

Calculation of Vertical Stability in an Inverse Aspect-Ratio Expanded Tokamak Plasma Equilibrium

Richard Fitzpatrick^a

*Institute for Fusion Studies, Department of Physics,
University of Texas at Austin, Austin, TX 78712*

I. PLASMA EQUILIBRIUM

All lengths are normalized to the major radius of the plasma magnetic axis, R_0 . All magnetic field-strengths are normalized to the toroidal field-strength at the magnetic axis, B_0 . All current densities are normalized to $B_0/(\mu_0 R_0)$. All plasma pressures are normalized to B_0^2/μ_0 .

Let R, ϕ, Z be right-handed cylindrical coordinates whose Jacobian is

$$(\nabla R \times \nabla \phi \cdot \nabla Z)^{-1} = R. \quad (1)$$

Note that $|\nabla \phi| = 1/R$.

Let r, θ, ϕ be right-handed flux-coordinates whose Jacobian is

$$\mathcal{J}(r, \theta) \equiv (\nabla r \times \nabla \theta \cdot \nabla \phi)^{-1} \equiv R \left(\frac{\partial R}{\partial \theta} \frac{\partial Z}{\partial r} - \frac{\partial R}{\partial r} \frac{\partial Z}{\partial \theta} \right) = r R^2. \quad (2)$$

Note that $r = r(R, Z)$ and $\theta = \theta(R, Z)$. The magnetic axis corresponds to $r = 0$. The plasma-vacuum interface corresponds to $r = a$. The inboard mid-plane corresponds to $\theta = 0$.

Consider an axisymmetric tokamak equilibrium whose magnetic field takes the form

$$\mathbf{B}(r, \theta) = f(r) \nabla \phi \times \nabla r + g(r) \nabla \phi = f \nabla(\phi - q\theta) \times \nabla r, \quad (3)$$

where

$$q(r) = \frac{r g}{f} \quad (4)$$

^a rfitzp@utexas.edu

is the safety-factor (i.e., the inverse of the rotational transform). Note that $\mathbf{B} \cdot \nabla r = 0$, which implies that r is a magnetic flux-surface label. We require $g = 1$ on the magnetic axis in order to ensure that the normalized toroidal magnetic field-strength at the axis is unity.

It is easily demonstrated that

$$B^r = \mathbf{B} \cdot \nabla r = 0, \quad (5)$$

$$B^\theta = \mathbf{B} \cdot \nabla \theta = \frac{f}{r R^2}, \quad (6)$$

$$B^\phi = \mathbf{B} \cdot \nabla \phi = \frac{g}{R^2}, \quad (7)$$

$$B_r = \mathcal{J} \nabla \theta \times \nabla \phi \cdot \mathbf{B} = -r f \nabla r \cdot \nabla \theta, \quad (8)$$

$$B_\theta = \mathcal{J} \nabla \phi \times \nabla r \cdot \mathbf{B} = r f |\nabla r|^2, \quad (9)$$

$$B_\phi = \mathcal{J} \nabla r \times \nabla \theta \cdot \mathbf{B} = g. \quad (10)$$

The Maxwell equation (neglecting the displacement current, because the plasma velocity perturbations due to axisymmetric modes are far smaller than the velocity of light in vacuum) $\mathbf{J} = \nabla \times \mathbf{B}$ yields

$$\mathcal{J} J^r = \frac{\partial B_\phi}{\partial \theta} = 0, \quad (11)$$

$$\mathcal{J} J^\theta = -\frac{\partial B_\phi}{\partial r} = -g', \quad (12)$$

$$\mathcal{J} J^\phi = \frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{\partial \theta} = \frac{\partial}{\partial r}(r f |\nabla r|^2) + \frac{\partial}{\partial \theta}(r f \nabla r \cdot \nabla \theta), \quad (13)$$

where \mathbf{J} is the equilibrium current density, $' \equiv d/dr$, and use has been made of Eqs. (8)–(10).

Equilibrium force balance requires that

$$\nabla P = \mathbf{J} \times \mathbf{B}, \quad (14)$$

where $P(r)$ is the equilibrium scalar plasma pressure. Here, for the sake of simplicity, we have neglected the small centrifugal modifications to force balance due to subsonic plasma rotation. It follows that

$$P' = \mathcal{J}(J^\theta B^\phi - J^\phi B^\theta) = -g' \frac{g}{R^2} - \frac{f}{r R^2} \left[\frac{\partial}{\partial r}(r f |\nabla r|^2) + \frac{\partial}{\partial \theta}(r f \nabla r \cdot \nabla \theta) \right], \quad (15)$$

where use has been made of Eqs. (5)–(7), and (11)–(13). The other two components of Eq. (14) are identically zero.

Equation (15) yields the *inverse Grad-Shafranov equation*:

$$\frac{f}{r} \frac{\partial}{\partial r} (r f |\nabla r|^2) + \frac{f}{r} \frac{\partial}{\partial \theta} (r f \nabla r \cdot \nabla \theta) + g g' + R^2 P' = 0. \quad (16)$$

It follows from Eqs. (4), (13), and (16) that

$$\mathcal{J} J^\phi = -q g' - \frac{r R^2 P'}{f}. \quad (17)$$

It is clear from Eqs. (12) and (17) that $g' = P' = 0$ in the current-free “vacuum” region surrounding the plasma, $r > a$. We shall also assume that $g' = P' = 0$ at the plasma-vacuum interface, so as to ensure that the equilibrium plasma current density is zero at the interface, $r = a$.

II. AXISYMMETRIC PLASMA PERTURBATION

A. Derivation of Axisymmetric Ideal-MHD P.D.E.s

Let us assume that all perturbed quantities are independent of the toroidal angle, ϕ . The perturbed plasma equilibrium satisfies the linearized, marginally-stable, ideal-MHD equations

$$\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \quad (18)$$

$$\nabla p = \mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b}, \quad (19)$$

$$\mathbf{j} = \nabla \times \mathbf{b}, \quad (20)$$

$$p = -\boldsymbol{\xi} \cdot \nabla P, \quad (21)$$

where $\boldsymbol{\xi}(r, \theta)$ is the plasma displacement, $\mathbf{b}(r, \theta)$ the perturbed magnetic field, $\mathbf{j}(r, \theta)$ the perturbed current density, and $p(r, \theta)$ the perturbed scalar pressure.

Now,

$$(\boldsymbol{\xi} \times \mathbf{B})_\theta = \mathcal{J} (\xi^\phi B^r - \xi^r B^\phi) = -\mathcal{J} B^\phi \xi^r, \quad (22)$$

$$(\boldsymbol{\xi} \times \mathbf{B})_\phi = \mathcal{J} (\xi^r B^\theta - \xi^\theta B^r) = \mathcal{J} B^\theta \xi^r, \quad (23)$$

where use has been made of the fact that $B^r = J^r = 0$. [See Eqs. (5) and (11).] Combining Eqs. (18) and (23), we obtain

$$\mathcal{J} b^r = \frac{\partial}{\partial \theta} (\mathcal{J} B^\theta \xi^r). \quad (24)$$

Thus, Eqs. (2), (4), and (6) give

$$r R^2 b^r = \frac{\partial y}{\partial \theta}, \quad (25)$$

where

$$y(r, \theta) = f \xi^r. \quad (26)$$

The constraint $\nabla \cdot \mathbf{b} = 0$, which follows from Eq. (18), immediately yields

$$r R^2 b^\theta = -\frac{\partial y}{\partial r}. \quad (27)$$

According to Eq. (21),

$$p = -P' \nabla r \cdot \boldsymbol{\xi} = -P' \xi^r. \quad (28)$$

So, the perturbed force balance equation, (19), yields

$$-\frac{\partial (P' \xi^r)}{\partial r} = (\mathbf{j} \times \mathbf{B})_r + (\mathbf{J} \times \mathbf{b})_r, \quad (29)$$

$$-\frac{\partial (P' \xi^r)}{\partial \theta} = (\mathbf{j} \times \mathbf{B})_\theta + (\mathbf{J} \times \mathbf{b})_\theta, \quad (30)$$

$$0 = (\mathbf{j} \times \mathbf{B})_\phi + (\mathbf{J} \times \mathbf{b})_\phi, \quad (31)$$

giving

$$-\frac{\partial (P' \xi^r)}{\partial r} = r R^2 (j^\theta B^\phi - j^\phi B^\theta) + r R^2 (J^\theta b^\phi - J^\phi b^\theta), \quad (32)$$

$$-\frac{\partial (P' \xi^r)}{\partial \theta} = r R^2 (j^\phi B^r - j^r B^\phi) + r R^2 (J^\phi b^r - J^r b^\phi), \quad (33)$$

$$0 = r R^2 (j^r B^\theta - j^\theta B^r) + r R^2 (J^r b^\theta - J^\theta b^r), \quad (34)$$

where use has been made of Eq. (2). Thus, according to Eqs. (5)–(7), (11), (12), and (17),

$$-\frac{\partial(P' \xi^r)}{\partial r} = f(q j^\theta - j^\phi) - g' b^\phi + \left(q g' + \frac{r R^2 P'}{f}\right) b^\theta, \quad (35)$$

$$-\frac{\partial(P' \xi^r)}{\partial \theta} = -r g j^r - \left(q g' + \frac{r R^2 P'}{f}\right) b^r, \quad (36)$$

$$0 = f j^r + g' b^r. \quad (37)$$

It follows from Eqs. (25) and (37) that

$$r R^2 j^r = -\alpha_g \frac{\partial y}{\partial \theta}, \quad (38)$$

where

$$\alpha_g(r) = \frac{g'}{f}. \quad (39)$$

Note that Eq. (36) is trivially satisfied. Hence, of the three components of the perturbed force balance equation, only Eq. (35) remains to be solved.

Equation (20) yields

$$r R^2 j^r = \frac{\partial b_\phi}{\partial \theta}, \quad (40)$$

$$r R^2 j^\theta = -\frac{\partial b_\phi}{\partial r}, \quad (41)$$

$$r R^2 j^\phi = \frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial \theta}, \quad (42)$$

where use has been made of Eq. (2). It follows from Eqs. (38), (40), and (41) that

$$b_\phi = -\alpha_g y, \quad (43)$$

$$r R^2 j^\theta = \frac{\partial(\alpha_g y)}{\partial r}. \quad (44)$$

Note that $\nabla \cdot \mathbf{j} = 0$, in accordance with Eq. (20).

Now,

$$\mathbf{b} = b_r \nabla r + b_\theta \nabla \theta + b_\phi \nabla \phi, \quad (45)$$

so

$$b^r = \mathbf{b} \cdot \nabla r = |\nabla r|^2 b_r + (\nabla r \cdot \nabla \theta) b_\theta, \quad (46)$$

$$b^\theta = \mathbf{b} \cdot \nabla \theta = (\nabla r \cdot \nabla \theta) b_r + |\nabla \theta|^2 b_\theta, \quad (47)$$

$$b^\phi = \mathbf{b} \cdot \nabla \phi = \frac{b_\phi}{R^2}. \quad (48)$$

Equations (2), (46), and (47) can be rearranged to give

$$b_r = \left(\frac{1}{|\nabla r|^2} \right) b^r - \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) b_\theta, \quad (49)$$

$$b^\theta = \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) b^r + \left(\frac{1}{r^2 R^2 |\nabla r|^2} \right) b_\theta. \quad (50)$$

Let

$$\mathcal{Z}(r, \theta) = |\nabla r|^2 r \frac{\partial y}{\partial r} + r \nabla r \cdot \nabla \theta \frac{\partial y}{\partial \theta}. \quad (51)$$

Equations (25), (27), (43), (49) and (50) yield

$$b_r = \frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} z, \quad (52)$$

$$b_\theta = -\mathcal{Z}, \quad (53)$$

$$b^\phi = -\frac{\alpha_g}{R^2} y. \quad (54)$$

Equations (42), (52), and (53) give

$$r R^2 j^\phi = -\frac{\partial \mathcal{Z}}{\partial r} - \frac{\partial}{\partial \theta} \left[\frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right]. \quad (55)$$

It follows from Eqs. (26), (27), (35), (44), (54), and (55) that

$$\begin{aligned} -\frac{\partial}{\partial r} \left(\frac{P'}{f} y \right) &= \frac{f q}{r R^2} \frac{\partial(\alpha_g y)}{\partial r} + \frac{f}{r R^2} \frac{\partial \mathcal{Z}}{\partial r} \\ &+ \frac{f}{r R^2} \frac{\partial}{\partial \theta} \left[\frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right] \\ &+ \frac{g' \alpha_g}{R^2} y - \left(q g' + \frac{r R^2 P'}{f} \right) \frac{1}{r R^2} \frac{\partial y}{\partial r}. \end{aligned} \quad (56)$$

Hence,

$$- \left[(\alpha_f \alpha_p + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2 \right] y = r \frac{\partial \mathcal{Z}}{\partial r} + \frac{\partial}{\partial \theta} \left[\frac{1}{|\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right], \quad (57)$$

where

$$\alpha_p(r) = \frac{r P'}{f^2}, \quad (58)$$

$$\alpha_f(r) = \frac{r^2}{f} \frac{d}{dr} \left(\frac{f}{r} \right). \quad (59)$$

Finally, Eqs. (51) and (57) yield the *axisymmetric ideal-MHD p.d.e.s*:

$$r \frac{\partial y}{\partial r} = \frac{\mathcal{Z}}{|\nabla r|^2} - \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \frac{\partial y}{\partial \theta}, \quad (60)$$

$$r \frac{\partial \mathcal{Z}}{\partial r} = - [(\alpha_f \alpha_p + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2] y - \frac{\partial}{\partial \theta} \left(\frac{1}{|\nabla r|^2} R^2 \frac{\partial y}{\partial \theta} \right) - \frac{\partial}{\partial \theta} \left(\frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right). \quad (61)$$

B. Derivation of the Axisymmetric Ideal-MHD O.D.E.s

Let

$$y(r, \theta) = \sum_m y_m(r) e^{im\theta}, \quad (62)$$

$$\mathcal{Z}(r, \theta) = \sum_m Z_m(r) e^{im\theta}. \quad (63)$$

Equations (60) and (61) yield the *axisymmetric ideal-MHD o.d.e.s*:

$$r \frac{dy_m}{dr} = \sum_{m'} \left(A_m^{m'} Z_{m'} + B_m^{m'} y_{m'} \right), \quad (64)$$

$$r \frac{dZ_m}{dr} = \sum_{m'} \left(C_m^{m'} Z_{m'} + D_m^{m'} y_{m'} \right), \quad (65)$$

where

$$A_m^{m'} = c_m^{m'}, \quad (66)$$

$$B_m^{m'} = -m' f_m^{m'}, \quad (67)$$

$$C_m^{m'} = -m f_m^{m'}, \quad (68)$$

$$D_m^{m'} = -(\alpha_f \alpha_p + r \alpha'_p) a_m^{m'} - (q r \alpha'_g + r^2 \alpha_g^2) \delta_m^{m'} + m m' b_m^{m'}, \quad (69)$$

and

$$a_m^{m'}(r) = \oint R^2 \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (70)$$

$$b_m^{m'}(r) = \oint |\nabla r|^{-2} R^{-2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (71)$$

$$c_m^{m'}(r) = \oint |\nabla r|^{-2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (72)$$

$$f_m^{m'}(r) = \oint \frac{i r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}. \quad (73)$$

C. Properties of Axisymmetric Ideal-MHD O.D.E.s

Note, from Eq. (25) that $\oint \mathcal{J} \mathbf{b} \cdot \nabla r d\theta = 0$. Note, from Eqs. (64)–(69), that $dZ_0/dr = 0$ in the vacuum region surrounding the plasma. Note also that $a_{m'}^m = a_m^{m'*}$, $b_{m'}^m = b_m^{m'*}$, $c_{m'}^m = c_m^{m'*}$, and $f_{m'}^m = -f_m^{m'*}$, which implies that

$$A_{m'}^m = A_{m'}^{m*}, \quad (74)$$

$$B_{m'}^m = -C_{m'}^{m*}, \quad (75)$$

$$C_{m'}^m = -B_{m'}^{m*}, \quad (76)$$

$$D_{m'}^m = D_{m'}^{m*}. \quad (77)$$

It follows from Eqs. (64), (65), and (74)–(77) that

$$r \frac{d}{dr} \left[\sum_m (Z_m y_m^* - y_m Z_m^*) \right] = 0. \quad (78)$$

D. Perturbed Electric Field

Let \mathbf{e} be the perturbed electric field, which satisfies

$$\nabla \times \mathbf{e} = i\omega \mathbf{b}. \quad (79)$$

Hence,

$$e_\phi = i\omega y, \quad (80)$$

and

$$\frac{\partial e_\theta}{\partial r} - \frac{\partial e_r}{\partial \theta} = -i\omega r \alpha_g y, \quad (81)$$

where use has been made of Eqs. (25), (27), and (54). We also expect $\nabla \cdot \mathbf{e} = 0$, which implies that $e_r = e_\theta = 0$ in the vacuum region, $r \geq a$, in which $\alpha_g = 0$.

E. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque exerted on the plasma lying within the magnetic flux-surface whose label is r is

$$T_\phi(r) = \oint \oint r R^2 b_\phi b^r d\theta d\phi. \quad (82)$$

It follows from Eqs. (25) and (43) that

$$T_\phi(r) = -\pi \alpha_g \oint \left(y^* \frac{\partial y}{\partial \theta} + y \frac{\partial y^*}{\partial \theta} \right) d\theta = -\pi \alpha_g \oint \frac{\partial |y|^2}{\partial \theta} d\theta = 0. \quad (83)$$

We conclude that an axisymmetric perturbation is incapable of exerting a net toroidal electromagnetic torque on the plasma.

F. Electromagnetic Energy Flux

The net electromagnetic energy flux across the plasma-vacuum interface is

$$\begin{aligned} \mathcal{E} &= \left[\oint \oint (\mathbf{e} \times \mathbf{b}) \cdot \nabla r \mathcal{J} d\theta d\phi \right]_{r=a} = \left[\oint \oint (e_\theta b_\phi - e_\phi b_\theta) d\theta d\phi \right]_{r=a} \\ &= i\pi\omega \oint (y \mathcal{Z}^* - y^* \mathcal{Z})_{r=a} d\theta, \\ &= i\pi^2\omega \sum_m (Z_m^* y_m - y_m^* Z_m)_{r=a}. \end{aligned} \quad (84)$$

Here, use has been made of Eqs. (53 and (80), as well as the fact that $e_\theta = 0$ for $r \geq a$.

G. Perturbed Plasma Potential Energy

The perturbed plasma potential energy in the region of the plasma lying within the magnetic flux-surface whose label is r is

$$\delta W_p = \frac{1}{2} \oint \oint r R^2 \xi^{r*} (-\mathbf{B} \cdot \mathbf{b} + \xi^r P') d\theta d\phi. \quad (85)$$

However,

$$\mathbf{B} \cdot \mathbf{b} - \xi^r P' = B^\theta b_\theta + B^\phi b_\phi - \xi^r P' = -\frac{f}{r R^2} (\mathcal{Z} + q \alpha_g y + \alpha_p R^2), \quad (86)$$

where use has been made of Eqs. (4)–(7), (26), (43), (53), and (58). Hence, we obtain

$$\delta W_p(r) = \frac{1}{2} \oint \oint y^* [\mathcal{Z} + (q \alpha_g + \alpha_p R^2) y] d\theta d\phi = \pi^2 \sum_m y_m^* \chi_m, \quad (87)$$

where

$$\chi_m(r) = Z_m + q \alpha_g y_m + \alpha_p \sum_{m'} a_m^{m'} y_{m'}. \quad (88)$$

III. INVERSE ASPECT-RATIO EXPANDED TOKAMAK EQUILIBRIUM

A. Equilibrium Magnetic Flux-Surfaces

Let us assume that the inverse aspect-ratio of the plasma, $\epsilon = a/R_0 = a$ (since R_0 is normalized to unity), is such that $0 < \epsilon \ll 1$. Let $r = \epsilon \hat{r}$, $\nabla = \epsilon^{-1} \hat{\nabla}$, and $' \rightarrow \epsilon^{-1}'$. Suppose that the loci of the equilibrium magnetic flux-surfaces can be written in the parametric form:

$$\begin{aligned} R(\hat{r}, \omega) = & 1 - \epsilon \hat{r} \cos \omega + \epsilon^2 \sum_{j>0} H_j(\hat{r}) \cos[(j-1)\omega] + \epsilon^2 \sum_{j>1} V_j(\hat{r}) \sin[(j-1)\omega] \\ & + \epsilon^3 L(\hat{r}) \cos \omega, \end{aligned} \quad (89)$$

$$\begin{aligned} Z(\hat{r}, \omega) = & \epsilon \hat{r} \sin \omega + \epsilon^2 \sum_{j>1} H_j(\hat{r}) \sin[(j-1)\omega] - \epsilon^2 \sum_{j>1} V_j(\hat{r}) \cos[(j-1)\omega] \\ & - \epsilon^3 L(\hat{r}) \sin \omega, \end{aligned} \quad (90)$$

where j is a positive integer. Here, $H_1(\hat{r})$ controls the relative horizontal locations of the flux-surface centroids, $H_2(\hat{r})$ and $V_2(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface ellipticities, $H_3(\hat{r})$ and $V_3(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface triangularities, et cetera, whereas $L(\hat{r})$ is a flux-surface re-labelling parameter. Moreover, $\omega(R, Z)$ is a poloidal angle that is distinct from θ . Note that V_1 does not appear in Eq. (90) because such a factor merely gives rise to a rigid vertical shift of the plasma that can be eliminated by a suitable choice of the origin of the flux-coordinate system.

Let

$$J(\hat{r}, \omega) = \frac{1}{\epsilon^2} \left(\frac{\partial R}{\partial \omega} \frac{\partial Z}{\partial \hat{r}} - \frac{\partial R}{\partial \hat{r}} \frac{\partial Z}{\partial \omega} \right) \quad (91)$$

be the Jacobian of the \hat{r} , ω coordinate system. We can transform to the \hat{r} , θ coordinate system by writing

$$\theta(\hat{r}, \omega) = 2\pi \int_0^\omega \frac{J(\hat{r}, \tilde{\omega})}{R(\hat{r}, \tilde{\omega})} d\tilde{\omega} \Big/ \oint \frac{J(\hat{r}, \omega)}{R(\hat{r}, \omega)} d\omega, \quad (92)$$

$$\hat{r} = \frac{1}{2\pi} \oint \frac{J(\hat{r}, \omega)}{R(\hat{r}, \omega)} d\omega. \quad (93)$$

This transformation ensures that

$$\frac{\partial \theta}{\partial \omega} = \frac{J}{\hat{r} R}, \quad (94)$$

and, hence, that

$$\mathcal{J} \equiv \frac{R}{\epsilon} \left(\frac{\partial R}{\partial \theta} \frac{\partial Z}{\partial \hat{r}} - \frac{\partial R}{\partial \hat{r}} \frac{\partial Z}{\partial \theta} \right) = \epsilon R J \frac{\partial \omega}{\partial \theta} = r R^2, \quad (95)$$

in accordance with Eq. (2).

B. Metric Elements

We can determine the metric elements of the flux-coordinate system by combining Eqs. (89)–(93). Evaluating the elements up to $\mathcal{O}(\epsilon)$, but retaining $\mathcal{O}(\epsilon^2)$ contributions to terms that are independent of ω , we obtain,

$$L(\hat{r}) = \frac{\hat{r}^3}{8} - \frac{\hat{r} H_1}{2} - \frac{1}{2} \sum_{j>1} (j-1) \frac{H_j^2}{\hat{r}} - \frac{1}{2} \sum_{j>1} (j-1) \frac{V_j^2}{\hat{r}}, \quad (96)$$

$$\begin{aligned} \theta &= \omega + \epsilon \hat{r} \sin \omega - \epsilon \sum_{j>0} \frac{1}{j} \left[H'_j - (j-1) \frac{H_j}{\hat{r}} \right] \sin(j\omega) \\ &+ \epsilon \sum_{j>1} \frac{1}{j} \left[V'_j - (j-1) \frac{V_j}{\hat{r}} \right] \cos(j\omega), \end{aligned} \quad (97)$$

$$\begin{aligned} |\hat{\nabla} \hat{r}|^2 &= 1 + 2\epsilon \sum_{j>0} H'_j \cos(j\theta) + 2\epsilon \sum_{j>1} V'_j \sin(j\theta) \\ &+ \epsilon^2 \left(\frac{3\hat{r}^2}{4} - H_1 + \frac{1}{2} \sum_{j>0} \left[H_j'^2 + (j^2-1) \frac{H_j^2}{\hat{r}^2} \right] \right. \\ &\left. + \frac{1}{2} \sum_{j>1} \left[V_j'^2 + (j^2-1) \frac{V_j^2}{\hat{r}^2} \right] \right), \end{aligned} \quad (98)$$

$$\begin{aligned}\hat{\nabla}\hat{r} \cdot \hat{\nabla}\theta &= \epsilon \sin \theta - \epsilon \sum_{j>0} \frac{1}{j} \left[H_j'' + \frac{H_j'}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \right] \sin(j\theta) \\ &+ \epsilon \sum_{j>1} \frac{1}{j} \left[V_j'' + \frac{V_j'}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \right] \cos(j\theta),\end{aligned}\quad (99)$$

$$R^2 = 1 - 2\epsilon \hat{r} \cos \theta - \epsilon^2 \left(\frac{\hat{r}^2}{2} - \hat{r} H_1' - 2 H_1 \right). \quad (100)$$

Here, $' \equiv d/d\hat{r}$. Moreover, we have made use of the fact that $V_j \propto H_j$, for $j > 1$, because V_j and H_j satisfy the identical differential equations, (106) and (107).

C. Expansion of Inverse Grad-Shafranov Equation

Let us write

$$f(\hat{r}) = \epsilon \frac{\hat{r} g}{q}, \quad (101)$$

$$g(\hat{r}) = 1 + \epsilon^2 g_2(\hat{r}) + \epsilon^4 g_4(\hat{r}), \quad (102)$$

$$P'(\hat{r}) = \epsilon^2 p_2'(\hat{r}), \quad (103)$$

where q , g_2 , g_4 , and p_2 are all $\mathcal{O}(1)$. Here, the safety-factor, $q(\hat{r})$, and the second-order plasma pressure gradient, $p_2'(\hat{r})$, are the two free flux-surface functions that characterize the plasma equilibrium.

Expanding the inverse Grad-Shafranov equation, (16), order by order in the small parameter ϵ , making use of Eqs. (98)–(103), we obtain

$$g_2' = -p_2' - \frac{\hat{r}}{q^2} (2 - s), \quad (104)$$

$$H_1'' = -(3 - 2s) \frac{H_1'}{\hat{r}} - 1 + \frac{2 p_2' q^2}{\hat{r}}, \quad (105)$$

$$H_j'' = -(3 - 2s) \frac{H_j'}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \quad \text{for } j > 1, \quad (106)$$

$$V_j'' = -(3 - 2s) \frac{V_j'}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \quad \text{for } j > 1, \quad (107)$$

$$g_4' = g_2 \left[p_2' - \frac{\hat{r}}{q^2} (2 - s) \right] - \frac{\hat{r}}{q} \Sigma + p_2' \left(\frac{\hat{r}^2}{2} + \frac{\hat{r}^2}{q^2} - 2 H_1 - 3 \hat{r} H_1' \right), \quad (108)$$

where $s = \hat{r} q' / q$ is the magnetic shear, and

$$\Sigma = \frac{S_2}{q} - \frac{2-s}{q} S_3 \quad (109)$$

$$S_1(\hat{r}) = \frac{1}{2} \sum_{j>0} \left[3 H_j'^2 - (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] + \frac{1}{2} \sum_{j>1} \left[3 V_j'^2 - (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right], \quad (110)$$

$$S_2(\hat{r}) = \frac{3 \hat{r}^2}{2} - 2 \hat{r} H_1' + \sum_{j>0} \left[H_j'^2 + 2 (j^2 - 1) \frac{H_j' H_j}{\hat{r}} - (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] \\ + \sum_{j>1} \left[V_j'^2 + 2 (j^2 - 1) \frac{V_j' V_j}{\hat{r}} - (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right], \quad (111)$$

$$S_3(\hat{r}) = -\frac{3 \hat{r}^2}{4} + \frac{\hat{r}^2}{q^2} + H_1 + S_1, \quad (112)$$

$$S_5(\hat{r}) = \frac{7 \hat{r}^2}{4} - H_1 - 3 \hat{r} H_1' + S_1. \quad (113)$$

Note that the relative horizontal shift of magnetic flux-surfaces, $-H_1$, otherwise known as the *Shafranov shift*, is driven by toroidicity [the second term on the right-hand side of Eq. (105)], and plasma pressure gradients (the third term). All of the other shaping terms (i.e., the H_j , for $j > 1$, and the V_j) are driven by axisymmetric currents flowing in external magnetic field-coils.

Equations (39), (58), (59), and (101)–(103) yield

$$\alpha_p(\hat{r}) = \frac{p_2' q^2}{\hat{r}} (1 - 2 \epsilon^2 g_2), \quad (114)$$

$$\alpha_g(\hat{r}) = \frac{q}{\hat{r}} (g_2' - \epsilon^2 g_2 g_2' + \epsilon^2 g_4'), \quad (115)$$

$$\alpha_f(\hat{r}) = -s + \epsilon^2 \hat{r} g_2'. \quad (116)$$

Finally, it follows from Eqs. (99) and (105)–(107) that

$$\hat{\nabla} \hat{r} \cdot \hat{\nabla} \theta = 2 \epsilon \left[1 - \frac{p_2' q^2}{\hat{r}} + (1 - s) \frac{H_1'}{\hat{r}} \right] \sin \theta \\ - 2 \epsilon \sum_{j>1} \frac{1}{j} \left[-(1 - s) \frac{H_j'}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \right] \sin(j \theta) \\ + 2 \epsilon \sum_{j>1} \frac{1}{j} \left[-(1 - s) \frac{V_j'}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \right] \cos(j \theta). \quad (117)$$

D. Calculation of Coupling Coefficients

Equations (98) and (110) yield

$$|\hat{\nabla}\hat{r}|^{-2} = 1 - 2\epsilon \sum_{j>0} H'_j \cos(j\theta) - 2\epsilon \sum_{j>0} V'_j \sin(j\theta) + \epsilon^2 \left(-\frac{3\hat{r}^2}{4} + H_1 + S_1 \right), \quad (118)$$

Equation (100) gives

$$R^{-2} = 1 + 2\epsilon \hat{r} \cos\theta + \epsilon^2 \left(\frac{5\hat{r}^2}{2} - \hat{r} H'_1 - 2H_1 \right). \quad (119)$$

The previous two equations imply that

$$\begin{aligned} |\hat{\nabla}\hat{r}|^{-2} R^{-2} &= 1 + 2\epsilon \hat{r} \cos\theta - 2\epsilon \sum_{j>0} H'_j \cos(j\theta) - 2\epsilon \sum_{j>1} V'_j \sin(j\theta) \\ &\quad + \epsilon^2 \left(\frac{7\hat{r}^2}{4} - H_1 - 3\hat{r} H'_1 + S_1 \right). \end{aligned} \quad (120)$$

Finally, Eqs. (117) and (118) give

$$\begin{aligned} \hat{\nabla}\hat{r} \cdot \hat{\nabla}\theta |\hat{\nabla}\hat{r}|^{-2} &= 2\epsilon \left[1 - \frac{p'_2 q^2}{\hat{r}} + (1-s) \frac{H'_1}{\hat{r}} \right] \sin\theta \\ &\quad - 2\epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) \frac{H'_j}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \right] \sin(j\theta) \\ &\quad + 2\epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) \frac{V'_j}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \right] \cos(j\theta), \end{aligned} \quad (121)$$

where use has been made of the fact that $V'_j \propto H'_j$ for $j > 1$.

Equations (70)–(73), (100), (118), (120), and (121) imply that

$$a_m^{m'} = \delta_m^{m'} - \epsilon \hat{r} (\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon^2 \left(\frac{\hat{r}^2}{2} - \hat{r} H'_1 - 2H_1 \right) \delta_m^{m'}, \quad (122)$$

$$\begin{aligned} b_m^{m'} &= \delta_m^{m'} + \epsilon \hat{r} (\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon \sum_{j>0} H'_j (\delta_{m'-m-j} + \delta_{m'-m+j}) \\ &\quad - \epsilon \sum_{j>1} i V'_j (\delta_{m'-m-j} - \delta_{m'-m+j}) + \epsilon^2 \left(\frac{7\hat{r}^2}{4} - H_1 - 3\hat{r} H'_1 + S_1 \right) \delta_m^{m'}, \end{aligned} \quad (123)$$

$$c_m^{m'} = \delta_m^{m'} - \epsilon \sum_{j>0} H'_j (\delta_{m'-m-j} + \delta_{m'-m+j}) - \epsilon \sum_{j>1} i V'_j (\delta_{m'-m-j} - \delta_{m'-m+j})$$

$$+ \epsilon^2 \left(-\frac{3\hat{r}^2}{4} + H_1 + S_1 \right) \delta_m^{m'}, \quad (124)$$

$$\begin{aligned} f_m^{m'} &= -\epsilon \left[\hat{r} - p'_2 q^2 + (1-s) H'_1 \right] (\delta_{m'-m-1} - \delta_{m'-m+1}) \\ &+ \epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) H'_j + (j^2 - 1) \frac{H_j}{\hat{r}} \right] (\delta_{m'-m-j} - \delta_{m'-m+j}) \\ &+ \epsilon \sum_{j>1} \frac{i}{j} \left[-(1-s) V'_j + (j^2 - 1) \frac{V_j}{\hat{r}} \right] (\delta_{m'-m-j} + \delta_{m'-m+j}). \end{aligned} \quad (125)$$

If we write

$$\alpha_g = \alpha_g^{(0)} + \epsilon^2 \alpha_g^{(2)}, \quad (126)$$

$$\alpha_p = \alpha_p^{(0)} + \epsilon^2 \alpha_p^{(2)}, \quad (127)$$

$$\alpha_f = \alpha_f^{(0)} + \epsilon^2 \alpha_f^{(2)}, \quad (128)$$

$$a_m^{m'} = 1 + \epsilon a_m^{m'(1)} + \epsilon^2 a_m^{m'(2)}, \quad (129)$$

$$b_m^{m'} = 1 + \epsilon b_m^{m'(1)} + \epsilon^2 b_m^{m'(2)}, \quad (130)$$

$$D_m^{m'} = D_m^{m'(0)} + \epsilon D_m^{m'(1)} + \epsilon^2 D_m^{m'(2)}, \quad (131)$$

where $\alpha_g^{(0)}$, $\alpha_g^{(2)}$, et cetera, are $\mathcal{O}(1)$, then it follows from Eq. (69) that

$$D_m^{m(0)} = -\alpha_f^{(0)} \alpha_p^{(0)} - \hat{r} \alpha_p'^{(0)} - q \hat{r} \alpha_g'^{(0)} + m^2, \quad (132)$$

$$D_m^{m'(1)} = -\left[\alpha_f^{(0)} \alpha_p^{(0)} + \hat{r} \alpha_p'^{(0)} \right] a_m^{m'(1)} + m m' b_m^{m'(1)}, \quad (133)$$

$$\begin{aligned} D_m^{m(2)} &= -\left[\alpha_f^{(0)} \alpha_p^{(0)} + \hat{r} \alpha_p'^{(0)} \right] a_m^{m'(2)} - \alpha_f^{(0)} \alpha_p^{(2)} - \alpha_f^{(2)} \alpha_p^{(0)} - \hat{r} \alpha_p'^{(2)} - q \hat{r} \alpha_g'^{(2)} \\ &- \hat{r}^2 \left[\alpha_g^{(0)} \right]^2 + m^2 b_m^{m(2)}. \end{aligned} \quad (134)$$

Finally, Eqs. (70)–(73), (122)–(125), and (132)–(134) give

$$A_m^m(\hat{r}) = 1 + \epsilon^2 \left(-\frac{3\hat{r}^2}{4} + H_1 + S_1 \right), \quad (135)$$

$$A_m^{m\pm 1}(\hat{r}) = -\epsilon H'_1, \quad (136)$$

$$A_m^{m\pm j}(\hat{r}) = -\epsilon (H'_j \pm i V'_j) \quad \text{for } j > 1, \quad (137)$$

$$B_m^m(\hat{r}) = 0, \quad (138)$$

$$B_m^{m\pm 1}(\hat{r}) = \pm \epsilon (m \pm 1) [\hat{r} - p'_2 q^2 + (1-s) H'_1], \quad (139)$$

$$B_m^{m\pm j}(\hat{r}) = \pm \epsilon \frac{m \pm j}{j} \left[(1-s) (H'_j \pm i V'_j) - (j^2 - 1) \frac{H_j \pm i V_j}{\hat{r}} \right] \quad \text{for } j > 1, \quad (140)$$

$$C_m^m(\hat{r}) = 0, \quad (141)$$

$$C_m^{m\pm 1}(\hat{r}) = \pm \epsilon m [\hat{r} - p'_2 q^2 + (1-s) H'_1], \quad (142)$$

$$C_m^{m\pm j}(\hat{r}) = \pm \epsilon \frac{m}{j} \left[(1-s) (H'_j \pm i V'_j) - (j^2 - 1) \frac{H_j \pm i V_j}{\hat{r}} \right] \quad \text{for } j > 1, \quad (143)$$

$$\begin{aligned} D_m^m(\hat{r}) = & m^2 + q \hat{r} \frac{d}{d\hat{r}} \left(\frac{2-s}{q} \right) + \epsilon^2 m^2 S_5 \\ & + \epsilon^2 \left\{ -\hat{r}^2 \frac{(2-s)^2}{q^2} + q \hat{r} \frac{d\Sigma}{d\hat{r}} - \hat{r} \frac{d}{d\hat{r}} (\hat{r} p'_2) - 2(1-s) \hat{r} p'_2 \right. \\ & \left. + 2 \hat{r} p'_2 q^2 \left(-2 + \frac{3 p'_2 q^2}{\hat{r}} \right) + 2 H'_1 q^2 \left[\frac{d}{d\hat{r}} (\hat{r} p'_2) - 4(1-s) p'_2 \right] \right\}, \end{aligned} \quad (144)$$

$$D_m^{m\pm 1}(\hat{r}) = \epsilon \left[\frac{d}{d\hat{r}} (\hat{r} p'_2) - (2-s) p'_2 \right] q^2 + \epsilon m (m \pm 1) (\hat{r} - H'_1), \quad (145)$$

$$D_m^{m\pm j}(\hat{r}) = -\epsilon m (m \pm j) (H'_j \pm i V'_j) \quad \text{for } j > 1. \quad (146)$$

E. Behavior Close to Magnetic Axis

When $\hat{r} \ll 1$, the well-behaved solution of the axisymmetric ideal-MHD o.d.e.s, (64) and (65), that is dominated by the poloidal harmonic whose poloidal mode number is m is such that

$$y_m(\hat{r}) = \hat{r}^{|m|}, \quad (147)$$

$$z_m(\hat{r}) = |m| \hat{r}^{|m|}, \quad (148)$$

with $y_{m'}(\hat{r}) = z_{m'}(\hat{r}) = 0$ for $m' \neq 0$.

IV. VACUUM SOLUTION

A. Toroidal Coordinates

Let μ, η, ϕ be right-handed toroidal coordinates defined such that

$$R = \frac{\sinh \mu}{\cosh \mu - \cos \eta}, \quad (149)$$

$$Z = \frac{\sin \eta}{\cosh \mu - \cos \eta}. \quad (150)$$

The scale-factors of the toroidal coordinate system are

$$h_\mu = h_\eta = \frac{1}{\cosh \mu - \cos \eta} \equiv h, \quad (151)$$

$$h_\phi = \frac{\sinh \mu}{\cosh \mu - \cos \eta} = h \sinh \mu. \quad (152)$$

Moreover,

$$\mathcal{J}' \equiv (\nabla \mu \times \nabla \eta \cdot \nabla \phi)^{-1} = h^3 \sinh \mu. \quad (153)$$

B. Perturbed Magnetic Field

The curl-free perturbed magnetic field in the vacuum region is written $\mathbf{b} = i \nabla V$, where $\nabla^2 V = 0$. The most general axisymmetric solution to Laplace's equation is

$$V(z, \eta) = \sum_m (z - \cos \eta)^{1/2} U_m(z) e^{-i m \eta}, \quad (154)$$

$$U_m(z) = p_m \hat{P}_{|m|-1/2}(z) + q_m \hat{Q}_{m-1/2}(z), \quad (155)$$

where $z = \cosh \mu$, the p_m and q_m are arbitrary complex coefficients, and

$$\hat{P}_{|m|-1/2}(z) = \cos(|m| \pi) \frac{\sqrt{\pi} \Gamma(|m| + 1/2) a^{|m|}}{2^{|m|-1/2} |m|!} P_{|m|-1/2}(z), \quad (156)$$

$$\hat{Q}_{|m|-1/2}(z) = \cos(|m| \pi) \frac{2^{|m|-1/2} |m|!}{\sqrt{\pi} \Gamma(|m| + 1/2) a^{|m|}} Q_{|m|-1/2}(z). \quad (157)$$

Here, the $P_{|m|-1/2}(z)$ and $Q_{|m|-1/2}(z)$ are toroidal functions, and $\Gamma(z)$ is a gamma function.

C. Toroidal Electromagnetic Angular Momentum Flux

The outward flux of toroidal angular momentum across a constant- z surface is

$$T_\phi(z) = - \oint \oint \mathcal{J}' b_\phi b^\mu d\eta d\phi = 0, \quad (158)$$

because $b_\phi = i \partial V / \partial \phi = 0$. Of course, this has to be the case because the flux of angular momentum across the plasma-vacuum interface is zero. (See Sect. II E.)

D. Electromagnetic Energy Flux

The outward flux of electromagnetic energy flux across a constant- z surface is

$$\mathcal{E}(z) = - \oint \oint \mathcal{J}' \mathbf{e} \times \mathbf{b} \cdot \nabla \mu d\eta d\phi = -i \pi \oint \left(e_\phi \frac{\partial V^*}{\partial \eta} - e_\phi^* \frac{\partial V}{\partial \eta} \right) d\eta, \quad (159)$$

given that $e_\mu = e_\eta = 0$ in the vacuum. However, $\nabla \times \mathbf{e} = i \omega \mathbf{b}$ implies that

$$\frac{\partial e_\phi}{\partial \eta} = -\omega h \sinh \mu \frac{\partial V}{\partial \mu} = -\omega h \sinh^2 \mu \frac{\partial V}{\partial z}. \quad (160)$$

Thus,

$$\begin{aligned} \mathcal{E}(z) &= i \pi \oint \left(\frac{\partial e_\phi}{\partial \eta} V^* - \frac{\partial e_\phi^*}{\partial \eta} V \right) d\eta = -i \pi \omega \oint h \sinh^2 \mu \left(\frac{\partial V}{\partial z} V^* - \frac{\partial V^*}{\partial z} V \right) d\eta \\ &= i \pi^2 \omega \sum_m (p_m q_m^* - q_m p_m^*) (z^2 - 1) \mathcal{W}(P_{|m|-1/2}, Q_{|m|-1/2}), \end{aligned} \quad (161)$$

where $\mathcal{W}(f, g) = f dg/dz - g df/dz$. However,

$$\mathcal{W}(P_{|m|-1/2}, Q_{|m|-1/2}) = \frac{1}{1 - z^2}, \quad (162)$$

so

$$\mathcal{E}(z) = -i \pi^2 \omega \sum_m (p_m q_m^* - q_m p_m^*). \quad (163)$$

Note that \mathcal{E} is independent of z , as must be the case because there are no energy sources in the vacuum region.

E. Solution in Vacuum Region

In the large-aspect ratio limit, $r \ll 1$, it can be demonstrated that

$$z \simeq \frac{1}{r}, \quad (164)$$

$$z^{1/2} \hat{P}_{-1/2}(z) \simeq \frac{1}{2} \ln(8z), \quad (165)$$

$$z^{1/2} \hat{P}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\pi) (az)^{|m|}}{|m|}, \quad (166)$$

$$z^{1/2} \hat{Q}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\pi) (az)^{-|m|}}{2}. \quad (167)$$

Note that Eq. (166) only applies to $|m| > 0$.

Now, according to Eqs. (25) and (38)

$$\frac{\partial y}{\partial \theta} = \mathcal{J} \mathbf{b} \cdot \nabla r = \mathbf{i} \mathcal{J} \nabla V \cdot \nabla r, \quad (168)$$

$$\mathcal{Z} = -\mathcal{J} \nabla \phi \times \nabla r \cdot \mathbf{b} = -\mathbf{i} \frac{\partial V}{\partial \theta}. \quad (169)$$

Thus,

$$\underline{V}(r) = \underline{\underline{P}}(r) \underline{p} + \underline{\underline{Q}}(r) \underline{q}, \quad (170)$$

$$\underline{\psi}(r) = \underline{\underline{R}}(r) \underline{p} + \underline{\underline{S}}(r) \underline{q}, \quad (171)$$

where $Z_m(r) = m V_m(r)$, $\psi_m(r) = m y_m(r)$, $\underline{V}(r)$ is the vector of the $V_m(r)$ values, $\underline{\psi}(r)$ is the vector of the $\psi_m(r)$ values, $\underline{\underline{P}}(r)$ is the matrix of the

$$\mathcal{P}_{mm'}(r) = \oint_r (z - \cos \eta)^{1/2} \hat{P}_{|m'|-1/2}(z) \exp[-\mathbf{i}(m\theta + m'\eta)] \frac{d\theta}{2\pi} \quad (172)$$

values, $\underline{\underline{Q}}(r)$ is the matrix of the

$$\mathcal{Q}_{mm'}(r) = \oint_r (z - \cos \eta)^{1/2} \hat{Q}_{|m'|-1/2}(z) \exp[-\mathbf{i}(m\theta + m'\eta)] \frac{d\theta}{2\pi} \quad (173)$$

values, $\underline{\underline{R}}(r)$ is the matrix of the

$$\mathcal{R}_{mm'}(r) = \oint_r \left\{ \left[\frac{1}{2} (z - \cos \eta)^{-1/2} \hat{P}_{|m'|-1/2}(z) + (z - \cos \eta)^{1/2} \frac{d\hat{P}_{|m'|-1/2}}{dz} \right] \mathcal{J} \nabla r \cdot \nabla z \right.$$

$$\begin{aligned}
& + \left[\frac{1}{2} (z - \cos \eta)^{-1/2} \sin \eta - i m' (z - \cos \eta)^{1/2} \right] \hat{P}_{|m'|-1/2}(z) \mathcal{J} \nabla r \cdot \nabla \eta \Big\} \\
& \times \exp[-i(m\theta + m'\eta)] \frac{d\theta}{2\pi}
\end{aligned} \tag{174}$$

values, $\underline{\underline{\mathcal{S}}}(r)$ is the matrix of the

$$\begin{aligned}
\mathcal{S}_{mm'}(r) = & \oint_r \left\{ \left[\frac{1}{2} (z - \cos \eta)^{-1/2} \hat{Q}_{|m'|-1/2}(z) + (z - \cos \eta)^{1/2} \frac{d\hat{Q}_{|m'|-1/2}}{dz} \right] \mathcal{J} \nabla r \cdot \nabla z \right. \\
& + \left. \left[\frac{1}{2} (z - \cos \eta)^{-1/2} \sin \eta - i m' (z - \cos \eta)^{1/2} \right] \hat{Q}_{|m'|-1/2}(z) \mathcal{J} \nabla r \cdot \nabla \eta \right\} \\
& \times \exp[-i(m\theta + m'\eta)] \frac{d\theta}{2\pi}
\end{aligned} \tag{175}$$

values, \underline{p} is the vector of the p_m coefficients, and \underline{q} is the vector of the q_m coefficients. Here, the subscript r on the integrals indicates that they are taken at constant r . Note from Eqs. (168) and (169), $\oint \mathcal{J} \nabla V \cdot \nabla r d\theta = Z_0 = 0$.

F. Energy Conservation

According to Eq. (84), the net flux of electromagnetic energy across the plasma-vacuum interface is

$$\mathcal{E} = i\pi^2 \omega (\underline{V}^\dagger \underline{\psi} - \underline{\psi}^\dagger \underline{V}). \tag{176}$$

However, this flux must be equal the energy flux through the vacuum region, so Eq. (163) gives

$$\mathcal{E} = -i\pi^2 \omega (\underline{q}^\dagger \underline{p} - \underline{p}^\dagger \underline{q}). \tag{177}$$

Equations (170), (171), and the previous two equations, yield

$$\underline{\underline{\mathcal{P}}}^\dagger \underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{R}}}^\dagger \underline{\underline{\mathcal{P}}}, \tag{178}$$

$$\underline{\underline{\mathcal{Q}}}^\dagger \underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{S}}}^\dagger \underline{\underline{\mathcal{Q}}}, \tag{179}$$

$$\underline{\underline{\mathcal{P}}}^\dagger \underline{\underline{\mathcal{S}}} - \underline{\underline{\mathcal{R}}}^\dagger \underline{\underline{\mathcal{Q}}} = \underline{\underline{1}}. \tag{180}$$

It can also be demonstrated that

$$\underline{\underline{\mathcal{Q}}} \underline{\underline{\mathcal{P}}}^\dagger = \underline{\underline{\mathcal{P}}} \underline{\underline{\mathcal{Q}}}^\dagger, \tag{181}$$

$$\underline{\underline{\mathcal{R}}}\underline{\underline{\mathcal{S}}}^\dagger = \underline{\underline{\mathcal{S}}}\underline{\underline{\mathcal{R}}}^\dagger. \quad (182)$$

The previous five equations hold throughout the vacuum region.

G. Ideal-Wall Matching Condition

Suppose that the plasma is surrounded by a wall that lies at $r = b_w a$, where $b_w \geq 1$. If the wall is perfectly conducting then $\underline{\underline{\psi}}(b_w a) = 0$. It follows from Eq. (171) that

$$\underline{\underline{q}} = \underline{\underline{I}}_b \underline{\underline{p}}, \quad (183)$$

where

$$\underline{\underline{I}}_b = -\underline{\underline{\mathcal{S}}}_b^{-1} \underline{\underline{\mathcal{R}}}_b \quad (184)$$

is termed the *wall matrix*. Here, $\underline{\underline{\mathcal{S}}}_b = \underline{\underline{\mathcal{S}}}(r = b_w a)$, et cetera. Equation (182) ensures that $\underline{\underline{I}}_b$ is Hermitian. It immediately follows from Eq. (177) that

$$\mathcal{E} = 0. \quad (185)$$

In other words, there is zero net electromagnetic energy flux out of a plasma surrounded by a perfectly conducting wall.

Making use of Eqs. (170) and (171), the matching condition at the plasma-vacuum interface for a perfectly-conducting wall becomes

$$\underline{\underline{V}}(r = a_+) = \underline{\underline{H}}\underline{\underline{\psi}}(r = a), \quad (186)$$

where

$$\underline{\underline{H}} = (\underline{\underline{\mathcal{P}}}_a + \underline{\underline{\mathcal{Q}}}_a \underline{\underline{I}}_b) (\underline{\underline{\mathcal{R}}}_a + \underline{\underline{\mathcal{S}}}_a \underline{\underline{I}}_b)^{-1} \quad (187)$$

is termed the *vacuum matrix*. Here, $\underline{\underline{\mathcal{P}}}_a = \underline{\underline{\mathcal{P}}}(r = a_+)$, et cetera. Making use of Eqs. (178)–(180), it is easily demonstrated that

$$\underline{\underline{H}} - \underline{\underline{H}}^\dagger = -[(\underline{\underline{\mathcal{R}}}_a + \underline{\underline{\mathcal{S}}}_a \underline{\underline{I}}_b)^{-1}]^\dagger (\underline{\underline{I}}_b - \underline{\underline{I}}_b^\dagger) (\underline{\underline{\mathcal{R}}}_a + \underline{\underline{\mathcal{S}}}_a \underline{\underline{I}}_b)^{-1}. \quad (188)$$

Thus, the vacuum matrix, $\underline{\underline{H}}$, is Hermitian because the wall matrix, $\underline{\underline{I}}_b$, is Hermitian.

H. Model Wall Matrix

Equations (164)–(167), (174), and (175) suggest that

$$\underline{\underline{\mathcal{R}}}_b = \underline{\underline{\mathcal{R}}}_a \underline{\underline{\rho}}^{-1}, \quad (189)$$

$$\underline{\underline{\mathcal{S}}}_b = \underline{\underline{\mathcal{S}}}_a \underline{\underline{\rho}}, \quad (190)$$

where

$$\rho_{mm'} = \delta_{mm'} \rho_m, \quad (191)$$

$$\rho_0 = 1 + \ln b_w, \quad (192)$$

$$\rho_{m \neq 0} = b_w^{|m|}. \quad (193)$$

Hence,

$$\underline{\underline{I}}_b = -\underline{\underline{\rho}}^{-1\dagger} \underline{\underline{\mathcal{S}}}_a^{-1} \underline{\underline{\mathcal{R}}}_a \underline{\underline{\rho}}^{-1}. \quad (194)$$

Note that our model wall matrix, $\underline{\underline{I}}_b$, is Hermitian given that $\underline{\underline{\mathcal{S}}}_a^{-1} \underline{\underline{\mathcal{R}}}_a$ is Hermitian. [See Eq. (182).] Our model wall matrix allows us to smoothly interpolate between a plasma with no wall (which corresponds to $b_w \rightarrow \infty$ and $\underline{\underline{H}} = \underline{\underline{\mathcal{P}}}_a \underline{\underline{\mathcal{R}}}_a^{-1}$), and a fixed boundary plasma (which corresponds to $b_w = 1$ and $\underline{\underline{H}}^{-1} = \underline{\underline{0}}$).

I. Perturbed Vacuum Potential Energy

Equation (25) implies that

$$\mathcal{J} \nabla r \cdot \nabla V = -i \frac{\partial y}{\partial \theta}. \quad (195)$$

The perturbed potential energy in the vacuum region is

$$\begin{aligned} \delta W_v &= \frac{1}{2} \int_{a_+}^{\infty} \oint \oint \mathbf{b}^* \cdot \mathbf{b} \mathcal{J} dr d\theta d\phi \\ &= \frac{1}{2} \int_{a_+}^{\infty} \oint \oint \nabla V^* \cdot \nabla V \mathcal{J} dr d\theta d\phi \\ &= -\frac{1}{2} \left(\oint \oint \mathcal{J} \nabla r \cdot \nabla V^* V d\theta d\phi \right)_{a_+} \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \left[\oint \oint \left(-i \frac{\partial y}{\partial \theta} \right)^* V d\theta d\phi \right]_{a_+} \\
&= -\pi^2 \left(\sum_m m y_m^* V_m \right)_{a_+}, \tag{196}
\end{aligned}$$

where use has been made of the facts that $\nabla^2 V = 0$ and $\nabla r \cdot \nabla V = 0$ at the ideal wall.

V. CALCULATION OF PERTURBED POTENTIAL ENERGY

A. Perturbed Plasma Potential Energy

Let the $y_{mm'}(r)$ and the $Z_{mm'}(r)$ be linearly independent solutions of the axisymmetric ideal-MHD o.d.e.s, (64) and (65), that are well-behaved at the magnetic axis. Here, m indexes the poloidal harmonic, whereas m' indexes the solution. Let

$$\chi_{mm'}(r) = Z_{mm'} + q \alpha_g y_{mm'} + \alpha_p \sum_{m''} a_m^{m''} y_{m''m'}. \tag{197}$$

The most general solution to the axisymmetric ideal-MHD o.d.e.s is written

$$y_m(r) = \sum_{m'} y_{mm'}(r) \alpha_{m'}, \tag{198}$$

$$\chi_m(r) = \sum_{m'} \chi_{mm'}(r) \alpha_{m'}, \tag{199}$$

where the α_m are arbitrary complex coefficients.

According to Eq. (87), the net perturbed plasma potential energy is

$$\delta W_p = \pi^2 \underline{y}^\dagger \underline{\chi}, \tag{200}$$

where \underline{y} is the vector of the $y_m(a)$ values and $\underline{\chi}$ the vector of the $\chi_m(a_-)$ values. It follows that

$$\delta W_p = \pi^2 \underline{\alpha}^\dagger \underline{\underline{y}}^\dagger \underline{\underline{\chi}} \underline{\alpha}, \tag{201}$$

where $\underline{\alpha}$ is the vector of the α_m values, $\underline{\underline{y}}$ the matrix of the $y_{mm'}(a)$ values, and $\underline{\underline{\chi}}$ the matrix of the $\chi_{mm'}(a_-)$ values. If

$$\underline{\underline{\chi}} = \underline{\underline{W}}_p \underline{\underline{y}}. \tag{202}$$

then

$$\delta W_p = \pi^2 \underline{\underline{\alpha}}^\dagger \underline{\underline{y}}^\dagger \underline{\underline{W}}_p \underline{\underline{y}} \underline{\underline{\alpha}}. \quad (203)$$

Equations (84) and (185) imply that $\underline{\underline{W}}_p$ is Hermitian. Here, we have made use of the fact that $Z_m(a_-) = \chi_m(a_-)$ because $\alpha_g(a) = \alpha_p(a) = 0$. [See Eq. (88).]

B. Perturbed Vacuum Potential Energy

According to Eq. (196), the perturbed vacuum potential energy is

$$\delta W_v = -\pi^2 \sum_m m y_m^*(a) V_m(a_+). \quad (204)$$

However, Eq. (186) implies that

$$V_m(a_+) = \sum_{m'} H_{mm'} m' y_{m'}(a). \quad (205)$$

Hence, we can write

$$\delta W_v = \pi^2 \underline{\underline{y}}^\dagger \underline{\underline{W}}_v \underline{\underline{y}}, \quad (206)$$

where $\underline{\underline{W}}_v$ is the matrix of the $-m H_{mm'} m'$ values. Given that $H_{mm'}$ is Hermitian, we deduce that $\underline{\underline{W}}_v$ is Hermitian. It follows that

$$\delta W_v = \pi^2 \underline{\underline{\alpha}}^\dagger \underline{\underline{y}}^\dagger \underline{\underline{W}}_v \underline{\underline{y}} \underline{\underline{\alpha}}. \quad (207)$$

C. Total Perturbed Potential Energy

The total perturbed potential energy is

$$\delta W = \delta W_p + \delta W_v = \pi^2 \underline{\underline{\alpha}}^\dagger \underline{\underline{y}}^\dagger \underline{\underline{W}} \underline{\underline{y}} \underline{\underline{\alpha}}, \quad (208)$$

where

$$\underline{\underline{W}} = \underline{\underline{W}}_p + \underline{\underline{W}}_v. \quad (209)$$

Given that $\underline{\underline{W}}_p$ and $\underline{\underline{W}}_v$ are both Hermitian, we deduce that $\underline{\underline{W}}$ is Hermitian.

The fact that $\underline{\underline{W}}$ is Hermitian allows us to write

$$\underline{\underline{W}} \underline{\underline{\beta}} = \underline{\underline{\beta}} \underline{\underline{\Lambda}}, \quad (210)$$

$$\underline{\underline{\beta}}^\dagger \underline{\underline{\beta}} = \underline{\underline{1}}, \quad (211)$$

where $\underline{\underline{\beta}}$ is real, and $\underline{\underline{\Lambda}}$ is the diagonal matrix of the real λ_m values. If $\hat{\underline{\underline{\alpha}}} = \underline{\underline{\beta}} \underline{\underline{y}} \underline{\underline{\alpha}}$ then

$$\delta W = \pi^2 \hat{\underline{\underline{\alpha}}}^\dagger \underline{\underline{\Lambda}} \hat{\underline{\underline{\alpha}}} = \pi^2 \sum_m |\hat{\alpha}_m|^2 \lambda_m. \quad (212)$$

Thus, if any of the λ_m are negative then solutions exist for which δW is negative, and the plasma is consequently unstable to an axisymmetric ideal mode.

Suppose that $\sum_m |\hat{\alpha}_m|^2 = 1$. The ideal energy of the m th mode, for which $\hat{\alpha}_{m'} = \delta_{mm'}$, is

$$\delta W_m = \pi^2 \lambda_m. \quad (213)$$

However,

$$\underline{\underline{\Lambda}} = \underline{\underline{\beta}}^\dagger \underline{\underline{W}} \underline{\underline{\beta}}, \quad (214)$$

Thus, the diagonal components of $\underline{\underline{\beta}}^\dagger \underline{\underline{W}}_p \underline{\underline{\beta}}$ and $\underline{\underline{\beta}}^\dagger \underline{\underline{W}}_v \underline{\underline{\beta}}$ are the plasma and vacuum contributions to the λ_m , respectively. The eigenfunction of the m th mode is such that $y_{m'}(a) = \beta_{mm'}$.