Time-Dependent Parallel Electron Energy Transport in a Magnetized Plasma of Arbitrary Collisonality

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I. INTRODUCTION

In Hazeltine (1998),[1] the steady-state transport of electron number density and energy parallel to the magnetic field of a magnetized, weakly-coupled, electron-ion plasma of arbitrary collisionality is investigated in slab geometry by solving a simplified one-dimensional kinetic equation for the electron distribution function that employs a Bhatnagar-Gross-Krook (BGK) electron-electron collision operator.[2] The resulting model is able to successfully reproduce standard results for the the electron heat flux in both the short mean-free-path and the long mean-free-path limits. In the short mean-free-path limit, electron energy transport is found to be local and diffusive in nature, whereas the transport is found to be non-local and convective in the long mean-free-path limit.

The aim of this paper is to generalize the analysis of Hazeltine (1998) by incorporating a model electron-ion collision operator, including a self-consistent calculation of the parallel electric field, and taking time dependence into account. The resulting enhanced model is used to investigate the transport of electron energy across a magnetic island chain in a tokamak plasma.

II. FUNDAMENTAL MODEL

A. Electron Distribution Function

Let $f_e(t,\mathbf{x},\mathbf{v})$ be the ensemble-averaged electron distribution function. Here, t denotes time, $\mathbf{x} = (x_1,\, x_2,\, x_3)$ is a position vector, x_1, x_2, x_3 are Cartesian coordinates that are defined such that the x_3-axis is parallel to the local equilibrium magnetic field, and $\mathbf{v}$ is the electron velocity. Let us write

$$f_e(t,\mathbf{x},\mathbf{v}) = n_e\, F(v_1)\, F(v_2)\, [F(v_3) + f(t,x_3,v_3)]\,, \tag{1}$$

where

$$F(v) = \frac{\exp(-v^2/v_{t\,e}^2)}{\pi^{1/2}\, v_{t\,e}}, \tag{2}$$

and $|f/F| \ll 1$. Here, n_e is the unperturbed electron number density,

$$v_{t\,e} = \sqrt{\frac{2\,T_e}{m_e}} \tag{3}$$

the electron thermal velocity, m_e the electron mass, and T_e the unperturbed electron temperature (measured in energy units). Note that we are assuming that the electron distribution function remains relatively close to a Maxwellian distribution.

B. Electron-Electron Collision Operator

The electron-electron collision operator is modeled as a BGK operator: [1,2]

$$C_{ee}(f_e) = -\nu_{ee}\,F(v_1)\,F(v_2)\left\{n_e\,F(v_3) + n_e\,f(t,x_3,v_3) - \frac{[n_e+\delta n_e(t,x_3)]\,m_e^{1/2}}{\pi^{1/2}\,(2\,[T_e+\delta T_e(t,x_3)])^{1/2}}\,\exp\left[-\frac{[v_3-V_e(t,x_3)]^2\,m_e}{2\,[T_e+\delta T_e(t,x_3)]}\right]\right\}. \tag{4}$$

Here, ν_{ee} is the electron-electron collision frequency. Moreover, $|\delta n_e/n_e| \ll 1$, $|V_e/v_3| \ll 1$, and $|\delta T_e/T_e| \ll 1$. It can be seen that the operator acts to relax the distribution function to the perturbed Maxwellian

$$F(v_1)\,F(v_2)\,\frac{[n_e+\delta n_e(t,x_3)]\,m_e^{1/2}}{\pi^{1/2}\,(2\,[T_e+\delta T_e(t,x_3)])^{1/2}}\,\exp\left[-\frac{[v_3-V_3(t,x_3)]^2\,m_e}{2\,[T_e+\delta T_e(t,x_3)]}\right]. \tag{5}$$

Note that we are working in an assumed common electron-ion rest frame. Expanding the collision operator, and only retaining terms that are first order in perturbed quantities, we obtain

$$C_{ee}(f_e) = -\nu_{ee}\,n_e\,F(v_1)\,F(v_2)\left\{f(t,x_3,v_3) - \left[\frac{\delta n_e(t,x_3)}{n_e} + \frac{V_e(t,x_3)}{v_{t\,e}}\,\frac{2\,v_3}{v_{t\,e}} + \frac{\delta T_e(t,x_3)}{T_e}\left(\frac{v_3^2}{v_{t\,e}^2} - \frac{1}{2}\right)\right]F(v_3)\right\}. \tag{6}$$

Now, in order for the electron-electron collision operator to conserve the number of electrons, we require that

$$\iiint C_{ee}(f_e)\,d^3\mathbf{v} = 0, \tag{7}$$

which yields

$$\frac{\delta n_e(t,x_3)}{n_e} = \int_{-\infty}^{\infty} f(t,x_3,v_3)\,dv_3. \tag{8}$$

Here, $\delta n_e(t,x_3)$ is the perturbed electron number density.

Likewise, in order for the electron-electron collision operator to conserve electron momentum, we need

$$\iiint \mathbf{v}\,C_{ee}(f_e)\,d^3\mathbf{v} = \mathbf{0}. \tag{9}$$

It is easily seen that $\iiint v_1\, C_{ee}(f_e)\, d^3\mathbf{v} = \iiint v_2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0$. Thus, we require

$$\iiint v_3\, C_{ee}(f_e)\, d^3\mathbf{v} = 0, \tag{10}$$

which yields

$$V_e(t, x_3) = \int_{-\infty}^{\infty} v_3\, f(t, x_3, v_3)\, dv_3. \tag{11}$$

Here, $V_e(t, x_3)$ is the perturbed parallel drift velocity of the electrons with respect to the ions.

Finally, in order for the electron-electron collision operator to conserve electron energy, we require

$$\iiint v^2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0. \tag{12}$$

It is easily seen that $\iiint v_1^2\, C_{ee}(f_e)\, d^3\mathbf{v} = \iiint v_2^2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0$. Thus, we need

$$\iiint v_3^2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0, \tag{13}$$

which yields

$$\frac{\delta T_e(t, x_3)}{T_e} = 2 \int_{-\infty}^{\infty} \left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right) f(t, x_3, v_3)\, dv_3. \tag{14}$$

Here, $\delta T_e(t, x_3)$ is the perturbed electron temperature. Thus, the electron-electron collision operator, (6), is now fully specified in terms of the perturbed electron distribution function, $f(t, x_3, v_3)$.

C. Electron-Ion Collision Operator

By analogy with the analysis in the previous subsection, our model electron-ion collision operator is written

$$C_{ei}(f_e) = -\nu_{ei}\, n_e\, F(v_1)\, F(v_2) \left\{ f(t, x_3, v_3) - \left[\frac{\delta n_e(t, x_3)}{n_e} + \frac{\delta T_e(t, x_3)}{T_e} \left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right)\right] F(v_3) \right\}, \tag{15}$$

where ν_{ei} is the electron-ion collision frequency. Note that this collision operator conserves the number of electrons, as well as the electron energy (because the ions are treated as infinitely massive with respect to the electrons), but does not conserve electron momentum (as a consequence of momentum transferred to the ions via collisions). Note, finally, that the ion fluid is stationary in the infinite mass limit.

D. Electron Kinetic Equation

The ensemble-averaged electron kinetic equation that governs the transport of electron number density and energy parallel to the magnetic field can be written [1,3]

$$\frac{\partial f_e}{\partial t} + v_3\,\frac{\partial f_e}{\partial x_3} - \frac{e}{m_e}\,E_3\,\frac{\partial f_e}{\partial v_3} = C_{ei}(f_e) + C_{ee}(f_e) + S(\mathbf{x},\mathbf{v}). \tag{16}$$

Here, we are assuming that the plasma is subject to a perturbed parallel electric field, $E_3(t, x_3)$. Moreover, the source term in the kinetic equation takes the form

$$S(t,\mathbf{x},\mathbf{v}) = n_e\,F(v_1)\,F(v_2)\,F(v_3)\left[S_0(t,x_3) + S_2(t,x_3)\left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right)\right], \tag{17}$$

where $S_0(t, x_3)$ represents a particle source, and $S_2(t, x_3)$ represents an energy source.

Linearizing the kinetic equation, and integrating over v_1 and v_2, we obtain

$$\frac{\partial f}{\partial t} + v_3\,\frac{\partial f}{\partial x_3} - \langle C_{ei}(f)\rangle - \langle C_{ee}(f)\rangle = \left[S_0 + S_1\,\frac{2\,v_3}{v_{te}} + S_2\left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right)\right]F(v_3), \tag{18}$$

where

$$\begin{aligned}
\langle C_{ee}(f)\rangle &= \frac{1}{n_e}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{ee}(f_e)\,dv_1\,dv_2 \\
&= -\nu_{ee}\left\{f - \left[\frac{\delta n_e}{n_e} + \frac{V_e}{v_{te}}\,\frac{2\,v_3}{v_{te}} + \frac{\delta T_e}{T_e}\left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right)\right]F(v_3)\right\}, && (19)\\
\langle C_{ei}(f)\rangle &= \frac{1}{n_e}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{ei}(f_e)\,dv_1\,dv_2 \\
&= -\nu_{ei}\left\{f - \left[\frac{\delta n_e}{n_e} + \frac{\delta T_e}{T_e}\left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right)\right]F(v_3)\right\}, && (20)\\
S_1(t,x_3) &= -\frac{e\,E_3(t,x_3)}{m_e\,v_{te}}. && (21)
\end{aligned}$$

E. Poisson-Maxwell Equation

Assuming that the ions constitute a uniform neutralizing background, the perturbed parallel electric field is related to the perturbed electron number density according to

$$\frac{\partial E_3}{\partial x_3} = -\frac{e\,\delta n_e(t,x_3)}{\epsilon_0}. \tag{22}$$

F. Heat Flux

The flux of parallel electron kinetic energy is defined [3]

$$\mathbf{q}_{\|}(t,\mathbf{x}) = \int\!\!\int\!\!\int \frac{1}{2}\, m_e\, (v_3 - V_e)^2\, (\mathbf{v}-\mathbf{V})\, f_e(t,\mathbf{x},\mathbf{v})\, d^3\mathbf{v}. \tag{23}$$

It is easily demonstrated that, to first order in small quantities, $q_{\|\,1} = q_{\|\,2} = 0$, and

$$q_{\|\,3}(t,x_3) = \frac{1}{2}\, m_e\, n_e \int_{-\infty}^{\infty} v_3 \left(v_3^2 - \frac{3}{2}\, v_{t\,e}^2\right) f(t,x_3,v_3)\, dv_3. \tag{24}$$

III. FOURIER-LAPLACE TRANSFORM SOLUTION OF ELECTRON KINETIC EQUATION

A. Normalization

Let

$$\nu_e = \nu_{ee} + \nu_{ei} \tag{25}$$

be the total electron collision frequency, and let

$$l_e = \frac{v_{t\,e}}{\nu_e} \tag{26}$$

be the electron mean-free-path between collisions. Let us adopt the following normalizations: $\hat{t} = \nu_e\, t$, $\hat{x} = x_3/l_e$, $u = v_3/v_{t\,e}$, $\hat{f} = v_{t\,e}\, f$, $\delta\hat{n}_e = \delta n_e/n_e$, $\hat{V}_e = V_e/v_{t\,e}$, $\delta\hat{T}_e = \delta T_e/T_e$, $\hat{S}_0 = S_0/\nu_e$, $\hat{S}_1 = S_1/\nu_e$, $\hat{S}_2 = S_2/\nu_e$, and $\hat{q}_e = q_{\|\,3}/(n_e\, T_e\, v_{t\,e})$.

The electron kinetic equation, (18), takes the normalized form

$$\frac{\partial \hat{f}}{\partial \hat{t}} + u\, \frac{\partial \hat{f}}{\partial \hat{x}} + \hat{f} = \left[(\delta\hat{n}_e + \hat{S}_0) + (\mu_e\, \hat{V}_e + \hat{S}_1)\, 2\, u + (\delta\hat{T}_e + \hat{S}_2)\left(u^2 - \frac{1}{2}\right)\right] F_M, \tag{27}$$

where

$$F_M(u) = \frac{\exp(-u^2)}{\pi^{1/2}}, \tag{28}$$

$$\mu_e = \frac{\nu_{ee}}{\nu_{ee} + \nu_{ei}}. \tag{29}$$

Here, use has been made of Eqs. (19) and (20). Furthermore, Eqs. (8), (11), (14), and (24) yield

$$\delta\hat{n}_e(\hat{t},\hat{x}) = \int_{-\infty}^{\infty} \hat{f}(\hat{t},\hat{x},u)\, du, \tag{30}$$

$$\hat{V}_e(\hat{t},\hat{x}) = \int_{-\infty}^{\infty} u\,\hat{f}(\hat{t},\hat{x},u)\,du, \tag{31}$$

$$\delta\hat{T}_e(\hat{t},\hat{x}) = 2\int_{-\infty}^{\infty}\left(u^2-\frac{1}{2}\right) f(\hat{t},\hat{x},u)\,du, \tag{32}$$

$$\hat{q}_e(\hat{t},\hat{x}) = \int_{-\infty}^{\infty} u\left(u^2-\frac{3}{2}\right)\hat{f}(\hat{t},\hat{x},u)\,du. \tag{33}$$

Finally, Eqs. (21) and (22) give

$$2\,\hat{\lambda}_D^{\,2}\,\frac{\partial\hat{S}_1}{\partial\hat{x}} = \delta\hat{n}_e, \tag{34}$$

where

$$\hat{\lambda}_D = \frac{\lambda_D}{l_e} \tag{35}$$

and

$$\lambda_D = \left(\frac{\epsilon_0\,T_e}{n_e\,e^2}\right)^{1/2} \tag{36}$$

is the Deybe length.[3] Note that $\hat{\lambda}_D$ is necessarily a small parameter in a weakly-coupled plasma.[3]

B. Fluid Equations

Taking $\int_{-\infty}^{\infty}(27)\,du$, we obtain the electron continuity equation,

$$\frac{\partial\delta\hat{n}_e}{\partial\hat{t}} + \frac{\partial\hat{V}_e}{\partial\hat{x}} = \hat{S}_0. \tag{37}$$

Taking $\int_{-\infty}^{\infty} u\,(27)\,du$, we obtain the electron momentum conservation equation,

$$\frac{\partial\hat{V}_e}{\partial\hat{t}} + \frac{1}{2}\frac{\partial}{\partial\hat{x}}(\delta\hat{n}_e+\delta\hat{T}_e) + (1-\mu_e)\,\hat{V}_e = \hat{S}_1. \tag{38}$$

Finally, taking $2\int_{-\infty}^{\infty}(u^2-1/2)\,(27)\,du$, we obtain the electron energy conservation equation,

$$\frac{\partial\delta T_e}{\partial\hat{t}} + 2\,\frac{\partial}{\partial\hat{x}}(\hat{V}_e+\hat{q}_e) = \hat{S}_2. \tag{39}$$

C. Fourier-Laplace Transformation

Let

$$\bar{f}(g,\hat{k},u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\hat{f}(\hat{t},\hat{x},u)\,\mathrm{e}^{-g\,\hat{t}}\,d\hat{t}\right)\mathrm{e}^{-\mathrm{i}\,\hat{k}\,\hat{x}}\,d\hat{x}, \tag{40}$$

$$\delta\bar{n}_e(g,\hat{k}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\delta\hat{n}_e(\hat{t},\hat{x})\,\mathrm{e}^{-g\,\hat{t}}\,d\hat{t}\right)\mathrm{e}^{-\mathrm{i}\,\hat{k}\,\hat{x}}\,d\hat{x}, \tag{41}$$

et cetera. Here, $\hat{k} = k\,l_e$, where k is the unormalized wavenumber. If we operate on Eqs. (27) and (34) with $\int_{-\infty}^{\infty}\int_0^{\infty}[(\cdots)\,\mathrm{e}^{-g\,\hat{t}}\,d\hat{t}]\,\mathrm{e}^{-\mathrm{i}\,\hat{k}\,\hat{x}}\,d\hat{x}$, and combine the resulting equations, then we obtain

$$(g+\mathrm{i}\,\hat{k}\,u+1)\,\bar{f} = \left[(\delta\bar{n}_e+\bar{S}_0) + \left(\mu_e\,\bar{V}_e + \frac{\delta\bar{n}_e}{2\,\mathrm{i}\,\hat{k}\,\hat{\lambda}_D^2}\right)2\,u + (\delta\bar{T}_e+\bar{S}_2)\left(u^2-\frac{1}{2}\right)\right]F_M. \tag{42}$$

Here, we are assuming that all perturbed quantities are zero for $\hat{t}<0$.

D. Fourier-Laplace Transformed Fluid Equations

Taking $\int_{-\infty}^{\infty}(42)\,du$, we obtain the Fourier-Laplace transformed electron continuity equation,

$$g\,\delta\bar{n}_e + \mathrm{i}\,\hat{k}\,\bar{V}_e = \bar{S}_0. \tag{43}$$

Taking $\int_{-\infty}^{\infty}u\,(42)\,du$, we obtain the Fourier-Laplace transformed electron momentum conservation equation,

$$(g+1-\mu_e)\,\bar{V}_e + \frac{\mathrm{i}\,\hat{k}}{2}\left(\varLambda_D\,\delta\bar{n}_e + \delta\bar{T}_e\right) = 0, \tag{44}$$

where

$$\varLambda_D(\hat{k}) = \frac{1+(\hat{k}\,\hat{\lambda}_D)^2}{(\hat{k}\,\hat{\lambda}_D)^2} = \frac{1+(k\,\lambda_D)^2}{(k\,\lambda_D)^2}. \tag{45}$$

Finally, taking $2\int_{-\infty}^{\infty}(u^2-1/2)\,(42)\,du$, we obtain the Fourier-Laplace transformed electron energy conservation equation,

$$g\,\delta\bar{T}_e + 2\,\mathrm{i}\,\hat{k}\,(\bar{V}_e+\bar{q}_e) = \bar{S}_2. \tag{46}$$

E. Reformulation

Equation (42) can be rearranged to give

$$\bar{f}(g,\hat{k},u) = (1+g)^{-1}\left\{\left(\delta\bar{n}_e+\bar{S}_0\right) + \left[\mu_e\,\bar{V}_e + \frac{(1+g)\,(\varLambda_D-1)\,\delta\bar{n}_e}{2\,\xi}\right]2\,u \right.$$
$$\left. + \left(\delta\bar{T}_e+\bar{S}_2\right)\left(u^2-\frac{1}{2}\right)\right\}\left(\frac{-\xi}{u-\xi}\right)F_M, \tag{47}$$

where

$$\xi(g,\hat{k}) = \frac{\mathrm{i}\,(1+g)}{\hat{k}} = \frac{\mathrm{i}\,(1+g)}{k\,l_e}. \tag{48}$$

Likewise, the fluid equations, (43), (44), and (46), can be re-expressed in the forms

$$\delta\bar{n}_e + \bar{S}_0 = (1+g)\left(\delta\bar{n}_e - \xi^{-1}\,\bar{V}_e\right), \tag{49}$$

$$\mu_e\,\bar{V}_e + \frac{(1+g)\,(\Lambda_D-1)\,\delta\bar{n}_e}{2\,\xi} = (1+g)\left[\bar{V}_e - \frac{\xi^{-1}}{2}\,(\delta\bar{n}_e + \delta\bar{T}_e)\right], \tag{50}$$

$$\delta\bar{T}_e + \bar{S}_2 = (1+g)\left[\delta\bar{T}_e - 2\,\xi^{-1}\,(\bar{V}_e + \bar{q}_e)\right]. \tag{51}$$

It follows that

$$\xi\left[\mu_e\,\bar{V}_e + \frac{(1+g)\,(\Lambda_D-1)\,\delta\bar{n}_e}{2\,\xi}\right] = g\,\xi^2\,\delta\bar{n}_e - \frac{1}{2}\,(1+g)\,(\delta\bar{n}_e + \delta\bar{T}_e) - \xi^2\,\hat{S}_0, \tag{52}$$

$$\bar{q}_e = (1+g)^{-1}\,\xi\left[-g\left(\delta\bar{n}_e - \frac{\delta\bar{T}_e}{2}\right) + \left(\bar{S}_0 - \frac{\bar{S}_2}{2}\right)\right]. \tag{53}$$

F. Modified Plasma Dispersion Function

Let

$$Z_n(\xi) = \int_{-\infty}^{\infty} u^n\left(\frac{-\xi}{u-\xi}\right) F_M(u)\,du. \tag{54}$$

It is easily demonstrated that

$$Z_{n+1} = \xi\,(Z_n - I_n), \tag{55}$$

where

$$I_n = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} u^n\,\exp(-u^2)\,du. \tag{56}$$

Now, $I_0 = 1$, and $I_1 = 0$, so

$$Z_1 = \xi\,(Z_0 - I_0) = \xi\,Z_0 - \xi, \tag{57}$$

$$Z_2 = \xi\,(Z_1 - I_1) = \xi^2\,Z_0 - \xi^2. \tag{58}$$

Note that

$$Z_0(\xi) = -\xi\,\bar{Z}(\xi), \tag{59}$$

where

$$\bar{Z}(\xi) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-u^2}}{u-\xi}\,du \tag{60}$$

is related to the plasma dispersion function.[3,4] In fact, it can be shown that[3]

$$\bar{Z}(\xi) = \mathrm{i}\,\pi^{1/2}\,w(\xi) \tag{61}$$

for $\mathrm{Im}(\xi) > 0$, and

$$\bar{Z}(\xi) = \mathrm{i}\,\pi^{1/2}\,w(\xi) - 2\,\mathrm{i}\,\pi^{1/2}\,\exp(-\xi^2) \tag{62}$$

for $\mathrm{Im}(\xi) < 0$. Here,

$$w(\xi) = \exp(-\xi^2)\,\mathrm{erfc}(-\mathrm{i}\,\xi) \tag{63}$$

is a so-called Faddeeva function (alternatively known as a Kramp function),[5] and $\mathrm{erfc}(z)$ is the complementary error function.[5]

Now,[3,5]

$$w(\xi) = 1 + \frac{2\,\mathrm{i}\,\xi}{\pi^{1/2}} + \mathcal{O}(\xi^2) \tag{64}$$

in the limit $|\xi| \ll 1$, whereas

$$w(\xi) = \sigma\,\exp(-\xi^2) + \frac{\mathrm{i}}{\pi^{1/2}}\left[\frac{1}{\xi} + \frac{1}{2\,\xi^3} + \frac{3}{4\,\xi^5} + \frac{15}{8\,\xi^7} + \frac{105}{16\,\xi^9} + \mathcal{O}\left(\frac{1}{\xi^{11}}\right)\right] \tag{65}$$

in the limit $|\xi| \to \infty$. Here,

$$\sigma = \begin{cases} 0 & \xi_i > |\xi_r|^{-1} \\ 1 & |\xi_i| < |\xi_r|^{-1} \\ 2 & \xi_i < -|\xi_r|^{-1} \end{cases}, \tag{66}$$

where $\xi = \xi_r + \mathrm{i}\,\xi_i$, and ξ_r and ξ_i are both real. It follows that

$$Z_0(\xi) = -\mathrm{i}\,\pi^{1/2}\,\mathrm{sgn}(\xi_i)\,\xi + 2\,\xi^2 + \mathcal{O}(\xi^3) \tag{67}$$

in the limit $|\xi| \ll 1$, whereas

$$Z_0(\xi) = -\mathrm{i}\,\pi^{1/2}\,\sigma'\,\xi\,\exp(-\xi^2) + 1 + \frac{1}{2\,\xi^2} + \frac{3}{4\,\xi^4} + \frac{15}{8\,\xi^6} + \frac{105}{16\,\xi^8} + \mathcal{O}\left(\frac{1}{\xi^{10}}\right) \tag{68}$$

in the limit $|\xi| \gg 1$, where

$$\sigma' = \begin{cases} 0 & |\xi_i| > |\xi_r|^{-1} \\ 1 & 0 < \xi_i < |\xi_r|^{-1} \\ -1 & -|\xi_r|^{-1} < \xi_i < 0 \end{cases}. \tag{69}$$

G. Fourier-Laplace Transformed Electron Heat Flux

Equations (30), (47), and (54) can be combined to give

$$(1+g)\,\delta\bar{n}_e = \left[(\delta\bar{n}_e+\bar{S}_0)-\frac{1}{2}\,(\delta\bar{T}_e+\bar{S}_2)\right]Z_0+\left[\mu_e\,\bar{V}_e+\frac{(1+g)\,(\Lambda_D-1)\,\delta\bar{n}_e}{2\,\xi}\right]2\,Z_1 + (\delta\bar{T}_e+\bar{S}_2)\,Z_2. \tag{70}$$

Equations (52), (57), (58), and (70) yield

$$\delta\bar{T}_e = \frac{[2\,\xi^2-(2\,\xi^2-1)\,Z_0]\,[\bar{S}_0-\bar{S}_2/2-g\,\delta\bar{n}_e]}{(\xi^2-1-g)-(\xi^2-3/2-g)\,Z_0}. \tag{71}$$

Finally, Eqs. (53) and (71) give [1]

$$\bar{q}_e(g,\hat{k}) = \xi\,G(\xi)\,\delta\bar{T}_e(g,\hat{k}), \tag{72}$$

where

$$G(\xi) = \frac{(\xi^2-1)-(\xi^2-3/2)\,Z_0}{2\,\xi^2-(2\,\xi^2-1)\,Z_0}. \tag{73}$$

Note that the electron heat flux only depends on the perturbed electron temperature, and is independent of both the perturbed electron number density and the electron drift velocity. It follows from Eqs. (67) and (68) that

$$G(\xi) = -\mathrm{i}\,\frac{\mathrm{sgn}(\xi_i)}{\pi^{1/2}\,\xi}+\mathcal{O}(1) \tag{74}$$

in the limit $|\xi|\ll 1$, and

$$G(\xi) = \frac{3}{4\,\xi^2}+\frac{3}{2\,\xi^4}+\mathcal{O}(\xi^{-6}) \tag{75}$$

in the limit $|\xi|\gg 1$.

H. Fourier-Laplace Transformed Electron Fluid Quantities

The Fourier-Laplace transformed fluid equations, (49)–(51), combined with Eq. (72), yield

$$g_1\,\delta\bar{n}_e-\xi^{-1}\,\bar{V}_e = \frac{\bar{S}_0}{1+g}, \tag{76}$$

$$-\Lambda_D\,\delta\bar{n}_e+g_2\,\xi^{-1}\,\bar{V}_e-\delta\bar{T}_e = 0, \tag{77}$$

$$-\xi^{-1}\,\bar{V}_e-\left(G-\frac{g_1}{2}\right)\delta\bar{T}_e = \frac{\bar{S}_2}{2\,(1+g)}, \tag{78}$$

where

$$g_1(g) = \frac{g}{1+g}, \tag{79}$$

$$g_2(g,\hat{k}) = \frac{2\,(1-\mu_e+g)\,\xi^2}{1+g}. \tag{80}$$

Finally, Eqs. (76)–(78) can be solved to give [1]

$$\delta\bar{n}_e(g,\hat{k}) = \frac{-[1+g_2\,(G-g_1/2)]\,\bar{S}_0(g,\hat{k})+\bar{S}_2(g,\hat{k})/2}{(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}, \tag{81}$$

$$\bar{V}_e(g,\hat{k}) = \frac{-(G-g_1/2)\,\Lambda_D\,\xi\,\bar{S}_0(g,\hat{k})+g_1\,\xi\,\bar{S}_2(g,\hat{k})/2}{(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}, \tag{82}$$

$$\delta\bar{T}_e(g,\hat{k}) = \frac{\Lambda_D\,\bar{S}_0(g,\hat{k})-(\Lambda_D-g_1\,g_2)\,\bar{S}_2(g,\hat{k})/2}{(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}. \tag{83}$$

The previous three equations specify the Fourier-Laplace transformed electron number density, drift velocity, and temperature directly in terms of the particle and energy sources.

IV. ELECTRON ENERGY TRANSPORT ACROSS A MAGNETIC ISLAND CHAIN

A. Introduction

Consider a tearing mode in a tokamak plasma.[6] Suppose that the mode possesses m periods in the poloidal direction, and n periods in the toroidal direction. As is well-known, the mode resonates with the equilibrium magnetic field at the so-called "rational surface" whose minor radius, r_s, satisfies $q(r_s) = m/n$, where $q(r)$ is the safety-factor profile, and r denotes the minor radius of equilibrium magnetic flux-surfaces.[7] The tearing mode reconnects magnetic flux at the rational surface, in the process opening up a magnetic island chain, centered on the surface, which also possesses m periods in the poloidal direction, and n periods in the toroidal direction.[8] Let W be the full radial width of the island chain.

As is well-known, if W exceeds a fairly small critical width, W_c, then the equilibrium electron temperature gradient is flattened in the region lying within the island chain's magnetic separatrix. In this case, the perpendicular electron energy flux across the island chain, driven by radial gradients in the electron temperature, is diverted along magnetic field-lines in a thin boundary layer that lies on the island separatrix. See Fig. 1. Let δ be the radial width of the boundary layer. The critical island width, W_c, is defined as the island width at which $\delta = W$. If $W > W_c$ then the electron temperature is flattened by the island chain, and the heat transport across the chain is predominately parallel to magnetic field-lines. On the other hand, if $W < W_c$ then the electron temperature is unaffected by the island chain, and the heat transport across the chain is predominately perpendicular to magnetic field-lines. We wish to employ the theory developed Sects. II and III to calculate the critical island width at arbitrary collisionality. (In reality, parallel electron heat transport in tokamak plasmas does not lie in the short mean-free-path regime.) We also wish to calculate the critical island width in the case in which the heat flow across the island chain oscillates at some frequency ω.

Our calculation takes place in the simplified island geometry pictured in Fig. 2. Here, the separatrix boundary layer has been split into two straight boundary layers which represent the inner (in r) and outer sides of the magnetic separatrix. Electron energy flows parallel to magnetic field-lines within these boundary layers. Furthermore, any energy that flows out of the ends of the inner boundary layer is fed into the corresponding end of the outer boundary layer, and vice versa.

B. Connection Length

Let x measure distance along magnetic field-lines in the separatrix boundary layers. The equation of the field-lines is [9]

$$\frac{dx}{d\zeta} = \frac{R_0\, r_s}{n\, s_s}\,\frac{1}{|r - r_s|}, \tag{84}$$

where $s_s = (d\ln q/\ln s)_{r_s}$ is the magnetic shear at the rational surface, and R_0 is the plasma minor radius. Here, $\zeta = m\,\theta - n\,\phi$ is a helical angle. Now, on the magnetic separatrix [9]

$$|r - r_s| = \frac{W}{2}\,\sin\left(\frac{\zeta}{2}\right). \tag{85}$$

Here, the island O-point lies at $\zeta = \pi$, whereas the X-points lie at $\zeta = 0,\, 2\pi$. The average value of $|r - r_s|$ in the boundary layers is

$$\langle |r - r_s|\rangle = \frac{W}{2}\int_0^{\pi}\sin\left(\frac{\zeta}{2}\right)\frac{d\zeta}{\pi} = \frac{W}{\pi}. \tag{86}$$

Hence, the so-called "connection length", which is defined as the length of the boundary layers parallel to magnetic field-lines, is

$$L = \int_0^{2\pi} \frac{dx}{d\zeta}\,d\zeta = \frac{R_0\,r_s}{n\,s_s}\,\frac{2\pi}{\langle|r-r_s|\rangle} = \frac{R_0\,r_s}{n\,s_s}\,\frac{2\pi^2}{W}. \tag{87}$$

Here, the island O-point lies at $x=0$, whereas the X-points lie at $x=\pm L/2$. Note that the two ends of the boundary layers lie at the X-points. See Fig. 2.

C. Temperature Perturbation

Consider the inner separatrix boundary layer. Suppose that the perturbed electron temperature on the inner (in r) side of the layer is

$$\delta T_{e\,\rm in}(x,t) = \delta T_{e\,0}\,\cos\left(\pi\,\frac{x}{L}\right)\cos(\omega\,t). \tag{88}$$

Suppose that the layer is much thinner than the island, which implies that the perturbed electron temperature is zero inside the island. Thus, the perturbed electron temperature on the outer side of the layer is

$$\delta T_{e\,\rm out}(t,x) = 0. \tag{89}$$

Finally, let

$$\delta T_e\,(t,x) = \delta T_{e\,0}\,\cos\left(\pi\,\frac{x}{L}\right)\cos(\omega\,t-\alpha) \tag{90}$$

be the average perturbed electron temperature in the layer. Here, α represents a phase-lag between the average temperature and the driving temperature on the inner side of the layer. Moreover, $\omega>0$ is the oscillation frequency of the temperature perturbation. Finally, $\delta T_{e\,0}$ represents the small temperature difference between the middle and the two ends of the layer that is responsible for driving the flow of energy around the island. Note that the perturbed temperature in the outer boundary layer is minus that in the inner boundary layer.

D. Parallel Electron Heat Flux

The normalized Fourier-Laplace transformed perturbed electron temperature in the inner boundary layer is

$$\delta\bar{T}_e(g,\hat{k}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\delta\hat{T}_e(\hat{x},\hat{t})\,{\rm e}^{-g\,\hat{t}}\right){\rm e}^{-{\rm i}\,\hat{k}\,\hat{x}}\,d\hat{x}$$

$$= \delta\hat{T}_{e\,0}\left[\delta(\hat{k}-\hat{k}_0)+\delta(\hat{k}+\hat{k}_0)\right]\frac{1}{2}\left(\frac{\mathrm{e}^{-\mathrm{i}\,\alpha}}{g-\mathrm{i}\,\hat{\omega}}+\frac{\mathrm{e}^{\mathrm{i}\,\alpha}}{g+\mathrm{i}\,\hat{\omega}}\right). \tag{91}$$

Here, $\hat{x} = x/l_e$, $\hat{L} = L/l_e$, $\hat{t} = \nu_e\,t$, $\hat{\omega} = \omega/\nu_e$, $\delta\hat{T} = \delta T_e/T_e$, $\delta\hat{T}_{e\,0} = \delta T_{e\,0}/T_e$, and $\hat{k}_0 = \pi/\hat{L}$. Moreover, ν_e, l_e, and T_e are the total electron collision frequency, the electron mean-free-path between collisions, and the equilibrium electron temperarture, respectively, at the rational surface.

Now, according to Eq. (72), the Fourier-Laplace transformed normalized parallel electron heat flux in the inner boundary layer is

$$\bar{q}_e(g,\hat{k}) = \xi\,G(\xi)\,\delta\bar{T}_e(g,\hat{k}), \tag{92}$$

where

$$\xi = \frac{\mathrm{i}\,(1+g)}{\hat{k}}, \tag{93}$$

and the function $G(\xi)$ is defined in Eq. (73).

Let

$$G(\xi) = G_r(\xi_r,\xi_i) + \mathrm{i}\,(\xi_r,\xi_i), \tag{94}$$

where

$$\xi_r = \frac{\hat{\omega}}{\hat{k}_0}, \tag{95}$$

$$\xi_i = \frac{1}{\hat{k}_0}, \tag{96}$$

and G_r and G_i are real. It is easily demonstrated that

$$G_r(\xi_r,\xi_i) = G_r(-\xi_r,\xi_i) = G_r(\xi_r,-\xi_i) = G_r(-\xi_r,-\xi_i), \tag{97}$$

$$G_i(\xi_r,\xi_i) = -G_i(-\xi_r,\xi_i) = -G_i(\xi_r,-\xi_i) = G_i(-\xi_r,-\xi_i). \tag{98}$$

The time-asymptotic normalized parallel heat flux in real space,

$$\hat{q}_e(\hat{t},\hat{x}) = \lim_{\hat{t}\rightarrow\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi\,\mathrm{i}}\int_C\bar{q}_e(g,\hat{k})\,\mathrm{e}^{g\,\hat{t}}\,dg\right)\mathrm{e}^{\mathrm{i}\,\hat{k}\,\hat{x}}\,d\hat{k}, \tag{99}$$

where C is the Bromwich contour, can be shown to take the form

$$\hat{q}_e(\hat{t},\hat{x}) = \delta\hat{T}_{e\,0}\,\{[\cos\alpha\,(-\xi_r\,G_r+\xi_i\,G_i)+\sin\alpha\,(\xi_i\,G_r+\xi_r\,G_i)]\sin(\hat{\omega}\,t)$$
$$+\,[\sin\alpha\,(-\xi_r\,G_r+\xi_i\,G_i)-\cos\alpha\,(\xi_i\,G_r+\xi_r\,G_i)]\cos(\hat{\omega}\,t)\}\sin(\hat{k}_0\,\hat{x}). \tag{100}$$

Here, $\hat{q}_e = q_e/(n_e\,T_e\,v_{t\,e})$, where n_e is the equilibrium electron number density at the rational surface, and $v_{t\,e} = (2\,T_e/m_e)^{1/2} = l_e\,\nu_e$. Moreover, $G_r = \mathrm{Re}[G(\xi_r,\xi_i)]$ and $G_i = \mathrm{Im}[G(\xi_r,\xi_i)]$

E. Energy Conservation

The net parallel electron heat flux in the inner separatrix boundary layer is

$$Q_\parallel(\hat{t},\hat{x}) = n_e\,T_e\,v_{t\,e}\,\hat{q}_e(\hat{x},\hat{t})\,\delta. \tag{101}$$

The perpendicular heat flux into the section of the layer that lies between $x=0$ and $x=x$ is

$$\begin{aligned} Q_\perp(\hat{t},\hat{x}) &= \frac{\kappa_\perp}{\delta}\int_0^x \delta T_{e\,\rm in}(\hat{x},\hat{t})\,dx \\ &= \frac{\kappa_\perp\,\delta T_{e\,0}}{\hat{\delta}\,\hat{k}_0}\,\cos(\hat{\omega}\,\hat{t})\,\sin(\hat{k}_0\,\hat{x}), \end{aligned} \tag{102}$$

where $\kappa_\perp$ is the perpendicular electron thermal conductivity at the rational surface, and $\hat{\delta} = \delta/l_e$. Finally, the electron thermal energy in the section of the layer is

$$\mathcal{E}(\hat{t},\hat{x}) = \frac{1}{2}\int_0^x n_e\,\delta T_e(\hat{x},\hat{t})\,dx\,\delta = \frac{n_e\,\delta\,l_e\,\delta T_{e\,0}}{2\,\hat{k}}\left[\cos\alpha\,\cos(\hat{\omega}\,\hat{t}) - \sin\alpha\,\sin(\hat{\omega}\,\hat{t})\right]\,\sin(\hat{k}_0\,\hat{x}) \tag{103}$$

Energy conservation in the layer requires that

$$\frac{\partial\mathcal{E}}{\partial t} = Q_\perp - Q_\parallel, \tag{104}$$

which implies that

$$\cos\alpha\left(\frac{1}{2}\,\xi_r - \xi_r\,G_r + \xi_i\,G_i\right) + \sin\alpha\,(\xi_i\,G_r + \xi_r\,G_i) = 0, \tag{105}$$

$$\sin\alpha\left(\frac{1}{2}\,\xi_r - \xi_r\,G_r + \xi_i\,G_i\right) - \cos\alpha\,(\xi_i\,G_r + \xi_r\,G_i) = \frac{\hat{\kappa}_\perp}{\hat{\delta}^{\,2}\,\hat{k}_0}, \tag{106}$$

where

$$\hat{\kappa}_\perp = \frac{\kappa_\perp}{n_e\,v_{t\,e}\,l_e}. \tag{107}$$

Thus, it follows that

$$\tan\alpha = \frac{\hat{\omega}/2 - \hat{\omega}\,G_r + G_i}{-G_r - \hat{\omega}\,G_i}, \tag{108}$$

and

$$\hat{\delta}^{\,2} = \frac{\hat{\kappa}_\perp}{[(\hat{\omega}/2 - \hat{\omega}\,G_r + G_i)^2 + (-G_r - \hat{\omega}\,G_i)^2]^{1/2}}. \tag{109}$$

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DATA AVAILABILITY STATEMENT

The digital data used in the figures in this paper can be obtained from the author upon reasonable request.

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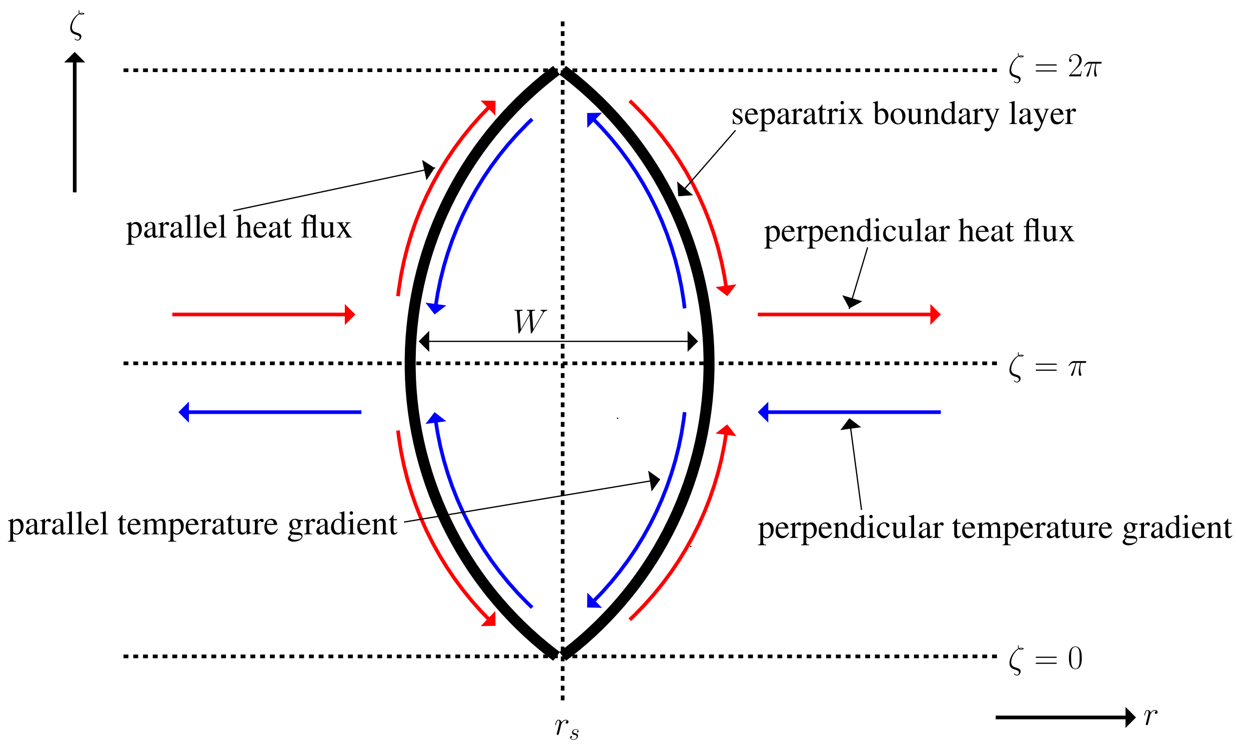

FIG. 1. True geometry of electron energy transport across a magnetic island. Here, $\zeta = m\,\theta - n\,\phi$, where θ is a poloidal angle, and ϕ a toroidal angle.

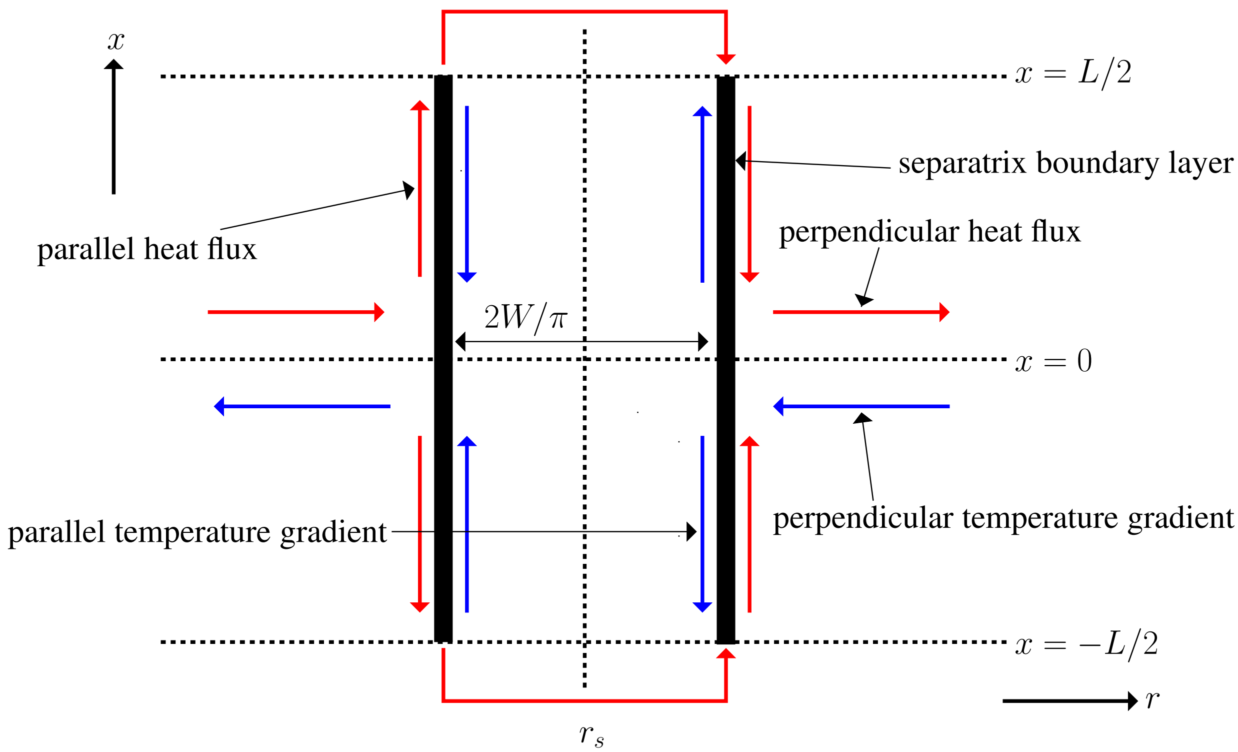

FIG. 2. Simplified geometry of electron energy transport across a magnetic island. Here, x measures distance along magnetic field-lines.