

Calculation of Vertical Stability in an Inverse Aspect-Ratio Expanded Tokamak Plasma Equilibrium

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I. PLASMA EQUILIBRIUM

All lengths are normalized to the major radius of the plasma magnetic axis, R_0 . All magnetic field-strengths are normalized to the toroidal field-strength at the magnetic axis, B_0 . All current densities are normalized to $B_0/(\mu_0 R_0)$. All plasma pressures are normalized to B_0^2/μ_0 .

Let R, ϕ, Z be right-handed cylindrical coordinates whose Jacobian is

$$(\nabla R \times \nabla \phi \cdot \nabla Z)^{-1} = R. \quad (1)$$

Note that $|\nabla \phi| = 1/R$.

Let r, θ, ϕ be right-handed flux-coordinates whose Jacobian is

$$\mathcal{J}(r, \theta) \equiv (\nabla r \times \nabla \theta \cdot \nabla \phi)^{-1} \equiv R \left(\frac{\partial R}{\partial \theta} \frac{\partial Z}{\partial r} - \frac{\partial R}{\partial r} \frac{\partial Z}{\partial \theta} \right) = r R^2. \quad (2)$$

Note that $r = r(R, Z)$ and $\theta = \theta(R, Z)$. The magnetic axis corresponds to $r = 0$. The inboard mid-plane corresponds to $\theta = 0$.

Consider an axisymmetric tokamak equilibrium whose magnetic field takes the form

$$\mathbf{B}(r, \theta) = f(r) \nabla \phi \times \nabla r + g(r) \nabla \phi = f \nabla(\phi - q \theta) \times \nabla r, \quad (3)$$

where

$$q(r) = \frac{r g}{f} \quad (4)$$

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is the safety-factor (i.e., the inverse of the rotational transform). Note that $\mathbf{B} \cdot \nabla r = 0$, which implies that r is a magnetic flux-surface label. We require $g = 1$ on the magnetic axis in order to ensure that the normalized toroidal magnetic field-strength at the axis is unity.

It is easily demonstrated that

$$B^r = \mathbf{B} \cdot \nabla r = 0, \quad (5)$$

$$B^\theta = \mathbf{B} \cdot \nabla \theta = \frac{f}{r R^2}, \quad (6)$$

$$B^\phi = \mathbf{B} \cdot \nabla \phi = \frac{g}{R^2}, \quad (7)$$

$$B_r = \mathcal{J} \nabla \theta \times \nabla \phi \cdot \mathbf{B} = -r f \nabla r \cdot \nabla \theta, \quad (8)$$

$$B_\theta = \mathcal{J} \nabla \phi \times \nabla r \cdot \mathbf{B} = r f |\nabla r|^2, \quad (9)$$

$$B_\phi = \mathcal{J} \nabla r \times \nabla \theta \cdot \mathbf{B} = g. \quad (10)$$

The Maxwell equation (neglecting the displacement current, because the plasma velocity perturbations due to axisymmetric modes are far smaller than the velocity of light in vacuum) $\mathbf{J} = \nabla \times \mathbf{B}$ yields

$$\mathcal{J} J^r = \frac{\partial B_\phi}{\partial \theta} = 0, \quad (11)$$

$$\mathcal{J} J^\theta = -\frac{\partial B_\phi}{\partial r} = -g', \quad (12)$$

$$\mathcal{J} J^\phi = \frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{\partial \theta} = \frac{\partial}{\partial r}(r f |\nabla r|^2) + \frac{\partial}{\partial \theta}(r f \nabla r \cdot \nabla \theta), \quad (13)$$

where \mathbf{J} is the equilibrium current density, $' \equiv d/dr$, and use has been made of Eqs. (8)–(10).

Equilibrium force balance requires that

$$\nabla P = \mathbf{J} \times \mathbf{B}, \quad (14)$$

where $P(r)$ is the equilibrium scalar plasma pressure. Here, for the sake of simplicity, we have neglected the small centrifugal modifications to force balance due to subsonic plasma rotation. It follows that

$$P' = \mathcal{J}(J^\theta B^\phi - J^\phi B^\theta) = -g' \frac{g}{R^2} - \frac{f}{r R^2} \left[\frac{\partial}{\partial r}(r f |\nabla r|^2) + \frac{\partial}{\partial \theta}(r f \nabla r \cdot \nabla \theta) \right], \quad (15)$$

where use has been made of Eqs. (5)–(7), and (11)–(13). The other two components of Eq. (14) are identically zero.

Equation (15) yields the *inverse Grad-Shafranov equation*,

$$\frac{f}{r} \frac{\partial}{\partial r} (r f |\nabla r|^2) + \frac{f}{r} \frac{\partial}{\partial \theta} (r f \nabla r \cdot \nabla \theta) + g g' + R^2 P' = 0. \quad (16)$$

It follows from Eqs. (4), (13), and (16) that

$$\mathcal{J} J^\phi = -q g' - \frac{r R^2 P'}{f}. \quad (17)$$

It is clear from Eqs. (12) and (17) that $g' = P' = 0$ in the current-free “vacuum” region surrounding the plasma. We shall also assume that $g' = P' = 0$ at the plasma-vacuum interface, so as to ensure that the equilibrium plasma current density is zero at the interface.

II. AXISYMMETRIC PLASMA PERTURBATION

A. Derivation of Axisymmetric Ideal-MHD P.D.E.s

Let us assume that all perturbed quantities are independent of the toroidal angle, ϕ . The perturbed plasma equilibrium satisfies the marginally-stable ideal-MHD equations

$$\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \quad (18)$$

$$\nabla p = \mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b}, \quad (19)$$

$$\mathbf{j} = \nabla \times \mathbf{b}, \quad (20)$$

$$p = -\boldsymbol{\xi} \cdot \nabla P, \quad (21)$$

where $\boldsymbol{\xi}(r, \theta)$ is the plasma displacement, $\mathbf{b}(r, \theta)$ the perturbed magnetic field, $\mathbf{j}(r, \theta)$ the perturbed current density, and $p(r, \theta)$ the perturbed scalar pressure.

Now,

$$(\boldsymbol{\xi} \times \mathbf{B})_\theta = \mathcal{J} (\xi^\phi B^r - \xi^r B^\phi) = -\mathcal{J} B^\phi \xi^r, \quad (22)$$

$$(\boldsymbol{\xi} \times \mathbf{B})_\phi = \mathcal{J} (\xi^r B^\theta - \xi^\theta B^r) = \mathcal{J} B^\theta \xi^r, \quad (23)$$

where use has been made of the fact that $B^r = J^r = 0$. [See Eqs. (5) and (11).] Combining Eqs. (18) and (23), we obtain

$$\mathcal{J} b^r = \frac{\partial}{\partial \theta} (\mathcal{J} B^\theta \xi^r). \quad (24)$$

Thus, Eqs. (2), (4), (6), and (7) give

$$r R^2 b^r = \frac{\partial y}{\partial \theta}, \quad (25)$$

where

$$y(r, \theta) = f \xi^r. \quad (26)$$

The constraint $\nabla \cdot \mathbf{b} = 0$, which follows from Eq. (18), immediately yields

$$r R^2 b^\theta = -\frac{\partial y}{\partial r}. \quad (27)$$

According to Eq. (21),

$$p = -P' \nabla r \cdot \boldsymbol{\xi} = -P' \xi^r. \quad (28)$$

So, the perturbed force balance equation, (19), yields

$$-\frac{\partial (P' \xi^r)}{\partial r} = (\mathbf{j} \times \mathbf{B})_r + (\mathbf{J} \times \mathbf{b})_r, \quad (29)$$

$$-\frac{\partial (P' \xi^r)}{\partial \theta} = (\mathbf{j} \times \mathbf{B})_\theta + (\mathbf{J} \times \mathbf{b})_\theta, \quad (30)$$

$$0 = (\mathbf{j} \times \mathbf{B})_\phi + (\mathbf{J} \times \mathbf{b})_\phi, \quad (31)$$

giving

$$-\frac{\partial (P' \xi^r)}{\partial r} = r R^2 (j^\theta B^\phi - j^\phi B^\theta) + r R^2 (J^\theta b^\phi - J^\phi b^\theta), \quad (32)$$

$$-\frac{\partial (P' \xi^r)}{\partial \theta} = r R^2 (j^\phi B^r - j^r B^\phi) + r R^2 (J^\phi b^r - J^r b^\phi), \quad (33)$$

$$0 = r R^2 (j^r B^\theta - j^\theta B^r) + r R^2 (J^r b^\theta - J^\theta b^r), \quad (34)$$

where use has been made of Eq. (2). Thus, according to Eqs. (5)–(7), (11), (12), and (17),

$$-\frac{\partial (P' \xi^r)}{\partial r} = f (q j^\theta - j^\phi) - g' b^\phi + \left(q g' + \frac{r R^2 P'}{f} \right) b^\theta, \quad (35)$$

$$-\frac{\partial(P'\xi^r)}{\partial\theta} = -r g j^r - \left(q g' + \frac{r R^2 P'}{f}\right) b^r, \quad (36)$$

$$0 = f j^r + g' b^r. \quad (37)$$

It follows from Eqs. (25) and (37) that

$$r R^2 j^r = -\alpha_g \frac{\partial y}{\partial\theta}, \quad (38)$$

where

$$\alpha_g(r) = \frac{g'}{f}. \quad (39)$$

Note that Eq. (36) is trivially satisfied. Hence, of the three components of the perturbed force balance equation, only Eq. (35) remains to be solved.

Equation (20) yields

$$r R^2 j^r = \frac{\partial b_\phi}{\partial\theta}, \quad (40)$$

$$r R^2 j^\theta = -\frac{\partial b_\phi}{\partial r}, \quad (41)$$

$$r R^2 j^\phi = \frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial\theta}, \quad (42)$$

where use has been made of Eq. (2). It follows from Eqs. (38), (40), and (41) that

$$b_\phi = -\alpha_g y, \quad (43)$$

$$r R^2 j^\theta = \frac{\partial(\alpha_g y)}{\partial r}. \quad (44)$$

Note that $\nabla \cdot \mathbf{j} = 0$, in accordance with Eq. (20).

Now,

$$\mathbf{b} = b_r \nabla r + b_\theta \nabla\theta + b_\phi \nabla\phi, \quad (45)$$

so

$$b^r = \mathbf{b} \cdot \nabla r = |\nabla r|^2 b_r + (\nabla r \cdot \nabla\theta) b_\theta, \quad (46)$$

$$b^\theta = \mathbf{b} \cdot \nabla\theta = (\nabla r \cdot \nabla\theta) b_r + |\nabla\theta|^2 b_\theta, \quad (47)$$

$$b^\phi = \mathbf{b} \cdot \nabla \phi = \frac{b_\phi}{R^2}. \quad (48)$$

Equations (2), (46), and (47) can be rearranged to give

$$b_r = \left(\frac{1}{|\nabla r|^2} \right) b^r - \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) b_\theta, \quad (49)$$

$$b^\theta = \left(\frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) b^r + \left(\frac{1}{r^2 R^2 |\nabla r|^2} \right) b_\theta. \quad (50)$$

Let

$$z = |\nabla r|^2 r \frac{\partial y}{\partial r} + r \nabla r \cdot \nabla \theta \frac{\partial y}{\partial \theta}. \quad (51)$$

Equations (25), (27), (43), (49) and (50) yield

$$b_r = \frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} z, \quad (52)$$

$$b_\theta = -z, \quad (53)$$

$$b^\phi = -\frac{\alpha_g}{R^2} y. \quad (54)$$

Equations (42), (52), and (53) give

$$r R^2 j^\phi = -\frac{\partial z}{\partial r} - \frac{\partial}{\partial \theta} \left[\frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} z \right]. \quad (55)$$

It follows from Eqs. (26), (27), (35), (44), (54), and (55) that

$$\begin{aligned} -\frac{\partial}{\partial r} \left(\frac{P'}{f} y \right) &= \frac{f q}{r R^2} \frac{\partial(\alpha_g y)}{\partial r} + \frac{f}{r R^2} \frac{\partial z}{\partial r} \\ &+ \frac{f}{r R^2} \frac{\partial}{\partial \theta} \left[\frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} z \right] \\ &+ \frac{g' \alpha_g}{R^2} y - \left(q g' + \frac{r R^2 P'}{f} \right) \frac{1}{r R^2} \frac{\partial y}{\partial r}. \end{aligned} \quad (56)$$

Hence,

$$-[(\alpha_p \alpha_f + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2] y = r \frac{\partial z}{\partial r} + \frac{\partial}{\partial \theta} \left[\frac{1}{|\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} z \right], \quad (57)$$

where

$$\alpha_p(r) = \frac{r P'}{f^2}, \quad (58)$$

$$\alpha_f(r) = \frac{r^2}{f} \frac{d}{dr} \left(\frac{f}{r} \right). \quad (59)$$

Finally, Eqs. (51) and (57) yield the *axisymmetric ideal-MHD p.d.e.s*:

$$r \frac{\partial y}{\partial r} = \frac{z}{|\nabla r|^2} - \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \frac{\partial y}{\partial \theta}, \quad (60)$$

$$\begin{aligned} r \frac{\partial z}{\partial r} = & - [(\alpha_p \alpha_f + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2] y - \frac{\partial}{\partial \theta} \left(\frac{1}{|\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} \right) \\ & - \frac{\partial}{\partial \theta} \left(\frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} z \right). \end{aligned} \quad (61)$$

B. Properties of Axisymmetric Ideal-MHD P.D.E.s

The previous two equations can be combined to give

$$\begin{aligned} r \frac{\partial(z y^*)}{\partial r} = & \frac{|z|^2}{|\nabla r|^2} - \frac{\partial y^*}{\partial \theta} \left(\frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} z \right) - [(\alpha_p \alpha_f + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2] |y|^2 \\ & - y^* \frac{\partial}{\partial \theta} \left(\frac{1}{|\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} \right) - y^* \frac{\partial}{\partial \theta} \left(\frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} z \right). \end{aligned} \quad (62)$$

Hence,

$$\begin{aligned} r \frac{d}{dr} \left(\oint z y^* \frac{d\theta}{2\pi} \right) = & \oint \left[\frac{|z|^2}{|\nabla r|^2} - [(\alpha_p \alpha_f + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2] |y|^2 \right. \\ & \left. + \frac{1}{|\nabla r|^2 R^2} \left| \frac{\partial y}{\partial \theta} \right|^2 \right] \frac{d\theta}{2\pi}, \end{aligned} \quad (63)$$

which implies that $\oint z y^* d\theta/2\pi$ is a real quantity, and also that

$$r \frac{d}{dr} \left[\oint (z y^* - y^* z) \frac{d\theta}{2\pi} \right] = 0. \quad (64)$$

C. Derivation of the Axisymmetric Ideal-MHD O.D.E.s

Let

$$y(r, \theta) = \sum_m y_m(r) e^{im\theta}, \quad (65)$$

$$z(r, \theta) = \sum_m z_m(r) e^{im\theta}. \quad (66)$$

Equations (60) and (61) yield the *axisymmetric ideal-MHD o.d.e.s*:

$$r \frac{dy_m}{dr} = \sum_{m'} \left(A_m^{m'} z_{m'} + B_m^{m'} y_{m'} \right), \quad (67)$$

$$r \frac{dz_m}{dr} = \sum_{m'} \left(C_m^{m'} z_{m'} + D_m^{m'} y_{m'} \right), \quad (68)$$

where

$$A_m^{m'} = c_m^{m'}, \quad (69)$$

$$B_m^{m'} = -m' f_m^{m'}, \quad (70)$$

$$C_m^{m'} = -m f_m^{m'}, \quad (71)$$

$$D_m^{m'} = -(\alpha_f \alpha_p + r \alpha_p') a_m^{m'} - (q r \alpha_g' + r^2 \alpha_g^2) \delta_m^{m'} + m m' b_m^{m'}, \quad (72)$$

where

$$a_m^{m'}(r) = \oint R^2 \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (73)$$

$$b_m^{m'}(r) = \oint |\nabla r|^{-2} R^{-2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (74)$$

$$c_m^{m'}(r) = \oint |\nabla r|^{-2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (75)$$

$$f_m^{m'}(r) = \oint \frac{i r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}. \quad (76)$$

D. Properties of Axisymmetric Ideal-MHD O.D.E.s

Note that $a_m^m = a_m^{m*}$, $b_m^m = b_m^{m*}$, $c_m^m = c_m^{m*}$, and $f_m^m = -f_m^{m*}$, which implies that

$$A_{m'}^m = A_{m'}^{m*}, \quad (77)$$

$$B_{m'}^m = -C_{m'}^{m*}, \quad (78)$$

$$C_{m'}^m = -B_{m'}^{m*}, \quad (79)$$

$$D_{m'}^m = D_{m'}^{m*}. \quad (80)$$

It follows from Eqs. (67), (68), and (77)–(80) that

$$r \frac{d}{dr} \left(\sum_m z_m y_m^* \right) = \sum_{m, m'} \left(z_m^* A_m^{m'} z_{m'} + y_m^* D_m^{m'} y_{m'} \right), \quad (81)$$

$$r \frac{d}{dr} \left[\sum_m (z_m y_m^* - y_m z_m^*) \right] = 0. \quad (82)$$

E. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque exerted on the plasma lying within the magnetic flux-surface whose label is r is

$$T_\phi(r) = \oint \oint r R^2 b_\phi b^r d\theta d\phi. \quad (83)$$

It follows from Eqs. (25) and (43) that

$$T_\phi(r) = -\pi \alpha_g \oint \left(y^* \frac{\partial y}{\partial \theta} + y \frac{\partial y^*}{\partial \theta} \right) d\theta = -\pi \alpha_g \oint \frac{\partial |y|^2}{\partial \theta} d\theta = 0. \quad (84)$$

We conclude that an axisymmetric perturbation is incapable of exerting a net toroidal electromagnetic torque on the plasma.

F. Perturbed Plasma Potential Energy

The perturbed plasma potential energy in the region of the plasma lying within the magnetic flux-surface whose label is r is

$$\delta W_p = \frac{1}{2} \oint \oint r R^2 \xi^{r*} (-\mathbf{B} \cdot \mathbf{b} + \xi^r P') d\theta d\phi. \quad (85)$$

However,

$$\mathbf{B} \cdot \mathbf{b} - \xi^r P' = B^\theta b_\theta + B^\phi b_\phi - \xi^r P' = -\frac{f}{r R^2} (z + q \alpha_g y + \alpha_p R^2), \quad (86)$$

where use has been made of Eqs. (4)–(7), (26), (43), (53), and (58). Hence, we obtain

$$\delta W_p(r) = \frac{1}{2} \oint \oint y^* [z + (q \alpha_g + \alpha_p R^2) y] d\theta d\phi = \pi^2 \sum_m y_m^* \chi_m, \quad (87)$$

where

$$\chi_m(r) = z_m + q \alpha_g y_m + \alpha_p \sum_{m'} a_m^{m'} y_{m'}. \quad (88)$$

III. INVERSE ASPECT-RATIO EXPANDED TOKAMAK EQUILIBRIUM

A. Equilibrium Magnetic Flux-Surfaces

Let us assume that the inverse aspect-ratio of the plasma, ϵ , is such that $0 < \epsilon \ll 1$. Let $r = \epsilon \hat{r}$, $\nabla = \epsilon^{-1} \hat{\nabla}$, and $' \rightarrow \epsilon^{-1} '$. Suppose that the loci of the equilibrium magnetic flux-surfaces can be written in the parametric form:

$$R(\hat{r}, \omega) = 1 - \epsilon \hat{r} \cos \omega + \epsilon^2 \sum_{j>0} H_j(\hat{r}) \cos[(j-1)\omega] + \epsilon^2 \sum_{j>1} V_j(\hat{r}) \sin[(j-1)\omega] + \epsilon^3 L(\hat{r}) \cos \omega, \quad (89)$$

$$Z(\hat{r}, \omega) = \epsilon \hat{r} \sin \omega + \epsilon^2 \sum_{j>1} H_j(\hat{r}) \sin[(j-1)\omega] - \epsilon^2 \sum_{j>1} V_j(\hat{r}) \cos[(j-1)\omega] - \epsilon^3 L(\hat{r}) \sin \omega, \quad (90)$$

where j is a positive integer. Here, $H_1(\hat{r})$ controls the relative horizontal locations of the flux-surface centroids, $H_2(\hat{r})$ and $V_2(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface ellipticities, $H_3(\hat{r})$ and $V_3(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface triangularities, et cetera, whereas $L(\hat{r})$ is a flux-surface re-labelling parameter. Moreover, $\omega(R, Z)$ is a poloidal angle that is distinct from θ . Note that V_1 does not appear in Eq. (90) because such a factor merely gives rise to a rigid vertical shift of the plasma that can be eliminated by a suitable choice of the origin of the flux-coordinate system.

Let

$$J(\hat{r}, \omega) = \frac{1}{\epsilon^2} \left(\frac{\partial R}{\partial \omega} \frac{\partial Z}{\partial \hat{r}} - \frac{\partial R}{\partial \hat{r}} \frac{\partial Z}{\partial \omega} \right) \quad (91)$$

be the Jacobian of the \hat{r}, ω coordinate system. We can transform to the \hat{r}, θ coordinate system by writing

$$\theta(\hat{r}, \omega) = 2\pi \int_0^\omega \frac{J(\hat{r}, \tilde{\omega})}{R(\hat{r}, \tilde{\omega})} d\tilde{\omega} \bigg/ \oint \frac{J(\hat{r}, \omega)}{R(\hat{r}, \omega)} d\omega, \quad (92)$$

$$\hat{r} = \frac{1}{2\pi} \oint \frac{J(\hat{r}, \omega)}{R(\hat{r}, \omega)} d\omega. \quad (93)$$

This transformation ensures that

$$\frac{\partial \theta}{\partial \omega} = \frac{J}{\hat{r} R}, \quad (94)$$

and, hence, that

$$\mathcal{J} \equiv \frac{R}{\epsilon} \left(\frac{\partial R}{\partial \theta} \frac{\partial Z}{\partial \hat{r}} - \frac{\partial R}{\partial \hat{r}} \frac{\partial Z}{\partial \theta} \right) = \epsilon R J \frac{\partial \omega}{\partial \theta} = r R^2, \quad (95)$$

in accordance with Eq. (2).

B. Metric Elements

We can determine the metric elements of the flux-coordinate system by combining Eqs. (89)–(93). Evaluating the elements up to $\mathcal{O}(\epsilon)$, but retaining $\mathcal{O}(\epsilon^2)$ contributions to terms that are independent of ω , we obtain,

$$L(\hat{r}) = \frac{\hat{r}^3}{8} - \frac{\hat{r} H_1}{2} - \frac{1}{2} \sum_{j>1} (j-1) \frac{H_j^2}{\hat{r}} - \frac{1}{2} \sum_{j>1} (j-1) \frac{V_j^2}{\hat{r}}, \quad (96)$$

$$\begin{aligned} \theta &= \omega + \epsilon \hat{r} \sin \omega - \epsilon \sum_{j>0} \frac{1}{j} \left[H_j' - (j-1) \frac{H_j}{\hat{r}} \right] \sin(j \omega) \\ &+ \epsilon \sum_{j>1} \frac{1}{j} \left[V_j' - (j-1) \frac{V_j}{\hat{r}} \right] \cos(j \omega), \end{aligned} \quad (97)$$

$$\begin{aligned} |\hat{\nabla} \hat{r}|^2 &= 1 + 2 \epsilon \sum_{j>0} H_j' \cos(j \theta) + 2 \epsilon \sum_{j>1} V_j' \sin(j \theta) \\ &+ \epsilon^2 \left(\frac{3 \hat{r}^2}{4} - H_1 + \frac{1}{2} \sum_{j>0} \left[H_j'^2 + (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] \right. \\ &\left. + \frac{1}{2} \sum_{j>1} \left[V_j'^2 + (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right] \right), \end{aligned} \quad (98)$$

$$\begin{aligned} \hat{\nabla} \hat{r} \cdot \hat{\nabla} \theta &= \epsilon \sin \theta - \epsilon \sum_{j>0} \frac{1}{j} \left[H_j'' + \frac{H_j'}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \right] \sin(j \theta) \\ &+ \epsilon \sum_{j>1} \frac{1}{j} \left[V_j'' + \frac{V_j'}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \right] \cos(j \theta), \end{aligned} \quad (99)$$

$$R^2 = 1 - 2 \epsilon \hat{r} \cos \theta - \epsilon^2 \left(\frac{\hat{r}^2}{2} - \hat{r} H_1' - 2 H_1 \right). \quad (100)$$

Here, $' \equiv d/d\hat{r}$. Moreover, we have made use of the fact that $V_j \propto H_j$, for $j > 1$, because V_j and H_j satisfy the identical differential equations, (106) and (107).

C. Expansion of Grad-Shafranov Equation

Let us write

$$f(\hat{r}) = \epsilon \frac{\hat{r} g}{q}, \quad (101)$$

$$g(\hat{r}) = 1 + \epsilon^2 g_2(\hat{r}) + \epsilon^4 g_4(\hat{r}), \quad (102)$$

$$P'(\hat{r}) = \epsilon^2 p'_2(\hat{r}), \quad (103)$$

where q , g_2 , g_4 , and p_2 are all $\mathcal{O}(1)$. Here, the safety-factor, $q(\hat{r})$, and the second-order plasma pressure gradient, $p'_2(\hat{r})$, are the two free flux-surface functions that characterize the plasma equilibrium.

Expanding the Grad-Shafranov equation, (16), order by order in the small parameter ϵ , making use of Eqs. (98)–(103), we obtain

$$g'_2 = -p'_2 - \frac{\hat{r}}{q^2} (2 - s), \quad (104)$$

$$H''_1 = -(3 - 2s) \frac{H'_1}{\hat{r}} - 1 + \frac{2p'_2 q^2}{\hat{r}}, \quad (105)$$

$$H''_j = -(3 - 2s) \frac{H'_j}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \quad \text{for } j > 1, \quad (106)$$

$$V''_j = -(3 - 2s) \frac{V'_j}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \quad \text{for } j > 1, \quad (107)$$

$$g'_4 = g_2 \left[p'_2 - \frac{\hat{r}}{q^2} (2 - s) \right] - \frac{\hat{r}}{q} \Sigma + p'_2 \left(\frac{\hat{r}^2}{2} + \frac{\hat{r}^2}{q^2} - 2H_1 - 3\hat{r}H'_1 \right), \quad (108)$$

where $s = \hat{r} q'/q$,

$$\Sigma = \frac{1}{q} \left(\frac{3\hat{r}^2}{2} - 2\hat{r}H'_1 + S_2 \right) - \frac{2-s}{q} \left(-\frac{3\hat{r}^2}{4} + \frac{\hat{r}^2}{q^2} + H_1 + S_1 \right), \quad (109)$$

$$S_1(\hat{r}) = \frac{1}{2} \sum_{j>0} \left[3H_j'^2 - (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] + \frac{1}{2} \sum_{j>1} \left[3V_j'^2 - (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right], \quad (110)$$

$$\begin{aligned} S_2(\hat{r}) = & \sum_{j>0} \left[H_j'^2 + 2(j^2 - 1) \frac{H'_j H_j}{\hat{r}} - (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] \\ & + \sum_{j>1} \left[V_j'^2 + 2(j^2 - 1) \frac{V'_j V_j}{\hat{r}} - (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right]. \end{aligned} \quad (111)$$

Note that the relative horizontal shift of magnetic flux-surfaces, H_1 , otherwise known as the *Shafranov shift*, is driven by toroidicity [the second term on the right-hand side of Eq. (105)], and plasma pressure gradients (the third term). All of the other shaping terms (i.e., the H_j , for $j > 1$, and the V_j) are driven by axisymmetric currents flowing in external magnetic field-coils.

Finally, it follows from Eqs. (39), (58), (59), and (101)–(103) that

$$\alpha_p(\hat{r}) = \frac{p'_2 q^2}{\hat{r}} (1 - 2\epsilon^2 g_2), \quad (112)$$

$$\alpha_g(\hat{r}) = \frac{q}{\hat{r}} (g'_2 - \epsilon^2 g_2 g'_2 + \epsilon^2 g'_4), \quad (113)$$

$$\alpha_f(\hat{r}) = -s + \epsilon^2 \hat{r} g'_2. \quad (114)$$

Finally, it follows from Eqs. (99) and (105)–(107) that

$$\begin{aligned} \hat{\nabla} \hat{r} \cdot \hat{\nabla} \theta &= 2\epsilon \left[1 - \frac{p'_2 q^2}{\hat{r}} + (1-s) \frac{H'_1}{\hat{r}} \right] \sin \theta \\ &\quad - 2\epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) \frac{H'_j}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \right] \sin(j\theta) \\ &\quad + 2\epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) \frac{V'_j}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \right] \cos(j\theta). \end{aligned} \quad (115)$$

D. Calculation of Coupling Coefficients

Equations (98) and (109) yield

$$|\hat{\nabla} \hat{r}|^{-2} = 1 - 2\epsilon \sum_{j>0} H'_j \cos(j\theta) - 2\epsilon \sum_{j>0} V'_j \sin(j\theta) + \epsilon^2 \left(-\frac{3\hat{r}^2}{4} + H_1 + S_1 \right), \quad (116)$$

Equation (100) gives

$$R^{-2} = 1 + 2\epsilon \hat{r} \cos \theta + \epsilon^2 \left(\frac{5\hat{r}^2}{2} - \hat{r} H'_1 - 2H_1 \right). \quad (117)$$

The previous two equations imply that

$$|\hat{\nabla} \hat{r}|^{-2} R^{-2} = 1 + 2\epsilon \hat{r} \cos \theta - 2\epsilon \sum_{j>0} H'_j \cos(j\theta) - 2\epsilon \sum_{j>1} V'_j \sin(j\theta)$$

$$+ \epsilon^2 \left(\frac{7 \hat{r}^2}{4} - H_1 - 3 \hat{r} H'_1 + S_1 \right). \quad (118)$$

Finally, Eqs. (115) and (116) give

$$\begin{aligned} \hat{\nabla} \hat{r} \cdot \hat{\nabla} \theta |\hat{\nabla} \hat{r}|^{-2} &= 2 \epsilon \left[1 - \frac{p'_2 q^2}{\hat{r}} + (1-s) \frac{H'_1}{\hat{r}} \right] \sin \theta \\ &\quad - 2 \epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) \frac{H'_j}{\hat{r}} + (j^2-1) \frac{H_j}{\hat{r}^2} \right] \sin(j \theta) \\ &\quad + 2 \epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) \frac{V'_j}{\hat{r}} + (j^2-1) \frac{V_j}{\hat{r}^2} \right] \cos(j \theta), \end{aligned} \quad (119)$$

where use has been made of the fact that $V'_j \propto H'_j$ for $j > 1$.

It follows from Eqs. (73)–(76), (100), (116), (118), and (119) that

$$a_m^{m'} = \delta_m^{m'} - \epsilon \hat{r} (\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon^2 \left(\frac{\hat{r}^2}{2} - \hat{r} H'_1 - 2 H_1 \right) \delta_m^{m'}, \quad (120)$$

$$\begin{aligned} b_m^{m'} &= \delta_m^{m'} + \epsilon \hat{r} (\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon \sum_{j>0} H'_j (\delta_{m'-m-j} + \delta_{m'-m+j}) \\ &\quad - \epsilon \sum_{j>1} i V'_j (\delta_{m'-m-j} - \delta_{m'-m+j}) + \epsilon^2 \left(\frac{7 \hat{r}^2}{4} - H_1 - 3 \hat{r} H'_1 + S_1 \right) \delta_m^{m'}, \end{aligned} \quad (121)$$

$$\begin{aligned} c_m^{m'} &= \delta_m^{m'} - \epsilon \sum_{j>0} H'_j (\delta_{m'-m-j} + \delta_{m'-m+j}) - \epsilon \sum_{j>1} i V'_j (\delta_{m'-m-j} - \delta_{m'-m+j}) \\ &\quad + \epsilon^2 \left(-\frac{3 \hat{r}^2}{4} + H_1 + S_1 \right) \delta_m^{m'}, \end{aligned} \quad (122)$$

$$\begin{aligned} f_m^{m'} &= -\epsilon \left[\hat{r} - p'_2 q^2 + (1-s) H'_1 \right] (\delta_{m'-m-1} - \delta_{m'-m+1}) \\ &\quad + \epsilon \sum_{j>1} \frac{1}{j} \left[-(1-s) H'_j + (j^2-1) \frac{H_j}{\hat{r}} \right] (\delta_{m'-m-j} - \delta_{m'-m+j}) \\ &\quad + \epsilon \sum_{j>1} \frac{i}{j} \left[-(1-s) V'_j + (j^2-1) \frac{V_j}{\hat{r}} \right] (\delta_{m'-m-j} + \delta_{m'-m+j}). \end{aligned} \quad (123)$$

If we write

$$\alpha_g = \alpha_g^{(0)} + \epsilon^2 \alpha_g^{(2)}, \quad (124)$$

$$\alpha_p = \alpha_p^{(0)} + \epsilon^2 \alpha_p^{(2)}, \quad (125)$$

$$\alpha_f = \alpha_f^{(0)} + \epsilon^2 \alpha_f^{(2)}, \quad (126)$$

$$a_m^{m'} = 1 + \epsilon a_m^{m'(1)} + \epsilon^2 a_m^{m'(2)}, \quad (127)$$

$$b_m^{m'} = 1 + \epsilon b_m^{m'(1)} + \epsilon^2 b_m^{m'(2)}, \quad (128)$$

$$D_m^{m'} = D_m^{m'(0)} + \epsilon D_m^{m'(1)} + \epsilon^2 D_m^{m'(2)} \quad (129)$$

then it follows from Eq. (72) that

$$D_m^{m(0)} = -\alpha_f^{(0)} \alpha_p^{(0)} - \hat{r} \alpha_p'^{(0)} - q \hat{r} \alpha_g'^{(0)} + m^2, \quad (130)$$

$$D_m^{m'(1)} = -\left[\alpha_f^{(0)} \alpha_p^{(0)} + \hat{r} \alpha_p'^{(0)}\right] a_m^{m'(1)} + m m' b_m^{m'(1)}, \quad (131)$$

$$\begin{aligned} D_m^{m(2)} = & -\left[\alpha_f^{(0)} \alpha_p^{(0)} + \hat{r} \alpha_p'^{(0)}\right] a_m^{m'(2)} - \alpha_f^{(0)} \alpha_p^{(2)} - \alpha_f^{(2)} \alpha_p^{(0)} - \hat{r} \alpha_p'^{(2)} - q \hat{r} \alpha_g'^{(2)} \\ & - \hat{r}^2 [\alpha_g^{(0)}]^2 + m^2 b_m^{m(2)}. \end{aligned} \quad (132)$$

It follows that

$$A_m^m(\hat{r}) = 1 + \epsilon^2 \left(-\frac{3\hat{r}^2}{4} + H_1 + S_1 \right), \quad (133)$$

$$A_m^{m\pm 1}(\hat{r}) = -\epsilon H_1', \quad (134)$$

$$A_m^{m\pm j}(\hat{r}) = -\epsilon (H_j' \pm i V_j'), \quad (135)$$

$$B_m^m(\hat{r}) = 0, \quad (136)$$

$$B_m^{m\pm 1}(\hat{r}) = \pm \epsilon (m \pm 1) [\hat{r} - p_2' q^2 + (1-s) H_1'], \quad (137)$$

$$B_m^{m\pm j}(\hat{r}) = \pm \epsilon \frac{m \pm j}{j} \left[(1-s) (H_j' \pm i V_j') - (j^2 - 1) \frac{H_j \pm i V_j}{\hat{r}} \right], \quad (138)$$

$$C_m^m(\hat{r}) = 0, \quad (139)$$

$$C_m^{m\pm 1}(\hat{r}) = \pm \epsilon m [\hat{r} - p_2' q^2 + (1-s) H_1'], \quad (140)$$

$$C_m^{m\pm j}(\hat{r}) = \pm \epsilon \frac{m}{j} \left[(1-s) (H_j' \pm i V_j') - (j^2 - 1) \frac{H_j \pm i V_j}{\hat{r}} \right], \quad (141)$$

$$\begin{aligned} D_m^m(\hat{r}) = & m^2 + q \hat{r} \frac{d}{d\hat{r}} \left(\frac{2-s}{q} \right) + \epsilon^2 m^2 \left(\frac{7\hat{r}^2}{4} - H_1 - 3\hat{r} H_1' + S_1 \right) \\ & + \epsilon^2 \left\{ -\hat{r}^2 \frac{(2-s)^2}{q^2} - \hat{r} \frac{d}{d\hat{r}} (\hat{r} p_2') - 2\hat{r} p_2' (2-s) + q \hat{r} \frac{d\Sigma}{d\hat{r}} + 2\hat{r} p_2' (1-q^2) \right. \\ & \left. + 2\hat{r} p_2' \left(-1 + \frac{3p_2' q^2}{\hat{r}} \right) + 2H_1' q^2 \left[\frac{d}{d\hat{r}} (\hat{r} p_2') + 4(s-1) p_2' \right] \right\}, \end{aligned} \quad (142)$$

$$D_m^{m\pm 1}(\hat{r}) = \epsilon [(\hat{r} p_2')' - (2 - s) p_2'] q^2 + \epsilon m (m \pm 1) (\hat{r} - H_1'), \quad (143)$$

$$D_m^{m\pm j}(\hat{r}) = -\epsilon m (m \pm j) (H_j' \pm \mathrm{i} V_j'). \quad (144)$$