

# Four-Field Resonant Layer Model

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## I. FOURIER-TRANSFORMED FOUR-FIELD EQUATIONS

In real space, the four-field resonant layer equations can be reduced to a set of ten coupled first-order differential equations. As we shall demonstrate, the equations can be reduced to a set of four coupled first-order differential equations in Fourier space. Clearly, it is advantageous to solve the equations in Fourier space.

The Fourier-transformed four-field layer equations take the form:

$$(g + iQ_e) \bar{\psi} = \frac{d(\bar{\phi} - \bar{N})}{dp} - p^2 \bar{\psi}, \quad (1)$$

$$g \bar{N} = -iQ_e \bar{\phi} - D^2 \frac{d(p^2 \bar{\psi})}{dp} + c_\beta^2 \frac{d\bar{V}}{dp} - P_\perp p^2 \bar{N}, \quad (2)$$

$$(g + iQ_i) p^2 \bar{\phi} = \frac{d(p^2 \bar{\psi})}{dp} - P_\varphi p^4 \left( \bar{\phi} + \frac{\bar{N}}{\iota} \right), \quad (3)$$

$$g \bar{V} = iQ_e \bar{\psi} + \frac{d\bar{N}}{dp} - P_\varphi p^2 \bar{V}. \quad (4)$$

It follows that

$$\bar{\psi} = \frac{1}{g + iQ_e + p^2} \frac{d(\bar{\phi} - \bar{N})}{dp}, \quad (5)$$

and

$$\frac{d(p^2 \bar{\psi})}{dp} = [(g + iQ_i) p^2 + P_\varphi p^4] \bar{\phi} + \frac{P_\varphi}{\iota} p^4 \bar{N}, \quad (6)$$

and

$$c_\beta^2 \frac{d\bar{V}}{dp} = (g + P_\perp p^2 + \iota^{-1} D^2 P_\varphi p^4) \bar{N} + [iQ_e + D^2 (g + iQ_i) p^2 + D^2 P_\varphi p^4] \bar{\phi}. \quad (7)$$

If we define

$$\bar{J} = p^2 \bar{\psi}, \quad (8)$$

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$$\bar{Y} = \bar{\phi} - \bar{N}, \quad (9)$$

then we can transform Eqs. (4)–(7) into the the following set of four coupled first-order differential equations:

$$\frac{d\bar{Y}}{dp} = \left( \frac{g + iQ_e + p^2}{p^2} \right) \bar{J}, \quad (10)$$

$$\frac{d\bar{N}}{dp} = \left( \frac{-iQ_e}{p^2} \right) \bar{J} + (g + P_\varphi p^2) \bar{V}, \quad (11)$$

$$\frac{d\bar{J}}{dp} = [(g + iQ_i) p^2 + P_\varphi p^4] \bar{Y} + [(g + iQ_i) p^2 + \iota_e^{-1} P_\varphi p^4] \bar{N}, \quad (12)$$

$$\begin{aligned} c_\beta^2 \frac{d\bar{V}}{dp} = & [iQ_e + D^2 (g + iQ_i) p^2 + D^2 P_\varphi p^4] \bar{Y} \\ & + (g + iQ_e + [P_\perp + D^2 (g + iQ_i)] p^2 + \iota_e^{-1} D^2 P_\varphi p^4) \bar{N}, \end{aligned} \quad (13)$$

where  $\iota_e = \iota/(1 + \iota)$ . Note that  $\iota = -Q_e/Q_i$  and  $\iota_e = -Q_e/(Q_i - Q_e)$ .

## II. SMALL- $p$ BEHAVIOR OF FOURIER-TRANSFORMED FOUR-FIELD EQUATIONS

### A. Introduction

Let us search for power-law solutions of Eqs. (10)–(13) at small values of  $p$ . Given that we have four coupled first-order differential equations, we expect to find four independent power-law solutions.

### B. First Solution

Suppose that

$$\bar{Y}(p) = y_{-1} p^{-1} + y_1 p + \mathcal{O}(p^3), \quad (14)$$

$$\bar{N}(p) = n_{-1} p^{-1} + n_1 p + \mathcal{O}(p^3), \quad (15)$$

$$\bar{J}(p) = j_0 + j_2 p^2 + \mathcal{O}(p^4), \quad (16)$$

$$\bar{V}(p) = v_2 p^2 + \mathcal{O}(p^4). \quad (17)$$

Equations (10)–(13) yield

$$-y_{-1} p^{-2} + y_1 = (g + i Q_e) (j_0 p^{-2} + j_2) + j_0 + \mathcal{O}(p^2), \quad (18)$$

$$-n_{-1} p^{-2} + n_1 = -i Q_e (j_0 p^{-2} + j_2) + \mathcal{O}(p^2), \quad (19)$$

$$2 j_2 p = (g + i Q_i) (y_{-1} + n_{-1}) p + \mathcal{O}(p^3), \quad (20)$$

$$\begin{aligned} 2 c_\beta^2 v_2 p &= i Q_e (y_{-1} p^{-1} + y_1 p) + (g + i Q_e) (n_{-1} p^{-1} + n_1 p) \\ &+ D^2 (g + i Q_i) y_{-1} p + [P_\perp + D^2 (g + i Q_i)] n_{-1} p + \mathcal{O}(p^3). \end{aligned} \quad (21)$$

It follows that

$$-y_{-1} = (g + i Q_e) j_0, \quad (22)$$

$$y_1 = (g + i Q_e) j_2 + j_0, \quad (23)$$

$$-n_{-1} = -i Q_e j_0, \quad (24)$$

$$n_1 = -i Q_e j_2, \quad (25)$$

$$2 j_2 = (g + i Q_i) (y_{-1} + n_{-1}), \quad (26)$$

$$0 = i Q_e y_{-1} + (g + i Q_e) n_{-1}, \quad (27)$$

$$\begin{aligned} 2 c_\beta^2 v_2 &= i Q_e y_1 + (g + i Q_e) n_1 \\ &+ D^2 (g + i Q_i) y_{-1} + [P_\perp + D^2 (g + i Q_i)] n_{-1}, \end{aligned} \quad (28)$$

which gives

$$y_{-1} = (g + i Q_e) a_{-1}, \quad (29)$$

$$y_1 = \left[ \frac{1}{2} g (g + i Q_e) (g + i Q_i) - 1 \right] a_{-1}, \quad (30)$$

$$n_{-1} = -i Q_e a_{-1}, \quad (31)$$

$$n_1 = -\frac{1}{2} g (i Q_e) (g + i Q_i) a_{-1}, \quad (32)$$

$$j_0 = -a_{-1}, \quad (33)$$

$$j_2 = \frac{1}{2} g (g + i Q_i) a_{-1}, \quad (34)$$

$$v_2 = \frac{[-i Q_e (1 + P_\perp) + g (g + i Q_i) D^2]}{2 c_\beta^2} a_{-1}, \quad (35)$$

where  $a_{-1}$  is an arbitrary constant.

### C. Second Solution

Suppose that

$$\bar{Y}(p) = y_0 + y_2 p^2 + \mathcal{O}(p^4), \quad (36)$$

$$\bar{N}(p) = n_0 + n_2 p^2 + \mathcal{O}(p^4), \quad (37)$$

$$\bar{J}(p) = j_3 p^3 + \mathcal{O}(p^5), \quad (38)$$

$$\bar{V}(p) = v_3 p^3 + \mathcal{O}(p^5). \quad (39)$$

Equations (10)–(13) give

$$2 y_2 p = (g + \mathrm{i} Q_e) j_3 p + \mathcal{O}(p^3), \quad (40)$$

$$2 n_2 p = -\mathrm{i} Q_e j_3 p + \mathcal{O}(p^3), \quad (41)$$

$$3 j_3 p^2 = (g + \mathrm{i} Q_i) (y_0 + n_0) p^2 + \mathcal{O}(p^4), \quad (42)$$

$$\begin{aligned} 3 c_\beta^2 v_3 p^2 &= \mathrm{i} Q_e (y_0 + y_2 p^2) + (g + \mathrm{i} Q_e) (n_0 + n_2 p^2) \\ &\quad + D^2 (g + \mathrm{i} Q_i) y_0 p^2 + [P_\perp + D^2 (g + \mathrm{i} Q_i)] n_0 p^2 + \mathcal{O}(p^4). \end{aligned} \quad (43)$$

It follows that

$$2 y_2 = (g + \mathrm{i} Q_e) j_3, \quad (44)$$

$$2 n_2 = -\mathrm{i} Q_e j_3, \quad (45)$$

$$3 j_3 = (g + \mathrm{i} Q_i) y_0 + (g + \mathrm{i} Q_i) n_0, \quad (46)$$

$$0 = \mathrm{i} Q_e y_0 + (g + \mathrm{i} Q_e) n_0, \quad (47)$$

$$\begin{aligned} 3 c_\beta^2 v_3 &= \mathrm{i} Q_e y_2 + (g + \mathrm{i} Q_e) n_2 \\ &\quad + D^2 (g + \mathrm{i} Q_e) n_0 + [P_\perp + D^2 (g + \mathrm{i} Q_i)] n_0. \end{aligned} \quad (48)$$

which gives

$$y_0 = (g + \mathrm{i} Q_e) a_0, \quad (49)$$

$$y_2 = \frac{1}{6} g (g + \mathrm{i} Q_e) (g + \mathrm{i} Q_i) a_0, \quad (50)$$

$$n_0 = -\mathrm{i} Q_e a_0, \quad (51)$$

$$n_2 = -\frac{1}{6} g (\mathrm{i} Q_e) (g + \mathrm{i} Q_i) a_0, \quad (52)$$

$$j_3 = \frac{1}{3} g (g + i Q_i) a_0, \quad (53)$$

$$v_3 = \frac{1}{3} \frac{[-i Q_e P_\perp + g (g + i Q_i) D^2]}{c_\beta^2} a_0, \quad (54)$$

where  $a_0$  is an arbitrary constant.

#### D. Third Solution

Suppose that

$$\bar{Y}(p) = y_2 p^2 + \mathcal{O}(p^4), \quad (55)$$

$$\bar{N}(p) = n_0 + n_2 p^2 + \mathcal{O}(p^4), \quad (56)$$

$$\bar{J}(p) = j_3 p^3 + \mathcal{O}(p^5), \quad (57)$$

$$\bar{V}(p) = v_1 p + \mathcal{O}(p^3). \quad (58)$$

Equations (10)–(13) give

$$2 y_2 p = (g + i Q_e) j_3 p + \mathcal{O}(p^3), \quad (59)$$

$$2 n_2 p = -i Q_e j_3 p + g v_1 p + \mathcal{O}(p^3), \quad (60)$$

$$3 j_3 p^2 = (g + i Q_i) n_0 p^2 + \mathcal{O}(p^4), \quad (61)$$

$$c_\beta^2 v_1 = (g + i Q_e) n_0 + \mathcal{O}(p^2). \quad (62)$$

It follows that

$$y_2 = \frac{1}{6} (g + i Q_e) (g + i Q_i) a_2, \quad (63)$$

$$n_0 = a_2, \quad (64)$$

$$n_2 = \frac{1}{2} (g + i Q_i) \left( -\frac{1}{3} i Q_e + \frac{g}{c_\beta^2} \right) a_2, \quad (65)$$

$$j_3 = \frac{1}{3} (g + i Q_i) a_2, \quad (66)$$

$$v_1 = \frac{(g + i Q_e)}{c_\beta^2} a_2, \quad (67)$$

where  $a_2$  is an arbitrary constant.

### E. Fourth Solution

Suppose that

$$\bar{Y}(p) = y_3 p^3 + \mathcal{O}(p^5), \quad (68)$$

$$\bar{N}(p) = n_1 p + \mathcal{O}(p^3), \quad (69)$$

$$\bar{J}(p) = j_4 p^4 + \mathcal{O}(p^6), \quad (70)$$

$$\bar{V}(p) = v_0 + v_2 p^2 + \mathcal{O}(p^4). \quad (71)$$

Equations (10)–(13) give

$$3 y_3 p^2 = (g + \mathrm{i} Q_e) j_4 p^2 + \mathcal{O}(p^4), \quad (72)$$

$$n_1 = g v_0 + \mathcal{O}(p^2), \quad (73)$$

$$4 j_4 p^3 = (g + \mathrm{i} Q_i) n_1 p^3 + \mathcal{O}(p^5), \quad (74)$$

$$2 c_\beta^2 v_2 p = (g + \mathrm{i} Q_e) n_1 p + \mathcal{O}(p^3). \quad (75)$$

It follows that

$$y_3 = \frac{1}{12} g (g + \mathrm{i} Q_e) (g + \mathrm{i} Q_i) a_3, \quad (76)$$

$$n_1 = g a_3, \quad (77)$$

$$j_4 = \frac{1}{4} g (g + \mathrm{i} Q_i) a_3, \quad (78)$$

$$v_2 = \frac{g (g + \mathrm{i} Q_e)}{2 c_\beta^2} a_3, \quad (79)$$

where  $a_3$  is an arbitrary constant.

### F. General Solution

We conclude that, at small values of  $p$ , the most general solution for  $\bar{Y}(p)$  and  $\bar{N}(p)$  takes the form

$$\bar{Y}(p) = (g + \mathrm{i} Q_e) p^{-1} a_{-1} + (g + \mathrm{i} Q_e) a_0 + \mathcal{O}(p), \quad (80)$$

$$\bar{N}(p) = (-\mathrm{i} Q_e) p^{-1} a_{-1} + (-\mathrm{i} Q_e) a_0 + a_2 + \mathcal{O}(p). \quad (81)$$

### III. MATRIX DIFFERENTIAL EQUATION

Let

$$\underline{u} = \begin{pmatrix} \bar{Y} \\ \bar{N} \end{pmatrix}, \quad (82)$$

$$\underline{v} = \begin{pmatrix} \bar{J} \\ c_\beta^2 \bar{V} \end{pmatrix}. \quad (83)$$

Equations (10)–(13) can be written in the form

$$\frac{d\underline{u}}{dp} = \underline{\underline{A}} \underline{v}, \quad (84)$$

$$\frac{d\underline{v}}{dp} = \underline{\underline{B}} \underline{u}, \quad (85)$$

where

$$A_{11} = \frac{g + i Q_e + p^2}{p^2}, \quad (86)$$

$$A_{12} = 0, \quad (87)$$

$$A_{21} = \frac{-i Q_e}{p^2}, \quad (88)$$

$$A_{22} = \frac{g + P_\varphi p^2}{c_\beta^2}, \quad (89)$$

$$B_{11} = (g + i Q_i) p^2 + P_\varphi p^4, \quad (90)$$

$$B_{12} = (g + i Q_i) p^2 + \iota_e^{-1} P_\varphi p^4, \quad (91)$$

$$B_{21} = i Q_e + D^2 (g + i Q_i) p^2 + D^2 P_\varphi p^4, \quad (92)$$

$$B_{22} = g + i Q_e + [P_\perp + D^2 (g + i Q_i)] p^2 + \iota_e^{-1} D^2 P_\varphi p^4. \quad (93)$$

Thus, we obtain the following matrix differential equation:

$$\frac{d}{dp} \left( \underline{\underline{A}}^{-1} \frac{d\underline{u}}{dp} \right) = \underline{\underline{B}} \underline{u}. \quad (94)$$

### IV. RICATTI MATRIX DIFFERENTIAL EQUATION

Let

$$p \frac{d\underline{u}}{dp} = \underline{\underline{W}} \underline{u}. \quad (95)$$

The previous equation can be combined with Eq. (94) to give

$$\left( p \frac{d\underline{\underline{W}}}{dp} - \underline{\underline{W}} + \underline{\underline{W}}\underline{\underline{W}} + \underline{\underline{A}}p \frac{d\underline{\underline{A}}^{-1}}{dp} \underline{\underline{W}} - p^2 \underline{\underline{A}}\underline{\underline{B}} \right) \underline{\underline{u}} = \underline{\underline{0}}, \quad (96)$$

which implies that

$$p \frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}}\underline{\underline{W}} - \underline{\underline{A}}p \frac{d\underline{\underline{A}}^{-1}}{dp} \underline{\underline{W}} + p^2 \underline{\underline{A}}\underline{\underline{B}}. \quad (97)$$

Now,

$$\underline{\underline{A}}^{-1} = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}, \quad (98)$$

where

$$C_{11} = \frac{p^2}{g + i Q_e + p^2}, \quad (99)$$

$$C_{21} = \frac{i c_\beta^2 Q_e}{(g + i Q_e + p^2)(g + P_\varphi p^2)}, \quad (100)$$

$$C_{22} = \frac{c_\beta^2}{g + P_\varphi p^2}. \quad (101)$$

So, if

$$p \frac{d\underline{\underline{A}}^{-1}}{dp} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix} \quad (102)$$

then

$$D_{11} = \frac{2 p^2 (g + i Q_e)}{(g + i Q_e + p^2)^2}, \quad (103)$$

$$D_{21} = -\frac{2 i c_\beta^2 Q_e p^2 [g + P_\varphi (g + i Q_e) + 2 P_\varphi p^2]}{(g + i Q_e + p^2)^2 (g + P_\varphi p^2)^2}, \quad (104)$$

$$D_{22} = -\frac{2 c_\beta^2 P_\varphi p^2}{(g + P_\varphi p^2)^2}. \quad (105)$$

Furthermore, if

$$\underline{\underline{A}}p \frac{d\underline{\underline{A}}^{-1}}{dp} = \begin{pmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{pmatrix} \quad (106)$$

then

$$E_{11} = \frac{2 (g + i Q_e)}{g + i Q_e + p^2}, \quad (107)$$

$$E_{21} = -\frac{2 i Q_e (g + 2 P_\varphi p^2)}{(g + i Q_e + p^2)(g + P_\varphi p^2)}, \quad (108)$$



$$E_{22} = -\frac{2P_\varphi p^2}{g + P_\varphi p^2}. \quad (109)$$

Finally, if

$$p^2 \underline{\underline{A}} \underline{\underline{B}} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad (110)$$

then

$$F_{11} = p^2 (g + iQ_e + p^2) (g + iQ_i + P_\varphi p^2), \quad (111)$$

$$F_{12} = p^2 (g + iQ_e + p^2) (g + iQ_i + \iota_e^{-1} P_\varphi p^2), \quad (112)$$

$$F_{21} = -iQ_e p^2 (g + iQ_i + P_\varphi p^2) + c_\beta^{-2} p^2 (g + P_\varphi p^2) [iQ_e + D^2 (g + iQ_i) p^2 + D^2 P_\varphi p^4], \quad (113)$$

$$F_{22} = -iQ_e p^2 (g + iQ_i + \iota_e^{-1} P_\varphi p^2) + c_\beta^{-2} p^2 (g + P_\varphi p^2) [g + iQ_e + [P_\perp + D^2 (g + iQ_i)] p^2 + \iota_e^{-1} D^2 P_\varphi p^4]. \quad (114)$$

Hence, Eq. (97) can be written as the following Riccati matrix differential equation:

$$p \frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}} \underline{\underline{W}} - \underline{\underline{E}} \underline{\underline{W}} + \underline{\underline{F}}. \quad (115)$$

Furthermore, if

$$\underline{\underline{W}} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad (116)$$

then

$$p \frac{dW_{11}}{dp} = W_{11} - W_{11} W_{11} - W_{12} W_{21} - E_{11} W_{11} + F_{11}, \quad (117)$$

$$p \frac{dW_{12}}{dp} = W_{12} - W_{11} W_{12} - W_{12} W_{22} - E_{11} W_{12} + F_{12}, \quad (118)$$

$$p \frac{dW_{21}}{dp} = W_{21} - W_{21} W_{11} - W_{22} W_{21} - E_{21} W_{11} - E_{22} W_{21} + F_{21}, \quad (119)$$

$$p \frac{dW_{22}}{dp} = W_{22} - W_{21} W_{12} - W_{22} W_{22} - E_{21} W_{12} - E_{22} W_{22} + F_{22}. \quad (120)$$

## V. SMALL- $p$ BEHAVIOR OF RICCATI MATRIX DIFFERENTIAL EQUATION

It follows from Eqs. (106)–(109) that  $\underline{\underline{E}} = \underline{\underline{E}}^{(0)} + \mathcal{O}(p^2)$  at small values of  $p$ , where

$$E_{11}^{(0)} = 2, \quad (121)$$

$$E_{12}^{(0)} = 0, \quad (122)$$

$$E_{21}^{(0)} = -\frac{2iQ_e}{g + iQ_e}, \quad (123)$$

$$E_{22}^{(0)} = 0. \quad (124)$$

Likewise, Eqs. (111)–(114) imply that  $\underline{\underline{F}} = \mathcal{O}(p^2)$ .

Suppose that  $\underline{\underline{W}} = \underline{\underline{W}}^{(0)} + p\underline{\underline{W}}^{(1)}$  at small values of  $p$ , where the elements of  $\underline{\underline{W}}^{(0)}$  and  $\underline{\underline{W}}^{(1)}$  are independent of  $p$ . Equation (115) gives

$$\underline{\underline{0}} = \underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)} \underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)} \underline{\underline{W}}^{(0)}, \quad (125)$$

$$\underline{\underline{0}} = -\underline{\underline{W}}^{(1)} \underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)} \underline{\underline{W}}^{(1)} - \underline{\underline{E}}^{(0)} \underline{\underline{W}}^{(1)}. \quad (126)$$

Suitable solutions are

$$\underline{\underline{W}}^{(0)} = \begin{pmatrix} -1, & 0 \\ -E_{21}^{(0)}/2, & 0 \end{pmatrix}, \quad (127)$$

$$\underline{\underline{W}}_{12}^{(1)} = 0, \quad (128)$$

$$\underline{\underline{W}}_{21}^{(1)} = -\frac{E_{21}^{(0)}}{2} \left[ \underline{\underline{W}}_{11}^{(1)} - \underline{\underline{W}}_{22}^{(1)} \right]. \quad (129)$$

At small values of  $p$ , let

$$\underline{\underline{u}}(p) = \underline{\underline{u}}_{-1} p^{-1} + \underline{\underline{u}}_0, \quad (130)$$

where the elements of  $\underline{\underline{u}}_{-1}$  (which are  $y_{-1}$  and  $n_{-1}$ , respectively) and the elements of  $\underline{\underline{u}}_0$  (which are  $y_0$  and  $n_0$ , respectively) are all constants. Equation (95) gives

$$\underline{\underline{W}}^{(0)} \underline{\underline{u}}_{-1} = -\underline{\underline{u}}_{-1}, \quad (131)$$

$$\underline{\underline{W}}^{(0)} \underline{\underline{u}}_0 + \underline{\underline{W}}^{(1)} \underline{\underline{u}}_{-1} = \underline{\underline{0}}. \quad (132)$$

Thus, making use of Eq. (127), we get

$$\begin{pmatrix} -1, & 0 \\ -E_{21}^{(0)}/2, & 0 \end{pmatrix} \begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix} = - \begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix}, \quad (133)$$

which implies that

$$\frac{E_{21}^{(0)}}{2} y_{-1} = -\frac{iQ_e}{g + iQ_e} y_{-1} = n_{-1}, \quad (134)$$

in accordance with Eqs. (29) and (31), where use has been made of Eq. (123). Thus, if we write

$$y_{-1} = (g + i Q_e) a_{-1}, \quad (135)$$

$$n_{-1} = -i Q_e a_{-1}, \quad (136)$$

$$y_0 = (g + i Q_e) a_0, \quad (137)$$

$$n_0 = -i Q_e a_0 + a_2, \quad (138)$$

in accordance with Eqs. (80) and (81), then we deduce from Eqs. (127), (129), and (132) that

$$\frac{\pi}{\hat{\Delta}_s} \equiv \frac{a_0}{a_{-1}} = W_{11}^{(1)}. \quad (139)$$

## VI. LARGE- $p$ BEHAVIOR OF RICCATI MATRIX DIFFERENTIAL EQUATION

At large values of  $p$ , it is clear from Eqs. (111)–(114) that  $\underline{\underline{F}}(p) = p^6 \underline{\underline{F}}^{(6)} + p^8 \underline{\underline{F}}^{(8)}$ , where the elements of  $\underline{\underline{F}}^{(6)}$  and  $\underline{\underline{F}}^{(8)}$  are constants. On the other hand, Eqs. (106)–(109) imply that  $\underline{\underline{E}}(p) = \underline{\underline{E}}^{(0)}$ , where the elements of  $\underline{\underline{E}}^{(0)}$  are constants. Thus, if we write  $\underline{\underline{W}}(p) = p^2 \underline{\underline{W}}^{(2)} + p^4 \underline{\underline{W}}^{(4)}$ , where the elements of  $\underline{\underline{W}}^{(2)}$  and  $\underline{\underline{W}}^{(4)}$  are constants, then Eq. (115) gives

$$\underline{\underline{W}}^{(4)} \underline{\underline{W}}^{(4)} = \underline{\underline{F}}^{(8)}, \quad (140)$$

$$\underline{\underline{W}}^{(2)} \underline{\underline{W}}^{(4)} + \underline{\underline{W}}^{(4)} \underline{\underline{W}}^{(2)} = \underline{\underline{F}}^{(6)}. \quad (141)$$

Now, according to Eqs. (111)–(114),

$$F_{11}^{(8)} = 0, \quad (142)$$

$$F_{12}^{(8)} = 0, \quad (143)$$

$$F_{21}^{(8)} = c_\beta^{-2} D^2 P_\varphi^2, \quad (144)$$

$$F_{22}^{(8)} = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2, \quad (145)$$

so Eq. (140) yields

$$W_{11}^{(4)} = 0, \quad (146)$$

$$W_{12}^{(4)} = 0, \quad (147)$$

$$W_{21}^{(4)} = -c_\beta^{-1} \iota_e^{1/2} D P_\varphi, \quad (148)$$

$$W_{22}^{(4)} = -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi, \quad (149)$$

where we have chosen the sign of the square root that is associated with well-behaved solutions at large values of  $p$ . Here, we are assuming that  $\iota_e > 0$ . Equations (111)–(114) also give

$$F_{11}^{(6)} = P_\varphi, \quad (150)$$

$$F_{12}^{(6)} = \iota_e^{-1} P_\varphi, \quad (151)$$

$$F_{21}^{(6)} = c_\beta^{-2} D^2 g P_\varphi + c_\beta^{-2} D^2 (g + i Q_i) P_\varphi, \quad (152)$$

$$F_{22}^{(6)} = c_\beta^{-2} \iota_e^{-1} D^2 g P_\varphi + c_\beta^{-2} [P_\perp + D^2 (g + i Q_i)] P_\varphi. \quad (153)$$

Thus, Eq. (141) yields

$$W_{12}^{(2)} W_{21}^{(4)} = F_{11}^{(6)}, \quad (154)$$

$$W_{12}^{(2)} W_{22}^{(4)} = F_{12}^{(6)}, \quad (155)$$

which gives

$$W_{12}^{(2)} = -c_\beta \iota_e^{-1/2} D^{-1}. \quad (156)$$

Now, if

$$\underline{\underline{W}} \underline{u} = \lambda(p) \underline{u} \quad (157)$$

then Eq. (95) yields

$$p \frac{d\underline{u}}{dp} = \lambda \underline{u}, \quad (158)$$

which implies that

$$\underline{u}(p) = \underline{u}(p_0) \exp \left[ \int_{p_0}^p \frac{\lambda_r(p')}{p'} dp' \right] \exp \left[ i \int_{p_0}^p \frac{\lambda_i(p')}{p'} dp' \right], \quad (159)$$

where  $\lambda_r$  and  $\lambda_i$  are the real and imaginary parts of  $\lambda$ , respectively. Of course, a solution that is well behaved at large values of  $p$  is such that  $\lambda_r$  is negative. As we have seen, the large- $p$  limit of Eq. (115) is

$$\underline{\underline{W}} \underline{\underline{W}} = \underline{\underline{F}}. \quad (160)$$

Hence, if

$$\underline{\underline{F}} \underline{u} = \Lambda \underline{u} \quad (161)$$

then Eqs. (157) and (161) imply that

$$\lambda^2 = \Lambda. \quad (162)$$

The eigenvalue problem for the  $F$ -matrix reduces to

$$\Lambda^2 - (F_{11} + F_{22}) \Lambda + F_{11} F_{22} - F_{12} F_{21} = 0. \quad (163)$$

Now,

$$F_{11} + F_{22} \simeq F_{22}^{(8)} p^8 = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2 p^8, \quad (164)$$

$$\begin{aligned} F_{11} F_{22} - F_{12} F_{21} &\simeq \left[ F_{11}^{(6)} F_{22}^{(8)} - F_{12}^{(6)} F_{21}^{(8)} \right] p^{14} \\ &\quad + \left[ F_{11}^{(6)} F_{22}^{(6)} - F_{12}^{(6)} F_{21}^{(6)} \right] p^{12} = c_\beta^{-2} R P_\varphi^2 p^{12}, \end{aligned} \quad (165)$$

where

$$R = P_\perp + (1 - \iota_e^{-1}) D^2 (g + i Q_i), \quad (166)$$

Hence, the two eigenvalues of the  $F$ -matrix are

$$\Lambda_1 \simeq F_{22}^{(8)} p^8 = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2 p^8, \quad (167)$$

$$\Lambda_2 \simeq \frac{[F_{11}^{(6)} F_{22}^{(6)} - F_{12}^{(6)} F_{21}^{(6)}]}{F_{22}^{(8)}} p^4 = \iota_e D^{-2} R p^4. \quad (168)$$

Thus, we deduce that the two eigenvalues of the  $W$ -matrix are

$$\lambda_1 = -\Lambda_1^{1/2} = -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi p^4, \quad (169)$$

$$\lambda_2 = -\Lambda_2^{1/2} = -\iota_e^{1/2} D^{-1} R^{1/2} p^2, \quad (170)$$

Here, the square root of  $R$  is taken such that the real part of  $\lambda_2$  is negative. Now, the eigenvalue problem for the  $W$ -matrix reduces to

$$\lambda^2 - W_{22}^{(4)} p^4 \lambda + \left[ W_{11}^{(2)} W_{22}^{(4)} - W_{12}^{(2)} W_{21}^{(4)} \right] p^6 = 0. \quad (171)$$

which yields

$$\lambda_1 \simeq W_{22}^{(4)} p^4, \quad (172)$$

which is satisfied, and

$$\lambda_2 \simeq \left[ W_{11}^{(2)} - \frac{W_{12}^{(2)} W_{21}^{(4)}}{W_{22}^{(4)}} \right] p^2, \quad (173)$$

which implies that

$$W_{11}^{(2)} = -\iota_e^{1/2} D^{-1} R^{1/2} - c_\beta \iota_e^{1/2} D^{-1}. \quad (174)$$

Hence, the large- $p$  boundary condition for the  $W$ -matrix is

$$\underline{\underline{W}}(p) = \begin{pmatrix} -\iota_e^{1/2} D^{-1} R^{1/2} p^2 - c_\beta \iota_e^{1/2} D^{-1} p^2, & -c_\beta \iota_e^{-1/2} D^{-1} p^2 \\ -c_\beta^{-1} \iota_e^{1/2} D P_\varphi p^4, & -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi p^4 \end{pmatrix}. \quad (175)$$

## VII. DETERMINATION OF PMAX

At large values of  $p$ , we have

$$F_{11}^{(4)} = g + \mathrm{i} Q_i + P_\varphi (g + \mathrm{i} Q_e), \quad (176)$$

$$F_{12}^{(4)} = g + \mathrm{i} Q_i + \iota_e^{-1} P_\varphi (g + \mathrm{i} Q_e), \quad (177)$$

$$F_{21}^{(4)} = -\mathrm{i} Q_e P_\varphi + c_\beta^{-2} [g D^2 (g + \mathrm{i} Q_i) + P_\varphi \mathrm{i} Q_e], \quad (178)$$

$$F_{22}^{(4)} = -\mathrm{i} Q_e \iota_e^{-1} P_\varphi + c_\beta^{-2} \{g [P_\perp + D^2 (g + \mathrm{i} Q_i)] + P_\varphi (g + \mathrm{i} Q_e)\}. \quad (179)$$

We require

$$p \gg \left| \frac{F_{21}^{(6)}}{F_{21}^{(8)}} \right|^{1/2}, \left| \frac{F_{22}^{(6)}}{F_{22}^{(8)}} \right|^{1/2}, \left| \frac{F_{11}^{(4)}}{F_{11}^{(6)}} \right|^{1/2}, \left| \frac{F_{12}^{(4)}}{F_{12}^{(6)}} \right|^{1/2}, \left| \frac{F_{21}^{(4)}}{F_{21}^{(6)}} \right|^{1/2}, \left| \frac{F_{22}^{(4)}}{F_{22}^{(6)}} \right|^{1/2}. \quad (180)$$