

# Magnetic Perturbations

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## I. MAGNETIC PERTURBATIONS IN FLUX COORDINATES

In the  $r, \theta, \phi$  flux coordinate system (where all lengths are normalized to  $R_0$ , and all magnetic field-strengths to  $B_0$ ), the perturbed magnetic field is written

$$\mathbf{b} = b^r \mathcal{J} \nabla \theta \times \nabla \phi + b^\theta \mathcal{J} \nabla \phi \times \nabla r + b^\phi \mathcal{J} \nabla r \times \nabla \theta. \quad (1)$$

T7 Eqs. (25) and (49) yield

$$b^r = \frac{1}{r R^2} \left( \frac{\partial}{\partial \theta} - i n q \right) y, \quad (2)$$

$$b^\phi = \frac{x}{R^2}, \quad (3)$$

whereas TJ Eqs. (78), (79), (80), (98), and (99) imply that

$$y(r, \theta) = \sum_m \frac{\psi_m(r)}{m - n q(r)} e^{i m \theta}, \quad (4)$$

$$x(r, \theta) = n \sum_m \frac{Z_m(r) + k_m(r) \psi_m(r)}{m - n q(r)} e^{i m \theta}. \quad (5)$$

It follows that

$$R^2 b^r = \frac{i}{r} \sum_m \psi_m e^{i m \theta}, \quad (6)$$

$$R^2 b^\phi = n \sum_m z_m e^{i m \theta}, \quad (7)$$

where

$$z_m = \frac{Z_m + k_m \psi_m}{m - n q}. \quad (8)$$

Now, TJ Eqs. (2) and (A10) yield

$$\mathcal{J} \nabla \cdot \mathbf{b} = \frac{\partial}{\partial r} (r R^2 b^r) + \frac{\partial}{\partial \theta} (r R^2 b^\theta) - i n r R^2 b^\phi = 0, \quad (9)$$

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so

$$\frac{\partial}{\partial \theta} (r^2 R^2 b^\theta) = i \sum_m \left[ -r \frac{d\psi_m}{dr} + \frac{n^2 r^2 (Z_m + k_m \psi_m)}{m - n q} \right] e^{i m \theta}. \quad (10)$$

But, TJ Eq. (102) gives

$$r \frac{d\psi_m}{dr} = \sum_{m'} \frac{L_m^{m'} Z_{m'} + M_m^{m'} \psi_{m'}}{m' - n q}, \quad (11)$$

so

$$\frac{\partial}{\partial \theta} (r^2 R^2 b^\theta) = -i \sum_m \chi_m e^{i m \theta}, \quad (12)$$

where

$$\chi_m(r) = \sum_{m'} \frac{\chi_m^{m'}}{m' - n q}, \quad (13)$$

$$\chi_m^{m'}(r) = (L_m^{m'} - n^2 r^2 \delta_m^{m'}) Z_{m'} + (M_m^{m'} - n^2 r^2 \delta_m^{m'} k_{m'}) \psi_{m'} \quad (14)$$

Note from TJ Eqs. (100), (262), (263), (266), (267), (268), and (269) that  $\chi_0^{m'} = 0$  for all  $m'$ , which implies that  $\chi_0(r) = 0$ . Thus,

$$r R^2 b^\theta = -\frac{1}{r} \sum_{m \neq 0} \hat{\chi}_m e^{i m \theta}, \quad (15)$$

where

$$\hat{\chi}_m(r) = \frac{\chi_m}{m}. \quad (16)$$

According to Eqs. (4), (26), (79), and (98) of TJ,

$$\xi^r = \frac{q}{r g} \sum_m \hat{\psi}_m e^{i m \theta}, \quad (17)$$

where

$$\hat{\psi}_m(r) = \frac{\psi_m}{m - n q}. \quad (18)$$

## II. REGULARIZATION

In the vicinity of the  $k$ th rational surface, whose resonant poloidal mode number is  $m_k$ , the resonant harmonic of the displacement varies as

$$\xi_{m_k}^r(x) = \frac{q}{r g} \frac{\psi_{m_k} \Psi_k}{m_k - n q} \simeq - \left( \frac{q}{g s} \right)_{r_k} \frac{\Psi_k}{x} = - \frac{W_k^2}{16 x}, \quad (19)$$

where  $x = r - r_k$ ,  $\Psi_k$  is the reconnected magnetic flux at the  $k$ th rational surface, and

$$W_k = 4 \left( \frac{q}{g s} \right)_{r_k}^{1/2} \Psi_k^{1/2} \quad (20)$$

is the corresponding magnetic island width. Here, we have made use of the fact that  $\psi_{m_k}(r_k) \simeq m_k$ . We wish  $\xi_{m_k}^r(x)$  to be zero at  $x = 0$ , and to reach a maximum value of  $-W_k/\sqrt{8}$  at  $x = W_k/\sqrt{8}$ . The function

$$F(y) = \frac{y^3}{3/4 - (5/4)y^2 + y^4} \quad (21)$$

is zero at  $y = 0$ , attains its maximum value 2 at  $x = 1$ , and is approximately  $1/y$  for  $y \gg 1$ .

Thus, we can write

$$\xi_{m_k}^r(r) = \frac{q}{r g} \frac{\psi_{m_k} \Psi_k (m_k - n q)^3}{(3/4) \delta_k^4 - (5/4) \delta_k^2 (m_k - n q)^2 + (m_k - n q)^4}, \quad (22)$$

where

$$\delta_k = \frac{m_k s(r_k)}{\sqrt{8} \hat{r}_k} \frac{W_k}{\epsilon_a}. \quad (23)$$

Note that

$$\frac{\Psi_k}{\epsilon_a} = \epsilon_a \left( \frac{g s}{q} \right)_{r_k} \left( \frac{W_k}{4 \epsilon_a} \right)^2. \quad (24)$$

For the special case of  $m_k = 1$ , we expect

$$\Delta \Psi_k = E_{kk} \Psi_k = -\frac{r_k}{m_k} A_{Sk}, \quad (25)$$

so that

$$\Psi_k \psi_{m_k}(x) \simeq \frac{m_k E_{kk} \Psi_k}{r_k} x \quad (26)$$

for  $x < 0$ . Thus,

$$\xi_{m_k}^r \simeq - \left( \frac{q}{g s} \right)_{r_k} \frac{E_{kk} \Psi_k}{r_k} \quad (27)$$

for  $x < 0$ . Hence, we can write

$$\frac{\Psi_k}{\epsilon_a} = -\epsilon_a \left( \frac{\hat{r} g s}{q} \right)_{r_k} \frac{1}{E_{kk}} \left( \frac{\xi_c}{\epsilon_a} \right), \quad (28)$$

where  $\xi_c$  is the plasma displacement in the core. We can also write

$$\xi_{m_k}^r(r) = \frac{q}{r g} \frac{\psi_{m_k} \Psi_k (m_k - n q)}{\delta_k^2 + (m_k - n q)^2}, \quad (29)$$

where

$$\delta_k = \frac{m_k s(r_k)}{2 \hat{r}_k} \frac{W_k}{\epsilon_a}. \quad (30)$$

### III. MAGNETIC PERTURBATIONS IN CYLINDRICAL COORDINATES

We can write the perturbed magnetic field associated with the tearing mode that reconnects magnetic flux at the  $k$ th rational surface as

$$b^R \equiv \mathbf{b} \cdot \nabla R = b^r \frac{\partial R}{\partial r} + b^\theta \frac{\partial R}{\partial \theta}, \quad (31)$$

$$b^Z \equiv \mathbf{b} \cdot \nabla Z = b^r \frac{\partial Z}{\partial r} + b^\theta \frac{\partial Z}{\partial \theta}, \quad (32)$$

$$R b^\phi \equiv R \mathbf{b} \cdot \nabla \phi = R b^\phi. \quad (33)$$

It follows that

$$b^R = \frac{\Psi_k}{\epsilon_a \hat{r} R^2} \sum_m \text{Re} \left[ \left( \frac{1}{\epsilon_a} \frac{\partial R}{\partial \hat{r}} i \psi_m - \frac{1}{\epsilon_a \hat{r}} \frac{\partial R}{\partial \theta} \hat{\chi}_m \right) e^{i(m\theta - n\phi)} \right], \quad (34)$$

$$b^Z = \frac{\Psi_k}{\epsilon_a \hat{r} R^2} \sum_m \text{Re} \left[ \left( \frac{1}{\epsilon_a} \frac{\partial Z}{\partial \hat{r}} i \psi_m - \frac{1}{\epsilon_a \hat{r}} \frac{\partial Z}{\partial \theta} \hat{\chi}_m \right) e^{i(m\theta - n\phi)} \right], \quad (35)$$

$$R b^\phi = \frac{n \Psi_k}{R} \sum_m \text{Re} [z_m e^{i(m\theta - n\phi)}]. \quad (36)$$

Thus,

$$b^R(r, \theta, \phi) = b_C^R(r, \theta) \cos(n\phi) + b_S^R(r, \theta) \sin(n\phi), \quad (37)$$

$$b^Z(r, \theta, \phi) = b_C^Z(r, \theta) \cos(n\phi) + b_S^Z(r, \theta) \sin(n\phi), \quad (38)$$

$$R b^\phi(r, \theta, \phi) = b_C^\phi(r, \theta) \cos(n\phi) + b_S^\phi(r, \theta) \sin(n\phi), \quad (39)$$

$$\xi^r(r, \theta, \phi) = \xi_C^r(r, \theta) \cos(n\phi) + \xi_S^r \sin(n\phi), \quad (40)$$

where

$$b_C^R(r, \theta) = -\frac{\Psi_k}{\epsilon_a \hat{r} R^2} \left\{ \frac{1}{\epsilon_a} \frac{\partial R}{\partial \hat{r}} \sum_m [\text{Re}(\psi_m) \sin(m\theta) + \text{Im}(\psi_m) \cos(m\theta)] \right. \quad (41)$$

$$\left. + \frac{1}{\epsilon_a \hat{r}} \frac{\partial R}{\partial \theta} \sum_{m \neq 0} [\text{Re}(\hat{\chi}_m) \cos(m\theta) - \text{Im}(\hat{\chi}_m) \sin(m\theta)] \right\}, \quad (42)$$

$$b_S^R(r, \theta) = -\frac{\Psi_k}{\epsilon_a \hat{r} R^2} \left\{ \frac{1}{\epsilon_a} \frac{\partial R}{\partial \hat{r}} \sum_m [-\text{Re}(\psi_m) \cos(m\theta) + \text{Im}(\psi_m) \sin(m\theta)] \right. \quad (43)$$

$$\left. + \frac{1}{\epsilon_a \hat{r}} \frac{\partial R}{\partial \theta} \sum_{m \neq 0} [\text{Re}(\hat{\chi}_m) \sin(m\theta) + \text{Im}(\hat{\chi}_m) \cos(m\theta)] \right\}, \quad (44)$$

$$b_C^Z(r, \theta) = -\frac{\Psi_k}{\epsilon_a \hat{r} R^2} \left\{ \frac{1}{\epsilon_a} \frac{\partial Z}{\partial r} \sum_m [\operatorname{Re}(\psi_m) \sin(m\theta) + \operatorname{Im}(\psi_m) \cos(m\theta)] \right. \quad (45)$$

$$\left. + \frac{1}{\epsilon_a \hat{r}} \frac{\partial Z}{\partial \theta} \sum_{m \neq 0} [\operatorname{Re}(\hat{\chi}_m) \cos(m\theta) - \operatorname{Im}(\hat{\chi}_m) \sin(m\theta)] \right\}, \quad (46)$$

$$b_S^Z(r, \theta) = -\frac{\Psi_k}{\epsilon_a \hat{r} R^2} \left\{ \frac{1}{\epsilon_a} \frac{\partial Z}{\partial \hat{r}} \sum_m [-\operatorname{Re}(\psi_m) \cos(m\theta) + \operatorname{Im}(\psi_m) \sin(m\theta)] \right. \quad (47)$$

$$\left. + \frac{1}{\epsilon_a \hat{r}} \frac{\partial Z}{\partial \theta} \sum_{m \neq 0} [\operatorname{Re}(\hat{\chi}_m) \sin(m\theta) + \operatorname{Im}(\hat{\chi}_m) \cos(m\theta)] \right\}, \quad (48)$$

$$b_C^\phi(r, \theta) = \frac{n \Psi_k}{R} \sum_m [\operatorname{Re}(z_m) \cos(m\theta) - \operatorname{Im}(z_m) \sin(m\theta)], \quad (49)$$

$$b_S^\phi(r, \theta) = \frac{n \Psi_k}{R} \sum_m [\operatorname{Re}(z_m) \sin(m\theta) + \operatorname{Im}(z_m) \cos(m\theta)], \quad (50)$$

$$\xi_C^r(r, \theta) = \frac{\Psi_k q}{\epsilon_a \hat{r} g} \sum_m [\operatorname{Re}(\hat{\psi}_m) \cos(m\theta) - \operatorname{Im}(\hat{\psi}_m) \sin(m\theta)], \quad (51)$$

$$\xi_S^r(r, \theta) = \frac{\Psi_k q}{\epsilon_a \hat{r} g} \sum_m [\operatorname{Re}(\hat{\psi}_m) \sin(m\theta) + \operatorname{Im}(\hat{\psi}_m) \cos(m\theta)]. \quad (52)$$

#### IV. VALUE OF $k_m$

Now,

$$\begin{aligned} \tilde{k}_m = & -\frac{2-s}{m} - \frac{\epsilon_a^2}{m} \left( -\hat{r} p'_2 + \frac{3\hat{r}^2}{2} - 2\hat{r} H'_1 + S_2 \right) \\ & + \epsilon_a^2 \frac{(2-s)}{m} \left( -\frac{3\hat{r}^2}{4} + \frac{\hat{r}^2}{q^2} + H_1 + S_1 \right) \\ & + \epsilon_a^2 \frac{n\hat{r}}{m^2} \left[ -q p'_2 + \frac{\hat{r}}{m q} (2-s)(m-nq) \right]. \end{aligned} \quad (53)$$

But, for the special case  $m = 0$ ,

$$\begin{aligned} \tilde{k}_0 = & -\frac{q p'_2}{n \hat{r}} - \frac{2-s}{n q} \\ & - \frac{\epsilon_a^2}{n q} \left( \frac{3\hat{r}^2}{2} - 2\hat{r} H_1 + S_2 \right) \\ & + \epsilon_a^2 \frac{(2-s)}{n q} \left( -\frac{3\hat{r}^2}{4} + \frac{\hat{r}^2}{q^2} + H_1 + S_1 \right) \\ & + \epsilon_a^2 \frac{q p'_2}{n \hat{r}} \left( 2g_2 + \frac{\hat{r}^2}{2} + \frac{\hat{r}^2}{q^2} - 2H_1 - 3\hat{r} H'_1 \right). \end{aligned} \quad (54)$$

It turns out that

$$k_m = \tilde{k}_m - \tilde{k}_0 \tag{55}$$

for all  $m$ .