

Pressure Flattening due to Asymmetric Magnetic Island

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I. MAGNETIC ISLAND

Let $x = r - r_s$, $X = x/W$, and $\zeta = m\theta - n\phi$, where W is the island width. The magnetic flux-surfaces of the magnetic island are contours of

$$\Omega(X, \zeta) = 8X^2 + \cos(\zeta - \delta^2 \sin \zeta) - 2\sqrt{8}\delta X \cos \zeta + \delta^2 \cos^2 \zeta, \quad (1)$$

where $|\delta| < 1$. The X-points lie at $X = \delta/\sqrt{8}$ and $\zeta = 0, 2\pi$, whereas the O-point lies at $X = -\delta/\sqrt{8}$ and $\zeta = \pi$. The O-point corresponds to $\Omega = -1$, whereas the magnetic separatrix corresponds to $\Omega = 1$.

Let

$$Y = X - \frac{\delta}{\sqrt{8}} \cos \zeta, \quad (2)$$

$$\xi = \zeta - \delta^2 \sin \zeta. \quad (3)$$

It follows that

$$\Omega(Y, \xi) = 8Y^2 + \cos \xi, \quad (4)$$

The X-points lie at $Y = 0$ and $\xi = 0, 2\pi$, whereas the O-point lies at $Y = 0$ and $\xi = \pi$. Moreover,

$$\zeta = \xi + 2 \sum_{n=1, \infty} \left[\frac{J_n(n\delta^2)}{n} \right] \sin(n\xi), \quad (5)$$

$$\cos \zeta = -\frac{\delta^2}{2} + \sum_{n=1, \infty} \left[\frac{J_{n-1}(n\delta^2) - J_{n+1}(n\delta^2)}{n} \right] \cos(n\xi), \quad (6)$$

$$\sin \zeta = \frac{2}{\delta^2} \sum_{n=1, \infty} \left[\frac{J_n(n\delta^2)}{n} \right] \sin(n\xi), \quad (7)$$

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$$\cos(m \zeta) = m \sum_{n=1, \infty} \left[\frac{J_{n-m}(n \delta^2) - J_{n+m}(n \delta^2)}{n} \right] \cos(n \xi), \quad (8)$$

$$\sin(m \zeta) = m \sum_{n=1, \infty} \left[\frac{J_{n-m}(n \delta^2) + J_{n+m}(n \delta^2)}{n} \right] \sin(n \xi), \quad (9)$$

for $m > 1$.

II. PLASMA DISPLACEMENT

Outside the separatrix, we can write

$$\Omega(X, \zeta) = 8 (X - \Xi)^2, \quad (10)$$

where $\Xi = \xi^r / W$. It follows that

$$\begin{aligned} \Xi(X, \zeta) &\simeq -\frac{[\Omega(X, \zeta) - 8 X^2]}{16 X} \\ &= -\frac{\cos(\zeta - \delta^2 \sin \zeta) + \delta^2 \cos^2 \zeta}{16 X} + \frac{\delta}{\sqrt{8}} \cos \zeta. \end{aligned} \quad (11)$$

Note that Ξ is an even function of ζ . Let us write

$$\Xi(X, \zeta) = \sum_{n=0, \infty} \Xi_n(X) \cos(n \zeta). \quad (12)$$

Thus,

$$\Xi_1(X) = 2 \oint \Xi(X, \zeta) \cos(\zeta) \frac{d\zeta}{2\pi} = -\frac{1}{8 X} \oint \cos(\zeta - \delta^2 \sin \zeta) \cos \zeta \frac{d\zeta}{2\pi} + \frac{\delta}{\sqrt{8}} \quad (13)$$

$$\begin{aligned} &= -\frac{1}{16 X} \oint \cos(-\delta^2 \sin \zeta) \cos \zeta \frac{d\zeta}{2\pi} \\ &\quad - \frac{1}{16 X} \oint \cos(2 \zeta - \delta^2 \sin \zeta) \cos \zeta \frac{d\zeta}{2\pi} + \frac{\delta}{\sqrt{8}}. \end{aligned} \quad (14)$$

But,

$$J_n(\delta^2) = \oint \cos(n \zeta - \delta^2 \sin \zeta) \frac{d\zeta}{2\pi}, \quad (15)$$

so

$$\Xi_1(X) = -\frac{J_0(\delta^2) + J_2(\delta^2)}{16 X} + \frac{\delta}{\sqrt{8}}, \quad (16)$$

and

$$\xi_1^r(x) = -\frac{W^2}{16 x} [J_0(\delta^2) + J_2(\delta^2)] + \frac{W \delta}{\sqrt{8}}. \quad (17)$$

Thus,

$$\begin{aligned}\psi_1(x) &= \frac{r}{q} (m - n q) \xi_1^r = -(n s g)_{r_s} x \xi_1^r \\ &= (n s g)_{r_k} \frac{W^2}{16} [J_0(\delta^2) + J_2(\delta^2)] - (n s g)_{r_s} \frac{W \delta}{\sqrt{8}} x,\end{aligned}\quad (18)$$

so

$$\frac{\psi_m(x)}{m} = J_0(\delta^2) + J_2(\delta^2) - 2\sqrt{8} \frac{\delta}{W} x. \quad (19)$$

It follows that

$$\delta = -\frac{W}{2\sqrt{8}m} \frac{d\psi_m(0)}{dr} \quad (20)$$

$$\simeq -\left[\frac{\psi_m(r_s + W) - \psi_m(r_s - W)}{4\sqrt{8}m} \right]. \quad (21)$$

III. FLUX-SURFACE AVERAGE OPERATOR

Now,

$$[A, B] \equiv \frac{\partial A}{\partial X} \Big|_{\zeta} \frac{\partial B}{\partial \zeta} \Big|_X - \frac{\partial B}{\partial X} \Big|_{\zeta} \frac{\partial A}{\partial \zeta} \Big|_X. \quad (22)$$

But,

$$\frac{\partial}{\partial X} \Big|_{\zeta} = \frac{\partial \Omega}{\partial X} \Big|_{\zeta} \frac{\partial}{\partial \Omega} \Big|_{\xi} + \frac{\partial \xi}{\partial X} \Big|_{\zeta} \frac{\partial}{\partial \xi} \Big|_{\Omega} = 16 Y \frac{\partial}{\partial \Omega} \Big|_{\xi}, \quad (23)$$

and

$$\frac{\partial}{\partial \zeta} \Big|_X = \frac{\partial \Omega}{\partial \zeta} \Big|_X \frac{\partial}{\partial \Omega} \Big|_{\xi} + \frac{\partial \xi}{\partial \zeta} \Big|_X \frac{\partial}{\partial \xi} \Big|_{\Omega}, \quad (24)$$

so

$$[A, B] \equiv \frac{16 Y}{\sigma} \left(\frac{\partial A}{\partial \Omega} \Big|_{\xi} \frac{\partial B}{\partial \xi} \Big|_{\Omega} - \frac{\partial B}{\partial \Omega} \Big|_{\xi} \frac{\partial A}{\partial \xi} \Big|_{\Omega} \right), \quad (25)$$

where

$$\sigma(\xi) \equiv \frac{d\zeta}{d\xi} = 1 + 2 \sum_{n=1, \infty} J_n(n \delta^2) \cos(n \xi). \quad (26)$$

In particular,

$$[A, \Omega] = -\frac{16 Y}{\sigma} \frac{\partial A}{\partial \xi} \Big|_{\Omega}. \quad (27)$$

The flux-surface average operator, $\langle \cdots \rangle$, is the annihilator of $[A, \Omega]$ for arbitrary $A(s, \Omega, \xi)$. Here, $s = +1$ for $Y > 0$ and $s = -1$ for $Y < 0$. It follows that

$$\langle A \rangle = \int_{\zeta_0}^{2\pi - \zeta_0} \frac{\sigma(\xi) A_+(\Omega, \xi)}{\sqrt{2(\Omega - \cos \xi)}} \frac{d\xi}{2\pi} \quad (28)$$

for $-1 \leq \Omega \leq 1$, and

$$\langle A \rangle = \int_0^{2\pi} \frac{\sigma(\xi) A(s, \Omega, \xi)}{\sqrt{2(\Omega - \cos \xi)}} \frac{d\xi}{2\pi} \quad (29)$$

for $\Omega > 1$. Here, $\xi_0 = \cos^{-1}(\Omega)$, and

$$A_+(\Omega, \xi) = \frac{1}{2} [A(+1, \Omega, \xi) + A(-1, \Omega, \xi)]. \quad (30)$$

IV. TEMPERATURE PERTURBATION

The electron temperature in the vicinity of the island can be written

$$T_e(X, \zeta) = T_{es} + s W T'_{es} \tilde{T}(\Omega). \quad (31)$$

Here, $\tilde{T}(\Omega)$ satisfies

$$\left\langle \frac{\partial^2 \tilde{T}}{\partial X^2} \right|_{\zeta} = 0, \quad (32)$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \rightarrow |X| \quad (33)$$

as $|X| \rightarrow \infty$. It follows that

$$\frac{d}{d\Omega} \left(\langle Y^2 \rangle \frac{d\tilde{T}}{d\Omega} \right) = 0 \quad (34)$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \rightarrow \frac{\Omega^{1/2}}{\sqrt{8}} \quad (35)$$

as $\Omega \rightarrow \infty$.

Outside the separatrix,

$$\langle Y^2 \rangle(\Omega) = \frac{1}{16} \int_0^{2\pi} \sigma(\xi) \sqrt{2(\Omega - \cos \xi)} \frac{d\xi}{2\pi}. \quad (36)$$

Let

$$k = \left(\frac{1 + \Omega}{2} \right)^{1/2}. \quad (37)$$

Thus, the O-point corresponds to $k = 0$ and the separatrix to $k = 1$. It follows that

$$\langle Y^2 \rangle(k) = \frac{k}{4\pi} \int_0^{\pi/2} \sigma(2\theta - \pi) (1 - \sin^2 \theta / k^2)^{1/2} d\theta. \quad (38)$$

Thus,

$$\langle Y^2 \rangle(k) = \frac{k}{4\pi} G(1/k), \quad (39)$$

where

$$G(p) = E_0(p) + 2 \cos(n\pi) \sum_{n=1,\infty} J_n(n\delta^2) E_n(p), \quad (40)$$

$$E_n(p) = \int_0^{\pi/2} \cos(2n\theta) (1 - p^2 \sin^2 \theta)^{1/2} d\theta. \quad (41)$$

Equation (34) yields

$$\tilde{T}(k) = 0 \quad (42)$$

for $0 \leq k \leq 1$, and

$$\frac{d}{dk} \left[G(1/k) \frac{d\tilde{T}}{dk} \right] = 0 \quad (43)$$

for $k > 1$. Thus,

$$\frac{d\tilde{T}}{dk} = \frac{c}{G(1/k)} \quad (44)$$

for $k > 1$, subject to the boundary condition that

$$\tilde{T}(k) \rightarrow \frac{k}{2} \quad (45)$$

as $k \rightarrow \infty$. In the limit that $p \rightarrow 0$,

$$E_0(p) = \frac{\pi}{2}, \quad (46)$$

$$E_{n>0}(p) = 0, \quad (47)$$

which implies that $c = \pi/4$. So

$$\frac{d\tilde{T}}{dk} = \frac{\pi}{4} \frac{1}{G(1/k)}, \quad (48)$$

$$\tilde{T}(k) = F(k), \quad (49)$$

$$F(k) = \frac{\pi}{4} \int_1^k \frac{dk'}{G(1/k')} \quad (50)$$

for $k > 1$.

V. HARMONICS OF TEMPERATURE PERTURBATION

We can write

$$\tilde{T}(X, \zeta) = \sum_{\nu=0, \infty} \delta T_\nu(X) \cos(\nu \zeta). \quad (51)$$

Now,

$$\delta T_0(X) = \oint \tilde{T}(X, \zeta) \frac{d\zeta}{2\pi}, \quad (52)$$

where the integral is at constant X . It follows that

$$\delta T_0(X) = \int_0^{\xi_c} F(k) \sigma(\xi) \frac{d\xi}{\pi}, \quad (53)$$

where

$$\xi_c = \cos^{-1}(1 - 8Y^2) \quad (54)$$

for $|Y| < 1/2$, and $\xi_c = \pi$ for $|Y| \geq 1/2$. Furthermore,

$$k = \left[4Y^2 + \cos^2\left(\frac{\xi}{2}\right) \right]^{1/2}. \quad (55)$$

Let

$$\delta T_{0\infty} = \lim_{X \rightarrow \infty} [X - \delta T_0(X)]. \quad (56)$$

For $\nu > 0$, we have

$$\delta T_\nu(X) = 2 \oint \tilde{T}(X, \zeta) \cos(\nu \zeta) \frac{d\zeta}{2\pi}. \quad (57)$$

Integrating by parts, we obtain

$$\delta T_\nu(X) = -\frac{2}{\nu} \oint \frac{\partial \tilde{T}}{\partial \zeta} \bigg|_X \sin(\nu \zeta) \frac{d\zeta}{2\pi}. \quad (58)$$

But,

$$\frac{\partial \tilde{T}}{\partial \zeta} \bigg|_X = \frac{d\tilde{T}}{dk} \frac{\partial k}{\partial \zeta} \bigg|_X = \frac{1}{4k} \frac{d\tilde{T}}{dk} \frac{\partial \Omega}{\partial \zeta} \bigg|_X = -\frac{1}{4k} \frac{d\tilde{T}}{dk} \kappa(\xi), \quad (59)$$

where

$$\kappa(\xi) = \sin \xi (1 - \delta^2 \cos \zeta) - 2\sqrt{8} \delta X \sin \zeta + \delta^2 \sin(2\zeta). \quad (60)$$

Hence,

$$\delta T_\nu(X) = \frac{1}{8\nu} \int_0^{\xi_c} \frac{\sin(\nu \zeta) \kappa(\xi) \sigma(\xi)}{k G(1/k)} d\xi. \quad (61)$$

VI. ASYMPTOTIC MATCHING

Consider the k th rational surface whose radius is r_k and whose resonant poloidal mode number is m_k . Let $x = r - r_k$ and $\zeta_k = m_k \theta - n \phi$.

In the outer region, we write the total electron temperature as

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) - \Psi_k \frac{q(r)}{r g(r)} \frac{T'_{e0}(r) \psi_{m_k}(r)}{m_k - n q(r)} e^{i\zeta_k}, \quad (62)$$

where $T'_{e0} = dT_{e0}/dr$, $T_{e0}(r)$ is the equilibrium electron temperature profile, Ψ_k is the reconnected flux, and

$$W_k = 4 \left(\frac{q}{g s} \right)_{r_k}^{1/2} \Psi_k^{1/2} \delta T_{0\infty} \quad (63)$$

is the island width. In the limit, $|x| \ll 1$, Eq. (62) yield

$$\tilde{T}_e(x, \theta, \phi) = T_{ek} + T'_{ek} x + T'_{ek} W_k \left(\frac{W_k}{16x} - \frac{\delta_k}{\sqrt{8}} \right) e^{i\zeta_k}, \quad (64)$$

Here, $T_{ek} = T_{e0}(r_k)$ and $T'_{ek} = (dT_{e0}/dr)_{r_k}$,

$$\delta_k = -\frac{W_k}{2\sqrt{8}m} \frac{d\psi_{m_k}(r_k)}{dr}, \quad (65)$$

and we have made use of the fact that $\psi_{m_k}(r_k) \simeq m_k$.

In the inner region, we write the total electron temperature as

$$\tilde{T}_e(x, \theta, \phi) = T_{ek} + T'_{ek} W_k \sum_{\nu=0,\infty} \delta T_\nu(x/W_k) e^{i\nu\zeta_k} + T'_{ek} W_k \delta T_{0\infty}, \quad (66)$$

The asymptotic matching process consists of writing

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e+} - \Psi_{k+} \frac{q(r)}{r g(r)} \frac{T'_{e0}(r) \psi_{m_k}(r)}{m_k - n q(r)} e^{i\zeta_k} \quad (67)$$

in the region $r > r_k + W_k$,

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e-} - \Psi_{k-} \frac{q(r)}{r g(r)} \frac{T'_{e0}(r) \psi_{m_k}(r)}{m_k - n q(r)} e^{i\zeta_k} \quad (68)$$

in the region $r < r_k - W_k$, and

$$\tilde{T}_e(r, \theta, \phi) = T_{ek} + T'_{ek} W_k \sum_{\nu=0,\infty} \delta T_\nu(x/W_k) e^{i\nu\zeta_k} + T'_{ek} W_k \delta T_{0\infty} \quad (69)$$

in the region $r_k - W_k \leq r \leq r_k + W_k$. Continuity of the solution at $r = r_k \pm W_k$ implies that

$$\delta T_{e+} = T'_{ek} W_k \delta T_{e0}(1) + T'_{ek} W_k \delta T_{0\infty} - T'_{ek} W_k, \quad (70)$$

$$\delta T_{e-} = T'_{ek} W_k \delta T_{e0}(-1) + T'_{ek} W_k \delta T_{0\infty} + T'_{ek} W_k, \quad (71)$$

$$\Psi_{k+} = -T'_{ek} W_k \delta T_1(1) \left(\frac{r g}{q} \frac{m_k - n q}{T'_{e0} \psi_{mk}} \right)_{r_k + W_k}, \quad (72)$$

$$\Psi_{k-} = -T'_{ek} W_k \delta T_1(-1) \left(\frac{r g}{q} \frac{m_k - n q}{T'_{e0} \psi_{mk}} \right)_{r_k - W_k}. \quad (73)$$

Finally, for the special case $m = 1$, we write

$$\tilde{T}_e(r, \theta, \phi) = -\xi^r(r, \theta, \phi) \frac{dT_{e0}}{dr}. \quad (74)$$

A. Normalized Quantities

Let $\hat{r} = r/\epsilon_a$, $\hat{r}_k = r_k/\epsilon_a$, $\hat{x} = x/\epsilon_a$, $\hat{T}'_{e0} = \epsilon_a T'_{e0}$, $\hat{T}'_{ek} = \epsilon_a T'_{ek}$, $\hat{W}_k = W_k/\epsilon_a$, and $\hat{\Psi}_k = \Psi_k/\epsilon_a^2$, etc., then

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e+} - \hat{\Psi}_{k+} \frac{q(\hat{r})}{\hat{r} g(\hat{r})} \frac{\hat{T}'_e(r) \psi_{mk}(r)}{m_k - n q(\hat{r})} e^{i\zeta_k} \quad (75)$$

in the region $\hat{r} > \hat{r}_k + \hat{W}_k$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e-} - \hat{\Psi}_{k-} \frac{q(\hat{r})}{\hat{r} g(\hat{r})} \frac{\hat{T}'_e(r) \psi_{mk}(r)}{m_k - n q(\hat{r})} e^{i\zeta_k} \quad (76)$$

in the region $\hat{r} < \hat{r}_k - \hat{W}_k$, and

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{ek} + \text{sgn}(\hat{x}) \hat{T}'_{ek} \hat{W}_k \sum_{\nu=0,\infty} \delta T_\nu(|\hat{x}|/\hat{W}_k) e^{i\nu\zeta_k} + \hat{T}'_{ek} \hat{W}_k \delta T_{0\infty} \quad (77)$$

in the region $\hat{r}_k - \hat{W}_k \leq \hat{r} \leq \hat{r}_k + \hat{W}_k$. Here,

$$\delta T_{e+} = \hat{T}'_{ek} \hat{W}_k \delta T_0(1) + \hat{T}'_{ek} \hat{W}_k \delta T_{0\infty} - \hat{T}'_{ek} \hat{W}_k, \quad (78)$$

$$\delta T_{e-} = \hat{T}'_{ek} \hat{W}_k \delta T_0(-1) + \hat{T}'_{ek} \hat{W}_k \delta T_{0\infty} + \hat{T}'_{ek} \hat{W}_k, \quad (79)$$

$$\hat{\Psi}_k = \left(\frac{\hat{W}_k}{4} \right)^2 \left(\frac{g s}{q} \right)_{\hat{r}_k}, \quad (80)$$

$$\hat{\Psi}_{k+} = -\hat{T}'_{ek} \hat{W}_k \delta T_1(1) \left(\frac{\hat{r} g}{q} \frac{m_k - n q}{\hat{T}'_{e0} \psi_{mk}} \right)_{\hat{r}_k + \hat{W}_k}, \quad (81)$$

$$\hat{\Psi}_{k-} = -\hat{T}'_{ek} \hat{W}_k \delta T_1(-1) \left(\frac{\hat{r} g}{q} \frac{m_k - n q}{\hat{T}'_{e0} \psi_{mk}} \right)_{\hat{r}_k - \hat{W}_k}. \quad (82)$$

For the special case $m = 1$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = -\frac{\xi^r(\hat{r}, \theta, \phi)}{\epsilon_a} \frac{dT_{e0}}{d\hat{r}}. \quad (83)$$