Time-Dependent Parallel Electron Particle and Energy Transport in a Magnetized Plasma of Arbitrary Collisonality

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I. INTRODUCTION

In Ref. 1, the steady-state transport of electron number density and energy parallel to the magnetic field of a magnetized, weakly-coupled, electron-ion plasma of arbitrary collisionality was investigated in slab geometry by solving a simplified one-dimensional kinetic equation for the electron distribution function that employed a Bhatnagar-Gross-Krook (BGK) electron-electron collision operator.[2] The resulting model was able to successfully reproduce standard results for the the electron heat flux in both the short mean-free-path and the long mean-free-path limits. In the short mean-free-path limit, electron energy transport was found to be local and diffusive in nature, whereas the transport was found to be non-local and convective in the long mean-free-path limit. Somewhat surprisingly, electron particle transport was found to be local and diffusive at all collisionalities. The aim of this paper is to generalize the analysis of Ref. 1 by incorporating a model electron-ion collision operator, including a self-consistent calculation of the parallel electric field, and taking time depedence into account.

II. BASIC MODEL

A. Electron Distribution Function

Let $f_e(t,\mathbf{x},\mathbf{v})$ be the ensemble-averaged electron distribution function. Here, t denotes time, $\mathbf{x} = (x_1,\, x_2,\, x_3)$ is a position vector, x_1, x_2, x_3 are Cartesian coordinates that are defined such that the x_3-axis is parallel to the local equilibrium magnetic field, and $\mathbf{v}$ is the electron velocity. Let us write

$$f_e(t,\mathbf{x},\mathbf{v}) = n_e\, F(v_1)\, F(v_2)\, [F(v_3) + f(t, x_3, v_3)]\,, \tag{1}$$

where

$$F(v) = \frac{\exp(-v^2/v_{t\,e}^2)}{\pi^{1/2}\, v_{t\,e}}, \tag{2}$$

and $|f/F| \ll 1$. Here, n_e is the unperturbed electron number density,

$$v_{t\,e} = \sqrt{\frac{2\,T_e}{m_e}} \tag{3}$$

the electron thermal velocity, m_e the electron mass, and T_e the unperturbed electron temperature (measured in energy units). Note that we are assuming that the electron distribution function remains relatively close to a Maxwellian distribution.

B. Electron-Electron Collision Operator

The electron-electron collision operator is modeled as a BGK operator: [1,2]

$$C_{ee}(f_e) = -\nu_{ee}\, F(v_1)\, F(v_2) \left\{ n_e\, F(v_3) + n_e\, f(t, x_3, v_3) \right.$$
$$\left. - \frac{[n_e + \delta n_e(t, x_3)]\, m_e^{1/2}}{\pi^{1/2}\, (2\, [T_e + \delta T_e(t, x_3)])^{1/2}}\, \exp\left[-\frac{[v_3 - V_e(t, x_3)]^2\, m_e}{2\, [T_e + \delta T_e(t, x_3)]}\right] \right\}. \tag{4}$$

Here, ν_{ee} is the electron-electron collision frequency. Moreover, $|\delta n_e/n_e| \ll 1$, $|V_e/v_3| \ll 1$, and $|\delta T_e/T_e| \ll 1$. It can be seen that the operator acts to relax the distribution function to the perturbed Maxwellian

$$F(v_1)\, F(v_2)\, \frac{[n_e + \delta n_e(t, x_3)]\, m_e^{1/2}}{\pi^{1/2}\, (2\, [T_e + \delta T_e(t, x_3)])^{1/2}}\, \exp\left[-\frac{[v_3 - V_3(t, x_3)]^2\, m_e}{2\, [T_e + \delta T_e(t, x_3)]}\right]. \tag{5}$$

Note that we are working in an assumed common electron-ion rest frame. Expanding the collision operator, and only retaining terms that are first order in perturbed quantities, we obtain

$$C_{ee}(f_e) = -\nu_{ee}\, n_e\, F(v_1)\, F(v_2) \left\{ f(t, x_3, v_3) - \left[\frac{\delta n_e(t, x_3)}{n_e} + \frac{V_e(t, x_3)}{v_{t\,e}}\, \frac{2\, v_3}{v_{t\,e}} \right.\right.$$
$$\left.\left. + \frac{\delta T_e(t, x_3)}{T_e} \left(\frac{v_3^2}{v_{t\,e}^2} - \frac{1}{2} \right) \right] F(v_3) \right\}. \tag{6}$$

Now, in order for the electron-electron collision operator to conserve the number of electrons, we require that

$$\iiint C_{ee}(f_e)\, d^3\mathbf{v} = 0, \tag{7}$$

which yields

$$\frac{\delta n_e(t, x_3)}{n_e} = \int_{-\infty}^{\infty} f(t, x_3, v_3)\, dv_3. \tag{8}$$

Here, δn_e is the perturbed electron number density.

Likewise, in order for the electron-electron collision operator to conserve electron momentum, we need

$$\iiint \mathbf{v}\, C_{ee}(f_e)\, d^3\mathbf{v} = 0. \tag{9}$$

It is easily seen that $\iiint v_1\, C_{ee}(f_e)\, d^3\mathbf{v} = \iiint v_2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0$. Thus, we require

$$\iiint v_3\, C_{ee}(f_e)\, d^3\mathbf{v} = 0, \tag{10}$$

which yields

$$V_e(t, x_3) = \int_{-\infty}^{\infty} v_3\, f(t, x_3, v_3)\, dv_3. \tag{11}$$

Here, V_e is the perturbed parallel drift velocity of the electrons with respect to the ions.

Finally, in order for the electron-electron collision operator to conserve electron energy, we require

$$\iiint v^2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0. \tag{12}$$

It is easily seen that $\iiint v_1^2\, C_{ee}(f_e)\, d^3\mathbf{v} = \iiint v_2^2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0$. Thus, we need

$$\iiint v_3^2\, C_{ee}(f_e)\, d^3\mathbf{v} = 0, \tag{13}$$

which yields

$$\frac{\delta T_e(t, x_3)}{T_e} = 2\int_{-\infty}^{\infty}\left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right) f(t, x_3, v_3)\, dv_3. \tag{14}$$

Here, δT_e is the perturbed electron temperature. Thus, the electron-electron collision operator, (6), is now fully specified in terms of the perturbed electron distribution function, $f(t, x_3, v_3)$.

C. Electron-Ion Collision Operator

By analogy with the analysis in the previous subsection, our model electron-ion collision operator is written

$$\begin{aligned} C_{ei}(f_e) = -\nu_{ei}\, n_e\, F(v_1)\, F(v_2)\Bigg\{ & f(t, x_3, v_3) \\ & - \left[\frac{\delta n_e(t, x_3)}{n_e} + \frac{\delta T_e(t, x_3)}{T_e}\left(\frac{v_3^2}{v_{te}^2} - \frac{1}{2}\right)\right] F(v_3)\Bigg\}, \end{aligned} \tag{15}$$

where ν_{ei} is the electron-ion collision frequency. Note that this collision operator conserves the number of electrons, as well as the electron energy (because the ions are treated as infinitely massive with respect to the electrons), but does not conserve electron momentum (as a consequence of momentum transferred to the ions via collisions). Note, finally, that the ion fluid is stationary in the infinite mass limit.

D. Electron Kinetic Equation

The ensemble-averaged electron kinetic equation that governs the transport of electron number density and energy parallel to the magnetic field can be written [1,3]

$$\frac{\partial f_e}{\partial t} + v_3\,\frac{\partial f_e}{\partial x_3} - \frac{e}{m_e}\,E_3\,\frac{\partial f_e}{\partial v_3} = C_{ei}(f_e) + C_{ee}(f_e) + S(\mathbf{x},\mathbf{v}). \tag{16}$$

Here, we are assuming that the plasma is subject to a perturbed parallel electric field, $E_3(t, x_3)$. Moreover, the source term in the kinetic equation takes the form

$$S(t,\mathbf{x},\mathbf{v}) = n_e\,F(v_1)\,F(v_2)\,F(v_3)\left[S_0(t,x_3) + S_2(t,x_3)\left(\frac{v_3^2}{v_{t\,e}^2} - \frac{1}{2}\right)\right], \tag{17}$$

where $S_0(t, x_3)$ represents a particle source, and $S_2(t, x_3)$ represents an energy source.

Linearizing the kinetic equation, and integrating over v_1 and v_2, we obtain

$$\frac{\partial f}{\partial t} + v_3\,\frac{\partial f}{\partial x_3} - \langle C_{ei}(f)\rangle - \langle C_{ee}(f)\rangle = \left[S_0 + S_1\,\frac{2\,v_3}{v_{t\,e}} + S_2\left(\frac{v_3^2}{v_{t\,e}^2} - \frac{1}{2}\right)\right]F(v_3), \tag{18}$$

where

$$\begin{aligned}
\langle C_{ee}(f)\rangle &= \frac{1}{n_e}\int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} C_{ee}(f_e)\,dv_1\,dv_2\\
&= -\nu_{ee}\left\{f - \left[\frac{\delta n_e}{n_e} + \frac{V_e}{v_{t\,e}}\,\frac{2\,v_3}{v_{t\,e}} + \frac{\delta T_e}{T_e}\left(\frac{v_3^2}{v_{t\,e}^2} - \frac{1}{2}\right)\right]F(v_3)\right\}, \qquad (19)\\
\langle C_{ei}(f)\rangle &= \frac{1}{n_e}\int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} C_{ei}(f_e)\,dv_1\,dv_2\\
&= -\nu_{ei}\left\{f - \left[\frac{\delta n_e}{n_e} + \frac{\delta T_e}{T_e}\left(\frac{v_3^2}{v_{t\,e}^2} - \frac{1}{2}\right)\right]F(v_3)\right\}, \qquad (20)\\
S_1(t,x_3) &= -\frac{e\,E_3(t,x_3)}{m_e\,v_{t\,e}}. \qquad (21)
\end{aligned}$$

E. Poisson-Maxwell Equation

Assuming that the ions constitute a uniform neutralizing background, the perturbed parallel electric field is related to the perturbed electron number density according to

$$\frac{\partial E_3}{\partial x_3} = -\frac{e\,\delta n_e(t,x_3)}{\epsilon_0}. \tag{22}$$

F. Heat Flux

The flux of parallel electron kinetic energy is defined[3]

$$\mathbf{q}_{\parallel}(t,\mathbf{x}) = \int\!\!\int\!\!\int \frac{1}{2}\, m_e\,(v_3 - V_e)^2\,(\mathbf{v}-\mathbf{V})\,f_e(t,\mathbf{x},\mathbf{v})\,d^3\mathbf{v}. \tag{23}$$

It is easily demonstrated that, to first order in small quantities, $q_{\parallel\,1} = q_{\parallel\,2} = 0$, and

$$q_{\parallel\,3}(t,x_3) = \frac{1}{2}\,m_e\,n_e\int_{-\infty}^{\infty} v_3\left(v_3^2 - \frac{3}{2}\,v_{t\,e}^2\right) f(t,x_3,v_3)\,dv_3. \tag{24}$$

III. FOURIER-LAPLACE TRANSFORM SOLUTION OF ELECTRON KINETIC EQUATION

A. Normalization

Let

$$\nu_e = \nu_{ee} + \nu_{ei} \tag{25}$$

be the total electron collision frequency, and let

$$l_e = \frac{v_{t\,e}}{\nu_e} \tag{26}$$

be the mean-free-path between collisions. Let us adopt the following normalizations: $\hat{t} = \nu_e\,t$, $\hat{x} = x_3/l_e$, $u = v_3/v_{t\,e}$, $\hat{f} = v_{t\,e}\,f$, $\delta\hat{n}_e = \delta n_e/n_e$, $\hat{V}_e = V_e/v_{t\,e}$, $\delta\hat{T}_e = \delta T_e/T_e$, $\hat{S}_0 = S_0/\nu_e$, $\hat{S}_1 = S_1/\nu_e$, $\hat{S}_2 = S_2/\nu_e$, and $\hat{q}_e = q_{\parallel\,3}/(n_e\,T_e\,v_{t\,e})$.

The electron kinetic equation, (18), takes the normalized form

$$\frac{\partial\hat{f}}{\partial\hat{t}} + u\,\frac{\partial\hat{f}}{\partial\hat{x}} + \hat{f} = \left[(\delta\hat{n}_e + \hat{S}_0) + (\mu_e\,\hat{V}_e + \hat{S}_1)\,2\,u + (\delta\hat{T}_e + \hat{S}_2)\left(u^2 - \frac{1}{2}\right)\right] F_M, \tag{27}$$

where

$$F_M(u) = \frac{\exp(-u^2)}{\pi^{1/2}}, \tag{28}$$

$$\mu_e = \frac{\nu_{ee}}{\nu_{ee} + \nu_{ei}}. \tag{29}$$

Here, use has been made of Eqs. (19) and (20). Furthermore, Eqs. (8), (11), (14), and (24) yield

$$\delta\hat{n}_e(\hat{t},\hat{x}) = \int_{-\infty}^{\infty} \hat{f}(\hat{t},\hat{x},u)\,du, \tag{30}$$

$$\hat{V}_e(\hat{t},\hat{x}) = \int_{-\infty}^{\infty} u\,\hat{f}(\hat{t},\hat{x},u)\,du, \tag{31}$$

$$\delta\hat{T}_e(\hat{t},\hat{x}) = 2\int_{-\infty}^{\infty}\left(u^2-\frac{1}{2}\right) f(\hat{t},\hat{x},u)\,du, \tag{32}$$

$$\hat{q}_e(\hat{t},\hat{x}) = \int_{-\infty}^{\infty} u\left(u^2-\frac{3}{2}\right)\hat{f}(\hat{t},\hat{x},u)\,du. \tag{33}$$

Finally, Eqs. (21) and (22) give

$$2\,\hat{\lambda}_D^{\,2}\,\frac{\partial\hat{S}_1}{\partial\hat{x}} = \delta\hat{n}_e, \tag{34}$$

where

$$\hat{\lambda}_D = \frac{\lambda_D}{l_e} \tag{35}$$

and

$$\lambda_D = \left(\frac{\epsilon_0\,T_e}{n_e\,e^2}\right)^{1/2} \tag{36}$$

is the Deybe length.[3] Note that $\hat{\lambda}_D$ is necessarily a small parameter in a weakly-coupled plasma.[3]

B. Fluid Equations

Taking $\int_{-\infty}^{\infty}(27)\,du$, we obtain the electron continuity equation,

$$\frac{\partial\delta\hat{n}_e}{\partial\hat{t}} + \frac{\partial\hat{V}_e}{\partial\hat{x}} = \hat{S}_0. \tag{37}$$

Taking $\int_{-\infty}^{\infty} u\,(27)\,du$, we obtain the electron momentum conservation equation,

$$\frac{\partial\hat{V}_e}{\partial\hat{t}} + \frac{1}{2}\,\frac{\partial}{\partial\hat{x}}(\delta\hat{n}_e+\delta\hat{T}_e) + (1-\mu_e)\,\hat{V}_e = \hat{S}_1. \tag{38}$$

Finally, taking $2\int_{-\infty}^{\infty}(u^2-1/2)\,(27)\,du$, we obtain the electron energy conservation equation,

$$\frac{\partial\delta T_e}{\partial\hat{t}} + 2\,\frac{\partial}{\partial\hat{x}}(\hat{V}_e+\hat{q}_e) = \hat{S}_2. \tag{39}$$

C. Fourier-Laplace Transformation

Let

$$\bar{f}(g,\hat{k},u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\hat{f}(\hat{t},\hat{x},u)\,{\rm e}^{-g\,\hat{t}}\,d\hat{t}\right){\rm e}^{-{\rm i}\,\hat{k}\,\hat{x}}\,d\hat{x}, \tag{40}$$

$$\delta\bar{n}_e(g,\hat{k}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\delta\hat{n}_e(\hat{t},\hat{x})\,\mathrm{e}^{-g\,\hat{t}}\,d\hat{t}\right)\mathrm{e}^{-\mathrm{i}\,\hat{k}\,\hat{x}}\,d\hat{x}, \tag{41}$$

et cetera. Here, $\hat{k} = k\,l_e$, where k is the unormalized wavenumber. If we operate on Eqs. (27) and (34) with $\int_{-\infty}^{\infty}\int_0^{\infty}[(\cdots)\,\mathrm{e}^{-g\,\hat{t}}\,d\hat{t}]\,\mathrm{e}^{-\mathrm{i}\,\hat{k}\,\hat{x}}\,d\hat{x}$, and combine the resulting equations, then we obtain

$$(g+\mathrm{i}\,\hat{k}\,u+1)\,\bar{f} = \left[(\delta\bar{n}_e+\bar{S}_0)+\left(\mu_e\,\bar{V}_e+\frac{\delta\bar{n}_e}{2\,\mathrm{i}\,\hat{k}\,\hat{\lambda}_D^2}\right)2\,u+(\delta\bar{T}_e+\bar{S}_2)\left(u^2-\frac{1}{2}\right)\right]F_M. \tag{42}$$

Here, we are assuming that all perturbed quantities are zero for $\hat{t}<0$.

D. Fourier-Laplace Transformed Fluid Equations

Taking $\int_{-\infty}^{\infty}(42)\,du$, we obtain the Fourier-Laplace transformed electron continuity equation,

$$g\,\delta\bar{n}_e+\mathrm{i}\,\hat{k}\,\bar{V}_e = \bar{S}_0. \tag{43}$$

Taking $\int_{-\infty}^{\infty}u\,(42)\,du$, we obtain the Fourier-Laplace transformed electron momentum conservation equation,

$$(g+1-\mu_e)\,\bar{V}_e+\frac{\mathrm{i}\,\hat{k}}{2}\left(\mathit{\Lambda}_D\,\delta\bar{n}_e+\delta\bar{T}_e\right) = 0, \tag{44}$$

where

$$\mathit{\Lambda}_D(\hat{k}) = \frac{1+(\hat{k}\,\hat{\lambda}_D)^2}{(\hat{k}\,\hat{\lambda}_D)^2} = \frac{1+(k\,\lambda_D)^2}{(k\,\lambda_D)^2}. \tag{45}$$

Finally, taking $2\int_{-\infty}^{\infty}(u^2-1/2)\,(42)\,du$, we obtain the Fourier-Laplace transformed electron energy conservation equation,

$$g\,\delta\bar{T}_e+2\,\mathrm{i}\,\hat{k}\,(\bar{V}_e+\bar{q}_e) = \bar{S}_2. \tag{46}$$

E. Reformulation

Equation (42) can be rearranged to give

$$\bar{f}(g,\hat{k},u) = (1+g)^{-1}\left\{\left(\delta\bar{n}_e+\bar{S}_0\right)+\left[\mu_e\,\bar{V}_e+\frac{(1+g)\,(\mathit{\Lambda}_D-1)\,\delta\bar{n}_e}{2\,\xi}\right]2\,u\right.$$
$$\left.+\left(\delta\bar{T}_e+\bar{S}_2\right)\left(u^2-\frac{1}{2}\right)\right\}\left(\frac{-\xi}{u-\xi}\right)F_M, \tag{47}$$

where

$$\xi(g, \hat{k}) = \frac{\mathrm{i}\,(1+g)}{\hat{k}} = \frac{\mathrm{i}\,(1+g)}{k\, l_e}. \tag{48}$$

Likewise, the fluid equations, (43), (44), and (46), can be re-expressed in the forms

$$\delta\bar{n}_e + \bar{S}_0 = (1+g)\left(\delta\bar{n}_e - \xi^{-1}\,\bar{V}_e\right), \tag{49}$$

$$\mu_e\,\bar{V}_e + \frac{(1+g)\,(\Lambda_D - 1)\,\delta\bar{n}_e}{2\,\xi} = (1+g)\left[\bar{V}_e - \frac{\xi^{-1}}{2}\,(\delta\bar{n}_e + \delta\bar{T}_e)\right], \tag{50}$$

$$\delta\bar{T}_e + \bar{S}_2 = (1+g)\left[\delta\bar{T}_e - 2\,\xi^{-1}\,(\bar{V}_e + \bar{q}_e)\right]. \tag{51}$$

It follows that

$$\xi\left[\mu_e\,\bar{V}_e + \frac{(1+g)\,(\Lambda_D - 1)\,\delta\bar{n}_e}{2\,\xi}\right] = g\,\xi^2\,\delta\bar{n}_e - \frac{1}{2}\,(1+g)\,(\delta\bar{n}_e + \delta\bar{T}_e) - \xi^2\,\hat{S}_0, \tag{52}$$

$$\bar{q}_e = (1+g)^{-1}\,\xi\left[-g\left(\delta\bar{n}_e - \frac{\delta\bar{T}_e}{2}\right) + \left(\bar{S}_0 - \frac{\bar{S}_2}{2}\right)\right]. \tag{53}$$

F. Modified Plasma Dispersion Function

Let

$$Z_n(\xi) = \int_{-\infty}^{\infty} u^n \left(\frac{-\xi}{u-\xi}\right) F_M(u)\,du. \tag{54}$$

It is easily demonstrated that

$$Z_{n+1} = \xi\,(Z_n - I_n), \tag{55}$$

where

$$I_n = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} u^n\,\exp(-u^2)\,du. \tag{56}$$

Now, $I_0 = 1$, and $I_1 = 0$, so

$$Z_1 = \xi\,(Z_0 - I_0) = \xi\,Z_0 - \xi, \tag{57}$$

$$Z_2 = \xi\,(Z_1 - I_1) = \xi^2\,Z_0 - \xi^2. \tag{58}$$

Note that

$$Z_0(\xi) = -\xi\,\bar{Z}(\xi), \tag{59}$$

where

$$\bar{Z}(\xi) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-u^2}}{u-\xi}\,du \tag{60}$$

is related to the plasma dispersion function.[3,4] In fact, it can be shown that[3]

$$\bar{Z}(\xi) = \mathrm{i}\,\pi^{1/2}\,w(\xi) \tag{61}$$

for $\mathrm{Im}(\xi) > 0$, and

$$\bar{Z}(\xi) = \mathrm{i}\,\pi^{1/2}\,w(\xi) - 2\,\mathrm{i}\,\pi^{1/2}\,\exp(-\xi^2) \tag{62}$$

for $\mathrm{Im}(\xi) < 0$. Here,

$$w(\xi) = \exp(-\xi^2)\,\mathrm{erfc}(-\mathrm{i}\,\xi) \tag{63}$$

is a so-called Faddeeva function (alternatively known as a Kramp function),[5] and $\mathrm{erfc}(z)$ is the complementary error function.[5]

Now,[3,5]

$$w(\xi) = 1 + \frac{2\,\mathrm{i}\,\xi}{\pi^{1/2}} + \mathcal{O}(\xi^2) \tag{64}$$

in the limit $|\xi| \ll 1$, whereas

$$w(\xi) = \sigma\,\exp(-\xi^2) + \frac{\mathrm{i}}{\pi^{1/2}}\left[\frac{1}{\xi} + \frac{1}{2\,\xi^3} + \frac{3}{4\,\xi^5} + \frac{15}{8\,\xi^7} + \frac{105}{16\,\xi^9} + \mathcal{O}\left(\frac{1}{\xi^{11}}\right)\right] \tag{65}$$

in the limit $|\xi| \to \infty$. Here,

$$\sigma = \begin{cases} 0 & \xi_i > |\xi_r|^{-1} \\ 1 & |\xi_i| < |\xi_r|^{-1} \\ 2 & \xi_i < -|\xi_r|^{-1} \end{cases}, \tag{66}$$

where $\xi = \xi_r + \mathrm{i}\,\xi_i$, and ξ_r and ξ_i are both real. It follows that

$$Z_0(\xi) = -\mathrm{i}\,\pi^{1/2}\,\mathrm{sgn}(\xi_i)\,\xi + 2\,\xi^2 + \mathcal{O}(\xi^3) \tag{67}$$

in the limit $|\xi| \ll 1$, whereas

$$Z_0(\xi) = -\mathrm{i}\,\pi^{1/2}\,\sigma'\,\xi\,\exp(-\xi^2) + 1 + \frac{1}{2\,\xi^2} + \frac{3}{4\,\xi^4} + \frac{15}{8\,\xi^6} + \frac{105}{16\,\xi^8} + \mathcal{O}\left(\frac{1}{\xi^{10}}\right) \tag{68}$$

in the limit $|\xi| \gg 1$, where

$$\sigma' = \begin{cases} 0 & |\xi_i| > |\xi_r|^{-1} \\ 1 & 0 < \xi_i < |\xi_r|^{-1} \\ -1 & -|\xi_r|^{-1} < \xi_i < 0 \end{cases}. \tag{69}$$

G. Fourier-Laplace Transformed Electron Heat Flux

Equations (30), (47), and (54) can be combined to give

$$(1+g)\,\delta\bar{n}_e = \left[(\delta\bar{n}_e+\bar{S}_0)-\frac{1}{2}\,(\delta\bar{T}_e+\bar{S}_2)\right]Z_0+\left[\mu_e\,\bar{V}_e+\frac{(1+g)\,(\varLambda_D-1)\,\delta\bar{n}_e}{2\,\xi}\right]2\,Z_1 + (\delta\bar{T}_e+\bar{S}_2)\,Z_2. \tag{70}$$

Equations (52), (57), (58), and (70) yield

$$\delta\bar{T}_e = \frac{[2\,\xi^2-(2\,\xi^2-1)\,Z_0]\,[\bar{S}_0-\bar{S}_2/2-g\,\delta\bar{n}_e]}{(\xi^2-1-g)-(\xi^2-3/2-g)\,Z_0}. \tag{71}$$

Finally, Eqs. (53) and (71) give [1]

$$\bar{q}_e(g,\hat{k}) = \xi\,G(\xi)\,\delta\bar{T}_e(g,\hat{k}), \tag{72}$$

where

$$G(\xi) = \frac{(\xi^2-1)-(\xi^2-3/2)\,Z_0}{2\,\xi^2-(2\,\xi^2-1)\,Z_0}. \tag{73}$$

Note that the electron heat flux only depends on the perturbed electron temperature, and is independent of both the perturbed electron number density and the electron drift velocity. It follows from Eqs. (67) and (68) that

$$G(\xi) = -\mathrm{i}\,\frac{\mathrm{sgn}(\xi_i)}{\pi^{1/2}\,\xi}+\mathcal{O}(1) \tag{74}$$

in the limit $|\xi|\ll 1$, and

$$G(\xi) = \frac{3}{4\,\xi^2}+\frac{3}{2\,\xi^4}+\mathcal{O}(\xi^{-6}) \tag{75}$$

in the limit $|\xi|\gg 1$.

H. Fourier-Laplace Transformed Electron Fluid Quantities

The Fourier-Laplace transformed fluid equations, (49)–(51), combined with Eq. (72), yield

$$g_1\,\delta\bar{n}_e-\xi^{-1}\,\bar{V}_e = \frac{\bar{S}_0}{1+g}, \tag{76}$$

$$-\varLambda_D\,\delta\bar{n}_e+g_2\,\xi^{-1}\,\bar{V}_e-\delta\bar{T}_e = 0, \tag{77}$$

$$-\xi^{-1}\,\bar{V}_e-\left(G-\frac{g_1}{2}\right)\delta\bar{T}_e = \frac{\bar{S}_2}{2\,(1+g)}, \tag{78}$$

where

$$g_1(g) = \frac{g}{1+g}, \tag{79}$$

$$g_2(g,\hat{k}) = \frac{2\,(1-\mu_e+g)\,\xi^2}{1+g}. \tag{80}$$

Finally, Eqs. (76)–(78) can be solved to give [1]

$$\delta\bar{n}_e(g,\hat{k}) = \frac{-[1+g_2\,(G-g_1/2)]\,\bar{S}_0(g,\hat{k})+\bar{S}_2(g,\hat{k})/2}{(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}, \tag{81}$$

$$\bar{V}_e(g,\hat{k}) = \frac{-(G-g_1/2)\,\Lambda_D\,\xi\,\bar{S}_0(g,\hat{k})+g_1\,\xi\,\bar{S}_2(g,\hat{k})/2}{(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}, \tag{82}$$

$$\delta\bar{T}_e(g,\hat{k}) = \frac{\Lambda_D\,\bar{S}_0(g,\hat{k})-(\Lambda_D-g_1\,g_2)\,\bar{S}_2(g,\hat{k})/2}{(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}. \tag{83}$$

The previous three equations specify the Fourier-Laplace transformed electron number density, drift velocity, and temperature directly in terms of the particle and energy sources.

IV. RESULTS

A. Model Electron Particle and Energy Sources

Suppose that the particle and energy sources have a Gaussian spatial dependence, with a maximum at $\hat{x}=0$, and a characteristic width $\sigma=\hat{\sigma}\,l_e$. Suppose, further, that the sources are switched on suddenly at $\hat{t}=0$. In other words,

$$S_0(\hat{t},\hat{x}) = \frac{A_0}{\sqrt{2\pi}\,\hat{\sigma}}\,\exp\left(-\frac{\hat{x}^2}{2\,\hat{\sigma}^2}\right)H(\hat{t}), \tag{84}$$

$$S_2(\hat{t},\hat{x}) = \frac{A_2}{\sqrt{2\pi}\,\hat{\sigma}}\,\exp\left(-\frac{\hat{x}^2}{2\,\hat{\sigma}^2}\right)H(\hat{t}), \tag{85}$$

where $H(\hat{t})$ is a Heaviside step function, and A_0 and A_2 are arbitrary constants. Note that

$$\int_{-\infty}^{\infty} S_0(\hat{t},\hat{x})\,d\hat{x} = A_0\,H(\hat{t}), \tag{86}$$

$$\int_{-\infty}^{\infty} S_2(\hat{t},\hat{x})\,d\hat{x} = A_2\,H(\hat{t}). \tag{87}$$

It follows from Eq. (41) that

$$\bar{S}_0(g,\hat{k}) = \frac{A_0}{\sqrt{2\pi}\,g}\,\exp\left[-\frac{(\hat{k}\,\hat{\sigma})^2}{2}\right], \tag{88}$$

$$\bar{S}_2(g,\hat{k}) = \frac{A_2}{\sqrt{2\pi}\,g}\,\exp\left[-\frac{(\hat{k}\,\hat{\sigma})^2}{2}\right]. \tag{89}$$

Let

$$K_{n,0}(g,\hat{k}) = \frac{-1-g_2\,(G-g_1/2)}{g\,(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}, \tag{90}$$

$$K_{T,0}(g,\hat{k}) = \frac{\Lambda_D}{g\,(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}, \tag{91}$$

$$K_{n,2}(g,\hat{k}) = \frac{1}{2\,g\,(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}, \tag{92}$$

$$K_{T,2}(g,\hat{k}) = \frac{-\Lambda_D+g_1\,g_2}{2\,g\,(1+g)\,[(G-g_1/2)\,(\Lambda_D-g_1\,g_2)-g_1]}. \tag{93}$$

and

$$F_{n,0}(g,\hat{x}) = \frac{1}{\pi}\int_0^\infty K_{n,0}(g,\hat{k})\,\exp\left[-\frac{(\hat{k}\,\hat{\sigma})^2}{2}\right]\cos(\hat{k}\,\hat{x})\,d\hat{k}, \tag{94}$$

$$F_{T,0}(g,\hat{x}) = \frac{1}{\pi}\int_0^\infty K_{T,0}(g,\hat{k})\,\exp\left[-\frac{(\hat{k}\,\hat{\sigma})^2}{2}\right]\cos(\hat{k}\,\hat{x})\,d\hat{k}, \tag{95}$$

$$F_{n,2}(g,\hat{x}) = \frac{1}{\pi}\int_0^\infty K_{n,2}(g,\hat{k})\,\exp\left[-\frac{(\hat{k}\,\hat{\sigma})^2}{2}\right]\cos(\hat{k}\,\hat{x})\,d\hat{k}, \tag{96}$$

$$F_{T,2}(g,\hat{x}) = \frac{1}{\pi}\int_0^\infty K_{T,2}(g,\hat{k})\,\exp\left[-\frac{(\hat{k}\,\hat{\sigma})^2}{2}\right]\cos(\hat{k}\,\hat{x})\,d\hat{k}. \tag{97}$$

Here, we have made use of the easily proved results that if $\hat{k}\to-\hat{k}$ then $\xi\to-\xi$, $Z\to-Z$, $Z_0\to Z_0$, $G\to G$, and $\Lambda_D\to\Lambda_D$.

If there is a unit amplitude particle source, but no energy source, so that $A_0=1$ and $A_2=0$, then Eqs. (41), (81), and (83) yield

$$\begin{aligned}\delta\hat{n}_e(\hat{t},\hat{x}) &= \frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Re}[F_{n,0}(\gamma+\mathrm{i}\,y,\hat{x})]\,\cos(y\,\hat{t})\,dy\\ &\quad-\frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Im}[F_{n,0}(\gamma+\mathrm{i}\,y),\hat{x})]\,\sin(y\,\hat{t})\,dy,\end{aligned} \tag{98}$$

$$\begin{aligned}\delta\hat{T}_e(\hat{t},\hat{x}) &= \frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Re}[F_{T,0}(\gamma+\mathrm{i}\,y,\hat{x})]\,\cos(y\,\hat{t})\,dy\\ &\quad-\frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Im}[F_{T,0}(\gamma+\mathrm{i}\,y),\hat{x})]\,\sin(y\,\hat{t})\,dy,\end{aligned} \tag{99}$$

where $\gamma>0$. Note $\delta n_e(\hat{t},-\hat{x})=\delta n_e(\hat{t},\hat{x})$, and $\delta\hat{T}_e(\hat{t},-\hat{x})=\delta\hat{T}_e(\hat{t},\hat{x})$. Here, use has been made of the easily proved results that if $g\to g^*$ then $K_{n,0}\to K_{n,0}^*$, and $K_{T,0}\to K_{T,0}^*$. Once we have determined $\delta\hat{n}_e(\hat{t},\hat{x})$ and $\delta\hat{T}_e(\hat{t},\hat{x})$ then the normalized particle flux, $\hat{V}_e(\hat{t},\hat{x})$, and the normalized heat flux, $\hat{q}_e(\hat{t},\hat{x})$ can be obtained from the fluid equations, (37) and (39).

On the other hand, if there is a unit amplitude energy source, but no particle source, so that $A_0=0$ and $A_2=1$, then

$$\delta\hat{n}_e(\hat{t},\hat{x}) = \frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Re}[F_{n,2}(\gamma+\mathrm{i}\,y,\hat{x})]\,\cos(y\,\hat{t})\,dy - \frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Im}[F_{n,2}(\gamma+\mathrm{i}\,y),\hat{x})]\,\sin(y\,\hat{t})\,dy, \tag{100}$$

$$\delta\hat{T}_e(\hat{t},\hat{x}) = \frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Re}[F_{T,2}(\gamma+\mathrm{i}\,y,\hat{x})]\,\cos(y\,\hat{t})\,dy - \frac{\mathrm{e}^{\gamma\,\hat{t}}}{\pi}\int_0^\infty \mathrm{Im}[F_{T,2}(\gamma+\mathrm{i}\,y),\hat{x})]\,\sin(y\,\hat{t})\,dy. \tag{101}$$

Here, use has been made of the easily proved results that if $g\rightarrow g^*$ then $K_{n,0}\rightarrow K_{n,0}^*$, $K_{n,2}\rightarrow K_{n,2}^*$, and $K_{T,0}\rightarrow K_{T,2}^*$. As before, once we have determined $\delta\hat{n}_e(\hat{t},\hat{x})$ and $\delta\hat{T}_e(\hat{t},\hat{x})$ then the normalized particle flux, $\hat{V}_e(\hat{t},\hat{x})$, and the normalized heat flux, $\hat{q}_e(\hat{t},\hat{x})$ can be determined from the fluid equations, (37) and (39).

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DATA AVAILABILITY STATEMENT

The digital data used in the figures in this paper can be obtained from the author upon reasonable request.

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