

Incorporation of Twisting Parity Response

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I. OUTER REGION

A. Behavior in Vicinity of Rational Surface

In the vicinity of the k th rational surface

$$\psi_k(r_k + x) = A_{Lk}^{\pm} |x|^{\nu_{Lk}} + \text{sgn}(x) A_{Sk}^{\pm} |x|^{\nu_{Sk}}, \quad (1)$$

where $+/-$ corresponds to $x > 0$ and $x < 0$ respectively, and

$$\nu_{Lk} = \frac{1}{2} - \sqrt{-D_{Ik}}, \quad (2)$$

$$\nu_{Sk} = \frac{1}{2} + \sqrt{-D_{Ik}}. \quad (3)$$

Let

$$\Psi_k^e = r_k^{\nu_{Lk}} \left(\frac{\nu_{Sk} - \nu_{Lk}}{L_k^k} \right)^{1/2} \frac{1}{2} (A_{Lk}^+ + A_{Lk}^-), \quad (4)$$

$$\Delta\Psi_k^e = r_k^{\nu_{Sk}} \left(\frac{\nu_{Sk} - \nu_{Lk}}{L_k^k} \right)^{1/2} (A_{Sk}^+ - A_{Sk}^-), \quad (5)$$

$$\Psi_k^o = r_k^{\nu_{Lk}} \left(\frac{\nu_{Sk} - \nu_{Lk}}{L_k^k} \right)^{1/2} \frac{1}{2} (A_{Lk}^+ - A_{Lk}^-), \quad (6)$$

$$\Delta\Psi_k^o = r_k^{\nu_{Sk}} \left(\frac{\nu_{Sk} - \nu_{Lk}}{L_k^k} \right)^{1/2} (A_{Sk}^+ + A_{Sk}^-). \quad (7)$$

B. Tearing Parity Solution

Let J be the number of poloidal harmonics included in the calculation. Let us launch J independent solution vectors from the magnetic axis. Let the solution vectors be denoted $\underline{\underline{\psi}}^{ae}(r)$ and $\underline{\underline{Z}}^{ae}(r)$. Here, the elements of $\underline{\underline{\psi}}^{ae}(r)$ are denoted $\psi_{j'j}(r)$, and the elements of

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$\underline{\underline{Z}}^{ae}(r)$ are denoted $Z_{j'j}(r)$, for $j', j = 1, J$. Furthermore, j' indexes the poloidal harmonic, whereas j indexes the solution vector launched from the axis. Let K be the number of rational surfaces in the plasma. The jump conditions imposed at the rational surfaces are

$$\Psi_k^o = 0, \quad (8)$$

$$\Delta\Psi_k^e = 0, \quad (9)$$

for $k = 1, K$, which implies that

$$A_{Lk}^+ = A_{Lk}^-, \quad (10)$$

$$A_{Sk}^+ = A_{Sk}^-. \quad (11)$$

Let Π_{kj}^{ae} be the value of Ψ_k^e at the k th rational surface associated with the j th solution launched from the axis. Likewise, let $\Delta\Pi_{kj}^{ae}$ be the value of $\Delta\Psi_k^o$ at the k th rational surface associated with the j th solution launched from the axis.

Let us launch K small solution vectors from each of the rational surfaces in the plasma. Let the solution vectors be denoted $\underline{\underline{\psi}}^{se}(r)$ and $\underline{\underline{Z}}^{se}(r)$. Here, the elements of $\underline{\underline{\psi}}^{se}(r)$ are denoted $\psi_{jk}(r)$, and the elements of $\underline{\underline{Z}}^{se}(r)$ are denoted $Z_{jk}(r)$, for $j = 1, J$ and $k = 1, K$. Furthermore, j indexes the poloidal harmonic, whereas k indexes the rational surface from which the solution is launched. The launch conditions are

$$\Psi_k^e = \Psi_k^o = 0, \quad (12)$$

$$\Delta\Psi_k^e = 1. \quad (13)$$

The jump conditions imposed at the other rational surfaces are

$$\Psi_{k'}^o = 0, \quad (14)$$

$$\Delta\Psi_{k'}^e = 0 \quad (15)$$

where $k' \neq k$. Let $\Pi_{k'k}^{se}$ be the value of $\Psi_{k'}^e$ at the k' th rational surface associated with the small solution vector launched from the k th rational surface. Likewise, let $\Delta\Pi_{k'k}^{se}$ be the value of $\Delta\Psi_{k'}^o$ at the k' th rational surface associated with the small solution vector launched from the k th rational surface. Here, k and k' run from 1 to K

The general tearing parity solution vectors are written

$$\underline{\underline{\psi}}^e(r) = \underline{\underline{\psi}}^{ae}(r) \underline{\underline{\alpha}}^e + \underline{\underline{\psi}}^{se}(r) \underline{\underline{\beta}}^e, \quad (16)$$

$$\underline{Z}^e(r) = \underline{\underline{Z}}^{ae}(r) \underline{\alpha}^e + \underline{\underline{Z}}^{se}(r) \underline{\beta}^e. \quad (17)$$

Here, $\underline{\alpha}^e$ is a $1 \times J$ vector of arbitrary coefficients, whereas $\underline{\beta}^e$ is a $1 \times K$ vector of arbitrary coefficients. However, the boundary condition at the plasma-vacuum interface is

$$\underline{\underline{I}}(a) \underline{\underline{Z}}^e(a) = \underline{\underline{H}} [\underline{\psi}^e(a) - \underline{\psi}^x(a)], \quad (18)$$

where $\underline{\underline{H}}$ is the vacuum matrix, $\underline{\psi}^x(r)$ is the RMP field, and

$$\underline{\underline{I}}_{jj'}(r) = \frac{\delta_{jj'}}{m_j - n q(r)}, \quad (19)$$

for $j, j' = 1, J$. It follows that

$$\underline{\underline{X}}^e \underline{\alpha}^e = \underline{\underline{Y}}^e \underline{\beta}^e - \underline{\Xi}, \quad (20)$$

where

$$\underline{\underline{X}}^e = \underline{\underline{I}}(a) \underline{\underline{Z}}^{ae}(a) - \underline{\underline{H}} \underline{\psi}^{ae}(a), \quad (21)$$

$$\underline{\underline{Y}}^e = \underline{\underline{H}} \underline{\psi}^{se}(a) - \underline{\underline{I}}(a) \underline{\underline{Z}}^{se}(a), \quad (22)$$

$$\underline{\Xi} = \underline{\underline{H}} \underline{\psi}^x(a). \quad (23)$$

Thus,

$$\underline{\alpha}^e = \underline{\underline{\Omega}}^e \underline{\beta}^e - \underline{\Upsilon}^e, \quad (24)$$

where

$$\underline{\underline{X}}^e \underline{\underline{\Omega}}^e = \underline{\underline{Y}}^e, \quad (25)$$

$$\underline{\underline{X}}^e \underline{\Upsilon}^e = \underline{\Xi}. \quad (26)$$

Note that $\underline{\underline{\psi}}^{ae}(a)$ is a $J \times J$ matrix, $\underline{\underline{I}}(a) \underline{\underline{Z}}^{ae}(a)$ is a $J \times J$ matrix, $\underline{\underline{\psi}}^{se}(a)$ is a $J \times K$ matrix, $\underline{\underline{I}}(a) \underline{\underline{Z}}^{se}(a)$ is a $J \times K$ matrix, $\underline{\underline{H}}$ is a $J \times J$ matrix, $\underline{\alpha}^e$ is a $1 \times J$ vector, $\underline{\beta}^e$ is a $1 \times K$ vector, and $\underline{\psi}^x$ and $\underline{\Xi}$ are $1 \times J$ vectors. Thus, $\underline{\underline{X}}^e$ is a $J \times J$ matrix, $\underline{\underline{Y}}^e$ is a $J \times K$ matrix, $\underline{\underline{\Omega}}^e$ is a $J \times K$ matrix, and $\underline{\Upsilon}^e$ is a $1 \times J$ vector.

It follows that

$$\underline{\Psi}^e = \underline{\underline{F}}^{ee} \underline{\beta}^e - \underline{\Lambda}^e, \quad (27)$$

$$\underline{\Delta \Psi}^e = \underline{\beta}^e, \quad (28)$$

$$\underline{\Psi}^o = 0, \quad (29)$$

$$\underline{\Delta\Psi}^o = \underline{F}^{oe} \underline{\beta}^e + \underline{\Delta\Lambda}^o, \quad (30)$$

where

$$\underline{F}^{ee} = \underline{\Pi}^{ae} \underline{\Omega}^e + \underline{\Pi}^{se}, \quad (31)$$

$$\underline{F}^{oe} = \underline{\Delta\Pi}^{ae} \underline{\Omega}^e + \underline{\Delta\Pi}^{se}, \quad (32)$$

$$\underline{\Lambda}^e = \underline{\Pi}^{ae} \underline{\Upsilon}^e, \quad (33)$$

$$\underline{\Delta\Lambda}^o = -\underline{\Delta\Pi}^{ae} \underline{\Upsilon}^e. \quad (34)$$

Note that $\underline{\Pi}^{ae}$ is a $K \times J$ matrix, $\underline{\Delta\Pi}^{ae}$ is a $K \times J$ matrix, $\underline{\Pi}^{se}$ is a $K \times K$ matrix, $\underline{\Delta\Pi}^{se}$ is a $K \times K$ matrix, and $\underline{\Lambda}^e$ is a $1 \times J$ vector. Thus, \underline{F}^{ee} is a $K \times K$ matrix, \underline{F}^{oe} is a $K \times K$ matrix, and $\underline{\Lambda}^e$ and $\underline{\Delta\Lambda}^o$ are $1 \times K$ vectors.

In the absence of an RMP, the fully-reconnected tearing parity eigenfunction associated with rational surface k is such that $\beta_{kk'}^e = \delta_{kk'}$. Thus,

$$\psi_{jk}^{fe}(r) = \psi_{jk}^{se}(r) + \sum_{j'=1,J} \psi_{jj'}^{ae}(r) \Omega_{j'k}^e, \quad (35)$$

$$Z_{jk}^{fe}(r) = Z_{jk}^{se}(r) + \sum_{j'=1,J} Z_{jj'}^{ae}(r) \Omega_{j'k}^e. \quad (36)$$

This eigenfunction is such that

$$\Psi_{k'}^e = F_{k'k}^{ee}, \quad (37)$$

$$\Delta\Psi_{kk'}^e = \delta_{k'k}, \quad (38)$$

$$\Psi_{k'}^o = 0, \quad (39)$$

$$\Delta\Psi_{k'}^o = F_{k'k}^{oe}. \quad (40)$$

C. Twisting Parity Solution

Let us launch J independent solution vectors from the magnetic axis. Let the j th solution vectors be denoted $\underline{\psi}^{ao}(r)$ and $\underline{Z}^{ao}(r)$. The jump conditions imposed at the rational surfaces are

$$\Psi_k^e = 0, \quad (41)$$

$$\Delta\Psi_k^o = 0, \quad (42)$$

for $k = 1, K$, which implies that

$$A_{Lk}^+ = -A_{Lk}^-, \quad (43)$$

$$A_{Sk}^+ = -A_{Sk}^-. \quad (44)$$

Let Π_{kj}^{ao} be the value of Ψ_k^o at the k th rational surface associated with the j th solution vector launched from the magnetic axis. Likewise, let $\Delta\Pi_{kj}^{ao}$ be the value of $\Delta\Psi_k^e$ at the k th rational surface associated with the j th solution vector launched from the magnetic axis.

Let us launch K small solution vectors from each of the rational surfaces in the plasma. Let the solution vectors be denoted $\underline{\underline{\psi}}^{so}(r)$ and $\underline{\underline{Z}}^{so}(r)$. The launch conditions are

$$\Psi_k^e = \Psi_k^o = 0, \quad (45)$$

$$\Delta\Psi_k^o = 1. \quad (46)$$

The jump conditions imposed at the other rational surfaces are

$$\Psi_{k'}^e = 0, \quad (47)$$

$$\Delta\Psi_{k'}^o = 0, \quad (48)$$

where $k' \neq k$. Let $\Pi_{k'k}^{so}$ be the value of $\Psi_{k'}^o$ at the k' th rational surface associated with the small solution vector launched from the k th rational surface. Likewise, let $\Delta\Pi_{k'k}^{so}$ be the value of $\Delta\Psi_{k'}^e$ at the k' th rational surface associated with the small solution vector launched from the k th rational surface.

The general twisting parity solution vectors are written

$$\underline{\underline{\psi}}^o(r) = \underline{\underline{\psi}}^{ao}(r) \underline{\underline{\alpha}}^o + \underline{\underline{\psi}}^{so}(r) \underline{\underline{\beta}}^o, \quad (49)$$

$$\underline{\underline{Z}}^o(r) = \underline{\underline{Z}}^{ao}(r) \underline{\underline{\alpha}}^o + \underline{\underline{Z}}^{so}(r) \underline{\underline{\beta}}^o, \quad (50)$$

Here, $\underline{\underline{\alpha}}^o$ is a $1 \times J$ vector of arbitrary coefficients, whereas $\underline{\underline{\beta}}^o$ is a $1 \times K$ vector of arbitrary coefficients. However, the boundary condition at the plasma-vacuum interface is again

$$\underline{\underline{I}}(a) \underline{\underline{Z}}^o(a) = \underline{\underline{H}} [\underline{\underline{\psi}}^o(a) - \underline{\underline{\psi}}^x(a)]. \quad (51)$$

It follows that

$$\underline{\underline{X}}^o \underline{\underline{\alpha}}^o = \underline{\underline{Y}}^o \underline{\underline{\beta}}^o - \underline{\underline{\Xi}}, \quad (52)$$

where

$$\underline{\underline{X}}^o = \underline{\underline{I}}(a) \underline{\underline{Z}}^{ao}(a) - \underline{\underline{H}} \underline{\underline{\psi}}^{ao}(a), \quad (53)$$

$$\underline{\underline{Y}}^o = \underline{\underline{H}} \underline{\underline{\psi}}^{so}(a) - \underline{\underline{I}}(a) \underline{\underline{Z}}^{so}(a). \quad (54)$$

Thus,

$$\underline{\underline{\alpha}}^o = \underline{\underline{\Omega}}^o \underline{\underline{\beta}}^o - \underline{\underline{\Upsilon}}^o, \quad (55)$$

where

$$\underline{\underline{X}}^o \underline{\underline{\Omega}}^o = \underline{\underline{Y}}^o, \quad (56)$$

$$\underline{\underline{X}}^o \underline{\underline{\Upsilon}}^o = \underline{\underline{\Xi}}. \quad (57)$$

Note that $\underline{\underline{\psi}}^{ao}(a)$ is a $J \times J$ matrix, $\underline{\underline{I}}(a) \underline{\underline{Z}}^{ao}(a)$ is a $J \times J$ matrix, $\underline{\underline{\psi}}^{so}(a)$ is a $J \times K$ matrix, $\underline{\underline{I}}(a) \underline{\underline{Z}}^{so}(a)$ is a $J \times K$ matrix, $\underline{\underline{H}}$ is a $J \times J$ matrix, $\underline{\underline{\alpha}}^o$ is a J vector, and $\underline{\underline{\beta}}^o$ is a K vector. Thus, $\underline{\underline{X}}^o$ is a $J \times J$ matrix, $\underline{\underline{Y}}^o$ is a $J \times K$ matrix, and $\underline{\underline{\Omega}}^o$ is a $J \times K$ matrix, and $\underline{\underline{\Upsilon}}^o$ is a $1 \times J$ vector.

It follows that

$$\underline{\underline{\Psi}}^e = \underline{\underline{0}}, \quad (58)$$

$$\underline{\underline{\Delta\Psi}}^e = \underline{\underline{F}}^{eo} \underline{\underline{\beta}}^o + \underline{\underline{\Delta\Lambda}}^e, \quad (59)$$

$$\underline{\underline{\Psi}}^o = \underline{\underline{F}}^{oo} \underline{\underline{\beta}}^o - \underline{\underline{\Lambda}}^o, \quad (60)$$

$$\underline{\underline{\Delta\Psi}}^o = \underline{\underline{\beta}}^o, \quad (61)$$

where

$$\underline{\underline{F}}^{oo} = \underline{\underline{\Pi}}^{ao} \underline{\underline{\Omega}}^o + \underline{\underline{\Pi}}^{so}, \quad (62)$$

$$\underline{\underline{F}}^{eo} = \underline{\underline{\Delta\Pi}}^{ao} \underline{\underline{\Omega}}^o + \underline{\underline{\Delta\Pi}}^{so}, \quad (63)$$

$$\underline{\underline{\Lambda}}^o = \underline{\underline{\Pi}}^{ao} \underline{\underline{\Upsilon}}^o, \quad (64)$$

$$\underline{\underline{\Delta\Lambda}}^e = -\underline{\underline{\Delta\Pi}}^{ao} \underline{\underline{\Upsilon}}^o. \quad (65)$$

Note that $\underline{\underline{\Pi}}^{ao}$ is a $K \times J$ matrix, $\underline{\underline{\Delta\Pi}}^{ao}$ is a $K \times J$ matrix, $\underline{\underline{\Pi}}^{so}$ is a $K \times K$ matrix, and $\underline{\underline{\Delta\Pi}}^{so}$ is a $K \times K$ matrix. Thus, $\underline{\underline{F}}^{oo}$ is a $K \times K$ matrix, $\underline{\underline{F}}^{eo}$ is a $K \times K$ matrix, and $\underline{\underline{\Lambda}}^o$ and $\underline{\underline{\Delta\Lambda}}^e$ are $1 \times K$ vectors.

In the absence of an RMP, the fully-reconnected twisting parity eigenfunction associated with rational surface k is such that $\beta_{k'}^o = \delta_{kk'}$. Thus,

$$\psi_{jk}^{fo}(r) = \psi_{jk}^{so}(r) + \sum_{j'=1,J} \psi_{jj'}^{ao}(r) \Omega_{j'k}^o, \quad (66)$$

$$Z_{jk}^{fo}(r) = Z_{jk}^{so}(r) + \sum_{j'=1,J} Z_{jj'}^{ao}(r) \Omega_{j'k}^o. \quad (67)$$

This eigenfunction is such that

$$\Psi_{k'}^e = 0, \quad (68)$$

$$\Delta \Psi_{kk'}^e = F_{k'k}^{eo}, \quad (69)$$

$$\Psi_{k'}^o = F_{k'k}^{oo}, \quad (70)$$

$$\Delta \Psi_{k'}^o = \delta_{k'k}. \quad (71)$$

D. General Dispersion Relation

The general dispersion relation is obtained by combining the tearing parity dispersion relation, (27)–(30), with the twisting parity dispersion relation, (58)–(61). We get

$$\underline{\Psi}^e = \underline{\underline{F}}^{ee} \underline{\beta}^e - \underline{\Lambda}^e, \quad (72)$$

$$\underline{\Delta \Psi}^e = \underline{\beta}^e + \underline{\underline{F}}^{eo} \underline{\beta}^o + \underline{\Delta \Lambda}^e, \quad (73)$$

$$\underline{\Psi}^o = \underline{\underline{F}}^{oo} \underline{\beta}^o - \underline{\Lambda}^o, \quad (74)$$

$$\underline{\Delta \Psi}^o = \underline{\beta}^o + \underline{\underline{F}}^{oe} \underline{\beta}^e + \underline{\Delta \Lambda}^o. \quad (75)$$

Hence, we obtain the general dispersion relation

$$\begin{pmatrix} \underline{\Delta \Psi}^e \\ \underline{\Delta \Psi}^o \end{pmatrix} = \begin{pmatrix} \underline{\underline{E}}^{ee} & \underline{\underline{E}}^{eo} \\ \underline{\underline{E}}^{oe} & \underline{\underline{E}}^{oo} \end{pmatrix} \begin{pmatrix} \underline{\Psi}^e \\ \underline{\Psi}^o \end{pmatrix} + \begin{pmatrix} \underline{\chi}^e \\ \underline{\chi}^o \end{pmatrix} \quad (76)$$

where

$$\underline{\underline{E}}^{ee} = (\underline{\underline{F}}^{ee})^{-1}, \quad (77)$$

$$\underline{\underline{E}}^{eo} = \underline{\underline{F}}^{eo} \underline{\underline{E}}^{oo}, \quad (78)$$

$$\underline{\underline{E}}^{oe} = \underline{\underline{F}}^{oe} \underline{\underline{E}}^{ee}, \quad (79)$$

$$\underline{\underline{E}}^{oo} = (\underline{\underline{F}}^{oo})^{-1}, \quad (80)$$

$$\underline{\chi}^e = \underline{\underline{E}}^{ee} \underline{\Lambda}^e + \underline{\underline{E}}^{eo} \underline{\Lambda}^o + \underline{\Delta} \underline{\Lambda}^e, \quad (81)$$

$$\underline{\chi}^o = \underline{\underline{E}}^{oe} \underline{\Lambda}^e + \underline{\underline{E}}^{oo} \underline{\Lambda}^o + \underline{\Delta} \underline{\Lambda}^o. \quad (82)$$

In the absence of an RMP, the tearing parity unreconnected eigenfunction associated with the k th rational surface is such that

$$\Psi_{k'}^e = \delta_{k'k}, \quad (83)$$

$$\Delta \Psi_{kk'}^e = E_{k'k}^{ee}, \quad (84)$$

$$\Psi_{k'}^o = 0, \quad (85)$$

$$\Delta \Psi_{k'}^o = E_{k'k}^{oe}. \quad (86)$$

Thus,

$$\psi_{jk}^{ue}(r) = \sum_{k'=1,k} \psi_{jk'}^{fe}(r) E_{k'k}^{ee} + \sum_{k'=1,k} \psi_{jk'}^{fo}(r) E_{k'k}^{oe}, \quad (87)$$

$$Z_{jk}^{ue}(r) = \sum_{k'=1,k} Z_{jk'}^{fe}(r) E_{k'k}^{ee} + \sum_{k'=1,k} Z_{jk'}^{fo}(r) E_{k'k}^{oe}. \quad (88)$$

In the absence of an RMP, the twisting parity unreconnected eigenfunction associated with the k th rational surface is such that

$$\Psi_{k'}^e = 0, \quad (89)$$

$$\Delta \Psi_{kk'}^e = E_{k'k}^{eo}, \quad (90)$$

$$\Psi_{k'}^o = \delta_{k'k}, \quad (91)$$

$$\Delta \Psi_{k'}^o = E_{k'k}^{oo}. \quad (92)$$

Thus,

$$\psi_{jk}^{uo}(r) = \sum_{k'=1,k} \psi_{jk'}^{fo}(r) E_{k'k}^{oo} + \sum_{k'=1,k} \psi_{jk'}^{fe}(r) E_{k'k}^{eo}, \quad (93)$$

$$Z_{jk}^{uo}(r) = \sum_{k'=1,k} Z_{jk'}^{fo}(r) E_{k'k}^{oo} + \sum_{k'=1,k} Z_{jk'}^{fe}(r) E_{k'k}^{eo}. \quad (94)$$

II. ANGULAR MOMENTUM CONSERVATION

The total toroidal electromagnetic torque acting on the plasma is

$$T_\varphi = 2 n \pi^2 \text{Im} \left(\underline{\Psi}^{e\dagger} \underline{\Delta} \underline{\Psi}^e + \underline{\Psi}^{o\dagger} \underline{\Delta} \underline{\Psi}^o \right), \quad (95)$$

which, in the absence of an RMP, gives

$$T_\varphi = 2 n \pi^2 \text{Im} \left(\underline{\Psi}^{e\dagger} \underline{E}^{ee} \underline{\Psi}^e + \underline{\Psi}^{e\dagger} \underline{E}^{eo} \underline{\Psi}^o + \underline{\Psi}^{o\dagger} \underline{E}^{oe} \underline{\Psi}^e + \underline{\Psi}^{o\dagger} \underline{E}^{oo} \underline{\Psi}^o \right), \quad (96)$$

or

$$\begin{aligned} T_\varphi = n \pi^2 \left[\underline{\Psi}^{e\dagger} (\underline{E}^{ee} - \underline{E}^{ee\dagger}) \underline{\Psi}^e + \underline{\Psi}^{e\dagger} (\underline{E}^{eo} - \underline{E}^{oe\dagger}) \underline{\Psi}^o \right. \\ \left. + \underline{\Psi}^{o\dagger} (\underline{E}^{oe} - \underline{E}^{eo\dagger}) \underline{\Psi}^e + \underline{\Psi}^{o\dagger} (\underline{E}^{oo} - \underline{E}^{oo\dagger}) \underline{\Psi}^o \right]. \end{aligned} \quad (97)$$

However, in the absence of an RMP, T_φ must be zero, irrespective of the values of the $\underline{\Psi}^e$ and the $\underline{\Psi}^o$. This is only possible if

$$\underline{E}^{ee\dagger} = \underline{E}^{ee}, \quad (98)$$

$$\underline{E}^{eo\dagger} = \underline{E}^{oe}, \quad (99)$$

$$\underline{E}^{oo\dagger} = \underline{E}^{oo}. \quad (100)$$

III. INNER LAYER EQUATIONS

In the vicinity of the k th rational surface, the inner layer equations are

$$(\hat{\gamma}_k + \text{i} Q_{Ek} + \text{i} Q_{ek}) \psi = -\text{i} X (\phi - N) + \frac{d^2 \psi}{dX^2}, \quad (101)$$

$$(\hat{\gamma}_k + \text{i} Q_{Ek}) N = -\text{i} Q_{ek} \phi - \text{i} c_{\beta k}^2 X V - \text{i} D_k^2 X \frac{d^2 \psi}{dX^2} + P_{\perp k} \frac{d^2 N}{dX^2}, \quad (102)$$

$$(\hat{\gamma}_k + \text{i} Q_{Ek} + \text{i} Q_{ik}) \frac{d^2 \phi}{dX^2} = -\text{i} X \frac{d^2 \psi}{dX^2} + P_{\varphi k} \frac{d^4}{dX^4} \left(\phi + \frac{N}{\iota_k} \right), \quad (103)$$

$$(\hat{\gamma}_k + \text{i} Q_{Ek}) V = \text{i} Q_{ek} \psi - \text{i} X N + P_{\varphi k} \frac{d^2 V}{dX^2}, \quad (104)$$

where $X = S_k^{1/3} (r - r_k) / r_k$. All quantities are as defined in TJ2025, except that

$$c_{\beta k} = \left(\frac{\beta_k}{1 + \beta_k} \right)^{1/2}, \quad (105)$$

$$\iota_k = -\frac{\omega_{*ek}}{\omega_{*ik}}, \quad (106)$$

$$Q_{ek} = -\left(\frac{\iota_k}{1 + \iota_k}\right) \tau_k \omega_{*k}, \quad (107)$$

$$Q_{ik} = \left(\frac{1}{1 + \iota_k}\right) \tau_k \omega_{*k}, \quad (108)$$

$$D_k = \left(\frac{\iota_k}{1 + \iota_k}\right)^{1/2} S_k^{1/3} \hat{d}_{\beta k}. \quad (109)$$

Equations (101)–(104) possess the trivial twisting parity solution:

$$\psi(X) = A X, \quad (110)$$

$$N(X) = A Q_{ek}, \quad (111)$$

$$\phi(X) = A (i \gamma_k - Q_{Ek}), \quad (112)$$

$$V(X) = 0. \quad (113)$$

IV. INTERMEDIATE LAYER EQUATIONS

The intermediate layer equation is

$$(1 + Y^2) \frac{d^2 \psi}{dY^2} = \nu_k (1 + \nu_k) \psi, \quad (114)$$

where $Y = (r - r_k)/\delta_k$, and

$$\nu_k = -\frac{1}{2} + \sqrt{-D_{Ik}} \simeq -\frac{1}{4} - D_{Ik}. \quad (115)$$

The general asymptotic behavior is

$$\psi(Y) = \hat{B}_{Lk} + \hat{B}_{Sk} |Y| \quad (116)$$

for $|Y| \ll 1$, and

$$\psi(Y) = \hat{A}_{Lk} |Y|^{-\nu_k} + \hat{A}_{Sk} |Y|^{1+\nu_k} \quad (117)$$

for $|Y| \gg 1$.

The tearing parity solution is such that

$$\psi(Y) = \hat{B}_{Lk}^e + \hat{B}_{Sk}^e |Y| \quad (118)$$

for $|Y| \ll 1$, and

$$\psi(Y) = \hat{A}_{Lk}^e |Y|^{-\nu_k} + \hat{A}_{Sk}^e |Y|^{1+\nu_k} \quad (119)$$

for $|Y| \gg 1$. Furthermore,

$$S_k^{1/3} \hat{\Delta}_k^e = \left(\frac{r_k}{\delta_k} \right) \frac{2 \hat{B}_{Sk}^e}{\hat{B}_{Lk}^e}, \quad (120)$$

where $\hat{\Delta}_k^e$ is the tearing parity layer response function, and

$$\Delta_k^e \equiv \frac{\Delta \Psi_k^e}{\Delta \Psi_k^e} = \left(\frac{r_k}{\delta_k} \right)^{1+2\nu_k} \frac{2 \hat{A}_{Sk}^e}{\hat{A}_{Lk}^e} \quad (121)$$

The twisting parity solution is such that

$$\psi(Y) = \hat{B}_{Sk}^o Y \quad (122)$$

for $|Y| \ll 1$, and

$$\psi(Y) = \text{sgn}(Y) \left(\hat{A}_{Lk}^o |Y|^{-\nu_k} + \hat{A}_{Sk}^o |Y|^{1+\nu_k} \right) \quad (123)$$

for $|Y| \gg 1$. Furthermore,

$$\Delta_k^o \equiv \frac{\Delta \Psi_k^o}{\Delta \Psi_k^o} = \left(\frac{r_k}{\delta_k} \right)^{1+2\nu_k} \frac{2 \hat{A}_{Sk}^o}{\hat{A}_{Lk}^o}. \quad (124)$$

The connection formulae are

$$\hat{B}_{Lk}^{e,o} = a_{LL} \hat{A}_{Lk}^{e,o} + a_{LS} \hat{A}_{Sk}^{e,o}, \quad (125)$$

$$\hat{B}_{Sk}^{e,o} = a_{SL} \hat{A}_{Lk}^{e,o} + a_{SS} \hat{A}_{Sk}^{e,o}. \quad (126)$$

It follows that

$$\left(\frac{\delta_k}{r_k} \right) \frac{S_k^{1/3} \hat{\Delta}_k^e}{2} = \frac{a_{SL} + a_{SS} (\delta_k/r_k)^{1+2\nu_k} (\Delta_k^e/2)}{a_{LL} + a_{LS} (\delta_k/r_k)^{1+2\nu_k} (\Delta_k^e/2)}, \quad (127)$$

$$\left(\frac{\delta_k}{r_k} \right)^{1+2\nu_k} \frac{\Delta_k^o}{2} = -\frac{a_{LL}}{a_{LS}}. \quad (128)$$

But, in the limit $|\nu_k| \rightarrow 0$, we find that $a_{LL} \rightarrow 1$, $a_{SL} \rightarrow -\nu_k \pi/2$, $a_{LS} \rightarrow -\nu_k \pi/2$, and $a_{SS} \rightarrow 1$. Thus, we obtain

$$\Delta_k^e \simeq S_k^{1/3} \hat{\Delta}_k^e + \frac{\pi \nu_k r_k}{\delta_k}, \quad (129)$$

$$\Delta_k^o \simeq \frac{4 r_k}{\pi \nu_k \delta_k}. \quad (130)$$

Let $\delta_k = \delta_{dk}/(2\sqrt{\pi})$, and let us identify ν_k with $-D_{Rk}$. It follows that

$$\Delta_k^e = S_k^{1/3} \hat{\Delta}_k^e + \Delta_{k \text{ crit}}^e, \quad (131)$$

$$\Delta_k^o = \Delta_{k \text{ crit}}^o, \quad (132)$$

where

$$\Delta_{k \text{ crit}}^e = \sqrt{2} \pi^{3/2} (-D_{Rk}) \frac{r_k}{\delta_{dk}}, \quad (133)$$

$$\Delta_{k \text{ crit}}^o = \frac{8}{\sqrt{\pi}} (-D_{Rk})^{-1} \frac{r_k}{\delta_{dk}}. \quad (134)$$

V. HOMOGENOUS DISPERSION RELATION

The homogeneous dispersion relation can be written

$$(S_k^{1/3} \hat{\Delta}_k^e + \Delta_{k \text{ crit}}^e) \Psi_k^e = \sum_{k'} (E_{kk'}^{ee} \Psi_{k'}^e + E_{kk'}^{eo} \Psi_{k'}^o), \quad (135)$$

$$0 = \sum_{k'} (\tilde{E}_{kk'}^{oo} \Psi_{k'}^o + E_{kk'}^{oe} \Psi_{k'}^e), \quad (136)$$

where

$$\tilde{E}_{kk'}^{oo} = E_{kk'}^{oo} - \Delta_{k \text{ crit}}^o \delta_{kk'}. \quad (137)$$

Hence,

$$\Psi_k^o = - \sum_{k', k''} (\tilde{E}_{kk'}^{oo})^{-1} E_{k'k''}^{oe} \Psi_{k''}^e, \quad (138)$$

and

$$(S_k^{1/3} \hat{\Delta}_k^e + \Delta_{k \text{ crit}}^e) \Psi_k^e = \sum_{k'} E_{kk'}^e \Psi_{k'}^e \quad (139)$$

where

$$E_{kk'}^e = E_{kk'}^{ee} - \sum_{k'', k'''} E_{kk''}^{eo} (\tilde{E}_{k''k'''}^{oo})^{-1} E_{k''k'}^{oe}. \quad (140)$$

Note that $\underline{\underline{E}}^e$ is Hermitian.

Suppose that $\hat{\Delta}_k$ is small, but $\hat{\Delta}_{k' \neq k}$ is order unity. In this case,

$$\Psi_{k'}^e \simeq \delta_{kk'} \Psi_k^e. \quad (141)$$

It follows that the growth rate of the mode that reconnects magnetic flux at the k th rational surface is governed by

$$S_k^{1/3} \hat{\Delta}_k^e \simeq E_{kk}^e - \Delta_{k \text{ crit}}^e. \quad (142)$$

The corresponding eigenfunction is

$$\psi_{jk}^u(r) = \psi_{jk}^{ue}(r) - \sum_{k',k''} \psi_{jk'}^{uo}(r) (\tilde{E}_{k'k''}^{oo})^{-1} E_{k''k}^{oe}, \quad (143)$$

$$Z_{jk}^u(r) = Z_{jk}^{ue}(r) - \sum_{k',k''} Z_{jk'}^{uo}(r) (\tilde{E}_{k'k''}^{oo})^{-1} E_{k''k}^{oe}, \quad (144)$$

and has the properties that

$$\Psi_{k'}^e = \delta_{kk'}, \quad (145)$$

$$\Delta \Psi_{k'}^e = E_{k'k}^e, \quad (146)$$

$$\Psi_{k'}^o = - \sum_{k',k''} (\tilde{E}_{k'k''}^{oo})^{-1} E_{k''k}^{oe}, \quad (147)$$

$$\Delta \Psi_{k'}^o = \Delta_{k' \text{ crit}}^o \Psi_{k'}^o. \quad (148)$$

Suppose that $\Psi_{k'}^e$ and $\Psi_{k''}^e$ are both non-zero, but that $\Psi_{k''}^e = 0$ for $k'' \neq k, k'$. The toroidal electromagnetic torque at a general rational surface labeled j is

$$\delta T_j = 2n\pi^2 \text{Im}(\Psi_j^{e*} \Delta \Psi_j^e + \Psi_j^{o*} \Delta \Psi_j^o). \quad (149)$$

Hence, we deduce that at the k th rational surface,

$$\delta T_k = 2n\pi^2 \text{Im}(\Psi_k^{e*} E_{kk'}^e \Psi_{k'}^e), \quad (150)$$

while, at the

$$\begin{aligned} \delta T_{k'} &= 2n\pi^2 \text{Im}(\Psi_{k'}^{e*} E_{k'k}^e \Psi_k^e) = -2n\pi^2 \text{Im}(\Psi_k^{e*} E_{k'k}^{e*} \Psi_{k'}^e) \\ &= -2n\pi^2 \text{Im}(\Psi_k^{e*} E_{kk'}^e \Psi_{k'}^e) = -\delta T_{k'}, \end{aligned} \quad (151)$$

with $\delta T_{k''} = 0$ for $k'' \neq k, k'$.

VI. INHOMOGENEOUS DISPERSION RELATION

The inhomogeneous dispersion relation can be written

$$(S_k^{1/3} \hat{\Delta}_k^e + \Delta_{k \text{ crit}}^e) \Psi_k^e = \sum_{k'} (E_{kk'}^{ee} \Psi_{k'}^e + E_{kk'}^{eo} \Psi_{k'}^o) + \chi_k^e, \quad (152)$$

$$0 = \sum_{k'} (\tilde{E}_{kk'}^{oo} \Psi_{k'}^o + E_{kk'}^{oe} \Psi_{k'}^e) + \chi_k^o. \quad (153)$$

It follows that

$$(S_k^{1/3} \hat{\Delta}_k^e + \Delta_{k \text{ crit}}^e) \Psi_k^e = \sum_{k'} E_{kk'}^e \Psi_{k'}^e + \chi_k, \quad (154)$$

where

$$\chi_k = \chi_k^e - \sum E_{k',k''}^{eo} (\tilde{E}_{k'k''}^{oo})^{-1} \chi_{k''}^o. \quad (155)$$