Benchmark of TJ Code Against STRIDE Code

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I. SYMMETRY RELATIONS

Let all lengths be normalized to the major radius of the magnetic axis, R_0, and let all magnetic field-strengths be normalized to the toroidal magnetic field-strength at the axis, B_0. Consider an axisymmetric toroidal plasma equilibrium. Let R, ϕ, Z be a conventional cylindrical coordinate system that is co-axial with the toroidal symmetry axis of the plasma. The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\phi\times\nabla\psi + g(\psi)\,\nabla\psi, \tag{1}$$

where ψ is the poloidal magnetic flux. We can also define

$$\Psi_N = \frac{\psi}{\psi_a}, \tag{2}$$

where ψ_a is the poloidal flux at the plasma boundary. Thus, $\Psi_N = 0$ at the magnetic axis, and $\Psi_N = 1$ on the plasma boundary. In the following, it is assumed that Ψ_N is the 'radial' coordinate in STRIDE.

Let

$$L_m^m(\Psi_N) = I(\Psi_N)\left(J(\Psi_N)\left[\frac{m\,g(\Psi_N)}{q(\Psi_N)}\right]^2 + [n\,\psi_a]^2\right), \tag{3}$$

$$I(\Psi_N) = \int_0^{\Psi_N}\frac{2\,q\,(\Psi_N')}{g(\Psi_N')}\,d\Psi_N', \tag{4}$$

$$J(\Psi_N) = \oint|\nabla\Psi_N|^{-2}\,\frac{d\theta}{2\pi}, \tag{5}$$

$$\rho(\Psi_N) = \frac{J(\Psi_N)\,g(\Psi_N)}{q(\Psi_N)}, \tag{6}$$

$$s(\Psi_N) = \frac{d\ln q}{d\Psi_N}, \tag{7}$$

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where m is a poloidal mode number, n is a toroidal mode number, $q(\Psi_N)$ is the safety-factor, and θ is a 'straight' poloidal angle defined such that

$$(\nabla\phi\times\nabla\theta\cdot\nabla\phi)^{-1} = \frac{q\,R^2}{g}, \tag{8}$$

Note that we are working in PEST coordinates.

Let m_j be the poloidal mode number of the jth rational surface, which is located at $\Psi_N = \Psi_{N\,j}$. Let $D_{I\,j}$ be the ideal Mercier index at the jth rational surface, and let

$$\nu_{L\,j} = \frac{1}{2} - \sqrt{-D_{I\,j}}, \tag{9}$$

$$\nu_{S\,j} = \frac{1}{2} + \sqrt{-D_{I\,j}}. \tag{10}$$

Let

$$f_{L\,j} = \left[\rho^{\,\nu_{L\,j}}\left(\frac{\nu_{S\,j}-\nu_{L\,j}}{L_{m_j}^{m_j}}\right)^{1/2} s\,m_j\right]_{\Psi_{N\,j}}, \tag{11}$$

$$f_{S\,j} = \left[\rho^{\,\nu_{S\,j}}\left(\frac{\nu_{S\,j}-\nu_{L\,j}}{L_{m_j}^{m_j}}\right)^{1/2} s\,m_j\right]_{\Psi_{N\,j}}. \tag{12}$$

Finally, let

$$\hat{A}_{ij} = f_{S\,j}^{-1}\,A_{ij}\,f_{L\,j'}, \tag{13}$$

$$\hat{B}_{ij} = f_{S\,j}^{-1}\,B_{ij}\,f_{L\,j'}, \tag{14}$$

$$\hat{\Gamma}_{ij} = f_{S\,j}^{-1}\,\Gamma_{ij}\,f_{L\,j'}, \tag{15}$$

$$\hat{\Delta}_{ij} = f_{S\,j}^{-1}\,\Delta_{ij}\,f_{L\,j'}, \tag{16}$$

where A_{ij}, B_{ij}, Γ_{ij}, and Δ_{ij} are the elements of the outer matching matrices calculated by STRIDE, whereas the hatted quantities are the corresponding matching matrices calculated by TJ. Equations (77) and (100) of Ref. 1 imply that

$$\hat{A}_{ji}^{*} = \hat{A}_{ij}, \tag{17}$$

$$\hat{\Delta}_{ji}^{*} = \hat{\Delta}_{ij}, \tag{18}$$

$$\hat{B}_{ji}^{*} = \hat{\Gamma}_{ij}. \tag{19}$$

These symmetries are ultimately due to the self-adjoint nature of the ideal-MHD force operator. However, as explained in Ref. 2, the symmetries can also be related to the conservation of toroidal electromagnetic angular momentum. The symmetries have to be respected by a toroidal tearing mode code, otherwise the code would predict that an isolated plasma could exert a net toroidal electromagnetic torque on itself. Note that, in the cylindrical limit, $\hat{\Delta}_{jj}$ is equivalent to $r_s\,\Delta'$: i.e., the tearing stability index normalized to the minor radius of the rational surface.

II. TJ CIRCULAR EQUILIBRIUM

The TJ circular equilibrium is characterized by a pressure profile

$$p(r) = \beta_0 \left[1 - \left(\frac{r}{a}\right)^2\right]^{p_p}, \tag{20}$$

and a parallel current profile, $\sigma = \mathbf{J}\cdot\mathbf{B}/B^2$,

$$\sigma(r) = \sigma_0 \left[1 - \left(\frac{r}{a}\right)^2\right]^{p_\sigma}. \tag{21}$$

These profiles are the same as the modified `lar` module in STRIDE.

III. BENCHMARK TESTS

A. Single Rational Surface

We use a zero-pressure, circular cross-section, plasma equilibrium that, in the cylindrical limit, has a Wesson-like current profile [3] characterized by the safety-factor on the magnetic axis, q_0, and the safety-factor at the plasma boundary, q_a. In fact, in the cylindrical limit, $j_\phi(r) = (2/q_0)\,(1 - r^2)^{q_a/q_0}$. We consider the stability of $n = 1$ tearing modes, and consider equilibria that only contain a single $n = 1$ rational surface: namely, the 2/1 surface.

1. Test 1

The first test has $q_0 = 1.1$, $q_a = 2.6$, and varies the inverse aspect-ratio of the plasma, ϵ_a. Figure 1 compares the $\hat{\Delta}_{11}$ (i.e., the tearing stability index of the 2/1 tearing mode) values calculated by the TJ code,[4] the TEAR code (which is a cylindrical tearing mode code), and the STRIDE code. It can be seen that the tearing stability index calculated by the TJ code asymptotes to that calculated by the TEAR code in the cylindrical limit, $\epsilon_a \to 0$. On the other hand, the stability index calculated by STRIDE exhibits wild oscillations in the cylindrical limit, and only becomes believable when $\epsilon_a > 0.2$. In the latter case, the STRIDE and TJ codes exhibit good agreement. Note that TJ has a dud data point at $\epsilon_a = 0.12$, which is under further investigation.

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