Ideal-MHD Energy Principle Analysis

I. INTRODUCTION

A. Fundamental Equations

Our fundamental equations are

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0, \tag{1}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] + \nabla p - \mathbf{j}\times\mathbf{B} = 0, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma\,p\,\nabla\cdot\mathbf{v} = 0, \tag{3}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \tag{4}$$

$$\mu_0\,\mathbf{j} = \nabla\times\mathbf{B}. \tag{5}$$

Note that Eq. (4) ensures that

$$\nabla\cdot\mathbf{B} = 0 \tag{6}$$

is automatically satisfied, provided that it is satisfied initially.

II. STABILITY ANALYSIS

A. Equilibrium

The equilibrium is such that

$$\rho(\mathbf{r},t) = \rho_0(\mathbf{r}), \tag{7}$$

$$\mathbf{v}(\mathbf{r},t) = \mathbf{0}, \tag{8}$$

$$p(\mathbf{r},t) = p_0(\mathbf{r}), \tag{9}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r}), \tag{10}$$

$$\mathbf{j}(\mathbf{r},t) = \mathbf{j}_0(\mathbf{r}), \tag{11}$$

where

$$\nabla p_0 = \mathbf{j}_0\times\mathbf{B}_0, \tag{12}$$

$$\mu_0 \, \mathbf{j}_0 = \nabla \times \mathbf{B}_0. \tag{13}$$

B. Perturbed Quantities

The perturbation is such that

$$\rho_1(\mathbf{r}, t) = \rho_1(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\,\omega\,t}, \tag{14}$$

$$\mathbf{v}_1(\mathbf{r}, t) = -\mathrm{i}\,\omega\,\boldsymbol{\xi}(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\,\omega\,t}, \tag{15}$$

$$p(\mathbf{r}, t) = p_1(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\,\omega\,t}, \tag{16}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_1(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\,\omega\,t}, \tag{17}$$

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_1(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\,\omega\,t}, \tag{18}$$

The linearized perturbed versions of Eqs. (1)–(5) are

$$\rho_1 = -\nabla \cdot (\rho_0 \, \boldsymbol{\xi}), \tag{19}$$

$$-\omega^2 \, \rho_0 \, \boldsymbol{\xi} = \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 - \nabla p_1, \tag{20}$$

$$p_1 = -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma \, p_0 \, \nabla \cdot \boldsymbol{\xi}, \tag{21}$$

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0), \tag{22}$$

$$\mu_0 \, \mathbf{j}_1 = \nabla \times \mathbf{B}_1. \tag{23}$$

Combining the previous four equations, we obtain

$$-\omega^2 \, \rho_0 \, \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{24}$$

where

$$\mathbf{F}(\boldsymbol{\xi}) = \mu_0^{-1} \, (\nabla \times \mathbf{Q}) \times \mathbf{B}_0 + \mu_0^{-1} \, (\nabla \times \mathbf{B}_0) \times \mathbf{Q} + \nabla(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma \, p_0 \, \nabla \cdot \boldsymbol{\xi}), \tag{25}$$

and

$$\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0). \tag{26}$$

C. Self-Adjointness of Force Operator

Suppose that the plasma is surrounded by a perfectly conducting wall whose outward unit normal is $\mathbf{n}$. We wish to demonstrate that

$$\int \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r} = \int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})\,d\mathbf{r}, \tag{27}$$

where $\boldsymbol{\xi}(\mathbf{r})$ and $\boldsymbol{\eta}(\mathbf{r})$ are two arbitrary vector fields that satisfy the boundary conditions

$$\mathbf{n}\cdot\boldsymbol{\xi} = \mathbf{n}\cdot\boldsymbol{\eta} = 0 \tag{28}$$

at the wall

According to Eq. (25), the integrand of the left-hand side of Eq. (27) takes the form

$$\boldsymbol{\eta}\cdot\mathbf{F}(\xi) = \boldsymbol{\eta}\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0) + \nabla(\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi})\right]. \tag{29}$$

The final term can be written

$$\boldsymbol{\eta}\cdot\nabla(\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi}) = \nabla\cdot(\boldsymbol{\eta}\,\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi}) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}). \tag{30}$$

However, according to Eq. (28), the divergence term integrates to zero. Hence, we obtain

$$\boldsymbol{\eta}\cdot\mathbf{F}(\xi) = \boldsymbol{\eta}\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0)\right] - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}). \tag{31}$$

Let us write

$$\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel\,\mathbf{b}, \tag{32}$$

$$\boldsymbol{\eta} = \boldsymbol{\eta}_\perp + \eta_\parallel\,\mathbf{b}, \tag{33}$$

where

$$\mathbf{b} = \frac{\mathbf{B}_0}{B_0}, \tag{34}$$

$$\mathbf{b}\cdot\boldsymbol{\xi}_\perp = \mathbf{b}\cdot\boldsymbol{\eta}_\perp = 0. \tag{35}$$

It follows from Eq. (26) that

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0). \tag{36}$$

Moreover, Eq. (12) implies that

$$\boldsymbol{\xi}\cdot\nabla p_0 = \boldsymbol{\xi}\cdot\mathbf{j}_0\times\mathbf{B}_0 = \boldsymbol{\xi}_\perp\cdot\mathbf{j}_0\times\mathbf{B}_0 = \boldsymbol{\xi}_\perp\cdot\nabla p_0. \tag{37}$$

Now,

$$\mathbf{B}_0 \cdot \left[\mu_0^{-1} (\nabla \times \mathbf{Q}) \times \mathbf{B}_0\right] = 0, \tag{38}$$

and

$$\mathbf{B}_0 \cdot \left[\mu_0^{-1} (\nabla \times \mathbf{B}_0) \times \mathbf{Q}\right] = \mathbf{B}_0 \cdot \mathbf{j}_0 \times \mathbf{Q} = -\mathbf{j}_0 \times \mathbf{B}_0 \cdot \mathbf{Q} = -\nabla p_0 \cdot \mathbf{Q}, \tag{39}$$

where use has been made of Eqs. (12) and (13). However, according to Eq. (36),

$$\begin{aligned} -\nabla p_0 \cdot \mathbf{Q} &= -\nabla p_0 \cdot \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B}_0) = \nabla \cdot [\nabla p_0 \times (\boldsymbol{\xi}_\perp \times \mathbf{B}_0)] = -\nabla \cdot [(\boldsymbol{\xi}_\perp \cdot \nabla p_0)\, \mathbf{B}_0] \\ &= -\mathbf{B}_0 \cdot \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p_0), \end{aligned} \tag{40}$$

where use has been made of Eqs. (6) and (12). The previous two equations imply that

$$\mathbf{B}_0 \cdot \left[\mu_0^{-1} (\nabla \times \mathbf{B}_0) \times \mathbf{Q} + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p_0)\right] = 0. \tag{41}$$