Pressure Flattening due to Magnetic Island

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I. MAGNETIC ISLAND

Let $x = r - r_s$, $X = x/W$, and $\zeta = m\,\theta - n\,\phi$, where W is the island width. The magnetic flux-surfaces of the magnetic island are contours of

$$\Omega(X,\zeta) = 8\,X^2 + \cos\zeta. \tag{1}$$

The X-points lie at $X = 0$ and $\zeta = 0,\,2\pi$, whereas the X-point lies at $X = 0$ and $\zeta = \pi$. The O-point corresponds to $\Omega = -1$, whereas the magnetic separatrix corresponds to $\Omega = 1$. Note that $\Omega \simeq 8\,X^2$ in the limit $|X| \gg 1$.

II. TEMPERATURE PERTURBATION IN INNER REGION

Let $T_0(r)$ be the unperturbed temperature profile. Let

$$T(X,\zeta) = T_s + \mathrm{sgn}(X)\,W\,T_s'\,\tilde{T}(\Omega) \tag{2}$$

be the temperature profile in the presence of the island, where $T_s = T_0(r_s)$, and $T_s' = (dT_0/dr)_{r=r_s}$. The normalized perturbed temperature profile, $\tilde{T}(\Omega)$, satisfies the energy conservation equation

$$\frac{d}{d\Omega}\left[\oint (\Omega - \cos\zeta)^{1/2}\,\frac{d\zeta}{2\pi}\,\frac{d\tilde{T}}{d\Omega}\right] = 0, \tag{3}$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \rightarrow |X| \tag{4}$$

as $|X| \rightarrow \infty$. Note that $T(X,\zeta) - T_s$ is an odd function of X.

Equations (2) and (3) imply that

$$\tilde{T}(\Omega) = 0 \tag{5}$$

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for $-1 \leq \Omega < 1$, and

$$\frac{d\tilde{T}}{d\Omega} = \frac{c}{\oint(\Omega - \cos\zeta)^{1/2}\,d\zeta/2\pi} \tag{6}$$

for $\Omega \geq 1$, where c is a constant. Let

$$k = \left(\frac{1+\Omega}{2}\right)^{1/2}. \tag{7}$$

The island O-point corresponds to $k = 0$, whereas the magnetic separatrix corresponds to $k = 1$. Note that $k \to 2\,|X|$ as $|X| \to \infty$.

Equation (6) yields

$$\frac{d\tilde{T}}{dk} = \frac{\sqrt{2}\,\pi\,c}{E(1/k)}, \tag{8}$$

where

$$E(p) \equiv \int_0^{\pi/2} (1 - p^2\,\sin^2\theta)^{1/2}\,d\theta \tag{9}$$

is a complete elliptic integral. The boundary condition (4) implies that

$$c = \frac{1}{4\sqrt{2}}. \tag{10}$$

Hence, we conclude that

$$\frac{d\tilde{T}}{dk} = \frac{\pi}{4}\,\frac{1}{E(1/k)} \tag{11}$$

for $k \geq 1$. Thus,

$$\tilde{T}(k) = 0 \tag{12}$$

for $0 \leq k < 1$, and

$$\tilde{T}(k) = F(k) \tag{13}$$

for $k \geq 1$, where

$$F(k) = \frac{\pi}{4}\int_1^k \frac{dk'}{E(1/k')}. \tag{14}$$

III. HARMONICS OF TEMPERATURE PERTURBATION

We can write

$$\tilde{T}(|X|,\zeta) = \sum_{\nu=0,\infty} \delta T_\nu(|X|)\,\cos(\nu\,\zeta). \tag{15}$$

Now,

$$\delta T_0(|X|) = \oint \tilde{T}(|X|,\zeta)\,\frac{d\zeta}{2\pi}, \tag{16}$$

where the integral is at constant $|X|$. It follows that

$$\delta T_0(|X|) = \int_0^{\zeta_c} F(k)\,\frac{d\zeta}{\pi}, \tag{17}$$

where

$$\zeta_c = \cos^{-1}\left(1 - 8\,X^2\right) \tag{18}$$

for $|X| < 1/2$, and $\zeta_c = \pi$ for $|X| \geq 1/2$. Furthermore,

$$k = \left[4\,|X|^2 + \cos^2\left(\frac{\zeta}{2}\right)\right]^{1/2}. \tag{19}$$

For $\nu > 0$, we have

$$\delta T_\nu(|X|) = 2\oint \tilde{T}(|X|,\zeta)\,\cos(\nu\,\zeta)\,\frac{d\zeta}{2\pi}, \tag{20}$$

where the integral is at constant $|X|$. Integrating by parts, we obtain

$$\delta T_\nu(|X|) = -\frac{2}{\nu}\oint \frac{\partial\tilde{T}}{\partial\zeta}\,\sin(\nu\,\zeta)\,\frac{d\zeta}{2\pi}. \tag{21}$$

But,

$$\frac{\partial\tilde{T}}{\partial\zeta} = \frac{d\tilde{T}}{dk}\,\frac{\partial k}{\partial\zeta} = -\frac{d\tilde{T}}{dk}\,\frac{\sin\zeta}{4\,k} = -\frac{\pi}{16}\,\frac{\sin\zeta}{k\,E(1/k)}, \tag{22}$$

so

$$\delta T_\nu(X) = \frac{1}{16\,\nu}\int_0^{\zeta_c} \frac{\cos[(\nu-1)\,\zeta] - \cos[(\nu+1)\,\zeta]}{k\,E(1/k)}\,d\zeta. \tag{23}$$

IV. ASYMPTOTIC BEHAVIOR

In the limit $|X| \ll 1$, we have

$$\zeta_c \simeq 4\,|X|, \tag{24}$$

$$k \simeq 1 + \frac{\zeta_c^2 - \zeta^2}{8}, \tag{25}$$

$$E(1/k) \simeq 1, \tag{26}$$

$$F(k) \simeq \frac{\pi}{4}\,(k-1). \tag{27}$$

It follows that

$$\delta T_0(|X|) \simeq \frac{4}{3}\,|X|^3, \tag{28}$$

$$\delta T_{\nu>0}(|X|) \simeq \frac{8}{3}\,|X|^3 \tag{29}$$

In the limit $|x|/W \gg 1$, we have

$$k \simeq 2\,|X|, \tag{30}$$

$$E(1/k) \simeq \frac{\pi}{2}. \tag{31}$$

It follows that

$$F(k) \simeq \frac{k}{2} - F_\infty, \tag{32}$$

$$\delta T_0(|X|) \simeq |X| - F_\infty, \tag{33}$$

$$\delta T_1(|X|) \simeq \frac{1}{16\,|X|}, \tag{34}$$

$$\delta T_{\nu>1}(|X|) \sim {\cal O}\left(\frac{1}{|X|^3}\right), \tag{35}$$

where

$$F_\infty = 0.3447. \tag{36}$$

V. ASYMPTOTIC MATCHING

Consider the kth rational surface whose radius is r_k and whose resonant poloidal mode number is m_k. Let $x = r - r_k$ and $\zeta_k = m_k\,\theta - n\,\phi$.

In the outer region, we write the total electron temperature as

$$\tilde{T}_e(r,\theta,\phi) = T_0(r) - \Psi_k\,\frac{q(r)}{r\,g(r)}\,\frac{T_0'(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\,{\rm e}^{\,{\rm i}\,\zeta_k}, \tag{37}$$

where $T_0' = dT_0/dr$, $T_0(r)$ is the equilibrium electron temperature profile, Ψ_k is the reconnected flux, and

$$W_k = 4\left(\frac{q}{g\,s}\right)^{1/2}_{r_k}\Psi_k^{1/2} \tag{38}$$

is the island width. Note that

$$T_0(r) = \frac{B_0^{\,2}}{2\,\mu_0\,e\,n_0}\,\epsilon_a^{\,2}\,p_c\,(1-\hat{r}^{\,2})^{\mu-\alpha} + T_{e\,{\rm ped}}, \tag{39}$$

giving

$$\frac{dT_0}{dr} = \frac{B_0^{\,2}}{\mu_0\,e\,n_0}\,\epsilon_a\,p_c\,(\mu-\alpha)\,\hat{r}\,(1-\hat{r}^{\,2})^{\mu-\alpha-1}. \tag{40}$$

In the limit, $|x| \ll 1$, Eq. (37) yield

$$\tilde{T}_e(x,\theta,\phi) = T_{e\,k} + T_{e\,k}'\,x + \frac{T_{e\,k}'\,W_k^{\,2}}{16\,x}\,{\rm e}^{\,{\rm i}\,\zeta_k}, \tag{41}$$

Here, $T_{e\,k} = T_0(r_k)$ and $T'_{e\,k} = (dT_0/dr)_{r_k}$.

In the inner region, we write the total electron temperature as

$$\tilde{T}_e(x,\theta,\phi) = T_{e\,k} + \mathrm{sgn}(x)\,T'_{e\,k}\,W_k \sum_{\nu=0,\infty} \delta T_\nu(|x|/W_k)\,\mathrm{e}^{\,\mathrm{i}\,\nu\,\zeta_k} + T'_{e\,k}\,W_k\,F_\infty, \tag{42}$$

In the limit $x \gg W_k$, the previous equation yields

$$\tilde{T}_e(x,\theta,\phi) \simeq T_{e\,k} + T'_{e\,k}\,x + \frac{T'_{e\,k}\,W_k^{\,2}}{16\,x}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k}, \tag{43}$$

On the other hand, in the limit $x \ll -W_k$, we get

$$\tilde{T}_e(x,\theta,\phi) \simeq T_{e\,k} + T'_{e\,k}\,x + 2\,T'_{e\,k}\,W_k\,F_\infty + \frac{T'_{e\,k}\,W_k^{\,2}}{16\,x}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k}. \tag{44}$$

The asymptotic matching process consists of writing

$$\tilde{T}_e(r,\theta,\phi) = T_0(r) + \delta T_{e\,+} - \Psi_{k+}\,\frac{q(r)}{r\,g(r)}\,\frac{T'_0(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k} \tag{45}$$

in the region $r > r_k + W_k$,

$$\tilde{T}_e(r,\theta,\phi) = T_0(r) + \delta T_{e\,-} - \Psi_{k-}\,\frac{q(r)}{r\,g(r)}\,\frac{T'_0(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k} \tag{46}$$

in the region $r < r_k - W_k$, and

$$\tilde{T}_e(r,\theta,\phi) = T_{e\,k} + \mathrm{sgn}(x)\,T'_{e\,k}\,W_k \sum_{\nu=0,\infty} \delta T_\nu(|x|/W_k)\,\mathrm{e}^{\,\mathrm{i}\,\nu\,\zeta_k} + T'_{e\,k}\,W_k\,F_\infty \tag{47}$$

in the region $r_k - W_k \leq r \leq r_k + W_k$. Continuity of the solution at $r = r_k \pm W_k$ implies that

$$\delta T_{e\,+} = T_{e\,k} + T'_{e\,k}\,W_k\,\delta T_0(1) + T'_{e\,k}\,W_k\,F_\infty - T_0(r_k + W_k), \tag{48}$$

$$\delta T_{e\,-} = T_{e\,k} - T'_{e\,k}\,W_k\,\delta T_0(1) + T'_{e\,k}\,W_k\,F_\infty - T_0(r_k - W_k), \tag{49}$$

$$\Psi_{k\,+} = -T'_{e\,k}\,W_k\,\delta T_1(1)\left(\frac{r\,g}{q}\,\frac{m_k - n\,q}{T'_0\,\psi_{m\,k}}\right)_{r_k+W_k}, \tag{50}$$

$$\Psi_{k\,-} = T'_{e\,k}\,W_k\,\delta T_1(1)\left(\frac{r\,g}{q}\,\frac{m_k - n\,q}{T'_0\,\psi_{m\,k}}\right)_{r_k-W_k}. \tag{51}$$

Finally, for the special case $m = 1$, we write

$$\tilde{T}(r,\theta,\phi) = -\xi^r(r,\theta,\phi)\,\frac{dT_0}{dr}. \tag{52}$$

A. Normalized Quantities

Let $\hat{r} = r/\epsilon_a$, $\hat{r}_k = r_k/\epsilon_a$, $\hat{x} = x/\epsilon_a$, $\hat{T}_0' = \epsilon_a\, T_0'$, $T_{e\,k}' = \epsilon_a\, T_{e\,k}$, $\hat{W}_k = W_k/\epsilon_a$, and $\hat{\Psi}_k = \Psi_k/\epsilon_a^2$, etc., then

$$\tilde{T}_e(\hat{r},\theta,\phi) = T_0(r) + \delta T_{e\,+} - \hat{\Psi}_{k+}\,\frac{q(\hat{r})}{\hat{r}\,g(\hat{r})}\,\frac{\hat{T}_e'(r)\,\psi_{m_k}(r)}{m_k - n\,q(\hat{r})}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k} \tag{53}$$

in the region $\hat{r} > \hat{r}_k + \hat{W}_k$,

$$\tilde{T}_e(\hat{r},\theta,\phi) = T_0(\hat{r}) + \delta T_{e\,-} - \hat{\Psi}_{k-}\,\frac{q(\hat{r})}{\hat{r}\,g(\hat{r})}\,\frac{\hat{T}_e'(r)\,\psi_{m_k}(r)}{m_k - n\,q(\hat{r})}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k} \tag{54}$$

in the region $\hat{r} < \hat{r}_k - \hat{W}_k$, and

$$\tilde{T}_e(\hat{r},\theta,\phi) = T_{e\,k} + \mathrm{sgn}(\hat{x})\,\hat{T}_{e\,k}'\,\hat{W}_k \sum_{\nu=0,\infty} \delta T_\nu(|\hat{x}|/\hat{W}_k)\,\mathrm{e}^{\,\mathrm{i}\,\nu\,\zeta_k} + \hat{T}_{e\,k}'\,\hat{W}_k\,F_\infty \tag{55}$$

in the region $\hat{r}_k - \hat{W}_k \leq \hat{r} \leq \hat{r}_k + \hat{W}_k$. Here,

$$\delta T_{e\,+} = T_{e\,k} + \hat{T}_{e\,k}'\,\hat{W}_k\,\delta T_0(1) + \hat{T}_{e\,k}'\,\hat{W}_k\,F_\infty - T_0(\hat{r}_k + \hat{W}_k), \tag{56}$$

$$\delta T_{e\,-} = T_{e\,k} - \hat{T}_{e\,k}'\,\hat{W}_k\,\delta T_0(1) + \hat{T}_{e\,k}'\,\hat{W}_k\,F_\infty - T_0(\hat{r}_k - \hat{W}_k), \tag{57}$$

$$\hat{\Psi}_k = \left(\frac{\hat{W}_k}{4}\right)^2 \left(\frac{g\,s}{q}\right)_{\hat{r}_k}, \tag{58}$$

$$\hat{\Psi}_{k\,+} = -\hat{T}_{e\,k}'\,\hat{W}_k\,\delta T_1(1) \left(\frac{\hat{r}\,g}{q}\,\frac{m_k - n\,q}{\hat{T}_0'\,\psi_{m\,k}}\right)_{\hat{r}_k+\hat{W}_k}, \tag{59}$$

$$\hat{\Psi}_{k\,-} = \hat{T}_{e\,k}'\,\hat{W}_k\,\delta T_1(1) \left(\frac{\hat{r}\,g}{q}\,\frac{m_k - n\,q}{\hat{T}_0'\,\psi_{m\,k}}\right)_{\hat{r}_k-\hat{W}_k}. \tag{60}$$