

# A four-field resonant response model for tokamak plasmas

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## 1 Introduction

Externally generated, static, *resonant magnetic perturbations* (RMPs) can drive magnetic reconnection in tokamak plasmas that are intrinsically stable to tearing perturbations [1, 2, 3, 4, 5]. RMPs arise naturally as a consequence of magnetic field-coil misalignments [6], but are sometimes deliberately applied to tokamak plasmas in order to suppress edge localized modes (ELMs) [7]. Driven reconnection is significant because it leads to the formation of magnetic island chains at so-called *rational* magnetic flux-surfaces within the plasma [8]. Such chains locally flatten the pressure profile, and, thereby, degrade the plasma energy confinement [9].

The analysis of the response of a tokamak plasma to an RMP is most efficiently formulated as an asymptotic matching problem in which the plasma is divided into two distinct regions [10]. In the so-called *outer region*, which comprises most of the plasma, the plasma perturbation is governed by the equations of linearized, marginally-stable, ideal-magnetohydrodynamics (MHD). However, these equations become singular on rational magnetic flux-surfaces at which the perturbed magnetic field resonates with the equilibrium field. In the *inner region*, which consists of a set of narrow layers centered on the various rational surfaces, non-ideal-MHD effects become important.

It is well known that single-fluid resistive-MHD offers a very poor description of the response of the inner region to the magnetic perturbation in the outer region. For instance, the strong diamagnetic flows present in tokamak plasmas imply that the electron and ion fluid velocities are significantly different from one another, necessitating a two-fluid treatment [11]. Moreover, resistive-MHD does not take into account the important ion sound-radius lengthscale below which electron and ion dynamics become decoupled from one another [12, 13]. Previously, Cole & Fitzpatrick [14] used the four-field model of Fitzpatrick & Waelbroeck [15], which consists of four coupled nonlinear partial differential equations, and which is based on the original four-field model of Hazeltine, Kotschenreuther & Morrison [16], to determine the linear two-fluid response of a resonant layer to the perturbation in the outer region. This treatment was extended in Ref. [17] to take into account the anomalously large perpendicular energy diffusivity present in tokamak plasmas.

In configuration space, the four-field models of Refs. [14] and [17] yield a set of resonant layer equations that can be expressed as *ten* coupled first-order linear ordinary differential equations (o.d.e.s) [18]. However, in Fourier space, the resonant layer equations can be written as *four* coupled first-order linear o.d.e.s. [14]. It is clearly advantageous to solve the equations in Fourier space. In Refs. [14] and [17], an approximation is made by which one of the terms in the Fourier-transformed layer equations is neglected. This approximation, which is valid in low- $\beta$  plasmas, is such that the layer equation that governs the parallel ion dynamics decouples from the other three equations, effectively converting a four-field resonant response model into a three-field model. Furthermore, the three remaining layer equations can be combined to give a single second-order linear o.d.e. This second-order equation is most conveniently solved by means of a Riccati transformation

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that converts it into a first-order nonlinear o.d.e. [19, 20]. The advantage of the Riccati approach is that it can deal with numerically problematic solutions that blow up as  $\exp(p^2)$ , or faster, at large  $p$ , where  $p$  is the Fourier-space variable.

Lee, Park & Na [18] recently demonstrated how to solve the full tenth-order four-field resonant layer equations in configuration space using a Riccati transformation. In the process, they discovered that, in a high- $\beta$  plasma, the RMP rotation frequency (as seen in the local  $\mathbf{E} \times \mathbf{B}$  frame at the rational surface) at which the electromagnetic torque exerted on the plasma in the immediate vicinity of the resonant layer passes through zero is shifted in the ion diamagnetic direction from the electron diamagnetic frequency. This result is significant because there is some experimental evidence for such a shift [21]. In the present paper, we demonstrate how the calculation of Lee et alia can be reimplemented in Fourier space. The Fourier version of the calculation is more convenient, from a numerical point of view, because it involves the solution of a fourth-order, rather than a tenth-order, system of equations.

## 2 Asymptotic matching

### 2.1 Plasma equilibrium

Consider a large aspect-ratio tokamak plasma equilibrium whose magnetic flux-surfaces map out (almost) concentric circles in the poloidal plane. Such an equilibrium can be approximated as a periodic cylinder [4]. Let  $r, \theta, z$  be right-handed cylindrical coordinates. The magnetic axis corresponds to  $r = 0$ , and the plasma boundary to  $r = a$ , where  $a$  is the simulated minor radius of the plasma. The system is assumed to be periodic in the  $z$ -direction with periodicity length  $2\pi R_0$ , where  $R_0$  is the simulated major radius of the plasma. The safety-factor profile takes the form  $q(r) = r B_z / [R_0 B_\theta(r)]$ , where  $B_z$  is the constant ‘‘toroidal’’ magnetic field-strength, and  $B_\theta(r)$  is the poloidal magnetic field-strength. The equilibrium poloidal and toroidal magnetic fluxes (divided by  $2\pi$ ) are written  $\psi_p(r) = B_z \int_0^r r'/q(r') dr'$  and  $\psi_t(r) = B_z r^2/2$ , respectively. The standard large aspect-ratio orderings,  $r/R_0 \ll 1$  and  $B_\theta/B_z \ll 1$ , are adopted.

### 2.2 Outer region

Consider a static (in the laboratory frame) magnetic perturbation that has  $m$  periods in the poloidal direction, and  $n$  periods in the toroidal direction. The response of the plasma to the perturbation is governed by the linearized equations of marginally-stable ideal-MHD everywhere in the plasma, apart from a (radially) narrow layer centered on the rational surface whose minor radius,  $r_s$ , is such that  $q(r_s) = m/n$  [10].

The perturbed magnetic field is written  $\delta\mathbf{B} \simeq \nabla\delta\psi \times \mathbf{e}_z$ , where  $\delta\psi(r, \theta, \varphi) = \delta\psi(r) \exp[i(m\theta - n\varphi)]$ , and  $\varphi = z/R_0$  is a simulated toroidal angle. In the outer region (i.e., everywhere in the plasma apart from the resonant layer), the perturbed helical magnetic flux,  $\delta\psi(r)$ , satisfies the *cylindrical tearing mode equation* [22, 23],

$$\frac{d^2\delta\psi}{dr^2} + \frac{1}{r} \frac{d\delta\psi}{dr} - \frac{m^2}{r^2} \delta\psi - \frac{J'_z \delta\psi}{r(1/q - n/m)} = 0, \quad (1)$$

where  $J_z(r) = R_0 \mu_0 j_z(r)/B_z$ , and  $j_z(r)$  is the equilibrium ‘‘toroidal’’ current density. Here,  $' \equiv d/dr$ .

In general, the solution of Eq. (1) that satisfies physical boundary conditions at the magnetic axis and the plasma boundary is such that  $\delta\psi(r)$  is continuous across the rational surface, whereas  $d\delta\psi/dr$  is discontinuous. The discontinuity of  $d\delta\psi/dr$  across the rational surface is indicative of the presence of a helical current sheet at the surface. The complex quantity  $\Psi_s = \delta\psi(r_s)$  determines the amplitude and phase of the reconnected helical magnetic flux at the rational surface, whereas the complex quantity

$$\Delta\Psi_s = \left[ r \frac{d\delta\psi}{dr} \right]_{r_s-}^{r_s+} \quad (2)$$

parameterizes the amplitude and phase of the helical current sheet [4].

Regime	$Q_e$	$Q_i$	$c_\beta$	$D$	$P_E$	$P_\varphi$
HRi	-1.0	1.0	1.0	10.0	1.0	$1.0 \times 10^{+0}$
HRii	-1.0	1.0	1.0	10.0	1.0	$1.0 \times 10^{-1}$
RIii	-0.15	0.15	1.0	1.0	1.0	$1.0 \times 10^{-4}$
VRii	-0.15	0.15	1.0	0.1	1.0	$1.0 \times 10^{+0}$

Table 1: The normalized layer parameters used in the benchmark comparisons shown in Figs. 1–4.

The solution of the cylindrical tearing mode equation in the outer region, in the presence of an externally generated RMP (with  $m$  periods in the poloidal direction and  $n$  periods in the toroidal direction), leads to the *tearing mode dispersion relation* [4, 10]

$$\Delta\Psi_s - E_{ss}\Psi_s = -E_{ss}\Psi_v, \quad (3)$$

where  $E_{ss}$  is a real dimensionless quantity known as the *tearing stability index* [10]. Moreover,  $\Psi_v$  is the so-called *vacuum flux*, and is defined as the reconnected magnetic flux that would be driven at the rational surface by the RMP were the plasma intrinsically tearing-stable (i.e.,  $E_{ss} < 0$ ), and were there no current sheet at the rational surface (i.e.,  $\Delta\Psi_s = 0$ ) [4].

### 2.3 Inner region

The current sheet at the rational surface is resolved by employing a extended-MHD resonant response model in the inner region (i.e., the region of the plasma in the immediate vicinity of the rational surface) to determine the complex *layer response index*,  $\Delta_s$ . Asymptotic matching between the solutions in the inner and the outer regions yields

$$\Delta_s = \frac{\Delta\Psi_s}{\Psi_s}. \quad (4)$$

The previous two equations lead to the *plasma response equation*,

$$\Psi_s = \frac{\Psi_v}{1 + \Delta_s/(-E_{ss})}. \quad (5)$$

Note that if  $|\Delta_s|/(-E_{ss}) \ll 1$  then  $\Psi_s \simeq \Psi_v$  and  $\Delta\Psi_s \simeq 0$ , which is termed a *vacuum response*. On the other hand, if  $|\Delta_s|/(-E_{ss}) \gg 1$  then  $|\Psi_s| \ll |\Psi_v|$ , which is termed an *ideal response*. In the vacuum response regime, the current sheet that is excited in the resonant layer is too feeble to prevent driven magnetic reconnection, and the reconnected flux at the rational surface is the same as that which would be driven if there were no plasma present at the surface. On the other hand, in the ideal response regime, the current sheet excited in the layer is large enough to almost completely suppress driven magnetic reconnection, which implies that the response of the plasma is equivalent to that which would occur if the ideal-MHD flux freezing constraint,  $\Psi_s = 0$ , were imposed at the rational surface [24].

## 3 Four-field resonant response model

### 3.1 Introduction

The Fourier-space four-field resonant response model employed in this paper is described in detail in Appendix A.

### 3.2 Dimensionless parameters

All four-field resonant response models are defined by the following seven dimensionless parameters:  $Q_E = -S^{1/3} n \omega_E \tau_H$ ,  $Q_e = -S^{1/3} n \omega_{*e} \tau_H$ ,  $Q_i = -S^{1/3} n \omega_{*i} \tau_H$ ,  $c_\beta = \sqrt{\beta/(1+\beta)}$ ,  $D = S^{1/3} \iota_e^{1/2} \hat{d}_\beta$ ,  $P_E =$

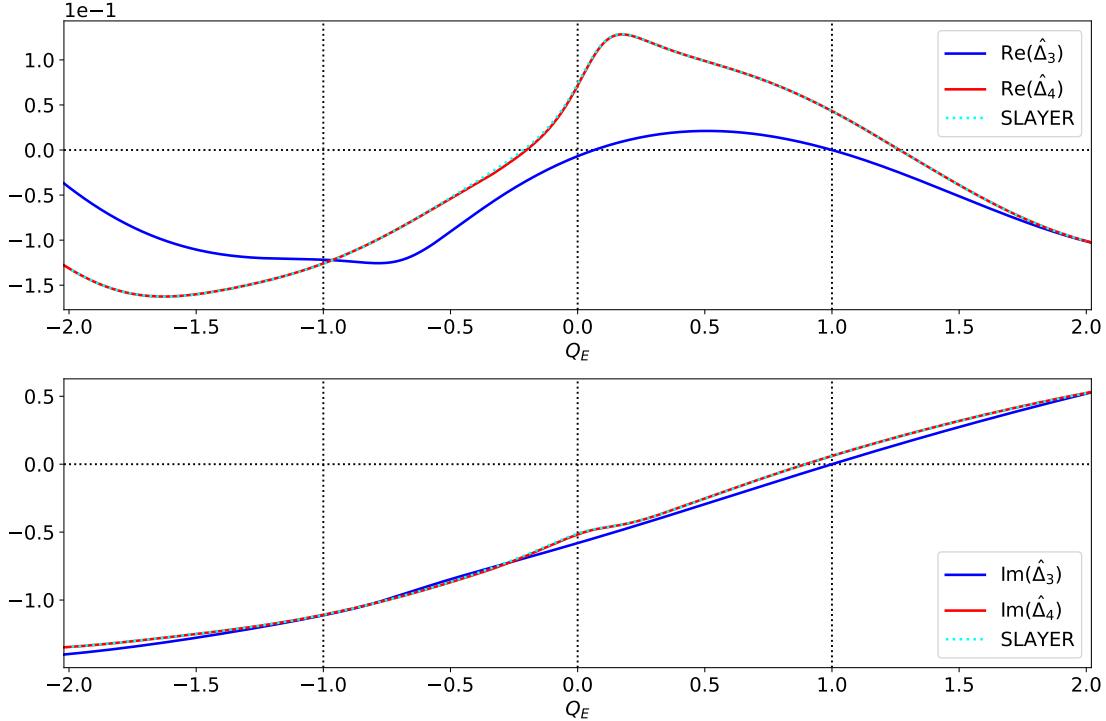


Figure 1: The scaled layer response index,  $\hat{\Delta}$ , plotted as a function of the normalized  $\mathbf{E} \times \mathbf{B}$  frequency,  $Q$ , in the first Hall-resistive (HRI) response regime of Ref. [14].  $\hat{\Delta}_3$  is the response index predicted by the three-field Fourier-space response model of Ref. [17].  $\hat{\Delta}_4$  is the response index predicted by the four-field Fourier-space response model specified in Appendix A. SLAYER is the response index predicted by the four-field configuration-space response model of Ref. [18]. The three vertical lines indicate the ion fluid resonance,  $Q_E + Q_i = 0$ , the MHD fluid resonance,  $Q_E = 0$ , and the electron fluid resonance,  $Q_E + Q_e = 0$ , respectively, in order from the left to the right.

$\tau_R/\tau_E$ , and  $P_\varphi = \tau_R/\tau_\varphi$ . Here,  $\tau_R$  is the resistive diffusion timescale,  $\tau_H$  the hydrodynamic timescale,  $\tau_E$  the energy confinement timescale,  $\tau_\varphi$  the momentum confinement timescale,  $S = \tau_R/\tau_H$  the Lundquist number,  $\omega_E$  the  $\mathbf{E} \times \mathbf{B}$  frequency,  $\omega_{*e}$  the electron diamagnetic frequency,  $\omega_{*i}$  the ion diamagnetic frequency,  $\sqrt{\beta} = \sqrt{(5/3)\mu_0 p/B_z}$  the ratio of the sound speed to the Alfvén speed,  $p$  the plasma pressure,  $d_\beta = c_\beta d_i$  the ion sound-radius, and  $d_i$  the collisionless ion skin-depth. Moreover,  $P_E$  and  $P_\varphi$  are magnetic Prandtl numbers,  $\iota_e = -\omega_{*e}/(\omega_{*i} - \omega_{*e})$ , and  $\hat{d}_\beta = d_\beta/r_s$ . All quantities are evaluated at the rational surface.

Note that the previous definitions are slightly different to those adopted in Ref. [18]. We can convert our dimensionless parameter into those used by Lee et alia by letting  $Q_E \rightarrow Q$ ,  $Q_e \rightarrow -Q_{*e}$ ,  $Q_i \rightarrow -Q_{*i}$ ,  $\iota_e \rightarrow 1/(1+\tau)$ ,  $P_E \rightarrow c_\beta^2$ , and  $P_\varphi \rightarrow P$ .

### 3.3 Method of solution

After linearization and Fourier transformation (with respect to  $x = r - r_s$ ), our four-field equations can be converted into the Riccati matrix differential equation

$$p \frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}}\underline{\underline{W}} - \underline{\underline{E}}\underline{\underline{W}} + \underline{\underline{F}}. \quad (6)$$

Here,  $p$  is the Fourier transform variable, and  $\underline{\underline{W}}(p)$  the  $2 \times 2$  solution matrix. The elements of the  $2 \times 2$  matrices  $\underline{\underline{E}}(p)$  and  $\underline{\underline{F}}(p)$  are specified in Eqs. (89)–(92) and (93)–(96), respectively. The method of solution

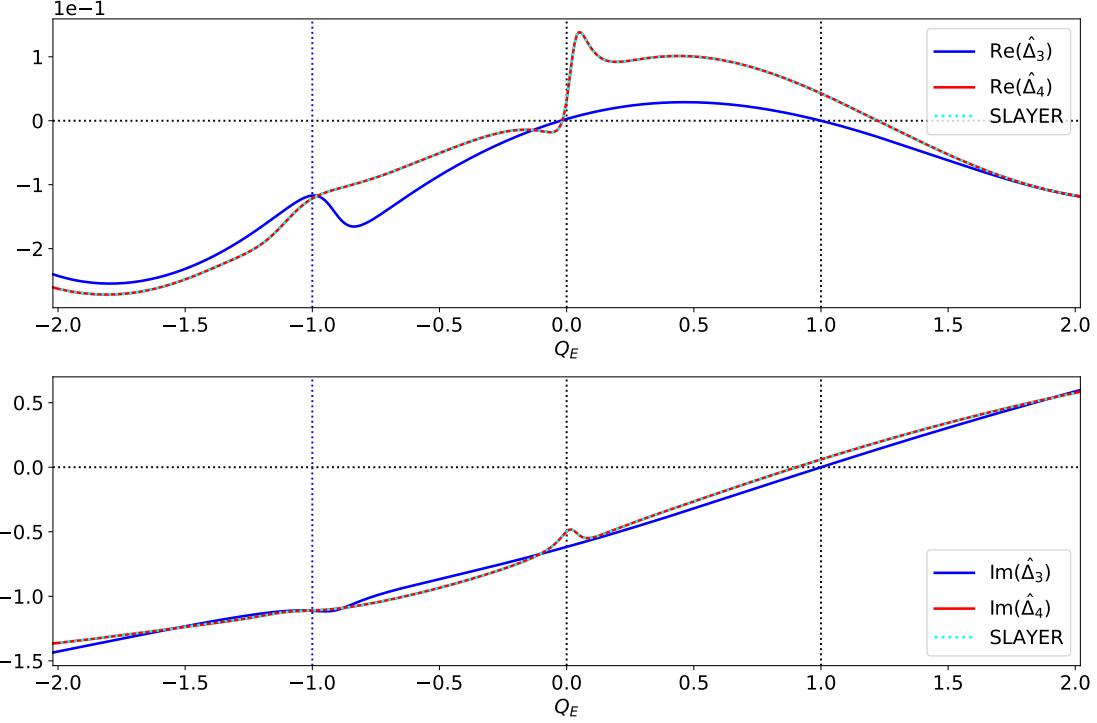


Figure 2: The scaled layer response index,  $\hat{\Delta}$ , plotted as a function of the normalized  $\mathbf{E} \times \mathbf{B}$  frequency,  $Q$ , in the second Hall-resistive (HRII) response regime of Ref. [14]. See caption to Fig. 1.

is as follows. A solution of the Riccati equation is launched from a large (with respect to unity) value of  $p$ , with the initial solution matrix specified in Eq. (153), and is then integrated to a small (with respect to unity) value of  $p$ . The scaled layer response index,  $\hat{\Delta}$ , is then deduced from the small- $p$  elements of the solution matrix by means of Eq. (117). The true layer response index is  $\Delta_s = S^{1/3} \hat{\Delta}$ .

### 3.4 Benchmarking against SLAYER

Our first task is to benchmark the results of our Fourier-space calculation of  $\hat{\Delta}$  against the results of the configuration-space calculation performed by Lee et alia. The latter calculation has been implemented in the SLAYER code [18, 20].

Figures 1–4 show the scaled layer response index,  $\hat{\Delta}$ , calculated as a function of the normalized  $\mathbf{E} \times \mathbf{B}$  frequency,  $Q_E$ , by three different solution methods, in four of the analytic layer response regimes identified in Ref. [14]. The dimensionless layer parameters used in the four calculations are specified in Table 1. The first solution method involves using the Fourier-space analysis of Ref. [17]. In this analysis, the term involving  $c_\beta^2$  is neglected in Eq. (44). Consequently, Eq. (46) decouples from Eqs. (43)–(45), which effectively converts a four-field response model (the four fields being  $\bar{\psi}$ ,  $\bar{N}$ ,  $\bar{\phi}$ , and  $\bar{V}$ ) into a three-field response model (the three fields being  $\bar{\psi}$ ,  $\bar{N}$ , and  $\bar{\phi}$ ). The second solution method involves using the Fourier-space analysis set out in Appendix A. This method solves Eqs. (43)–(46), with no neglected terms, and thus constitutes a four-field response model. The first two methods have been implemented in the TJ code [25, 26]. The third solution method involves using the configuration-space analysis of Ref. [18], which is implemented in the SLAYER code [20]. Solution methods two and three constitute equivalent four-field response models.

It is clear from Figs. 1–4 that the second and third solution methods yield exactly the same results. This is remarkable because the second method involves solving four coupled first-order o.d.e.s in Fourier space,

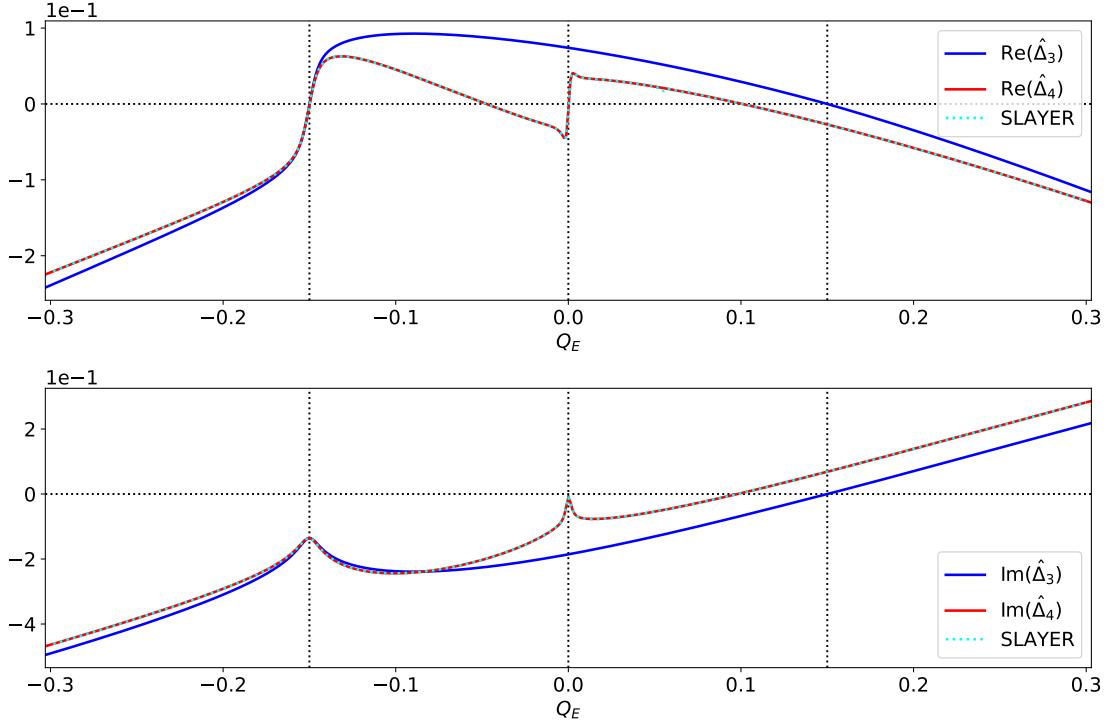


Figure 3: The scaled layer response index,  $\hat{\Delta}$ , plotted as a function of the normalized  $\mathbf{E} \times \mathbf{B}$  frequency,  $Q$ , in the second resistive-inertial (RII) response regime of Ref. [14]. See caption to Fig. 1.

whereas the third involves solving ten coupled first-order o.d.e.s in configuration space. Moreover, as is clear from a comparison of Appendix A and Ref. [18], the analysis that leads to the four coupled equations in the former method is radically different from, and completely independent of, that which leads to the ten coupled equations in the latter method. It follows that the two solution methods constitute strong independent checks on one another. Hence, we can be confident that the two methods are valid, and also that they have been implemented correctly in the TJ and SLAYER codes.

Figures 1–4 allow us to assess the consequences of neglecting the term involving  $c_\beta^2$  in Eq. (44). Obviously, this neglect is not going to make a difference if the ion sound parameter,  $c_\beta$ , is much less than unity, which explains why the rather large value,  $c_\beta = 1$ , is used in the calculations shown in the figures. (See Table 1.) It can be seen from the figures that the neglect of the  $c_\beta$  term has a larger relative effect on the real part of the scaled layer response index than on the imaginary part. Furthermore, Figs. 2 and 3 indicate that if the  $c_\beta$  term is neglected then the plasma response model fails to capture resonant behavior that occurs, in some response regimes, close to the MHD-fluid resonance ( $Q_E = 0$ ). All four figures indicate that if the  $c_\beta$  term is neglected then the imaginary part of the response index passes through zero exactly at the electron fluid resonance ( $Q_E + Q_e = 0$ ). This implies that the electromagnetic torque exerted by the RMP on the plasma in the resonant layer is zero at the electron fluid resonance. However, as is clear from the figures, when the  $c_\beta$  term is retained then the  $Q_E$  value at which  $\text{Im}(\hat{\Delta}) = 0$  is shifted from the electron fluid resonance in the ion diamagnetic direction [i.e.,  $\text{Im}(\hat{\Delta}) = 0$  when  $Q_E + Q_e < 0$ ] [18].

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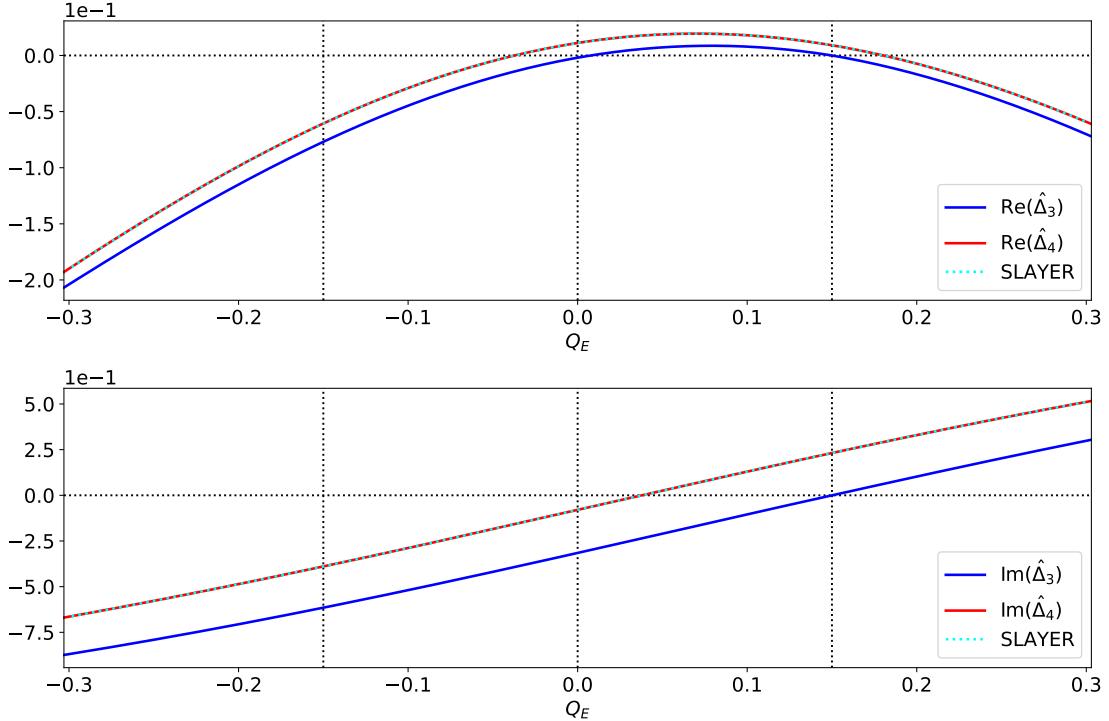


Figure 4: The scaled layer response index,  $\hat{\Delta}$ , plotted as a function of the normalized  $\mathbf{E} \times \mathbf{B}$  frequency,  $Q$ , in the second viscous-resistive (VRii) response regime of Ref. [14]. See caption to Fig. 1.

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## Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## A Four-field resonant plasma response model

### A.1 Fundamental definitions

The plasma is assumed to consist of two species. First, electrons of mass  $m_e$ , electrical charge  $-e$ , number density  $n_e$ , and temperature  $T_e$ . Second, ions of mass  $m_i$ , electrical charge  $+e$ , number density  $n_i$ , and temperature  $T_i$ . Let  $p = n_e(T_e + T_i)$  be the total plasma pressure.

Let  $r_s$  be the minor radius of the rational surface. It is helpful to define  $n_0 = n_e(r_s)$ ,  $p_0 = p(r_s)$ ,

$$\eta_e = \frac{d \ln T_e}{d \ln n_e} \Big|_{r=r_s}, \quad (7)$$

$$\eta_i = \frac{d \ln T_i}{d \ln n_e} \Big|_{r=r_s}, \quad (8)$$

$$\iota = \left( \frac{T_e}{T_i} \right)_{r=r_s} \left( \frac{1 + \eta_e}{1 + \eta_i} \right) = \left( \frac{dp_e}{dp_i} \right)_{r_s}, \quad (9)$$

where  $n_e(r)$ ,  $p(r)$ ,  $p_e(r)$ ,  $p_i(r)$ ,  $T_e(r)$ , and  $T_i(r)$  refer to electron number density, total pressure, electron pressure, ion pressure, electron temperature, and ion temperature profiles, respectively, in the absence of the magnetic perturbation.

For the sake of simplicity, the perturbed electron and ion temperature profiles are assumed to be functions of the perturbed electron number density profile in the immediate vicinity of the rational surface. In other words,  $T_e = T_e(n_e)$  and  $T_i = T_i(n_e)$ . This implies that  $p = p(n_e)$ . The “MHD velocity”, which is the velocity of a fictional MHD fluid [27], is defined  $\mathbf{V} = \mathbf{V}_E + V_{\parallel i} \mathbf{b}$ , where  $\mathbf{V}_E$  is the  $\mathbf{E} \times \mathbf{B}$  drift velocity,  $V_{\parallel i}$  is the parallel component of the ion fluid velocity,  $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ , and  $\mathbf{B}$  is the magnetic field-strength.

## A.2 Fundamental fields

The four fundamental fields in our four-field model—namely,  $\psi$ ,  $N$ ,  $\phi$ , and  $V$ —have the following definitions:

$$\nabla \psi = \frac{\mathbf{e}_{\parallel} \times \mathbf{B}}{r_s B_z}, \quad (10)$$

$$N = -\hat{d}_i \left( \frac{p - p_0}{B_z^2/\mu_0} \right), \quad (11)$$

$$\nabla \phi = \frac{\mathbf{e}_{\parallel} \times \mathbf{V}}{r_s V_A}, \quad (12)$$

$$V = \hat{d}_i \left( \frac{\mathbf{e}_{\parallel} \cdot \mathbf{V}}{V_A} \right). \quad (13)$$

Here,  $\mathbf{e}_{\parallel} = (0, \epsilon/q_s, 1)$ ,  $\epsilon = r/R_0$ ,  $q_s = m/n$ ,  $V_A = (B_z/\sqrt{\mu_0 n_0 m_i})_{r_s}$ ,  $d_i = \sqrt{m_i/(n_0 e^2 \mu_0)}_{r_s}$ , and  $\hat{d}_i = d_i/r_s$ . Our model also employs the auxiliary field

$$J = -\frac{2\epsilon_s}{q_s} + \hat{\nabla}^2 \psi, \quad (14)$$

where  $\epsilon_s = r_s/R_0$ , and  $\hat{\nabla} = r_s \nabla$ . Note that  $V_A$  is the *Alfvén speed* at the rational surface, whereas  $d_i$  is the *collisionless ion skin-depth*.

## A.3 Fundamental equations

For the case of a static (in the laboratory frame) magnetic perturbation, the four-field model takes the form [15, 14, 17]:

$$0 = [\phi, \psi] - \iota_e [N, \psi] + \hat{\eta}_{\parallel} J + \hat{E}_{\parallel}, \quad (15)$$

$$0 = [\phi, N] + \hat{d}_{\beta}^2 [J, \psi] + c_{\beta}^2 [V, \psi] + \hat{\chi}_E \hat{\nabla}^2 N, \quad (16)$$

$$0 = [\phi, \hat{\nabla}^2 \phi] - \frac{\iota_i}{2} \left( \hat{\nabla}^2 [\phi, N] + [\hat{\nabla}^2 \phi, N] + [\hat{\nabla}^2 N, \phi] \right) + [J, \psi] + \hat{\chi}_{\varphi} \hat{\nabla}^4 (\phi + \iota_i N), \quad (17)$$

$$0 = [\phi, V] + [N, \psi] + \hat{\chi}_{\varphi} \hat{\nabla}^2 V. \quad (18)$$

Here,  $[A, B] \equiv \hat{\nabla} A \times \hat{\nabla} B \cdot \mathbf{e}_{\parallel}$ ,  $\iota_e = \iota/(1 + \iota)$ ,  $\iota_i = 1/(1 + \iota)$ ,  $\hat{t} = t/(r_s/V_A)$ ,  $\hat{\eta}_{\parallel} = \eta_{\parallel}/(\mu_0 r_s V_A)$ ,  $\hat{E}_{\parallel} = E_{\parallel}/(B_z V_A)$ ,  $\hat{\chi}_E = \chi_E/(r_s V_A)$ ,  $\hat{\chi}_{\varphi} = \chi_{\varphi}/(r_s V_A)$ , where  $\eta_{\parallel}$  is the parallel plasma electrical resistivity at the rational surface,  $E_{\parallel}$  the parallel inductive electric field that maintains the equilibrium toroidal plasma current in the vicinity of the rational surface,  $\chi_E$  the anomalous perpendicular heat diffusivity at the rational surface, and  $\chi_{\varphi}$  the anomalous perpendicular ion momentum diffusivity at the rational surface. Moreover,  $d_{\beta} = c_{\beta} d_i$ , and  $\hat{d}_{\beta} = d_{\beta}/r_s$ , where  $c_{\beta} = [\beta/(1 + \beta)]_{r_s}^{1/2}$ , and  $\beta = (5/3) \mu_0 p_0 / B_z^2$ . Here,  $d_{\beta}$  is usually referred to as the *ion sound-radius*.

#### A.4 Matching to plasma equilibrium

The unperturbed plasma equilibrium is such that  $\mathbf{B} = (0, B_{\theta}(r), B_z)$ ,  $p = p(r)$ ,  $\mathbf{V} = (0, V_E(r), V_z(r))$ , where  $V_E(r) \simeq -E_r/B_z$  is the (dominant  $\theta$ -component of the)  $\mathbf{E} \times \mathbf{B}$  velocity. Now, the resonant layer is assumed to have a radial thickness that is much smaller than  $r_s$ . Hence, we only need to evaluate plasma equilibrium quantities in the immediate vicinity of the rational surface. Equations (10)–(13) suggest that

$$\psi(\hat{x}) = \frac{\hat{x}^2}{2 \hat{L}_s}, \quad (19)$$

$$N(\hat{x}) = -\hat{V}_* \hat{x}, \quad (20)$$

$$\phi(\hat{x}) = -\hat{V}_E \hat{x}, \quad (21)$$

$$V(\hat{x}) = \hat{V}_{\parallel}, \quad (22)$$

where  $\hat{x} = (r - r_s)/r_s$ ,  $\hat{L}_s = L_s/r_s$ ,  $L_s = R_0 q_s/s_s$ ,  $\hat{V}_E = V_E(r_s)/V_A$ ,  $\hat{V}_* = V_*(r_s)/V_A$ ,  $V_*(r) = (dp/dr)/(e n_0 B_z)$  is the (dominant  $\theta$ -component of the) diamagnetic velocity, and  $\hat{V}_{\parallel} = \hat{d}_i V_z(r_s)/V_A$ . Here,  $s_s = s(r_s)$  and  $s(r) = d \ln q / d \ln r$ . We also have

$$J(\hat{x}) = -\left(\frac{2}{s_s} - 1\right) \frac{1}{\hat{L}_s}, \quad (23)$$

and  $\hat{E}_{\parallel}(\hat{x}) = (2/s_s - 1)(\hat{\eta}_{\parallel}/\hat{L}_s)$ .

#### A.5 Derivation of linear layer equations

In accordance with Eqs. (19)–(23), let us write

$$\psi(\hat{x}, \zeta) = \frac{\hat{x}^2}{2 \hat{L}_s} + \tilde{\psi}(\hat{x}) e^{i\zeta}, \quad (24)$$

$$\phi(\hat{x}, \zeta) = -\hat{V}_E \hat{x} + \tilde{\phi}(\hat{x}) e^{i\zeta}, \quad (25)$$

$$N(\hat{x}, \zeta) = -\hat{V}_* \hat{x} + \iota_e \tilde{N}(\hat{x}) e^{i\zeta}, \quad (26)$$

$$V(\hat{x}, \zeta) = \hat{V}_{\parallel} + \iota_e \tilde{V}(\hat{x}) e^{i\zeta}, \quad (27)$$

$$J(\hat{x}, \zeta) = -\left(\frac{2}{s_s} - 1\right) \frac{1}{\hat{L}_s} + \hat{\nabla}^2 \tilde{\psi}(\hat{x}) e^{i\zeta}, \quad (28)$$

where  $\zeta = m\theta - n\varphi$ . Substituting Eqs. (24)–(28) into Eqs. (14)–(18), and only retaining terms that are first order in perturbed quantities, we obtain the following set of linear equations:

$$-\mathrm{i} n (\omega_E + \omega_{*e}) \tau_H \tilde{\psi} = -\mathrm{i} \hat{x} (\tilde{\phi} - \tilde{N}) + S^{-1} \hat{\nabla}^2 \tilde{\psi}, \quad (29)$$

$$-\mathrm{i} n \omega_E \tau_H \tilde{N} = \mathrm{i} n \omega_{*e} \tau_H \tilde{\phi} - \mathrm{i} \iota_e \hat{d}_\beta^2 \hat{x} \hat{\nabla}^2 \tilde{\psi} - \mathrm{i} c_\beta^2 \hat{x} \tilde{V} + S^{-1} P_E \hat{\nabla}^2 \tilde{N}, \quad (30)$$

$$-\mathrm{i} n (\omega_E + \omega_{*i}) \tau_H \hat{\nabla}^2 \tilde{\phi} = -\mathrm{i} \hat{x} \hat{\nabla}^2 \tilde{\psi} + S^{-1} P_\varphi \hat{\nabla}^4 \left( \tilde{\phi} + \frac{\tilde{N}}{\iota} \right), \quad (31)$$

$$-\mathrm{i} n \omega_E \tau_H \tilde{V} = -\mathrm{i} n \omega_{*e} \tau_H \tilde{\psi} - \mathrm{i} \hat{x} \tilde{N} + S^{-1} P_\varphi \hat{\nabla}^2 \tilde{V}. \quad (32)$$

Here,  $\tau_H = L_s/(m V_A)$  is the hydromagnetic time,  $\omega_E = -(q_s/r_s) V_E(r_s) = -(d\Phi/d\psi_p)_{r_s}$  the  $\mathbf{E} \times \mathbf{B}$  frequency,  $\omega_{*e} = \iota_e (q_s/r_s) V_*(r_s) = [(dp_e/d\psi_p)/(e n_e)]_{r_s}$  the electron diamagnetic frequency,  $\omega_{*i} = -\iota_i (m/r_s) V_*(r_s) = -[(dp_i/d\psi_p)/(e n_e)]_{r_s}$  the ion diamagnetic frequency,  $\Phi(r)$  the equilibrium electrostatic potential,  $S = \tau_R/\tau_H$  the Lundquist number,  $\tau_R = \mu_0 r_s^2/\eta_\parallel$  the resistive diffusion time,  $\tau_E = r_s^2/\chi_E$  the energy confinement time, and  $\tau_\varphi = r_s^2/\chi_\varphi$  the toroidal momentum confinement time. Furthermore,  $P_E = \tau_R/\tau_E$  and  $P_\varphi = \tau_R/\tau_\varphi$  are magnetic Prandtl numbers.

Let us define the stretched radial variable  $X = S^{1/3} \hat{x}$ . Assuming that  $X \sim \mathcal{O}(1)$  in the layer (i.e., assuming that the layer thickness is roughly of order  $S^{-1/3} r_s$ ), and making use of the fact that  $S \gg 1$  in conventional tokamak plasmas, Eqs. (29)–(32) reduce to the following set of linear layer equations [14, 17]:

$$\mathrm{i} (Q_E + Q_e) \tilde{\psi} = -\mathrm{i} X \left( \tilde{\phi} - \tilde{N} \right) + \frac{d^2 \tilde{\psi}}{dX^2}, \quad (33)$$

$$\mathrm{i} Q_E \tilde{N} = -\mathrm{i} Q_e \tilde{\phi} - \mathrm{i} D^2 X \frac{d^2 \tilde{\psi}}{dX^2} - \mathrm{i} c_\beta^2 X \tilde{V} + P_E \frac{d^2 \tilde{N}}{dX^2}, \quad (34)$$

$$\mathrm{i} (Q_E + Q_i) \frac{d^2 \tilde{\phi}}{dX^2} = -\mathrm{i} X \frac{d^2 \tilde{\psi}}{dX^2} + P_\varphi \frac{d^4}{dX^4} \left( \tilde{\phi} + \frac{\tilde{N}}{\iota} \right), \quad (35)$$

$$\mathrm{i} Q_E \tilde{V} = \mathrm{i} Q_e \tilde{\psi} - \mathrm{i} X \tilde{N} + P_\varphi \frac{d^2 \tilde{V}}{dX^2}. \quad (36)$$

Here,  $Q_E = -S^{1/3} n \omega_E \tau_H$ ,  $Q_{e,i} = -S^{1/3} n \omega_{*e,i} \tau_H$ , and  $D = S^{1/3} \iota_e^{1/2} \hat{d}_\beta$ .

If we write  $P_E = c_\beta^2$  then Eqs. (33)–(36) become equivalent to the set of layer equations solved by Lee et alia. (To be slightly more exact, we get from our equations to those of Lee et alia by making the following transformation:  $Q_E \rightarrow Q$ ,  $Q_e \rightarrow -Q_{*e}$ ,  $Q_i \rightarrow -Q_{*i}$ ,  $\iota \rightarrow 1/\tau$ ,  $P_E \rightarrow c_\beta^2$ ,  $P_\varphi \rightarrow P$ ,  $\tilde{\psi} \rightarrow -\tilde{\psi}$ ,  $\tilde{N} \rightarrow \tilde{Z}$ ,  $\tilde{\phi} \rightarrow \tilde{\phi}$ , and  $\tilde{V} \rightarrow -\tilde{V}_z$ .)

The  $\omega\text{-}\beta$  approximation used in Refs. [14] and [17] involves neglecting the term containing  $c_\beta^2$  in Eq. (34). This approximation decouples Eq. (36) from the three preceding equations, and effectively converts a four-field resonant response model into a three-field model. In the following, we shall not make use of this approximation.

## A.6 Asymptotic matching

The linear layer equations, (33)–(36), possess tearing parity solutions characterized by the symmetry  $\tilde{\psi}(-X) = \tilde{\psi}(X)$ ,  $\tilde{N}(-X) = -\tilde{N}(X)$ ,  $\tilde{\phi}(-X) = -\tilde{\phi}(X)$ ,  $\tilde{V}(-X) = \tilde{V}(X)$ . As is easily demonstrated, the asymptotic

behavior of the tearing parity solutions to Eqs. (33)–(36) are such that

$$\tilde{\psi}(X) \rightarrow \psi_0 \left[ \frac{\hat{\Delta}}{2} |X| + 1 + \mathcal{O}\left(\frac{1}{X^2}\right) \right], \quad (37)$$

$$\tilde{\phi}(X) \rightarrow -\psi_0 Q_E \left[ \frac{\hat{\Delta}}{2} \operatorname{sgn}(X) + \frac{1}{X} + \mathcal{O}\left(\frac{1}{X^2}\right) \right], \quad (38)$$

$$\tilde{N}(X) \rightarrow \psi_0 Q_e \left[ \frac{\hat{\Delta}}{2} \operatorname{sgn}(X) + \frac{1}{X} + \mathcal{O}\left(\frac{1}{X^2}\right) \right], \quad (39)$$

$$\tilde{V}(X) \rightarrow \mathcal{O}\left(\frac{1}{X^3}\right) \quad (40)$$

as  $|X| \rightarrow \infty$ , where  $\psi_0$  is an arbitrary constant. It follows from Eq. (4) that layer response index is

$$\Delta_s = S^{1/3} \hat{\Delta}. \quad (41)$$

## A.7 Fourier transformation

Equations (33)–(36) are most conveniently solved in Fourier transform space [14]. Let

$$\bar{\phi}(p) = \int_{-\infty}^{\infty} \tilde{\phi}(X) e^{-ipX} dX, \quad (42)$$

et cetera. The Fourier transformed linear layer equations become

$$i(Q_E + Q_e) \bar{\psi} = \frac{d}{dp} (\bar{\phi} - \bar{N}) - p^2 \bar{\psi}, \quad (43)$$

$$iQ_E \bar{N} = -iQ_e \bar{\phi} - D^2 \frac{d(p^2 \bar{\psi})}{dp} + c_\beta^2 \frac{d\bar{V}}{dp} - P_E p^2 \bar{N}, \quad (44)$$

$$i(Q_E + Q_i) p^2 \bar{\phi} = \frac{d(p^2 \bar{\psi})}{dp} - P_\varphi p^4 \left( \bar{\phi} + \frac{\bar{N}}{\iota} \right), \quad (45)$$

$$iQ_E \bar{V} = iQ_e \bar{\psi} + \frac{d\bar{N}}{dp} - P_\varphi p^2 \bar{V}, \quad (46)$$

where, for a tearing parity solution,

$$\bar{\phi}(p) - \bar{N}(p) \equiv \bar{Y}(p) \rightarrow \bar{Y}_0 \left[ \frac{\hat{\Delta}}{\pi p} + 1 + \mathcal{O}(p) \right] \quad (47)$$

as  $p \rightarrow 0$ . Here,  $\bar{Y}_0$  is an arbitrary constant.

Finally, if we define

$$\bar{J}(p) = p^2 \bar{\psi}, \quad (48)$$

then Eqs. (43)–(46) can be converted into the following equivalent set of four coupled first-order o.d.e.s:

$$\frac{d\bar{Y}}{dp} = \left[ \frac{i(Q_E + Q_e) + p^2}{p^2} \right] \bar{J}, \quad (49)$$

$$\frac{d\bar{N}}{dp} = - \left( \frac{iQ_e}{p^2} \right) \bar{J} + (iQ_E + P_\varphi p^2) \bar{V}, \quad (50)$$

$$\frac{d\bar{J}}{dp} = [i(Q_E + Q_i)p^2 + P_\varphi p^4] \bar{Y} + [i(Q_E + Q_i)p^2 + \iota_e^{-1} P_\varphi p^4] \bar{N}, \quad (51)$$

$$c_\beta^2 \frac{d\bar{V}}{dp} = [iQ_e + i(Q_E + Q_i)D^2 p^2 + D^2 P_\varphi p^4] \bar{Y} \\ + \{i(Q_E + Q_e) + [P_E + i(Q_E + Q_i)D^2] p^2 + \iota_e^{-1} D^2 P_\varphi p^4\} \bar{N}, \quad (52)$$

Note that  $\iota_e = -Q_e/(Q_i - Q_e)$ .

## A.8 Small argument expansion

Let us search for power-law solutions of Eqs. (49)–(52) at small values of  $p$ . Given that we have four coupled first-order o.d.e.s, we expect to find four independent power-law solutions. The first solution is such that

$$\bar{Y}(p) = i(Q_E + Q_e) a_{-1} p^{-1} - \left[ \frac{i}{2} Q_E (Q_E + Q_e) (Q_E + Q_i) + 1 \right] a_{-1} p + \mathcal{O}(p^3), \quad (53)$$

$$\bar{N}(p) = -iQ_e a_{-1} p^{-1} + \frac{i}{2} Q_E Q_e (Q_E + Q_i) a_{-1} p + \mathcal{O}(p^3), \quad (54)$$

$$\bar{J}(p) = -a_{-1} - \frac{1}{2} Q_E (Q_E + Q_i) a_{-1} p^2 + \mathcal{O}(p^4), \quad (55)$$

$$\bar{V}(p) = - \frac{[iQ_e (1 + P_E) + Q_E (Q_E + Q_i) D^2]}{2 c_\beta^2} a_{-1} p^2 + \mathcal{O}(p^4), \quad (56)$$

where  $a_{-1}$  is an arbitrary constant. The second solution is such that

$$\bar{Y}(p) = i(Q_E + Q_e) a_0 - \frac{i}{6} Q_E (Q_E + Q_e) (Q_E + Q_i) a_0 p^2 + \mathcal{O}(p^4), \quad (57)$$

$$\bar{N}(p) = -iQ_e a_0 + \frac{i}{6} Q_E Q_e (Q_E + Q_i) a_0 p^2 + \mathcal{O}(p^4), \quad (58)$$

$$\bar{J}(p) = -\frac{1}{3} Q_E (Q_E + Q_i) a_0 p^3 + \mathcal{O}(p^5), \quad (59)$$

$$\bar{V}(p) = -\frac{1}{3} \frac{[iQ_e P_E + Q_E (Q_E + Q_i) D^2]}{c_\beta^2} a_0 p^3 + \mathcal{O}(p^5), \quad (60)$$

where  $a_0$  is an arbitrary constant. The third solution is such that

$$\bar{Y}(p) = -\frac{1}{6} (Q_E + Q_e) (Q_E + Q_i) a_2 p^2 + \mathcal{O}(p^4), \quad (61)$$

$$\bar{N}(p) = a_2 + \frac{i}{2} (Q_E + Q_i) \left( -\frac{i}{3} Q_e + \frac{g}{c_\beta^2} \right) a_2 p^2 + \mathcal{O}(p^4), \quad (62)$$

$$\bar{J}(p) = \frac{i}{3} (Q_E + Q_i) a_2 p^3 + \mathcal{O}(p^5), \quad (63)$$

$$\bar{V}(p) = i \frac{(Q_E + Q_e)}{c_\beta^2} a_2 p + \mathcal{O}(p^3), \quad (64)$$

where  $a_2$  is an arbitrary constant. The final solution is such that

$$\bar{Y}(p) = -\frac{i}{12} Q_E (Q_E + Q_e) (Q_E + Q_i) a_3 p^3 + \mathcal{O}(p^5), \quad (65)$$

$$\bar{N}(p) = -i Q_E a_3 p + \mathcal{O}(p^3), \quad (66)$$

$$\bar{J}(p) = -\frac{1}{4} Q_E (Q_E + Q_i) a_3 p^4 + \mathcal{O}(p^6), \quad (67)$$

$$\bar{V}(p) = -\frac{Q_E (Q_E + Q_e)}{2 c_\beta^2} a_3 p^2 + \mathcal{O}(p^4), \quad (68)$$

where  $a_3$  is an arbitrary constant. Note, however, that the third solution is not consistent with Eqs. (37)–(40), which mandate that  $\bar{N}(p) \rightarrow -[Q_e/(Q_E + Q_e)] \bar{Y}(p) + \mathcal{O}(p)$  and  $\bar{V}(p) \rightarrow \mathcal{O}(p^2)$  as  $p \rightarrow 0$ . Hence, we deduce that  $a_2 = 0$ .

We conclude that, at small values of  $p$ , the most general solution for  $\bar{Y}(p)$  and  $\bar{N}(p)$  takes the form

$$\bar{Y}(p) = i (Q_E + Q_e) (a_{-1} p^{-1} + a_0) + \mathcal{O}(p), \quad (69)$$

$$\bar{N}(p) = -i Q_e (a_{-1} p^{-1} + a_0) + \mathcal{O}(p), \quad (70)$$

which is consistent with Eqs. (37)–(40).

## A.9 Riccati matrix differential equation

Let

$$\underline{u} = \begin{pmatrix} \bar{Y} \\ \bar{N} \end{pmatrix}, \quad (71)$$

$$\underline{v} = \begin{pmatrix} \bar{J} \\ c_\beta^2 \bar{V} \end{pmatrix}. \quad (72)$$

Equations (49)–(52) can be written in the form

$$\frac{d\underline{u}}{dp} = \underline{\underline{A}} \underline{v}, \quad (73)$$

$$\frac{d\underline{v}}{dp} = \underline{\underline{B}} \underline{u}, \quad (74)$$

where

$$A_{11} = \frac{i (Q_E + Q_e) + p^2}{p^2}, \quad (75)$$

$$A_{12} = 0, \quad (76)$$

$$A_{21} = -\frac{i Q_e}{p^2}, \quad (77)$$

$$A_{22} = \frac{i Q_E + P_\varphi p^2}{c_\beta^2}, \quad (78)$$

$$B_{11} = i (Q_E + Q_i) p^2 + P_\varphi p^4, \quad (79)$$

$$B_{12} = i (Q_E + Q_i) p^2 + \iota_e^{-1} P_\varphi p^4, \quad (80)$$

$$B_{21} = i Q_e + i (Q_E + Q_i) D^2 p^2 + D^2 P_\varphi p^4, \quad (81)$$

$$B_{22} = i (Q_E + Q_e) + [P_E + i (Q_E + Q_i) D^2] p^2 + \iota_e^{-1} D^2 P_\varphi p^4. \quad (82)$$

Thus, we obtain the following matrix differential equation:

$$\frac{d}{dp} \left( \underline{\underline{A}}^{-1} \frac{du}{dp} \right) = \underline{\underline{B}} u. \quad (83)$$

Let

$$p \frac{du}{dp} = \underline{\underline{W}} u. \quad (84)$$

The previous two equations can be combined to give

$$\left( p \frac{d\underline{\underline{W}}}{dp} - \underline{\underline{W}} + \underline{\underline{W}} \underline{\underline{W}} + \underline{\underline{A}} p \frac{d\underline{\underline{A}}^{-1}}{dp} \underline{\underline{W}} - p^2 \underline{\underline{A}} \underline{\underline{B}} \right) u = 0, \quad (85)$$

which yields the Riccati matrix differential equation,

$$p \frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}} \underline{\underline{W}} - \underline{\underline{E}} \underline{\underline{W}} + \underline{\underline{F}}, \quad (86)$$

where

$$\underline{\underline{E}}(p) = \underline{\underline{A}} p \frac{d\underline{\underline{A}}^{-1}}{dp}, \quad (87)$$

$$\underline{\underline{F}}(p) = p^2 \underline{\underline{A}} \underline{\underline{B}}. \quad (88)$$

In fact, it is easily demonstrated that

$$E_{11} = \frac{2i(Q_E + Q_e)}{i(Q_E + Q_e) + p^2}, \quad (89)$$

$$E_{12} = 0, \quad (90)$$

$$E_{21} = -\frac{2iQ_e(iQ_E + 2P_\varphi p^2)}{[i(Q_E + Q_e) + p^2](iQ_E + P_\varphi p^2)}, \quad (91)$$

$$E_{22} = -\frac{2P_\varphi p^2}{iQ_E + P_\varphi p^2}, \quad (92)$$

and

$$F_{11} = p^2 [i(Q_E + Q_e) + p^2] [i(Q_E + Q_i) + P_\varphi p^2], \quad (93)$$

$$F_{12} = p^2 [i(Q_E + Q_e) + p^2] [i(Q_E + Q_i) + \iota_e^{-1} P_\varphi p^2], \quad (94)$$

$$F_{21} = -iQ_e p^2 [i(Q_E + Q_i) + P_\varphi p^2] + c_\beta^{-2} p^2 (iQ_E + P_\varphi p^2) [iQ_e + i(Q_E + Q_i) D^2 p^2 + D^2 P_\varphi p^4], \quad (95)$$

$$F_{22} = -iQ_e p^2 [i(Q_E + Q_i) + \iota_e^{-1} P_\varphi p^2] + c_\beta^{-2} p^2 (iQ_E + P_\varphi p^2) \{ i(Q_E + Q_e) + [P_E + i(Q_E + Q_i) D^2] p^2 + \iota_e^{-1} D^2 P_\varphi p^4 \}. \quad (96)$$

Finally, if

$$\underline{\underline{W}}(p) = \begin{pmatrix} W_{11}, & W_{12} \\ W_{21}, & W_{22} \end{pmatrix} \quad (97)$$

then Eq. (86) yields

$$p \frac{dW_{11}}{dp} = W_{11} - W_{11}W_{11} - W_{12}W_{21} - E_{11}W_{11} + F_{11}, \quad (98)$$

$$p \frac{dW_{12}}{dp} = W_{12} - W_{11}W_{12} - W_{12}W_{22} - E_{11}W_{12} + F_{12}, \quad (99)$$

$$p \frac{dW_{21}}{dp} = W_{21} - W_{21}W_{11} - W_{22}W_{21} - E_{21}W_{11} - E_{22}W_{21} + F_{21}, \quad (100)$$

$$p \frac{dW_{22}}{dp} = W_{22} - W_{21}W_{12} - W_{22}W_{22} - E_{21}W_{12} - E_{22}W_{22} + F_{22}. \quad (101)$$

Thus, our final system of equations consists of a set of four coupled nonlinear differential equations.

### A.10 Small-argument behavior of Riccati matrix differential equation

It follows from Eqs. (89)–(92) that  $\underline{\underline{E}}(p) = \underline{\underline{E}}^{(0)} + \mathcal{O}(p^2)$  at small values of  $p$ , where

$$\underline{E}_{11}^{(0)} = 2, \quad (102)$$

$$\underline{E}_{12}^{(0)} = 0, \quad (103)$$

$$\underline{E}_{21}^{(0)} = -\frac{2Q_e}{Q_E + Q_e}, \quad (104)$$

$$\underline{E}_{22}^{(0)} = 0. \quad (105)$$

Likewise, Eqs. (93)–(96) imply that  $\underline{\underline{F}}(p) = \mathcal{O}(p^2)$ .

Suppose that  $\underline{\underline{W}}(p) = \underline{\underline{W}}^{(0)} + \underline{\underline{W}}^{(1)}p$  at small values of  $p$ , where the elements of  $\underline{\underline{W}}^{(0)}$  and  $\underline{\underline{W}}^{(1)}$  are independent of  $p$ . Equation (86) gives

$$\underline{0} = \underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)}\underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)}\underline{\underline{W}}^{(0)}, \quad (106)$$

$$\underline{0} = -\underline{\underline{W}}^{(1)}\underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)}\underline{\underline{W}}^{(1)} - \underline{\underline{E}}^{(0)}\underline{\underline{W}}^{(1)}. \quad (107)$$

Suitable solutions are

$$\underline{\underline{W}}^{(0)} = \begin{pmatrix} -1, & 0 \\ -E_{21}^{(0)}/2, & 0 \end{pmatrix}, \quad (108)$$

$$W_{12}^{(1)} = 0, \quad (109)$$

$$W_{21}^{(1)} = \frac{E_{21}^{(0)}}{2} [W_{11}^{(1)} - W_{22}^{(1)}]. \quad (110)$$

At small values of  $p$ , let

$$\underline{u}(p) = \underline{u}_{-1} p^{-1} + \underline{u}_0, \quad (111)$$

where the elements of  $\underline{u}_{-1}$  (which are  $y_{-1}$  and  $n_{-1}$ , respectively) and the elements of  $\underline{u}_0$  (which are  $y_0$  and  $n_0$ , respectively) are all constants. Equation (84) gives

$$\underline{\underline{W}}^{(0)}\underline{u}_{-1} = -\underline{u}_{-1}, \quad (112)$$

$$\underline{\underline{W}}^{(0)}\underline{u}_0 + \underline{\underline{W}}^{(1)}\underline{u}_{-1} = \underline{0}. \quad (113)$$

Thus, making use of Eq. (108), we get

$$\begin{pmatrix} -1, & 0 \\ -E_{21}^{(0)}/2, & 0 \end{pmatrix} \begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix} = - \begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix}, \quad (114)$$

which implies that

$$\frac{E_{21}^{(0)}}{2} y_{-1} = -\frac{Q_e}{Q_E + Q_e} y_{-1} = n_{-1}, \quad (115)$$

in accordance with Eqs. (69) and (70), where use has been made of Eq. (104). Equations (108)–(110) and (113) yield

$$\frac{y_0}{y_{-1}} = W_{11}^{(1)}, \quad (116)$$

with  $n_0$  undetermined. It follows from Equation (47) that

$$\frac{\pi}{\hat{\Delta}} \equiv \frac{y_0}{y_{-1}} = W_{11}^{(1)} = \frac{dW_{11}(0)}{dp}. \quad (117)$$

Note, incidentally, that the result  $\pi/\hat{\Delta} = y_0/y_{-1}$  follows directly from Eq. (33), and would hold even if  $a_2$  were non-zero.

### A.11 Large-argument behavior of Riccati matrix differential equation

At large values of  $p$ , it is clear from Eqs. (93)–(96) that  $\underline{F}(p) = \underline{F}^{(6)} p^6 + \underline{F}^{(8)} p^8$ , where the elements of  $\underline{F}^{(6)}$  and  $\underline{F}^{(8)}$  are independent of  $p$ . On the other hand, Eqs. (89)–(92) imply that  $\underline{E}(p) = \underline{E}^{(0)}$ , where the elements of  $\underline{E}^{(0)}$  are independent of  $p$ . Thus, if we write  $\underline{W}(p) = \underline{W}^{(2)} p^2 + \underline{W}^{(4)} p^4$ , where the elements of  $\underline{W}^{(2)}$  and  $\underline{W}^{(4)}$  are independent of  $p$ , then Eq. (86) gives

$$\underline{W}^{(4)} \underline{W}^{(4)} = \underline{F}^{(8)}, \quad (118)$$

$$\underline{W}^{(2)} \underline{W}^{(4)} + \underline{W}^{(4)} \underline{W}^{(2)} = \underline{F}^{(6)}. \quad (119)$$

Now, according to Eqs. (93)–(96),

$$F_{11}^{(8)} = 0, \quad (120)$$

$$F_{12}^{(8)} = 0, \quad (121)$$

$$F_{21}^{(8)} = c_\beta^{-2} D^2 P_\varphi^2, \quad (122)$$

$$F_{22}^{(8)} = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2, \quad (123)$$

so Eq. (118) yields

$$W_{11}^{(4)} = 0, \quad (124)$$

$$W_{12}^{(4)} = 0, \quad (125)$$

$$W_{21}^{(4)} = -c_\beta^{-1} \iota_e^{1/2} D P_\varphi, \quad (126)$$

$$W_{22}^{(4)} = -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi, \quad (127)$$

where we have chosen the sign of the square root that is associated with well-behaved solutions at large values of  $p$ . Here, we are assuming that  $\iota_e > 0$ . Equations (93)–(96) also give

$$F_{11}^{(6)} = P_\varphi, \quad (128)$$

$$F_{12}^{(6)} = \iota_e^{-1} P_\varphi, \quad (129)$$

$$F_{21}^{(6)} = i c_\beta^{-2} Q_E D^2 P_\varphi + i c_\beta^{-2} (Q_E + Q_i) D^2 P_\varphi, \quad (130)$$

$$F_{22}^{(6)} = i c_\beta^{-2} \iota_e^{-1} Q_E D^2 P_\varphi + c_\beta^{-2} [P_E + i (Q_E + Q_i) D^2] P_\varphi. \quad (131)$$

Thus, Eq. (119) yields

$$W_{12}^{(2)} W_{21}^{(4)} = F_{11}^{(6)}, \quad (132)$$

$$W_{12}^{(2)} W_{22}^{(4)} = F_{12}^{(6)}, \quad (133)$$

which gives

$$W_{12}^{(2)} = -c_\beta \iota_e^{-1/2} D^{-1}. \quad (134)$$

Now, if

$$\underline{W} \underline{u} = \lambda(p) \underline{u} \quad (135)$$

then Eq. (84) yields

$$p \frac{d\underline{u}}{dp} = \lambda \underline{u}, \quad (136)$$

which implies that

$$\underline{u}(p) = \underline{u}(p_0) \exp \left[ \int_{p_0}^p \frac{\lambda_r(p')}{p'} dp' \right] \exp \left[ i \int_{p_0}^p \frac{\lambda_i(p')}{p'} dp' \right], \quad (137)$$

where  $\lambda_r$  and  $\lambda_i$  are the real and imaginary parts of  $\lambda$ , respectively. Of course, a solution that is well behaved at large values of  $p$  is such that  $\lambda_r$  is negative. As we have seen, the large- $p$  limit of Eq. (86) is

$$\underline{\underline{W}} \underline{\underline{u}} = \underline{\underline{F}}. \quad (138)$$

Hence, if

$$\underline{\underline{F}} \underline{\underline{u}} = \Lambda \underline{\underline{u}} \quad (139)$$

then Eqs. (135) and (139) imply that

$$\lambda^2 = \Lambda. \quad (140)$$

The eigenvalue problem for the  $F$ -matrix reduces to

$$\Lambda^2 - (F_{11} + F_{22}) \Lambda + F_{11} F_{22} - F_{12} F_{21} = 0. \quad (141)$$

Now,

$$F_{11} + F_{22} \simeq F_{22}^{(8)} p^8 = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2 p^8, \quad (142)$$

$$\begin{aligned} F_{11} F_{22} - F_{12} F_{21} &\simeq \left[ F_{11}^{(6)} F_{22}^{(8)} - F_{12}^{(6)} F_{21}^{(8)} \right] p^{14} \\ &\quad + \left[ F_{11}^{(6)} F_{22}^{(6)} - F_{12}^{(6)} F_{21}^{(6)} \right] p^{12} = c_\beta^{-2} R P_\varphi^2 p^{12}, \end{aligned} \quad (143)$$

where

$$R = P_E + i (1 - \iota_e^{-1}) (Q_E + Q_i) D^2, \quad (144)$$

Hence, the two eigenvalues of the  $F$ -matrix are

$$\Lambda_1 \simeq F_{22}^{(8)} p^8 = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2 p^8, \quad (145)$$

$$\Lambda_2 \simeq \frac{[F_{11}^{(6)} F_{22}^{(6)} - F_{12}^{(6)} F_{21}^{(6)}]}{F_{22}^{(8)}} p^4 = \iota_e D^{-2} R p^4. \quad (146)$$

Thus, we deduce that the two eigenvalues of the  $W$ -matrix are

$$\lambda_1 = -\Lambda_1^{1/2} = -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi p^4, \quad (147)$$

$$\lambda_2 = -\Lambda_2^{1/2} = -\iota_e^{1/2} D^{-1} R^{1/2} p^2, \quad (148)$$

Here, the square root of  $R$  is taken such that the real part of  $\lambda_2$  is negative. Now, the eigenvalue problem for the  $W$ -matrix reduces to

$$\lambda^2 - W_{22}^{(4)} p^4 \lambda + \left[ W_{11}^{(2)} W_{22}^{(4)} - W_{12}^{(2)} W_{21}^{(4)} \right] p^6 = 0. \quad (149)$$

which yields

$$\lambda_1 \simeq W_{22}^{(4)} p^4, \quad (150)$$

which is in agreement with Eq. (147), and

$$\lambda_2 \simeq \left[ W_{11}^{(2)} - \frac{W_{12}^{(2)} W_{21}^{(4)}}{W_{22}^{(4)}} \right] p^2, \quad (151)$$

which implies that

$$W_{11}^{(2)} = -\iota_e^{1/2} D^{-1} R^{1/2} - c_\beta \iota_e^{1/2} D^{-1}. \quad (152)$$

Hence, the large- $p$  boundary condition for the  $W$ -matrix is

$$\underline{\underline{W}}(p) = \begin{pmatrix} -\iota_e^{1/2} D^{-1} R^{1/2} p^2 - c_\beta \iota_e^{1/2} D^{-1} p^2, & -c_\beta \iota_e^{-1/2} D^{-1} p^2 \\ -c_\beta^{-1} \iota_e^{1/2} D P_\varphi p^4, & -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi p^4 \end{pmatrix}. \quad (153)$$

## A.12 Method of solution

The method of solution is to launch the well-behaved asymptotic solution (153) of Eqs. (98)–(101) from large  $p$ , and then integrate the equations backward to small  $p$ . The scaled layer response index,  $\hat{\Delta}$ , is then determined from Eq. (117). Note that, because there are no free parameters in expression (153), the scaled layer response index is uniquely determined by this procedure. Finally, the true layer response index is given by  $\Delta_s = S^{1/3} \hat{\Delta}$ . [See Eq. (41).]

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