

Pressure Flattening due to Magnetic Island

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I. MAGNETIC ISLAND

Let $x = r - r_s$, $X = x/W$, and $\zeta = m\theta - n\phi$, where W is the island width. The magnetic flux-surfaces of the magnetic island are contours of

$$\Omega(X, \zeta) = 8X^2 + \cos \zeta. \quad (1)$$

The X-points lie at $X = 0$ and $\zeta = 0, 2\pi$, whereas the O-point lies at $X = 0$ and $\zeta = \pi$. The O-point corresponds to $\Omega = -1$, whereas the magnetic separatrix corresponds to $\Omega = 1$. Note that $\Omega \simeq 8X^2$ in the limit $|X| \gg 1$.

II. TEMPERATURE PERTURBATION IN INNER REGION

Let $T_{e0}(r)$ be the unperturbed electron temperature profile. Let

$$T_e(X, \zeta) = T_{es} + \text{sgn}(X) W T'_{es} \tilde{T}(\Omega) \quad (2)$$

be the temperature profile in the presence of the island, where $T_{es} = T_{e0}(r_s)$, and $T'_{es} = (dT_{e0}/dr)_{r=r_s}$. The normalized perturbed electron temperature profile, $\tilde{T}(\Omega)$, satisfies the energy conservation equation

$$\frac{d}{d\Omega} \left[\oint (\Omega - \cos \zeta)^{1/2} \frac{d\zeta}{2\pi} \frac{d\tilde{T}}{d\Omega} \right] = 0, \quad (3)$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \rightarrow |X| \quad (4)$$

as $|X| \rightarrow \infty$. Note that $T(X, \zeta) - T_{es}$ is an odd function of X .

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Equations (2) and (3) imply that

$$\tilde{T}(\Omega) = 0 \quad (5)$$

for $-1 \leq \Omega < 1$, and

$$\frac{d\tilde{T}}{d\Omega} = \frac{c}{\oint (\Omega - \cos \zeta)^{1/2} d\zeta / 2\pi} \quad (6)$$

for $\Omega \geq 1$, where c is a constant. Let

$$k = \left(\frac{1 + \Omega}{2} \right)^{1/2}. \quad (7)$$

The island O-point corresponds to $k = 0$, whereas the magnetic separatrix corresponds to $k = 1$. Note that $k \rightarrow 2|X|$ as $|X| \rightarrow \infty$.

Equation (6) yields

$$\frac{d\tilde{T}}{dk} = \frac{\sqrt{2} \pi c}{E(1/k)}, \quad (8)$$

where

$$E(p) \equiv \int_0^{\pi/2} (1 - p^2 \sin^2 \theta)^{1/2} d\theta \quad (9)$$

is a complete elliptic integral. The boundary condition (4) implies that

$$c = \frac{1}{4\sqrt{2}}. \quad (10)$$

Hence, we conclude that

$$\frac{d\tilde{T}}{dk} = \frac{\pi}{4} \frac{1}{E(1/k)} \quad (11)$$

for $k \geq 1$. Thus,

$$\tilde{T}(k) = 0 \quad (12)$$

for $0 \leq k < 1$, and

$$\tilde{T}(k) = F(k) \quad (13)$$

for $k \geq 1$, where

$$F(k) = \frac{\pi}{4} \int_1^k \frac{dk'}{E(1/k')}. \quad (14)$$

III. HARMONICS OF TEMPERATURE PERTURBATION

We can write

$$\tilde{T}(|X|, \zeta) = \sum_{\nu=0, \infty} \delta T_\nu(|X|) \cos(\nu \zeta). \quad (15)$$

Now,

$$\delta T_0(|X|) = \oint \tilde{T}(|X|, \zeta) \frac{d\zeta}{2\pi}, \quad (16)$$

where the integral is at constant $|X|$. It follows that

$$\delta T_0(|X|) = \int_0^{\zeta_c} F(k) \frac{d\zeta}{\pi}, \quad (17)$$

where

$$\zeta_c = \cos^{-1}(1 - 8X^2) \quad (18)$$

for $|X| < 1/2$, and $\zeta_c = \pi$ for $|X| \geq 1/2$. Furthermore,

$$k = \left[4|X|^2 + \cos^2\left(\frac{\zeta}{2}\right) \right]^{1/2}. \quad (19)$$

For $\nu > 0$, we have

$$\delta T_\nu(|X|) = 2 \oint \tilde{T}(|X|, \zeta) \cos(\nu \zeta) \frac{d\zeta}{2\pi}, \quad (20)$$

where the integral is at constant $|X|$. Integrating by parts, we obtain

$$\delta T_\nu(|X|) = -\frac{2}{\nu} \oint \frac{\partial \tilde{T}}{\partial \zeta} \sin(\nu \zeta) \frac{d\zeta}{2\pi}. \quad (21)$$

But,

$$\frac{\partial \tilde{T}}{\partial \zeta} = \frac{d\tilde{T}}{dk} \frac{\partial k}{\partial \zeta} = -\frac{d\tilde{T}}{dk} \frac{\sin \zeta}{4k} = -\frac{\pi}{16} \frac{\sin \zeta}{k E(1/k)}, \quad (22)$$

so

$$\delta T_\nu(X) = \frac{1}{16\nu} \int_0^{\zeta_c} \frac{\cos[(\nu - 1)\zeta] - \cos[(\nu + 1)\zeta]}{k E(1/k)} d\zeta. \quad (23)$$

IV. ASYMPTOTIC BEHAVIOR

In the limit $|X| \ll 1$, we have

$$\zeta_c \simeq 4|X|, \quad (24)$$

$$k \simeq 1 + \frac{\zeta_c^2 - \zeta^2}{8}, \quad (25)$$

$$E(1/k) \simeq 1, \quad (26)$$

$$F(k) \simeq \frac{\pi}{4} (k - 1). \quad (27)$$

It follows that

$$\delta T_0(|X|) \simeq \frac{4}{3} |X|^3, \quad (28)$$

$$\delta T_{\nu>0}(|X|) \simeq \frac{8}{3} |X|^3 \quad (29)$$

In the limit $|x|/W \gg 1$, we have

$$k \simeq 2 |X|, \quad (30)$$

$$E(1/k) \simeq \frac{\pi}{2}. \quad (31)$$

It follows that

$$F(k) \simeq \frac{k}{2} - F_\infty, \quad (32)$$

$$\delta T_0(|X|) \simeq |X| - F_\infty, \quad (33)$$

$$\delta T_1(|X|) \simeq \frac{1}{16 |X|}, \quad (34)$$

$$\delta T_{\nu>1}(|X|) \sim \mathcal{O}\left(\frac{1}{|X|^3}\right), \quad (35)$$

where

$$F_\infty = 0.3447. \quad (36)$$

V. ASYMPTOTIC MATCHING

Consider the k th rational surface whose radius is r_k and whose resonant poloidal mode number is m_k . Let $x = r - r_k$ and $\zeta_k = m_k \theta - n \phi$.

In the outer region, we write the total electron temperature as

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) - \Psi_k \frac{q(r)}{r g(r)} \frac{T'_{e0}(r) \psi_{m_k}(r)}{m_k - n q(r)} e^{i\zeta_k}, \quad (37)$$

where $T'_{e0} = dT_{e0}/dr$, $T_{e0}(r)$ is the equilibrium electron temperature profile, Ψ_k is the reconnected flux, and

$$W_k = 4 \left(\frac{q}{g s} \right)_{r_k}^{1/2} \Psi_k^{1/2} \quad (38)$$

is the island width. In the limit, $|x| \ll 1$, Eq. (37) yield

$$\tilde{T}_e(x, \theta, \phi) = T_{ek} + T'_{ek} x + \frac{T'_{ek} W_k^2}{16 x} e^{i\zeta_k}, \quad (39)$$

Here, $T_{ek} = T_{e0}(r_k)$ and $T'_{ek} = (dT_{e0}/dr)_{r_k}$, and we have made use of the fact that $\psi_{m_k}(r_k) \simeq m_k$.

In the inner region, we write the total electron temperature as

$$\tilde{T}_e(x, \theta, \phi) = T_{ek} + \text{sgn}(x) T'_{ek} W_k \sum_{\nu=0, \infty} \delta T_\nu(|x|/W_k) e^{i\nu\zeta_k} + T'_{ek} W_k F_\infty, \quad (40)$$

In the limit $x \gg W_k$, the previous equation yields

$$\tilde{T}_e(x, \theta, \phi) \simeq T_{ek} + T'_{ek} x + \frac{T'_{ek} W_k^2}{16x} e^{i\zeta_k}, \quad (41)$$

On the other hand, in the limit $x \ll -W_k$, we get

$$\tilde{T}_e(x, \theta, \phi) \simeq T_{ek} + T'_{ek} x + 2T'_{ek} W_k F_\infty + \frac{T'_{ek} W_k^2}{16x} e^{i\zeta_k}. \quad (42)$$

The asymptotic matching process consists of writing

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e+} - \Psi_{k+} \frac{q(r)}{r g(r)} \frac{T'_{e0}(r) \psi_{m_k}(r)}{m_k - n q(r)} e^{i\zeta_k} \quad (43)$$

in the region $r > r_k + W_k$,

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e-} - \Psi_{k-} \frac{q(r)}{r g(r)} \frac{T'_{e0}(r) \psi_{m_k}(r)}{m_k - n q(r)} e^{i\zeta_k} \quad (44)$$

in the region $r < r_k - W_k$, and

$$\tilde{T}_e(r, \theta, \phi) = T_{ek} + \text{sgn}(x) T'_{ek} W_k \sum_{\nu=0, \infty} \delta T_\nu(|x|/W_k) e^{i\nu\zeta_k} + T'_{ek} W_k F_\infty \quad (45)$$

in the region $r_k - W_k \leq r \leq r_k + W_k$. Continuity of the solution at $r = r_k \pm W_k$ implies that

$$\delta T_{e+} = T'_{ek} W_k \delta T_{e0}(1) + T'_{ek} W_k F_\infty - T'_{ek} W_k, \quad (46)$$

$$\delta T_{e-} = -T'_{ek} W_k \delta T_{e0}(1) + T'_{ek} W_k F_\infty + T'_{ek} W_k, \quad (47)$$

$$\Psi_{k+} = -T'_{ek} W_k \delta T_1(1) \left(\frac{r g}{q} \frac{m_k - n q}{T'_{e0} \psi_{m_k}} \right)_{r_k + W_k}, \quad (48)$$

$$\Psi_{k-} = T'_{ek} W_k \delta T_1(1) \left(\frac{r g}{q} \frac{m_k - n q}{T'_{e0} \psi_{m_k}} \right)_{r_k - W_k}. \quad (49)$$

Finally, for the special case $m = 1$, we write

$$\tilde{T}_e(r, \theta, \phi) = -\xi^r(r, \theta, \phi) \frac{dT_{e0}}{dr}. \quad (50)$$

A. Normalized Quantities

Let $\hat{r} = r/\epsilon_a$, $\hat{r}_k = r_k/\epsilon_a$, $\hat{x} = x/\epsilon_a$, $\hat{T}'_{e0} = \epsilon_a T'_{e0}$, $\hat{T}'_{ek} = \epsilon_a \hat{T}'_{ek}$, $\hat{W}_k = W_k/\epsilon_a$, and $\hat{\Psi}_k = \Psi_k/\epsilon_a^2$, etc., then

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e+} - \hat{\Psi}_{k+} \frac{q(\hat{r})}{\hat{r} g(\hat{r})} \frac{\hat{T}'_e(r) \psi_{m_k}(r)}{m_k - n q(\hat{r})} e^{i\zeta_k} \quad (51)$$

in the region $\hat{r} > \hat{r}_k + \hat{W}_k$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e-} - \hat{\Psi}_{k-} \frac{q(\hat{r})}{\hat{r} g(\hat{r})} \frac{\hat{T}'_e(r) \psi_{m_k}(r)}{m_k - n q(\hat{r})} e^{i\zeta_k} \quad (52)$$

in the region $\hat{r} < \hat{r}_k - \hat{W}_k$, and

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{ek} + \text{sgn}(\hat{x}) \hat{T}'_{ek} \hat{W}_k \sum_{\nu=0,\infty} \delta T_\nu(|\hat{x}|/\hat{W}_k) e^{i\nu\zeta_k} + \hat{T}'_{ek} \hat{W}_k F_\infty \quad (53)$$

in the region $\hat{r}_k - \hat{W}_k \leq \hat{r} \leq \hat{r}_k + \hat{W}_k$. Here,

$$\delta T_{e+} = \hat{T}'_{ek} \hat{W}_k \delta T_0(1) + \hat{T}'_{ek} \hat{W}_k F_\infty - \hat{T}'_{ek} \hat{W}_k, \quad (54)$$

$$\delta T_{e-} = -\hat{T}'_{ek} \hat{W}_k \delta T_0(1) + \hat{T}'_{ek} \hat{W}_k F_\infty + \hat{T}'_{ek} \hat{W}_k, \quad (55)$$

$$\hat{\Psi}_k = \left(\frac{\hat{W}_k}{4} \right)^2 \left(\frac{g s}{q} \right)_{\hat{r}_k}, \quad (56)$$

$$\hat{\Psi}_{k+} = -\hat{T}'_{ek} \hat{W}_k \delta T_1(1) \left(\frac{\hat{r} g}{q} \frac{m_k - n q}{\hat{T}'_{e0} \psi_{m_k}} \right)_{\hat{r}_k + \hat{W}_k}, \quad (57)$$

$$\hat{\Psi}_{k-} = \hat{T}'_{ek} \hat{W}_k \delta T_1(1) \left(\frac{\hat{r} g}{q} \frac{m_k - n q}{\hat{T}'_{e0} \psi_{m_k}} \right)_{\hat{r}_k - \hat{W}_k}. \quad (58)$$

For the special case $m = 1$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = -\frac{\xi^r(\hat{r}, \theta, \phi)}{\epsilon_a} \frac{dT_{e0}}{d\hat{r}}. \quad (59)$$