Neoclassical Toroidal Viscosity

R. Fitzpatrick [a]

Institute for Fusion Studies, Department of Physics,

University of Texas at Austin, Austin TX 78712, USA

I. CURVILINEAR COORDINATES

A. Basis Vectors

Let $\mathbf{x} = (x_1,\, x_2,\, x_3)$ be a position vector, where the x_i are Cartesian coordinates. (In the following, all latin indices, i, j, k, l are assumed to run from 1 to 3. Moreover, use is made of the Einstein summation convention.) Let the $q_i(x_1, x_2, x_3)$ be curvilinear coordinates. We can define the contravariant basis vectors,

$$\mathbf{e}^i = \frac{\partial q^i}{\partial \mathbf{x}}, \tag{1}$$

and the covariant basis vectors,

$$\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial q^i}. \tag{2}$$

Note that

$$\mathbf{e}^i\cdot\mathbf{e}_j = \frac{\partial q^i}{\partial \mathbf{x}}\,\frac{\partial \mathbf{x}}{\partial q^j} = \frac{\partial q^i}{\partial x_k}\,\frac{\partial x_k}{\partial q^j} = \frac{\partial q^i}{\partial q^j} = \delta^i_j, \tag{3}$$

where δ^i_j is the Kronecker delta symbol, and use has been made of the chain rule.

The Jacobian, $\mathcal{J}$, of the curvilinear coordinate system is defined as

$$\mathcal{J} = \frac{\partial \mathbf{x}}{\partial q^1}\cdot\frac{\partial \mathbf{x}}{\partial q^2}\times\frac{\partial \mathbf{x}}{\partial q^3} = \mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3. \tag{4}$$

Note that

$$\mathcal{J}^{\,-1} = \frac{\partial q^1}{\partial \mathbf{x}}\cdot\frac{\partial q^2}{\partial \mathbf{x}}\times\frac{\partial q^3}{\partial \mathbf{x}} = \mathbf{e}^1\cdot\mathbf{e}^2\times\mathbf{e}^3. \tag{5}$$

It is easily seen that

$$\epsilon_{ijk} \equiv \mathbf{e}_i\cdot\mathbf{e}_j\times\mathbf{e}_k = \mathcal{J}\,\varepsilon_{ijk}, \tag{6}$$

[a] rfitzp@utexas.edu

$$\epsilon^{ijk} \equiv \mathbf{e}^i \cdot \mathbf{e}^j \times \mathbf{e}^k = \mathcal{J}^{-1}\, \varepsilon_{ijk}, \tag{7}$$

where ε_{ijk} is the Levi-Civita symbol, and the ϵ_{ijk} and the ϵ^{ijk} are the covariant and contravaiant components of the Levi-Civita tensor, respectively.

The covariant basis vectors can be expressed in terms of the contravariant basis vectors, as follows

$$\mathbf{e}_i = \mathcal{J}\, \frac{\partial q^j}{\partial \mathbf{x}} \times \frac{\partial q^k}{\partial \mathbf{x}} = \mathcal{J}\, \mathbf{e}^j \times \mathbf{e}^k, \tag{8}$$

where i, j, k are cyclic. To demonstrate the validity of this expression we need to show that $\mathbf{e}^l \cdot \mathbf{e}_i = \delta^l_i$, which implies that

$$\mathcal{J}\, \mathbf{e}^l \cdot \mathbf{e}^j \times \mathbf{e}^k = \varepsilon_{ljk} = \delta^l_i, \tag{9}$$

which is obviously satisfied. The contravariant basis vectors can also be expressed in terms of the covariant basis vectors as follows,

$$\mathbf{e}^i = \mathcal{J}^{-1}\, \mathbf{e}_j \times \mathbf{e}_k, \tag{10}$$

where i, j, k are cyclic.

B. Vectors

The contravariant components, a^i, of a vector $\mathbf{a}$ are defined via

$$\mathbf{a} = a^i\, \mathbf{e}_i. \tag{11}$$

The covariant components, a_i, are defined via

$$\mathbf{a} = a_i\, \mathbf{e}^i. \tag{12}$$

Thus,

$$\mathbf{a} \cdot \mathbf{b} = a^i\, b_j\, \mathbf{e}_i \cdot \mathbf{e}^j = a^i\, b_j\, \delta^j_i = a^i\, b_i, \tag{13}$$

where use has been made of Eq. (3). Similarly,

$$\mathbf{a} \cdot \mathbf{b} = a_i\, b^j\, \mathbf{e}^i \cdot \mathbf{e}_j = a_i\, b^j\, \delta^i_j = a_i\, b^i, \tag{14}$$

Note that

$$a^i = \mathbf{a} \cdot \mathbf{e}^i, \tag{15}$$

$$a_i = \mathbf{a}\cdot\mathbf{e}_i. \tag{16}$$

Now,

$$(\mathbf{a}\times\mathbf{b})^i = \mathbf{a}\times\mathbf{b}\cdot\mathbf{e}^i = a_j\,b_k\,\mathbf{e}^j\times\mathbf{e}^k\cdot\mathbf{e}^i = \mathcal{J}^{-1}\,\varepsilon^{ijk}\,a_j\,b_k, \tag{17}$$

where use has been made of Eq. (7). Likewise,

$$(\mathbf{a}\times\mathbf{b})_i = \mathbf{a}\times\mathbf{b}\cdot\mathbf{e}_i = a^j\,b^k\,\mathbf{e}_j\times\mathbf{e}_k\cdot\mathbf{e}_i = \mathcal{J}\,\varepsilon_{ijk}\,a^j\,b^k, \tag{18}$$

where use has been made of Eq. (6).

C. Metric Tensor

The contravariant components of the metric tensor, $\overleftrightarrow{\mathbf{g}}$, are defined

$$g^{ij} = \mathbf{e}^i\cdot\mathbf{e}^j. \tag{19}$$

Note that $g^{ji} = g^{ij}$. Likewise, the covariant components of the metric tensor are

$$g_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j. \tag{20}$$

Note that $g_{ji} = g_{ij}$. Furthermore,

$$dx^2 = \frac{\partial\mathbf{x}}{\partial q^i}\cdot\frac{\partial\mathbf{x}}{\partial q^j}\,dq^i\,dq^j = \mathbf{e}_i\cdot\mathbf{e}_j\,dq^i\,dq^j = g_{ij}\,dq^i\,dq^j, \tag{21}$$

where use has been made of Eq. (2).

Now,

$$\mathbf{e}^i = (\mathbf{e}^i\cdot\mathbf{e}^j)\,\mathbf{e}_j, \tag{22}$$

so

$$\mathbf{a}\cdot\mathbf{e}^i = \mathbf{a}\cdot[(\mathbf{e}^i\cdot\mathbf{e}^j)\,\mathbf{e}_j] = (\mathbf{e}^i\cdot\mathbf{e}^j)\,(\mathbf{a}\cdot\mathbf{e}_j), \tag{23}$$

which yields

$$a^i = g^{ij}\,a_j. \tag{24}$$

Likewise,

$$\mathbf{e}_i = (\mathbf{e}_i\cdot\mathbf{e}_j)\,\mathbf{e}^j, \tag{25}$$

so

$$\mathbf{a}\cdot\mathbf{e}_i = \mathbf{a}\cdot[(\mathbf{e}_i\cdot\mathbf{e}_j)\,\mathbf{e}^j] = (\mathbf{e}_i\cdot\mathbf{e}_j)\,(\mathbf{a}\cdot\mathbf{e}^j), \tag{26}$$

which yields

$$a_i = g_{ij}\, a^{\,j}. \tag{27}$$

Combining Eqs. (24) and (27), we get

$$a^i = g^{ij}\, g_{jk}\, a^k = \delta^i_k\, a^k, \tag{28}$$

which implies that

$$g^{ik}\, g_{kj} = \delta^i_j. \tag{29}$$

Likewise,

$$a_i = g_{ij}\, g^{\,jk}\, a_k = \delta^k_i\, a_k, \tag{30}$$

which implies that

$$g_{ik}\, g^{\,kj} = \delta^j_i. \tag{31}$$

D. Chistoffel Symbol

The Christoffel symbols, Γ_{kij} and $\Gamma^k_{\,ij}$, are defined such that

$$\frac{\partial \mathbf{e}_i}{\partial q^{\,j}} = \Gamma_{kij}\, \mathbf{e}^k = \Gamma^k_{\,ij}\, \mathbf{e}_k. \tag{32}$$

It follows that

$$\Gamma_{kij} = \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_i}{\partial q^{\,j}}, \tag{33}$$

$$\Gamma^k_{\,ij} = \mathbf{e}^k \cdot \frac{\partial \mathbf{e}_i}{\partial q^{\,j}}. \tag{34}$$

However,

$$\frac{\partial(\mathbf{e}^k \cdot \mathbf{e}_i)}{\partial q^j} = \mathbf{e}^k \cdot \frac{\partial \mathbf{e}_i}{\partial q_k} + \mathbf{e}_i \cdot \frac{\partial \mathbf{e}^k}{\partial q^j} = 0, \tag{35}$$

which implies that

$$\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^k}{\partial q^j} = -\Gamma^k_{\,ij}. \tag{36}$$

Now,

$$\frac{\partial g_{ij}}{\partial q^k} = \frac{\partial(\mathbf{e}_i \cdot \mathbf{e}_j)}{\partial q^k} = \frac{\partial \mathbf{e}_i}{\partial q^k} \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial q^k} = \Gamma_{jik} + \Gamma_{ijk}, \tag{37}$$

$$\frac{\partial g_{ik}}{\partial q^{\,j}} = \frac{\partial(\mathbf{e}_i \cdot \mathbf{e}_k)}{\partial q^{\,j}} = \frac{\partial \mathbf{e}_i}{\partial q^{\,j}} \cdot \mathbf{e}_k + \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_k}{\partial q^{\,j}} = \Gamma_{kij} + \Gamma_{ikj}, \tag{38}$$

$$\frac{\partial g_{jk}}{\partial q^i} = \frac{\partial(\mathbf{e}_j\cdot\mathbf{e}_k)}{\partial q^i} = \frac{\partial \mathbf{e}_j}{\partial q^i}\cdot\mathbf{e}_k + \mathbf{e}_j\cdot\frac{\partial \mathbf{e}_k}{\partial q^i} = \Gamma_{kji} + \Gamma_{jki}. \tag{39}$$

However,

$$\frac{\partial \mathbf{e}_i}{\partial q^j} = \frac{\partial^2 \mathbf{x}}{\partial q^j\,\partial q^i} = \frac{\partial^2 \mathbf{x}}{\partial q^i\,\partial q^j} = \frac{\partial \mathbf{e}_j}{\partial q^i}, \tag{40}$$

where use has been made of Eq. (2). Hence, we deduce that

$$\Gamma_{kji} = \Gamma_{kij}. \tag{41}$$

It follows from Eqs. (37)–(39) and (41) that

$$\begin{aligned}\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^j} &= \Gamma_{jik} + \Gamma_{ijk} + \Gamma_{k,ji} + \Gamma_{jki} - \Gamma_{kij} - \Gamma_{ikj}\\ &= \Gamma_{jik} + \cancel{\Gamma_{ijk}} + \cancel{\Gamma_{kji}} + \Gamma_{jik} - \cancel{\Gamma_{kji}} - \cancel{\Gamma_{i,jk}},\end{aligned} \tag{42}$$

which implies that

$$\Gamma_{jik} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^j}\right), \tag{43}$$

or

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right). \tag{44}$$

The previous two equations yield

$$\frac{\partial g_{ij}}{\partial q^k} = \Gamma_{ijk} + \Gamma_{jik}. \tag{45}$$

E. Covariant Derivative

Let $\mathbf{A}(\mathbf{x})$ be a vector field. We can write

$$\frac{\partial \mathbf{A}}{\partial q^j} = \frac{\partial\left(A^k\,\mathbf{e}_k\right)}{\partial q^j} = \frac{\partial A^k}{\partial q^j}\,\mathbf{e}_k + A^k\,\frac{\partial \mathbf{e}_k}{\partial q^j} = \frac{\partial A^k}{\partial q^j}\,\mathbf{e}_k + A^k\,\Gamma^l_{\ kj}\,\mathbf{e}_l, \tag{46}$$

where use has been made of Eq. (32). Let

$$\partial_j A^i \equiv \mathbf{e}^i\cdot\frac{\partial \mathbf{A}}{\partial q^j}, \tag{47}$$

where ∂_j is the covariant derivative. It follows that

$$\partial_j A^i = \frac{\partial A^i}{\partial q^j} + \Gamma^i_{\ jk}\,A^k. \tag{48}$$

We can also write

$$\frac{\partial \mathbf{A}}{\partial q^{\,j}} = \frac{\partial\,(A_k\,\mathbf{e}^k)}{\partial q^{\,j}} = \frac{\partial A_k}{\partial q^{\,j}}\,\mathbf{e}^k + A_k\,\frac{\partial \mathbf{e}^k}{\partial q^{\,j}} = \frac{\partial A_k}{\partial q^{\,j}}\,\mathbf{e}^k - A_k\,\Gamma^k_{jl}\,\mathbf{e}_l, \tag{49}$$

where use has been made of Eq. (36). Let

$$\partial_j A_i \equiv \mathbf{e}_i\cdot\frac{\partial \mathbf{A}}{\partial q^{\,j}}. \tag{50}$$

It follows that

$$\partial_j A_i = \frac{\partial A_i}{\partial q_j} - \Gamma^k_{ij}\,A_k. \tag{51}$$

F. Jacobian Matrix

Let us define the Jacobian matrix

$$J_{ij} = \frac{\partial x_j}{\partial q^i}. \tag{52}$$

Of course, the Jacobian is the determinant of the Jacobian matrix

$$\mathcal{J} = ||J_{ij}|| = \frac{\partial \mathbf{x}}{\partial q^1}\cdot\frac{\partial \mathbf{x}}{\partial q^2}\times\frac{\partial \mathbf{x}}{\partial q^3}. \tag{53}$$

See Eq. (4). Now,

$$g_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j = \frac{\partial \mathbf{x}}{\partial q^i}\cdot\frac{\partial \mathbf{x}}{\partial q^{\,j}} = \frac{\partial x_k}{\partial q^i}\,\frac{\partial x_k}{\partial q^{\,j}} = J_{ik}\,J_{jk} = J_{ij}\,J^T_{kj}, \tag{54}$$

where use has been made of Eq. (2), and $J^T_{ij} = J_{ji}$. However, $||A\,B|| = ||A||\,||B||$ and $||A^T|| = ||A||$. Hence, we deduce that

$$||g_{ij}|| = \mathcal{J}^{\,2}. \tag{55}$$

Jacobi's formula states that

$$\frac{\partial ||A||}{\partial q^k} = ||A||\,\mathrm{Tr}\left(A^{-1}\,\frac{\partial A}{\partial q^k}\right) \tag{56}$$

for any square differentiable matrix, $A_{ij}(q_1, q_2, q_3)$. Let $A_{ij} = g_{ij}$. Now, the inverse of g_{ij} is g^{ij}. Hence, making use of Eq. (55), we deduce that

$$\frac{\partial \mathcal{J}^{\,2}}{\partial q^k} = \mathcal{J}^{\,2}\,g^{ij}\,\frac{\partial g_{ji}}{\partial q^k}, \tag{57}$$

which reduces to

$$\frac{\partial g_{ij}}{\partial q^k}\,g^{ij} = \frac{2}{\mathcal{J}}\,\frac{\partial \mathcal{J}}{\partial q^k}. \tag{58}$$

G. Scalar Fields

Let $\phi(\mathbf{x})$ be a scalar field. It follows that

$$\nabla\phi \equiv \frac{\partial\phi}{\partial\mathbf{x}} = \frac{\partial\phi}{\partial q^i}\,\frac{\partial q^i}{\partial\mathbf{x}} = \frac{\partial\phi}{\partial q^i}\,\mathbf{e}^i, \tag{59}$$

where use has been made of Eq. (1). Thus,

$$(\nabla\phi)_i = \frac{\partial\phi}{\partial q^i}. \tag{60}$$

H. Vector Fields

Let $\mathbf{A}(\mathbf{x})$ be a vector field. Now,

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{\partial\mathbf{A}}{\partial\mathbf{x}} = \frac{\partial q^i}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial q^i}\left(A^j\,\mathbf{e}_j\right) = \mathbf{e}^i\cdot\left(\frac{\partial A^j}{\partial q^i}\,\mathbf{e}_j + A^j\,\frac{\partial\mathbf{e}_j}{\partial q^i}\right)\\ &= \mathbf{e}^i\cdot\left(\frac{\partial A^j}{\partial q^i}\,\mathbf{e}_j + A^j\,\Gamma^k_{\ ji}\,\mathbf{e}_k\right) = \frac{\partial A^i}{\partial q^i} + A^j\,\Gamma^i_{\ ij},\end{aligned} \tag{61}$$

where use has been made of Eqs. (1), (3), (32), and (41). Now,

$$\Gamma^i_{\ ij} = g^{ik}\,\Gamma_{kij} = \frac{1}{2}\,g^{ik}\left(\frac{\partial g_{ki}}{\partial q^j} + \cancel{\frac{\partial g_{kj}}{\partial q^i}} - \cancel{\frac{\partial g_{ij}}{\partial q_k}}\right) = \frac{1}{2}\,g^{ik}\,\frac{\partial g_{ik}}{\partial q^j} = \frac{1}{\mathcal{J}}\,\frac{\partial\mathcal{J}}{\partial q^j}, \tag{62}$$

where use has been made of Eqs. (44) and (58). Thus,

$$\nabla\cdot\mathbf{A} = \frac{\partial A^i}{\partial q^i} + \frac{A^i}{\mathcal{J}}\,\frac{\partial\mathcal{J}}{\partial q^i} = \frac{1}{\mathcal{J}}\,\frac{\partial(\mathcal{J}\,A^i)}{\partial q^i}. \tag{63}$$

It is clear from Eqs. (48) and (61) that

$$\nabla\cdot\mathbf{A} = \partial_i A^i. \tag{64}$$

Let us assume that

$$(\nabla\times\mathbf{A})^i = \epsilon^{ijk}\,\partial_j A_k = \mathcal{J}^{-1}\,\varepsilon_{ijk}\left(\frac{\partial A_k}{\partial q_j} - \Gamma^l_{\ jk}\,A_l\right), \tag{65}$$

where use has been made of Eqs. (7) and (48). However, given that $\Gamma^l_{\ jk} = \Gamma^l_{\ kj}$, it is clear that $\varepsilon_{ijk}\,\Gamma^l_{\ jk} = 0$. Thus, we obtain

$$(\nabla\times\mathbf{A})^i = \mathcal{J}^{-1}\,\varepsilon_{ijk}\,\frac{\partial A_k}{\partial q_j}. \tag{66}$$

II. EQUILIBRIUM MAGNETIC FIELD

Let ψ, θ, α be magnetic coordinates, where ψ is a flux-surface label, θ a poloidal angle, and α a helical angle. Suppose that the geometric toroidal angle is $\varphi = \alpha + q(\psi)\,\theta$. Let $\mathcal{J}^{-1} = \nabla\psi\cdot\nabla\theta\times\nabla\alpha$ be the Jacobian of the coordinate system. The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\alpha\times\nabla\psi = \mathcal{J}^{-1}\,\mathbf{e}_\theta, \tag{67}$$

where use has been made of Eq. (8). It follows that

$$\mathbf{b} \equiv \frac{\mathbf{B}}{B} = (\mathcal{J}\,B)^{-1}\,\mathbf{e}_\theta. \tag{68}$$

Thus,

$$(\mathbf{b}\,\mathbf{b})^{ij} = (\mathcal{J}\,B)^{-2}\,(\mathbf{e}_\theta\cdot\mathbf{e}^i)\,(\mathbf{e}_\theta\cdot\mathbf{e}^j) = (\mathcal{J}\,B)^{-2}\,\delta^i_\theta\,\delta^j_\theta. \tag{69}$$

Furthermore,

$$\mathbf{b}\cdot\nabla f = \frac{1}{\mathcal{J}\,B}\,\frac{\partial f}{\partial\theta}. \tag{70}$$

III. PLASMA VISCOSITY TENSOR

Let $\overleftrightarrow{\mathbf{1}}$ be the identity tensor. It follows that

$$1^{ij} = \mathbf{e}^i\cdot\mathbf{e}^j = g^{ij}. \tag{71}$$

Thus, if

$$\overleftrightarrow{\mathbf{\Pi}} = (\delta p_\parallel - \delta p_\perp)\,\mathbf{b}\,\mathbf{b} + \delta p_\perp\,\overleftrightarrow{\mathbf{1}} \tag{72}$$

is the perturbed plasma viscosity tensor then

$$\mathit{\Pi}^{\,ij} = (\delta p_\parallel - \delta p_\perp)\,(\mathcal{J}\,B)^{-2}\,\delta^i_\theta\,\delta^j_\theta + \delta p_\perp\,g^{ij}, \tag{73}$$

where use has been made of Eqs. (69) and (71).

Consider

$$\nabla\cdot\left(\overleftrightarrow{\mathbf{\Pi}}\cdot\mathbf{e}_k\right) = \nabla\cdot\overleftrightarrow{\mathbf{\Pi}}\cdot\mathbf{e}_k + \overleftrightarrow{\mathbf{\Pi}} : \frac{\partial\mathbf{e}_k}{\partial\mathbf{x}}. \tag{74}$$

Now,

$$\frac{\partial\mathbf{e}_k}{\partial\mathbf{x}} = \frac{\partial\mathbf{e}_k}{\partial q^j}\,\frac{\partial q^j}{\partial\mathbf{x}} = \mathit{\Gamma}_{ikj}\,\mathbf{e}^i\,\mathbf{e}^j = \mathit{\Gamma}_{ijk}\,\mathbf{e}^i\,\mathbf{e}^j, \tag{75}$$

where use has been made of Eqs. (1), (32), and (41). Thus,

$$\overset{\leftrightarrow}{\mathbf{\Pi}} : \frac{\partial \mathbf{e}_k}{\partial \mathbf{x}} = \mathit{\Pi}^{\,ij}\, \mathit{\Gamma}_{ijk}. \tag{76}$$

Now, $\mathit{\Pi}^{\,ij} = \mathit{\Pi}^{\,ji}$, so

$$\overset{\leftrightarrow}{\mathbf{\Pi}} : \frac{\partial \mathbf{e}_k}{\partial \mathbf{x}} = \frac{1}{2}\, \mathit{\Pi}^{\,ij}\,(\mathit{\Gamma}_{ijk} + \mathit{\Gamma}_{jik}) = \frac{1}{2}\, \mathit{\Pi}^{\,ij}\, \frac{\partial g_{ij}}{\partial q^k}, \tag{77}$$

where use has been made of Eq. (45).

If we integrate Eq. (74) over all space, and neglect surface terms, then we find that

$$\int \nabla\cdot\overset{\leftrightarrow}{\mathbf{\Pi}}\cdot\mathbf{e}_k\, d^3\mathbf{x} = -\frac{1}{2}\int \mathit{\Pi}^{\,ij}\, \frac{\partial g_{ij}}{\partial q^k}\, d^3\mathbf{x}, \tag{78}$$

where use has been made of Eq. (77). It follows from Eq. (73) that

$$\int \nabla\cdot\overset{\leftrightarrow}{\mathbf{\Pi}}\cdot\mathbf{e}_k\, d^3\mathbf{x} = -\frac{1}{2}\int \left[\frac{(\delta p_\parallel - \delta p_\perp)}{(\mathcal{J}\,B)^2}\, \frac{\partial g_{\theta\theta}}{\partial q^k} + \delta p_\perp\, g^{ij}\, \frac{\partial g_{ij}}{\partial q^k}\right] d^3\mathbf{x}. \tag{79}$$

However, Eq. (68) implies that

$$g_{\theta\theta} = \mathbf{e}_\theta\cdot\mathbf{e}_\theta = (\mathcal{J}\,B)^2. \tag{80}$$

Finally, making use of Eq. (58), we get

$$\begin{aligned}\int \nabla\cdot\overset{\leftrightarrow}{\mathbf{\Pi}}\cdot\mathbf{e}_k\, d^3\mathbf{x} &= -\frac{1}{2}\int \left[\frac{(\delta p_\parallel - \delta p_\perp)}{(\mathcal{J}\,B)^2}\, \frac{\partial(\mathcal{J}\,B)^2}{\partial q^k} + \delta p_\perp\, \frac{2}{\mathcal{J}}\, \frac{\partial \mathcal{J}}{\partial q^k}\right] d^3\mathbf{x}\\ &= -\int \left[(\delta p_\parallel - \delta p_\perp)\, \frac{1}{B}\, \frac{\partial B}{\partial q^k} + \delta p_\parallel\, \frac{1}{\mathcal{J}}\, \frac{\partial \mathcal{J}}{\partial q^k}\right] d^3\mathbf{x}.\end{aligned} \tag{81}$$

IV. TOROIDAL VISCOUS TORQUE

Let R, φ, Z be cylindrical coordinates. It follows that $x_1 = R\cos\varphi$, $x_2 = R\sin\varphi$, $x_3 = Z$. Thus,

$$\frac{\partial \mathbf{x}}{\partial \varphi} = -R\,\sin\varphi\,\hat{\mathbf{e}}_x + R\,\cos\varphi\,\hat{\mathbf{e}}_y = R\,\hat{\mathbf{e}}_\varphi, \tag{82}$$

where $\hat{\mathbf{e}}_x = \nabla x/|\nabla x|$, $\hat{\mathbf{e}}_y = \nabla y/|\nabla y|$, and $\hat{\mathbf{e}}_\varphi = \nabla\varphi/|\nabla\varphi|$. It follows that the net toroidal viscous torque acting on the plasma is

$$T_\varphi = \int \frac{\partial \mathbf{x}}{\partial \varphi}\cdot\nabla\cdot\overset{\leftrightarrow}{\mathbf{\Pi}}\, d^3\mathbf{x} = \int \frac{\partial \mathbf{x}}{\partial \alpha}\cdot\nabla\cdot\overset{\leftrightarrow}{\mathbf{\Pi}}\, d^3\mathbf{x} = \int \mathbf{e}_\alpha\cdot\nabla\cdot\overset{\leftrightarrow}{\mathbf{\Pi}}\, d^3\mathbf{x}. \tag{83}$$

Here, use has been made of Eq. (2), as well as

$$\left.\frac{\partial}{\partial \varphi}\right|_{\psi,\theta} = \left.\frac{\partial}{\partial \alpha}\right|_{\psi,\theta}. \tag{84}$$

Thus, Eqs. (81) and (83) imply that

$$T_\varphi = -\int \left[(\delta p_\parallel - \delta p_\perp)\,\frac{1}{B}\,\frac{\partial B}{\partial \alpha} + \delta p_\parallel\,\frac{1}{\mathcal{J}}\,\frac{\partial \mathcal{J}}{\partial \alpha}\right] d^3\mathbf{x}. \tag{85}$$

The unperturbed equilibrium is such that $\partial B/\partial\alpha = \partial\mathcal{J}/\partial\alpha = 0$. In the perturbed equilibrium, $B \rightarrow B + \delta B$ and $dV = \mathcal{J}\,d\psi\,d\theta\,d\alpha \rightarrow \mathcal{J}\,(1 + \nabla\cdot\boldsymbol{\xi})\,d\psi\,d\theta\,d\alpha$, where $\boldsymbol{\xi}(\mathbf{x})$ is the plasma displacement. Hence, to lowest order in perturbed quantities,

$$T_\varphi = -\int \left[(\delta p_\parallel - \delta p_\perp)\,\frac{\partial}{\partial\alpha}\left(\frac{\delta B}{B}\right) + \delta p_\parallel\,\frac{\partial(\nabla\cdot\boldsymbol{\xi})}{\partial\alpha}\right] d^3\mathbf{x}. \tag{86}$$

Let $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ be right-handed orthogonal unit vectors such that $\hat{\mathbf{e}}_3 = \mathbf{b}$ at a given point in space. The ion velocity is written

$$\mathbf{v} = v_1\,\hat{\mathbf{e}}_1 + v_2\,\hat{\mathbf{e}}_2 + v_3\,\hat{\mathbf{e}}_3 = v_\perp\,\cos\zeta\,\hat{\mathbf{e}}_1 + v_\perp\,\sin\zeta\,\hat{\mathbf{e}}_2 + v_\parallel\,\hat{\mathbf{e}}_3, \tag{87}$$

where ζ is the gyro-angle. Now,

$$\delta p_\parallel(\mathbf{x}) = \int m\,v_\parallel^2\,f_1\,d^3\mathbf{v}, \tag{88}$$

$$\delta p_\perp(\mathbf{x}) = \frac{1}{2}\int m\,v_\perp^2\,f_1\,d^3\mathbf{v}, \tag{89}$$

where $f_1(\mathbf{x},\mathbf{v})$ is the perturbed, gyro-averaged, ion distribution function, and m the ion mass. Let $E = m\,(v_\perp^2 + v_\parallel^2)/2$ be the ion kinetic energy, and $\mu = m\,v_\perp^2/(2\,B)$ the ion magnetic moment. It is easily demonstrated that

$$\frac{\partial(E,\mu,\zeta)}{\partial(v_1,v_2,v_3)} = \frac{m^2\,v_\parallel}{B}. \tag{90}$$

Equations (86), (88), and (89) yield

$$T_\varphi = -\iint \left[(2\,E - 3\,\mu\,B)\,\frac{\partial}{\partial\alpha}\left(\frac{\delta B}{B}\right) + (2\,E - \,2\,\mu\,B)\,\frac{\partial(\nabla\cdot\boldsymbol{\xi})}{\partial\alpha}\right] f_1\,d^3\mathbf{x}\,d^3\mathbf{v}. \tag{91}$$

Finally, making use of Eq. (90), we get

$$T_\varphi = -\frac{2\pi}{m^2}\int \frac{\mathcal{J}\,B}{v_\parallel}\left[(2\,E - 3\,\mu\,B)\,\frac{\partial}{\partial\alpha}\left(\frac{\delta B}{B}\right) + (2\,E - \,2\,\mu\,B)\,\frac{\partial(\nabla\cdot\boldsymbol{\xi})}{\partial\alpha}\right] f_1\,dE\,d\mu\,d\psi\,d\theta\,d\alpha. \tag{92}$$

V. ION DRIFT-KINETIC EQUATION

The ion drift-kinetic equation is written

$$v_\parallel\,\mathbf{b}\cdot\nabla f_1 + v_d^{\,\alpha}\,\frac{\partial f_1}{\partial\alpha} - C_i(f_1) = -v_d^{\,\psi}\,\frac{\partial f_0}{\partial\psi}, \tag{93}$$

where $\mathbf{v}_d$ is the ion drift-velocity, C_i the ion collision operator, and $f_0(\psi, E, \mu)$ the unperturbed ion distribution function. Let us employ a Krook collision operator, such that $C_i(f_1) = -\nu_i\, f_1$, where ν_i is the ion collision frequency. It follows that

$$\frac{v_\parallel}{\mathcal{J}\, B}\,\frac{\partial f_1}{\partial\theta} + v_d^{\,\alpha}\,\frac{\partial f_1}{\partial\alpha} + \nu_i\, f_1 = -v_d^{\,\psi}\,\frac{\partial f_0}{\partial\psi}, \tag{94}$$

where use has been made of Eq. (70). Note that we are neglecting $v_d^{\,\theta}$ with respect to $v_\parallel$. This is reasonable because the grad-B and curvature drift velocities are order ρ/L smaller than the thermal velocity, where ρ is the ion gyro-radius, and L a typical variation length-scale. We also expect the $\mathbf{E}\times\mathbf{B}$ drift velocity to be much smaller than the thermal velocity (otherwise ions would be drifting supersonically).

Now, motion at constant ψ and α corresponds to motion along magnetic field-lines. An element of length, ds, along a field-line is such that

$$(ds)^2 = g_{\theta\theta}\,(d\theta)^2 = (\mathcal{J}\,B)^2\,(d\theta)^2, \tag{95}$$

where use has been made of Eq. (80). Thus, $ds = \mathcal{J}\,B\,d\theta$. The time interval required for an ion to move along a magnetic field-line between points θ and $\theta + d\theta$ is

$$dt = \frac{ds}{v_\parallel} = \frac{\mathcal{J}\,B}{v_\parallel}\,d\theta. \tag{96}$$

During this time interval, the ion drifts perpendicular to the field-line (within the magnetic flux-surface) such that its α coordinate increases by

$$d\alpha = v_d^{\,\alpha}\,dt = \frac{\mathcal{J}\,B\,v_d^{\,\alpha}}{v_\parallel}\,d\theta. \tag{97}$$

It follows that

$$\alpha(\theta) = \alpha(0) + \int_0^\theta \frac{\mathcal{J}\,B\,v_d^{\,\alpha}}{v_\parallel}\,d\theta'. \tag{98}$$

In one poloidal transit, α increases by

$$\Delta\alpha = \int_{-\theta_t}^{\theta_t}\frac{\mathcal{J}\,B\,v_d^{\,\alpha}}{v_\parallel}\,d\theta, \tag{99}$$

where θ_t is the angular position of a turning point for a trapped ion, and $\theta_t = \pi$ for a passing ion The corresponding mean increase in the toroidal angle is

$$\Delta\varphi = \Delta\alpha \tag{100}$$

for a trapped ion,

$$\mathit{\Delta}\varphi = \mathit{\Delta}\alpha + 2\pi\, q \tag{101}$$

for a passing ion.

Suppose that perturbed quantities vary with α as $\exp(-{\rm i}\, n\, \alpha)$, where the positive integer n is the toroidal mode number of the magnetic perturbation. It follows that perturbed quantities are periodic in the toroidal angle with periodicity interval $2\pi/n$. If

$$\mathit{\Delta}\varphi = \frac{2\pi\, \ell}{n}, \tag{102}$$

where ℓ is an integer, then trapped ions experience the same perturbed field during each bounce orbit, whereas passing ions experience the same perturbed field during each poloidal transit. We would expect ions that have this property to resonate with the magnetic perturbation. Such resonant ions have

$$(\mathit{\Delta}\alpha)_{res} = \frac{2\pi\, \ell}{n} \tag{103}$$

if they are trapped, and

$$(\mathit{\Delta}\alpha)_{res} = \frac{2\pi\, (\ell - n\, q)}{n} \tag{104}$$

if they are passing. Thus, we can write

$$(\mathit{\Delta}\alpha)_{res} = \frac{2\pi\, (\ell - \sigma\, n\, q)}{n}, \tag{105}$$

where $\sigma = 0$ for trapped ions, and $\sigma = 1$ for passing ions. Let us suppose that

$$f_1(\mathbf{x}, \mathbf{v}) = \delta f_\ell(\psi, E, \mu)\, \mathcal{P}_\ell(\psi, \theta, \alpha, E, \mu), \tag{106}$$

where

$$\begin{aligned}\mathcal{P}_\ell(\psi, \theta, \alpha, E, \mu) &= \exp\left[-{\rm i}\, n\left(\alpha + \frac{(\mathit{\Delta}\alpha)_{res}}{\mathit{\Delta}\alpha}\int_0^\theta \frac{\mathcal{J}\, B\, v_d^\alpha}{v_\parallel}\, d\theta'\right)\right]\\ &= \exp\left[-{\rm i}\, n\, \alpha - {\rm i}\, (\ell - \sigma\, n\, q)\, h\right],\end{aligned} \tag{107}$$

and

$$h(\psi, \theta, E, \mu) = 2\pi \int_0^\theta \frac{\mathcal{J}\, B\, v_d^\alpha}{v_\parallel}\, d\theta' \bigg/ \int_{-\theta_t}^{\theta_t} \frac{\mathcal{J}\, B\, v_d^\alpha}{v_\parallel}\, d\theta. \tag{108}$$

The previous three equations ensure that $\mathit{\Delta}\alpha = (\mathit{\Delta}\alpha)_{res}$. Note that, for passing ions, f_1 is periodic in both θ and φ. For trapped ions, f_1 is periodic at $\theta = \pm\theta_t$ on a given magnetic field-line.

Combining Eqs. (94) and (106)–(108), we get

$$\left[-\mathrm{i}\,(\ell-\sigma\,n\,q)\,2\pi\,v_d^{\alpha}\Big/\!\!\int_{-\theta_t}^{\theta_t}\frac{\mathcal{J}\,B\,v_d^{\alpha}}{v_\parallel}\,d\theta\;-\mathrm{i}\,n\,v_d^{\alpha}+\nu_i\right]\delta f_\ell\,\mathcal{P}_\ell=-v_d^{\psi}\,\frac{\partial f_0}{\partial\psi}. \tag{109}$$

The bounce average operator, $\langle\cdots\rangle_b$, is defined such that

$$\langle A\rangle_b=\int_{-\theta_t}^{\theta_t}\frac{\mathcal{J}\,B\,A}{v_\parallel}\,d\theta\Big/\int_{-\theta_t}^{\theta_t}\frac{\mathcal{J}\,B}{v_\parallel}\,d\theta. \tag{110}$$

Note, from Eq. (96), that the bounce average is merely a time average over the bounce motion. Moreover, $\langle 1\rangle_b=1$. The bounce average of Eq. (109) divided by $\mathcal{P}_\ell$ gives

$$[-\mathrm{i}\,(\ell-\sigma\,n\,q)\,\omega_b-\mathrm{i}\,n\,\langle v_d^{\alpha}\rangle_b+\nu_i]\,\delta f_\ell=-\left\langle v_d^{\psi}\,\mathcal{P}_\ell^{-1}\right\rangle_b\frac{\partial f_0}{\partial\psi}, \tag{111}$$

where

$$\omega_b(\psi,E,\mu)=2\pi\Big/\!\!\int_{-\theta_t}^{\theta_t}\frac{\mathcal{J}\,B}{v_\parallel}\,d\theta \tag{112}$$

is the bounce frequency.

VI. ION DRIFT-VELOCITY

The ion drift-velocity takes the form

$$\mathbf{v}_d=\frac{\mathbf{b}}{m\,\Omega}\times\left[e\,\nabla\Phi+\mu\,\nabla B+m\,v_\parallel^2\,(\mathbf{b}\cdot\nabla)\,\mathbf{b}\right], \tag{113}$$

where e is the magnitude of the electron charge, $\Omega=e\,B/m$ the ion gyro-frequency, and $\Phi(\psi)$ the electrostatic potential. If we assume ion motion takes place at constant magnetic moment, μ, and total energy, $\mathcal{E}=(1/2)\,m\,v_\parallel^2+\mu\,B+e\,\Phi$, then it is easily demonstrated that

$$\mathbf{v}_d=\frac{v_\parallel}{\Omega}\,\nabla\times(v_\parallel\,\mathbf{b})-\frac{v_\parallel^2}{\Omega}\,[\mathbf{b}\cdot(\nabla\times\mathbf{b})]\,\mathbf{b}. \tag{114}$$

Now, from Eq. (66),

$$v_d^{\alpha}=\frac{v_\parallel}{\mathcal{J}\,\Omega}\left[\frac{\partial(v_\parallel\,b_\theta)}{\partial\psi}-\frac{\partial(v_\parallel\,b_\psi)}{\partial\theta}\right], \tag{115}$$

so

$$\langle v_d^{\alpha}\rangle_b=\frac{\omega_b}{2\pi}\,\frac{B}{\Omega}\int_{-\theta_t}^{\theta_t}\frac{\partial(v_\parallel\,\mathcal{J}\,B)}{\partial\psi}\,d\theta, \tag{116}$$

where use has been made of Eq. (110), as well as

$$b^{\theta}=(\mathcal{J}\,B)^{-1}\,\mathbf{e}_\theta\cdot\mathbf{e}_\theta=\mathcal{J}\,B. \tag{117}$$

The latter equation follows from Eqs. (68) and (80). Thus, we obtain

$$\langle v_d^{\alpha}\rangle_b = \omega_E + \omega_D \tag{118}$$

where

$$\omega_E(\psi) = -\frac{d\Phi}{d\psi} \tag{119}$$

is the electric precession frequency, whereas

$$\begin{aligned}\omega_D(\psi, E, \mu) &= \left\langle -\frac{\mu}{e}\frac{\partial B}{\partial \psi} + \frac{(2\,E - 2\,\mu\,B)}{e}\,\frac{\partial \ln(\mathcal{J}\,B)}{\partial \psi}\right\rangle_b \\ &= \left\langle \frac{(2\,E-3\,\mu\,B)}{e}\,\frac{1}{B}\,\frac{\partial B}{\partial\psi} + \frac{(2\,E-2\,\mu\,B)}{e}\,\frac{1}{\mathcal{J}}\,\frac{\partial\mathcal{J}}{\partial\psi}\right\rangle_b\end{aligned} \tag{120}$$

is the magnetic precession frequency.

Now, from Eq. (66),

$$v_d^{\psi} = \frac{v_\parallel}{\mathcal{J}\,\Omega}\left[\frac{\partial(v_\parallel\,b_\alpha)}{\partial\theta} - \frac{\partial(v_\parallel\,b_\theta)}{\partial\alpha}\right], \tag{121}$$

which implies that

$$\left\langle v_d^{\psi}\,\mathcal{P}_\ell^{-1}\right\rangle_b = -\left\langle \mathcal{P}_\ell^{-1}\,\frac{v_\parallel}{\mathcal{J}\,\Omega}\,\frac{\partial\,(v_\parallel\,\mathcal{J}\,B)}{\partial\alpha}\right\rangle_b, \tag{122}$$

where we have ignored the term involving b_α on the assumption that it will average to zero when integrated in α. Thus,

$$\left\langle v_d^{\psi}\,\mathcal{P}_\ell^{-1}\right\rangle_b = -\frac{\omega_b}{2\pi\,e}\,J_\ell^{*} \tag{123}$$

where

$$J_\ell(\psi,\alpha,E,\mu) = \int_{-\theta_t}^{\theta_t}\frac{\mathcal{J}\,B}{v_\parallel}\,\mathcal{P}_\ell\left[(2\,E-3\,\mu\,B)\,\frac{1}{B}\,\frac{\partial B}{\partial\alpha} + (2\,E-2\,\mu\,B)\,\frac{1}{\mathcal{J}}\,\frac{\partial\mathcal{J}}{\partial\alpha}\right]d\theta, \tag{124}$$

which reduces to

$$J_\ell(\psi,\alpha,E,\mu) = \int_{-\theta_t}^{\theta_t}\frac{\mathcal{J}\,B}{v_\parallel}\,\mathcal{P}_\ell\left[(2\,E-3\,\mu\,B)\,\frac{\partial}{\partial\alpha}\left(\frac{\delta B}{B}\right) + (2\,E-2\,\mu\,B)\,\frac{\partial(\nabla\cdot\boldsymbol{\xi})}{\partial\alpha}\right]d\theta \tag{125}$$

to lowest order in perturbed quantities.

Finally, Eqs. (111), (118), and (123) yield

$$[-\mathrm{i}\,(\ell-\sigma\,n\,q)\,\omega_b - \mathrm{i}\,n\,(\omega_E+\omega_D)+\nu_i]\,\delta f_\ell = \frac{\omega_b}{2\pi\,e}\,J_\ell^{*}\,\frac{\partial f_0}{\partial\psi}. \tag{126}$$

VII. NEOCLASSICAL TOROIDAL VISCOUS TORQUE

The equilibrium ion distribution function is assumed to take the form

$$f_0(\psi, E, \mu) = \frac{N(\psi)}{[2\pi\, T(\psi)/m]^{3/2}}\,\exp\left[-\frac{\mathcal{E} - e\,\Phi(\psi)}{T(\psi)}\right], \tag{127}$$

where $N(\psi)$ is the ion number density, and $T(\psi)$ the ion temperature. The derivative of this distribution function with respect to ψ is taken at constant $\mathcal{E}$. Thus,

$$\left.\frac{\partial f_0}{\partial\psi}\right|_{\mathcal{E}} = -\frac{e\,f_0}{T}\left[\omega_E + \omega_{*N} + \left(\frac{E}{T} - \frac{3}{2}\right)\omega_{*T}\right], \tag{128}$$

where

$$\omega_{*N}(\psi) = -\frac{T}{N\,e}\,\frac{dN}{d\psi}, \tag{129}$$

$$\omega_{*T}(\psi) = -\frac{1}{e}\,\frac{dT}{d\psi} \tag{130}$$

are the density gradient and temperature gradient contributions to the ion diamagnetic frequency, respectively.

Equations (92), (106), (125), (126), and (128) give the following expression for the neoclassical toroidal viscous torque,

$$T_\varphi = -\frac{1}{m^2}\int\!\!\int\!\!\int K_l\,\frac{\omega_b\,f_0}{T}\,\frac{[\omega_E + \omega_{*N} + (E/T - 3/2)\,\omega_{*T}]}{\mathrm{i}\,(\ell - \sigma\,n\,q)\,\omega_b + \mathrm{i}\,n\,(\omega_E + \omega_D) - \nu_i}\,d\psi\,dE\,d\mu, \tag{131}$$

where

$$K_\ell = \oint |J_\ell|^2\,d\alpha. \tag{132}$$

Let $x = E/T$, $\Lambda = \mu\,B_0/E$, $\bar{\omega}_b(\psi, \Lambda) = R_0\,\omega_b/(2\,x\,T\,)^{1/2}$, $\bar{J}_\ell(\psi, \Lambda) = J_\ell/(2\,x\,T\,m\,R_0^2)^{1/2}$, and $\bar{K}_\ell(\psi, \Lambda) = \oint |\bar{J}_\ell|^2\,d\alpha$, where all quantities are dimensionless. We find that

$$T_\varphi = -\frac{R_0}{\pi^{3/2}\,B_0}\int d\psi\,N\,T\int d\Lambda\,\bar{\omega}_b\,\bar{K}_\ell\int dx\,\mathcal{R}_\ell, \tag{133}$$

where

$$\mathcal{R}_\ell(\psi, \Lambda, x) = \frac{[\omega_E + \omega_{*N} + (x - 3/2)\,\omega_{*T}]\,x^{5/2}\,\mathrm{e}^{-x}}{\mathrm{i}\,(\ell - \sigma\,n\,q)\,\omega_b + \mathrm{i}\,n\,(\omega_E + \omega_D) - \nu_i}, \tag{134}$$