Incorporation of Twisting Parity Response

R. Fitzpatrick [a]

Institute for Fusion Studies, Department of Physics,

University of Texas at Austin, Austin TX 78712, USA

I. OUTER REGION

A. Behavior in Vicinity of Rational Surface

In the vicinity of the kth rational surface

$$\psi_k(r_k + x) = A^{\pm}_{L\,k}\,|x|^{\,\nu_{L\,k}} + \mathrm{sgn}(x)\,A^{\pm}_{S\,k}\,|x|^{\,\nu_{S\,k}}, \tag{1}$$

where $+/-$ corresponds to $x > 0$ and $x < 0$ respectively, and

$$\nu_{L\,k} = \frac{1}{2} - \sqrt{-D_{I\,k}}, \tag{2}$$

$$\nu_{S\,k} = \frac{1}{2} + \sqrt{-D_{I\,k}}. \tag{3}$$

Let

$$\Psi^e_k = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}\frac{1}{2}\,(A^+_{L\,k} + A^-_{L\,k}), \tag{4}$$

$$\Delta\Psi^e_k = r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}(A^+_{S\,k} - A^-_{S\,k}), \tag{5}$$

$$\Psi^o_k = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}\frac{1}{2}\,(A^+_{L\,k} - A^-_{L\,k}), \tag{6}$$

$$\Delta\Psi^o_k = r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L^k_k}\right)^{1/2}(A^+_{S\,k} + A^-_{S\,k}). \tag{7}$$

B. Tearing Parity Solution

Let J be the number of poloidal harmonics included in the calculation. Let us launch J independent solution vectors from the magnetic axis. Let the solution vectors be denoted $\underline{\underline{\psi}}^{\,a\,e}(r)$ and $\underline{\underline{Z}}^{\,a\,e}(r)$. Here, the elements of $\underline{\underline{\psi}}^{\,a\,e}(r)$ are denoted $\psi_{j'j}(r)$, and the elements of

---

[a] rfitzp@utexas.edu

$\underline{\underline{Z}}^{\,a\,e}(r)$ are denoted $Z_{j'j}(r)$, for $j', j = 1, J$. Furthermore, j' indexes the poloidal harmonic, whereas j indexes the solution vector launched from the axis. Let K be the number of rational surfaces in the plasma. The jump conditions imposed at the rational surfaces are

$$\Psi_k^o = 0, \tag{8}$$

$$\Delta\Psi_k^e = 0, \tag{9}$$

for $k = 1, K$, which implies that

$$A_{L\,k}^{+} = A_{L\,k}^{-}, \tag{10}$$

$$A_{S\,k}^{+} = A_{S\,k}^{-}. \tag{11}$$

Let $\Pi_{kj}^{a\,e}$ be the value of Ψ_k^e at the kth rational surface associated with the jth solution launched from the axis. Likewise, let $\Delta\Pi_{kj}^{a\,e}$ be the value of $\Delta\Psi_k^o$ at the kth rational surface associated with the jth solution launched from the axis.

Let us launch K small solution vectors from each of the rational surfaces in the plasma. Let the solution vectors be denoted $\underline{\underline{\psi}}^{\,s\,e}(r)$ and $\underline{\underline{Z}}^{\,s\,e}(r)$. Here, the elements of $\underline{\underline{\psi}}^{\,s\,e}(r)$ are denoted $\psi_{jk}(r)$, and the elements of $\underline{\underline{Z}}^{\,a\,e}(r)$ are denoted $Z_{jk}(r)$, for $j = 1, J$ and $k = 1, K$. Furthermore, j indexes the poloidal harmonic, whereas k indexes the rational surface from which the solution is launched. The launch conditions are

$$\Psi_k^e = \Psi_k^o = 0, \tag{12}$$

$$\Delta\Psi_k^e = 1. \tag{13}$$

The jump conditions imposed at the other rational surfaces are

$$\Psi_{k'}^o = 0, \tag{14}$$

$$\Delta\Psi_{k'}^e = 0 \tag{15}$$

where $k' \neq k$. Let $\Pi_{k'k}^{s\,e}$ be the value of $\Psi_{k'}^e$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface. Likewise, let $\Delta\Pi_{k'k}^{s\,e}$ be the value of $\Delta\Psi_{k'}^o$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface. Here, k and k' run from 1 to K

The general tearing parity solution vectors are written

$$\underline{\psi}^{\,e}(r) = \underline{\underline{\psi}}^{\,a\,e}(r)\,\underline{\alpha}^{e} + \underline{\underline{\psi}}^{\,s\,e}(r)\,\underline{\beta}^{\,e}, \tag{16}$$

$$\underline{Z}^{\,e}(r) = \underline{\underline{Z}}^{\,a\,e}(r)\,\underline{\alpha}^{e} + \underline{\underline{Z}}^{\,s\,e}(r)\,\underline{\beta}^{\,e}. \tag{17}$$

Here, $\underline{\alpha}^{e}$ is a $1\times J$ vector of arbitrary coefficients, whereas $\underline{\beta}^{\,e}$ is a $1\times K$ vector of arbitrary coefficients. However, the boundary condition at the plasma-vacuum interface is

$$\underline{\underline{I}}(a)\,\underline{Z}^{\,e}(a) = \underline{\underline{H}}\;[\underline{\psi}^{\,e}(a) - \underline{\psi}^{\,x}(a)], \tag{18}$$

where $\underline{\underline{H}}$ is the vacuum matrix, $\underline{\psi}^{\,x}(r)$ is the RMP field, and

$$\underline{\underline{I}}_{jj'}(r) = \frac{\delta_{jj'}}{m_j - n\,q(r)}, \tag{19}$$

for $j, j' = 1, J$. It follows that

$$\underline{\underline{X}}^{e}\,\underline{\alpha}^{e} = \underline{\underline{Y}}^{e}\,\underline{\beta}^{\,e} - \underline{\Xi}, \tag{20}$$

where

$$\underline{\underline{X}}^{e} = \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,a\,e}(a) - \underline{\underline{H}}\;\underline{\underline{\psi}}^{\,a\,e}(a), \tag{21}$$

$$\underline{\underline{Y}}^{e} = \underline{\underline{H}}\;\underline{\underline{\psi}}^{\,s\,e}(a) - \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,s\,e}(a), \tag{22}$$

$$\underline{\Xi} = \underline{\underline{H}}\;\underline{\psi}^{\,x}(a). \tag{23}$$

Thus,

$$\underline{\alpha}^{e} = \underline{\underline{\Omega}}^{\,e}\,\underline{\beta}^{\,e} - \underline{\Upsilon}^{\,e}, \tag{24}$$

where

$$\underline{\underline{X}}^{e}\,\underline{\underline{\Omega}}^{\,e} = \underline{\underline{Y}}^{e}, \tag{25}$$

$$\underline{\underline{X}}^{e}\,\underline{\Upsilon}^{\,e} = \underline{\Xi}. \tag{26}$$

Note that $\underline{\underline{\psi}}^{\,a\,e}(a)$ is a $J\times J$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,a\,e}(a)$ is a $J\times J$ matrix, $\underline{\underline{\psi}}^{\,s\,e}(a)$ is a $J\times K$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{\,s\,e}(a)$ is a $J\times K$ matrix, $\underline{\underline{H}}$ is a $J\times J$ matrix, $\underline{\alpha}^{e}$ is a $1\times J$ vector, $\underline{\beta}^{\,e}$ is a $1\times K$ vector, and $\underline{\psi}^{\,x}$ and $\underline{\Xi}$ are all $1\times J$ vectors. Thus, $\underline{\underline{X}}^{e}$ is a $J\times J$ matrix, $\underline{\underline{Y}}^{e}$ is a $J\times K$ matrix, $\underline{\underline{\Omega}}^{\,e}$ is a $J\times K$ matrix, and $\underline{\Upsilon}^{\,e}$ is a $1\times J$ vector.

It follows that

$$\underline{\Psi}^{\,e} = \underline{\underline{F}}^{\,ee}\,\underline{\beta}^{\,e} - \underline{\Lambda}^{\,e}, \tag{27}$$

$$\underline{\Delta\Psi}^{\,e} = \underline{\beta}^{\,e}, \tag{28}$$

$$\underline{\Psi}^{o} = \underline{0}, \tag{29}$$

$$\underline{\Delta\Psi}^{o} = \underline{\underline{F}}^{oe}\,\underline{\beta}^{e} + \underline{\Delta\Lambda}^{o}, \tag{30}$$

where

$$\underline{\underline{F}}^{ee} = \underline{\underline{\Pi}}^{ae}\,\underline{\underline{\Omega}}^{e} + \underline{\underline{\Pi}}^{se}, \tag{31}$$

$$\underline{\underline{F}}^{oe} = \underline{\underline{\Delta\Pi}}^{ae}\,\underline{\underline{\Omega}}^{e} + \underline{\underline{\Delta\Pi}}^{se}, \tag{32}$$

$$\underline{\Lambda}^{e} = \underline{\underline{\Pi}}^{ae}\,\underline{\Upsilon}^{e}, \tag{33}$$

$$\underline{\Delta\Lambda}^{o} = -\underline{\underline{\Delta\Pi}}^{ae}\,\underline{\Upsilon}^{e}. \tag{34}$$

Note that $\underline{\underline{\Pi}}^{ae}$ is a $K\times J$ matrix, $\underline{\underline{\Delta\Pi}}^{ae}$ is a $K\times J$ matrix, $\underline{\underline{\Pi}}^{se}$ is a $K\times K$ matrix, $\underline{\underline{\Delta\Pi}}^{se}$ is a $K\times K$ matrix, and $\underline{\Lambda}^{e}$ is a $1\times J$ vector. Thus, $\underline{\underline{F}}^{ee}$ is a $K\times K$ matrix, $\underline{\underline{F}}^{oe}$ is a $K\times K$ matrix, and $\underline{\Lambda}^{e}$ and $\underline{\Delta\Lambda}^{o}$ are $1\times K$ vectors.

In the absence of an RMP, the fully-reconnected tearing parity eigenfunction associated with rational surface k is such that $\beta^{e}_{k'} = \delta_{kk'}$. Thus,

$$\psi^{fe}_{jk}(r) = \psi^{se}_{jk}(r) + \sum_{j'=1,J}\psi^{ae}_{jj'}(r)\,\Omega^{e}_{j'k}, \tag{35}$$

$$Z^{fe}_{jk}(r) = Z^{se}_{jk}(r) + \sum_{j'=1,J}Z^{ae}_{jj'}(r)\,\Omega^{e}_{j'k}. \tag{36}$$

This eigenfunction is such that

$$\Psi^{e}_{k'} = F^{ee}_{k'k}, \tag{37}$$

$$\Delta\Psi^{e}_{kk'} = \delta_{k'k}, \tag{38}$$

$$\Psi^{o}_{k'} = 0, \tag{39}$$

$$\Delta\Psi^{o}_{k'} = F^{oe}_{k'k}. \tag{40}$$

C. Twisting Parity Solution

Let us launch J independent solution vectors from the magnetic axis. Let the jth solution vectors be denoted $\underline{\underline{\psi}}^{ao}(r)$ and $\underline{\underline{Z}}^{ao}(r)$. The jump conditions imposed at the rational surfaces are

$$\Psi^{e}_{k} = 0, \tag{41}$$

$$\Delta\Psi_k^{o} = 0, \tag{42}$$

for $k = 1, K$, which implies that

$$A_{L\,k}^{+} = -A_{L\,k}^{-}, \tag{43}$$

$$A_{S\,k}^{+} = -A_{S\,k}^{-}. \tag{44}$$

Let $\Pi_{kj}^{a\,o}$ be the value of Ψ_k^{o} at the kth rational surface associated with the jth solution vector launched from the magnetic axis. Likewise, let $\Delta\Pi_{kj}^{a\,o}$ be the value of $\Delta\Psi_k^{e}$ at the kth rational surface associated with the jth solution vector launched from the magnetic axis.

Let us launch K small solution vectors from each of the rational surfaces in the plasma. Let the solution vectors be denoted $\underline{\underline{\psi}}^{\,s\,o}(r)$ and $\underline{\underline{Z}}^{\,s\,o}(r)$. The launch conditions are

$$\Psi_k^{e} = \Psi_k^{o} = 0, \tag{45}$$

$$\Delta\Psi_k^{o} = 1. \tag{46}$$

The jump conditions imposed at the other rational surfaces are

$$\Psi_{k'}^{e} = 0, \tag{47}$$

$$\Delta\Psi_{k'}^{o} = 0, \tag{48}$$

where $k' \neq k$. Let $\Pi_{k'k}^{s\,o}$ be the value of $\Psi_{k'}^{o}$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface. Likewise, let $\Delta\Pi_{k'k}^{s\,o}$ be the value of $\Delta\Psi_{k'}^{e}$ at the k'th rational surface associated with the small solution vector launched from the kth rational surface.

The general twisting parity solution vectors are written

$$\underline{\psi}^{\,o}(r) = \underline{\underline{\psi}}^{\,a\,o}(r)\,\underline{\alpha}^{o} + \underline{\underline{\psi}}^{\,s\,o}(r)\,\underline{\beta}^{\,o}, \tag{49}$$

$$\underline{Z}^{\,o}(r) = \underline{\underline{Z}}^{\,a\,o}(r)\,\underline{\alpha}^{o} + \underline{\underline{Z}}^{\,s\,o}(r)\,\underline{\beta}^{\,o}, \tag{50}$$

Here, $\underline{\alpha}^{o}$ is a $1\times J$ vector of arbitrary coefficients, whereas $\underline{\beta}^{\,o}$ is a $1\times K$ vector of arbitrary coefficients. However, the boundary condition at the plasma-vacuum interface is again

$$\underline{\underline{I}}(a)\,\underline{Z}^{\,o}(a) = \underline{\underline{H}}\,[\underline{\psi}^{\,o}(a) - \underline{\psi}^{\,x}(a)]. \tag{51}$$

It follows that

$$\underline{\underline{X}}^{o}\,\underline{\alpha}^{o} = \underline{\underline{Y}}^{o}\,\underline{\beta}^{\,o} - \underline{\Xi}, \tag{52}$$

where

$$\underline{\underline{X}}^{o} = \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{a\,o}(a) - \underline{\underline{H}}\,\underline{\underline{\psi}}^{a\,o}(a), \tag{53}$$

$$\underline{\underline{Y}}^{o} = \underline{\underline{H}}\,\underline{\underline{\psi}}^{s\,o}(a) - \underline{\underline{I}}(a)\,\underline{\underline{Z}}^{s\,o}(a). \tag{54}$$

Thus,

$$\underline{\alpha}^{o} = \underline{\underline{\Omega}}^{o}\,\underline{\beta}^{o} - \underline{\Upsilon}^{o}, \tag{55}$$

where

$$\underline{\underline{X}}^{o}\,\underline{\underline{\Omega}}^{o} = \underline{\underline{Y}}^{o}, \tag{56}$$

$$\underline{\underline{X}}^{o}\,\underline{\Upsilon}^{o} = \underline{\Xi}. \tag{57}$$

Note that $\underline{\underline{\psi}}^{a\,o}(a)$ is a $J\times J$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{a\,o}(a)$ is a $J\times J$ matrix, $\underline{\underline{\psi}}^{s\,o}(a)$ is a $J\times K$ matrix, $\underline{\underline{I}}(a)\,\underline{\underline{Z}}^{s\,o}(a)$ is a $J\times K$ matrix, $\underline{\underline{H}}$ is a $J\times J$ matrix, $\underline{\alpha}^{o}$ is a J vector, and $\underline{\beta}^{o}$ is a K vector. Thus, $\underline{\underline{X}}^{o}$ is a $J\times J$ matrix, $\underline{\underline{Y}}^{o}$ is a $J\times K$ matrix, and $\underline{\underline{\Omega}}^{o}$ is a $J\times K$ matrix, and $\underline{\Upsilon}^{o}$ is a $1\times J$ vector.

It follows that

$$\underline{\Psi}^{e} = \underline{0}, \tag{58}$$

$$\underline{\Delta\Psi}^{e} = \underline{\underline{F}}^{eo}\,\underline{\beta}^{o} + \underline{\Delta\Lambda}^{e}, \tag{59}$$

$$\underline{\Psi}^{o} = \underline{\underline{F}}^{oo}\,\underline{\beta}^{o} - \underline{\Lambda}^{o}, \tag{60}$$

$$\underline{\Delta\Psi}^{o} = \underline{\beta}^{o}, \tag{61}$$

where

$$\underline{\underline{F}}^{oo} = \underline{\underline{\Pi}}^{a\,o}\,\underline{\underline{\Omega}}^{o} + \underline{\underline{\Pi}}^{s\,o}, \tag{62}$$

$$\underline{\underline{F}}^{eo} = \underline{\underline{\Delta\Pi}}^{a\,o}\,\underline{\underline{\Omega}}^{o} + \underline{\underline{\Delta\Pi}}^{s\,o}, \tag{63}$$

$$\underline{\Lambda}^{o} = \underline{\underline{\Pi}}^{a\,o}\,\underline{\Upsilon}^{o}, \tag{64}$$

$$\underline{\Delta\Lambda}^{e} = -\underline{\underline{\Delta\Pi}}^{a\,o}\,\underline{\Upsilon}^{o}. \tag{65}$$

Note that $\underline{\underline{\Pi}}^{a\,o}$ is a $K\times J$ matrix, $\underline{\underline{\Delta\Pi}}^{a\,o}$ is a $K\times J$ matrix, $\underline{\underline{\Pi}}^{s\,o}$ is a $K\times K$ matrix, and $\underline{\underline{\Delta\Pi}}^{s\,o}$ is a $K\times K$ matrix. Thus, $\underline{\underline{F}}^{oo}$ is a $K\times K$ matrix, $\underline{\underline{F}}^{eo}$ is a $K\times K$ matrix, and $\underline{\Lambda}^{o}$ and $\underline{\Delta\Lambda}^{e}$ are $1\times K$ vectors.

In the absence of an RMP, the fully-reconnected twisting parity eigenfunction associated with rational surface k is such that $\beta^o_{k'} = \delta_{kk'}$. Thus,

$$\psi^{f\,o}_{jk}(r) = \psi^{s\,o}_{jk}(r) + \sum_{j'=1,J} \psi^{a\,o}_{jj'}(r)\,\Omega^o_{j'k}, \tag{66}$$

$$Z^{f\,o}_{jk}(r) = Z^{s\,o}_{jk}(r) + \sum_{j'=1,J} Z^{a\,o}_{jj'}(r)\,\Omega^o_{j'k}. \tag{67}$$

This eigenfunction is such that

$$\Psi^e_{k'} = 0, \tag{68}$$

$$\Delta\Psi^e_{kk'} = F^{eo}_{k'k}, \tag{69}$$

$$\Psi^o_{k'} = F^{oo}_{k'k}, \tag{70}$$

$$\Delta\Psi^o_{k'} = \delta_{k'k}. \tag{71}$$

D. General Dispersion Relation

The general dispersion relation is obtained by combining the tearing parity dispersion relation, (27)–(30), with the twisting parity dispersion relation, (58)–(61). We get

$$\underline{\Psi}^{\,e} = \underline{\underline{F}}^{\,ee}\,\underline{\beta}^{\,e} - \underline{\Lambda}^{\,e}, \tag{72}$$

$$\underline{\Delta\Psi}^{\,e} = \underline{\beta}^{\,e} + \underline{\underline{F}}^{\,eo}\,\underline{\beta}^{\,o} + \underline{\Delta\Lambda}^{\,e}, \tag{73}$$

$$\underline{\Psi}^{\,o} = \underline{\underline{F}}^{\,oo}\,\underline{\beta}^{\,o} - \underline{\Lambda}^{\,o}, \tag{74}$$

$$\underline{\Delta\Psi}^{\,o} = \underline{\beta}^{\,o} + \underline{\underline{F}}^{\,oe}\,\underline{\beta}^{\,e} + \underline{\Delta\Lambda}^{\,o}. \tag{75}$$

Hence, we obtain the general dispersion relation

$$\begin{pmatrix} \underline{\Delta\Psi}^{\,e} \\ \underline{\Delta\Psi}^{\,o} \end{pmatrix} = \begin{pmatrix} \underline{\underline{E}}^{\,ee} & \underline{\underline{E}}^{\,eo} \\ \underline{\underline{E}}^{\,oe} & \underline{\underline{E}}^{\,oo} \end{pmatrix} \begin{pmatrix} \underline{\Psi}^{\,e} \\ \underline{\Psi}^{\,o} \end{pmatrix} + \begin{pmatrix} \underline{\chi}^{\,e} \\ \underline{\chi}^{\,o} \end{pmatrix} \tag{76}$$

where

$$\underline{\underline{E}}^{\,ee} = (\underline{\underline{F}}^{\,ee})^{-1}, \tag{77}$$

$$\underline{\underline{E}}^{\,eo} = \underline{\underline{F}}^{\,eo}\,\underline{\underline{E}}^{\,oo}, \tag{78}$$

$$\underline{\underline{E}}^{\,oe} = \underline{\underline{F}}^{\,oe}\,\underline{\underline{E}}^{\,ee}, \tag{79}$$

$$\underline{\underline{E}}^{\,oo} = (\underline{\underline{F}}^{\,oo})^{-1}, \tag{80}$$

$$\underline{\chi}^{\,e} = \underline{\underline{E}}^{\,ee}\,\underline{\Lambda}^{\,e} + \underline{\underline{E}}^{\,eo}\,\underline{\Lambda}^{\,o} + \underline{\Delta\Lambda}^{\,e}, \tag{81}$$

$$\underline{\chi}^{\,o} = \underline{\underline{E}}^{\,oe}\,\underline{\Lambda}^{\,e} + \underline{\underline{E}}^{\,oo}\,\underline{\Lambda}^{\,o} + \underline{\Delta\Lambda}^{\,o}. \tag{82}$$

In the absence of an RMP, the tearing parity unreconnected eigenfunction associated with the kth rational surface is such that

$$\Psi^{\,e}_{k'} = \delta_{k'k}, \tag{83}$$

$$\Delta\Psi^{\,e}_{kk'} = E^{ee}_{k'k}, \tag{84}$$

$$\Psi^{\,o}_{k'} = 0, \tag{85}$$

$$\Delta\Psi^{\,o}_{k'} = E^{oe}_{k'k}. \tag{86}$$

Thus,

$$\psi^{\,u\,e}_{jk}(r) = \sum_{k'=1,k} \psi^{\,f\,e}_{jk'}(r)\,E^{ee}_{k'k} + \sum_{k'=1,k} \psi^{\,f\,o}_{jk'}(r)\,E^{oe}_{k'k}, \tag{87}$$

$$Z^{\,u\,e}_{jk}(r) = \sum_{k'=1,k} Z^{\,f\,e}_{jk'}(r)\,E^{ee}_{k'k} + \sum_{k'=1,k} Z^{\,f\,o}_{jk'}(r)\,E^{oe}_{k'k}. \tag{88}$$

In the absence of an RMP, the twisting parity unreconnected eigenfunction associated with the kth rational surface is such that

$$\Psi^{\,e}_{k'} = 0, \tag{89}$$

$$\Delta\Psi^{\,e}_{kk'} = E^{eo}_{k'k}, \tag{90}$$

$$\Psi^{\,o}_{k'} = \delta_{k'k}, \tag{91}$$

$$\Delta\Psi^{\,o}_{k'} = E^{oo}_{k'k}. \tag{92}$$

Thus,

$$\psi^{\,u\,o}_{jk}(r) = \sum_{k'=1,k} \psi^{\,f\,o}_{jk'}(r)\,E^{oo}_{k'k} + \sum_{k'=1,k} \psi^{\,f\,e}_{jk'}(r)\,E^{eo}_{k'k}, \tag{93}$$

$$Z^{\,u\,o}_{jk}(r) = \sum_{k'=1,k} Z^{\,f\,o}_{jk'}(r)\,E^{oo}_{k'k} + \sum_{k'=1,k} Z^{\,f\,e}_{jk'}(r)\,E^{eo}_{k'k}. \tag{94}$$

II. ANGULAR MOMENTUM CONSERVATION

In the absence of an RMP, the total toroidal electromagnetic torque acting on the plasma is

$$T_\varphi = 2\,n\,\pi^2\,{\rm Im}\left(\underline{\Psi}^{\,e\,\dagger}\,\underline{\Delta\Psi}^{\,e} + \underline{\Psi}^{\,o\,\dagger}\,\underline{\Delta\Psi}^{\,o}\right), \tag{95}$$

which gives

$$T_\varphi = 2\,n\,\pi^2\,{\rm Im}\left(\underline{\Psi}^{\,e\,\dagger}\,\underline{\underline{E}}^{\,ee}\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,e\,\dagger}\,\underline{\underline{E}}^{\,eo}\,\underline{\Psi}^{\,o} + \underline{\Psi}^{\,o\,\dagger}\,\underline{\underline{E}}^{\,oe}\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,o\,\dagger}\,\underline{\underline{E}}^{\,oo}\,\underline{\Psi}^{\,o}\right), \tag{96}$$

or

$$\begin{aligned} T_\varphi = n\,\pi^2\,\Big[&\underline{\Psi}^{\,e\,\dagger}\,(\underline{\underline{E}}^{\,ee} - \underline{\underline{E}}^{\,ee\,\dagger})\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,e\,\dagger}\,(\underline{\underline{E}}^{\,eo} - \underline{\underline{E}}^{\,oe\,\dagger})\,\underline{\Psi}^{\,o}\\ &+ \underline{\Psi}^{\,o\,\dagger}\,(\underline{\underline{E}}^{\,oe} - \underline{\underline{E}}^{\,eo\,\dagger})\,\underline{\Psi}^{\,e} + \underline{\Psi}^{\,o\,\dagger}\,(\underline{\underline{E}}^{\,oo} - \underline{\underline{E}}^{\,oo\,\dagger})\,\underline{\Psi}^{\,o}\Big]. \end{aligned} \tag{97}$$

However, T_φ must be zero, irrespective of the values of the $\underline{\Psi}^{\,e}$ and the $\underline{\Psi}^{\,o}$. This is only possible if

$$\underline{\underline{E}}^{\,ee\,\dagger} = \underline{\underline{E}}^{\,ee}, \tag{98}$$

$$\underline{\underline{E}}^{\,eo\,\dagger} = \underline{\underline{E}}^{\,oe}, \tag{99}$$

$$\underline{\underline{E}}^{\,oo\,\dagger} = \underline{\underline{E}}^{\,oo}. \tag{100}$$

III. INNER LAYER EQUATIONS

In the vicinity of the kth rational surface, the inner layer equations are

$$(\hat{\gamma}_k + {\rm i}\,Q_{E\,k} + {\rm i}\,Q_{e\,k})\,\psi = -{\rm i}\,X\,(\phi - N) + \frac{d^2\psi}{dX^2}, \tag{101}$$

$$(\hat{\gamma}_k + {\rm i}\,Q_{E\,k})\,N = -{\rm i}\,Q_{e\,k}\,\phi - {\rm i}\,c_{\beta\,k}^2\,X\,V - {\rm i}\,D_k^2\,X\,\frac{d^2\psi}{dX^2} + P_{\perp\,k}\,\frac{d^2N}{dX^2}, \tag{102}$$

$$(\hat{\gamma}_k + {\rm i}\,Q_{E\,k} + {\rm i}\,Q_{i\,k})\,\frac{d^2\phi}{dX^2} = -{\rm i}\,X\,\frac{d^2\psi}{dX^2} + P_{\varphi\,k}\,\frac{d^4}{dX^4}\left(\phi + \frac{N}{\iota_k}\right), \tag{103}$$

$$(\hat{\gamma}_k + {\rm i}\,Q_{E\,k})\,V = {\rm i}\,Q_{e\,k}\,\psi - {\rm i}\,X\,N + P_{\varphi\,k}\,\frac{d^2V}{dX^{\,2}}, \tag{104}$$

where $X = S_k^{1/3}\,(r - r_k)/r_k$. All quantities are as defined in TJ2025, except that

$$c_{\beta\,k} = \left(\frac{\beta_k}{1+\beta_k}\right)^{1/2}, \tag{105}$$

$$\iota_k = -\frac{\omega_{\ast\,e\,k}}{\omega_{\ast\,i\,k}}, \tag{106}$$

$$Q_{e\,k} = -\left(\frac{\iota_k}{1+\iota_k}\right)\tau_k\,\omega_{\ast\,k}, \tag{107}$$

$$Q_{i\,k} = \left(\frac{1}{1+\iota_k}\right)\tau_k\,\omega_{\ast\,k}, \tag{108}$$

$$D_k = \left(\frac{\iota_k}{1+\iota_k}\right)^{1/2} S_k^{1/3}\,\hat{d}_{\beta\,k}. \tag{109}$$

Equations (101)–(104) possess the trivial twisting parity solution:

$$\psi(X) = A\,X, \tag{110}$$

$$N(X) = A\,Q_{e\,k}, \tag{111}$$

$$\phi(X) = A\,({\rm i}\,\gamma_k - Q_{E\,k}), \tag{112}$$

$$V(X) = 0. \tag{113}$$

IV. INTERMEDIATE LAYER EQUATIONS

The intermediate layer equation is

$$(1+Y^2)\,\frac{d^2\psi}{dY^2} = \nu_k\,(1+\nu_k)\,\psi, \tag{114}$$

where $Y = (r-r_k)/\delta_k$, and

$$\nu_k = -\frac{1}{2} + \sqrt{-D_{I\,k}} \simeq -\frac{1}{4} - D_{I\,k}. \tag{115}$$

The general asymptotic behavior is

$$\psi(Y) = \hat{B}_{L\,k} + \hat{B}_{S\,k}\,|Y| \tag{116}$$

for $|Y|\ll 1$, and

$$\psi(Y) = \hat{A}_{L\,k}\,|Y|^{-\nu_k} + \hat{A}_{S\,k}\,|Y|^{1+\nu_k} \tag{117}$$

for $|Y|\gg 1$.

The tearing parity solution is such that

$$\psi(Y) = \hat{B}^{\,e}_{L\,k} + \hat{B}^{\,e}_{S\,k}\,|Y| \tag{118}$$

for $|Y| \ll 1$, and

$$\psi(Y) = \hat{A}^{e}_{L\,k}\,|Y|^{-\nu_k} + \hat{A}^{e}_{S\,k}\,|Y|^{1+\nu_k} \tag{119}$$

for $|Y| \gg 1$. Furthermore,

$$S_k^{1/3}\,\hat{\varDelta}_k^{e} = \left(\frac{r_k}{\delta_k}\right)\frac{2\,\hat{B}^{e}_{S\,k}}{\hat{B}^{e}_{L\,k}}, \tag{120}$$

where $\hat{\varDelta}_k^e$ is the tearing parity layer response function, and

$$\varDelta_k^{e} \equiv \frac{\Delta\Psi_k^{e}}{\Delta\Psi_k^{e}} = \left(\frac{r_k}{\delta_k}\right)^{1+2\,\nu_k}\frac{2\,\hat{A}^{e}_{S\,k}}{\hat{A}^{e}_{L\,k}} \tag{121}$$

The twisting parity solution is such that

$$\psi(Y) = \hat{B}^{o}_{S\,k}\,Y \tag{122}$$

for $|Y| \ll 1$, and

$$\psi(Y) = \mathrm{sgn}(Y)\left(\hat{A}^{o}_{L\,k}\,|Y|^{-\nu_k} + \hat{A}^{o}_{S\,k}\,|Y|^{1+\nu_k}\right) \tag{123}$$

for $|Y| \gg 1$. Furthermore,

$$\varDelta_k^{o} \equiv \frac{\Delta\Psi_k^{o}}{\Delta\Psi_k^{o}} = \left(\frac{r_k}{\delta_k}\right)^{1+2\,\nu_k}\frac{2\,\hat{A}^{o}_{S\,k}}{\hat{A}^{o}_{L\,k}}. \tag{124}$$

The connection formulae are

$$\hat{B}^{e,o}_{L\,k} = a_{LL}\,\hat{A}^{e,o}_{L\,k} + a_{LS}\,\hat{A}^{e,o}_{S\,k}, \tag{125}$$

$$\hat{B}^{e,o}_{S\,k} = a_{SL}\,\hat{A}^{e,o}_{L\,k} + a_{SS}\,\hat{A}^{e,o}_{S\,k}. \tag{126}$$

It follows that

$$\left(\frac{\delta_k}{r_k}\right)\frac{S_k^{1/3}\,\hat{\varDelta}_k^{e}}{2} = \frac{a_{SL} + a_{SS}\,(\delta_k/r_k)^{1+2\,\nu_k}\,(\varDelta_k^e/2)}{a_{LL} + a_{LS}\,(\delta_k/r_k)^{1+2\,\nu_k}\,(\varDelta_k^e/2)}, \tag{127}$$

$$\left(\frac{\delta_k}{r_k}\right)^{1+2\,\nu_k}\frac{\varDelta_k^{o}}{2} = -\frac{a_{LL}}{a_{LS}}. \tag{128}$$

But, in the limit $|\nu_k| \to 0$, we find that $a_{LL} \to 1$, $a_{SL} \to -\nu_k\,\pi/2$, $a_{LS} \to -\nu_k\,\pi/2$, and $a_{SS} \to 1$. Thus, we obtain

$$\varDelta_k^{e} \simeq S_k^{1/3}\,\hat{\varDelta}_k^{e} + \frac{\pi\,\nu_k\,r_k}{\delta_k}, \tag{129}$$

$$\varDelta_k^{o} \simeq \frac{4\,r_k}{\pi\,\nu_k\,\delta_k}. \tag{130}$$

Let $\delta_k = \delta_{d\,k}/(2\sqrt{\pi})$, and let us identify ν_k with $-D_{R\,k}$. It follows that

$$\varDelta_k^e = S_k^{1/3}\,\hat{\varDelta}_k^e + \varDelta_{k\,\rm crit}^e, \tag{131}$$

$$\varDelta_k^o = \varDelta_{k\,\rm crit}^o, \tag{132}$$

where

$$\varDelta_{k\,\rm crit}^e = \sqrt{2}\,\pi^{3/2}\,(-D_{R\,k})\,\frac{r_k}{\delta_{d\,k}}, \tag{133}$$

$$\varDelta_{k\,\rm crit}^o = \frac{8}{\sqrt{\pi}}\,(-D_{R\,k})^{-1}\,\frac{r_k}{\delta_{d\,k}}. \tag{134}$$

V. HOMOGENOUS DISPERSION RELATION

The homogeneous dispersion relation can be written

$$(S_k^{1/3}\,\hat{\varDelta}_k^e + \varDelta_{k\,\rm crit}^e)\,\varPsi_k^e = \sum_{k'}(E_{kk'}^{ee}\,\varPsi_{k'}^e + E_{kk'}^{eo}\,\varPsi_{k'}^o), \tag{135}$$

$$0 = \sum_{k'}(\tilde{E}_{kk'}^{oo}\,\varPsi_{k'}^o + E_{kk'}^{oe}\,\varPsi_{k'}^e), \tag{136}$$

where

$$\tilde{E}_{kk'}^{oo} = E_{kk'}^{oo} - \varDelta_{k\,\rm crit}^o\,\delta_{kk'}. \tag{137}$$

Hence,

$$\varPsi_k^o = -\sum_{k',k''}(\tilde{E}_{kk'}^{oo})^{-1}\,E_{k'k''}^{oe}\,\varPsi_{k''}^e, \tag{138}$$

and

$$(S_k^{1/3}\,\hat{\varDelta}_k^e + \varDelta_{k\,\rm crit}^e)\,\varPsi_k^e = \sum_{k'}E_{kk'}^e\,\varPsi_{k'}^e \tag{139}$$

where

$$E_{kk'}^e = E_{kk'}^{ee} - \sum_{k'',k'''}E_{kk''}^{eo}\,(\tilde{E}_{k''k'''}^{oo})^{-1}\,E_{k'''k'}^{oe}. \tag{140}$$

Note that $\underline{\underline{E}}^{\,e}$ is Hermitian.

Suppose that $\hat{\varDelta}_k$ is small, but $\hat{\varDelta}_{k'\neq k}$ is order unity. In this case,

$$\varPsi_{k'}^e \simeq \delta_{kk'}\,\varPsi_k^e. \tag{141}$$

It follows that the growth rate of the mode that reconnects magnetic flux at the kth rational surface is governed by

$$S_k^{1/3}\,\hat{\varDelta}_k^e \simeq E_{kk}^e - \varDelta_{k\,\rm crit}^e. \tag{142}$$

The corresponding eigenfunction is

$$\psi^u_{jk}(r) = \psi^{u\,e}_{jk}(r) - \sum_{k',k''} \psi^{u\,o}_{jk'}(r)\,(\tilde{E}^{oo}_{k'k''})^{-1}\,E^{oe}_{k''k}, \tag{143}$$

$$Z^u_{jk}(r) = Z^{u\,e}_{jk}(r) - \sum_{k',k''} Z^{u\,o}_{jk'}(r)\,(\tilde{E}^{oo}_{k'k''})^{-1}\,E^{oe}_{k''k}, \tag{144}$$

and has the properties that

$$\Psi^e_{k'} = \delta_{kk'}, \tag{145}$$

$$\Delta\Psi^e_{k'} = E^e_{k'k}, \tag{146}$$

$$\Psi^o_{k'} = -\sum_{k',k''} (\tilde{E}^{oo}_{k'k''})^{-1}\,E^{oe}_{k''k}, \tag{147}$$

$$\Delta\Psi^o_{k'} = \Delta^o_{k'\,\mathrm{crit}}\,\Psi^o_{k'}. \tag{148}$$

Suppose that Ψ^e_k and $\Psi^e_{k'}$ are both non-zero, but that $\Psi^e_{k''} = 0$ for $k'' \neq k, k'$. The toroidal electromagnetic torque at the kth rational surface is

$$\delta T_k = 2\,n\,\pi^2\,\mathrm{Im}\,(\Psi^{e\,*}_k\,\Delta\Psi^e_k + \Psi^{o\,*}_k\,\Delta\Psi^o_k)\,. \tag{149}$$

Hence, we deduce that

$$\delta T_k = 2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_k\,E^e_{kk'}\,\Psi^e_{k'}), \tag{150}$$

and

$$\begin{aligned} \delta T_{k'} &= 2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_{k'}\,E^e_{k'k}\,\Psi^e_k) = -2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_k\,E^{e\,*}_{k'k}\,\Psi'^e_k) \\ &= -2\,n\,\pi^2\,\mathrm{Im}(\Psi^{e\,*}_k\,E^e_{kk'}\,\Psi'^e_k) = -\delta T_{k'}, \end{aligned} \tag{151}$$

with $\delta T_{k''} = 0$ for $k'' \neq k, k'$.

VI. INHOMOGENOUS DISPERSION RELATION

The inhomogeneous dispersion relation can be written

$$(S_k^{1/3}\,\hat{\Delta}^e_k + \Delta^e_{k\,\mathrm{crit}})\,\Psi^e_k = \sum_{k'} (E^{ee}_{kk'}\,\Psi^e_{k'} + E^{eo}_{kk'}\,\Psi^o_{k'}) + \chi^e_k, \tag{152}$$

$$0 = \sum_{k'} (\tilde{E}^{oo}_{kk'}\,\Psi^o_{k'} + E^{oe}_{kk'}\,\Psi^e_{k'}) + \chi^o_k. \tag{153}$$

It follows that

$$(S_k^{1/3}\,\hat{\Delta}_k^e + \Delta_{k\,\mathrm{crit}}^e)\,\Psi_k^e = \sum_{k'} E_{kk'}^e\,\Psi_{k'}^e + \chi_k, \tag{154}$$

where

$$\chi_k = \chi_k^e - \sum E_{k',k''}^{eo}\,(\tilde{E}_{k'k''}^{oo})^{-1}\,\chi_{k''}^o. \tag{155}$$