Ideal-MHD Energy Principle Analysis

I. IDEAL-MHD STABILITY ANALAYIS

A. Ideal-MHD Equations

The fundamental equations of ideal magnetohydrodynamics (ideal-MHD) are

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0, \tag{1}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] + \nabla p - \mathbf{j}\times\mathbf{B} = 0, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma\,p\,\nabla\cdot\mathbf{v} = 0, \tag{3}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{4}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \tag{5}$$

$$\mu_0\,\mathbf{j} = \nabla\times\mathbf{B}, \tag{6}$$

where ρ is the plasma mass density, $\mathbf{v}$ the plasma velocity, p the (scalar) plasma pressure, $\gamma = 5/3$ the ratio of specific heats, $\mathbf{B}$ the magnetic field-strength, and $\mathbf{j}$ the electric current density.

B. Plasma Equilibrium

The plasma equilibrium is such that

$$\rho(\mathbf{r},t) = \rho_0(\mathbf{r}), \tag{7}$$

$$\mathbf{v}(\mathbf{r},t) = \mathbf{0}, \tag{8}$$

$$p(\mathbf{r},t) = p_0(\mathbf{r}), \tag{9}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r}), \tag{10}$$

$$\mathbf{j}(\mathbf{r},t) = \mathbf{j}_0(\mathbf{r}), \tag{11}$$

where

$$\nabla p_0 = \mathbf{j}_0\times\mathbf{B}_0, \tag{12}$$

$$\nabla \cdot \mathbf{B}_0 = 0, \tag{13}$$

$$\mu_0\, \mathbf{j}_0 = \nabla \times \mathbf{B}_0. \tag{14}$$

Note that we are neglecting equilibrium plasma flows, because these are generally unimportant provided that they remain sub-sonic and sub-Alfvénic.

C. Plasma Equation of Motion

Let us write

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \tilde{\rho}_1(\mathbf{r}, t), \tag{15}$$

$$\mathbf{v}(\mathbf{r}, t) = \tilde{\mathbf{v}}_1(\mathbf{r}, t), \tag{16}$$

$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + \tilde{p}_1(\mathbf{r}, t), \tag{17}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \widetilde{\mathbf{B}}_1(\mathbf{r}, t), \tag{18}$$

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_0(\mathbf{r}) + \tilde{\mathbf{j}}(\mathbf{r}, t), \tag{19}$$

where the perturbed quantities (denoted by the subscript 1) are all assumed to be much smaller than the corresponding equilibrium quantities (denoted by the subscript 0).

The linearized perturbed versions of Eqs. (1)–(6) are

$$\frac{\partial \tilde{\rho}_1}{\partial t} + \nabla \cdot (\rho_0\, \tilde{\mathbf{v}}_1) = 0, \tag{20}$$

$$\rho_0\, \frac{\partial \tilde{\mathbf{v}}_1}{\partial t} + \nabla \tilde{p}_1 - \tilde{\mathbf{j}}_1 \times \mathbf{B}_0 - \mathbf{j}_0 \times \widetilde{\mathbf{B}}_1 = 0, \tag{21}$$

$$\frac{\partial \tilde{p}_1}{\partial t} + \tilde{\mathbf{v}}_1 \cdot \nabla p_0 + \gamma\, p_0\, \nabla \cdot \tilde{\mathbf{v}}_1 = 0, \tag{22}$$

$$\nabla \cdot \widetilde{\mathbf{B}}_1 = 0, \tag{23}$$

$$\frac{\partial \widetilde{\mathbf{B}}_1}{\partial t} = \nabla \times (\tilde{\mathbf{v}}_1 \times \mathbf{B}_0), \tag{24}$$

$$\mu_0\, \tilde{\mathbf{j}}_1 = \nabla \times \widetilde{\mathbf{B}}_1, \tag{25}$$

respectively. Let us write

$$\tilde{\mathbf{v}}_1 = \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t}, \tag{26}$$

where $\tilde{\boldsymbol{\xi}}(\mathbf{r}, t)$ represents the displacement of the plasma from its equilibrium position. If we assume that $\tilde{\rho}_1(\mathbf{r}, 0) = \tilde{p}_1(\mathbf{r}, 0) = \tilde{\mathbf{B}}_1(\mathbf{r}, 0) = 0$ then Eqs. (20), (22), and (24) can be integrated to give

$$\tilde{\rho}_1 = -\nabla\cdot(\rho_0\,\tilde{\boldsymbol{\xi}}), \tag{27}$$

$$\tilde{p}_1 = -\tilde{\boldsymbol{\xi}}\cdot\nabla p_0 - \gamma\,p_0\,\nabla\cdot\tilde{\boldsymbol{\xi}}, \tag{28}$$

$$\tilde{\mathbf{B}}_1 = \nabla\times(\tilde{\boldsymbol{\xi}}\times\mathbf{B}_0), \tag{29}$$

respectively. Substitution of these equations into Eq. (21), making use of Eq. (25), yields the linearized perturbed plasma *equation of motion*,

$$\rho_0\,\frac{\partial^2\tilde{\boldsymbol{\xi}}}{\partial t^2} = \mathbf{F}(\tilde{\boldsymbol{\xi}}), \tag{30}$$

where

$$\mathbf{F}(\tilde{\boldsymbol{\xi}}) = \mu_0^{-1}\,(\nabla\times\widetilde{\mathbf{Q}})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\widetilde{\mathbf{Q}} + \nabla(\tilde{\boldsymbol{\xi}}\cdot\nabla p_0 + \gamma\,p_0\,\nabla\cdot\tilde{\boldsymbol{\xi}}) \tag{31}$$

is known as the *force operator*, and

$$\widetilde{\mathbf{Q}} \equiv \widetilde{\mathbf{B}}_1 = \nabla\times(\tilde{\boldsymbol{\xi}}\times\mathbf{B}_0) \tag{32}$$

is the perturbed magnetic field. Note that the previous equation automatically satisfies Eq. (23). The force operator clearly specifies the perturbed force density that develops in the plasma in response to the displacement $\tilde{\boldsymbol{\xi}}$. Note that Eq. (27) does not actually contribute to equation of motion.

D. Normal Mode Analysis

The most efficient way to investigate linear stability is to formulate the problem in terms of normal modes. This goal is achieved by letting all perturbed quantities vary in time as $\exp(-{\rm i}\,\omega\,t)$. Thus,

$$\tilde{\mathbf{v}}_1(\mathbf{r},t) = \mathbf{v}_1(\mathbf{r})\,\exp(-{\rm i}\,\omega\,t), \tag{33}$$

$$\tilde{\boldsymbol{\xi}}(\mathbf{r},t) = \boldsymbol{\xi}(\mathbf{r})\,\exp(-{\rm i}\,\omega\,t), \tag{34}$$

$$\widetilde{\mathbf{Q}}(\mathbf{r},t) = \mathbf{Q}(\mathbf{r})\,\exp(-{\rm i}\,\omega\,t), \tag{35}$$

et cetera. Equation (26) transforms to give

$$\mathbf{v}_1 = -{\rm i}\,\omega\,\boldsymbol{\xi}, \tag{36}$$

whereas the linearized perturbed plasma equation of motion, (30), yields

$$-\omega^2\,\rho_0\,\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{37}$$

where

$$\mathbf{F}(\boldsymbol{\xi}) = \mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0 + \gamma\,p_0\,\nabla\cdot\boldsymbol{\xi}), \tag{38}$$

and

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0). \tag{39}$$

E. Plasma Boundary

The plasma is assumed to be confined on a set of nested toroidal magnetic flux-surfaces. Let the plasma occupy the volume V, and let S be its bounding surface. Furthermore, let $\mathbf{n}$ be a unit outward-directed normal to S. Because S corresponds to an equilibrium magnetic flux-surface, it follows that

$$\mathbf{n}\cdot\mathbf{B}_0 = 0 \tag{40}$$

on S.

F. Useful Analysis

Consider the integral

$$\int_V \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r}, \tag{41}$$

where $\boldsymbol{\xi}(\mathbf{r})$ and $\boldsymbol{\eta}(\mathbf{r})$ are two arbitrary vector fields. According to Eq. (38), the integrand takes the form

$$\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = \boldsymbol{\eta}\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0) + \nabla(\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi})\right]. \tag{42}$$

The final term can be written

$$\boldsymbol{\eta}\cdot\nabla(\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi}) = \nabla\cdot[\gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,\boldsymbol{\eta}] - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}). \tag{43}$$

Hence,

$$\boldsymbol{\eta}\cdot\mathbf{F}(\xi) = \nabla\cdot[\gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,\boldsymbol{\eta}] + \boldsymbol{\eta}\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0)\right]$$

$$- \gamma\, p_0\, (\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}). \tag{44}$$

Let us write

$$\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel\,\mathbf{b}, \tag{45}$$

$$\boldsymbol{\eta} = \boldsymbol{\eta}_\perp + \eta_\parallel\,\mathbf{b}, \tag{46}$$

where

$$\mathbf{b} = \frac{\mathbf{B}_0}{B_0}, \tag{47}$$

$$\mathbf{b}\cdot\boldsymbol{\xi}_\perp = \mathbf{b}\cdot\boldsymbol{\eta}_\perp = 0. \tag{48}$$

Thus, $\xi_\parallel\,\mathbf{b}$ and $\boldsymbol{\xi}_\perp$ are the components of $\boldsymbol{\xi}$ that are parallel to and perpendicular to the equilibrium magnetic field, et cetera. It follows from Eq. (39) that

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0). \tag{49}$$

Moreover, Eq. (12) implies that

$$\boldsymbol{\xi}\cdot\nabla p_0 = \boldsymbol{\xi}\cdot\mathbf{j}_0\times\mathbf{B}_0 = \boldsymbol{\xi}_\perp\cdot\mathbf{j}_0\times\mathbf{B}_0 = \boldsymbol{\xi}_\perp\cdot\nabla p_0. \tag{50}$$

Now,

$$\mathbf{B}_0\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0\right] = 0, \tag{51}$$

and

$$\mathbf{B}_0\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q}\right] = \mathbf{B}_0\cdot\mathbf{j}_0\times\mathbf{Q} = -\mathbf{j}_0\times\mathbf{B}_0\cdot\mathbf{Q} = -\nabla p_0\cdot\mathbf{Q}, \tag{52}$$

where use has been made of Eqs. (12) and (14). However, according to Eq. (49),

$$\begin{aligned} -\nabla p_0\cdot\mathbf{Q} &= -\nabla p_0\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0) = \nabla\cdot[\nabla p_0\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0)] = -\nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,\mathbf{B}_0]\\ &= -\mathbf{B}_0\cdot\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0), \end{aligned} \tag{53}$$

where use has been made of Eqs. (12) and (13). The previous two equations, combined with Eq. (50), imply that

$$\mathbf{B}_0\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0)\right] = 0. \tag{54}$$

Thus, Eqs. (44), (46), (50), (51), and (54) yield

$$\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = \nabla\cdot[\gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,\boldsymbol{\eta}] + \boldsymbol{\eta}_\perp\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\right]$$
$$-\gamma\,p_0\,(\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}). \tag{55}$$

Let

$$I = \boldsymbol{\eta}_\perp\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\right]. \tag{56}$$

Now,

$$\boldsymbol{\eta}_\perp\cdot\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0) = \nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,\boldsymbol{\eta}_\perp] - (\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,(\nabla\cdot\boldsymbol{\eta}_\perp). \tag{57}$$

Furthermore,

$$(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + (\nabla\times\mathbf{B}_0)\times\mathbf{Q} = \mathbf{Q}\cdot\nabla\mathbf{B}_0 + \mathbf{B}_0\cdot\nabla\mathbf{Q} - \nabla(\mathbf{B}_0\cdot\mathbf{Q}). \tag{58}$$

Hence,

$$I = \nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,\boldsymbol{\eta}_\perp] + \mu_0^{-1}\,\boldsymbol{\eta}_\perp\cdot[\mathbf{Q}\cdot\nabla\mathbf{B}_0 + \mathbf{B}_0\cdot\nabla\mathbf{Q} - \nabla(\mathbf{B}_0\cdot\mathbf{Q})] - (\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,(\nabla\cdot\boldsymbol{\eta}_\perp). \tag{59}$$

Note that

$$Q_i = \frac{\partial}{\partial x_j}\,(\xi_{\perp\,i}\,B_{0\,j} - \xi_{\perp\,j}\,B_{0\,i}). \tag{60}$$

It follows that

$$\begin{aligned}(\mathbf{Q}\cdot\nabla\mathbf{B}_0)_i &= Q_j\,\frac{\partial B_{0\,i}}{\partial x_j} = \frac{\partial}{\partial x_k}\,(\xi_{\perp\,j}\,B_{0\,k} - \xi_{\perp\,k}\,B_{0\,j})\,\frac{\partial B_{0\,i}}{\partial x_j}\\ &= B_{0\,k}\,\frac{\partial \xi_{\perp\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} - \xi_{\perp\,k}\,\frac{\partial B_{0\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} - B_{0\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,\frac{\partial \xi_{\perp\,k}}{\partial x_k}\\ &= \left[(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}_0 - (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot\nabla\mathbf{B}_0 - (\mathbf{B}_0\cdot\nabla\mathbf{B}_0)\,(\nabla\cdot\boldsymbol{\xi}_\perp)\right]_i,\end{aligned} \tag{61}$$

where use has been made of Eq. (13). However,

$$(\mathbf{B}_0\cdot\nabla\mathbf{B}_0) = B_0^2\,\boldsymbol{\kappa} + (\mathbf{B}_0\cdot\nabla B_0)\,\mathbf{b}, \tag{62}$$

where

$$\boldsymbol{\kappa} = \mathbf{b}\cdot\nabla\mathbf{b} = \frac{\mathbf{R}_c}{R_c^2} \tag{63}$$

is the curvature vector of the equilibrium magnetic field (i.e., $\mathbf{R}_c$ is the local radius of curvature of equilibrium magnetic field-lines). Hence, we deduce that

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{Q}\cdot\nabla\mathbf{B}_0) = \boldsymbol{\eta}_\perp\cdot\left[(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}_0 - (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot\nabla\mathbf{B}_0 - B_0^2\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,\boldsymbol{\kappa}\right]. \tag{64}$$

Now,

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{Q}) = \eta_{\perp\,i}\,B_{0\,j}\,\frac{\partial Q_i}{\partial x_j} = \frac{\partial}{\partial x_j}\,(\eta_{\perp i}\,Q_i\,B_{0\,j}) - Q_i\,B_{0\,j}\,\frac{\partial \eta_{\perp\,i}}{\partial x_j}$$
$$= \nabla\cdot[(\boldsymbol{\eta}_\perp\cdot\mathbf{Q})\,\mathbf{B}_0] - \mathbf{Q}\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) = -\mathbf{Q}\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp), \tag{65}$$

where we have used Eq. (13). Note that the divergence term would integrate to zero in Eq. (41), because of the boundary condition Eq. (40). Hence, this term has been neglected. However, from Eqs. (13) and (60),

$$\mathbf{Q} = \mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0 - (\nabla\cdot\boldsymbol{\xi}_\perp)\,\mathbf{B}_0. \tag{66}$$

Thus,

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{Q}) = -(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) + (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp)$$
$$+ (\nabla\cdot\boldsymbol{\xi}_\perp)\,\mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp). \tag{67}$$

But,

$$\mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) = B_{0\,i}\,B_{0\,j}\,\frac{\partial\eta_{\perp\,i}}{\partial x_j} = \frac{\partial}{\partial x_j}\,(B_{0\,i}\,B_{0\,j}\,\eta_{\perp\,i}) - B_{0\,i}\,\frac{\partial B_{0\,j}}{\partial x_j}\,\eta_{\perp\,i}$$
$$= \nabla\cdot[(\mathbf{B}_0\cdot\boldsymbol{\eta}_\perp)\,\mathbf{B}_0] - B_0^{\,2}\,\boldsymbol{\kappa}\cdot\boldsymbol{\eta}_\perp = -B_0^{\,2}\,\boldsymbol{\kappa}\cdot\boldsymbol{\eta}_\perp, \tag{68}$$

where we have made use of Eq. (13), as well as Eq. (62), and the divergence term has integrated to zero because of Eq. (40). Thus, we arrive at

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{Q}) = -(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) + (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - B_0^{\,2}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp). \tag{69}$$

Now,

$$-\boldsymbol{\eta}_\perp\cdot\nabla(\mathbf{B}_0\cdot\mathbf{Q}) = -\eta_{\perp i}\,\frac{\partial}{\partial x_i}\,(B_{0\,j}\,Q_j) = -\frac{\partial}{\partial x_i}\,(\eta_{\perp\,i}\,B_{0\,j}\,Q_j) + B_{0\,j}\,Q_j\,\frac{\partial\eta_{\perp i}}{\partial x_i}$$
$$= -\nabla\cdot[(\mathbf{B}_0\cdot\mathbf{Q})\,\boldsymbol{\eta}_\perp] + (\mathbf{B}_0\cdot\mathbf{Q})\,(\nabla\cdot\boldsymbol{\eta}_\perp). \tag{70}$$

But, from Eq. (66),

$$\mathbf{B}_0\cdot\mathbf{Q} = -B_0^{\,2}\,\nabla\cdot\boldsymbol{\xi}_\perp + \mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp) - \mathbf{B}_0\cdot(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)$$
$$= -B_0^{\,2}\,\nabla\cdot\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\cdot\nabla(B_0^{\,2}/2) + \mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp). \tag{71}$$

However,

$$\mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp) = B_{0\,i}\,B_{0\,j}\,\frac{\partial\xi_{\perp\,i}}{\partial x_j} = \frac{\partial}{\partial x_j}\,(B_{0\,i}\,B_{0\,j}\,\xi_{\perp\,i}) - B_{0\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,\xi_{\perp\,i}$$
$$= \nabla\cdot[(\mathbf{B}_0\cdot\boldsymbol{\xi}_\perp)\,\mathbf{B}_0] - \boldsymbol{\xi}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{B}_0) = -B_0^{\,2}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}), \tag{72}$$

where we have used Eqs. (13) and (62), and the divergence term has integrated to zero because of Eq. (40). Thus, we deduce that

$$-\boldsymbol{\eta}_\perp\cdot\nabla(\mathbf{B}_0\cdot\mathbf{Q}) = -\nabla\cdot[(\mathbf{B}_0\cdot\mathbf{Q})\,\boldsymbol{\eta}_\perp] - B_0^{\,2}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp)$$
$$- [\boldsymbol{\xi}_\perp\cdot\nabla(B_0^{\,2}/2) + B_0^{\,2}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})]\,(\nabla\cdot\boldsymbol{\eta}_\perp). \tag{73}$$

It follows from Eqs. (55), (56), (59), (64), (69), and (73) that

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) &= \nabla\cdot[\gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,\boldsymbol{\eta} + (\boldsymbol{\xi}_\perp\cdot\nabla p_0 - \mathbf{B}_0\cdot\mathbf{Q})\,\boldsymbol{\eta}_\perp]\\ &\quad + \frac{B_0^{\,2}}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \mu_0^{-1}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta})\\ &\quad - \left[\boldsymbol{\xi}_\perp\cdot\nabla\left(p_0 + \frac{B_0^{\,2}}{2\,\mu_0}\right) + \frac{B_0^{\,2}}{\mu_0}\,\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right]\nabla\cdot\boldsymbol{\eta}_\perp\\ &\quad - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp) + R,\end{aligned} \tag{74}$$

where

$$\mu_0\,R = \boldsymbol{\eta}_\perp\cdot[(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}_0 - (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot\nabla\mathbf{B}_0] + (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp). \tag{75}$$

However, from Eqs. (12) and (14),

$$\nabla p_0 = \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{B}_0 = \mu_0^{-1}\left[(\mathbf{B}_0\cdot\nabla)\,\mathbf{B}_0 - \nabla(B_0^{\,2}/2)\right]. \tag{76}$$

Thus, Eq. (62) yields

$$\boldsymbol{\xi}_\perp\cdot\nabla\left(p_0 + \frac{B_0^{\,2}}{2}\right) = \frac{B_0^{\,2}}{\mu_0}\,\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}. \tag{77}$$

Hence, Eq. (74) simplifies to give

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) &= \nabla\cdot[\gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,\boldsymbol{\eta} + (\boldsymbol{\xi}_\perp\cdot\nabla p_0 - \mathbf{B}_0\cdot\mathbf{Q})\,\boldsymbol{\eta}_\perp]\\ &\quad + \frac{B_0^{\,2}}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \mu_0^{-1}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi})\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta})\\ &\quad - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp) + R.\end{aligned} \tag{78}$$

Now,

$$
\begin{aligned}
\mu_0\,R &= \eta_{\perp\,i}\,B_{0\,k}\,\frac{\partial\xi_{\perp\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} - \eta_{\perp\,i}\,\xi_{\perp\,k}\,\frac{\partial B_{0\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} + \xi_{\perp\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,B_{0\,k}\,\frac{\partial\eta_{\perp\,i}}{\partial x_k}\\
&= \frac{\partial}{\partial x_k}\left(\eta_{\perp\,i}\,B_{0\,k}\,\xi_{\perp\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\right) - \xi_{\perp\,j}\,\frac{\partial}{\partial x_k}\left(\eta_{\perp\,i}\,B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_j}\right)\\
&\quad - \eta_{\perp\,i}\,\xi_{\perp\,k}\,\frac{\partial B_{0\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} + \xi_{\perp\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,B_{0\,k}\,\frac{\partial\eta_{\perp\,i}}{\partial x_k}\\
&= -\eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial}{\partial x_k}\left(B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_j}\right) - \eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial B_{0\,k}}{\partial x_j}\,\frac{\partial B_{0\,i}}{\partial x_k}\\
&= -\eta_{\perp\,i}\,\xi_{\perp\,j}\,B_{0\,k}\,\frac{\partial^2 B_{0\,i}}{\partial x_j\,\partial x_k} - \eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial B_{0\,k}}{\partial x_j}\,\frac{\partial B_{0\,i}}{\partial x_k}\\
&= -\eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial}{\partial x_j}\left(B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_k}\right),
\end{aligned}
\tag{79}
$$

where the divergence term has integrated to zero because of Eq. (40), and use has been made of Eq. (13). However, from Eq. (76),

$$
\mu_0^{-1}\,B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_k} = \frac{\partial}{\partial x_i}\left(p_0 + \frac{B_0^{\,2}}{2\,\mu_0}\right).
\tag{80}
$$

Thus,

$$
R = -\eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial^2}{\partial x_i\,\partial x_j}\left(p_0 + \frac{B_0^{\,2}}{2\,\mu_0}\right) = -(\boldsymbol{\eta}_\perp\,\boldsymbol{\xi}_\perp : \nabla\nabla)\left(p_0 + \frac{B_0^{\,2}}{2\,\mu_0}\right).
\tag{81}
$$

It follows from Eq. (78), and the previous equation, that

$$
\begin{aligned}
\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) &= \nabla\cdot\left[\gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,\boldsymbol{\eta} + (\boldsymbol{\xi}_\perp\cdot\nabla p_0 - \mathbf{B}_0\cdot\mathbf{Q})\,\boldsymbol{\eta}_\perp\right]\\
&\quad - \mu_0^{-1}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta}) + \frac{B_0^{\,2}}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp)\\
&\quad - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp)\\
&\quad - (\boldsymbol{\eta}_\perp\,\boldsymbol{\xi}_\perp : \nabla\nabla)\left(p_0 + \frac{B_0^{\,2}}{2\,\mu_0}\right).
\end{aligned}
\tag{82}
$$

Finally, we obtain

$$
\begin{aligned}
\int_V \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r} = \int_V &\left[-\mu_0^{-1}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta})\right.\\
&+ \frac{B_0^{\,2}}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\eta}_\perp)\\
&\left.-\frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp) - (\boldsymbol{\eta}_\perp\,\boldsymbol{\xi}_\perp : \nabla\nabla)\left(p_0 + \frac{B_0^{\,2}}{2\,\mu_0}\right)\right]d\mathbf{r}
\end{aligned}
$$

$$+\int_S \mathbf{n}\cdot\boldsymbol{\eta}_\perp\,(\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi}+\boldsymbol{\xi}_\perp\cdot\nabla p_0-\mathbf{B}_0\cdot\mathbf{Q})\,dS, \tag{83}$$

where use has been made of Eq. (40).

G. Perfectly Conducting Wall

Suppose that the plasma is immediately surrounded by a perfectly conducting wall. In other words, suppose that the plasma's bounding surface, S, corresponds to the inner boundary of the wall. Standard electromagnetic boundary conditions require that

$$\mathbf{n}\times\mathbf{E}=0, \tag{84}$$

$$\mathbf{n}\cdot\mathbf{B}=0, \tag{85}$$

on S. Here, $\mathbf{E}$ is the electric field.

The plasma velocity is assumed to be dominated by the $\mathbf{E}\times\mathbf{B}$ drift velocity. In other words,

$$\mathbf{v}=\frac{\mathbf{E}\times\mathbf{B}}{B^2}. \tag{86}$$

If we write $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0(\mathbf{r})+\mathbf{E}_1(\mathbf{r})\,\exp(-\mathrm{i}\,\omega\,t)$ then it is clear from Eq. (8) that $\mathbf{E}_0=\mathbf{0}$. It follows from the previous equation that

$$\mathbf{v}_1=\frac{\mathbf{E}_1\times\mathbf{B}_0}{B_0^2}. \tag{87}$$

Now, Eqs. (84) and (85) imply that

$$\mathbf{n}\cdot\mathbf{B}_0=0 \tag{88}$$

on S, in accordance with Eq. (40), as well as

$$\mathbf{n}\times\mathbf{E}_1=0, \tag{89}$$

$$\mathbf{n}\cdot\mathbf{B}_1=0. \tag{90}$$

Equations (87) and (89) yield

$$\mathbf{n}\cdot\mathbf{v}_1=0 \tag{91}$$

on S. Hence, it follows from Eqs. (36) and (88) that

$$\mathbf{n}\cdot\boldsymbol{\xi}=\mathbf{n}\cdot\boldsymbol{\xi}_\perp=0 \tag{92}$$

on S.

Making use of Eqs. (39), (49) and (90), we also require

$$\mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0) = 0 \tag{93}$$

on S. Now, Eqs. (88) and (92) imply that

$$\boldsymbol{\xi}_\perp\times\mathbf{B}_0 = f\,\mathbf{n} \tag{94}$$

on S, where f is some scalar. Thus, the boundary condition (93) becomes

$$\mathbf{n}\cdot\nabla\times(f\,\mathbf{n}) = \mathbf{n}\cdot[\nabla f\times\mathbf{n} + f\,\nabla\times\mathbf{n}) = f\,\mathbf{n}\cdot\nabla\times\mathbf{n} = 0. \tag{95}$$

Now, because S is an equilibrium magnetic flux-surface, it corresponds to a contour of the equilibrium poloidal magnetic flux, $\psi(\mathbf{r})$. It follows that

$$\mathbf{n} = \frac{\nabla\psi}{|\nabla\psi|}. \tag{96}$$

Thus,

$$\mathbf{n}\cdot\nabla\times\mathbf{n} = \frac{\nabla\psi}{|\nabla\psi|}\cdot\left[\nabla\left(\frac{1}{|\nabla\psi|}\right)\times\nabla\psi\right] = 0. \tag{97}$$

Hence, we deduce that the boundary condition (93) is automatically satisfied provided that the boundary condition (92) is satisfied.

H. Self-Adjoint Property of Force Operator

We wish to demonstrate that the force operator is *self-adjoint*. In other words, we wish to prove that

$$\int_V \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r} = \int_V \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})\,d\mathbf{r}, \tag{98}$$

where $\boldsymbol{\xi}(\mathbf{r})$ and $\boldsymbol{\eta}(\mathbf{r})$ are two arbitrary vector fields that satisfy the physical boundary conditions

$$\mathbf{n}\cdot\boldsymbol{\xi}_\perp = \mathbf{n}\cdot\boldsymbol{\eta}_\perp = 0 \tag{99}$$

on S. However, it follows from Eq. (83), and the previous boundary conditions, that

$$\begin{aligned}\int_V \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r} = \int_V \Big[&-\mu_0^{-1}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta})\\ &+\frac{B_0^{\,2}}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp)\\ &-(\boldsymbol{\eta}_\perp\,\boldsymbol{\xi}_\perp:\nabla\nabla)\left(p_0+\frac{B_0^{\,2}}{2\,\mu_0}\right)\Big]\,d\mathbf{r} = \int_V \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})\,d\mathbf{r},\end{aligned} \tag{100}$$

which immediately proves the result.

I. Reality of ω^2

Consider a discrete normal mode with frequency ω and displacment $\boldsymbol{\xi}(\mathbf{r})$. Taking the scalar product of Eq. (37) with $\boldsymbol{\xi}^*(\mathbf{r})$, and integrating over the plasma volume, we obtain

$$\omega^2 \int_V \rho_0\, |\boldsymbol{\xi}|^2\, d\mathbf{r} = -\int_V \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, d\mathbf{r}. \tag{101}$$

Likewise, taking the scalar product of the complex conjugate of Eq. (37) with $\boldsymbol{\xi}(\mathbf{r})$, and integrating over the plasma volume, we get

$$(\omega^2)^* \int_V \rho_0\, |\boldsymbol{\xi}|^2\, d\mathbf{r} = -\int_V \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}^*)\, d\mathbf{r}. \tag{102}$$

Here, we have made use of the fact that $[\mathbf{F}(\boldsymbol{\xi})]^* = \mathbf{F}(\boldsymbol{\xi}^*)$. Taking the difference between the previous two equations, we obtain

$$\left[\omega^2 - (\omega^2)^*\right] \int_V \rho_0\, |\boldsymbol{\xi}|^2\, d\mathbf{r} = -\int_V \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, d\mathbf{r} + \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}^*)\, d\mathbf{r}. \tag{103}$$

However, the self-adjoint property of the force operator, (98), yields

$$\left[\omega^2 - (\omega^2)^*\right] \int_V \rho_0\, |\boldsymbol{\xi}|^2\, d\mathbf{r} = 0. \tag{104}$$

Given that the integral in the previous expression is positive-definite, we deduce that

$$\omega^2 = (\omega^2)^*. \tag{105}$$

In other words, ω^2 is a real quantity. It immediately follows from the eigenmode equation, (37), as well as the fact that all of the differential operators appearing in $\mathbf{F}(\boldsymbol{\xi})$ are real, that $\boldsymbol{\xi}$ can be chosen to be real.

In terms of the usual definition of exponential stability, a discrete normal mode with $\omega^2 > 0$ corresponds to a pure oscillation, and would, therefore, be considered stable. Conversely, a discrete mode with $\omega^2 < 0$ has one branch that grows exponentially in time, and would, therefore, be considered unstable. Clearly, the transition from stability to instability occurs when $\omega^2 = 0$.

J. Orthogonality of Normal Modes

Consider two discrete normal modes. Let the first have the frequency ω_a and the displacement $\boldsymbol{\xi}_a(\mathbf{r})$. Let the second have the frequency ω_b and the displacement $\boldsymbol{\xi}_b(\mathbf{r})$. Let the modes satisfy the physical boundary conditions

$$\mathbf{n}\cdot\boldsymbol{\xi}_{a\,\perp} = \mathbf{n}\cdot\boldsymbol{\xi}_{b\,\perp} = 0 \tag{106}$$

at the wall. It follows from Eq. (37) that

$$\omega_a^{\,2}\,\rho_0\,\boldsymbol{\xi}_a = -\mathbf{F}(\boldsymbol{\xi}_a), \tag{107}$$

$$\omega_b^{\,2}\,\rho_0\,\boldsymbol{\xi}_b = -\mathbf{F}(\boldsymbol{\xi}_b). \tag{108}$$

Forming the scalar product of the first equation with $\boldsymbol{\xi}_b$, and the second with $\boldsymbol{\xi}_a$, taking the difference, and integrating over the plasma volume, we obtain

$$(\omega_a^{\,2}-\omega_b^{\,2})\int_V \rho_0\,\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}_b\,d\mathbf{r} = -\int_V [\boldsymbol{\xi}_b\cdot\mathbf{F}(\boldsymbol{\xi}_a) - \boldsymbol{\xi}_a\cdot\mathbf{F}(\boldsymbol{\xi}_b)]\,d\mathbf{r}. \tag{109}$$

However, the self-adjoint property of the force operator, (98), yields

$$(\omega_a^{\,2}-\omega_b^{\,2})\int_V \rho_0\,\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}_b\,d\mathbf{r} = 0. \tag{110}$$

Hence, we deduce that two discrete normal modes with different frequencies are orthogonal, in the sense that

$$\int_V \rho_0\,\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}_b\,d\mathbf{r} = 0. \tag{111}$$

Of course, the previous proof fails if we encounter multiple distinct normal modes that share the same frequency, ω. However, any linear combination of such degenerate modes is also a valid normal mode with frequency ω, and it is always possible to form linear combinations that are mutually orthogonal. With this caveat, we can state that discrete normal modes are mutually orthogonal.

K. Variational Principle

Taking the scalar product of the eigenmode equation, (37), with $(1/2)\,\boldsymbol{\xi}^*$, integrating over the plasma volume, and rearranging we obtain

$$\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})} \tag{112}$$

where

$$\delta K(\boldsymbol{\xi}^*,\boldsymbol{\xi}) = -\omega^2\,K(\boldsymbol{\xi}^*,\boldsymbol{\xi}), \tag{113}$$

$$K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \frac{1}{2}\int_V \rho_0\,|\boldsymbol{\xi}|^2\,d\mathbf{r}, \tag{114}$$

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2}\int_V \boldsymbol{\xi}^*\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r}. \tag{115}$$

Here, the quantity δW represents the change in potential energy associated with the perturbation, because it is equal to the work done against the force density, $\mathbf{F}(\boldsymbol{\xi})$, in displacing the plasma by an amount $\boldsymbol{\xi}$. The quantity K is proportional to the kinetic energy.

We wish to demonstrate that any allowable $\boldsymbol{\xi}(\mathbf{r})$ function for which ω^2 becomes an extremum satisfies the eigenmode equation, (37). The proof follows by letting $\boldsymbol{\xi}\rightarrow\boldsymbol{\xi}+\delta\boldsymbol{\xi}$ and $\omega^2\rightarrow\omega^2+\delta\omega^2$, and setting $\delta\omega^2 = 0$ (corresponding to ω^2 being an extremum). Neglecting terms that are quadratic in small quantities, we obtain

$$\omega^2+\delta\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})+\delta W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi})+K(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi})+K(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})}. \tag{116}$$

Rearranging the previous equation, making use of Eq. (112), and again neglecting terms that are quadratic in small quantities, we get

$$\delta\omega^2 = \frac{\delta W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})-\omega^2\,[K(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi})+K(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})]}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi})}. \tag{117}$$

Setting $\delta\omega^2$ to zero yields

$$\omega^2\,K(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi})-\delta W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi})+\omega^2\,K(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi}) = 0. \tag{118}$$

Making use of Eqs. (114) and (115), as well as the self-adjoint property of the force operator, (98), we get

$$\int\left\{\delta\boldsymbol{\xi}^*\cdot[\omega^2\,\rho_0\,\boldsymbol{\xi}+\mathbf{F}(\boldsymbol{\xi})]+\delta\boldsymbol{\xi}\cdot[\omega^2\,\rho_0\,\boldsymbol{\xi}^*+\mathbf{F}(\boldsymbol{\xi}^*)]\right\}d\mathbf{r} = 0. \tag{119}$$

However, in order for ω^2 to be an extremum, the previous equation must hold for arbitrary $\delta\boldsymbol{\xi}$. Hence, we obtain

$$-\omega^2\,\rho_0\,\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{120}$$

which is identical to Eq. (37).

L. Energy Principle

The ideal-MHD *energy principle* states that if

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \geq 0 \tag{121}$$

for all allowable plasma displacements (i.e., bounded in energy, and satisfying appropriate boundary conditions) then the plasma is ideally stable. In other words, there exist no normal modes with $\omega^2 < 0$. Conversely, if δW is negative for any allowable displacement then the plasma is ideally unstable. In other words, there exists at least one normal mode with $\omega^2 < 0$.

The proof of the energy principle is straightforward if one assumes that the normal modes are discrete, and form a complete set of basis functions, $\boldsymbol{\xi}_n(\mathbf{r})$, each satisfying

$$-\omega_n^2\,\rho_0\,\boldsymbol{\xi}_n = \mathbf{F}(\boldsymbol{\xi}_n). \tag{122}$$

In this case, any arbitrary trial function, $\boldsymbol{\xi}(\mathbf{r})$, can be represented as

$$\boldsymbol{\xi}(\mathbf{r}) = \sum_n a_n\,\boldsymbol{\xi}_n(\mathbf{r}). \tag{123}$$

Now, we demonstrated in Sect. I J that the normal modes are orthogonal with respect to the weight function $\rho_0(\mathbf{r})$. Let us normalize them such that

$$\int_V \rho_0\,\boldsymbol{\xi}_n^{\,*}\cdot\boldsymbol{\xi}_m\,d\mathbf{r} = \delta_{nm}. \tag{124}$$

It follows that

$$\begin{aligned}
\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi}) &= -\frac{1}{2}\int_V \boldsymbol{\xi}^*\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r}\\
&= -\frac{1}{2}\sum_{n,m} a_n^*\,a_m\int_V \boldsymbol{\xi}_n^{\,*}\cdot\mathbf{F}(\boldsymbol{\xi}_m)\,d\mathbf{r} = \frac{1}{2}\sum_{n,m} a_n^*\,a_m\,\omega_m^2\int_V \rho_0\,\boldsymbol{\xi}_n^{\,*}\cdot\boldsymbol{\xi}_m\,d\mathbf{r}\\
&= \frac{1}{2}\sum_n |a_n|^2\,\omega_n^2.
\end{aligned} \tag{125}$$

The previous equation implies that if some $\boldsymbol{\xi}(\mathbf{r})$ can be found for which $\delta W < 0$ then at least one of the ω_n^2 is negative, indicating instability. Conversely, if $\delta W \geq 0$ for all $\boldsymbol{\xi}(\mathbf{r})$ then all of the ω_n^2 are non-negative, indicating stability.

Unfortunately, the previous proof of the energy principle is not completely valid because it assumes that the normal modes of the plasma have discrete frequencies. In reality, while purely growing or decaying normal modes, characterized by $\omega^2 < 0$, do indeed have discrete frequencies, oscillatory modes, characterized by $\omega^2 > 0$, can have frequencies that lie in a continuous range. Hence, we need a more general proof of the energy principle. Let us work in the real time domain. The perturbed kinetic energy of the plasma is [see Eq. (114)]

$$K\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t},\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right), \tag{126}$$

whereas the peturbed potential energy is [see Eq. (115)]

$$\delta W(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}) = -\frac{1}{2}\int_V \tilde{\boldsymbol{\xi}}\cdot\mathbf{F}(\tilde{\boldsymbol{\xi}})\,d\mathbf{r}, \tag{127}$$

Thus, the total perturbed plasma energy is

$$H(t) = K\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t},\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right) + \delta W(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}) = \frac{1}{2}\int_V\left[\rho_0\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right)^2 - \tilde{\boldsymbol{\xi}}\cdot\mathbf{F}(\tilde{\boldsymbol{\xi}})\right]d\mathbf{r}. \tag{128}$$

Note that the kinetic energy, K, is positive definite. It follows that

$$\frac{dH}{dt} = \int\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\cdot\left[\rho_0\,\frac{\partial^2\tilde{\boldsymbol{\xi}}}{\partial t^2} - \mathbf{F}(\tilde{\boldsymbol{\xi}})\right] = 0, \tag{129}$$

where use has been made of Eq. (30), as well as the self-adjoint property of the force operator. The previous equation demonstrates that $H(t) = H_0$, where H_0 is constant in time. In other words,

$$H_0 = K\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t},\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right) + \delta W(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}). \tag{130}$$

Let us assume that δW is positive for all allowable $\tilde{\boldsymbol{\xi}}(\mathbf{r},t)$. The previous equation implies that

$$\delta W(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}) = H_0 - K\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t},\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right) > 0. \tag{131}$$

However, exponential growth of the kinetic energy K would eventually violate the previous equation. Hence, we deduce that the requirement that $\delta W > 0$ for all allowable plasma displacements is a sufficient condition to prohibit exponential growth.

To show the necessity of the energy principle, let us assume that there exists a perturbation, $\boldsymbol{\eta}(\mathbf{r})$, that satisfies the physical boundary conditions, and is such that $\delta W(\boldsymbol{\eta},\boldsymbol{\eta}) < 0$. Consider a plasma displacement, $\tilde{\boldsymbol{\xi}}(\mathbf{r},t)$, that satisfies the initial conditions

$$\tilde{\boldsymbol{\xi}}(\mathbf{r},0) = \boldsymbol{\eta}(\mathbf{r}), \tag{132}$$

$$\frac{\partial\tilde{\boldsymbol{\xi}}(\mathbf{r},0)}{\partial t} = \mathbf{0}. \tag{133}$$

Energy conservation implies that

$$H_0 = (\delta W + K)_{t=0} = \delta W(\boldsymbol{\eta},\boldsymbol{\eta}) < 0. \tag{134}$$

Consider

$$I(t) \equiv K(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}) = \frac{1}{2}\int_V \rho_0\,\tilde{\boldsymbol{\xi}}^2\,d\mathbf{r}. \tag{135}$$

It follows that

$$\frac{d^2 I}{dt^2} = \int_V \rho_0 \left[\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right)^2 + \tilde{\boldsymbol{\xi}}\cdot\frac{\partial^2\tilde{\boldsymbol{\xi}}}{\partial t^2}\right] d\mathbf{r} = \int_V \rho_0 \left[\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right)^2 + \tilde{\boldsymbol{\xi}}\cdot\mathbf{F}(\tilde{\boldsymbol{\xi}})\right] d\mathbf{r}$$
$$= 2\,K\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t},\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right) - 2\,\delta W = 4\,K\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t},\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right) - 2\,H_0 \geq -2\,H_0 > 0, \tag{136}$$

where use has been made of Eq. (30) and the energy conservation equation, (130). Equation (136) implies that if $\delta W(\boldsymbol{\eta},\boldsymbol{\eta}) < 0$ then $I(t)$ grows without bound as $t\rightarrow\infty$, at least as fast as t^2, showing that $\tilde{\boldsymbol{\xi}}$ increases at least as fast as t. More complete analysis indicates that if $\delta W(\boldsymbol{\eta},\boldsymbol{\eta}) < 0$ then there exists a $\tilde{\boldsymbol{\xi}}$ that grows exponentially as $\tilde{\boldsymbol{\xi}}\sim\exp(\lambda\,t)$ where $\lambda > [-\delta W(\boldsymbol{\eta},\boldsymbol{\eta})/K(\boldsymbol{\eta},\boldsymbol{\eta})]^{1/2}$. The implication is that any allowed perturbation that makes $\delta W < 0$ will grow exponentially in time. Consequently, $\delta W > 0$ for all allowed perturbations is a necessary condition to prohibit exponential growth.