

Program FLUX

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1 Construction of Flux Coordinate System

Let us adopt a normalization scheme in which all lengths are normalized to R_0 , all magnetic field-strengths to B_0 , and all pressures to B_0^2/μ_0 . In the following, all quantities are assumed to be normalized.

Let R, ϕ, Z be conventional right-handed cylindrical coordinates, such that

$$\nabla R \times \nabla \phi \cdot \nabla Z = \frac{1}{R}, \quad (1)$$

where $|\nabla \phi| = 1/R$.

We can write the equilibrium magnetic field in the form

$$\mathbf{B} = \nabla \phi \times \nabla \psi_p + g(\psi_p) \nabla \phi = \nabla(\phi - q\theta) \times \nabla \psi_p, \quad (2)$$

where the poloidal flux, $\psi_p(R, Z)$, and the toroidal flux function, $g(\psi_p)$, are both given. Furthermore,

$$\nabla \psi_p \times \nabla \theta \cdot \nabla \phi = \frac{g}{q R^2}, \quad (3)$$

where $q = q(\psi_p)$. Here, θ is a so-called “straight” poloidal angle, and $q(\psi_p)$ is the safety-factor.

Let $\Psi = \psi_p/\psi_c = 1 - \Psi_N$, where ψ_c is the value of ψ_p on the magnetic axis. (It is assumed that $\psi_p = 0$ on the magnetic separatrix.) Thus, $\Psi = 1$ on the magnetic axis, and $\Psi = 0$ on the magnetic separatrix. The previous equation implies that

$$\frac{d\theta}{dl} = \frac{g}{q |\psi_c| R \sqrt{\Psi_R^2 + \Psi_Z^2}}, \quad (4)$$

where dl is an element of poloidal path-length along a constant- Ψ surface, and $\Psi_R \equiv \partial\Psi/\partial R$, et cetera. Furthermore,

$$dR = -\frac{\Psi_Z dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}, \quad (5)$$

$$dZ = \frac{\Psi_R dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}}. \quad (6)$$

It follows that

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi |\psi_c|} \oint \frac{dl}{R \sqrt{\Psi_R^2 + \Psi_Z^2}}. \quad (7)$$

If we define

$$\tan \zeta = \frac{Z - Z_{\text{axis}}}{R_{\text{axis}} - R} \quad (8)$$

then

$$\frac{dR}{d\zeta} = -\Psi_Z F, \quad (9)$$

$$\frac{dZ}{d\zeta} = \Psi_R F, \quad (10)$$

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{|\psi_c|} \oint \frac{F}{R} \frac{d\zeta}{2\pi}, \quad (11)$$

where

$$F = \frac{(R_{\text{axis}} - R)^2 + (Z - Z_{\text{axis}})^2}{-(Z - Z_{\text{axis}}) \Psi_Z + (R_{\text{axis}} - R) \Psi_R}. \quad (12)$$

Note that $\zeta = 0$ on the inboard mid-plane.

It is helpful to define the length-like flux-surface label $r(\Psi)$, such that

$$\nabla r \times \nabla \theta \cdot \nabla \phi = \frac{1}{r R^2}. \quad (13)$$

It follows that

$$r(\Psi) = \left[2 |\psi_c| \int_{\Psi}^1 \frac{q(\Psi')}{g(\Psi')} d\Psi' \right]^{1/2}. \quad (14)$$

Let

$$a = r(\Psi_v), \quad (15)$$

where Ψ_v is the value of Ψ at the plasma/vacuum boundary. Note that $\Psi_v > 0$ and $f(r) = d\psi_p/dr$.

We can calculate $R(r, \theta)$ and $Z(r, \theta)$ by integrating

$$\frac{dR}{d\theta} = -|\psi_c| \frac{q}{g} R \Psi_Z, \quad (16)$$

$$\frac{dZ}{d\theta} = |\psi_c| \frac{q}{g} R \Psi_R, \quad (17)$$

along constant- r surfaces. Here, $\theta = 0$ on the inboard mid-plane.

2 Profile Functions

Program TJ also needs the following profile functions:

$$s(r) = r \frac{d \ln q}{dr}, \quad (18)$$

$$\alpha_g(r) = \frac{dg}{d\psi_p}, \quad (19)$$

$$\alpha_p(r) = \frac{q}{g} \frac{dP}{d\psi_p}, \quad (20)$$

$$\alpha_f(r) = -r \frac{d \ln(q/g)}{dr}, \quad (21)$$

$$\alpha'_g(r) = \frac{d\alpha_g}{dr}, \quad (22)$$

$$\alpha'_p(r) = \frac{d\alpha_p}{dr}. \quad (23)$$

where $P(r)$ is the equilibrium pressure profile.

3 Metric Elements

Let $R_r = \partial R / \partial r|_\theta$, et cetera. It follows that

$$R_\theta = -|\psi_c| \frac{q}{g} R \Psi_Z, \quad (24)$$

$$Z_\theta = |\psi_c| \frac{q}{g} R \Psi_R, \quad (25)$$

$$r R = R_\theta Z_r - R_r Z_\theta, \quad (26)$$

$$|\nabla r|^{-2} = \frac{r^2 R^2}{R_\theta^2 + Z_\theta^2}, \quad (27)$$

$$\frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} = -\frac{r (R_r R_\theta + Z_r Z_\theta)}{R_\theta^2 + Z_\theta^2}. \quad (28)$$

Let

$$a(r, \theta) = R^2, \quad (29)$$

$$b(r, \theta) = |\nabla r|^{-2} R^{-2}, \quad (30)$$

$$c(r, \theta) = |\nabla r|^{-2}, \quad (31)$$

$$d(r, \theta) = |\nabla r|^{-2} R^2, \quad (32)$$

$$e(r, \theta) = |\nabla r|^{-2} R^4, \quad (33)$$

$$f(r, \theta) = \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2}, \quad (34)$$

$$g(r, \theta) = \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} R^2. \quad (35)$$

Program TJ requires the following metric functions:

$$a_j^c(r) = \oint a(r, \theta) \cos(j \theta) \frac{d\theta}{2\pi}, \quad (36)$$

$$b_j^c(r) = \oint b(r, \theta) \cos(j \theta) \frac{d\theta}{2\pi}, \quad (37)$$

$$c_j^c(r) = \oint c(r, \theta) \cos(j \theta) \frac{d\theta}{2\pi}, \quad (38)$$

$$d_j^c(r) = \oint d(r, \theta) \cos(j \theta) \frac{d\theta}{2\pi}, \quad (39)$$

$$e_j^c(r) = \oint e(r, \theta) \cos(j \theta) \frac{d\theta}{2\pi}, \quad (40)$$

$$f_j^c(r) = \oint f(r, \theta) \cos(j \theta) \frac{d\theta}{2\pi}, \quad (41)$$

$$g_j^c(r) = \oint g(r, \theta) \cos(j \theta) \frac{d\theta}{2\pi}, \quad (42)$$

for $j = 0, J$,

$$a_j^s(r) = \oint a(r, \theta) \sin(j \theta) \frac{d\theta}{2\pi}, \quad (43)$$

$$b_j^s(r) = \oint b(r, \theta) \sin(j \theta) \frac{d\theta}{2\pi}, \quad (44)$$

$$c_j^s(r) = \oint c(r, \theta) \sin(j \theta) \frac{d\theta}{2\pi}, \quad (45)$$

$$d_j^s(r) = \oint d(r, \theta) \sin(j \theta) \frac{d\theta}{2\pi}, \quad (46)$$

$$e_j^s(r) = \oint e(r, \theta) \sin(j \theta) \frac{d\theta}{2\pi}, \quad (47)$$

$$f_j^s(r) = \oint f(r, \theta) \sin(j \theta) \frac{d\theta}{2\pi}, \quad (48)$$

$$g_j^s(r) = \oint g(r, \theta) \sin(j \theta) \frac{d\theta}{2\pi}, \quad (49)$$

for $j = 1, J$. The program also requires da_j^c/dr , for $j = 0, J$, da_j^s/dr , for $j = 1, J$, dc_0^c/dr , c_0^s/dr , dd_0^c/dr , and dd_0^s/dr .

4 Neoclassical Coordinate System

The neoclassical poloidal angle is defined

$$\mathbf{b} \cdot \nabla \Theta = \frac{1}{\gamma(\Psi)}. \quad (50)$$

It follows that

$$\frac{d\Theta}{dl} = \frac{B R}{|\psi_c| \gamma \sqrt{\Psi_R^2 + \Psi_Z^2}}, \quad (51)$$

where

$$B R = [g^2 + |\psi_c|^2 (\Psi_R^2 + \Psi_Z^2)]^{1/2}. \quad (52)$$

Hence,

$$\gamma(\Psi) = \frac{1}{2\pi |\psi_c|} \oint \frac{B R dl}{\sqrt{\Psi_R^2 + \Psi_Z^2}} = \frac{1}{|\psi_c|} \oint B R F \frac{d\zeta}{2\pi}. \quad (53)$$

We can calculate $R(r, \Theta)$ and $Z(r, \Theta)$ from

$$\frac{dR}{d\Theta} = -|\psi_c| \frac{\gamma \Psi_Z}{B R}, \quad (54)$$

$$\frac{dZ}{d\Theta} = |\psi_c| \frac{\gamma \Psi_R}{B R}. \quad (55)$$

Note that

$$\frac{d\theta}{d\Theta} = \left(\frac{\gamma g}{q} \right) \frac{1}{B R^2}. \quad (56)$$

5 Vacuum Solution

The plasma/vacuum interface lie at $\Psi = \Psi_v$. Let us solve

$$\nabla^2 \Phi = 0 \quad (57)$$

subject to the boundary conditions $\Phi = 0$ on the plasma/vacuum boundary, and $\Phi = 1$ on the bounding box. Let

$$\nabla \Phi \times \nabla \hat{\theta} \cdot \nabla \phi = \frac{1}{\Gamma(\Phi) R^2}. \quad (58)$$

It follows that

$$\frac{d\hat{\theta}}{dl} = \frac{1}{\Gamma R \sqrt{\Phi_R^2 + \Phi_Z^2}}. \quad (59)$$

$$\frac{dR}{d\zeta} = \Phi_Z \hat{F}, \quad (60)$$

$$\frac{dZ}{d\zeta} = -\Phi_R \hat{F}, \quad (61)$$

$$\Gamma = \oint \frac{\hat{F}}{R} \frac{d\zeta}{2\pi}, \quad (62)$$

where

$$\hat{F} = \frac{(R_{\text{axis}} - R)^2 + (Z - Z_{\text{axis}})^2}{(Z - Z_{\text{axis}})\Phi_Z - (R_{\text{axis}} - R)\Phi_R}. \quad (63)$$

We wish to define the length-like flux surface label $\hat{r}(\Phi)$ such that

$$\nabla \hat{r} \times \nabla \hat{\theta} \cdot \nabla \phi = \frac{1}{r R^2}, \quad (64)$$

and $\hat{r} = a$ on the plasma/vacuum boundary. It follows that

$$\hat{r}^2 = a^2 + 2 \int_0^\Phi \Gamma(\Phi') d\Phi'. \quad (65)$$

We can calculate $R(\hat{r}, \hat{\theta})$ and $Z(\hat{r}, \hat{\theta})$ by integrating

$$\frac{dR}{d\hat{\theta}} = \Gamma R \Phi_Z, \quad (66)$$

$$\frac{dZ}{d\hat{\theta}} = -\Gamma R \Phi_R, \quad (67)$$

along constant- \hat{r} surfaces. Here, $\hat{\theta} = 0$ on the inboard mid-plane.