Pressure Flattening at Rational Surface

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I. FUNDAMENTAL EQUATION

Consider a rational surface whose resonant poloidal mode number is m, and that is located at the magnetic flux-surface $r = r_s$. Let $x = r - r_s$. The resonant harmonics of the perturbed magnetic field satisfy

$$x\,\frac{d\psi_m}{dx} = L_0\,Z_m, \tag{1}$$

$$x\,\frac{dZ_m}{dx} = P_0\,\psi_m + Z_m. \tag{2}$$

The previous equations can be combined to give

$$\frac{d^2\psi_m}{dx^2} = \frac{\nu\,(\nu+1)}{x^2}\,\psi_m, \tag{3}$$

where

$$\nu\,(\nu+1) = -D_I - \frac{1}{4} = L_0\,P_0 = \left[\frac{2\,r\,\mu_0\,p'\,(1-q^2)}{B_0^2\,s^2}\right]_{r_s}. \tag{4}$$

Let

$$\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} + L_0\,P_0} = -\frac{1}{2} + \sqrt{-D_I} \geq -\frac{1}{2}. \tag{5}$$

It follows that $\nu_L = -\nu$ and $\nu_S = 1 + \nu$. In the limit that $|L_0\,P_0| \ll 1$,

$$\nu \simeq L_0\,P_0. \tag{6}$$

The most general tearing parity solution of Eq. (3) is

$$\psi(x) = A_L\,|x|^{-\nu} + A_S\,\mathrm{sgn}(x)\,|x|^{1+\nu}. \tag{7}$$

Making the definitions

$$\Psi = r_s^{-\nu}\left(\frac{1+2\,\nu}{L_m^m}\right)^{1/2} A_L, \tag{8}$$

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$$\mathit{\Delta}\Psi = r_s^{1+\nu}\left(\frac{1+2\,\nu}{L_m^m}\right)^{1/2} 2\,A_S, \tag{9}$$

$$\mathit{\Delta}_{\rm out} = \frac{\mathit{\Delta}\Psi}{\Psi}, \tag{10}$$

it is clear that

$$\mathit{\Delta}_{\rm out} = r_s^{1+2\,\nu}\,\frac{2\,A_S}{A_L}. \tag{11}$$

II. PRESSURE FLATTENING

Suppose that the pressure gradient is locally flattened in the vicinity of the rational surface in such a manner that

$$p'(x) = p'_{\rm out}\,\frac{x^2}{x^2+\delta^2}. \tag{12}$$

Here, it is assumed that $\delta \ll r_s$. Equation (3) becomes

$$\frac{d^2\psi_m}{dx^2} = \frac{\nu\,(\nu+1)}{x^2+\delta^2}\,\psi_m, \tag{13}$$

where

$$\nu\,(\nu+1) = \left[\frac{2\,r\,\mu_0\,p'_{\rm out}\,(1-q^2)}{B_0^2\,s^2}\right]_{r_s}. \tag{14}$$

Let $X = x/\delta$. Equation (13) becomes

$$(1+X^2)\,\frac{d^2\psi_m}{dX^2} = \nu\,(\nu+1)\,\psi_m \tag{15}$$

Consider the small-$|X|$ behavior of the solution to Eq. (15). If we write

$$\psi_m(X) = \sum_{m=0,2,4} a_m\,X^{\mu+m} \tag{16}$$

then we obtain

$$a_0\,\mu\,(\mu-1) = 0, \tag{17}$$

$$a_2 = \frac{\nu\,(\nu+1)-\mu\,(\mu-1)}{(\mu+2)\,(\mu+1)}\,a_0. \tag{18}$$

The solutions are $\mu = 0$ with

$$a_2 = \frac{\nu\,(1+\nu)}{2}\,a_0 \tag{19}$$

and $\mu = 1$ with

$$a_2 = \frac{\nu\,(1+\nu)}{6}\,a_0. \tag{20}$$

It follows that the most general tearing parity small-$|X|$ solution takes the form

$$\psi_m(X) \simeq \hat{A}_{L\,\rm in}\left[1+\frac{\nu\,(1+\nu)}{2}\,X^2\right]+\hat{A}_{S\,\rm in}\,|X|\left[1+\frac{\nu\,(1+\nu)}{6}\,X^2\right]. \tag{21}$$

We can define

$$\varDelta_{\rm in} = \left(\frac{r_s}{\delta}\right)\frac{2\,\hat{A}_{S\,\rm in}}{\hat{A}_{L\,\rm in}}. \tag{22}$$

Consider the large-$|X|$ behavior of the solution to Eq. (15). Let $Y=1/X$. Equation (15) transforms into

$$(1+Y^2)\,\frac{d}{dY}\left(Y^2\,\frac{d\psi_m}{dY}\right) = \nu\,(\nu+1)\,\psi_m. \tag{23}$$

If we write

$$\psi_m(Y) = \sum_{m=0,2,4} a_m\,Y^{\mu+m} \tag{24}$$

then we obtain

$$a_0\left[\mu\,(\mu+1)-\nu\,(\nu+1)\right] = 0, \tag{25}$$

$$a_2 = -\frac{\mu\,(\mu+1)}{(\mu+2)\,(\mu+3)-\nu\,(1+\nu)}\,a_0. \tag{26}$$

The solutions are $\mu=\nu$ with

$$a_2 = -\frac{\nu\,(1+\nu)}{2\,(3+2\,\nu)}\,a_0 \tag{27}$$

and $\mu=-1-\nu$ with

$$a_2 = -\frac{\nu\,(1+\nu)}{2\,(1-2\,\nu)}\,a_0. \tag{28}$$

It follows that the most general tearing parity large-$|X|$ solution takes the form

$$\psi_m(X) = \hat{A}_{L\,\rm out}\,|X|^{-\nu}\left[1-\frac{\nu\,(1+\nu)}{2\,(3+2\,\nu)}\,\frac{1}{X^2}\right] + \hat{A}_{S\,\rm out}\,{\rm sgn}(X)\,|X|^{1+\nu}\left[1-\frac{\nu\,(1+\nu)}{2\,(1-2\,\nu)}\,\frac{1}{X^2}\right] \tag{29}$$

By analogy with Eq. (11), we can write

$$\varDelta_{\rm out} = \left(\frac{r_s}{\delta}\right)^{1+2\,\nu}\frac{2\,\hat{A}_{S\,\rm out}}{\hat{A}_{L\,\rm out}}. \tag{30}$$

III. CONNECTION FORMULAE

Suppose that we launch the ‘large’ solution

$$\psi_L(X) = X^{-\nu}\left[1-\frac{\nu\,(1+\nu)}{2\,(3+2\,\nu)}\,\frac{1}{X^2}\right] \tag{31}$$

from large X and integrate to $X=0$. Suppose that

$$\psi_L(0) = a_{LL}, \tag{32}$$

$$\frac{d\psi_L(0)}{dx} = a_{LS}. \tag{33}$$

Suppose that we launch the 'small' solution

$$\psi_S(X) = X^{1+\nu}\left[1-\frac{\nu\,(1+\nu)}{2\,(1-2\,\nu)}\,\frac{1}{X^2}\right] \tag{34}$$

from large X and integrate to $X=0$. Suppose that

$$\psi_L(0) = a_{SL}, \tag{35}$$

$$\frac{d\psi_L(0)}{dx} = a_{SS}. \tag{36}$$

The most general solution is

$$\psi_m(X) = \hat{A}_{L\,\rm out}\,\psi_L(X) + \hat{A}_{S\,\rm out}\,\psi_S(X). \tag{37}$$

It follows that

$$\hat{A}_{L\,\rm in} = a_{LL}\,\hat{A}_{L\,\rm out} + a_{LS}\,\hat{A}_{S\,\rm out}, \tag{38}$$

$$\hat{A}_{S\,\rm in} = a_{SL}\,\hat{A}_{L\,\rm out} + a_{SS}\,\hat{A}_{S\,\rm out}. \tag{39}$$

It follows from Eqs. (11) and (22) that

$$\left(\frac{\delta}{r_s}\right)\frac{\mathit{\Delta}_{\rm in}}{2} = \frac{a_{SL} + a_{SS}\,(\delta/r_s)^{1+2\,\nu}\,(\mathit{\Delta}_{\rm out}/2)}{a_{LL} + a_{LS}\,(\delta/r_s)^{1+2\,\nu}\,(\mathit{\Delta}_{\rm out}/2)}. \tag{40}$$

IV. ANALYTIC SOLUTION

Let $\psi_m = (1+X^2)\,\phi$. Equation (15) transforms to give

$$(1+X^2)\,\frac{d^2\phi}{dX^2} + 4\,X\,\frac{d\phi}{dX} + [2-\nu\,(\nu+1)]\,\phi = 0. \tag{41}$$

Let $z = {\rm i}\,X$. The previous equation becomes

$$(1-z^2)\,\frac{d^2\phi}{dz^2} - 4\,z\,\frac{d\phi}{dz} - [2-\nu\,(\nu+1)]\,\phi = 0. \tag{42}$$

Now, $Q_\nu(z)$ and $Q_{-1-\nu}(z)$ satisfy

$$(1-z^2)\,\frac{d^2w}{dz^2} - 2\,z\,\frac{dw}{dz} + \nu\,(\nu+1)\,w = 0. \tag{43}$$

Let $w' = dw/dz$. Differentiation of the previous equation yields

$$(1-z^2)\,\frac{d^2w'}{dz^2} - 2\,z\,\frac{dw'}{dz} - [2-\nu\,(\nu+1)]\,w' = 0. \tag{44}$$

It is clear from a comparison of Eqs. (42) and (44) that the two independent solutions of Eq. (15) can be written $(1+X^2)\,Q_\nu'(\mathrm{i}\,X)$ and $(1+X^2)\,Q_{-1-\nu}'(\mathrm{i}\,X)$, where $'$ denotes differentiation with respect to argument.

Now [Erdelyi, p. 134, (40)],

$$\begin{aligned} Q_\nu(z) &= \frac{\pi^{1/2}\,\Gamma(1/2+\nu/2)\,\mathrm{e}^{-\mathrm{i}\,(\pi/2)\,(1+\nu)}}{2\,\Gamma(1+\nu/2)}\,F\left(-\frac{\nu}{2},\frac{1}{2}+\frac{\nu}{2};\frac{1}{2};z^2\right)\\ &\quad+\frac{\pi^{1/2}\,\Gamma(1+\nu/2)\,\mathrm{e}^{-\mathrm{i}\,(\pi/2)\,\nu}}{\Gamma(1/2+\nu/2)}\,z\,F\left(\frac{1}{2}-\frac{\nu}{2},1+\frac{\nu}{2};\frac{3}{2};z^2\right), \end{aligned} \tag{45}$$

which yields

$$\begin{aligned} Q_\nu(z) &\simeq \frac{\pi^{1/2}\,\Gamma(1/2+\nu/2)\,\mathrm{e}^{-\mathrm{i}\,(\pi/2)\,(1+\nu)}}{2\,\Gamma(1+\nu/2)}\left[1-\frac{\nu\,(1+\nu)\,z^2}{2}\right]\\ &\quad+\frac{\pi^{1/2}\,\Gamma(1+\nu/2)\,\mathrm{e}^{-\mathrm{i}\,(\pi/2)\,\nu}}{\Gamma(1/2+\nu/2)}\,z, \end{aligned} \tag{46}$$

and

$$\begin{aligned} Q_\nu'(z) &\simeq -\frac{\pi^{1/2}\,\nu\,(1+\nu)\,\Gamma(1/2+\nu/2)\,\mathrm{e}^{-\mathrm{i}\,(\pi/2)\,(1+\nu)}}{2\,\Gamma(1+\nu/2)}\,z\\ &\quad+\frac{\pi^{1/2}\,\Gamma(1+\nu/2)\,\mathrm{e}^{-\mathrm{i}\,(\pi/2)\,\nu}}{\Gamma(1/2+\nu/2)}. \end{aligned} \tag{47}$$

So at small-X,

$$\begin{aligned} (1+X^2)\,Q_\nu'(\mathrm{i}\,X) = \mathrm{e}^{-\mathrm{i}\,\nu\,\pi/2}\,&\left[\frac{\pi^{1/2}\,\Gamma(1+\nu/2)}{\Gamma(1/2+\nu/2)}\right.\\ &\left.-\frac{\pi^{1/2}\,\nu\,(1+\nu)\,\Gamma(1/2+\nu/2)}{2\,\Gamma(1+\nu/2)}\,X\right] \end{aligned} \tag{48}$$

Furthermore, [Erdelyi, p. 134, (41)],

$$Q_\nu(z) = \frac{\pi^{1/2}\,\Gamma(1+\nu)}{2^{1+\nu}\,\Gamma(3/2+\nu)}\,z^{-1-\nu}\,F\left(1+\frac{\nu}{2},\frac{1}{2}+\frac{\nu}{2};\frac{3}{2}+\nu;\frac{1}{z^2}\right), \tag{49}$$

which yields

$$Q_\nu'(z) \simeq -\frac{\pi^{1/2}\,(1+\nu)\,\Gamma(1+\nu)}{2^{1+\nu}\,\Gamma(3/2+\nu)}\,z^{-2-\nu} \tag{50}$$

So, at large-X,

$$(1+X^2)\,Q'_\nu(\mathrm{i}\,X) \simeq \mathrm{e}^{-\mathrm{i}\,\nu\,\pi/2}\,\frac{\pi^{1/2}\,(1+\nu)\,\Gamma(1+\nu)}{2^{1+\nu}\,\Gamma(3/2+\nu)}\,X^{-\nu}. \tag{51}$$

It is clear from a comparison of Eqs. (21), (29), (38), (39), (48), and (51) that

$$\hat{A}_{L\,\mathrm{in}} = a_{LL}\,\hat{A}_{L\,\mathrm{out}}, \tag{52}$$

$$\hat{A}_{S\,\mathrm{in}} = a_{SL}\,\hat{A}_{L\,\mathrm{out}}, \tag{53}$$

where

$$a_{LL} = \frac{2^{1+\nu}\,\Gamma(1+\nu/2)\,\Gamma(3/2+\nu)}{(1+\nu)\,\Gamma(1/2+\nu/2)\,\Gamma(1+\nu)}, \tag{54}$$

$$a_{SL} = -\frac{2^{\nu}\,\nu\,\Gamma(1/2+\nu/2)\,\Gamma(3/2+\nu)}{\Gamma(1+\nu/2)\,\Gamma(1+\nu)}. \tag{55}$$

In the limit $\nu\rightarrow 0$,

$$a_{LL} \rightarrow 1, \tag{56}$$

$$a_{SL} \rightarrow -\frac{\nu\,\pi}{2}. \tag{57}$$

Now, according to Eq. (48),

$$(1+X^2)\,Q'_{-1-\nu}(\mathrm{i}\,X) = \mathrm{e}^{\,\mathrm{i}\,(1+\nu)\,\pi/2}\left[\frac{\pi^{1/2}\,\Gamma(1/2-\nu/2)}{\Gamma(-\nu/2)} - \frac{\pi^{1/2}\,\nu\,(1+\nu)\,\Gamma(-\nu/2)}{2\,\Gamma(1/2-\nu/2)}\,X\right]. \tag{58}$$

Furthermore, according to Eq. (51),

$$(1+X^2)\,Q'_{-1-\nu}(\mathrm{i}\,X) \simeq -\mathrm{e}^{\,\mathrm{i}\,(1+\nu)\,\pi/2}\,\frac{\pi^{1/2}\,\nu\,\Gamma(-\nu)}{2^{-\nu}\,\Gamma(1/2-\nu)}\,X^{1+\nu}. \tag{59}$$

A comparison of Eqs. (21), (29), (38), (39), (58), and (59) yields

$$\hat{A}_{L\,\mathrm{in}} = a_{LS}\,\hat{A}_{S\,\mathrm{out}}, \tag{60}$$

$$\hat{A}_{S\,\mathrm{in}} = a_{SS}\,\hat{A}_{S\,\mathrm{out}}, \tag{61}$$

where

$$a_{LS} = -\frac{\Gamma(1/2-\nu/2)\,\Gamma(1/2-\nu)}{2^{\nu}\,\nu\,\Gamma(-\nu)\,\Gamma(-\nu/2)} = -\frac{\nu\,\Gamma(1/2-\nu/2)\,\Gamma(1/2-\nu)}{2^{1+\nu}\,\Gamma(1-\nu)\,\Gamma(1-\nu/2)}, \tag{62}$$

$$a_{SS} = \frac{(1+\nu)\,\Gamma(-\nu/2)\,\Gamma(1/2-\nu)}{2^{1+\nu}\,\Gamma(1/2-\nu/2)\,\Gamma(-\nu)} = \frac{(1+\nu)\,\Gamma(1-\nu/2)\,\Gamma(1/2-\nu)}{2^{\nu}\,\Gamma(1/2-\nu/2)\,\Gamma(1-\nu)}. \tag{63}$$

In the limit $\nu \to 0$,

$$a_{LS} \to -\frac{\nu\,\pi}{2}, \tag{64}$$

$$a_{SS} \to 1. \tag{65}$$