

Resistive Wall

R. Fitzpatrick^a

*Institute for Fusion Studies, Department of Physics,
University of Texas at Austin, Austin TX 78712, USA*

I. RESISTIVE WALL PHYSICS

A. Resistive Wall

Let the inner surface of the resistive wall surrounding the plasma lie at $\mu = \mu_w$, and let the outer surface lie at $\mu = \mu_w - \bar{d}_w \sinh \mu_w$, where $\bar{d}_w \ll 1$ is a positive constant. The physical wall thickness is

$$d(\eta) = \frac{\bar{d}_w \sinh \mu_w}{|\nabla \mu|} = h_w(\eta) \sinh \mu_w \bar{d}_w, \quad (1)$$

where

$$h_w(\eta) = \frac{1}{z_w - \cos \eta}, \quad (2)$$

and $z_w = \cosh \mu_w$. Let the electrical conductivity of the wall material vary as

$$\sigma(\eta) = \frac{\bar{\sigma}_w}{h_w^2(\eta) \sinh^2 \mu_w}, \quad (3)$$

where $\bar{\sigma}_w$ is a positive constant. It follows that $\sigma d^2 = \bar{\sigma}_w \bar{d}_w^2$.

B. Wall Matching Conditions

If we write

$$\mathbf{b} = \nabla \times \mathbf{A} \quad (4)$$

in the vacuum region then the boundary conditions at the wall are

$$\mathbf{n}_w \times \mathbf{A}|_{z_{w-}} = \frac{1}{\cosh \lambda} \mathbf{n}_w \times \mathbf{A}|_{z_{w+}} \quad (5)$$

$$\mathbf{n}_w \times (\nabla \times \mathbf{A})|_{z_{w+}} = -\frac{\lambda \tanh \lambda}{\bar{d}_w h_w \sinh \mu_w} \mathbf{n}_w \times (\mathbf{n}_w \times \mathbf{A})|_{z_{w+}} + \frac{\mathbf{n}_w \times (\nabla \times \mathbf{A})|_{z_{w-}}}{\cosh \lambda}, \quad (6)$$

^a rfitzp@utexas.edu

$$\lambda = \sqrt{\hat{\gamma} \bar{d}_w}, \quad (7)$$

$$\hat{\gamma} = \gamma \bar{\tau}_w, \quad (8)$$

$$\bar{\tau}_w = \mu_0 R_0^2 \bar{\sigma}_w \bar{d}_w, \quad (9)$$

where γ is the growth-rate of the magnetic perturbation, and $\bar{\tau}_w$ is the effective L/R time of the wall. Here, $\mathbf{n}_w = -\mathbf{e}_\mu$ is an outward unit normal vector to the wall. Now,

$$\begin{aligned} \nabla \times \mathbf{A} = & \frac{1}{h^2 \sinh \mu} \left(\frac{\partial \hat{A}_\phi}{\partial \eta} - \frac{\partial \hat{A}_\eta}{\partial \phi} \right) \mathbf{e}_\mu + \frac{1}{h^2 \sinh \mu} \left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu} \right) \mathbf{e}_\eta \\ & + \frac{1}{h^2} \left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta} \right) \mathbf{e}_\phi, \end{aligned} \quad (10)$$

where

$$\hat{A}_\mu = h A_\mu, \quad (11)$$

$$\hat{A}_\eta = h A_\eta, \quad (12)$$

$$\hat{A}_\phi = h \sinh \mu A_\phi. \quad (13)$$

Furthermore,

$$\mathbf{n}_w \times \mathbf{A} = -\mathbf{e}_\mu \times \mathbf{A} = A_\phi \mathbf{e}_\eta - A_\eta \mathbf{e}_\phi, \quad (14)$$

$$\mathbf{n}_w \times (\mathbf{n}_w \times \mathbf{A}) = -\mathbf{e}_\mu \times (\mathbf{n}_w \times \mathbf{A}) = -A_\eta \mathbf{e}_\eta - A_\phi \mathbf{e}_\phi, \quad (15)$$

$$\begin{aligned} \mathbf{n}_w \times (\nabla \times \mathbf{A}) = & -\mathbf{e}_\mu \times (\nabla \times \mathbf{A}) = \frac{1}{h^2} \left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta} \right) \mathbf{e}_\eta - \frac{1}{h^2 \sinh \mu} \left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu} \right) \mathbf{e}_\phi. \end{aligned} \quad (16)$$

Thus, the wall matching conditions become

$$\hat{A}_\eta \Big|_{z_{w-}} = \frac{1}{\cosh \lambda} \hat{A}_\eta \Big|_{z_{w+}}, \quad (17)$$

$$\hat{A}_\phi \Big|_{z_{w+}} = \frac{1}{\cosh \lambda} \hat{A}_\phi \Big|_{z_{w-}}, \quad (18)$$

$$\left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta} \right) \Big|_{z_{w+}} = \frac{\lambda \tanh \lambda}{\bar{d}_w \sinh \mu_w} \hat{A}_\eta \Big|_{z_{w+}} + \frac{1}{\cosh \lambda} \left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta} \right) \Big|_{z_{w-}}, \quad (19)$$

$$\left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu} \right) \Big|_{z_{w+}} = -\frac{\lambda \tanh \lambda}{\bar{d}_w \sinh \mu_w} \hat{A}_\phi \Big|_{\mu_{z+}} + \frac{1}{\cosh \lambda} \left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu} \right) \Big|_{z_{w-}}. \quad (20)$$

Let

$$C(z, \eta, \phi) = \frac{\partial \hat{A}_\eta}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \eta}. \quad (21)$$

The wall matching conditions reduce to

$$C(z_{w-}, \eta, \phi) = \frac{1}{\cosh \lambda} C(z_{w+}, \eta, \phi), \quad (22)$$

$$\frac{\partial C(z_{w+}, \eta, \phi)}{\partial z} = \frac{\lambda \tanh \lambda}{\bar{d}_w \sinh^2 \mu_w} C(z_{w+}, \eta, \phi) + \frac{1}{\cosh \lambda} \frac{\partial C(z_{w-}, \eta, \phi)}{\partial z}. \quad (23)$$

However, if

$$\mathbf{b} = \mathbf{i} \nabla V = \nabla \times \mathbf{A} \quad (24)$$

then

$$C = -\mathbf{i} h \sinh \mu \frac{\partial V}{\partial \mu} = -\mathbf{i} h (z^2 - 1) \frac{\partial V}{\partial z}. \quad (25)$$

Thus,

$$C = -\mathbf{i} \frac{z^2 - 1}{z - \cos \eta} \sum_m \left[\frac{U_m}{2(z - \cos \eta)^{1/2}} + (z - \cos \eta)^{1/2} \frac{dU_m}{dz} \right] e^{-\mathbf{i}(m\eta + n\phi)}, \quad (26)$$

$$\frac{\partial C}{\partial z} = -\mathbf{i} \sum_m \left[\frac{(3/4) \sin^2 \eta}{(z - \cos \eta)^{5/2}} - \frac{(1/2) \cos \eta}{(z - \cos \eta)^{3/2}} + \frac{m^2 + n^2/(z^2 - 1)}{(z - \cos \eta)^{1/2}} \right] U_m e^{-\mathbf{i}(m\eta + n\phi)}. \quad (27)$$

It follows that

$$\begin{aligned} \sum_m \left[\frac{U_m}{2} + (z - \cos \eta) \frac{dU_m}{dz} \right]_{z_{w-}} e^{-\mathbf{i}m\eta} &= \frac{1}{\cosh \lambda} \sum_m \left[\frac{U_m}{2} + (z - \cos \eta) \frac{dU_m}{dz} \right]_{z_{w+}} e^{-\mathbf{i}m\eta}, \\ \sum_m \left[\frac{3}{4} \sin^2 \eta - \frac{1}{2} (z - \cos \eta) \cos \eta + (z - \cos \eta)^2 \left(m^2 + \frac{n^2}{z^2 - 1} \right) \right] U_m e^{-\mathbf{i}m\eta} \Big|_{z_{w+}} &= \\ f_w \sum_m (z - \cos \eta) \left[\frac{U_m}{2} + (z - \cos \eta) \frac{dU_m}{dz} \right]_{z_{w+}} e^{-\mathbf{i}m\eta} \\ + \frac{1}{\cosh \lambda} \sum_m \left[\frac{3}{4} \sin^2 \eta - \frac{1}{2} (z - \cos \eta) \cos \eta + (z - \cos \eta)^2 \left(m^2 + \frac{n^2}{z^2 - 1} \right) \right] U_m e^{-\mathbf{i}m\eta} \Big|_{z_{w-}}, \end{aligned} \quad (28)$$

where

$$f_w = \frac{\lambda \tanh \lambda}{\bar{d}_w}. \quad (30)$$

Thus, we can write

$$\sum_{m'} I_{mm'} U_{m'}(z_{w-}) = \frac{1}{\cosh \lambda} \sum_{m'} I_{mm'} U_{m'}(z_{w+}), \quad (31)$$

$$\sum_{m'} J_{mm'} U_{m'}(z_{w+}) = f_w \sum_{m', m''} k_{mm''} I_{m''m'} U_{m'}(z_{w+}) + \frac{1}{\cosh \lambda} \sum_{m'} J_{mm'} U_{m'}(z_{w-}), \quad (32)$$

where

$$I_{mm'} = \left(\frac{1}{2} + z \frac{d}{dz} \right) \delta_{mm'} - \frac{1}{2} \frac{d}{dz} (\delta_{m+1m'} + \delta_{m-1m'}), \quad (33)$$

$$\begin{aligned} J_{mm'} = & \left[\frac{5}{8} + \left(\frac{1}{2} + z^2 \right) \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] \delta_{mm'} - z \left[\frac{1}{4} + \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] (\delta_{m+1m'} + \delta_{m-1m'}) \\ & + \left[-\frac{1}{16} + \frac{1}{4} \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] (\delta_{m+2m'} + \delta_{m-2m'}), \end{aligned} \quad (34)$$

$$k_{mm'} = z \delta_{mm'} - \frac{1}{2} (\delta_{m+1m'} + \delta_{m-1m'}). \quad (35)$$

C. Vacuum Solution

Now,

$$U_m(z) = p_{m-} \hat{P}_{|m|-1/2}^n(z) \quad (36)$$

in the region $z < z_w$, whereas

$$U_m(z) = p_{m+} \hat{P}_{|m|-1/2}^n(z) + q_{m+} \hat{Q}_{|m|-1/2}^n(z) \quad (37)$$

in the region $z > z_w$. Let \underline{I}_p be the matrix of the

$$\left\{ \left[\left(\frac{1}{2} + z \frac{d}{dz} \right) \delta_{mm'} - \frac{1}{2} \frac{d}{dz} (\delta_{m+1m'} + \delta_{m-1m'}) \right] \hat{P}_{|m'|-1/2}^n(z) \right\}_{z_w} \quad (38)$$

values. Let \underline{I}_q be the matrix of the

$$\left\{ \left[\left(\frac{1}{2} + z \frac{d}{dz} \right) \delta_{mm'} - \frac{1}{2} \frac{d}{dz} (\delta_{m+1m'} + \delta_{m-1m'}) \right] \hat{Q}_{|m'|-1/2}^n(z) \right\}_{z_w} \quad (39)$$

values. Let \underline{J}_p be the matrix of the

$$\begin{aligned} & \left\{ \left[\frac{5}{8} + \left(\frac{1}{2} + z^2 \right) \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] \delta_{mm'} - z \left[\frac{1}{4} + \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] (\delta_{m+1m'} + \delta_{m-1m'}) \right. \\ & \left. + \left[-\frac{1}{16} + \frac{1}{4} \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] (\delta_{m+2m'} + \delta_{m-2m'}) \right\} \hat{P}_{|m'|-1/2}^n(z_w) \end{aligned} \quad (40)$$

values. Let \underline{J}_q be the matrix of the

$$\left\{ \left[\frac{5}{8} + \left(\frac{1}{2} + z^2 \right) \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] \delta_{mm'} - z \left[\frac{1}{4} + \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] (\delta_{m+1m'} + \delta_{m-1m'}) \right\}$$

$$+ \left[-\frac{1}{16} + \frac{1}{4} \left(m'^2 + \frac{n^2}{z^2 - 1} \right) \right] (\delta_{m+2m'} + \delta_{m-2m'}) \} \hat{Q}_{|m'|-1/2}^n(z_w) \quad (41)$$

values. Let \underline{k} be the matrix of the $k_{mm'}$ values. Finally, let \underline{p}_+ be the vector of the p_{m+} values, et cetera. Thus, we obtain

$$\underline{I}_p \underline{p}_- = \frac{1}{\cosh \lambda} \left(\underline{I}_p \underline{p}_+ + \underline{I}_q \underline{q}_+ \right), \quad (42)$$

$$\underline{J}_p \underline{p}_+ + \underline{J}_q \underline{q}_+ = f_w \underline{k} \left(\underline{I}_p \underline{p}_+ + \underline{I}_q \underline{q}_+ \right) + \frac{1}{\cosh \lambda} \underline{J}_p \underline{p}_-, \quad (43)$$

which can be rearranged to give

$$\left(\tanh^2 \lambda \underline{J}_p - f_w \hat{\underline{I}}_p \right) \underline{p}_+ + \left(\underline{J}_{pq} + \tanh^2 \lambda \underline{J}_{qp} - f_w \hat{\underline{I}}_q \right) \underline{q}_+, \quad (44)$$

where

$$\hat{\underline{I}}_p = \underline{k} \underline{I}_p, \quad (45)$$

$$\hat{\underline{I}}_q = \underline{k} \underline{I}_q, \quad (46)$$

$$\underline{J}_{pq} = \underline{J}_q - \underline{J}_p \hat{\underline{I}}_p^{-1} \hat{\underline{I}}_q, \quad (47)$$

$$\underline{J}_{qp} = \underline{J}_p \hat{\underline{I}}_p^{-1} \hat{\underline{I}}_q. \quad (48)$$

Now, $z_w \sim 1/\bar{b}_w$, where \bar{b}_w is the mean wall minor radius. In the large aspect-ratio limit, $\bar{b}_w \ll 1$, we have $\underline{I}_p \sim \mathcal{O}(1)$, $\underline{I}_q \sim \mathcal{O}(1)$, $\underline{J}_p \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{J}_q \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{K}_p \sim \mathcal{O}(1/\bar{b}_w)$, $\underline{K}_q \sim \mathcal{O}(1/\bar{b}_w)$, and $\underline{k} \sim \mathcal{O}(1/\bar{b}_w)$. It follows that $\hat{\underline{I}}_p \sim \mathcal{O}(1/\bar{b}_w)$, $\hat{\underline{I}}_q \sim \mathcal{O}(1/\bar{b}_w)$, $\underline{J}_{pq} \sim \mathcal{O}(1/\bar{b}_w^2)$ and $\underline{J}_{qp} \sim \mathcal{O}(1/\bar{b}_w^2)$. Thus, the ratio of the first to the second term multiplying \underline{p}_+ in Eq. (44) is

$$\tanh \lambda \frac{\bar{d}_w}{\lambda \bar{b}_w}. \quad (49)$$

However, the wall analysis is premised on the assumption that

$$\frac{\bar{d}_w}{\lambda \bar{b}_w} \ll 1. \quad (50)$$

Hence, the first term is negligible with respect to the second, irrespective of the value of λ .

The ratios of the three terms multiplying \underline{q}_+ in Eq. (44) are

$$\frac{\bar{d}_w}{\lambda \bar{b}_w}, \quad \tanh^2 \lambda \frac{\bar{d}_w}{\lambda \bar{b}_w}, \quad \tanh \lambda. \quad (51)$$

Thus, in the thin-shell limit, $\lambda \ll 1$, the second term is negligible with respect to the first. In the thick-shell limit, $\lambda \gg 1$, the third term is dominant. Thus, we can neglect the second term. Hence, we deduce that

$$\underline{q}_+ = \underline{\mathcal{F}} \underline{p}_+, \quad (52)$$

where

$$\underline{\mathcal{F}} = f_w \underline{I} (\underline{J} + f_w \underline{1})^{-1}, \quad (53)$$

$$\underline{I} = -\hat{\underline{I}}_q^{-1} \hat{\underline{I}}_p, \quad (54)$$

$$\underline{J} = \hat{\underline{I}}_p^{-1} (\underline{J}_q \underline{I} + \underline{J}_p). \quad (55)$$

Note that $\underline{I} \sim \mathcal{O}(1)$ and $\underline{J} \sim \mathcal{O}(1/\bar{b}_w)$.

D. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque acting on the plasma is

$$T_\phi = -2\pi^2 n \operatorname{Im}(\underline{p}_+^\dagger \underline{q}_+) = -2\pi^2 n \operatorname{Im}(\underline{p}_+^\dagger \underline{\mathcal{F}} \underline{p}_+) = -\pi^2 n \operatorname{Im}[\underline{p}_+^\dagger (\underline{\mathcal{F}} - \underline{\mathcal{F}}^\dagger) \underline{p}_+]. \quad (56)$$

However, we expect this torque to be zero if f_w is real, which implies that $\underline{\mathcal{F}} = \underline{\mathcal{F}}^\dagger$ when f_w is real. In other words,

$$f_w \underline{I} (\underline{J} + f_w \underline{1})^{-1} = f_w (\underline{J}^\dagger + f_w \underline{1})^{-1} \underline{I}^\dagger, \quad (57)$$

which implies that

$$f_w (\underline{J}^\dagger + f_w \underline{1}) \underline{I} = f_w \underline{I}^\dagger (\underline{J} + f_w \underline{1}). \quad (58)$$

However, the previous equation holds for arbitrary real f_w , so we can separately equate the coefficients of f_w and f_w^2 to give

$$\underline{J}^\dagger \underline{I} = \underline{I}^\dagger \underline{J}, \quad (59)$$

$$\underline{I} = \underline{I}^\dagger. \quad (60)$$

It follows that \underline{I} and

$$\underline{K} = \underline{I} \underline{J} \quad (61)$$

are both real symmetric matrices. In general,

$$\underline{\mathcal{F}} - \underline{\mathcal{F}}^\dagger = (f_w - f_w^*) [(\underline{J} + f_w \underline{1})^{-1}]^\dagger \underline{K} (\underline{J} + f_w \underline{1})^{-1}, \quad (62)$$

$$T_\phi = -2\pi^2 n \operatorname{Im}(f_w) [(\underline{\underline{J}} + f_w \underline{\underline{1}})^{-1} \underline{\underline{p}}_+]^\dagger \underline{\underline{K}} [(\underline{\underline{J}} + f_w \underline{\underline{1}})^{-1} \underline{\underline{p}}_+]. \quad (63)$$

Thus, $\underline{\underline{\mathcal{F}}}$ is clearly Hermitian, and T_ϕ is zero, if f_w is real.