Resistive Wall Calculations

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I. VACUUM SOLUTION

A. Normalization

Let all lengths be normalized to the major radius of the axisymmetric plasma equilibrium's magnetic axis, R_0. Let all magnetic field-strengths be normalized to the toroidal magnetic field-strength at the magnetic axis, B_0.

B. Toroidal Coordinates

Let μ, η, ϕ be right-handed toroidal coordinates defined such that

$$R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}, \tag{1}$$

$$Z = \frac{\sin\eta}{\cosh\mu - \cos\eta}, \tag{2}$$

where R, ϕ, Z are right-handed cylindrical coordinates whose symmetry axis corresponds to that of the plasma equilibrium. Note that $(\nabla R\times\nabla\phi\cdot\nabla Z)^{-1} = R$. The scale-factors of the toroidal coordinate system are

$$h_\mu = h_\eta = \frac{1}{\cosh\mu - \cos\eta} \equiv h, \tag{3}$$

$$h_\phi = \frac{\sinh\mu}{\cosh\mu - \cos\eta} = h\,\sinh\mu. \tag{4}$$

Moreover,

$$\mathcal{J}' \equiv (\nabla\mu\times\nabla\eta\cdot\nabla\phi) = h^3\,\sinh\mu. \tag{5}$$

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C. Perturbed Magnetic Field

The curl-free perturbed magnetic field in the vacuum region is written

$$\mathbf{b} = \mathrm{i}\,\nabla\left[V(\mu,\eta)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi}\right], \tag{6}$$

where the toroidal mode number, n, is a positive integer. Given that $\nabla\cdot\mathbf{b}=0$, we deduce that

$$\begin{aligned}\nabla^2 V \equiv (z-\cos\eta)^3 &\left\{\frac{\partial}{\partial z}\left[\frac{z^2-1}{z-\cos\eta}\,\frac{\partial V}{\partial z}\right]\right.\\ &\left.+\frac{\partial}{\partial\eta}\left[\frac{1}{z-\cos\eta}\,\frac{\partial V}{\partial\eta}\right]-\frac{n^2\,V}{(z^2-1)\,(z-\cos\eta)}\right\}=0.\end{aligned} \tag{7}$$

Here, $z=\cosh\mu$.

Let

$$f_z = z^2-1, \tag{8}$$

$$f_\eta = (z-\cos\eta)^{1/2}, \tag{9}$$

which implies that

$$\frac{df_z}{dz} = 2\,z, \tag{10}$$

$$\frac{\partial f_\eta}{\partial z} = \frac{1}{2\,f_\eta}, \tag{11}$$

$$\frac{\partial f_\eta}{\partial\eta} = \frac{\sin\eta}{2\,f_\eta} \tag{12}$$

Suppose that

$$V(z,\eta) = \sum_m (z-\cos\eta)^{1/2}\,U_m(z)\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}. \tag{13}$$

Taking the sum and eikonal as read, and letting $'=d/dz$, we get

$$\frac{\partial V}{\partial z} = \frac{U_m}{2\,f_\eta}+f_\eta\,U_m', \tag{14}$$

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^2}\,\frac{\partial V}{\partial z}\right) &= \frac{\partial}{\partial z}\left(\frac{f_z\,U_m}{2\,f_\eta^3}+\frac{f_z\,U_m'}{f_\eta}\right)=\frac{z\,U_m}{f_\eta^3}-\frac{3\,f_z\,U_m}{4\,f_\eta^5}+\frac{f_z\,U_m'}{2\,f_\eta^3}+\frac{2\,z\,U_m'}{f_\eta}-\frac{f_z\,U_m'}{2\,f_\eta^3}+\frac{f_z\,U_m''}{f_\eta}\\ &=\frac{z\,U_m}{f_\eta^3}-\frac{3\,(z^2-1)\,U_m}{4\,f_\eta^5}+\frac{2\,z\,U_m'}{f_\eta}+\frac{(z^2-1)\,U_m''}{f_\eta},\end{aligned} \tag{15}$$

$$\frac{\partial V}{\partial\eta} = \frac{\sin\eta\,U_m}{2\,f_\eta}-\mathrm{i}\,m f_\eta\,U_m, \tag{16}$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{f_\eta^2}\,\frac{\partial V}{\partial\eta}\right)=\frac{\partial}{\partial\eta}\left(\frac{\sin\eta\,U_m}{2\,f_\eta^3}-\frac{\mathrm{i}\,m\,U_m}{f_\eta}\right)=\frac{\cos\eta\,U_m}{2\,f_\eta^3}-\frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5}-\frac{\mathrm{i}\,m\,\sin\eta\,U_m}{2\,f_\eta^3}$$
$$+\frac{\mathrm{i}\,m\,\sin\eta\,U_m}{2\,f_\eta^3}-\frac{m^2\,U_m}{f_\eta}$$
$$=\frac{\cos\eta\,U_m}{2\,f_\eta^3}-\frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5}-\frac{m^2\,U_m}{f_\eta}, \tag{17}$$
$$-\frac{n^2\,V}{f_z\,f_\eta^2}=-\frac{n^2\,U_m}{(z^2-1)\,f_\eta}. \tag{18}$$

Thus, Eq. (7) becomes

$$0=\frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^2}\,\frac{\partial V}{\partial z}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{f_\eta^2}\,\frac{\partial V}{\partial\eta}\right)-\frac{n^2\,V}{f_z\,f_\eta^2}$$
$$=\frac{z\,U_m}{f_\eta^3}-\frac{3\,(z^2-1)\,U_m}{4\,f_\eta^5}+\frac{2\,z\,U_m'}{f_\eta}+\frac{(z^2-1)\,U_m''}{f_\eta}$$
$$+\frac{\cos\eta\,U_m}{2\,f_\eta^3}-\frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5}-\frac{m^2\,U_m}{f_\eta}-\frac{n^2\,U_m}{(z^2-1)\,f_\eta}$$
$$=\frac{1}{f_\eta}\left[(z^2-1)\,U_m''+2\,z\,U_m'+\left(\frac{1}{4}-m^2\right)U_m-\frac{n^2\,U_m}{z^2-1}\right]. \tag{19}$$

The most general solution of the previous equation is

$$U_m(z)=p_m\,\hat{P}^n_{|m|-1/2}(z)+q_m\,\hat{Q}^n_{m-1/2}(z), \tag{20}$$

where

$$\hat{P}^n_{|m|-1/2}(z)=\cos(|m|\,\pi)\,\frac{\sqrt{\pi}\,\Gamma(|m|+1/2-n)\,\epsilon^{\,|m|}}{2^{\,|m|-1/2}\,|m|!}\,P^n_{|m|-1/2}(z), \tag{21}$$

$$\hat{Q}^n_{|m|-1/2}(z)=\cos(n\,\pi)\,\cos(|m|\,\pi)\,\frac{2^{\,|m|-1/2}\,|m|!}{\sqrt{\pi}\,\Gamma(|m|+1/2+n)\,\epsilon^{\,|m|}}\,Q^n_{|m|-1/2}. \tag{22}$$

Here, ϵ is the inverse-aspect ratio of the plasma equilibrium, and p_m and q_m are arbitrary complex coefficients. Moreover, we have made use of the fact that

$$P^n_{-m-1/2}(z)=P^n_{m-1/2}(z), \tag{23}$$

$$Q^n_{-m-1/2}(z)=Q^n_{m-1/2}(z). \tag{24}$$

D. Toroidal Electromagnetic Angular Momentum Flux

The outward flux of toroidal angular momentum across a constant-z surface is

$$T_\phi(z)=-\oint\oint\mathcal{J}'\,b_\phi\,b^\mu\,d\eta\,d\phi. \tag{25}$$

Now,

$$b^\mu \equiv \mathbf{b}\cdot\nabla\mu = \mathrm{i}\,\frac{\partial V}{\partial\mu}\,|\nabla\mu|^2 = \mathrm{i}\,\frac{\sinh\mu}{h^2}\,\frac{\partial V}{\partial z}, \tag{26}$$

$$b_\phi \equiv \mathcal{J}'\,\nabla\mu\times\nabla\eta\cdot\nabla V = n\,V, \tag{27}$$

so

$$\begin{aligned}
T_\phi(z) &= -\frac{\mathrm{i}\,n\,\pi}{2}\oint\frac{z^2-1}{z-\cos\eta}\left(\frac{\partial V}{\partial z}\,V^\ast-\frac{\partial V^\ast}{\partial z}\,V\right)d\eta\\
&= -\mathrm{i}\,n\,\pi^2\sum_m (z^2-1)\left(\frac{dU_m}{dz}\,U_m^\ast-\frac{dU_m^\ast}{dz}\,U_m\right)\\
&= -\mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast-q_m\,p_m^\ast)\,(z^2-1)\left(\frac{d\hat{P}|^n_{|m|-1/2}}{dz}\,\hat{Q}^n_{|m|-1/2}-\frac{d\hat{Q}^n_{|m|-1/2}}{dz}\,\hat{P}^n_{|m|-1/2}\right)\\
&= \mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast-q_m\,p_m^\ast)\,(z^2-1)\,\mathcal{W}(\hat{P}^n_{|m|-1/2},\hat{Q}^n_{|m|-1/2}),
\end{aligned} \tag{28}$$

where use has been made of Eqs. (13) and (20). But,

$$\begin{aligned}
\mathcal{W}(\hat{P}^n_{|m|-1/2},\hat{Q}^n_{|m|-1/2}) &= \cos(n\,\pi)\,\frac{\Gamma(|m|+1/2-n)}{\Gamma(|m|+1/2+n)}\,\mathcal{W}(P^n_{|m|-1/2},Q^n_{|m|-1/2})\\
&= \cos(n\,\pi)\,\frac{\Gamma(|m|+1/2-n)}{\Gamma(|m|+1/2+n)}\,\frac{\cos(n\,\pi)}{1-z^2}\,\frac{\Gamma(|m|+1/2+n)}{\Gamma(|m|+1/2-n)}\\
&= \frac{1}{1-z^2},
\end{aligned} \tag{29}$$

where use has been made of Eqs. (21) and (22), so

$$T_\phi(z) = 2\pi^2\,n\sum_m \mathrm{Im}(q_m^\ast\,p_m). \tag{30}$$

Note that T_ϕ is independent of z.

II. MATCHING AT PLASMA-VACUUM INTERFACE

A. Solution in Vacuum Region

Let r, θ, ϕ be right-handed flux coordinates, where r is a flux-surface label, θ is a poloidal angle that is zero on the inboard mid-plane, and

$$\mathcal{J} \equiv (\nabla r\times\nabla\theta\cdot\nabla\phi)^{-1} = r\,R^2. \tag{31}$$

In the large-aspect ratio limit $r \ll 1$, it can be demonstrated that

$$z \simeq \frac{1}{r}, \tag{32}$$

$$z^{\,1/2}\,\hat{P}^{\,n}_{-1/2}(z) \simeq \frac{1}{2}\,\ln\left(\frac{8\,z}{\zeta_n}\right), \tag{33}$$

$$z^{1/2}\,\hat{P}^{\,n}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\,\pi)\,(\epsilon\,z)^{|m|}}{|m|}, \tag{34}$$

$$z^{1/2}\,\hat{Q}^{\,n}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\,\pi)\,(\epsilon\,z)^{-|m|}}{2}, \tag{35}$$

$$\zeta_n = \exp\left(\sum_{j=1,n}\frac{2}{2\,j-1}\right). \tag{36}$$

Note that Eq. (34) only applies to $|m| > 0$.

The plasma-vacuum interface lies at $r = \epsilon$. The wall lies at $r = b_w\,\epsilon$, where $b_w > 1$. In the vacuum region, $\epsilon \leq r \leq b_w\,\epsilon$, lying between the plasma and the wall, we can write

$$\underline{V}(r) = \underline{\underline{\mathcal{P}}}(r)\,\underline{p} + \underline{\underline{\mathcal{Q}}}(r)\,\underline{q}, \tag{37}$$

$$\underline{\psi}(r) = \underline{\underline{\mathcal{R}}}(r)\,\underline{p} + \underline{\underline{\mathcal{S}}}(r)\,\underline{q}, \tag{38}$$

where $\underline{V}(r)$ is the vector of the $V_m(r)$ values, $\underline{\psi}(r)$ is the vector of the $\psi_m(r)$ values, $\underline{\underline{\mathcal{P}}}(r)$ is the matrix of the

$$\mathcal{P}_{mm'}(r) = \oint_r (z-\cos\eta)^{1/2}\,\hat{P}^{\,n}_{|m'|-1/2}(z)\,\exp[-{\rm i}\,(m\,\theta + m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{39}$$

values, $\underline{\underline{\mathcal{Q}}}(r)$ is the matrix of the

$$\mathcal{Q}_{mm'}(r) = \oint_r (z-\cos\eta)^{1/2}\,\hat{Q}^{\,n}_{|m'|-1/2}(z)\,\exp[-{\rm i}\,(m\,\theta + m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{40}$$

values, $\underline{\underline{\mathcal{R}}}(r)$ is the matrix of the

$$\begin{aligned}\mathcal{R}_{mm'}(r) = \oint_r \Bigg\{&\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\hat{P}^{\,n}_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{d\hat{P}^{\,n}_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z\\ &+\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta - {\rm i}\,m'\,(z-\cos\eta)^{1/2}\right]\hat{P}^{\,n}_{|m'|-1/2}(z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\Bigg\}\\ &\times\exp[-{\rm i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi}\end{aligned} \tag{41}$$

values, $\underline{\underline{\mathcal{S}}}(r)$ is the matrix of the

$$\mathcal{S}_{mm'}(r) = \oint_r \Bigg\{\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\hat{Q}^{\,n}_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{d\hat{Q}^{\,n}_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z$$

$$+\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta-{\rm i}\,m'\,(z-\cos\eta)^{1/2}\right]\hat{Q}^{\,n}_{|m'|-1/2}(z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\Bigg\}$$
$$\times\,\exp[-{\rm i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{42}$$

values, $\underline{p}$ is the vector of the p_m coefficients, and $\underline{q}$ is the vector of the q_m coefficients. Here, the subscript r on the integrals indicates that they are taken at constant r.

B. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque acting on the plasma is

$$T_\phi(r) = -2\pi^2\,n\,{\rm Im}(\underline{V}^\dagger\,\underline{\psi}) = -\pi^2\,n\,(\underline{V}^\dagger\,\underline{\psi}-\underline{\psi}^\dagger\,\underline{V}). \tag{43}$$

However, Eq. (30) implies that

$$T_\phi = 2\pi^2\,n\,{\rm Im}(\underline{q}^\dagger\,\underline{p}) = \pi^2\,n\,(\underline{q}^\dagger\,\underline{p}-\underline{p}^\dagger\,\underline{q}). \tag{44}$$

Now, Eqs. (37), (38), and (43) give

$$T_\phi = -\pi^2\,n\,\Big[\underline{p}^\dagger\,(\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{R}}}-\underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{P}}})\,\underline{p}+\underline{p}^\dagger\,(\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{S}}}-\underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{Q}}})\,\underline{q}$$
$$-\underline{q}^\dagger\,(\underline{\underline{\mathcal{S}}}^\dagger\,\underline{\underline{\mathcal{P}}}-\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{R}}})\,\underline{p}+\underline{q}^\dagger\,(\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{S}}}-\underline{\underline{\mathcal{S}}}^\dagger\,\underline{\underline{\mathcal{R}}})\,\underline{q}\Big] \tag{45}$$

The previous equation is consistent with Eq. (44) provided that

$$\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{P}}}, \tag{46}$$
$$\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{S}}}^\dagger\,\underline{\underline{\mathcal{Q}}}, \tag{47}$$
$$\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{S}}}-\underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{Q}}} = \underline{\underline{1}}. \tag{48}$$

The previous three equations can be combined with Eqs. (37) and (38) to give

$$\underline{p} = \underline{\underline{\mathcal{S}}}^{\,\dagger}\,\underline{V}-\underline{\underline{\mathcal{Q}}}^{\,\dagger}\,\underline{\psi}, \tag{49}$$
$$\underline{q} = -\underline{\underline{\mathcal{R}}}^{\,\dagger}\,\underline{V}+\underline{\underline{\mathcal{P}}}^{\,\dagger}\,\underline{\psi}. \tag{50}$$

However, the previous two equations are only consistent with Eqs. (37) and (38) provided

$$\underline{\underline{\mathcal{Q}}}\,\underline{\underline{\mathcal{P}}}^\dagger = \underline{\underline{\mathcal{P}}}\,\underline{\underline{\mathcal{Q}}}^\dagger, \tag{51}$$
$$\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{S}}}^\dagger = \underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{R}}}^\dagger, \tag{52}$$
$$\underline{\underline{\mathcal{P}}}\,\underline{\underline{\mathcal{S}}}^\dagger-\underline{\underline{\mathcal{Q}}}\,\underline{\underline{\mathcal{R}}}^\dagger = \underline{\underline{1}}. \tag{53}$$

Note that Eqs. (46)–(48) and (51)–(53) hold throughout the vacuum region.

C. Ideal-Wall Matching Condition

If the wall is perfectly-conducting then $\underline{\psi}(b_w\,\epsilon) = 0$. It follows from Eq. (38) that

$$\underline{q} = \underline{\underline{I}}\,\underline{p}, \tag{54}$$

where

$$\underline{\underline{I}} = -\underline{\underline{\mathcal{S}}}_b^{-1}\,\underline{\underline{\mathcal{R}}}_b \tag{55}$$

is termed the wall matrix. Here, $\underline{\underline{\mathcal{S}}}_b = \underline{\underline{\mathcal{S}}}(b_w\,\epsilon)$, et cetera. Equation (52) ensures that $\underline{\underline{I}}$ is Hermitian. Making use of Eqs. (37) and (38), the matching condition at the plasma-vacuum interface for an ideal wall becomes

$$\underline{V}(\epsilon) = \underline{\underline{H}}\,\underline{\psi}(\epsilon), \tag{56}$$

where

$$\underline{\underline{H}} = (\underline{\underline{\mathcal{P}}}_\epsilon + \underline{\underline{\mathcal{Q}}}_\epsilon\,\underline{\underline{I}})\,(\underline{\underline{\mathcal{R}}}_\epsilon + \underline{\underline{\mathcal{S}}}_\epsilon\,\underline{\underline{I}})^{-1} \tag{57}$$

is termed the vacuum matrix. Here, $\underline{\underline{\mathcal{P}}}_\epsilon = \underline{\underline{\mathcal{P}}}(\epsilon)$, et cetera. Making use of Eqs. (46)–(48), it is easily demonstrated that

$$\underline{\underline{H}} - \underline{\underline{H}}^\dagger = -[(\underline{\underline{\mathcal{R}}}_\epsilon + \underline{\underline{\mathcal{S}}}_\epsilon\,\underline{\underline{I}})^{-1}]^\dagger\,(\underline{\underline{I}} - \underline{\underline{I}}^\dagger)\,(\underline{\underline{\mathcal{R}}}_\epsilon + \underline{\underline{\mathcal{S}}}_\epsilon\,\underline{\underline{I}})^{-1} \tag{58}$$

Thus, $\underline{\underline{H}}$ is Hermitian because $\underline{\underline{I}}$ is Hermitian.

D. Model Wall Matrix

Equations (32)–(36), (41), and (42) suggest that

$$\underline{\underline{\mathcal{R}}}_b = \underline{\underline{\mathcal{R}}}_\epsilon\,\underline{\underline{\rho}}^{-1}, \tag{59}$$

$$\underline{\underline{\mathcal{S}}}_b = \underline{\underline{\mathcal{S}}}_\epsilon\,\underline{\underline{\rho}}, \tag{60}$$

where

$$\rho_{mm'} = \delta_{mm'}\,\rho_m, \tag{61}$$

$$\rho_0 = 1 + \ln b_w, \tag{62}$$

$$\rho_{m\neq 0} = b_w^{|m|}. \tag{63}$$

Hence,

$$\underline{\underline{I}} = -\underline{\underline{\rho}}^{-1}\,\underline{\underline{\mathcal{S}}}_{\epsilon}^{-1}\,\underline{\underline{\mathcal{R}}}_{\epsilon}\,\underline{\underline{\rho}}^{-1}. \tag{64}$$

Note that $\underline{\underline{I}}$ is Hermitian, given that $\underline{\underline{\mathcal{S}}}_{\epsilon}^{-1}\,\underline{\underline{\mathcal{R}}}_{\epsilon}$ is Hermitian.

E. Resistive-Wall Matching Condition

If the wall is resistive then

$$\underline{q} = g_w\,\underline{\underline{I}}\,\underline{p}, \tag{65}$$

where

$$g_w = \frac{f_w}{1+f_w}, \tag{66}$$

$$f_w = \frac{\lambda\,\tanh\lambda}{d_w}, \tag{67}$$

$$\lambda = (\hat{\gamma}\,d_w)^{1/2}, \tag{68}$$

$$\hat{\gamma} = \gamma\,\tau_w, \tag{69}$$

$$\tau_w = \mu_0\,R_0^2\,\sigma_w\,d_w. \tag{70}$$

Here, d_w is the wall thickness (normalized to R_0), the perturbed magnetic field is assumed to vary in time as $\exp(\gamma\,t)$, σ_w is the electrical conductivity of the wall material, and τ_w is the L/R time of the wall.

Making use of Eqs. (37), (38), and (65), the matching condition at the plasma-vacuum interface for an resistive wall becomes

$$\underline{V}(\epsilon) = \underline{\underline{H}}\,\underline{\psi}(\epsilon), \tag{71}$$

where

$$\underline{\underline{H}} = (\underline{\underline{\mathcal{P}}}_{\epsilon} + g_w\,\underline{\underline{\mathcal{Q}}}_{\epsilon}\,\underline{\underline{I}})\,(\underline{\underline{\mathcal{R}}}_{\epsilon} + g_w\,\underline{\underline{\mathcal{S}}}_{\epsilon}\,\underline{\underline{I}})^{-1}. \tag{72}$$

Assuming that g_w is real, and making use of Eqs. (46)–(48), it is easily demonstrated that

$$\underline{\underline{H}} - \underline{\underline{H}}^{\dagger} = -g_w\,[(\underline{\underline{\mathcal{R}}}_{\epsilon} + \underline{\underline{\mathcal{S}}}_{\epsilon}\,\underline{\underline{I}})^{-1}]^{\dagger}\,(\underline{\underline{I}} - \underline{\underline{I}}^{\dagger})\,(\underline{\underline{\mathcal{R}}}_{\epsilon} + \underline{\underline{\mathcal{S}}}_{\epsilon}\,\underline{\underline{I}})^{-1} \tag{73}$$

Thus, given that $\underline{\underline{I}}$ is Hermitian, we deduce that $\underline{\underline{H}}$ is Hermitian, as long as g_w is real.

III. VACUUM MATRIX

A. No-Wall Vacuum Matrix

In the no-wall limit, $g_w = 0$, and so Eq. (72) yields the following expression for the vacuum matrix:

$$\underline{\underline{H}}_{nw} = \underline{\underline{\mathcal{P}}}_{\epsilon}\,\underline{\underline{\mathcal{R}}}_{\epsilon}^{-1}. \tag{74}$$

Equation (46) implies that $\underline{\underline{H}}_{nw}$ is Hermitian.

B. Perfect-Wall Vacuum Matrix

In the perfect-wall limit, $g_w = 1$, and so Eq. (72) yields the following expression for the vacuum matrix:

$$\underline{\underline{H}}_{pw} = (\underline{\underline{\mathcal{P}}}_{\epsilon} + \underline{\underline{\mathcal{Q}}}_{\epsilon}\,\underline{\underline{I}})\,(\underline{\underline{\mathcal{R}}}_{\epsilon} + \underline{\underline{\mathcal{S}}}_{\epsilon}\,\underline{\underline{I}})^{-1}. \tag{75}$$

Equation (58) ensures that $\underline{\underline{H}}_{pw}$ is Hermitian, given that that $\underline{\underline{I}}$ is Hermitian.

C. General Vacuum Matrix

The expression, (72), for the general vacuum matrix can be rewritten in the form

$$\underline{\underline{H}} = \underline{\underline{H}}_{nw} + f_w\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1}, \tag{76}$$

where

$$\underline{\underline{B}} = \underline{\underline{\mathcal{R}}}_{\epsilon}\,(\underline{\underline{\mathcal{R}}}_{\epsilon} + \underline{\underline{\mathcal{S}}}_{\epsilon}\,\underline{\underline{I}})^{-1}. \tag{77}$$

Suppose that f_w is real. It follows that

$$\underline{\underline{H}} - \underline{\underline{H}}^{\dagger} = f_w\left[(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1} - (\underline{\underline{B}}^{\dagger} + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\right]. \tag{78}$$

However, we know that $\underline{\underline{H}}$ is Hermitian when f_w is real. Hence, we deduce that

$$(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1} = (\underline{\underline{B}}^{\dagger} + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw}), \tag{79}$$

which implies that

$$\underline{\underline{B}}^{\dagger}\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw}) = (\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,\underline{\underline{B}}. \tag{80}$$

In other words, the matrix

$$\underline{\underline{C}} = (\underline{\underline{H}}_{nw} - \underline{\underline{H}}_{pw})\,\underline{\underline{B}} \tag{81}$$

is Hermitian. It is easily demonstrated from Eq. (80) that Eq. (79) holds even when f_w is complex.

To check that $\underline{\underline{C}}$ is Hermitian, Eq. (80) yields

$$\begin{aligned}
&(\underline{\underline{\mathcal{R}}}_\epsilon^{\dagger} + \underline{\underline{I}}\,\underline{\underline{\mathcal{S}}}_\epsilon^{\dagger})^{-1}\,\underline{\underline{\mathcal{R}}}_\epsilon^{\dagger}\left[\underline{\underline{\mathcal{R}}}_\epsilon^{-1\,\dagger}\,\underline{\underline{P}}_\epsilon^{\dagger} - (\underline{\underline{\mathcal{P}}}_\epsilon + \underline{\underline{\mathcal{Q}}}_\epsilon\,\underline{\underline{I}})\,(\underline{\underline{\mathcal{R}}}_\epsilon + \underline{\underline{\mathcal{S}}}_\epsilon\,\underline{\underline{I}})^{-1}\right]\\
&= \left[\underline{\underline{\mathcal{P}}}_\epsilon\,\underline{\underline{\mathcal{R}}}_\epsilon^{-1} - (\underline{\underline{\mathcal{R}}}_\epsilon^{\dagger} + \underline{\underline{I}}\,\underline{\underline{\mathcal{S}}}_\epsilon^{\dagger})^{-1}\,(\underline{\underline{\mathcal{P}}}_\epsilon^{\dagger} + \underline{\underline{I}}\,\underline{\underline{\mathcal{Q}}}_\epsilon^{\dagger})\right]\underline{\underline{\mathcal{R}}}_\epsilon\,(\underline{\underline{\mathcal{R}}}_\epsilon + \underline{\underline{\mathcal{S}}}_\epsilon\,\underline{\underline{I}})^{-1}.
\end{aligned} \tag{82}$$

where use has been made of Eqs. (74), (75), and (77), as well as the fact that $\underline{\underline{H}}_{pw}$ and $\underline{\underline{H}}_{nw}$ are both Hermitian. Making use of Eqs. (46) and (48), the previous equation reduces to

$$\underline{\underline{I}} = \underline{\underline{I}}, \tag{83}$$

which is obviously satisfied.

Finally, Eqs. (76) and (79) can be combined to give

$$\begin{aligned}
\underline{\underline{H}} &= \underline{\underline{H}}_{nw} + \frac{f_w}{2}\left[(\underline{\underline{B}}^{\dagger} + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw}) + (\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1}\right]\\
&= \underline{\underline{H}}_{nw} + f_w\,(\underline{\underline{B}}^{\dagger} + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw}),\\
&= \underline{\underline{H}}_{nw} + f_w\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1}.
\end{aligned} \tag{84}$$

Note, from the first line of the previous equation, that $\underline{\underline{H}}$ is clearly Hermitian if f_w is real.

IV. APPLICATIONS

A. Wall Torque on Rotating Tearing Mode

Consider an unreconnected tearing mode, resonant at the kth rational surface, and rotating at the angular phase velocity ω_k. It follows that $\gamma = -\mathrm{i}\,\omega_k$. Thus,

$$f_w = \frac{\zeta}{2\,d_w}\left[\left(\frac{\sinh\zeta - \sin\zeta}{\cosh\zeta + \cos\zeta}\right) - \mathrm{i}\left(\frac{\sinh\zeta + \sin\zeta}{\cosh\zeta + \cos\zeta}\right)\right], \tag{85}$$

where

$$\zeta = (2\,\hat{\omega}_k\,d_w)^{1/2}, \tag{86}$$

and $\hat{\omega}_k = \omega_k\,\tau_w$. The matching condition at the plasma-vacuum interface becomes

$$\underline{V}(\epsilon) = \underline{\underline{H}}(\zeta)\;\underline{\psi}(\epsilon). \tag{87}$$

Note that $\underline{\underline{H}}(\zeta)$ is not generally Hermitian, because f_w is complex, which implies that the E-matrix is not Hermitian. The toroidal electromagnetic torque acting at the kth rational surface is

$$\delta T_k = 2\pi^2\,n\,\mathrm{Im}(E_{kk})\,|\Psi_k|^2. \tag{88}$$

B. Ideal-Plasma Resistive-Wall Mode

We can write

$$\underline{V}(\epsilon) = \underline{\underline{V}}_{\,i}\,\underline{\alpha}, \tag{89}$$

$$\underline{\psi}(\epsilon) = \underline{\underline{\psi}}_{\,i}\,\underline{\alpha}, \tag{90}$$

where the $\underline{\underline{V}}_{\,i}$ and $\underline{\underline{\psi}}_{\,i}$ are ideal solutions at the plasma-vacuum interface. The net toroidal electromagnetic torque acting on the plasma is

$$T_\phi = -2\pi^2\,n\,\mathrm{Im}(\underline{V}^\dagger\,\underline{\psi}) = -2\pi^2\,n\,\mathrm{Im}(\underline{\alpha}^\dagger\,\underline{\underline{V}}_{\,i}^\dagger\,\underline{\underline{\psi}}_{\,i}\,\underline{\alpha}). \tag{91}$$

However, the net torque acting on an ideal plasma is zero, so

$$\underline{\underline{V}}_{\,i}^\dagger\,\underline{\underline{\psi}}_{\,i} = \underline{\underline{\psi}}_{\,i}^\dagger\,\underline{\underline{V}}_{\,i}. \tag{92}$$

Equations (71), (89), and (90) imply that

$$\underline{\underline{V}}_{\,i}\,\underline{\alpha} = \underline{\underline{H}}\,\underline{\underline{\psi}}_{\,i}\,\underline{\alpha}. \tag{93}$$

Writing

$$\underline{\underline{\psi}}_{\,i}\,\underline{\alpha} = \underline{x}, \tag{94}$$

we obtain

$$\underline{\underline{W}}_{\,p}\,\underline{x} = \underline{\underline{H}}\,\underline{x}, \tag{95}$$

where

$$\underline{\underline{W}}_{\,p} = \underline{\underline{V}}_{\,i}\,\underline{\underline{\psi}}_{\,i}^{-1}. \tag{96}$$

Equation (92) ensures that $\underline{\underline{W}}_{\,p}$ is Hermitian.

Equation (84) can be combined with Eq. (95) to give

$$\begin{aligned}\underline{\underline{W}}_{nw}\,\underline{x} &= f_w\,(\underline{\underline{B}}^{\dagger} + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{W}}_{nw} - \underline{\underline{W}}_{pw})\,\underline{x}\\ &= f_w\,(\underline{\underline{W}}_{nw} - \underline{\underline{W}}_{pw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1}\,\underline{x}\end{aligned} \tag{97}$$

where

$$\underline{\underline{W}}_{nw} = \underline{\underline{W}}_{p} - \underline{\underline{H}}_{nw}, \tag{98}$$

$$\underline{\underline{W}}_{pw} = \underline{\underline{W}}_{p} - \underline{\underline{H}}_{pw}. \tag{99}$$

Note that $\underline{\underline{W}}_{nw}$ and $\underline{\underline{W}}_{pw}$ are Hermitian. Writing

$$(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1}\,\underline{x} = \underline{y}, \tag{100}$$

we obtain

$$\underline{\underline{B}}^{\dagger}\,\underline{\underline{W}}_{nw}\,\underline{x} = \lambda\,\underline{\underline{W}}_{pw}\,\underline{x}, \tag{101}$$

$$\underline{\underline{W}}_{nw}\,\underline{\underline{B}}\,\underline{y} = \lambda\,\underline{\underline{W}}_{pw}\,\underline{y}, \tag{102}$$

where $\lambda = -f_w$. Finally, making use of Eqs. (81), (98), (99), (100), and (102), we get

$$\underline{\underline{W}}_{pw}\,\underline{x} = (\underline{\underline{W}}_{pw} - \underline{\underline{W}}_{nw})\,\underline{\underline{B}}\,\underline{y} = \underline{\underline{C}}\,\underline{y}. \tag{103}$$

Let $\underline{\underline{W}}_{pw}^{\,1/2}$ be the unique positive definite Hermitian square root of the positive definite Hermitian matrix $\underline{\underline{W}}_{pw}$, and let $\underline{\underline{W}}_{pw}^{-1/2}$ be its inverse. Let

$$\hat{\underline{x}} = \underline{\underline{W}}_{pw}^{\,1/2}\,\underline{x}, \tag{104}$$

$$\hat{\underline{y}} = \underline{\underline{W}}_{pw}^{\,1/2}\,\underline{y}. \tag{105}$$

Equations (101)–(103) transform to give

$$\underline{\underline{E}}\,\hat{\underline{x}} = \lambda\,\hat{\underline{x}}, \tag{106}$$

$$\underline{\underline{E}}^{\dagger}\,\hat{\underline{y}} = \lambda\,\hat{\underline{y}}, \tag{107}$$

$$\hat{\underline{x}} = \underline{\underline{D}}\,\hat{\underline{y}}, \tag{108}$$

where

$$\underline{\underline{E}} = \underline{\underline{W}}_{pw}^{-1/2}\,\underline{\underline{B}}^{\dagger}\,\underline{\underline{W}}_{nw}\,\underline{\underline{W}}_{pw}^{-1/2}, \tag{109}$$

$$\underline{\underline{D}} = \underline{\underline{W}}_{pw}^{-1/2}\,\underline{\underline{C}}\,\underline{\underline{W}}_{pw}^{-1/2}. \tag{110}$$

Let $\underline{\underline{D}}^{1/2}$ be the unique positive definite Hermitian square root of the positive definite Hermitian matrix $\underline{\underline{D}}$, and let $\underline{\underline{D}}^{-1/2}$ be its inverse. It is easily demonstated that

$$\underline{\underline{E}}\,\underline{\underline{D}} = \underline{\underline{D}}\,\underline{\underline{E}}^{\dagger}, \tag{111}$$

which implies that

$$\underline{\underline{D}}^{-1/2}\,\underline{\underline{E}}\,\underline{\underline{D}}^{1/2} = \underline{\underline{D}}^{1/2}\,\underline{\underline{E}}^{\dagger}\,\underline{\underline{D}}^{-1/2}. \tag{112}$$

In other words,

$$\underline{\underline{F}} = \underline{\underline{D}}^{-1/2}\,\underline{\underline{E}}\,\underline{\underline{D}}^{1/2} \tag{113}$$

is Hermitian.

Let

$$\underline{\tilde{x}} = \underline{\underline{D}}^{-1/2}\,\underline{\hat{x}}, \tag{114}$$

$$\underline{\tilde{y}} = \underline{\underline{D}}^{1/2}\,\underline{\hat{y}}. \tag{115}$$

Equations (106)–(108) transform to give

$$\underline{\underline{F}}\,\underline{\tilde{x}} = \lambda\,\underline{\tilde{x}}, \tag{116}$$

$$\underline{\tilde{x}} = \underline{\tilde{y}}. \tag{117}$$

Let the λ_m and the $\underline{\beta}_m$ be the eigenvalues and eigenvectors of the Hermitian matrix $\underline{\underline{F}}$. Thus, the λ_m are real, and the $\underline{\beta}_m$ are orthonormal. To be more exact,

$$\underline{\underline{F}}\,\underline{\underline{\beta}} = \underline{\underline{\beta}}\,\underline{\underline{\Lambda}}, \tag{118}$$

$$\underline{\underline{\beta}}^{\dagger}\,\underline{\underline{\beta}} = \underline{\underline{1}}, \tag{119}$$

where $\underline{\underline{\beta}}$ is the matrix of the $\underline{\beta}_m$, and $\underline{\underline{\Lambda}}$ is the diagonal matrix of the λ_m.

The mth resistive wall mode has the dispersion relation

$$f_w = -\lambda_m \tag{120}$$

and the eigenfunction

$$\underline{\psi}_m(\epsilon) = \underline{\underline{W}}_{pw}^{-1/2}\,\underline{\underline{D}}^{1/2}\,\underline{\beta}_m, \tag{121}$$

$$\underline{V}_m(\epsilon) = \underline{\underline{W}}_p\, \underline{\psi}_m(\epsilon), \tag{122}$$

$$\underline{\Xi}_m(\epsilon) = \underline{\underline{Q}}^{-1}\, \underline{\psi}_m(\epsilon), \tag{123}$$

$$\underline{Z}_m(\epsilon) = \underline{\underline{Q}}\, \underline{V}_m(\epsilon), \tag{124}$$

where $\underline{\underline{Q}}$ is the diagonal matrix of the $m - n\, q(\epsilon)$ values.

Alternatively, we can write

$$\underline{\underline{\widetilde{W}}}_{nw} = \underline{\underline{Q}}\, \underline{\underline{W}}_{nw}\, \underline{\underline{Q}}, \tag{125}$$

$$\underline{\underline{\widetilde{W}}}_{pw} = \underline{\underline{Q}}\, \underline{\underline{W}}_{pw}\, \underline{\underline{Q}}, \tag{126}$$

$$\underline{\underline{\widetilde{B}}} = \underline{\underline{Q}}^{-1}\, \underline{\underline{B}}\, \underline{\underline{Q}}, \tag{127}$$

$$\underline{\underline{\widetilde{C}}} = \underline{\underline{Q}}\, \underline{\underline{C}}\, \underline{\underline{Q}}, \tag{128}$$

$$\underline{\underline{\widetilde{D}}} = \underline{\underline{\widetilde{W}}}_{pw}^{-1/2}\, \underline{\underline{\widetilde{C}}}\, \underline{\underline{\widetilde{W}}}_{pw}^{-1/2}, \tag{129}$$

$$\underline{\underline{\widetilde{E}}} = \underline{\underline{\widetilde{W}}}_{pw}^{-1/2}\, \underline{\underline{\widetilde{B}}}^{\dagger}\, \underline{\underline{\widetilde{W}}}_{nw}\, \underline{\underline{\widetilde{W}}}_{pw}^{-1/2}, \tag{130}$$

$$\underline{\underline{\widetilde{F}}} = \underline{\underline{\widetilde{D}}}^{-1/2}\, \underline{\underline{\widetilde{E}}}\, \underline{\underline{\widetilde{D}}}^{1/2}. \tag{131}$$

Let the λ_m and the $\underline{\beta}_m$ be the eigenvalues and eigenvectors of the Hermitian matrix $\underline{\underline{\widetilde{F}}}$. The mth resistive wall mode has the dispersion relation $f_w = -\lambda_m$ and the eigenfunction

$$\underline{\Xi}_m(\epsilon) = \underline{\underline{\widetilde{W}}}_{pw}^{-1/2}\, \underline{\underline{\widetilde{D}}}^{1/2}\, \underline{\beta}_m, \tag{132}$$

$$\underline{\psi}_m(\epsilon) = \underline{\underline{Q}}\, \underline{\Xi}_m(\epsilon), \tag{133}$$

$$\underline{V}_m(\epsilon) = \underline{\underline{W}}_p\, \underline{\psi}_m(\epsilon), \tag{134}$$

$$\underline{Z}_m(\epsilon) = \underline{\underline{Q}}\, \underline{V}_m(\epsilon). \tag{135}$$

C. Resistive-Plasma Resistive-Wall Mode

The m'th ideal eigenfunction has the form

$$\psi^i_{mm'}(r) = \psi^a_{mm'}(r) - \sum_{k'} \psi^u_{mk'}(r)\, \Pi^a_{k'm'}, \tag{136}$$

$$V^i_{mm'}(r) = V^a_{mm'}(r) - \sum_{k'} V^u_{mk'}(r)\, \Pi^a_{k'm'}. \tag{137}$$

Now, the $\psi^a_{mm'}(r)$ and the $V^a_{mm'}(r)$ have no current sheets at the various rational surfaces in the plasma. On the other hand, the $\psi^u_{mk'}(r)$ and the $V^u_{mk'}(r)$ are such that $\Delta\Psi_k = E_{kk'}$. Hence, the current sheet generated at the kth rational surface the plasma by the m'th ideal eigenfunction is

$$\Delta\Psi_{km'} = -\sum_{k'} E_{kk'}\,\Pi^a_{k'm}. \tag{138}$$

Thus, the reconnected flux driven at the kth rational surface is

$$\Psi_{km'} = -\sum_{k',k''} \left[\underline{\underline{\Delta}}(0) - \underline{\underline{E}}\right]^{-1}_{kk'} E_{k'k''}\,\Pi^a_{k''m}, \tag{139}$$

where the layer response matrix is evaluated at zero frequency, on the assumption that the resistive wall mode evolves in time very slowly. To take into account the reconnected flux in the plasma, we write

$$\psi_{mm'}(r) = \psi^i_{mm'}(r) + \sum_k \psi^u_{mk}(r)\,\Psi_{km'}, \tag{140}$$

$$V_{mm'}(r) = V^i_{mm'}(r) + \sum_k V^u_{mk}(r)\,\Psi_{km'}, \tag{141}$$

which implies that

$$\psi_{mm'}(r) = \psi^a_{mm'}(r) - \sum_{k,k'} \psi^u_{mk}(r)\left(\delta_{kk'} + \sum_{k''}\left[\underline{\underline{\Delta}}(0) - \underline{\underline{E}}\right]^{-1}_{kk''} E_{k''k'}\right)\Pi^a_{k'm'}, \tag{142}$$

$$V_{mm'}(r) = V^a_{mm'}(r) - \sum_{k,k'} V^u_{mk}(r)\left(\delta_{kk'} + \sum_{k''}\left[\underline{\underline{\Delta}}(0) - \underline{\underline{E}}\right]^{-1}_{kk''} E_{k''k'}\right)\Pi^a_{k'm'}. \tag{143}$$

Let $\underline{\underline{\psi}}$ be the matrix of the $\psi_{mm'}(\epsilon)$ values and let $\underline{\underline{V}}$ be the matrix of the $V_{mm'}(\epsilon)$ values. The plasma energy matrix is

$$\underline{\underline{W}}_p = \underline{\underline{V}}\,\underline{\underline{\psi}}^{-1}. \tag{144}$$

However, the energy matrix is not necessarily Hermitian. The net toroidal electromagnetic torque acting on the plasma is

$$T_\phi = -2\pi^2\,n\,\mathrm{Im}(\underline{x}^\dagger\,\underline{\underline{W}}_p\,\underline{x}). \tag{145}$$

Given that $\underline{\underline{W}}_p$ is not necessarily Hermitian, the torque is not necessarily zero.

As before, the resistive wall mode eigenmode equation is

$$\underline{\underline{B}}^\dagger\,\underline{\underline{W}}_{nw}\,\underline{x} = \lambda\,\underline{\underline{W}}_{pw}\,\underline{x}, \tag{146}$$

$$\underline{\underline{W}}_{nw}\,\underline{\underline{B}}\,\underline{y} = \lambda\,\underline{\underline{W}}_{pw}\,\underline{y}, \tag{147}$$

$$\underline{\underline{W}}_{pw}\,\underline{x} = \underline{\underline{C}}\,\underline{y}, \tag{148}$$

where $\lambda = -f_w$. These equations can be transformed to give

$$\underline{\underline{F}}\,\underline{\tilde{x}} = \lambda\,\underline{\tilde{x}}, \tag{149}$$

where

$$\underline{\underline{F}} = \underline{\underline{D}}^{-1/2}\,\underline{\underline{E}}\,\underline{\underline{D}}^{1/2}, \tag{150}$$

$$\underline{\underline{E}} = \underline{\underline{W}}_{pw}^{-1/2}\,\underline{\underline{B}}^{\dagger}\,\underline{\underline{W}}_{nw}\,\underline{\underline{W}}_{pw}^{-1/2}, \tag{151}$$

$$\underline{\underline{D}} = \underline{\underline{W}}_{pw}^{-1/2}\,\underline{\underline{C}}\,\underline{\underline{W}}_{pw}^{-1/2}, \tag{152}$$

and

$$\underline{x} = \underline{\underline{W}}_{pw}^{-1/2}\,\underline{\underline{D}}^{\,1/2}\,\underline{\tilde{x}}, \tag{153}$$

$$\underline{y} = \underline{\underline{W}}_{pw}^{-1/2}\,\underline{\underline{D}}^{-1/2}\,\underline{\tilde{x}}. \tag{154}$$

Note, however, that $\underline{\underline{W}}_{nw}$, $\underline{\underline{W}}_{nw}$, $\underline{\underline{D}}$, and $\underline{\underline{F}}$ are not necessarily Hermitian. Hence, λ is not necessarily real.