Advanced Divertors

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1 Plasma Equilibrium

Let x, y, z be right-handed Cartesian coordinates. The system is assumed to be periodic in the z-direction with period $2\pi\,R_0$, where R_0 is the simulated major radius of the plasma. Let $\phi = z/R_0$ be a simulated toroidal angle. The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\phi\times\nabla\psi_p + B_0\,R_0\,\nabla\phi, \tag{1}$$

where B_0 is the toroidal magnetic field-strength, and ψ_p the poloidal magnetic flux (divided by 2π).

It is convenient to re-express the magnetic field in the standard Clebsch form

$$\mathbf{B} = \nabla(\phi - q\,\theta)\times\nabla\psi_p, \tag{2}$$

where θ is a poloidal angle, and $q = q(\psi_p)$ the (dimensionless) safety-factor. Equations (1) and (2) can be reconciled provided

$$\nabla\psi_p\times\nabla\theta\cdot\nabla\phi = \frac{B_0}{R_0\,q}. \tag{3}$$

Note, from Eq. (2), that $\mathbf{B}\cdot\nabla\psi_p = 0$, which implies that ψ_p is a magnetic flux-surface label. Furthermore, $\mathbf{B}\cdot\nabla(\phi - q\,\theta) = 0$, which implies that magnetic field-lines within a given flux-surface appear as straight lines, with gradient $d\phi/d\theta = q$, when plotted in the θ-ϕ plane.

Let a be the mean minor radius of the plasma, $\hat{\nabla} = a\,\nabla$, I_p the toroidal plasma current, $X = x/a$, $Y = y/a$, and

$$\psi_p(x,y) = \frac{\mu_0\,I_p\,R_0}{2\pi}\,\psi(X,Y). \tag{4}$$

The safety-factor on a given magnetic flux-surface is

$$q = \frac{q_\ast}{2\pi}\oint\frac{dL}{|\hat{\nabla}\psi|}, \tag{5}$$

where

$$q_\ast = \frac{2\pi\,B_0\,a^2}{\mu_0\,I_p\,R_0}. \tag{6}$$

Here, dL is an element of normalized length, in the X-Y plane, along the flux-surface.

We require $\hat{\nabla}^2\psi = 0$ everywhere that a current does not flow, where $\hat{\nabla}^2 \equiv \partial^2/\partial X^2 + \partial^2/\partial Y^2$. If we let $z = X + {\rm i}\,Y$ then we can automatically ensure this by writing

$$F(z) = \psi(X,Y) + {\rm i}\,\chi(X,Y), \tag{7}$$

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where $\psi(X, Y)$ and $\chi(X, Y)$ are real functions. The Cauchy-Reiman relations yield

$$\frac{\partial\psi}{\partial X} = \frac{\partial\chi}{\partial Y}, \tag{8}$$

$$\frac{\partial\psi}{\partial Y} = -\frac{\partial\chi}{\partial X}, \tag{9}$$

from which it follows that $\hat{\nabla}^2\psi = 0$. The poloidal magnetic field is

$$\mathbf{B}_p \equiv \nabla\phi\times\nabla\psi_p = \frac{\mu_0\,I_p}{2\pi\,a}\,\hat{\mathbf{B}}_p, \tag{10}$$

where

$$\hat{B}_{p\,X} = -\frac{\partial\psi}{\partial Y}, \tag{11}$$

$$\hat{B}_{p\,Y} = \frac{\partial\psi}{\partial X}. \tag{12}$$

Now,

$$\frac{dF}{dz} = \frac{\partial\psi}{\partial X} + {\rm i}\,\frac{\partial\chi}{\partial X} = \frac{\partial\psi}{\partial X} - {\rm i}\,\frac{\partial\psi}{\partial Y}, \tag{13}$$

which implies that

$$\hat{B}_{p\,X} = {\rm Im}\left(\frac{dF}{dz}\right), \tag{14}$$

$$\hat{B}_{p\,Y} = {\rm Re}\left(\frac{dF}{dz}\right). \tag{15}$$

2 First-Order Magnetic Null

Let

$$F(z) = \ln z + \zeta\,\ln(z+{\rm i}), \tag{16}$$

where ζ is real and positive. The first term corresponds to the plasma current filament located at the origin, whereas the second corresponds to the divertor coil filament located at $X = 0$, $Y = -1$. The current in the plasma filament is I_p, whereas that in the divertor coil filament is $\zeta\,I_p$. It follows that

$$\psi(X, Y) = \frac{1}{2}\,\ln\left(X^2+Y^2\right) + \frac{\zeta}{2}\,\ln\left[X^2+(Y+1)^2\right], \tag{17}$$

$$\psi_X(X, Y) = \frac{X}{X^2+Y^2} + \frac{\zeta\,X}{X^2+(Y+1)^2}, \tag{18}$$

$$\psi_Y(X, Y) = \frac{Y}{X^2+Y^2} + \frac{\zeta\,(Y+1)}{X^2+(Y+1)^2}. \tag{19}$$

Here, $\psi_X \equiv \partial\psi/\partial X$, et cetera.

Now,

$$\frac{dF}{dz} = \frac{1}{z} + \frac{\zeta}{z+{\rm i}}. \tag{20}$$

If $z = z_0$ corresponds to a null in the poloidal magnetic field, at which $|\hat{\mathbf{B}}_p| = 0$, then we require $(dF/dz)_{z_0} = 0$. We obtain

$$z_0 = -\frac{\rm i}{1+\zeta}. \tag{21}$$

In other words, the magnetic null is located at $X = 0$, $Y = Y_0$, where $Y_0 = -1/(1+\zeta)$. The magnetic separatrix corresponds to the curve $\psi(X,Y) = \psi_x$, where

$$\psi_x \equiv \psi(0, Y_0) = \ln\left[\frac{\zeta^{\,\zeta}}{(1+\zeta)^{1+\zeta}}\right]. \tag{22}$$

Now,

$$\frac{d^2F}{dz^2} = -\frac{1}{z^2} - \frac{\zeta}{(z+\mathrm{i})^2}. \tag{23}$$

So, at the magnetic null,

$$\left(\frac{d^2F}{dz^2}\right)_{z_0} = \frac{(1+\zeta)^3}{\zeta}. \tag{24}$$

Thus, in the vicinity of the magnetic null

$$F(z) \simeq F(z_0) + \frac{1}{2}\,\frac{(1+\zeta)^3}{\zeta}\,(z - z_0)^2. \tag{25}$$

Let

$$z - z_0 = u + \mathrm{i}\,v. \tag{26}$$

It follows that

$$\psi(u,v) - \psi_x = \mathrm{Re}[F(z) - F(z_0)] = \frac{1}{2}\,\frac{(1+\zeta)^3}{\zeta}\,(u^2 - v^2). \tag{27}$$

Now, the magnetic separatrix corresponds to $\psi = \psi_x$. Thus, the separatrix curves are the two straight-lines $u = v$ and $u = -v$ which cross at right-angles at the null point.

Consider the flux-surface

$$v^2 - u^2 = \epsilon^2, \tag{28}$$

whose distance of closest approach to the null point is ϵ. Here, $0 < \epsilon \ll 1$. Note that

$$\frac{dv}{du} = \frac{u}{v}. \tag{29}$$

This flux-surface corresponds to the contour

$$\psi(u,v) - \psi_x = -\frac{1}{2}\,\frac{(1+\zeta)^3}{\zeta}\,\epsilon^2. \tag{30}$$

On the flux-surface,

$$|\hat{\nabla}\psi| = \left[\left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2\right]^{1/2} = \frac{1}{2}\,\frac{(1+\zeta)^3}{\zeta}\,\left[(2\,u)^2 + (2\,v)^2\right]^{1/2} = \frac{(1+\zeta)^3}{\zeta}\,(u^2+v^2)^{1/2}, \tag{31}$$

and

$$dL = \left[1 + \left(\frac{dv}{du}\right)^2\right]^{1/2} du = \left(1 + \frac{u^2}{v^2}\right)^{1/2} du = \frac{(u^2+v^2)^{1/2}\,du}{v} = \frac{(u^2+v^2)^{1/2}\,du}{(u^2+\epsilon^2)^{1/2}}. \tag{32}$$

Thus,

$$\frac{q(\epsilon)}{q_\ast} = \frac{1}{\pi}\,\frac{\zeta}{(1+\zeta)^3}\int_0^L \frac{du}{(u^2+\epsilon^2)^{1/2}} = \frac{1}{\pi}\,\frac{\zeta}{(1+\zeta)^3}\left[\ln\left\{(u^2+\epsilon^2)^{1/2} + u\right\}\right]_0^L \simeq \frac{1}{\pi}\,\frac{\zeta}{(1+\zeta)^3}\,\ln\left(\frac{2\,L}{\epsilon}\right), \tag{33}$$

where L is arbitrary.

3 Second-Order Magnetic Null

Let

$$F(z) = \ln z + \frac{\zeta}{2}\,\ln(z+\mathrm{i}+\Delta) + \frac{\zeta}{2}\,\ln(z+\mathrm{i}-\Delta), \tag{34}$$

where ζ and Δ are real and positive. The first term corresponds to the plasma current filament located at the origin, whereas the second and third terms corresponds to two divertor coil filaments, carrying equal currents, located at $X=\pm\Delta$, $Y=-1$. The plasma current filament carries the current I_p, whereas the two divertor coils carry the currents $\zeta\,I_p/2$. It follows that

$$\psi(X,Y) = \frac{1}{2}\,\ln\left(X^2+Y^2\right) + \frac{\zeta}{4}\,\ln\left[(X+\Delta)^2+(Y+1)^2\right] + \frac{\zeta}{4}\,\ln\left[(X-\Delta)^2+(Y+1)^2\right], \tag{35}$$

$$\psi_X(X,Y) = \frac{X}{X^2+Y^2} + \frac{\zeta\,(X+\Delta)}{2\,[(X+\Delta)^2+(Y+1)^2]} + \frac{\zeta\,(X-\Delta)}{2\,[(X-\Delta)^2+(Y+1)^2]}, \tag{36}$$

$$\psi_Y(X,Y) = \frac{Y}{X^2+Y^2} + \frac{\zeta\,(Y+1)}{2\,[(X+\Delta)^2+(Y+1)^2]} + \frac{\zeta\,(Y+1)}{2\,[(X-\Delta)^2+(Y+1)^2]}. \tag{37}$$

Now,

$$\frac{dF}{dz} = \frac{1}{z} + \frac{\zeta}{2\,(z+\mathrm{i}+\Delta)} + \frac{\zeta}{2\,(z+\mathrm{i}-\Delta)}. \tag{38}$$

If $z=z_0$ corresponds to a null in the poloidal magnetic field, at which $|\hat{\mathbf{B}}|=0$, then we require $(dF/dz)_{z_0}=0$. It follows that

$$(1+\zeta)\,z_0^{\,2} + \mathrm{i}\,(2+\zeta)\,z_0 - 1 - \Delta^2 = 0, \tag{39}$$

which gives

$$z_0 = \frac{-\mathrm{i}\,(2+\zeta)\pm\mathrm{i}\sqrt{\zeta^2-4\,(1+\zeta)\,\Delta^2}}{2\,(1+\zeta)}. \tag{40}$$

Thus, in general, there are two magnetic null points.

Now,

$$\frac{d^2F}{dz^2} = -\frac{1}{z^2} - \frac{\zeta}{2\,(z+\mathrm{i}+\Delta)^2} - \frac{\zeta}{2\,(z+\mathrm{i}-\Delta)^2}. \tag{41}$$

If $z=z_0$ corresponds to a second-order null point then we require $(d^2F/dz^2)_{z_0}=0$. It follows that

$$\Delta = \frac{\zeta}{2\sqrt{1+\zeta}}, \tag{42}$$

which is, of course, the condition for the two solutions in Eq. (40) to merge. The second-order null is located at $X=0$, $Y=Y_0$, where

$$Y_0 = -\frac{(2+\zeta)}{2\,(1+\zeta)}. \tag{43}$$

The magnetic separatrix corresponds to the curve $\psi(X,Y)=\psi_x$, where

$$\psi_x \equiv \psi(0,Y_0) = \ln\left[\frac{\zeta^{\,\zeta}\,(2+\zeta)^{1+\zeta/2}}{2^{1+\zeta}\,(1+\zeta)^{1+\zeta}}\right]. \tag{44}$$

Now,

$$\frac{d^3F}{dz^3} = \frac{2}{z^3} + \frac{\zeta}{(z+\mathrm{i}+\Delta)^3} + \frac{\zeta}{(z+\mathrm{i}-\Delta)^3}. \tag{45}$$

So, at the magnetic null,

$$\left(\frac{d^3F}{dz^3}\right)_{z_0} = -\frac{16\,\mathrm{i}\,(1+\zeta)^4}{(2+\zeta)^2\,\zeta^2}. \tag{46}$$

Thus, in the vicinity of the magnetic null

$$F(z) \simeq F(z_0) - \frac{1}{6}\,\frac{16\,\mathrm{i}\,(1+\zeta)^4}{(2+\zeta)^2\,\zeta^2}\,(z-z_0)^3. \tag{47}$$

Let

$$z - z_0 = u + \mathrm{i}\,v. \tag{48}$$

It follows that

$$\psi(u,v) - \psi_x = \mathrm{Re}(F-F_0) = \frac{8}{3}\,\frac{(1+\zeta)^4}{(2+\zeta)^2\,\zeta^2}\,(3\,u^2 - v^2)\,v \tag{49}$$

Now, the magnetic separatrix curves corresponds to $\psi = \psi_x$. Thus, the separatrix curves are the three straight-lines $u = \pm v/\sqrt{3}$ and $v = 0$ which subtend 60° with respect to one another.

Consider the flux-surface

$$(v^2 - 3\,u^2)\,v = \epsilon^3 \tag{50}$$

whose distance of closest approach to the null point is ϵ. Here, $0 < \epsilon \ll 1$. Note that

$$\frac{du}{dv} = \frac{v^2 - u^2}{2\,u\,v}. \tag{51}$$

This flux-surface corresponds to the contour

$$\psi(u,v) - \psi_x = -\frac{8}{3}\,\frac{(1+\zeta)^4}{(2+\zeta)^2\,\zeta^2}\,\epsilon^3. \tag{52}$$

On the flux-surface,

$$\begin{aligned}|\hat{\nabla}\psi| &= \left[\left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2\right]^{1/2} = \frac{8}{3}\,\frac{(1+\zeta)^4}{(2+\zeta)^2\,\zeta^2}\left[(6\,u\,v)^2 + (3\,u^2 - 3\,v^2)^2\right]^{1/2}\\ &= \frac{8\,(1+\zeta)^4}{(2+\zeta)^2\,\zeta^2}\,(u^2+v^2),\end{aligned} \tag{53}$$

and

$$dL = \left[\left(\frac{du}{dv}\right)^2 + 1\right]^{1/2} dv = \left[\left(\frac{v^2-u^2}{2\,u\,v}\right)^2 + 1\right]^{1/2} dv = \frac{(u^2+v^2)\,dv}{2\,u\,v}. \tag{54}$$

Thus,

$$\begin{aligned}\frac{q(\epsilon)}{q_\ast} &= \frac{1}{8\,\pi}\,\frac{(2+\zeta)^2\,\zeta^2}{(1+\zeta)^4}\int_\epsilon^\infty \frac{dv}{2\,u\,v} = \frac{\sqrt{3}}{16\,\pi}\,\frac{(2+\zeta)^2\,\zeta^2}{(1+\zeta)^4}\int_\epsilon^\infty \frac{dv}{v^{1/2}\,(v^3-\epsilon^3)^{1/2}}\\ &= \frac{\sqrt{3}}{16\,\pi}\,\frac{(2+\zeta)^2\,\zeta^2}{(1+\zeta)^4}\,\frac{1}{\epsilon}\int_1^\infty \frac{dx}{x^{1/2}\,(x^3-1)^{1/2}} = \frac{\sqrt{3}}{16\,\pi^{1/2}}\,\frac{\Gamma(4/3)}{\Gamma(5/6)}\,\frac{(2+\zeta)^2\,\zeta^2}{(1+\zeta)^4}\,\frac{1}{\epsilon}.\end{aligned} \tag{55}$$