A Four-Field Resonant Response Model for Tokamak Plasmas

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I. INTRODUCTION

Tearing modes are slowly growing instabilities of ideally-stable tokamak plasmas that reconnect magnetic field-lines at various rational surfaces within the plasma, in the process forming magnetic island chains that locally flatten the pressure profile and, thereby, degrade the plasma confinement.[1] If tearing modes grow to sufficiently large amplitude then they can trigger major disruptions.[2] In fact, tokamak plasmas are observed to be particularly disruption-prone when tearing modes lock (i.e., become stationary in the laboratory frame) to externally generated, resonant magnetic perturbations.[3]

The analysis of tearing mode dynamics in tokamak plasmas is most efficiently formulated as an asymptotic matching problem in which the plasma is divided into two distinct regions.[4] In the so-called *outer region*, which comprises most of the plasma, the perturbation is governed by the equations of linearized, marginally-stable, ideal-magnetohydrodynamics (MHD). However, these equations become singular on *rational* magnetic flux-surfaces at which the perturbed magnetic field resonates with the equilibrium field. In the *inner region*, which consists of a set of narrow layers centered on the various rational surfaces, non-ideal-MHD effects become important.

It is well known that single-fluid, resistive magnetohydrodynamics (MHD) offers a very poor description of the response of the inner region to the tearing perturbation in the outer region. For instance, the strong diamagnetic flows present in tokamak plasmas imply that the electron and ion fluid velocities are significantly different from one another, necessitating a two-fluid treatment.[5] Moreover, resistive-MHD does not take into account the important ion sound radius lengthscale below which electron and ion dynamics become decoupled from one another.[6,7] Previously, Cole & Fitzpatrick[8] used the four-field model of Fitzpatrick & Waelbroeck[9] (which is based on the original four-field model of Hazeltine, Kotschenreuther & Morrison[10]) to determine the two-fluid response of a linear tearing layer to the perturbation in the outer region. This treatment was extended in Ref. 11 to take into account the anomalously large perpendicular particle diffusivity present in tokamak plasmas.

In configuration space, the four-field models of Refs. 8 and 11 yield a set of resonant layer equations that can be expressed as *ten* coupled first-order linear differential equations.[12] However, in Fourier space, the resonant layer equations can be written as *four* first-order linear differential equatons.[8] It is clearly advantageous to solve the equations in Fourier space. In Refs. 8 and 11, an approximation is made by which one of the terms in the Fourier-transformed layer equations is neglected. This approximation, which is valid in low-β plasmas, is such that the layer equation that governs the parallel ion dynamics decouples from the other three equations, effectively converting a four-field resonant response model into a three-field model. Furthermore, the three remaining equations can be combined to give a single second-order linear ordinary differential equation. This second-order equation is most conveniently solved numerically by means of a Riccati transformation that converts it into a first-order nonlinear differential equation.[13,14] The advantage of the Riccati approach is that it can deal with numerically problematic solutions that blow up as $\exp(p^2)$, or faster, at large p, where p is the Fourier-space variable.

Lee, Park & Na[12] recently demonstrated how to solve the full tenth-order four-field resonant layer equations in configuration space using a Riccati transformation. In the process, they discovered that, in a high-β plasma, the frequency (in the local $\mathbf{E}\times\mathbf{B}$ frame) at which resonant response of a tearing layer attains its maximum value is shifted in the ion diamagnetic direction from the electron diamagnetic frequency. This result is significant because there is some experimental evidence for such a shift.[15] In the present paper, we demonstrate how the calculation of Lee et alia can be reimplemented in Fourier space. The Fourier version of the calculation is more convenient, from a numerical point of view, because it involves the solution of a fourth-order, rather than a tenth-order, system of equations.

II. PRELIMINARY ANALYSIS

A. Plasma Equilibrium

Consider a large aspect-ratio tokamak plasma equilibrium whose magnetic flux-surfaces map out (almost) concentric circles in the poloidal plane. Such an equilibrium can be approximated as a periodic cylinder.[16] Let r, θ, z be right-handed cylindrical coordinates. The magnetic axis corresponds to $r = 0$, and the plasma boundary to $r = a$. The system is assumed to be periodic in the z direction with periodicity length $2\pi\,R_0$, where R_0 is the simulated major radius of the plasma. The safety-factor profile takes the form $q(r) = r\,B_z/[R_0\,B_\theta(r)]$, where B_z is the constant "toroidal" magnetic field-strength, and $B_\theta(r)$ is the poloidal magnetic field-strength. The standard large aspect-ratio orderings, $r/R_0 \ll 1$

and $B_\theta/B_z \ll 1$, are adopted.

B. Asymptotic Matching

Consider a tearing mode perturbation that has m periods in the poloidal direction, and n periods in the toroidal direction. The response of the plasma to the tearing mode is governed by marginally stable, ideal-MHD everywhere in the plasma, apart from a (radially) narrow layer centered on the rational surface, minor radius r_s, at which $q(r_s) = m/n$.[4]

The perturbed magnetic field associated with the tearing mode is written $\delta\mathbf{B} \simeq \nabla\delta\psi\times\mathbf{e}_z$, where $\delta\psi(r,\theta,\varphi,t) = \delta\psi(r,t)\,\exp[\,{\rm i}\,(m\,\theta - n\,\varphi)]$, and $\varphi = z/R_0$ is a simulated toroidal angle.

In the outer region (i.e., everywhere in the plasma apart from the resonant layer), the perturbed helical magnetic flux, $\delta\psi(r,t)$, satisfies the *cylindrical tearing mode equation*,[1]

$$\frac{\partial^2\delta\psi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\delta\psi}{\partial r} - \frac{m^2}{r^2}\,\delta\psi - \frac{J_z'\,\delta\psi}{r\,(1/q - n/m)} = 0, \tag{1}$$

where $J_z(r) = R_0\,\mu_0\,j_z(r)/B_z$, and $j_z(r)$ is the equilibrium "toroidal" current density. Here, $' \equiv d/dr$. In general, the solution of Eq. (1) that satisfies physical boundary conditions at the magnetic axis and the plasma boundary is such that $\delta\psi$ is continuous across the rational surface, whereas $\partial\delta\psi/\partial r$ is discontinuous. The discontinuity of $\partial\delta\psi/\partial r$ across the rational surface is indicative of the presence of a helical current sheet at the surface. The complex quantity $\Psi_s(t) = \delta\psi(r_s,t)$ determines the amplitude and phase of the reconnected helical magnetic flux at the rational surface, whereas the complex quantity[16]

$$\Delta\Psi_s = \left[r\,\frac{\partial\delta\psi}{\partial r}\right]_{r_{s-}}^{r_{s+}} \tag{2}$$

parameterizes the amplitude and phase of the helical current sheet flowing at the surface. The solution of the cylindrical tearing mode equation in the outer region, in the presence of an externally generated, resonant magnetic perturbation of the same helicity as the tearing mode, leads to the relation[4,16]

$$\Delta\Psi_s = (\Delta_s')\,\Psi_s + (-\Delta_s')\,\Psi_v, \tag{3}$$

where Δ_s' is a real dimensionless quantity known as the *tearing stability index*. Moreover, Ψ_v is the so-called *vacuum flux*, and is defined as the reconnected magnetic flux that would be driven at the rational surface by the resonant magnetic perturbation were the plasma intrinsically tearing stable (i.e., $\Delta'_s < 0$), and were there no current sheet at the rational surface (i.e., $\Delta\Psi_s = 0$).

The current sheet at the rational surface can only be resolved by solving a resistive-MHD plasma response model in the inner region (i.e., the region of the plasma in the immediate vicinity of the rational surface), and asymptotically matching the solution so obtained to the ideal-MHD solution in the outer region. The four-field plasma response model used in this paper is described in the following section.

III. FOUR-FIELD RESONANT PLASMA RESPONSE MODEL

A. Useful Definitions

The plasma is assumed to consist of two species. First, electrons of mass m_e, electrical charge $-e$, number density n, and temperature T_e. Second, ions of mass m_i, electrical charge $+e$, number density n, and temperature T_i. Let $p = n\,(T_e + T_i)$ be the total plasma pressure.

It is helpful to define $n_0 = n(r_s)$, $p_0 = p(r_s)$,

$$\eta_e = \left.\frac{d\ln T_e}{d\ln n}\right|_{r=r_s}, \tag{4}$$

$$\eta_i = \left.\frac{d\ln T_i}{d\ln n}\right|_{r=r_s}, \tag{5}$$

$$\iota = \left(\frac{T_e}{T_i}\right)_{r=r_s}\left(\frac{1+\eta_e}{1+\eta_i}\right), \tag{6}$$

where $n(r)$, $p(r)$, $T_e(r)$, and $T_i(r)$ refer to number density, pressure, and temperature profiles that are unperturbed by the tearing mode.

For the sake of simplicity, the perturbed electron and ion temperature profiles are assumed to be functions of the perturbed electron number density profile in the immediate vicinity of the rational surface. In other words, $T_e = T_e(n)$ and $T_i = T_i(n)$. This implies that $p = p(n)$. The "MHD velocity", which is the velocity of a fictional MHD fluid, is defined $\mathbf{V} = \mathbf{V}_E + V_{\parallel\,i}\,\mathbf{b}$, where $\mathbf{V}_E$ is the $\mathbf{E}\times\mathbf{B}$ drift velocity, $V_{\parallel\,i}$ is the parallel component of the ion fluid velocity, $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$, and $\mathbf{B}$ is the magnetic field-strength.

B. Fundamental Fields

The four fundamental fields in our four-field model—namely, ψ, N, ϕ, and V—have the following definitions:

$$\nabla\psi = \frac{\mathbf{n}\times\mathbf{B}}{r_s\,B_z}, \tag{7}$$

$$N = -\hat{d}_i\left(\frac{p-p_0}{B_z^{\,2}/\mu_0}\right), \tag{8}$$

$$\nabla\phi = \frac{\mathbf{n}\times\mathbf{V}}{r_s\,V_A}, \tag{9}$$

$$V = \hat{d}_i\left(\frac{\mathbf{n}\cdot\mathbf{V}}{V_A}\right). \tag{10}$$

Here, $\mathbf{n} = (0,\,\epsilon/q_s,\,1)$, $\epsilon = r/R_0$, $q_s = m/n$, $V_A = B_z/\sqrt{\mu_0\,n_0\,m_i}$, $d_i = \sqrt{m_i/(n_0\,e^{\,2}\,\mu_0)}$, and $\hat{d}_i = d_i/r_s$. Our model also employs the auxiliary field

$$J = -\frac{2\,\epsilon_s}{q_s} + \hat{\nabla}^2\psi, \tag{11}$$

where $\epsilon_s = r_s/R_0$, and $\hat{\nabla} = r_s\,\nabla$. Note that V_A is the Alfvén speed, whereas d_i is the collisionless ion skin-depth.

C. Fundamental Equations

The four-field model takes the form:[8,9,11]

$$\frac{\partial\psi}{\partial\hat{t}} = [\phi,\psi] - \iota_e\,[N,\psi] + \hat{\eta}_\parallel\,J + \hat{E}_\parallel, \tag{12}$$

$$\frac{\partial N}{\partial\hat{t}} = [\phi,N] + \hat{d}_\beta^{\,2}\,[J,\psi] + c_\beta^{\,2}\,[V,\psi] + \hat{D}_\perp\,\hat{\nabla}_\perp^{\,2}N, \tag{13}$$

$$\frac{\partial\hat{\nabla}^2\phi}{\partial\hat{t}} = [\phi,\hat{\nabla}^2\phi] - \frac{\iota_i}{2}\left(\hat{\nabla}^2[\phi,N] + [\hat{\nabla}^2\phi,N] + [\hat{\nabla}^2 N,\phi]\right) + [J,\psi] + \hat{\chi}_\varphi\,\hat{\nabla}^4(\phi + \iota_i\,N)\,, \tag{14}$$

$$\frac{\partial V}{\partial\hat{t}} = [\phi,V] + [N,\psi] + \hat{\chi}_\varphi\,\hat{\nabla}^2 V. \tag{15}$$

Here, $[A,B] \equiv \hat{\nabla}A\times\hat{\nabla}B\cdot\mathbf{n}$, $\iota_e = \iota/(1+\iota)$, $\iota_i = 1/(1+\iota)$, $\hat{t} = t/(r_s/V_A)$, $\hat{\eta}_\parallel = \eta_\parallel/(\mu_0\,r_s\,V_A)$, $\hat{E}_\parallel = E_\parallel/(B_z\,V_A)$, $\hat{\chi}_\varphi = \chi_\varphi/(r_s\,V_A)$, where $\eta_\parallel$ is the parallel plasma electrical resistivity at the rational surface, $E_\parallel$ the parallel inductive electric field that maintains the equilibrium toroidal plasma current in the vicinity of the rational surface, χ_φ the anomalous perpendicular ion momentum diffusivity at the rational surface, and $D_\perp$ the anomalous perpendicular electron/ion energy diffusivity in the vicinity of the rational surface. Moreover, $d_\beta = c_\beta\, d_i$, and $\hat{d}_\beta = d_\beta/r_s$, where $c_\beta = \sqrt{\beta/(1+\beta)}$, and $\beta = (5/3)\,\mu_0\, p_0/B_z^{\,2}$. Here, d_β is usually referred to as the *ion sound radius*.

D. Matching to Plasma Equilibrium

The unperturbed plasma equilibrium is such that $\mathbf{B} = (0,\, B_\theta(r),\, B_z)$, $p = p(r)$, $\mathbf{V} = (0,\, V_E(r),\, V_z(r))$, where $V_E(r) \simeq E_r/B_z$ is the (dominant θ-component of the) $\mathbf{E}\times\mathbf{B}$ velocity. Now, the resonant layer is assumed to have a radial thickness that is much smaller than r_s. Hence, we only need to evaluate plasma equilibrium quantities in the immediate vicinity of the rational surface. Equations (7)–(10) suggest that

$$\psi(\hat{x}) = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s}, \tag{16}$$

$$N(\hat{x}) = -\hat{V}_*\,\hat{x}, \tag{17}$$

$$\phi(\hat{x}) = -\hat{V}_E\,\hat{x}, \tag{18}$$

$$V(\hat{x}) = \hat{V}_\parallel, \tag{19}$$

where $\hat{x} = (r - r_s)/r_s$, $\hat{L}_s = L_s/r_s$, $L_s = R_0\, q_s/s_s$, $\hat{V}_E = V_E(r_s)/V_A$, $\hat{V}_* = V_*(r_s)/V_A$, $V_*(r) = (dp/dr)/(e\, n_0\, B_z)$ is the (dominant θ-component of the) diamagnetic velocity, and $\hat{V}_\parallel = \hat{d}_i\, V_z(r_s)/V_A$. Here, $s_s = s(r_s)$ and $s(r) = d\ln q/d\ln r$. We also have

$$J(\hat{x}) = -\left(\frac{2}{s_s} - 1\right)\frac{1}{\hat{L}_s}, \tag{20}$$

and $\hat{E}_\parallel(\hat{x}) = (2/s_s - 1)\,(\hat{\eta}_\parallel/\hat{L}_s)$.

IV. LINEAR RESONANT PLASMA RESPONSE MODEL

A. Introduction

The aim of this section is to obtain a set of linear layer equations from the four-field model introduced in Sect. III.

B. Derivation of Linear Layer Equations

In accordance with Eqs. (16)–(20), let us write

$$\psi(\hat{x},\zeta,\hat{t}) = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s} + \tilde{\psi}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{21}$$

$$\phi(\hat{x},\zeta,\hat{t}) = -\hat{V}_E\,\hat{x} + \tilde{\phi}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{22}$$

$$N(\hat{x},\zeta,\hat{t}) = -\hat{V}_*\,\hat{x} + \iota_e\,\tilde{N}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{23}$$

$$V(\hat{x},\zeta,\hat{t}) = \hat{V}_\parallel + \iota_e\,\tilde{V}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{24}$$

$$J(\hat{x},\zeta,\hat{t}) = -\left(\frac{2}{s_s}-1\right)\frac{1}{\hat{L}_s} + \hat{\nabla}^2\tilde{\psi}(\hat{x})\,{\rm e}^{\,{\rm i}\,(\zeta-\hat{\omega}\,\hat{t})}, \tag{25}$$

where $\zeta = m\,\theta - n\,\varphi$, $\hat{\omega} = r_s\,\omega/V_A$, and ω is the frequency of the tearing mode in the laboratory frame. Substituting Eqs. (21)–(25) into Eqs. (11)–(15), and only retaining terms that are first order in perturbed quantities, we obtain the following set of linear equations:

$$-{\rm i}\,(\omega-\omega_E-\omega_{*\,e})\,\tau_H\,\tilde{\psi} = -{\rm i}\,\hat{x}\,(\tilde{\phi}-\tilde{N}) + S^{-1}\,\hat{\nabla}^2\tilde{\psi}, \tag{26}$$

$$\begin{aligned}-{\rm i}\,(\omega-\omega_E)\,\tau_H\,\tilde{N} &= -{\rm i}\,\omega_{*\,e}\,\tau_H\,\tilde{\phi} - {\rm i}\,\iota_e\,\hat{d}_\beta^{\,2}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} - {\rm i}\,c_\beta^{\,2}\,\hat{x}\,\tilde{V}\\ &\quad + S^{-1}\,P_\perp\,\hat{\nabla}_\perp^2\tilde{N},\end{aligned} \tag{27}$$

$$-{\rm i}\,(\omega-\omega_E-\omega_{*\,i})\,\tau_H\,\hat{\nabla}^2\tilde{\phi} = -{\rm i}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\varphi\,\hat{\nabla}^4\left(\tilde{\phi}+\frac{\tilde{N}}{\iota}\right), \tag{28}$$

$$-{\rm i}\,(\omega-\omega_E)\,\tau_H\,\tilde{V} = {\rm i}\,\omega_{*\,e}\,\tau_H\,\tilde{\psi} - {\rm i}\,\hat{x}\,\tilde{N} + S^{-1}\,P_\varphi\,\hat{\nabla}^2\tilde{V}. \tag{29}$$

Here, $\tau_H = L_s/(m\,V_A)$ is the hydromagnetic time, $\omega_E = (m/r_s)\,V_E(r_s)$ the $\mathbf{E}\times\mathbf{B}$ frequency, $\omega_{*\,e} = -\iota_e\,(m/r_s)V_*(r_s)$ the electron diamagnetic frequency, $\omega_{*\,i} = \iota_i\,(m/r_s)V_*(r_s)$ the ion diamagnetic frequency, $S = \tau_R/\tau_H$ the Lundquist number, $\tau_R = \mu_0\,r_s^{\,2}/\eta_\parallel$ the resistive diffusion time, $\tau_\varphi = r_s^{\,2}/\chi_\varphi$ the toroidal momentum confinement time, and $\tau_\perp = r_s^{\,2}/D_\perp$. Furthermore, $P_\varphi = \tau_R/\tau_\varphi$ and $P_\perp = \tau_R/\tau_\perp$ are magnetic Prandtl numbers.

Let us define the stretched radial variable $X = S^{\,1/3}\,\hat{x}$. Assuming that $X\sim{\cal O}(1)$ in the layer (i.e., assuming that the layer thickness is roughly of order $S^{-1/3}\,r_s$), and making use of the fact that $S\gg 1$ in conventional tokamak plasmas, Eqs. (26)–(29) reduce to the following set of linear layer equations: [8]

$$(g + \mathrm{i}\, Q_e)\, \tilde{\psi} = -\mathrm{i}\, X \left(\tilde{\phi} - \tilde{N}\right) + \frac{d^2\tilde{\psi}}{dX^2}, \tag{30}$$

$$g\, \tilde{N} = -\mathrm{i}\, Q_e\, \tilde{\phi} - \mathrm{i}\, D^2\, X\, \frac{d^2\tilde{\psi}}{dX^2} - \mathrm{i}\, c_\beta^2\, X\, \tilde{V} + P_\perp\, \frac{d^2\tilde{N}}{dX^2}, \tag{31}$$

$$(g + \mathrm{i}\, Q_i)\, \frac{d^2\tilde{\phi}}{dX^2} = -\mathrm{i}\, X\, \frac{d^2\tilde{\psi}}{dX^2} + P_\varphi\, \frac{d^4}{dX^4}\left(\tilde{\phi} + \frac{\tilde{N}}{\iota}\right), \tag{32}$$

$$g\, \tilde{V} = \mathrm{i}\, Q_e\, \tilde{\psi} - \mathrm{i}\, X\, \tilde{N} + P_\varphi\, \frac{d^2\tilde{V}}{dX^2}. \tag{33}$$

Here, $g = -\mathrm{i}\,(Q - Q_E) = -\mathrm{i}\, S^{1/3}\,(\omega - \omega_E)\,\tau_H$, $Q_{e,i} = S^{1/3}\,\omega_{\ast\, e,i}\,\tau_H$, and $D = S^{1/3}\,\iota_e^{1/2}\,\hat{d}_\beta$. If we write $P_\perp = c_\beta^2$ then Eqs. (30)–(33) become equivalent to the set of layer equations solved by Lee et alia.

The previously mentioned low-β approximation used in Refs. 8 and 11 involves neglecting the term containing c_β^2 in Eq. (31). This approximation decouples Eq. (33) from the three preceding equations, and effectively converts a four-field resonant response model into a three-field model. In the following, we shall not use this approximation.

C. Asymptotic Matching

The linear layer equations, (30)–(33), possess tearing parity solutions characterized by the symmetry $\tilde{\psi}(-X) = \tilde{\psi}(X)$, $\tilde{N}(-X) = -\tilde{N}(X)$, $\tilde{\phi}(-X) = -\tilde{\phi}(X)$, $\tilde{V}(-X) = \tilde{V}(X)$. If we assume that the asymptotic behavior of the tearing parity layer solutions is such that

$$\tilde{\psi}(X) \to \psi_0 \left[1 + \frac{\hat{\Delta}}{2}\,|X| + {\cal O}(X^2)\right] \tag{34}$$

as $|X| \to \infty$, where ψ_0 is an arbitrary constant, then asymptotic matching to the outer solution yields

$$\Delta\Psi_s = (S^{1/3}\,\hat{\Delta})\;\Psi_s. \tag{35}$$

V. FOURIER-TRANSFORMED FOUR-FIELD EQUATIONS

A. Fourier Transformation

Equations (30)–(33) are most conveniently solved in Fourier transform space.[8] Let

$$\bar{\phi}(p) = \int_{-\infty}^{\infty} \tilde{\phi}(X)\, \mathrm{e}^{-\mathrm{i}\, p\, X}\, dX, \tag{36}$$

et cetera. The Fourier transformed linear layer equations become

$$(g + \mathrm{i}\, Q_e)\, \bar{\psi} = \frac{d}{dp}\big(\bar{\phi} - \bar{N}\big) - p^2\, \bar{\psi}, \tag{37}$$

$$g\, \bar{N} = -\mathrm{i}\, Q_e\, \bar{\phi} - D^2\, \frac{d(p^2\, \bar{\psi})}{dp} + c_\beta^2\, \frac{d\bar{V}}{dp} - P_\perp\, p^2 \bar{N}, \tag{38}$$

$$(g + \mathrm{i}\, Q_i)\, p^2\, \bar{\phi} = \frac{d(p^2\, \bar{\psi})}{dp} - P_\varphi\, p^4 \left(\bar{\phi} + \frac{\bar{N}}{\iota}\right), \tag{39}$$

$$g\, \bar{V} = \mathrm{i}\, Q_e\, \bar{\psi} + \frac{d\bar{N}}{dp} - P_\varphi\, p^2\, \bar{V}, \tag{40}$$

where, for a tearing parity solution,

$$\bar{\phi}(p) \to \bar{\phi}_0 \left[\frac{\hat{\Delta}}{\pi\, p} + 1 + \mathcal{O}(p)\right] \tag{41}$$

as $p \to 0$.

Finally, if we define

$$\bar{J}(p) = p^2\, \bar{\psi}, \tag{42}$$

$$\bar{Y}(p) = \bar{\phi} - \bar{N}, \tag{43}$$

then we obtain the following set of four coupled first-order differential equations:

$$\frac{d\bar{Y}}{dp} = \left(\frac{g + \mathrm{i}\, Q_e + p^2}{p^2}\right) \bar{J}, \tag{44}$$

$$\frac{d\bar{N}}{dp} = \left(\frac{-\mathrm{i}\, Q_e}{p^2}\right) \bar{J} + (g + P_\varphi\, p^2)\, \bar{V}, \tag{45}$$

$$\frac{d\bar{J}}{dp} = [(g + \mathrm{i}\, Q_i)\, p^2 + P_\varphi\, p^4]\, \bar{Y} + [(g + \mathrm{i}\, Q_i)\, p^2 + \iota_e^{-1}\, P_\varphi\, p^4]\, \bar{N}, \tag{46}$$

$$\begin{aligned} c_\beta^2\, \frac{d\bar{V}}{dp} &= [\mathrm{i}\, Q_e + D^2\, (g + \mathrm{i}\, Q_i)\, p^2 + D^2\, P_\varphi\, p^4]\, \bar{Y} \\ &\quad + (g + \mathrm{i}\, Q_e + [P_\perp + D^2\, (g + \mathrm{i}\, Q_i)]\, p^2 + \iota_e^{-1}\, D^2\, P_\varphi\, p^4)\, \bar{N}, \end{aligned} \tag{47}$$

Note that $\iota_e = -Q_e/(Q_i - Q_e)$.

B. Small Argument Expansion

Let us search for power-law solutions of Eqs. (44)–(47) at small values of p. Given that we have four coupled first-order differential equations, we expect to find four independent power-law solutions. The first solution is such that

$$\bar{Y}(p) = (g + \mathrm{i}\,Q_e)\,a_{-1}\,p^{-1} + \left[\frac{1}{2}\,g\,(g + \mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i) - 1\right] a_{-1}\,p + \mathcal{O}(p^3), \tag{48}$$

$$\bar{N}(p) = -\mathrm{i}\,Q_e\,a_{-1}\,p^{-1} - \frac{1}{2}\,g\,(\mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i)\,a_{-1}\,p + \mathcal{O}(p^3), \tag{49}$$

$$\bar{J}(p) = -a_{-1} + \frac{1}{2}\,g\,(g + \mathrm{i}\,Q_i)\,a_{-1}\,p^2 + \mathcal{O}(p^4), \tag{50}$$

$$\bar{V}(p) = \frac{[-\mathrm{i}\,Q_e\,(1 + P_\perp) + g\,(g + \mathrm{i}\,Q_i)\,D^2]}{2\,c_\beta^2}\,a_{-1}\,p^2 + \mathcal{O}(p^4), \tag{51}$$

where a_{-1} is an arbitrary constant. The second solution is such that

$$\bar{Y}(p) = (g + \mathrm{i}\,Q_e)\,a_0 + \frac{1}{6}\,g\,(g + \mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i)\,a_0\,p^2 + \mathcal{O}(p^4), \tag{52}$$

$$\bar{N}(p) = -\mathrm{i}\,Q_e\,a_0 - \frac{1}{6}\,g\,(\mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i)\,a_0\,p^2 + \mathcal{O}(p^4), \tag{53}$$

$$\bar{J}(p) = \frac{1}{3}\,g\,(g + \mathrm{i}\,Q_i)\,a_0\,p^3 + \mathcal{O}(p^5), \tag{54}$$

$$\bar{V}(p) = \frac{1}{3}\,\frac{[-\mathrm{i}\,Q_e\,P_\perp + g\,(g + \mathrm{i}\,Q_i)\,D^2]}{c_\beta^2}\,a_0\,p^3 + \mathcal{O}(p^5), \tag{55}$$

where a_0 is an arbitrary constant. The third solution is such that

$$\bar{Y}(p) = \frac{1}{6}\,(g + \mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i)\,a_2\,p^2 + \mathcal{O}(p^4), \tag{56}$$

$$\bar{N}(p) = a_2 + \frac{1}{2}\,(g + \mathrm{i}\,Q_i)\left(-\frac{1}{3}\,\mathrm{i}\,Q_e + \frac{g}{c_\beta^2}\right) a_2\,p^2 + \mathcal{O}(p^4), \tag{57}$$

$$\bar{J}(p) = \frac{1}{3}\,(g + \mathrm{i}\,Q_i)\,a_2\,p^3 + \mathcal{O}(p^5), \tag{58}$$

$$\bar{V}(p) = \frac{(g + \mathrm{i}\,Q_e)}{c_\beta^2}\,a_2\,p + \mathcal{O}(p^3), \tag{59}$$

where a_2 is an arbitrary constant. The final solution is such that

$$\bar{Y}(p) = \frac{1}{12}\,g\,(g+\mathrm{i}\,Q_e)\,(g+\mathrm{i}\,Q_i)\,a_3\,p^3 + \mathcal{O}(p^5), \tag{60}$$

$$\bar{N}(p) = g\,a_3\,p + \mathcal{O}(p^3), \tag{61}$$

$$\bar{J}(p) = \frac{1}{4}\,g\,(g+\mathrm{i}\,Q_i)\,a_3\,p^4 + \mathcal{O}(p^6), \tag{62}$$

$$\bar{V}(p) = \frac{g\,(g+\mathrm{i}\,Q_e)}{2\,c_\beta^2}\,a_3\,p^2 + \mathcal{O}(p^4), \tag{63}$$

where a_3 is an arbitrary constant.

We conclude that, at small values of p, the most general solution for $\bar{Y}(p)$ and $\bar{N}(p)$ takes the form

$$\bar{Y}(p) = (g+\mathrm{i}\,Q_e)\,a_{-1}\,p^{-1} + (g+\mathrm{i}\,Q_e)\,a_0 + \mathcal{O}(p), \tag{64}$$

$$\bar{N}(p) = (-\mathrm{i}\,Q_e)\,a_{-1}\,p^{-1} + (-\mathrm{i}\,Q_e)\,a_0 + a_2 + \mathcal{O}(p). \tag{65}$$

C. Ricatti Matrix Differential Equation

Let

$$\underline{u} = \begin{pmatrix} \bar{Y} \\ \bar{N} \end{pmatrix}, \tag{66}$$

$$\underline{v} = \begin{pmatrix} \bar{J} \\ c_\beta^2\,\bar{V} \end{pmatrix}. \tag{67}$$

Equations (44)–(47) can be written in the form

$$\frac{d\underline{u}}{dp} = \underline{\underline{A}}\,\underline{v}, \tag{68}$$

$$\frac{d\underline{v}}{dp} = \underline{\underline{B}}\,\underline{u}, \tag{69}$$

where

$$A_{11} = \frac{g + \mathrm{i}\, Q_e + p^2}{p^2}, \tag{70}$$

$$A_{12} = 0, \tag{71}$$

$$A_{21} = \frac{-\mathrm{i}\, Q_e}{p^2}, \tag{72}$$

$$A_{22} = \frac{g + P_\varphi\, p^2}{c_\beta^2}, \tag{73}$$

$$B_{11} = (g + \mathrm{i}\, Q_i)\, p^2 + P_\varphi\, p^4, \tag{74}$$

$$B_{12} = (g + \mathrm{i}\, Q_i)\, p^2 + \iota_e^{-1}\, P_\varphi\, p^4, \tag{75}$$

$$B_{21} = \mathrm{i}\, Q_e + D^{\,2}\, (g + \mathrm{i}\, Q_i)\, p^2 + D^{\,2}\, P_\varphi\, p^4, \tag{76}$$

$$B_{22} = g + \mathrm{i}\, Q_e + [P_\perp + D^{\,2}\, (g + \mathrm{i}\, Q_i)]\, p^2 + \iota_e^{-1}\, D^{\,2}\, P_\varphi\, p^4. \tag{77}$$

Thus, we obtain the following matrix differential equation:

$$\frac{d}{dp}\left(\underline{\underline{A}}^{-1}\,\frac{d\underline{u}}{dp}\right) = \underline{\underline{B}}\,\underline{u}. \tag{78}$$

Let

$$p\,\frac{d\underline{u}}{dp} = \underline{\underline{W}}\,\underline{u}. \tag{79}$$

The previous two equations can be combined to give

$$\left(p\,\frac{d\underline{\underline{W}}}{dp} - \underline{\underline{W}} + \underline{\underline{W}}\,\underline{\underline{W}} + \underline{\underline{A}}\,p\,\frac{d\underline{\underline{A}}^{-1}}{dp}\,\underline{\underline{W}} - p^2\,\underline{\underline{A}}\,\underline{\underline{B}}\right)\underline{u} = \underline{0}, \tag{80}$$

which yields the Riccati matrix differential equation,

$$p\,\frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}}\,\underline{\underline{W}} - \underline{\underline{E}}\,\underline{\underline{W}} + \underline{\underline{F}}, \tag{81}$$

where

$$\underline{\underline{E}}(p) = \underline{\underline{A}}\,p\,\frac{d\underline{\underline{A}}^{-1}}{dp}, \tag{82}$$

$$\underline{\underline{F}}(p) = p^2\,\underline{\underline{A}}\,\underline{\underline{B}}. \tag{83}$$

In fact, it is easily demonstrated that

$$E_{11} = \frac{2\,(g+{\rm i}\,Q_e)}{g+{\rm i}\,Q_e+p^2}, \tag{84}$$

$$E_{12} = 0, \tag{85}$$

$$E_{21} = -\frac{2\,{\rm i}\,Q_e\,(g+2\,P_\varphi\,p^2)}{(g+{\rm i}\,Q_e+p^2)\,(g+P_\varphi\,p^2)}, \tag{86}$$

$$E_{22} = -\frac{2\,P_\varphi\,p^2}{g+P_\varphi\,p^2}, \tag{87}$$

and

$$F_{11} = p^2\,(g+{\rm i}\,Q_e+p^2)\,(g+{\rm i}\,Q_i+P_\varphi\,p^2), \tag{88}$$

$$F_{12} = p^2\,(g+{\rm i}\,Q_e+p^2)\,(g+{\rm i}\,Q_i+\iota_e^{-1}\,P_\varphi\,p^2), \tag{89}$$

$$\begin{aligned} F_{21} &= -{\rm i}\,Q_e\,p^2\,(g+{\rm i}\,Q_i+P_\varphi\,p^2)\\ &\quad+c_\beta^{-2}\,p^2\,(g+P_\varphi\,p^2)\,[{\rm i}\,Q_e+D^{\,2}\,(g+{\rm i}\,Q_i)\,p^2+D^{\,2}\,P_\varphi\,p^4], \end{aligned} \tag{90}$$

$$\begin{aligned} F_{22} &= -{\rm i}\,Q_e\,p^2\,(g+{\rm i}\,Q_i+\iota_e^{-1}\,P_\varphi\,p^2)\\ &\quad+c_\beta^{-2}\,p^2\,(g+P_\varphi\,p^2)\,[g+{\rm i}\,Q_e+[P_\perp+D^{\,2}\,(g+{\rm i}\,Q_i)]\,p^2+\iota_e^{-1}\,D^{\,2}\,P_\varphi\,p^4]. \end{aligned} \tag{91}$$

Finally, if

$$\underline{\underline{W}}(p) = \begin{pmatrix} W_{11}, & W_{12}\\ W_{21}, & W_{22}\end{pmatrix} \tag{92}$$

then Eq. (81) yields

$$p\,\frac{dW_{11}}{dp} = W_{11} - W_{11}\,W_{11} - W_{12}\,W_{21} - E_{11}\,W_{11} + F_{11}, \tag{93}$$

$$p\,\frac{dW_{12}}{dp} = W_{12} - W_{11}\,W_{12} - W_{12}\,W_{22} - E_{11}\,W_{12} + F_{12}, \tag{94}$$

$$p\,\frac{dW_{21}}{dp} = W_{21} - W_{21}\,W_{11} - W_{22}\,W_{21} - E_{21}\,W_{11} - E_{22}\,W_{21} + F_{21}, \tag{95}$$

$$p\,\frac{dW_{22}}{dp} = W_{22} - W_{21}\,W_{12} - W_{22}\,W_{22} - E_{21}\,W_{12} - E_{22}\,W_{22} + F_{22}. \tag{96}$$

Thus, our final system of equations consists of a set of four coupled nonlinear differential equations.

D. Small Argument Behavior of Riccati Matrix Differential Equation

It follows from Eqs. (84)–(87) that $\underline{\underline{E}}(p) = \underline{\underline{E}}^{(0)} + {\cal O}(p^2)$ at small values of p, where

$$E_{11}^{(0)} = 2, \tag{97}$$

$$E_{12}^{(0)} = 0, \tag{98}$$

$$E_{21}^{(0)} = -\frac{2\,{\rm i}\,Q_e}{g+{\rm i}\,Q_e}, \tag{99}$$

$$E_{22}^{(0)} = 0. \tag{100}$$

Likewise, Eqs. (88)–(91) imply that $\underline{\underline{F}}(p) = {\cal O}(p^2)$.

Suppose that $\underline{\underline{W}}(p) = \underline{\underline{W}}^{(0)} + \underline{\underline{W}}^{(1)}\,p$ at small values of p, where the elements of $\underline{\underline{W}}^{(0)}$ and $\underline{\underline{W}}^{(1)}$ are independent of p. Equation (81) gives

$$\underline{\underline{0}} = \underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)}\,\underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)}\,\underline{\underline{W}}^{(0)}, \tag{101}$$

$$\underline{\underline{0}} = -\underline{\underline{W}}^{(1)}\,\underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)}\,\underline{\underline{W}}^{(1)} - \underline{\underline{E}}^{(0)}\,\underline{\underline{W}}^{(1)}. \tag{102}$$

Suitable solutions are

$$\underline{\underline{W}}^{(0)} = \left(\begin{array}{cc} -1, & 0\\ -E_{21}^{(0)}/2, & 0\end{array}\right), \tag{103}$$

$$W_{12}^{(1)} = 0, \tag{104}$$

$$W_{21}^{(1)} = -\frac{E_{21}^{(0)}}{2}\left[W_{11}^{(1)} - W_{22}^{(1)}\right]. \tag{105}$$

At small values of p, let

$$\underline{u}(p) = \underline{u}_{-1}\,p^{-1} + \underline{u}_0, \tag{106}$$

where the elements of $\underline{u}_{-1}$ (which are y_{-1} and n_{-1}, respectively) and the elements of $\underline{u}_0$ (which are y_0 and n_0, respectively) are all constants. Equation (79) gives

$$\underline{\underline{W}}^{(0)}\,\underline{u}_{-1} = -\underline{u}_{-1}, \tag{107}$$

$$\underline{\underline{W}}^{(0)}\,\underline{u}_0 + \underline{\underline{W}}^{(1)}\,\underline{u}_{-1} = \underline{0}. \tag{108}$$

Thus, making use of Eq. (103), we get

$$\left(\begin{array}{cc} -1, & 0\\ -E_{21}^{(0)}/2, & 0\end{array}\right)\left(\begin{array}{c} y_{-1}\\ n_{-1}\end{array}\right) = -\left(\begin{array}{c} y_{-1}\\ n_{-1}\end{array}\right), \tag{109}$$

which implies that

$$\frac{E_{21}^{(0)}}{2}\,y_{-1} = -\frac{\mathrm{i}\,Q_e}{g+\mathrm{i}\,Q_e}\,y_{-1} = n_{-1}, \tag{110}$$

in accordance with Eqs. (64) and (65), where use has been made of Eq. (99). Thus, if we write $y_{-1} = (g+\mathrm{i}\,Q_e)\,a_{-1}$, $n_{-1} = -\mathrm{i}\,Q_e\,a_{-1}$, $y_0 = (g+\mathrm{i}\,Q_e)\,a_0$, and $n_0 = -\mathrm{i}\,Q_e\,a_0 + a_2$, in accordance with Eqs. (64) and (65), then we deduce from Eqs. (41), (43), (103)–(105), and (108) that

$$\frac{\pi}{\hat{\Delta}} \equiv \frac{a_0}{a_{-1}} = W_{11}^{(1)} = \frac{dW_{11}(0)}{dp}. \tag{111}$$

E. Large Argument Behavior of Riccati Matrix Differential Equation

At large values of p, it is clear from Eqs. (88)–(91) that $\underline{\underline{F}}(p) = \underline{\underline{F}}^{(6)}\,p^6 + \underline{\underline{F}}^{(8)}\,p^8$, where the elements of $\underline{\underline{F}}^{(6)}$ and $\underline{\underline{F}}^{(8)}$ are constants. On the other hand, Eqs. (84)–(87) imply that $\underline{\underline{E}}(p) = \underline{\underline{E}}^{(0)}$, where the elements of $\underline{\underline{E}}^{(0)}$ are constants. Thus, if we write $\underline{\underline{W}}(p) = \underline{\underline{W}}^{(2)}\,p^2 + \underline{\underline{W}}^{(4)}\,p^4$, where the elements of $\underline{\underline{W}}^{(2)}$ and $\underline{\underline{W}}^{(4)}$ are constants, then Eq. (81) gives

$$\underline{\underline{W}}^{(4)}\,\underline{\underline{W}}^{(4)} = \underline{\underline{F}}^{(8)}, \tag{112}$$

$$\underline{\underline{W}}^{(2)}\,\underline{\underline{W}}^{(4)} + \underline{\underline{W}}^{(4)}\,\underline{\underline{W}}^{(2)} = \underline{\underline{F}}^{(6)}. \tag{113}$$

Now, according to Eqs. (88)–(91),

$$F_{11}^{(8)} = 0, \tag{114}$$

$$F_{12}^{(8)} = 0, \tag{115}$$

$$F_{21}^{(8)} = c_\beta^{-2}\,D^2\,P_\varphi^{\,2}, \tag{116}$$

$$F_{22}^{(8)} = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^{\,2}, \tag{117}$$

so Eq. (112) yields

$$W_{11}^{(4)} = 0, \tag{118}$$

$$W_{12}^{(4)} = 0, \tag{119}$$

$$W_{21}^{(4)} = -c_\beta^{-1}\,\iota_e^{1/2}\,D\,P_\varphi, \tag{120}$$

$$W_{22}^{(4)} = -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi, \tag{121}$$

where we have chosen the sign of the square root that is associated with well-behaved solutions at large values of p. Here, we are assuming that $\iota_e > 0$. Equations (88)–(91) also give

$$F_{11}^{(6)} = P_\varphi, \tag{122}$$

$$F_{12}^{(6)} = \iota_e^{-1}\,P_\varphi, \tag{123}$$

$$F_{21}^{(6)} = c_\beta^{-2}\,D^{\,2}\,g\,P_\varphi + c_\beta^{-2}\,D^{\,2}\,(g+\mathrm{i}\,Q_i)\,P_\varphi, \tag{124}$$

$$F_{22}^{(6)} = c_\beta^{-2}\,\iota_e^{-1}\,D^{\,2}\,g\,P_\varphi + c_\beta^{-2}\,[P_\perp + D^{\,2}\,(g+\mathrm{i}\,Q_i)]\,P_\varphi. \tag{125}$$

Thus, Eq. (113) yields

$$W_{12}^{(2)}\,W_{21}^{(4)} = F_{11}^{(6)}, \tag{126}$$

$$W_{12}^{(2)}\,W_{22}^{(4)} = F_{12}^{(6)}, \tag{127}$$

which gives

$$W_{12}^{(2)} = -c_\beta\,\iota_e^{-1/2}\,D^{-1}. \tag{128}$$

Now, if

$$\underline{\underline{W}}\,\underline{u} = \lambda(p)\,\underline{u} \tag{129}$$

then Eq. (79) yields

$$p\,\frac{d\underline{u}}{dp} = \lambda\,\underline{u}, \tag{130}$$

which implies that

$$\underline{u}(p) = \underline{u}(p_0)\,\exp\left[\int_{p_0}^{p}\frac{\lambda_r(p')}{p'}\,dp'\right]\exp\left[\mathrm{i}\int_{p_0}^{p}\frac{\lambda_i(p')}{p'}\,dp'\right], \tag{131}$$

where λ_r and λ_i are the real and imaginary parts of λ, respectively. Of course, a solution that is well behaved at large values of p is such that λ_r is negative. As we have seen, the large-p limit of Eq. (81) is

$$\underline{\underline{W}}\,\underline{\underline{W}} = \underline{\underline{F}}. \tag{132}$$

Hence, if

$$\underline{\underline{F}}\,\underline{u} = \Lambda\,\underline{u} \tag{133}$$

then Eqs. (129) and (133) imply that

$$\lambda^2 = \Lambda. \tag{134}$$

The eigenvalue problem for the F-matrix reduces to

$$\Lambda^2 - (F_{11} + F_{22})\,\Lambda + F_{11}\,F_{22} - F_{12}\,F_{21} = 0. \tag{135}$$

Now,

$$F_{11} + F_{22} \simeq F_{22}^{(8)}\,p^8 = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^2\,p^8, \tag{136}$$

$$\begin{aligned} F_{11}\,F_{22} - F_{12}\,F_{21} \simeq & \left[F_{11}^{(6)}\,F_{22}^{(8)} - F_{12}^{(6)}\,F_{21}^{(8)}\right] p^{14} \\ & + \left[F_{11}^{(6)}\,F_{22}^{(6)} - F_{12}^{(6)}\,F_{21}^{(6)}\right] p^{12} = c_\beta^{-2}\,R\,P_\varphi^2\,p^{12}, \end{aligned} \tag{137}$$

where

$$R = P_\perp + (1 - \iota_e^{-1})\,D^2\,(g + {\rm i}\,Q_i), \tag{138}$$

Hence, the two eigenvalues of the F-matrix are

$$\Lambda_1 \simeq F_{22}^{(8)}\,p^8 = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^2\,p^8, \tag{139}$$

$$\Lambda_2 \simeq \frac{[F_{11}^{(6)}\,F_{22}^{(6)} - F_{12}^{(6)}\,F_{21}^{(6)}]}{F_{22}^{(8)}}\,p^4 = \iota_e\,D^{-2}\,R\,p^4. \tag{140}$$

Thus, we deduce that the two eigenvalues of the W-matrix are

$$\lambda_1 = -\Lambda_1^{1/2} = -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi\,p^4, \tag{141}$$

$$\lambda_2 = -\Lambda_2^{1/2} = -\iota_e^{1/2}\,D^{-1}\,R^{1/2}\,p^2, \tag{142}$$

Here, the square root of R is taken such that the real part of λ_2 is negative. Now, the eigenvalue problem for the W-matrix reduces to

$$\lambda^2 - W_{22}^{(4)}\,p^4\,\lambda + \left[W_{11}^{(2)}\,W_{22}^{(4)} - W_{12}^{(2)}\,W_{21}^{(4)}\right] p^6 = 0. \tag{143}$$

which yields

$$\lambda_1 \simeq W_{22}^{(4)}\,p^4, \tag{144}$$

which is satisfied, and

$$\lambda_2 \simeq \left[W_{11}^{(2)} - \frac{W_{12}^{(2)}\,W_{21}^{(4)}}{W_{22}^{(4)}}\right] p^2, \tag{145}$$

which implies that

$$W_{11}^{(2)} = -\iota_e^{1/2}\,D^{-1}\,R^{1/2} - c_\beta\,\iota_e^{1/2}\,D^{-1}. \tag{146}$$

Hence, the large-p boundary condition for the W-matrix is

$$\underline{\underline{W}}(p) = \begin{pmatrix} -\iota_e^{1/2}\,D^{-1}\,R^{1/2}\,p^2 - c_\beta\,\iota_e^{1/2}\,D^{-1}\,p^2, & -c_\beta\,\iota_e^{-1/2}\,D^{-1}\,p^2 \\ -c_\beta^{-1}\,\iota_e^{1/2}\,D\,P_\varphi\,p^4, & -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi\,p^4 \end{pmatrix}. \tag{147}$$

F. Method of Solution

The method of solution is to launch the well-behaved asymptotic solution (147) of Eqs. (93)–(96) from large p, and then integrate the equations backward to small p. The complex layer response parameter, $\hat{\Delta}$, is then determined from Eq. (111).

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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