Incorporation of Twisting Parity Response

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I. OUTER REGION

A. Behavior in Vicinity of Rational Surface

In the vicinity of the kth rational surface

$$\psi_k(r_k + x) = A^{\pm}_{L\,k}\,|x|^{\,\nu_{L\,k}} + \mathrm{sgn}(x)\,A^{\pm}_{S\,k}\,|x|^{\,\nu_{S\,k}}, \tag{1}$$

where $+/-$ corresponds to $x>0$ and $x<0$ respectively, and

$$\nu_{L\,k} = \frac{1}{2} - \sqrt{-D_{I\,k}}, \tag{2}$$

$$\nu_{S\,k} = \frac{1}{2} + \sqrt{-D_{I\,k}}. \tag{3}$$

Let

$$\Psi^e_k = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}\frac{1}{2}\,(A^+_{L\,k} + A^-_{L\,k}), \tag{4}$$

$$\Delta\Psi^e_k = r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}(A^+_{S\,k} - A^-_{S\,k}), \tag{5}$$

$$\Psi^o_k = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}\frac{1}{2}\,(A^+_{L\,k} - A^-_{L\,k}), \tag{6}$$

$$\Delta\Psi^o_k = r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}(A^+_{S\,k} + A^-_{S\,k}). \tag{7}$$

B. Tearing Parity Solution

Let J be the number of poloidal harmonics included in the calculation. Let us launch J independent solution vectors from the magnetic axis. Let the solution vectors be denoted $\underline{\underline{\psi}}^{\,a\,e}(r)$ and $\underline{\underline{Z}}^{\,a\,e}(r)$. Here, the elements of $\underline{\underline{\psi}}^{\,a\,e}(r)$ are denoted $\psi_{j'j}(r)$, and the elements of

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$\underline{\underline{Z}}^{\,a\,e}(r)$ are denoted $Z_{j'j}(r)$, for $j', j = 1, J$. Furthermore, j' indexes the poloidal harmonic, whereas j indexes the solution vector launched from the axis. Let K be the number of rational surfaces in the plasma. The jump conditions imposed at the rational surfaces are

$$\Psi_k^o = 0, \tag{8}$$

$$\Delta\Psi_k^e = 0, \tag{9}$$

for $k = 1, K$, which implies that

$$A_{L\,k}^{+} = A_{L\,k}^{-}, \tag{10}$$

$$A_{S\,k}^{+} = A_{S\,k}^{-}. \tag{11}$$

Let $\Pi_{kj}^{a\,e}$ be the value of Ψ_k^e at the kth rational surface associated with the jth solution launched from the axis. Likewise, let $\Delta\Pi_{kj}^{a\,e}$ be the value of $\Delta\Psi_k^o$ at the kth rational surface associated with the jth solution launched from the axis.

Let us launch K small solution vectors from each of the rational surfaces in the plasma. Let the solution vectors be denoted $\underline{\underline{\psi}}^{\,s\,e}(r)$ and $\underline{\underline{Z}}^{\,s\,e}(r)$. Here, the elements of $\underline{\underline{\psi}}^{\,s\,e}(r)$ are denoted $\psi_{jk}(r)$, and the elements of $\underline{\underline{Z}}^{\,a\,e}(r)$ are denoted $Z_{jk}(r)$, for $j = 1, J$ and $k = 1, K$. Furthermore, j indexes the poloidal harmonic, whereas k indexes the rational surface from which the solution is launched. The launch conditions are

$$\Psi_k^e = \Psi_k^o = 0, \tag{12}$$

$$\Delta\Psi_k^e = 1. \tag{13}$$

The jump conditions imposed at the other rational surfaces are

$$\Psi_{k'}^o = 0, \tag{14}$$

$$\Delta\Psi_{k'}^e = 0 \tag{15}$$

where $k' \neq k$. Let $\Pi_{k'k}^{s\,e}$ be the value of $\Psi_{k'}^e$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface. Likewise, let $\Delta\Pi_{k'k}^{s\,e}$ be the value of $\Delta\Psi_{k'}^o$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface. Here, k and k' run from 1 to K

The general tearing parity solution vectors are written

$$\underline{\psi}^{\,e}(r) = \underline{\underline{\psi}}^{\,a\,e}(r)\,\underline{\alpha}^e + \underline{\underline{\psi}}^{\,s\,e}(r)\,\underline{\beta}^{\,e}, \tag{16}$$

$$\underline{Z}^{\,e}(r) = \underline{\underline{Z}}^{\,a\,e}(r)\,\underline{\alpha}^{e} + \underline{\underline{Z}}^{\,s\,e}(r)\,\underline{\beta}^{\,e}. \tag{17}$$

Here, $\underline{\alpha}^e$ is a $1\times J$ vector of arbitrary coefficients, whereas $\underline{\beta}^{\,e}$ is a $1\times K$ vector of arbitrary coefficients. However, the boundary condition at the plasma-vacuum interface is

$$\underline{\underline{I}}(a)\,\underline{Z}^{\,e}(a) = \underline{\underline{H}}\,[\underline{\psi}^{\,e}(a) - \underline{\psi}^{\,x}(a)], \tag{18}$$

where $\underline{\underline{H}}$ is the vacuum matrix, $\underline{\psi}^{\,x}(r)$ is the RMP field, and

$$\underline{\underline{I}}_{jj'}(r) = \frac{\delta_{jj'}}{m_j - n\,q(r)}, \tag{19}$$

for $j, j' = 1, J$. It follows that

$$\underline{\underline{X}}^{e}\,\underline{\alpha}^{e} = \underline{\underline{Y}}^{e}\,\underline{\beta}^{\,e} - \underline{\Xi}, \tag{20}$$

where

$$\underline{\underline{X}}^{e} = \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,a\,e}(a) - \underline{\underline{H}}\,\underline{\underline{\psi}}^{\,a\,e}(a), \tag{21}$$

$$\underline{\underline{Y}}^{e} = \underline{\underline{H}}\,\underline{\underline{\psi}}^{\,s\,e}(a) - \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,s\,e}(a), \tag{22}$$

$$\underline{\Xi} = \underline{\underline{H}}\,\underline{\psi}^{\,x}(a). \tag{23}$$

Thus,

$$\underline{\alpha}^{e} = \underline{\underline{\Omega}}^{\,e}\,\underline{\beta}^{\,e} - \underline{\Upsilon}^{\,e}, \tag{24}$$

where

$$\underline{\underline{X}}^{e}\,\underline{\underline{\Omega}}^{\,e} = \underline{\underline{Y}}^{e}, \tag{25}$$

$$\underline{\underline{X}}^{e}\,\underline{\Upsilon}^{\,e} = \underline{\Xi}. \tag{26}$$

Note that $\underline{\underline{\psi}}^{\,a\,e}(a)$ is a $J\times J$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,a\,e}(a)$ is a $J\times J$ matrix, $\underline{\underline{\psi}}^{\,s\,e}(a)$ is a $J\times K$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,s\,e}(a)$ is a $J\times K$ matrix, $\underline{\underline{H}}$ is a $J\times J$ matrix, $\underline{\alpha}^e$ is a $1\times J$ vector, $\underline{\beta}^{\,e}$ is a $1\times K$ vector, and $\underline{\psi}^{\,x}$ and $\underline{\Xi}$ are $1\times J$ vectors. Thus, $\underline{\underline{X}}^e$ is a $J\times J$ matrix, $\underline{\underline{Y}}^e$ is a $J\times K$ matrix, $\underline{\underline{\Omega}}^{\,e}$ is a $J\times K$ matrix, and $\underline{\Upsilon}^{\,e}$ is a $1\times J$ vector.

It follows that

$$\underline{\Psi}^{\,e} = \underline{\underline{F}}^{\,ee}\,\underline{\beta}^{\,e} - \underline{\Lambda}^{\,e}, \tag{27}$$

$$\underline{\Delta\Psi}^{\,e} = \underline{\beta}^{\,e}, \tag{28}$$

$$\underline{\Psi}^{\,o} = \underline{0}, \tag{29}$$

$$\underline{\Delta\Psi}^{\,o} = \underline{\underline{F}}^{\,oe}\,\underline{\beta}^{\,e} + \underline{\Delta\Lambda}^{\,o}, \tag{30}$$

where

$$\underline{\underline{F}}^{\,ee} = \underline{\underline{\Pi}}^{\,a\,e}\,\underline{\underline{\Omega}}^{\,e} + \underline{\underline{\Pi}}^{\,s\,e}, \tag{31}$$

$$\underline{\underline{F}}^{\,oe} = \underline{\underline{\Delta\Pi}}^{\,a\,e}\,\underline{\underline{\Omega}}^{\,e} + \underline{\underline{\Delta\Pi}}^{\,s\,e}, \tag{32}$$

$$\underline{\Lambda}^{\,e} = \underline{\underline{\Pi}}^{\,a\,e}\,\underline{\Upsilon}^{\,e}, \tag{33}$$

$$\underline{\Delta\Lambda}^{\,o} = -\underline{\underline{\Delta\Pi}}^{\,a\,e}\,\underline{\Upsilon}^{\,e}. \tag{34}$$

Note that $\underline{\underline{\Pi}}^{\,a\,e}$ is a $K\times J$ matrix, $\underline{\underline{\Delta\Pi}}^{\,a\,e}$ is a $K\times J$ matrix, $\underline{\underline{\Pi}}^{\,s\,e}$ is a $K\times K$ matrix, $\underline{\underline{\Delta\Pi}}^{\,s\,e}$ is a $K\times K$ matrix, and $\underline{\Lambda}^{\,e}$ is a $1\times J$ vector. Thus, $\underline{\underline{F}}^{\,ee}$ is a $K\times K$ matrix, $\underline{\underline{F}}^{\,oe}$ is a $K\times K$ matrix, and $\underline{\Lambda}^{e}$ and $\underline{\Delta\Lambda}^{o}$ are $1\times K$ vectors.

In the absence of an RMP, the fully-reconnected tearing parity eigenfunction associated with rational surface k is such that $\beta^e_{k'} = \delta_{kk'}$. Thus,

$$\psi^{\,f\,e}_{jk}(r) = \psi^{\,s\,e}_{jk}(r) + \sum_{j'=1,J}\psi^{\,a\,e}_{jj'}(r)\,\Omega^{\,e}_{j'k}, \tag{35}$$

$$Z^{\,f\,e}_{jk}(r) = Z^{\,s\,e}_{jk}(r) + \sum_{j'=1,J}Z^{\,a\,e}_{jj'}(r)\,\Omega^{\,e}_{j'k}. \tag{36}$$

This eigenfunction is such that

$$\Psi^{\,e}_{k'} = F^{\,ee}_{k'k}, \tag{37}$$

$$\Delta\Psi^{\,e}_{kk'} = \delta_{k'k}, \tag{38}$$

$$\Psi^{\,o}_{k'} = 0, \tag{39}$$

$$\Delta\Psi^{\,o}_{k'} = F^{\,oe}_{k'k}. \tag{40}$$

C. Twisting Parity Solution

Let us launch J independent solution vectors from the magnetic axis. Let the jth solution vectors be denoted $\underline{\underline{\psi}}^{\,a\,o}(r)$ and $\underline{\underline{Z}}^{\,a\,o}(r)$. The jump conditions imposed at the rational surfaces are

$$\Psi^{\,e}_{k} = 0, \tag{41}$$

$$\Delta\Psi_k^o = 0, \tag{42}$$

for $k = 1, K$, which implies that

$$A_{L\,k}^{+} = -A_{L\,k}^{-}, \tag{43}$$

$$A_{S\,k}^{+} = -A_{S\,k}^{-}. \tag{44}$$

Let $\Pi_{kj}^{a\,o}$ be the value of Ψ_k^o at the kth rational surface associated with the jth solution vector launched from the magnetic axis. Likewise, let $\Delta\Pi_{kj}^{a\,o}$ be the value of $\Delta\Psi_k^e$ at the kth rational surface associated with the jth solution vector launched from the magnetic axis.

Let us launch K small solution vectors from each of the rational surfaces in the plasma. Let the solution vectors be denoted $\underline{\underline{\psi}}^{\,s\,o}(r)$ and $\underline{\underline{Z}}^{\,s\,o}(r)$. The launch conditions are

$$\Psi_k^e = \Psi_k^o = 0, \tag{45}$$

$$\Delta\Psi_k^o = 1. \tag{46}$$

The jump conditions imposed at the other rational surfaces are

$$\Psi_{k'}^e = 0, \tag{47}$$

$$\Delta\Psi_{k'}^o = 0, \tag{48}$$

where $k' \neq k$. Let $\Pi_{k'k}^{s\,o}$ be the value of $\Psi_{k'}^o$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface. Likewise, let $\Delta\Pi_{k'k}^{s\,o}$ be the value of $\Delta\Psi_{k'}^e$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface.

The general twisting parity solution vectors are written

$$\underline{\psi}^{\,o}(r) = \underline{\underline{\psi}}^{\,a\,o}(r)\,\underline{\alpha}^o + \underline{\underline{\psi}}^{\,s\,o}(r)\,\underline{\beta}^{\,o}, \tag{49}$$

$$\underline{Z}^{\,o}(r) = \underline{\underline{Z}}^{\,a\,o}(r)\,\underline{\alpha}^o + \underline{\underline{Z}}^{\,s\,o}(r)\,\underline{\beta}^{\,o}, \tag{50}$$

Here, $\underline{\alpha}^o$ is a $1\times J$ vector of arbitrary coefficients, whereas $\underline{\beta}^{\,o}$ is a $1\times K$ vector of arbitrary coefficients. However, the boundary condition at the plasma-vacuum interface is again

$$\underline{\underline{I}}(a)\,\underline{Z}^{\,o}(a) = \underline{\underline{H}}\,[\underline{\psi}^{\,o}(a) - \underline{\psi}^{\,x}(a)]. \tag{51}$$

It follows that

$$\underline{\underline{X}}^o\,\underline{\alpha}^o = \underline{\underline{Y}}^o\,\underline{\beta}^{\,o} - \underline{\Xi}, \tag{52}$$

where

$$\underline{\underline{X}}^{o} = \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,a\,o}(a) - \underline{\underline{H}}\ \underline{\underline{\psi}}^{\,a\,o}(a), \tag{53}$$

$$\underline{\underline{Y}}^{o} = \underline{\underline{H}}\ \underline{\underline{\psi}}^{\,s\,o}(a) - \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,s\,o}(a). \tag{54}$$

Thus,

$$\underline{\alpha}^{o} = \underline{\underline{\Omega}}^{\,o}\,\underline{\beta}^{\,o} - \underline{\Upsilon}^{\,o}, \tag{55}$$

where

$$\underline{\underline{X}}^{o}\,\underline{\underline{\Omega}}^{\,o} = \underline{\underline{Y}}^{o}, \tag{56}$$

$$\underline{\underline{X}}^{o}\,\underline{\Upsilon}^{\,o} = \underline{\Xi}. \tag{57}$$

Note that $\underline{\underline{\psi}}^{\,a\,o}(a)$ is a $J\times J$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,a\,o}(a)$ is a $J\times J$ matrix, $\underline{\underline{\psi}}^{\,s\,o}(a)$ is a $J\times K$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,s\,o}(a)$ is a $J\times K$ matrix, $\underline{\underline{H}}$ is a $J\times J$ matrix, $\underline{\alpha}^{o}$ is a J vector, and $\underline{\beta}^{\,o}$ is a K vector. Thus, $\underline{\underline{X}}^{o}$ is a $J\times J$ matrix, $\underline{\underline{Y}}^{o}$ is a $J\times K$ matrix, and $\underline{\underline{\Omega}}^{\,o}$ is a $J\times K$ matrix, and $\underline{\Upsilon}^{\,o}$ is a $1\times J$ vector.

It follows that

$$\underline{\Psi}^{\,e} = \underline{0}, \tag{58}$$

$$\underline{\Delta\Psi}^{\,e} = \underline{\underline{F}}^{\,eo}\,\underline{\beta}^{\,o} + \underline{\Delta\Lambda}^{\,e}, \tag{59}$$

$$\underline{\Psi}^{\,o} = \underline{\underline{F}}^{\,oo}\,\underline{\beta}^{\,o} - \underline{\Lambda}^{\,o}, \tag{60}$$

$$\underline{\Delta\Psi}^{\,o} = \underline{\beta}^{\,o}, \tag{61}$$

where

$$\underline{\underline{F}}^{\,oo} = \underline{\underline{\Pi}}^{\,a\,o}\,\underline{\underline{\Omega}}^{\,o} + \underline{\underline{\Pi}}^{\,s\,o}, \tag{62}$$

$$\underline{\underline{F}}^{\,eo} = \underline{\underline{\underline{\Delta\Pi}}}^{\,a\,o}\,\underline{\underline{\Omega}}^{\,o} + \underline{\underline{\underline{\Delta\Pi}}}^{\,s\,o}, \tag{63}$$

$$\underline{\Lambda}^{\,o} = \underline{\underline{\Pi}}^{\,a\,o}\,\underline{\Upsilon}^{\,o}, \tag{64}$$

$$\underline{\Delta\Lambda}^{\,e} = -\underline{\underline{\underline{\Delta\Pi}}}^{\,a\,o}\,\underline{\Upsilon}^{\,o}. \tag{65}$$

Note that $\underline{\underline{\Pi}}^{\,a\,o}$ is a $K\times J$ matrix, $\underline{\underline{\underline{\Delta\Pi}}}^{\,a\,o}$ is a $K\times J$ matrix, $\underline{\underline{\Pi}}^{\,s\,o}$ is a $K\times K$ matrix, and $\underline{\underline{\underline{\Delta\Pi}}}^{\,s\,o}$ is a $K\times K$ matrix. Thus, $\underline{\underline{F}}^{\,oo}$ is a $K\times K$ matrix, $\underline{\underline{F}}^{\,eo}$ is a $K\times K$ matrix, and $\underline{\Lambda}^{o}$ and $\underline{\Delta\Lambda}^{e}$ are $1\times K$ vectors.

In the absence of an RMP, the fully-reconnected twisting parity eigenfunction associated with rational surface k is such that $\beta^o_{k'} = \delta_{kk'}$. Thus,

$$\psi^{f\,o}_{jk}(r) = \psi^{s\,o}_{jk}(r) + \sum_{j'=1,J} \psi^{a\,o}_{jj'}(r)\,\Omega^o_{j'k}, \tag{66}$$

$$Z^{f\,o}_{jk}(r) = Z^{s\,o}_{jk}(r) + \sum_{j'=1,J} Z^{a\,o}_{jj'}(r)\,\Omega^o_{j'k}. \tag{67}$$

This eigenfunction is such that

$$\Psi^e_{k'} = 0, \tag{68}$$

$$\Delta\Psi^e_{kk'} = F^{eo}_{k'k}, \tag{69}$$

$$\Psi^o_{k'} = F^{oo}_{k'k}, \tag{70}$$

$$\Delta\Psi^o_{k'} = \delta_{k'k}. \tag{71}$$

D. General Dispersion Relation

The general dispersion relation is obtained by combining the tearing parity dispersion relation, (27)–(30), with the twisting parity dispersion relation, (58)–(61). We get

$$\underline{\Psi}^{\,e} = \underline{\underline{F}}^{\,ee}\,\underline{\beta}^{\,e} - \underline{\Lambda}^{\,e}, \tag{72}$$

$$\underline{\Delta\Psi}^{\,e} = \underline{\beta}^{\,e} + \underline{\underline{F}}^{\,eo}\,\underline{\beta}^{\,o} + \underline{\Delta\Lambda}^{\,e}, \tag{73}$$

$$\underline{\Psi}^{\,o} = \underline{\underline{F}}^{\,oo}\,\underline{\beta}^{\,o} - \underline{\Lambda}^{\,o}, \tag{74}$$

$$\underline{\Delta\Psi}^{\,o} = \underline{\beta}^{\,o} + \underline{\underline{F}}^{\,oe}\,\underline{\beta}^{\,e} + \underline{\Delta\Lambda}^{\,o}. \tag{75}$$

Hence, we obtain the general dispersion relation

$$\begin{pmatrix} \underline{\Delta\Psi}^{\,e} \\ \underline{\Delta\Psi}^{\,o} \end{pmatrix} = \begin{pmatrix} \underline{\underline{E}}^{\,ee} & \underline{\underline{E}}^{\,eo} \\ \underline{\underline{E}}^{\,oe} & \underline{\underline{E}}^{\,oo} \end{pmatrix} \begin{pmatrix} \underline{\Psi}^{\,e} \\ \underline{\Psi}^{\,o} \end{pmatrix} + \begin{pmatrix} \underline{\chi}^{\,e} \\ \underline{\chi}^{\,o} \end{pmatrix} \tag{76}$$

where

$$\underline{\underline{E}}^{\,ee} = (\underline{\underline{F}}^{\,ee})^{-1}, \tag{77}$$

$$\underline{\underline{E}}^{\,eo} = \underline{\underline{F}}^{\,eo}\,\underline{\underline{E}}^{\,oo}, \tag{78}$$

$$\underline{\underline{E}}^{\,oe} = \underline{\underline{F}}^{\,oe}\,\underline{\underline{E}}^{\,ee}, \tag{79}$$

$$\underline{\underline{E}}^{\,oo} = (\underline{\underline{F}}^{\,oo})^{-1}, \tag{80}$$

$$\underline{\chi}^{\,e} = \underline{\underline{E}}^{\,ee}\,\underline{\Lambda}^{\,e} + \underline{\underline{E}}^{\,eo}\,\underline{\Lambda}^{\,o} + \underline{\Delta\Lambda}^{\,e}, \tag{81}$$

$$\underline{\chi}^{\,o} = \underline{\underline{E}}^{\,oe}\,\underline{\Lambda}^{\,e} + \underline{\underline{E}}^{\,oo}\,\underline{\Lambda}^{\,o} + \underline{\Delta\Lambda}^{\,o}. \tag{82}$$

In the absence of an RMP, the tearing parity unreconnected eigenfunction associated with the kth rational surface is such that

$$\Psi_{k'}^{e} = \delta_{k'k}, \tag{83}$$

$$\Delta\Psi_{kk'}^{e} = E_{k'k}^{ee}, \tag{84}$$

$$\Psi_{k'}^{o} = 0, \tag{85}$$

$$\Delta\Psi_{k'}^{o} = E_{k'k}^{oe}. \tag{86}$$

Thus,

$$\psi_{jk}^{u\,e}(r) = \sum_{k'=1,k} \psi_{jk'}^{f\,e}(r)\,E_{k'k}^{ee} + \sum_{k'=1,k} \psi_{jk'}^{f\,o}(r)\,E_{k'k}^{oe}, \tag{87}$$

$$Z_{jk}^{u\,e}(r) = \sum_{k'=1,k} Z_{jk'}^{f\,e}(r)\,E_{k'k}^{ee} + \sum_{k'=1,k} Z_{jk'}^{f\,o}(r)\,E_{k'k}^{oe}. \tag{88}$$

In the absence of an RMP, the twisting parity unreconnected eigenfunction associated with the kth rational surface is such that

$$\Psi_{k'}^{e} = 0, \tag{89}$$

$$\Delta\Psi_{kk'}^{e} = E_{k'k}^{eo}, \tag{90}$$

$$\Psi_{k'}^{o} = \delta_{k'k}, \tag{91}$$

$$\Delta\Psi_{k'}^{o} = E_{k'k}^{oo}. \tag{92}$$

Thus,

$$\psi_{jk}^{u\,o}(r) = \sum_{k'=1,k} \psi_{jk'}^{f\,o}(r)\,E_{k'k}^{oo} + \sum_{k'=1,k} \psi_{jk'}^{f\,e}(r)\,E_{k'k}^{eo}, \tag{93}$$

$$Z_{jk}^{u\,o}(r) = \sum_{k'=1,k} Z_{jk'}^{f\,o}(r)\,E_{k'k}^{oo} + \sum_{k'=1,k} Z_{jk'}^{f\,e}(r)\,E_{k'k}^{eo}. \tag{94}$$

II. ANGULAR MOMENTUM CONSERVATION

The total toroidal electromagnetic torque acting on the plasma is

$$T_\varphi = 2\,n\,\pi^2\,\mathrm{Im}\left(\underline{\Psi}^{\,e\,\dagger}\,\underline{\Delta\Psi}^{\,e} + \underline{\Psi}^{\,o\,\dagger}\,\underline{\Delta\Psi}^{\,o}\right), \tag{95}$$

which, in the absence of an RMP, gives

$$T_\varphi = 2\,n\,\pi^2\,\mathrm{Im}\left(\underline{\Psi}^{\,e\,\dagger}\,\underline{\underline{E}}^{\,ee}\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,e\,\dagger}\,\underline{\underline{E}}^{\,eo}\,\underline{\Psi}^{\,o} + \underline{\Psi}^{\,o\,\dagger}\,\underline{\underline{E}}^{\,oe}\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,o\,\dagger}\,\underline{\underline{E}}^{\,oo}\,\underline{\Psi}^{\,o}\right), \tag{96}$$

or

$$\begin{aligned} T_\varphi = n\,\pi^2\,\Big[&\underline{\Psi}^{\,e\,\dagger}\,(\underline{\underline{E}}^{\,ee} - \underline{\underline{E}}^{\,ee\,\dagger})\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,e\,\dagger}\,(\underline{\underline{E}}^{\,eo} - \underline{\underline{E}}^{\,oe\,\dagger})\,\underline{\Psi}^{\,o} \\ &+ \underline{\Psi}^{\,o\,\dagger}\,(\underline{\underline{E}}^{\,oe} - \underline{\underline{E}}^{\,eo\,\dagger})\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,o\,\dagger}\,(\underline{\underline{E}}^{\,oo} - \underline{\underline{E}}^{\,oo\,\dagger})\,\underline{\Psi}^{\,o}\Big]. \end{aligned} \tag{97}$$

However, in the absence of an RMP, T_φ must be zero, irrespective of the values of the $\underline{\Psi}^{\,e}$ and the $\underline{\Psi}^{\,o}$. This is only possible if

$$\underline{\underline{E}}^{\,ee\,\dagger} = \underline{\underline{E}}^{\,ee}, \tag{98}$$

$$\underline{\underline{E}}^{\,eo\,\dagger} = \underline{\underline{E}}^{\,oe}, \tag{99}$$

$$\underline{\underline{E}}^{\,oo\,\dagger} = \underline{\underline{E}}^{\,oo}. \tag{100}$$

III. INNER LAYER EQUATIONS

In the vicinity of the kth rational surface, the inner layer equations are

$$(\hat{\gamma}_k + \mathrm{i}\,Q_{E\,k} + \mathrm{i}\,Q_{e\,k})\,\psi = -\mathrm{i}\,X\,(\phi - N) + \frac{d^2\psi}{dX^2}, \tag{101}$$

$$(\hat{\gamma}_k + \mathrm{i}\,Q_{E\,k})\,N = -\mathrm{i}\,Q_{e\,k}\,\phi - \mathrm{i}\,c_{\beta\,k}^2\,X\,V - \mathrm{i}\,D_k^2\,X\,\frac{d^2\psi}{dX^2} + P_{\perp\,k}\,\frac{d^2N}{dX^2}, \tag{102}$$

$$(\hat{\gamma}_k + \mathrm{i}\,Q_{E\,k} + \mathrm{i}\,Q_{i\,k})\,\frac{d^2\phi}{dX^2} = -\mathrm{i}\,X\,\frac{d^2\psi}{dX^2} + P_{\varphi\,k}\,\frac{d^4}{dX^4}\left(\phi + \frac{N}{\iota_k}\right), \tag{103}$$

$$(\hat{\gamma}_k + \mathrm{i}\,Q_{E\,k})\,V = \mathrm{i}\,Q_{e\,k}\,\psi - \mathrm{i}\,X\,N + P_{\varphi\,k}\,\frac{d^2V}{dX^{\,2}}, \tag{104}$$

where $X = S_k^{1/3}\,(r - r_k)/r_k$. All quantities are as defined in TJ2025, except that

$$c_{\beta\,k} = \left(\frac{\beta_k}{1+\beta_k}\right)^{1/2}, \tag{105}$$

$$\iota_k = -\frac{\omega_{\ast\, e\, k}}{\omega_{\ast\, i\, k}}, \tag{106}$$

$$Q_{e\,k} = -\left(\frac{\iota_k}{1+\iota_k}\right)\tau_k\,\omega_{\ast\,k}, \tag{107}$$

$$Q_{i\,k} = \left(\frac{1}{1+\iota_k}\right)\tau_k\,\omega_{\ast\,k}, \tag{108}$$

$$D_k = \left(\frac{\iota_k}{1+\iota_k}\right)^{1/2} S_k^{1/3}\,\hat{d}_{\beta\,k}. \tag{109}$$

Equations (101)–(104) possess the trivial twisting parity solution:

$$\psi(X) = A\,X, \tag{110}$$

$$N(X) = A\,Q_{e\,k}, \tag{111}$$

$$\phi(X) = A\,({\rm i}\,\gamma_k - Q_{E\,k}), \tag{112}$$

$$V(X) = 0. \tag{113}$$

IV. INTERMEDIATE LAYER EQUATIONS

The intermediate layer equation is

$$(1+Y^2)\,\frac{d^2\psi}{dY^2} = \nu_k\,(1+\nu_k)\,\psi, \tag{114}$$

where $Y=(r-r_k)/\delta_k$, and

$$\nu_k = -\frac{1}{2}+\sqrt{-D_{I\,k}} \simeq -\frac{1}{4} - D_{I\,k}. \tag{115}$$

The general asymptotic behavior is

$$\psi(Y) = \hat{B}_{L\,k} + \hat{B}_{S\,k}\,|Y| \tag{116}$$

for $|Y|\ll 1$, and

$$\psi(Y) = \hat{A}_{L\,k}\,|Y|^{-\nu_k} + \hat{A}_{S\,k}\,|Y|^{1+\nu_k} \tag{117}$$

for $|Y|\gg 1$.

The tearing parity solution is such that

$$\psi(Y) = \hat{B}^{\,e}_{L\,k} + \hat{B}^{\,e}_{S\,k}\,|Y| \tag{118}$$

for $|Y| \ll 1$, and

$$\psi(Y) = \hat{A}^{e}_{L\,k}\,|Y|^{-\nu_k} + \hat{A}^{e}_{S\,k}\,|Y|^{1+\nu_k} \tag{119}$$

for $|Y| \gg 1$. Furthermore,

$$S_k^{1/3}\,\hat{\Delta}_k^e = \left(\frac{r_k}{\delta_k}\right)\frac{2\,\hat{B}^{e}_{S\,k}}{\hat{B}^{e}_{L\,k}}, \tag{120}$$

where $\hat{\Delta}_k^e$ is the tearing parity layer response function, and

$$\Delta_k^e \equiv \frac{\Delta\Psi_k^e}{\Delta\Psi_k^e} = \left(\frac{r_k}{\delta_k}\right)^{1+2\,\nu_k}\frac{2\,\hat{A}^{e}_{S\,k}}{\hat{A}^{e}_{L\,k}} \tag{121}$$

The twisting parity solution is such that

$$\psi(Y) = \hat{B}^{o}_{S\,k}\,Y \tag{122}$$

for $|Y| \ll 1$, and

$$\psi(Y) = {\rm sgn}(Y)\left(\hat{A}^{o}_{L\,k}\,|Y|^{-\nu_k} + \hat{A}^{o}_{S\,k}\,|Y|^{1+\nu_k}\right) \tag{123}$$

for $|Y| \gg 1$. Furthermore,

$$\Delta_k^o \equiv \frac{\Delta\Psi_k^o}{\Delta\Psi_k^o} = \left(\frac{r_k}{\delta_k}\right)^{1+2\,\nu_k}\frac{2\,\hat{A}^{o}_{S\,k}}{\hat{A}^{o}_{L\,k}}. \tag{124}$$

The connection formulae are

$$\hat{B}^{e,o}_{L\,k} = a_{LL}\,\hat{A}^{e,o}_{L\,k} + a_{LS}\,\hat{A}^{e,o}_{S\,k}, \tag{125}$$

$$\hat{B}^{e,o}_{S\,k} = a_{SL}\,\hat{A}^{e,o}_{L\,k} + a_{SS}\,\hat{A}^{e,o}_{S\,k}. \tag{126}$$

It follows that

$$\left(\frac{\delta_k}{r_k}\right)\frac{S_k^{1/3}\,\hat{\Delta}_k^e}{2} = \frac{a_{SL} + a_{SS}\,(\delta_k/r_k)^{1+2\,\nu_k}\,(\Delta_k^e/2)}{a_{LL} + a_{LS}\,(\delta_k/r_k)^{1+2\,\nu_k}\,(\Delta_k^e/2)}, \tag{127}$$

$$\left(\frac{\delta_k}{r_k}\right)^{1+2\,\nu_k}\frac{\Delta_k^o}{2} = -\frac{a_{LL}}{a_{LS}}. \tag{128}$$

But, in the limit $|\nu_k| \to 0$, we find that $a_{LL} \to 1$, $a_{SL} \to -\nu_k\,\pi/2$, $a_{LS} \to -\nu_k\,\pi/2$, and $a_{SS} \to 1$. Thus, we obtain

$$\Delta_k^e \simeq S_k^{1/3}\,\hat{\Delta}_k^e + \frac{\pi\,\nu_k\,r_k}{\delta_k}, \tag{129}$$

$$\Delta_k^o \simeq \frac{4\,r_k}{\pi\,\nu_k\,\delta_k}. \tag{130}$$

Let $\delta_k = \delta_{d\,k}/(2\sqrt{\pi})$, and let us identify ν_k with $-D_{R\,k}$. It follows that

$$\Delta_k^e = S_k^{1/3}\,\hat{\Delta}_k^e + \Delta_{k\,\mathrm{crit}}^e, \tag{131}$$

$$\Delta_k^o = \Delta_{k\,\mathrm{crit}}^o, \tag{132}$$

where

$$\Delta_{k\,\mathrm{crit}}^e = \sqrt{2}\,\pi^{3/2}\,(-D_{R\,k})\,\frac{r_k}{\delta_{d\,k}}, \tag{133}$$

$$\Delta_{k\,\mathrm{crit}}^o = \frac{8}{\sqrt{\pi}}\,(-D_{R\,k})^{-1}\,\frac{r_k}{\delta_{d\,k}}. \tag{134}$$

V. HOMOGENOUS DISPERSION RELATION

The homogeneous dispersion relation can be written

$$(S_k^{1/3}\,\hat{\Delta}_k^e + \Delta_{k\,\mathrm{crit}}^e)\,\Psi_k^e = \sum_{k'}(E_{kk'}^{ee}\,\Psi_{k'}^e + E_{kk'}^{eo}\,\Psi_{k'}^o), \tag{135}$$

$$0 = \sum_{k'}(\tilde{E}_{kk'}^{oo}\,\Psi_{k'}^o + E_{kk'}^{oe}\,\Psi_{k'}^e), \tag{136}$$

where

$$\tilde{E}_{kk'}^{oo} = E_{kk'}^{oo} - \Delta_{k\,\mathrm{crit}}^o\,\delta_{kk'}. \tag{137}$$

Hence,

$$\Psi_k^o = -\sum_{k',k''}(\tilde{E}_{kk'}^{oo})^{-1}\,E_{k'k''}^{oe}\,\Psi_{k''}^e, \tag{138}$$

and

$$(S_k^{1/3}\,\hat{\Delta}_k^e + \Delta_{k\,\mathrm{crit}}^e)\,\Psi_k^e = \sum_{k'} E_{kk'}^e\,\Psi_{k'}^e \tag{139}$$

where

$$E_{kk'}^e = E_{kk'}^{ee} - \sum_{k'',k'''} E_{kk''}^{eo}\,(\tilde{E}_{k''k'''}^{oo})^{-1}\,E_{k'''k'}^{oe}. \tag{140}$$

Note that $\underline{\underline{E}}^{\,e}$ is Hermitian.

Suppose that $\hat{\Delta}_k$ is small, but $\hat{\Delta}_{k'\neq k}$ is order unity. In this case,

$$\Psi_{k'}^e \simeq \delta_{kk'}\,\Psi_k^e. \tag{141}$$

It follows that the growth rate of the mode that reconnects magnetic flux at the kth rational surface is governed by

$$S_k^{1/3}\,\hat{\Delta}_k^e \simeq E_{kk}^e - \Delta_{k\,\mathrm{crit}}^e. \tag{142}$$

The corresponding eigenfunction is

$$\psi^{u}_{jk}(r) = \psi^{u\,e}_{jk}(r) - \sum_{k',k''} \psi^{u\,o}_{jk'}(r)\,(\tilde{E}^{oo}_{k'k''})^{-1}\,E^{oe}_{k''k}, \tag{143}$$

$$Z^{u}_{jk}(r) = Z^{u\,e}_{jk}(r) - \sum_{k',k''} Z^{u\,o}_{jk'}(r)\,(\tilde{E}^{oo}_{k'k''})^{-1}\,E^{oe}_{k''k}, \tag{144}$$

and has the properties that

$$\Psi^{e}_{k'} = \delta_{kk'}, \tag{145}$$

$$\Delta\Psi^{e}_{k'} = E^{e}_{k'k}, \tag{146}$$

$$\Psi^{o}_{k'} = -\sum_{k',k''} (\tilde{E}^{oo}_{k'k''})^{-1}\,E^{oe}_{k''k}, \tag{147}$$

$$\Delta\Psi^{o}_{k'} = \Delta^{o}_{k'\,\mathrm{crit}}\,\Psi^{o}_{k'}. \tag{148}$$

Suppose that Ψ^{e}_{k} and $\Psi^{e}_{k'}$ are both non-zero, but that $\Psi^{e}_{k''} = 0$ for $k'' \neq k, k'$. The toroidal electromagnetic torque at a general rational surface labeled j is

$$\delta T_j = 2\,n\,\pi^2\,\mathrm{Im}\left(\Psi^{e\,*}_{j}\,\Delta\Psi^{e}_{j} + \Psi^{o\,*}_{j}\,\Delta\Psi^{o}_{j}\right). \tag{149}$$

Hence, we deduce that at the kth rational surface,

$$\delta T_k = 2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_{k}\,E^{e}_{kk'}\,\Psi^{e}_{k'}), \tag{150}$$

while, at the

$$\begin{aligned}\delta T_{k'} &= 2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_{k'}\,E^{e}_{k'k}\,\Psi^{e}_{k}) = -2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_{k}\,E^{e\,*}_{k'k}\,\Psi^{e}_{'k})\\ &= -2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_{k}\,E^{e}_{kk'}\,\Psi^{e}_{'k}) = -\delta T_{k'},\end{aligned} \tag{151}$$

with $\delta T_{k''} = 0$ for $k'' \neq k, k'$.

VI. INHOMOGENOUS DISPERSION RELATION

The inhomogeneous dispersion relation can be written

$$(S^{1/3}_{k}\,\hat{\Delta}^{e}_{k} + \Delta^{e}_{k\,\mathrm{crit}})\,\Psi^{e}_{k} = \sum_{k'} (E^{ee}_{kk'}\,\Psi^{e}_{k'} + E^{eo}_{kk'}\,\Psi^{o}_{k'}) + \chi^{e}_{k}, \tag{152}$$

$$0 = \sum_{k'} (\tilde{E}^{oo}_{kk'}\,\Psi^{o}_{k'} + E^{oe}_{kk'}\,\Psi^{e}_{k'}) + \chi^{o}_{k}. \tag{153}$$

It follows that

$$(S_k^{1/3}\,\hat{\varDelta}_k^e + \varDelta_{k\,\mathrm{crit}}^e)\,\varPsi_k^e = \sum_{k'} E_{kk'}^e\,\varPsi_{k'}^e + \chi_k, \tag{154}$$

where

$$\chi_k = \chi_k^e - \sum E_{k',k''}^{eo}\,(\tilde{E}_{k'k''}^{oo})^{-1}\,\chi_{k''}^o. \tag{155}$$