# Separatrix

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## I. SAFETY-FACTOR

Let $\hat{r} = r/a$. Suppose that the last closed magnetic flux surface is at $\hat{r} = 1$. Suppose that the safety-factor profile takes the form

$$q(\hat{r}) = q_0 + (q_{95} - q_0)\,\frac{\ln(1-\hat{r}^{\,2})}{\ln(1-\hat{r}_{95}^{\,2})}. \tag{1}$$

## II. POLOIDAL MAGNETIC FLUX

Now,

$$\frac{d\Psi_p}{dr} = B_0\,R_0\,f = B_0\,a\,\frac{\hat{r}}{q}, \tag{2}$$

assuming that $g(\hat{r}) = 1$. Let $\psi_p = \Psi_p/(B_0\,a^2)$. It follows that

$$\frac{d\psi_p}{d\hat{r}} = \frac{\hat{r}}{q} = \frac{\hat{r}}{q_0 - \alpha\,\ln(1-\hat{r}^{\,2})}, \tag{3}$$

where

$$\alpha = -\frac{q_{95} - q_0}{\ln(1-\hat{r}_{95}^{\,2})}. \tag{4}$$

So,

$$\begin{aligned}
\psi_p(\hat{r}) &= \int_0^{\hat{r}} \frac{d\hat{\psi}_p}{d\hat{r}}\,d\hat{r} = \frac{1}{2}\int_0^{\hat{r}^{\,2}} \frac{dx}{q_0 - \alpha\,\ln(1-x)} = \frac{1}{2}\int_{1-\hat{r}^{\,2}}^{1} \frac{dy}{q_0 - \alpha\,\ln y}\\
&= \frac{\mathrm{e}^{\,q_0/\alpha}}{2\,\alpha}\int_{q_0/\alpha}^{q_0/\alpha-\ln(1-\hat{r}^{\,2})} \frac{\mathrm{e}^{-z}\,dz}{z} = \frac{\mathrm{e}^{\,q_0/\alpha}}{2\,\alpha}\left[\int_{q_0/\alpha}^{\infty}\frac{\mathrm{e}^{-z}\,dz}{z} - \int_{q_0/\alpha-\ln(1-\hat{r}^{\,2})}^{\infty}\frac{\mathrm{e}^{-z}\,dz}{z}\right]\\
&= \frac{\mathrm{e}^{\,q_0/\alpha}}{2\,\alpha}\left[\int_1^{\infty}\frac{\mathrm{e}^{-(q_0/\alpha)\,t}\,dt}{t} - \int_1^{\infty}\frac{\mathrm{e}^{-[q_0/\alpha-\ln(1-\hat{r}^{\,2})]\,t}\,dt}{t}\right],
\end{aligned} \tag{5}$$

giving

$$\psi_p(\hat{r}) = \frac{\mathrm{e}^{\,q_0/\alpha}}{2\,\alpha}\left\{E_1(q_0/\alpha) - E_1[q_0/\alpha - \ln(1-\hat{r}^{\,2})]\right\}. \tag{6}$$

Thus,

$$\psi_p(1) = \frac{\mathrm{e}^{\,q_0/\alpha}}{2\,\alpha}\,E_1(q_0/\alpha). \tag{7}$$

Let $\hat{\psi}_p(\hat{r}) = \psi_p(\hat{r})/\psi_p(1)$. It follows that

$$\hat{\psi}_p(\hat{r}) = 1 - \frac{E_1[q_0/\alpha - \ln(1-\hat{r}^{\,2})]}{E_1(q_0/\alpha)}. \tag{8}$$

By definition, $\hat{\psi}_p(\hat{r}_{95}) = 0.95$, so

$$0.95 = 1 - \frac{E_1[q_0/\alpha - \ln(1-\hat{r}_{95}^{\,2})]}{E_1(q_0/\alpha)} = 1 - \frac{E_1(q_{95}/\alpha)}{E_1(q_0/\alpha)}, \tag{9}$$

giving

$$0.05 = \frac{E_1(q_{95}/\alpha)}{E_1(q_0/\alpha)}. \tag{10}$$

Assuming that $q_0$ and $q_{95}$ are specified, the previous equation can be solved to give $\alpha$, which then determines $\hat{r}_{95}$.

## III. RATIONAL SURFACES

Suppose that $n$ is the toroidal mode number. The $m$, $n$ rational surface lies at $\hat{r} = \hat{r}_m$, where $q(\hat{r}_m) = q_m$ and $q_m = m/n$. Thus,

$$\hat{r}_m = \left[1 - \exp\left(\frac{q_0 - q_m}{\alpha}\right)\right]^{1/2}. \tag{11}$$

Hence, if $\hat{\psi}_m = \hat{\psi}_p(\hat{r}_m)$ then

$$\hat{\psi}_m = 1 - \frac{E_1(q_m/\alpha)}{E_1(q_0/\alpha)}. \tag{12}$$

The spacing (in $\hat{r}$) between successive rational surfaces is

$$\delta_m = \frac{d\hat{r}_m}{dm} = \frac{1-\hat{r}_m^{\,2}}{2\,\hat{r}_m\,n\,\alpha}. \tag{13}$$

In the continuum limit, assuming that $\hat{r}_m \simeq 1$, we get

$$\hat{\delta}_{\rm rational}(\hat{r}) \simeq \frac{1-\hat{r}}{n\,\alpha} = \frac{x}{n\,\alpha}, \tag{14}$$

where $x = 1 - \hat{r}$.

## IV. MAGNETIC SHEAR

The magnetic shear is

$$s(\hat{r}) = \frac{\hat{r}}{q}\,\frac{dq}{d\hat{r}} = \frac{2\,\alpha\,\hat{r}^{\,2}}{(1-\hat{r}^{\,2})\,q} \simeq \frac{1}{f(x)}, \tag{15}$$

where

$$f(x) = -x\,\ln(2\,x). \tag{16}$$

## V. PROFILES

Let

$$n_e(\hat{\psi}_p) = 20\,n_{e\,95}\,(1-\hat{\psi}_p) + n_{e\,100}, \tag{17}$$

$$T_e(\hat{\psi}_p) = 20\,T_{e\,95}\,(1-\hat{\psi}_p) + T_{e\,100}, \tag{18}$$

$$T_i(\hat{\psi}_p) = 20\,T_{i\,95}\,(1-\hat{\psi}_p) + T_{i\,100}. \tag{19}$$

Thus,

$$\begin{aligned}
\omega_{*\,e}(\hat{\psi}_p) &= \frac{T_e}{e}\left(\frac{d\ln n_e}{d\Psi_p} + \frac{d\ln T_e}{d\Psi_p}\right) = \frac{T_e}{e\,B_0\,a^2\,\psi_p(1)}\left(\frac{d\ln n_e}{d\hat{\psi}_p} + \frac{d\ln T_e}{d\hat{\psi}_p}\right)\\
&\simeq -\frac{20\,T_{e\,100}}{e\,B_0\,a^2\,\psi_p(1)}\left(\frac{n_{e\,95}}{n_{e\,100}} + \frac{T_{e\,95}}{T_{e\,100}}\right), \qquad (20)\\
\omega_{*\,i}(\hat{\psi}_p) &= -\frac{T_i}{e}\left(\frac{d\ln n_e}{d\Psi_p} + \frac{d\ln T_i}{d\Psi_p}\right) = -\frac{T_i}{e\,B_0\,a^2\,\psi_p(1)}\left(\frac{d\ln n_e}{d\hat{\psi}_p} + \frac{d\ln T_i}{d\hat{\psi}_p}\right)\\
&\simeq -\frac{20\,T_{i\,100}}{e\,B_0\,a^2\,\psi_p(1)}\left(\frac{n_{e\,95}}{n_{e\,100}} + \frac{T_{i\,95}}{T_{i\,100}}\right). \qquad (21)
\end{aligned}$$

## VI. LAYER QUANTITIES

Now,

$$\tau_{ee}(\hat{\psi}_p) = \frac{6\sqrt{2}\,\pi^{3/2}\,\epsilon_0^2\,m_e^{1/2}\,T_e^{3/2}}{\ln\Lambda\,e^4\,n_e} \simeq \frac{6\sqrt{2}\,\pi^{3/2}\,\epsilon_0^2\,m_e^{1/2}\,T_{e\,100}^{3/2}}{\ln\Lambda\,e^4\,n_{e\,100}}, \tag{22}$$

$$\sigma_\parallel = 1.96\,\frac{n_e\,e^2\,\tau_{ee}}{m_e}, \tag{23}$$

$$\tau_R = \mu_0\,a^2\,\sigma_\parallel\,\hat{r}^{\,2} \simeq \mu_0\,a^2\,\sigma_\parallel, \tag{24}$$

$$\tau_H = \frac{R_0}{B_0}\,\frac{\sqrt{\mu_0\,m_i\,n_e}}{n\,s} \simeq \tau_A\,f(x), \tag{25}$$

$$\tau_A = \frac{R_0}{B_0}\,\frac{\sqrt{\mu_0\,m_i\,n_{e\,100}}}{n}, \tag{26}$$

$$\tau_\perp = \frac{a^2\,\hat{r}^{\,2}}{D_\perp} \simeq \frac{a^2}{D_\perp}, \tag{27}$$

$$\tau_\varphi = \frac{a^2\,\hat{r}^{\,2}}{\chi_\varphi} \simeq \frac{a^2}{\chi_\varphi}, \tag{28}$$

$$\tau = -\frac{\omega_{\ast\,e}}{\omega_{\ast\,i}}, \tag{29}$$

$$d_\beta = \frac{\sqrt{(5/3)\,m_i\,(T_e+T_i)}}{e\,B_0} \simeq \frac{\sqrt{(5/3)\,m_i\,(T_{e\,100}+T_{i\,100})}}{e\,B_0}, \tag{30}$$

$$S = \frac{\tau_R}{\tau_H} = \frac{\mathcal{S}}{f(x)}, \tag{31}$$

$$\mathcal{S} = \frac{\tau_R}{\tau_A}, \tag{32}$$

$$P_\varphi = \frac{\tau_R}{\tau_\varphi}, \tag{33}$$

$$P_\perp = \frac{\tau_R}{\tau_\perp}, \tag{34}$$

$$D = S^{1/3}\left(\frac{\tau}{1+\tau}\right)^{1/2}\,\frac{d_\beta}{a\,\hat{r}} \simeq \mathcal{D}\,[f(x)]^{-1/3}, \tag{35}$$

$$\mathcal{D} = \mathcal{S}^{1/3}\left(\frac{\tau}{1+\tau}\right)^{1/2}\,\frac{d_\beta}{a}, \tag{36}$$

$$Q_E = -S^{1/3}\,n\,\omega_E\,\tau_H \simeq \mathcal{Q}_E\,[f(x)]^{2/3}, \tag{37}$$

$$\mathcal{Q}_E = -\mathcal{S}^{1/3}\,n\,\omega_E\,\tau_A, \tag{38}$$

$$\mathcal{Q}_{e,i} = -S^{1/3}\,n\,\omega_{\ast\,e,i}\,\tau_H \simeq \mathcal{Q}_{\ast\,e,i}\,[f(x)]^{2/3}, \tag{39}$$

$$\mathcal{Q}_{e\,i} = -\mathcal{S}^{1/3}\,n\,\omega_{\ast\,e\,i}\,\tau_A, \tag{40}$$

$$Q = S^{1/3}\,\omega\,\tau_H = \mathcal{Q}\,[f(x)]^{2/3}, \tag{41}$$

$$\mathcal{Q} = \mathcal{S}^{1/3}\,\omega\,\tau_A. \tag{42}$$

## VII. LAYER EQUATION

The layer equation is written

$$\frac{d}{dp}\left[A(p)\,\frac{dY_e}{dp}\right] - \frac{B(p)}{C(p)}\,p^2\,Y_e = 0, \tag{43}$$

where

$$A = \frac{p^2}{-\mathrm{i}\,(Q - Q_E - Q_e) + p^2}, \tag{44}$$

$$B = -\mathrm{i}\,(Q - Q_E)\,(Q - Q_E - Q_i) - \mathrm{i}\,(Q - Q_E - Q_i)\,(P_\varphi + P_\perp)\,p^2 + P_\varphi\,P_\perp\,p^4, \tag{45}$$

$$C = -\mathrm{i}\,(Q - Q_E - Q_e) + [P_\perp - \mathrm{i}\,(Q - Q_E - Q_i)\,D^{\,2}]\,p^2 + (1 + 1/\tau)\,P_\varphi\,D^{\,2}\,p^4. \tag{46}$$

Let

$$p = f^{\,-1/9}\,\hat{p}. \tag{47}$$

It follows that

$$\frac{d}{d\hat{p}}\left[\mathcal{A}\,\frac{dY_e}{d\hat{p}}\right] - \frac{\mathcal{B}}{\mathcal{C}}\,\hat{p}^{\,2}\,Y_e = 0, \tag{48}$$

where

$$\mathcal{A} = \frac{\hat{p}^{\,2}}{-\mathrm{i}\,(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_e)\,f^{\,8/9} + \hat{p}^{\,2}}, \tag{49}$$

$$\mathcal{B} = -\mathrm{i}\,(\mathcal{Q} - \mathcal{Q}_E)\,(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_i)\,f^{\,16/9} - \mathrm{i}\,(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_i)\,(P_\varphi + P_\perp)\,\hat{p}^{\,2}\,f^{\,8/9} + P_\varphi\,P_\perp\,\hat{p}^{\,4}, \tag{50}$$

$$\mathcal{C} = -\mathrm{i}\,(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_e)\,f^{\,12/9} + [P_\perp - \mathrm{i}\,(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_i)\,\mathcal{D}^{2}]\,\hat{p}^{\,2}\,f^{\,8/9} + (1 + 1/\tau)\,P_\varphi\,\mathcal{D}^{\,2}\,\hat{p}^{\,4}. \tag{51}$$

Thus, Eq. (48) reduces to

$$\frac{d^2\,Y_e}{d\hat{p}^{\,2}} - \frac{P_\perp}{(1 + 1/\tau)\,\mathcal{D}^{\,2}}\,\hat{p}^{\,2}\,Y_e \simeq= 0. \tag{52}$$

The characteristic layer width in $\hat{p}$ space is

$$\hat{p}_\ast = \left[\frac{\mathcal{D}^{\,2}\,(1 + 1/\tau)}{P_\perp}\right]^{1/4}. \tag{53}$$

Thus, the characteristic layer width in $\hat{r}$ space is

$$\hat{\delta}_{\rm layer} = \frac{S^{-1/3}}{f^{-1/9}\,\hat{p}_\ast} = \mathcal{S}^{-1/3}\left[\frac{P_\perp}{\mathcal{D}^{\,2}\,(1+1/\tau)}\right]^{1/4} f^{-2/9} = \mathit{\Delta}_{\rm layer}\, f^{-2/9}, \tag{54}$$

where

$$\mathit{\Delta}_{\rm layer} = \left(\frac{\tau_A^{\,1/6}\,\tau_\perp^{\,1/4}}{\tau_R^{\,5/6}}\right)\hat{d}_\beta^{\,1/2}, \tag{55}$$

$$\hat{d}_\beta = \frac{\sqrt{(5/3)\,m_i\,(T_{e\,100}+T_{i\,100})}}{e\,B_0\,a}. \tag{56}$$

## VIII. OVERLAP CRITERION

The resistive layer width exceeds the spacing between rational surfaces when

$$\hat{\delta}_{\rm layer} > \hat{\delta}_{\rm rational}, \tag{57}$$

or

$$x > x_c \tag{58}$$

where

$$x_c^{\,11/4}\,[-\ln(2\,x_c)]^{2/9} = n\,\alpha\,\mathit{\Delta}_{\rm layer}. \tag{59}$$