι*.

κέντρον, τό : *no*, center.

κλάω : *vb*, break off, inflect.

κλίνω, κλίνω, ἔκλινα, κέκλικα, κέκλιμαι, ἐκλίθην : *vb*, lean, incline.

κλίσις -εως, ἡ : *no*, inclination, bending.

κοῖλος -η -ον : *adj*, hollow, concave.

κορυφή, ἡ : *no*, top, summit, apex; *κατὰ κορυφήν*, vertically opposite (of angles).

κρίνω, κρινῶ, ἔκρῖνα, κέκρικα, κέκριμαι, ἐκρίθην : *vb*, judge.

κύβος, ὁ : *no*, cube.

κύκλος, ὁ : *no*, circle.

κύλινδρος, ὁ : *no*, cylinder.

κυρτός -ή -όν : *adj*, convex.

κῶνος, ὁ : *no*, cone.

λαμβάνω, λήψομαι, ἔλαβον, εἴληφα εἴλημμαι, ἐλήφθην : *vb*, take.

λέγω : *vb*, say; *pres pass part*, *λεγόμενος -η -ον*, *adj*, so-called; see *ἔιρω*.

λείπω, λείψω, ἔλιπον, λέλοιπα, λέλειμμαι, ἐλείφθην : *vb*, leave, leave behind.

λημμάτιον, τό : *no*, diminutive of *λῆμμα*.

λῆμμα -ατος, τό : *no*, lemma.

λῆψις -εως, ἡ : *no*, taking, catching.

λόγος, ὁ : *no*, ratio, proportion, argument.

λοιπός -ή -όν : *adj*, remaining.

μανθάνω, μαθήσομαι, ἔμαθον, μεμάθηκα, —, — : *vb*, learn.

μέγεθος -εος, τό : *no*, magnitude, size.

μείζων -ον : *adj*, greater.

μένω, μενῶ, ἔμεινα, μεμένηκα, —, — : *vb*, stay, remain.

μέρος -ους, τό : *no*, part, direction, side.

μέσος -η -ον : *adj*, middle, mean, medial; *ἐκ δύο μέσων*, bimedial.

μεταλαμβάνω : *vb*, take up.

μεταξύ : *adv*, between.

μετέωρος -ον : *adj*, raised off the ground.

μετρέω : *vb*, measure.

μέτρον, τό : *no*, measure.

μηδείς, μηδεμία, μηδέν : *adj*, not even one, (neut.) nothing.

μηδέποτε : *adv*, never.

μηδέτερος -α -ον : *pro*, neither (of two).

μῆκος -εος, τό : *no*, length.

μήν : *par*, truely, indeed.

μονάς -άδος, ἡ : *no*, unit, unity.

μοναχός -ή -όν : *adj*, unique.

μοναχῶς : *adv*, uniquely.

μόνος -η -ον : *adj*, alone.

νοέω, —, νόησα, νενόηκα, νενόημαι, ἐνοήθην : *vb*, apprehend, conceive.

οἷος -α -ον : *pre*, such as, of what sort.

ὀκτάεδρος -ον : *adj*, eight-sided.

ὅλος -η -ον : *adj*, whole.

ὁμογενής -ές : *adj*, of the same kind.

ὅμοιος -α -ον : *adj*, similar.

ὁμοιοπληθής -ές : *adj*, similar in number.

ὁμοιοταγής -ές : *adj*, similarly arranged.

ὁμοιότης -ητος, ἡ : *no* similarity.

ὁμοίως : *adv*, similarly.

ὁμόλογος -ον : *adj*, corresponding, homologous.

ὁμοταγής -ές : *adj*, ranged in the same row or line.

ὁμώνυμος -ον : *adj*, having the same name.

ὄνομα -ατος, τό : *no*, name; *ἐκ δύο ὀνομάτων*, binomial.

ὀξυγώνιος -ον : *adj*, acute-angled; *τὸ ὀξυγώνιον*, *no*, acute angle.

ὀξύς -εῖα -ύ : *adj*, acute.

ὁποιοσοῦν = ὁποῖος -α -ον + οὖν : *adj*, of whatever kind, any kind whatsoever.

ὁπόσος -η -ον : *pro*, as many, as many as.

ὁποσοσδηποτοῦν = ὁπόσος -η -ον + δή + ποτέ + οὖν : *adj*, of whatever number, any number whatsoever.

ὁποσοσοῦν = ὁπόσος -η -ον + οὖν : *adj*, of whatever number, any number whatsoever.

ὁπότερος -α -ον : *pro*, either (of two), which (of two).

ὀρθογώνιον, τό : *no*, rectangle, right-angle.

ὀρθός -ή -όν : *adj*, straight, right-angled, perpendicular; *πρὸς ὀρθὰς γωνίας*, at right-angles.

ὅρος, ὁ : *no*, boundary, definition, term (of a ratio).

ὁσαδηποτοῦν = ὅσα + δή + ποτέ + οὖν : *ind*, any number whatsoever.

ὁσάκις : *ind*, as many times as, as often as.
ὁσαπλάσιος -ον : *pro*, as many times as.
ὅσος -η -ον : *pro*, as many as.
ὅσπερ, ἥπερ, ὅπερ : *pro*, the very man who, the very thing which.
ὅστις, ἥτις, ὅ τι : *pro*, anyone who, anything which.
ὅταν : *adv*, when, whenever.
ὁτιοῦν : *ind*, whatsoever.
οὐδείς, οὐδεμία, οὐδέν : *pro*, not one, nothing.
οὐδέτερος -α -ον : *pro*, not either.
οὐθέτερος : see *οὐδέτερος*.
οὐθέν : *ind*, nothing.
οὖν : *adv*, therefore, in fact.
οὕτως : *adv*, thusly, in this case.
πάλιν : *adv*, back, again.
πάντως : *adv*, in all ways.
παρὰ : *prep* + *acc*, parallel to.
παραβάλλω : *vb*, apply (a figure); see *βάλλω*.
παραβολή, ἡ : *no*, application.
παράκειμαι : *vb*, lie beside, apply (a figure); see *κεῖμαι*.
παραλλάσσω, παραλλάξω, —, παρήλλαχα, —, — : *vb*, miss, fall awry.
παραλληλεπίπεδος, -ον : *adj*, with parallel surfaces; *τὸ παραλληλεπίπεδον*, *no*, parallelepiped.
παραλληλόγραμμος -ον : *adj*, bounded by parallel lines; *τὸ παραλληλόγραμμον*, *no*, parallelogram.
παράλληλος -ον : *adj*, parallel; *τὸ παράλληλον*, *no*, parallel, parallel-line.
παραπλήρωμα -ατος, τό : *no*, complement (of a parallelogram).
παρατέλυετος -ον : *adj*, penultimate.
παρέκ : *prep* + *gen*, except.
παρεμπίπτω : *vb*, insert; see *πίπτω*.
πάσχω, πείσομαι, ἔπαθον, πέπονθα, —, — : *vb*, suffer.
πεντάγωνος -ον : *adj*, pentagonal; *τὸ πεντάγωνον*, *no*, pentagon.
πενταπλάσιος -α -ον : *adj*, five-fold, five-times.
πεντεκαιδεκάγωνον, τό : *no*, fifteen-sided figure.
πεπερασμένος -η -ον : *adj*, finite, limited; see *περαίνω*.
περαίνω, περανῶ, ἐπέρανα, —, πεπεράσμαι, ἐπερανάνθην : *vb*, bring to end, finish, complete; *pass*, be finite.
πέρας -ατος, τό : *no*, end, extremity.
περατόω, —, —, —, —, — : *vb*, bring to an end.
περιγράφω : *vb*, circumscribe; see *γράφω*.
περιέχω : *vb*, encompass, surround, contain, comprise; see *ἔχω*.
περιλαμβάνω : *vb*, enclose; see *λαμβάνω*.
περισσάκις : *adv*, an odd number of times.
περισσός -ή -όν : *adj*, odd.
περιφέρεια, ἡ : *no*, circumference.
περιφέρω : *vb*, carry round; see *φέρω*.
πηλικότης -ητος, ἡ : *no*, magnitude, size.
πίπτω, πεσοῦμαι, ἔπεσον, πέπτωκα, —, — : *vb*, fall.
πλάτος -εος, τό : *no*, breadth, width.
πλείων -ον : *adj*, more, several.
πλευρά, ἡ : *no*, side.
πλῆθος -εος, τό : *no*, great number, multitude, number.
πλήν : *adv & prep* + *gen*, more than.
ποιός -ά -όν : *adj*, of a certain nature, kind, quality, type.
πολλαπλασιάζω : *vb*, multiply.
πολλαπλασιασμός, ὁ : *no*, multiplication.
πολλαπλάσιον, τό : *no*, multiple.
πολύεδρος -ον : *adj*, polyhedral; *τό πολύεδρον*, *no*, polyhedron.
πολύγωνος -ον : *adj*, polygonal; *τό πολύγωνον*, *no*, polygon.
πολύπλευρος -ον : *adj*, multilateral.
πόρισμα -ατος, τό : *no*, corollary.
ποτέ : *ind*, at some time.
πρῖσμα -ατος, τό : *no*, prism.
προβαίνω : *vb*, step forward, advance.
προδείκνυμι : *vb*, show previously; see *δείκνυμι*.
προεκτίθημι : *vb*, set forth beforehand; see *τίθημι*.
προερέω : *vb*, say beforehand; *perf pass part*, *προειρημένος -η -ον*, *adj*, aforementioned; see *εἴρω*.
προσαναπληρόω : *vb*, fill up, complete.
προσαναγράφω : *vb*, complete (tracing of); see *γράφω*.
προσαρμόζω : *vb*, fit to, attach to.
προσεκβάλλω : *vb*, produce (a line); see *ἐκβάλλω*.
προσευρίσκω : *vb*, find besides, find; see *εὑρίσκω*.
προσλαμβάμω : *vb*, add.
πρόκειμαι : *vb*, set before, prescribe; see *κεῖμαι*.
πρόσκειμαι : *vb*, be laid on, have been added to; see *κεῖμαι*.
προσπίπτω : *vb*, fall on, fall toward, meet; see *πίπτω*.
προτασις -εως, ἡ : *no*, proposition.
προστάσσω : *vb*, prescribe, enjoin; *τὸ τροσταχθέν*, *no*, the thing prescribed; see *τάσσω*.
προστίθημι : *vb*, add; see *τίθημι*.
πρότερος -α -ον : *adj*, first (comparative), before, former.
προτίθημι : *vb*, assign; see *τίθημι*.
προχωρέω : *vb*, go/come forward, advance.
πρῶτος -α -ον : *adj*, first, prime.
πυραμίς -ίδος, ἡ : *no*, pyramid.
ῥητός -ή -όν : *adj*, expressible, rational.
ῥομβοειδής -ές : *adj*, rhomboidal; *τὸ ῥομβοειδές*, *no*, romboid.
ῥόμβος, ὁ *no*, rhombus.
σημεῖον, τό : *no*, point.
σκαληνός -ή -όν : *adj*, scalene.
στερεός -ά -όν : *adj*, solid; *τὸ στερεόν*, *no*, solid, solid body.
στοιχεῖον, τό : *no*, element.
στρέφω, -στρέψω, ἔστρεψα, —, ἐσταμμαι, ἐστάφην : *vb*, turn.
σύγκειμαι : *vb*, lie together, be the sum of, be composed; *συγκείμενος -η -ον*, *adj*, composed (ratio), compounded; see *κεῖμαι*.

σύγκρίνω : *vb*, compare; see *κρίνω*.

συμβαίνω : *vb*, come to pass, happen, follow; see *βαίνω*.

συμβάλλω : *vb*, throw together, meet; see *βάλλω*.

σύμμετρος -ον : *adj*, commensurable.

σύμπας -αντος, ὁ : *no*, sum, whole.

συμπίπτω : *vb*, meet together (of lines); see *πίπτω*.

συμπληρόω : *vb*, complete (a figure), fill in.

συνάγω : *vb*, conclude, infer; see *ἄγω*.

συναμφότεροι -αι -α : *adj*, both together; *ὁ συναμφότερος*, *no*, sum (of two things).

συναποδείκνυμι : *no*, demonstrate together; see *δείκνυμι*.

συναφή, ἡ : *no*, point of junction.

σύνδυο, οἱ, αἱ, τά : *no*, two together, in pairs.

συνεχής -ές : *adj*, continuous; *κατὰ τὸ συνεχές*, continuously.

σύνθεσις -εως, ἡ : *no*, putting together, composition.

σύνθετος -ον : *adj*, composite.

συ[ν]ίστημι : *vb*, construct (a figure), set up together; *perf imperat pass 3rd sg*, *συνεστάτω*; see *ἵστημι*.

συντίθημι : *vb*, put together, add together, compound (ratio); see *τίθημι*.

σχέσις -εως, ἡ : *no*, state, condition.

σχῆμα -ατος, τό : *no*, figure.

σφαῖρα -ας, ἡ : *no*, sphere.

τάξις -εως, ἡ : *no*, arrangement, order.

ταράσσω, ταράξω, —, —, τετάραγμαι, ἐταράχθην : *vb*, stir, trouble, disturbe; *τεταραγμένος -η -ον*, *adj*, disturbed, perturbed.

τάσσω, τάξω, ἔταξα, τέταχα, τέταγμαι, ἐτάχθην : *vb*, arrange, draw up.

τέλειος -α -ον : *adj*, perfect.

τέμνω, τεμνῶ, ἔτεμον, -τέτμηκα, τέτμημαι, ἐτμήθην : *vb*, cut; *pres/fut indic act 3rd sg*, *τέμει*.

τεταρτημοριον, τό : *no*, quadrant.

τετράγωνος -ον : *adj*, square; *τὸ τετράγωνον*, *no*, square.

τετράκις : *adv*, four times.

τετραπλάσιος -α -ον : *adj*, quadruple.

τετράπλευρος -ον : *adj*, quadrilateral.

τετραπλόος -η -ον : *adj*, fourfold.

τίθημι, θήσω, ἔθηκα, τέθηκα, κεῖμαι, ἐτέθην : *vb*, place, put.

τμῆμα -ατος, τό : *no*, part cut off, piece, segment.

τοίνυν : *par*, accordingly.

τοιοῦτος -αύτη -οῦτο : *pro*, such as this.

τομεύς -έως, ὁ : *no*, sector (of circle).

τομή, ἡ : *no*, cutting, stump, piece.

τόπος, ὁ : *no*, place, space.

τοσαυτάκις : *adv*, so many times.

τοσαυταπλάσιος -α -ον : *pro*, so many times.

τοσοῦτος -αύτη -οῦτο : *pro*, so many.

τουτέστι = τοῦτ᾽ ἔστι : *par*, that is to say.

τραπέζιον, τό : *no*, trapezium.

τρίγωνος -ον : *adj*, triangular; *τὸ τρίγωνον*, *no*, triangle.

τριπλάσιος -α -ον : *adj*, triple, threefold.

τρίπλευρος -ον : *adj*, trilateral.

τριπλ -όος -η -ον : *adj*, triple.

τρόπος, ὁ : *no*, way.

τυγχάνω, *τεύξομαι, ἔτυχον, τετύχηκα, τέτευγμαι, ἐτεύχθην* : *vb*, hit, happen to be at (a place).

ὑπάρχω : *vb*, begin, be, exist; see *ἄρχω*.

ὑπεξαίρεσις -εως, ἡ : *no*, removal.

ὑπερβάλλω : *vb*, overshoot, exceed; see *βάλλω*.

ὑπεροχή, ἡ : *no*, excess, difference.

ὑπερέχω : *vb*, exceed; see *ἔχω*.

ὑπόθεσις -εως, ἡ : *no*, hypothesis.

ὑπόκειμαι : *vb*, underlie, be assumed (as hypothesis); see *κεῖμαι*.

ὑπολείπω : *vb*, leave remaining.

ὑποτείνω, ὑποτενῶ, ὑπέτεινα, ὑποτέτακα, ὑποτέταμαι, ὑπετάθην : *vb*, subtend.

ὕψος -εος, τό : *no*, height.

φανερός -ά -όν : *adj*, visible, manifest.

φημί, φήσω, ἔφην, —, —, — : *vb*, say; *ἔφαμεν*, we said.

φέρω, οἴσω, ἤνεγκον, ἐνήνοχα, ἐνήνεγμαι, ἠνέχθην : *vb*, carry.

χώριον, τό : *no*, place, spot, area, figure.

χωρίς : *pre + gen*, apart from.

ψαύω : *vb*, touch.

ὡς : *par*, as, like, for instance.

ὡς ἔτυχεν : *par*, at random.

ὡσαύτως : *adv*, in the same manner, just so.

ὥστε : *conj*, so that (causal), hence.