Calculation of Vertical Stability in an Inverse Aspect-Ratio Expanded Tokamak Plasma Equilibrium

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I. PLASMA EQUILIBRIUM

All lengths are normalized to the major radius of the plasma magnetic axis, R_0. All magnetic field-strengths are normalized to the toroidal field-strength at the magnetic axis, B_0. All current densities are normalized to $B_0/(\mu_0\,R_0)$. All plasma pressures are normalized to B_0^2/μ_0.

Let R, ϕ, Z be right-handed cylindrical coordinates whose Jacobian is

$$(\nabla R\times\nabla\phi\cdot\nabla Z)^{-1} = R. \tag{1}$$

Note that $|\nabla\phi| = 1/R$.

Let r, θ, ϕ be right-handed flux-coordinates whose Jacobian is

$$\mathcal{J}(r,\theta)\equiv(\nabla r\times\nabla\theta\cdot\nabla\phi)^{-1}\equiv R\left(\frac{\partial R}{\partial\theta}\frac{\partial Z}{\partial r}-\frac{\partial R}{\partial r}\frac{\partial Z}{\partial\theta}\right) = r\,R^2. \tag{2}$$

Note that $r = r(R,Z)$ and $\theta=\theta(R,Z)$. The magnetic axis corresponds to $r=0$. The inboard mid-plane corresponds to $\theta = 0$.

Consider an axisymmetric tokamak equilibrium whose magnetic field takes the form

$$\mathbf{B}(r,\theta) = f(r)\,\nabla\phi\times\nabla r + g(r)\,\nabla\phi = f\,\nabla(\phi-q\,\theta)\times\nabla r, \tag{3}$$

where

$$q(r) = \frac{r\,g}{f} \tag{4}$$

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is the safety-factor (i.e., the inverse of the rotational transform). Note that $\mathbf{B}\cdot\nabla r = 0$, which implies that r is a magnetic flux-surface label. We require $g=1$ on the magnetic axis in order to ensure that the normalized toroidal magnetic field-strength at the axis is unity.

It is easily demonstrated that

$$B^{\,r} = \mathbf{B}\cdot\nabla r = 0, \tag{5}$$

$$B^{\,\theta} = \mathbf{B}\cdot\nabla\theta = \frac{f}{r\,R^2}, \tag{6}$$

$$B^{\,\phi} = \mathbf{B}\cdot\nabla\phi = \frac{g}{R^2}, \tag{7}$$

$$B_r = \mathcal{J}\,\nabla\theta\times\nabla\phi\cdot\mathbf{B} = -r\,f\,\nabla r\cdot\nabla\theta, \tag{8}$$

$$B_\theta = \mathcal{J}\,\nabla\phi\times\nabla r\cdot\mathbf{B} = r\,f\,|\nabla r|^2, \tag{9}$$

$$B_\phi = \mathcal{J}\,\nabla r\times\nabla\theta\cdot\mathbf{B} = g. \tag{10}$$

The Maxwell equation (neglecting the displacement current, because axisymmetric modes are comparatively low-frequency phenomena) $\mathbf{J} = \nabla\times\mathbf{B}$ yields

$$\mathcal{J}\,J^{\,r} = \frac{\partial B_\phi}{\partial\theta} = 0, \tag{11}$$

$$\mathcal{J}\,J^{\,\theta} = -\frac{\partial B_\phi}{\partial r} = -g', \tag{12}$$

$$\mathcal{J}\,J^{\,\phi} = \frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{\partial\theta} = \frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\,, \tag{13}$$

where $\mathbf{J}$ is the equilibrium current density, $'\equiv d/dr$, and use has been made of Eqs. (8)–(10).

Equilibrium force balance requires that

$$\nabla P = \mathbf{J}\times\mathbf{B}, \tag{14}$$

where $P(r)$ is the equilibrium scalar plasma pressure. Here, for the sake of simplicity, we have neglected the small centrifugal modifications to force balance due to subsonic plasma rotation. It follows that

$$P' = \mathcal{J}(J^{\,\theta}\,B^{\,\phi} - J^{\,\phi}\,B^{\,\theta}) = -g'\,\frac{g}{R^2} - \frac{f}{r\,R^2}\left[\frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\right], \tag{15}$$

where use has been made of Eqs. (5)–(7), and (11)–(13). The other two components of Eq. (14) are identically zero.

Equation (15) yields the *Grad-Shafranov equation*,

$$\frac{f}{r}\frac{\partial}{\partial r}\left(r\,f\,|\nabla r|^2\right)+\frac{f}{r}\frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)+g\,g'+R^2\,P'=0. \tag{16}$$

It follows from Eqs. (4), (13), and (16) that

$$\mathcal{J}\,J^{\phi}=-q\,g'-\frac{r\,R^2\,P'}{f}. \tag{17}$$

It is clear from Eqs. (12) and (17) that $g'=P'=0$ in the current-free "vacuum" region surrounding the plasma. We shall also assume that $g'=P'=0$ at the plasma-vacuum interface, so as to ensure that the equilibrium plasma current density is zero at the interface.

II. AXISYMMETRIC PLASMA PERTURBATION

Let us assume that all perturbed quantities are independent of the toroidal angle, ϕ. The perturbed plasma equilibrium satisfies the marginally-stable ideal-MHD equations

$$\mathbf{b}=\nabla\times(\boldsymbol{\xi}\times\mathbf{B}), \tag{18}$$

$$\nabla p=\mathbf{j}\times\mathbf{B}+\mathbf{J}\times\mathbf{b}, \tag{19}$$

$$\mathbf{j}=\nabla\times\mathbf{b}, \tag{20}$$

$$p=-\boldsymbol{\xi}\cdot\nabla P, \tag{21}$$

where $\boldsymbol{\xi}(r,\theta)$ is the plasma displacement, $\mathbf{b}(r,\theta)$ the perturbed magnetic field, $\mathbf{j}(r,\theta)$ the perturbed current density, and $p(r,\theta)$ the perturbed scalar pressure.

Now,

$$(\boldsymbol{\xi}\times\mathbf{B})_{\theta}=\mathcal{J}\,(\xi^{\phi}\,B^{r}-\xi^{r}\,B^{\phi})=-\mathcal{J}\,B^{\phi}\,\xi^{r}, \tag{22}$$

$$(\boldsymbol{\xi}\times\mathbf{B})_{\phi}=\mathcal{J}\,(\xi^{r}\,B^{\theta}-\xi^{\theta}\,B^{r})=\mathcal{J}\,B^{\theta}\,\xi^{r}, \tag{23}$$

where use has been made of the fact that $B^{r}=J^{r}=0$. [See Eqs. (5) and (11).] Combining Eqs. (18) and (23), we obtain

$$\mathcal{J}\,b^{r}=\frac{\partial}{\partial\theta}\left(\mathcal{J}\,B^{\theta}\,\xi^{r}\right). \tag{24}$$

Thus, Eqs. (2), (4), (6), and (7) give

$$r\,R^2\,b^r = \frac{\partial y}{\partial\theta}, \tag{25}$$

where

$$y(r,\theta) = f\,\xi^{\,r}. \tag{26}$$

The constraint $\nabla\cdot\mathbf{b}=0$, which follows from Eq. (18), immediately yields

$$r\,R^2\,b^\theta = -\frac{\partial y}{\partial r}. \tag{27}$$

According to Eq. (21),

$$p = -P'\,\nabla r\cdot\boldsymbol{\xi} = -P'\,\xi^{\,r}. \tag{28}$$

So, the perturbed force balance equation, (19), yields

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = (\mathbf{j}\times\mathbf{B})_r + (\mathbf{J}\times\mathbf{b})_r, \tag{29}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial\theta} = (\mathbf{j}\times\mathbf{B})_\theta + (\mathbf{J}\times\mathbf{b})_\theta, \tag{30}$$

$$0 = (\mathbf{j}\times\mathbf{B})_\phi + (\mathbf{J}\times\mathbf{b})_\phi, \tag{31}$$

giving

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = r\,R^2\,(j^{\,\theta}\,B^\phi - j^{\,\phi}\,B^\theta) + r\,R^2\,(J^\theta\,b^\phi - J^\phi\,b^\theta), \tag{32}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial\theta} = r\,R^2\,(j^{\,\phi}\,B^r - j^{\,r}\,B^\phi) + r\,R^2\,(J^\phi\,b^r - J^r\,b^\phi), \tag{33}$$

$$0 = r\,R^2\,(j^{\,r}\,B^\theta - j^{\,\theta}\,B^r) + r\,R^2\,(J^r\,b^\theta - J^\theta\,b^r), \tag{34}$$

where use has been made of Eq. (2). Thus, according to Eqs. (5)–(7), (11), (12), and (17),

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = f\,(q\,j^{\,\theta} - j^{\,\phi}) - g'\,b^\phi + \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)b^\theta, \tag{35}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial\theta} = -r\,g\,j^{\,r} - \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)b^r, \tag{36}$$

$$0 = f\,j^{\,r} + g'\,b^r. \tag{37}$$

It follows from Eqs. (25) and (37) that

$$r\,R^2\,j^{\,r} = -\alpha_g\,\frac{\partial y}{\partial\theta}, \tag{38}$$

where

$$\alpha_g(r) = \frac{g'}{f}. \tag{39}$$

Note that Eq. (36) is trivially satisfied. Hence, of the three components of the perturbed force balance equation, only Eq. (35) remains to be solved.

Equation (20) yields

$$r\,R^2\,j^{\,r} = \frac{\partial b_\phi}{\partial\theta}, \tag{40}$$

$$r\,R^2\,j^{\,\theta} = -\frac{\partial b_\phi}{\partial r}, \tag{41}$$

$$r\,R^2\,j^{\,\phi} = \frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial\theta}, \tag{42}$$

where use has been made of Eq. (2). It follows from Eqs. (38), (40), and (41) that

$$b_\phi = -\alpha_g\,y, \tag{43}$$

$$r\,R^2\,j^{\,\theta} = \frac{\partial(\alpha_g\,y)}{\partial r}. \tag{44}$$

Now,

$$\mathbf{b} = b_r\,\nabla r + b_\theta\,\nabla\theta + b_\phi\,\nabla\phi, \tag{45}$$

so

$$b^{\,r} = \mathbf{b}\cdot\nabla r = |\nabla r|^2\,b_r + (\nabla r\cdot\nabla\theta)\,b_\theta, \tag{46}$$

$$b^{\,\theta} = \mathbf{b}\cdot\nabla\theta = (\nabla r\cdot\nabla\theta)\,b_r + |\nabla\theta|^2\,b_\theta, \tag{47}$$

$$b^{\,\phi} = \mathbf{b}\cdot\nabla\phi = \frac{b_\phi}{R^2}. \tag{48}$$

Equations (46) and (47) can be rearranged to give

$$b_r = \left(\frac{1}{|\nabla r|^2}\right)b^{\,r} - \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right)b_\theta, \tag{49}$$

$$b^{\theta} = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right) b^{r} + \left[|\nabla\theta|^2 - \frac{(\nabla r\cdot\nabla\theta)^2}{|\nabla r|^2}\right] b_{\theta}. \tag{50}$$

But, from Eq. (2),

$$|\nabla r|^2\,|\nabla\theta|^2 - (\nabla r\cdot\nabla\theta)^2 = \frac{1}{r^2\,R^{\,2}}. \tag{51}$$

Thus, Eq. (50) reduces to

$$b^{\theta} = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right) b^{r} + \left(\frac{1}{r^2\,R^{\,2}\,|\nabla r|^2}\right) b_{\theta}. \tag{52}$$

Let

$$z = |\nabla r|^2\, r\,\frac{\partial y}{\partial r} + r\,\nabla r\cdot\nabla\theta\,\frac{\partial y}{\partial\theta}. \tag{53}$$

Equations (25), (27), (43), (49) and (52) yield

$$b_r = \frac{1}{r\,|\nabla r|^2\,R^{\,2}}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z, \tag{54}$$

$$b_\theta = -z, \tag{55}$$

$$b^{\,\phi} = -\frac{\alpha_g}{R^{\,2}}\,y. \tag{56}$$

Equations (42), (54), and (55) give

$$r\,R^{\,2}\,j^{\,\phi} = -\frac{\partial z}{\partial r} - \frac{\partial}{\partial\theta}\left[\frac{1}{r\,|\nabla r|^2\,R^{\,2}}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right]. \tag{57}$$

It follows from Eqs. (26), (27), (35), (44), (56), and (57) that

$$\begin{aligned} -\frac{\partial}{\partial r}\left(\frac{P'}{f}\,y\right) &= \frac{f\,q}{r\,R^{\,2}}\,\frac{\partial(\alpha_g\,y)}{\partial r} + \frac{f}{r\,R^{\,2}}\,\frac{\partial z}{\partial r} \\ &\quad + \frac{f}{r\,R^{\,2}}\,\frac{\partial}{\partial\theta}\left[\frac{1}{r\,|\nabla r|^2\,R^{\,2}}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right] \\ &\quad + \frac{g'\,\alpha_g}{R^{\,2}}\,y - \left(q\,g' + \frac{r\,R^{\,2}\,P'}{f}\right)\frac{1}{r\,R^{\,2}}\,\frac{\partial y}{\partial r}. \end{aligned} \tag{58}$$

Hence,

$$-\left[(\alpha_p\,\alpha_f + r\,\alpha_p')R^{\,2} + q\,r\,\alpha_g' + r^2\,\alpha_g^2\right] y = r\,\frac{\partial z}{\partial r} + \frac{\partial}{\partial\theta}\left[\frac{1}{|\nabla r|^2\,R^{\,2}}\,\frac{\partial y}{\partial\theta} + \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right], \tag{59}$$

where

$$\alpha_p(r) = \frac{r\,P'}{f^2}, \tag{60}$$

$$\alpha_f(r) = \frac{r^2}{f}\,\frac{d}{dr}\!\left(\frac{f}{r}\right). \tag{61}$$

Finally, it follows from Eqs. (53) and (59) that

$$r\,\frac{\partial y}{\partial r} = \frac{z}{|\nabla r|^2} - \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\frac{\partial y}{\partial\theta}, \tag{62}$$

$$r\,\frac{\partial z}{\partial r} = -\left[(\alpha_p\,\alpha_f + r\,\alpha_p')R^2 + q\,r\,\alpha_g' + r^2\,\alpha_g^{\,2}\right]y - \frac{\partial}{\partial\theta}\!\left(\frac{1}{|\nabla r|^2\,R^2}\,\frac{\partial y}{\partial\theta}\right) - \frac{\partial}{\partial\theta}\!\left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right). \tag{63}$$

The previous two equations can be combined to give

$$r\,\frac{\partial(z\,y^\ast)}{\partial r} = \frac{|z|^2}{|\nabla r|^2} - \frac{\partial y^\ast}{\partial\theta}\left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right) - \left[(\alpha_p\,\alpha_f + r\,\alpha_p')R^2 + q\,r\,\alpha_g' + r^2\,\alpha_g^{\,2}\right]|y|^2 - y^\ast\,\frac{\partial}{\partial\theta}\!\left(\frac{1}{|\nabla r|^2\,R^2}\,\frac{\partial y}{\partial\theta}\right) - y^\ast\,\frac{\partial}{\partial\theta}\!\left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right). \tag{64}$$

Hence,

$$r\,\frac{d}{dr}\!\left(\oint z\,y^\ast\,\frac{d\theta}{2\pi}\right) = \oint\left[\frac{|z|^2}{|\nabla r|^2} - \left[(\alpha_p\,\alpha_f + r\,\alpha_p')R^2 + q\,r\,\alpha_g' + r^2\,\alpha_g^{\,2}\right]|y|^2 + \frac{1}{|\nabla r|^2\,R^2}\left|\frac{\partial y}{\partial\theta}\right|^2\right]\frac{d\theta}{2\pi}, \tag{65}$$

which implies that $\oint z\,y^\ast\,d\theta/2\pi$ is a real quantity, and also that

$$r\,\frac{d}{dr}\left[\oint(z\,y^\ast - y^\ast\,z)\,\frac{d\theta}{2\pi}\right] = 0. \tag{66}$$

Let

$$y(r,\theta) = \sum_m y_m(r)\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{67}$$

$$z(r,\theta) = \sum_m z_m(r)\,{\rm e}^{\,{\rm i}\,m\,\theta}. \tag{68}$$

Equations (62) and (63) yield

$$r\,\frac{dy_m}{dr} = \sum_{m'}\left(A_m^{\,m'}\,z_{m'} + B_m^{\,m'}\,y_{m'}\right), \tag{69}$$

$$r\,\frac{dz_m}{dr} = \sum_{m'}\left(C_m^{m'}\,z_{m'} + D_m^{m'}\,y_{m'}\right), \tag{70}$$

where

$$A_m^{m'} = c_m^{m'}, \tag{71}$$

$$B_m^{m'} = -m'\,f_m^{m'}, \tag{72}$$

$$C_m^{m'} = -m\,f_m^{m'}, \tag{73}$$

$$D_m^{m'} = -(\alpha_f\,\alpha_p + r\,\alpha_p')\,a_m^{m'} - (q\,r\,\alpha_g' + r^{\,2}\,\alpha_g^2)\,\delta_m^{m'} + m\,m'\,b_m^{m'}, \tag{74}$$

where

$$a_m^{m'}(r) = \oint R^2\,\exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{75}$$

$$b_m^{m'}(r) = \oint |\nabla r|^{-2}\,R^{-2}\,\exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{76}$$

$$c_m^{m'}(r) = \oint |\nabla r|^{-2}\,\exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{77}$$

$$f_m^{m'}(r) = \oint \frac{\mathrm{i}\,r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}. \tag{78}$$

Note that $a_{m'}^{m} = a_m^{m'*}$, $b_{m'}^{m} = b_m^{m'*}$, $c_{m'}^{m} = c_m^{m'*}$, and $f_{m'}^{m} = -f_m^{m'*}$, which implies that

$$A_{m'}^{m} = A_{m'}^{m\,*}, \tag{79}$$

$$B_{m'}^{m} = -C_{m'}^{m\,*}, \tag{80}$$

$$C_{m'}^{m} = -B_{m'}^{m\,*}, \tag{81}$$

$$D_{m'}^{m} = A_{m'}^{m\,*}. \tag{82}$$

It follows from Eqs. (69), (79), and (79)–(82) that

$$r\,\frac{d}{dr}\left(\sum_m z_m\,y_m^*\right) = \sum_{m,m'}\left(z_m^*\,A_m^{m'}\,z_{m'} + y_m^*\,D_m^{m'}\,y_{m'}\right), \tag{83}$$

$$r\,\frac{d}{dr}\left[\sum_m (z_m\,y_m^* - y_m\,z_m^*)\right] = 0. \tag{84}$$

III. TOROIDAL ELECTROMAGNETIC TORQUE

The net toroidal electromagnetic torque exerted on the plasma lying within the magnetic flux-surface whose label is r is

$$T_\phi(r) = \oint\oint r\,R^2\,b_\phi\,b^r\,d\theta\,d\phi. \tag{85}$$

It follows from Eqs. (25) and (43) that

$$T_\phi(r) = -\pi\,\alpha_g \oint\left(y^\ast\,\frac{\partial y}{\partial\theta} + y\,\frac{\partial y^\ast}{\partial\theta}\right)d\theta = -\pi\,\alpha_g\oint\frac{\partial |y|^2}{\partial\theta}\,d\theta = 0. \tag{86}$$

We conclude that an axisymmetric perturbation is incapable of exerting a net toroidal electromagnetic torque on the plasma.

IV. PERTURBED PLASMA POTENTIAL ENERGY

The perturbed plasma potential energy in the region of the plasma lying within the magnetic flux-surface whose label is r is

$$\delta W_p = \frac{1}{2}\oint\oint r\,R^2\,\xi_r^\ast(-\mathbf{B}\cdot\mathbf{b} + \xi^{\,r}\,P')\,d\theta\,d\phi. \tag{87}$$

However,

$$\mathbf{B}\cdot\mathbf{b} - \xi^{\,r}\,P' = B^{\,\theta}\,b_\theta + B^{\,\phi}\,b_\phi - \xi^{\,r}\,P' = -\frac{f}{r\,R^2}\left(z + q\,\alpha_g\,y + \alpha_p\,R^2\right), \tag{88}$$

where use has been made of Eqs. (4), (6), (7), (26), (43), (55), and (60). Hence, we obtain

$$\delta W_p(r) = \frac{1}{2}\oint\oint y^\ast\left[z + (q\,\alpha_g + \alpha_p\,R^2)\,y\right]d\theta\,d\phi = \pi^2\sum_m y_m^\ast\,\chi_m, \tag{89}$$

where

$$\chi_m(r) = z_m + q\,\alpha_g\,y_m + \alpha_p\sum_{m'} a_m^{m'}\,y_{m'}. \tag{90}$$