

Four-Field Resonant Layer Model

R. Fitzpatrick^a

*Institute for Fusion Studies, Department of Physics,
University of Texas at Austin, Austin TX 78712, USA*

I. FOURIER-TRANSFORMED FOUR-FIELD EQUATIONS

In real space, the four-field resonant layer equations can be reduced to a set of ten coupled first-order differential equations. As we shall demonstrate, the equations can be reduced to a set of four coupled first-order differential equations in Fourier space. Clearly, it is advantageous to solve the equations in Fourier space.

The Fourier-transformed four-field layer equations take the form:

$$(g + iQ_e) \bar{\psi} = \frac{d(\bar{\phi} - \bar{N})}{dp} - p^2 \bar{\psi}, \quad (1)$$

$$g \bar{N} = -iQ_e \bar{\phi} - D^2 \frac{d(p^2 \bar{\psi})}{dp} + c_\beta^2 \frac{d\bar{V}}{dp} - P_\perp p^2 \bar{N}, \quad (2)$$

$$(g + iQ_i) p^2 \bar{\phi} = \frac{d(p^2 \bar{\psi})}{dp} - P_\varphi p^4 \left(\bar{\phi} + \frac{\bar{N}}{\iota} \right), \quad (3)$$

$$g \bar{V} = iQ_e \bar{\psi} + \frac{d\bar{N}}{dp} - P_\varphi p^2 \bar{V}. \quad (4)$$

It follows that

$$\bar{\psi} = \frac{1}{g + iQ_e + p^2} \frac{d(\bar{\phi} - \bar{N})}{dp}, \quad (5)$$

and

$$\frac{d(p^2 \bar{\psi})}{dp} = [(g + iQ_i) p^2 + P_\varphi p^4] \bar{\phi} + \frac{P_\varphi}{\iota} p^4 \bar{N}, \quad (6)$$

and

$$c_\beta^2 \frac{d\bar{V}}{dp} = (g + P_\perp p^2 + \iota^{-1} D^2 P_\varphi p^4) \bar{N} + [iQ_e + D^2 (g + iQ_i) p^2 + D^2 P_\varphi p^4] \bar{\phi}. \quad (7)$$

If we define

$$\bar{J} = p^2 \bar{\psi}, \quad (8)$$

^a rfitzp@utexas.edu

$$\bar{Y} = \bar{\phi} - \bar{N}, \quad (9)$$

then we can transform Eqs. (4)–(7) into the the following set of four coupled first-order differential equations:

$$\frac{d\bar{Y}}{dp} = \left(\frac{g + iQ_e + p^2}{p^2} \right) \bar{J}, \quad (10)$$

$$\frac{d\bar{N}}{dp} = \left(\frac{-iQ_e}{p^2} \right) \bar{J} + (g + P_\varphi p^2) \bar{V}, \quad (11)$$

$$\frac{d\bar{J}}{dp} = [(g + iQ_i) p^2 + P_\varphi p^4] \bar{Y} + [(g + iQ_i) p^2 + \iota_e^{-1} P_\varphi p^4] \bar{N}, \quad (12)$$

$$\begin{aligned} c_\beta^2 \frac{d\bar{V}}{dp} = & [iQ_e + D^2 (g + iQ_i) p^2 + D^2 P_\varphi p^4] \bar{Y} \\ & + (g + iQ_e + [P_\perp + D^2 (g + iQ_i)] p^2 + \iota_e^{-1} D^2 P_\varphi p^4) \bar{N}, \end{aligned} \quad (13)$$

where $\iota_e = \iota/(1 + \iota)$. Note that $\iota = -Q_e/Q_i$ and $\iota_e = -Q_e/(Q_i - Q_e)$.

II. SMALL- p BEHAVIOR OF FOURIER-TRANSFORMED FOUR-FIELD EQUATIONS

A. Introduction

Let us search for power-law solutions of Eqs. (10)–(13) at small values of p . Given that we have four coupled first-order differential equations, we expect to find four independent power-law solutions.

B. First Solution

Suppose that

$$\bar{Y}(p) = y_{-1} p^{-1} + y_1 p + \mathcal{O}(p^3), \quad (14)$$

$$\bar{N}(p) = n_{-1} p^{-1} + n_1 p + \mathcal{O}(p^3), \quad (15)$$

$$\bar{J}(p) = j_0 + j_2 p^2 + \mathcal{O}(p^4), \quad (16)$$

$$\bar{V}(p) = v_2 p^2 + \mathcal{O}(p^4). \quad (17)$$

Equations (10)–(13) yield

$$-y_{-1} p^{-2} + y_1 = (g + i Q_e) (j_0 p^{-2} + j_2) + j_0 + \mathcal{O}(p^2), \quad (18)$$

$$-n_{-1} p^{-2} + n_1 = -i Q_e (j_0 p^{-2} + j_2) + \mathcal{O}(p^2), \quad (19)$$

$$2 j_2 p = (g + i Q_i) (y_{-1} + n_{-1}) p + \mathcal{O}(p^3), \quad (20)$$

$$\begin{aligned} 2 c_\beta^2 v_2 p &= i Q_e (y_{-1} p^{-1} + y_1 p) + (g + i Q_e) (n_{-1} p^{-1} + n_1 p) \\ &+ D^2 (g + i Q_i) y_{-1} p + [P_\perp + D^2 (g + i Q_i)] n_{-1} p + \mathcal{O}(p^3). \end{aligned} \quad (21)$$

It follows that

$$-y_{-1} = (g + i Q_e) j_0, \quad (22)$$

$$y_1 = (g + i Q_e) j_2 + j_0, \quad (23)$$

$$-n_{-1} = -i Q_e j_0, \quad (24)$$

$$n_1 = -i Q_e j_2, \quad (25)$$

$$2 j_2 = (g + i Q_i) (y_{-1} + n_{-1}), \quad (26)$$

$$0 = i Q_e y_{-1} + (g + i Q_e) n_{-1}, \quad (27)$$

$$\begin{aligned} 2 c_\beta^2 v_2 &= i Q_e y_1 + (g + i Q_e) n_1 \\ &+ D^2 (g + i Q_i) y_{-1} + [P_\perp + D^2 (g + i Q_i)] n_{-1}, \end{aligned} \quad (28)$$

which gives

$$y_{-1} = (g + i Q_e) a_{-1}, \quad (29)$$

$$y_1 = \left[\frac{1}{2} g (g + i Q_e) (g + i Q_i) - 1 \right] a_{-1}, \quad (30)$$

$$n_{-1} = -i Q_e a_{-1}, \quad (31)$$

$$n_1 = -\frac{1}{2} g (i Q_e) (g + i Q_i) a_{-1}, \quad (32)$$

$$j_0 = -a_{-1}, \quad (33)$$

$$j_2 = \frac{1}{2} g (g + i Q_i) a_{-1}, \quad (34)$$

$$v_2 = \frac{[-i Q_e (1 + P_\perp) + g (g + i Q_i) D^2]}{2 c_\beta^2} a_{-1}, \quad (35)$$

where a_{-1} is an arbitrary constant.

C. Second Solution

Suppose that

$$\bar{Y}(p) = y_0 + y_2 p^2 + \mathcal{O}(p^4), \quad (36)$$

$$\bar{N}(p) = n_0 + n_2 p^2 + \mathcal{O}(p^4), \quad (37)$$

$$\bar{J}(p) = j_3 p^3 + \mathcal{O}(p^5), \quad (38)$$

$$\bar{V}(p) = v_3 p^3 + \mathcal{O}(p^5). \quad (39)$$

Equations (10)–(13) give

$$2 y_2 p = (g + \mathrm{i} Q_e) j_3 p + \mathcal{O}(p^3), \quad (40)$$

$$2 n_2 p = -\mathrm{i} Q_e j_3 p + \mathcal{O}(p^3), \quad (41)$$

$$3 j_3 p^2 = (g + \mathrm{i} Q_i) (y_0 + n_0) p^2 + \mathcal{O}(p^4), \quad (42)$$

$$\begin{aligned} 3 c_\beta^2 v_3 p^2 &= \mathrm{i} Q_e (y_0 + y_2 p^2) + (g + \mathrm{i} Q_e) (n_0 + n_2 p^2) \\ &\quad + D^2 (g + \mathrm{i} Q_i) y_0 p^2 + [P_\perp + D^2 (g + \mathrm{i} Q_i)] n_0 p^2 + \mathcal{O}(p^4). \end{aligned} \quad (43)$$

It follows that

$$2 y_2 = (g + \mathrm{i} Q_e) j_3, \quad (44)$$

$$2 n_2 = -\mathrm{i} Q_e j_3, \quad (45)$$

$$3 j_3 = (g + \mathrm{i} Q_i) y_0 + (g + \mathrm{i} Q_i) n_0, \quad (46)$$

$$0 = \mathrm{i} Q_e y_0 + (g + \mathrm{i} Q_e) n_0, \quad (47)$$

$$\begin{aligned} 3 c_\beta^2 v_3 &= \mathrm{i} Q_e y_2 + (g + \mathrm{i} Q_e) n_2 \\ &\quad + D^2 (g + \mathrm{i} Q_e) n_0 + [P_\perp + D^2 (g + \mathrm{i} Q_i)] n_0. \end{aligned} \quad (48)$$

which gives

$$y_0 = (g + \mathrm{i} Q_e) a_0, \quad (49)$$

$$y_2 = \frac{1}{6} g (g + \mathrm{i} Q_e) (g + \mathrm{i} Q_i) a_0, \quad (50)$$

$$n_0 = -\mathrm{i} Q_e a_0, \quad (51)$$

$$n_2 = -\frac{1}{6} g (\mathrm{i} Q_e) (g + \mathrm{i} Q_i) a_0, \quad (52)$$

$$j_3 = \frac{1}{3} g (g + i Q_i) a_0, \quad (53)$$

$$v_3 = \frac{1}{3} \frac{[-i Q_e P_\perp + g (g + i Q_i) D^2]}{c_\beta^2} a_0, \quad (54)$$

where a_0 is an arbitrary constant.

D. Third Solution

Suppose that

$$\bar{Y}(p) = y_2 p^2 + \mathcal{O}(p^4), \quad (55)$$

$$\bar{N}(p) = n_0 + n_2 p^2 + \mathcal{O}(p^4), \quad (56)$$

$$\bar{J}(p) = j_3 p^3 + \mathcal{O}(p^5), \quad (57)$$

$$\bar{V}(p) = v_1 p + \mathcal{O}(p^3). \quad (58)$$

Equations (10)–(13) give

$$2 y_2 p = (g + i Q_e) j_3 p + \mathcal{O}(p^3), \quad (59)$$

$$2 n_2 p = -i Q_e j_3 p + g v_1 p + \mathcal{O}(p^3), \quad (60)$$

$$3 j_3 p^2 = (g + i Q_i) n_0 p^2 + \mathcal{O}(p^4), \quad (61)$$

$$c_\beta^2 v_1 = (g + i Q_e) n_0 + \mathcal{O}(p^2). \quad (62)$$

It follows that

$$y_2 = \frac{1}{6} (g + i Q_e) (g + i Q_i) a_2, \quad (63)$$

$$n_0 = a_2, \quad (64)$$

$$n_2 = \frac{1}{2} (g + i Q_i) \left(-\frac{1}{3} i Q_e + \frac{g}{c_\beta^2} \right) a_2, \quad (65)$$

$$j_3 = \frac{1}{3} (g + i Q_i) a_2, \quad (66)$$

$$v_1 = \frac{(g + i Q_e)}{c_\beta^2} a_2, \quad (67)$$

where a_2 is an arbitrary constant.

E. Fourth Solution

Suppose that

$$\bar{Y}(p) = y_3 p^3 + \mathcal{O}(p^5), \quad (68)$$

$$\bar{N}(p) = n_1 p + \mathcal{O}(p^3), \quad (69)$$

$$\bar{J}(p) = j_4 p^4 + \mathcal{O}(p^6), \quad (70)$$

$$\bar{V}(p) = v_0 + v_2 p^2 + \mathcal{O}(p^4). \quad (71)$$

Equations (10)–(13) give

$$3 y_3 p^2 = (g + \mathrm{i} Q_e) j_4 p^2 + \mathcal{O}(p^4), \quad (72)$$

$$n_1 = g v_0 + \mathcal{O}(p^2), \quad (73)$$

$$4 j_4 p^3 = (g + \mathrm{i} Q_i) n_1 p^3 + \mathcal{O}(p^5), \quad (74)$$

$$2 c_\beta^2 v_2 p = (g + \mathrm{i} Q_e) n_1 p + \mathcal{O}(p^3). \quad (75)$$

It follows that

$$y_3 = \frac{1}{12} g (g + \mathrm{i} Q_e) (g + \mathrm{i} Q_i) a_3, \quad (76)$$

$$n_1 = g a_3, \quad (77)$$

$$j_4 = \frac{1}{4} g (g + \mathrm{i} Q_i) a_3, \quad (78)$$

$$v_2 = \frac{g (g + \mathrm{i} Q_e)}{2 c_\beta^2} a_3, \quad (79)$$

where a_3 is an arbitrary constant.

F. General Solution

We conclude that, at small values of p , the most general solution for $\bar{Y}(p)$ and $\bar{N}(p)$ takes the form

$$\bar{Y}(p) = (g + \mathrm{i} Q_e) p^{-1} a_{-1} + (g + \mathrm{i} Q_e) a_0 + \mathcal{O}(p), \quad (80)$$

$$\bar{N}(p) = (-\mathrm{i} Q_e) p^{-1} a_{-1} + (-\mathrm{i} Q_e) a_0 + a_2 + \mathcal{O}(p). \quad (81)$$

III. MATRIX DIFFERENTIAL EQUATION

Let

$$\underline{u} = \begin{pmatrix} \bar{Y} \\ \bar{N} \end{pmatrix}, \quad (82)$$

$$\underline{v} = \begin{pmatrix} \bar{J} \\ c_\beta^2 \bar{V} \end{pmatrix}. \quad (83)$$

Equations (10)–(13) can be written in the form

$$\frac{d\underline{u}}{dp} = \underline{\underline{A}} \underline{v}, \quad (84)$$

$$\frac{d\underline{v}}{dp} = \underline{\underline{B}} \underline{u}, \quad (85)$$

where

$$A_{11} = \frac{g + i Q_e + p^2}{p^2}, \quad (86)$$

$$A_{21} = \frac{-i Q_e}{p^2}, \quad (87)$$

$$A_{22} = \frac{g + P_\varphi p^2}{c_\beta^2}, \quad (88)$$

$$B_{11} = (g + i Q_i) p^2 + P_\varphi p^4, \quad (89)$$

$$B_{12} = (g + i Q_i) p^2 + \iota_e^{-1} P_\varphi p^4, \quad (90)$$

$$B_{21} = i Q_e + D^2 (g + i Q_i) p^2 + D^2 P_\varphi p^4, \quad (91)$$

$$B_{22} = g + i Q_e + [P_\perp + D^2 (g + i Q_i)] p^2 + \iota_e^{-1} D^2 P_\varphi p^4. \quad (92)$$

Thus, we obtain the following matrix differential equation:

$$\frac{d}{dp} \left(\underline{\underline{A}}^{-1} \frac{d\underline{u}}{dp} \right) = \underline{\underline{B}} \underline{u}. \quad (93)$$

IV. RICCATI MATRIX DIFFERENTIAL EQUATION

Let

$$p \frac{d\underline{u}}{dp} = \underline{\underline{W}} \underline{u}. \quad (94)$$

The previous equation can be combined with Eq. (93) to give

$$\left(p \frac{d\underline{\underline{W}}}{dp} - \underline{\underline{W}} + \underline{\underline{W}}\underline{\underline{W}} + \underline{\underline{A}}p \frac{d\underline{\underline{A}}^{-1}}{dp} \underline{\underline{W}} - p^2 \underline{\underline{A}}\underline{\underline{B}} \right) \underline{\underline{u}} = \underline{\underline{0}}, \quad (95)$$

which implies that

$$p \frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}}\underline{\underline{W}} - \underline{\underline{A}}p \frac{d\underline{\underline{A}}^{-1}}{dp} \underline{\underline{W}} + p^2 \underline{\underline{A}}\underline{\underline{B}}. \quad (96)$$

Now,

$$\underline{\underline{A}}^{-1} = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}, \quad (97)$$

where

$$C_{11} = \frac{p^2}{g + i Q_e + p^2}, \quad (98)$$

$$C_{21} = \frac{i c_\beta^2 Q_e}{(g + i Q_e + p^2)(g + P_\varphi p^2)}, \quad (99)$$

$$C_{22} = \frac{c_\beta^2}{g + P_\varphi p^2}. \quad (100)$$

So, if

$$p \frac{d\underline{\underline{A}}^{-1}}{dp} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix} \quad (101)$$

then

$$D_{11} = \frac{2 p^2 (g + i Q_e)}{(g + i Q_e + p^2)^2}, \quad (102)$$

$$D_{21} = -\frac{2 i c_\beta^2 Q_e p^2 [g + P_\varphi (g + i Q_e) + 2 P_\varphi p^2]}{(g + i Q_e + p^2)^2 (g + P_\varphi p^2)^2}, \quad (103)$$

$$D_{22} = -\frac{2 c_\beta^2 P_\varphi p^2}{(g + P_\varphi p^2)^2}. \quad (104)$$

Furthermore, if

$$\underline{\underline{A}}p \frac{d\underline{\underline{A}}^{-1}}{dp} = \begin{pmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{pmatrix} \quad (105)$$

then

$$E_{11} = \frac{2 (g + i Q_e)}{g + i Q_e + p^2}, \quad (106)$$

$$E_{21} = -\frac{2 i Q_e (g + 2 P_\varphi p^2)}{(g + i Q_e + p^2)(g + P_\varphi p^2)}, \quad (107)$$

$$E_{22} = -\frac{2P_\varphi p^2}{g + P_\varphi p^2}. \quad (108)$$

Finally, if

$$p^2 \underline{\underline{A}} \underline{\underline{B}} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad (109)$$

then

$$F_{11} = p^2 (g + iQ_e + p^2) (g + iQ_i + P_\varphi p^2), \quad (110)$$

$$F_{12} = p^2 (g + iQ_e + p^2) (g + iQ_i + \iota_e^{-1} P_\varphi p^2), \quad (111)$$

$$F_{21} = -iQ_e p^2 (g + iQ_i + P_\varphi p^2) + c_\beta^{-2} p^2 (g + P_\varphi p^2) [iQ_e + D^2 (g + iQ_i) p^2 + D^2 P_\varphi p^4], \quad (112)$$

$$F_{22} = -iQ_e p^2 (g + iQ_i + \iota_e^{-1} P_\varphi p^2) + c_\beta^{-2} p^2 (g + P_\varphi p^2) [g + iQ_e + [P_\perp + D^2 (g + iQ_i)] p^2 + \iota_e^{-1} D^2 P_\varphi p^4]. \quad (113)$$

Hence, Eq. (96) can be written as the following Riccati matrix differential equation:

$$p \frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}} \underline{\underline{W}} - \underline{\underline{E}} \underline{\underline{W}} + \underline{\underline{F}}. \quad (114)$$

Furthermore, if

$$\underline{\underline{W}} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad (115)$$

then

$$p \frac{dW_{11}}{dp} = W_{11} - W_{11} W_{11} - W_{12} W_{21} - E_{11} W_{11} + F_{11}, \quad (116)$$

$$p \frac{dW_{12}}{dp} = W_{12} - W_{11} W_{12} - W_{12} W_{22} - E_{11} W_{12} + F_{12}, \quad (117)$$

$$p \frac{dW_{21}}{dp} = W_{21} - W_{21} W_{11} - W_{22} W_{21} - E_{21} W_{11} - E_{22} W_{21} + F_{21}, \quad (118)$$

$$p \frac{dW_{22}}{dp} = W_{22} - W_{21} W_{12} - W_{22} W_{22} - E_{21} W_{12} - E_{22} W_{22} + F_{22}. \quad (119)$$

V. SMALL- p BEHAVIOR OF RICCATI MATRIX DIFFERENTIAL EQUATION

Let $\underline{\underline{E}} = \underline{\underline{E}}^{(0)}$ at $p = 0$. It follows from Eqs. (105)–(108) that

$$E_{11}^{(0)} = 2, \quad (120)$$

$$E_{12}^{(0)} = 0, \quad (121)$$

$$E_{21}^{(0)} = -\frac{2iQ_e}{g + iQ_e}, \quad (122)$$

$$E_{22}^{(0)} = 0. \quad (123)$$

Likewise, at small values of p , we can write $\underline{\underline{F}} = p^2 \underline{\underline{F}}^{(2)}$, where the elements of $\underline{\underline{F}}^{(2)}$ are constants, and where use has been made of Eqs. (110)–(113).

Suppose that $\underline{\underline{W}} = \underline{\underline{W}}^{(0)}$ at $p = 0$. Equation (114) gives

$$\underline{\underline{0}} = \underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)} \underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)} \underline{\underline{W}}^{(0)}, \quad (124)$$

at $p = 0$, which yields

$$(\underline{\underline{1}} - \underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)}) \underline{\underline{W}}^{(0)} = \underline{\underline{0}}. \quad (125)$$

Hence, we deduce that

$$\underline{\underline{W}}^{(0)} = \underline{\underline{1}} - \underline{\underline{E}}^{(0)} = \begin{pmatrix} -1 & 0 \\ -E_{21}^{(0)} & 1 \end{pmatrix}. \quad (126)$$

At small values of p , let

$$\underline{u}(p) = \underline{u}_{-1} p^{-1} + \underline{u}_0, \quad (127)$$

$$\underline{\underline{W}}(p) = \underline{\underline{W}}^{(0)} + p \underline{\underline{W}}^{(1)}, \quad (128)$$

where the elements of \underline{u}_{-1} (which are y_{-1} and n_{-1} , respectively), the elements of \underline{u}_0 (which are y_0 and n_0 , respectively), and the elements of $\underline{\underline{W}}^{(1)}$, are all constants. Equation (94) gives

$$\underline{\underline{W}}^{(0)} \underline{u}_{-1} = -\underline{u}_{-1}, \quad (129)$$

$$\underline{\underline{W}}^{(0)} \underline{u}_0 + \underline{\underline{W}}^{(1)} \underline{u}_{-1} = \underline{\underline{0}}. \quad (130)$$

Thus, making use of Eq. (126), we get

$$\begin{pmatrix} -1 & 0 \\ -E_{21}^{(0)} & 1 \end{pmatrix} \begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix} = - \begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix}, \quad (131)$$

which implies that

$$E_{21}^{(0)} y_{-1} = -\frac{2iQ_e}{g + iQ_e} y_{-1} = 2n_{-1}, \quad (132)$$

in accordance with Eqs. (29) and (31), where use has been made of Eq. (122). Thus, if we write

$$y_{-1} = (g + i Q_e) a_{-1}, \quad (133)$$

$$n_{-1} = -i Q_e a_{-1}, \quad (134)$$

$$y_0 = (g + i Q_e) a_0, \quad (135)$$

$$n_0 = -i Q_e a_0 + a_2, \quad (136)$$

in accordance with Eqs. (80) and (81), then we deduce from Eqs. (126) and (130) that

$$\frac{\pi}{\hat{\Delta}_s} \equiv \frac{a_0}{a_{-1}} = W_{11}^{(1)} - W_{12}^{(1)} \frac{(i Q_e)}{g + i Q_e}, \quad (137)$$

and

$$\frac{a_2}{a_{-1}} = i Q_e \left[W_{22}^{(1)} - W_{11}^{(1)} \right] + \frac{(i Q_e)^2}{g + i Q_e} W_{12}^{(1)} - (g + i Q_e) W_{21}^{(1)}. \quad (138)$$

VI. LARGE- p BEHAVIOR OF RICCATI MATRIX DIFFERENTIAL EQUATION

At large values of p , it is clear from Eqs. (110)–(113) that $\underline{\underline{F}}(p) = p^6 \underline{\underline{F}}^{(6)} + p^8 \underline{\underline{F}}^{(8)}$, where the elements of $\underline{\underline{F}}^{(6)}$ and $\underline{\underline{F}}^{(8)}$ are constants. On the other hand, Eqs. (105)–(108) imply that $\underline{\underline{E}}(p) = \underline{\underline{E}}^{(0)}$, where the elements of $\underline{\underline{E}}^{(0)}$ are constants. Thus, if we write $\underline{\underline{W}}(p) = p^2 \underline{\underline{W}}^{(2)} + p^4 \underline{\underline{W}}^{(4)}$, where the elements of $\underline{\underline{W}}^{(2)}$ and $\underline{\underline{W}}^{(4)}$ are constants, then Eq. (114) gives

$$\underline{\underline{W}}^{(4)} \underline{\underline{W}}^{(4)} = \underline{\underline{F}}^{(8)}, \quad (139)$$

$$\underline{\underline{W}}^{(2)} \underline{\underline{W}}^{(4)} + \underline{\underline{W}}^{(4)} \underline{\underline{W}}^{(2)} = \underline{\underline{F}}^{(6)}. \quad (140)$$

Now, according to Eqs. (110)–(113),

$$F_{11}^{(8)} = 0, \quad (141)$$

$$F_{12}^{(8)} = 0, \quad (142)$$

$$F_{21}^{(8)} = c_\beta^{-2} D^2 P_\varphi^2, \quad (143)$$

$$F_{22}^{(8)} = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2, \quad (144)$$

so Eq. (139) yields

$$W_{11}^{(4)} = 0, \quad (145)$$

$$W_{12}^{(4)} = 0, \quad (146)$$

$$W_{21}^{(4)} = -c_\beta^{-1} \iota_e^{1/2} D P_\varphi, \quad (147)$$

$$W_{22}^{(4)} = -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi, \quad (148)$$

where we have chosen the sign of the square root that is associated with well-behaved solutions at large values of p . Here, we are assuming that $\iota_e > 0$. Equations (110)–(113) also give

$$F_{11}^{(6)} = P_\varphi, \quad (149)$$

$$F_{12}^{(6)} = \iota_e^{-1} P_\varphi, \quad (150)$$

$$F_{21}^{(6)} = c_\beta^{-2} D^2 g P_\varphi + c_\beta^{-2} D^2 (g + i Q_i) P_\varphi, \quad (151)$$

$$F_{22}^{(6)} = c_\beta^{-2} \iota_e^{-1} D^2 g P_\varphi + c_\beta^{-2} [P_\perp + D^2 (g + i Q_i)] P_\varphi. \quad (152)$$

Thus, Eq. (140) yields

$$W_{12}^{(2)} W_{21}^{(4)} = F_{11}^{(6)}, \quad (153)$$

$$W_{12}^{(2)} W_{22}^{(4)} = F_{12}^{(6)}, \quad (154)$$

which gives

$$W_{12}^{(2)} = -c_\beta \iota_e^{-1/2} D^{-1}. \quad (155)$$

Now, if

$$\underline{\underline{W}} \underline{u} = \lambda(p) \underline{u} \quad (156)$$

then Eq. (94) yields

$$p \frac{d\underline{u}}{dp} = \lambda \underline{u}, \quad (157)$$

which implies that

$$\underline{u}(p) = \underline{u}(p_0) \exp \left[\int_{p_0}^p \frac{\lambda_r(p')}{p'} dp' \right] \exp \left[i \int_{p_0}^p \frac{\lambda_i(p')}{p'} dp' \right], \quad (158)$$

where λ_r and λ_i are the real and imaginary parts of λ , respectively. Of course, a solution that is well behaved at large values of p is such that λ_r is negative. As we have seen, the large- p limit of Eq. (114) is

$$\underline{\underline{W}} \underline{\underline{W}} = \underline{\underline{F}}. \quad (159)$$

Hence, if

$$\underline{\underline{F}} \underline{u} = \Lambda \underline{u} \quad (160)$$

then Eqs. (156) and (160) imply that

$$\lambda^2 = \Lambda. \quad (161)$$

The eigenvalue problem for the F -matrix reduces to

$$\Lambda^2 - (F_{11} + F_{22}) \Lambda + F_{11} F_{22} - F_{12} F_{21} = 0. \quad (162)$$

Now,

$$F_{11} + F_{22} \simeq F_{22}^{(8)} p^8 = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2 p^8, \quad (163)$$

$$\begin{aligned} F_{11} F_{22} - F_{12} F_{21} &\simeq \left[F_{11}^{(6)} F_{22}^{(8)} - F_{12}^{(6)} F_{21}^{(8)} \right] p^{14} \\ &+ \left[F_{11}^{(6)} F_{22}^{(6)} - F_{12}^{(6)} F_{21}^{(6)} \right] p^{12} = c_\beta^{-2} R P_\varphi^2 p^{12}, \end{aligned} \quad (164)$$

where

$$R = P_\perp + (1 - \iota_e^{-1}) D^2 (g + i Q_i), \quad (165)$$

Hence, the two eigenvalues of the F -matrix are

$$\Lambda_1 \simeq F_{22}^{(8)} p^8 = c_\beta^{-2} \iota_e^{-1} D^2 P_\varphi^2 p^8, \quad (166)$$

$$\Lambda_2 \simeq \frac{[F_{11}^{(6)} F_{22}^{(6)} - F_{12}^{(6)} F_{21}^{(6)}]}{F_{22}^{(8)}} p^4 = \iota_e D^{-2} R p^4. \quad (167)$$

Thus, we deduce that the two eigenvalues of the W -matrix are

$$\lambda_1 = -\Lambda_1^{1/2} = -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi p^4, \quad (168)$$

$$\lambda_2 = -\Lambda_2^{1/2} = -\iota_e^{1/2} D^{-1} R^{1/2} p^2, \quad (169)$$

Here, the square root of R is taken such that the real part of λ_2 is negative. Now, the eigenvalue problem for the W -matrix reduces to

$$\lambda^2 - W_{22}^{(4)} p^4 \lambda + \left[W_{11}^{(2)} W_{22}^{(4)} - W_{12}^{(2)} W_{21}^{(4)} \right] p^6 = 0. \quad (170)$$

which yields

$$\lambda_1 \simeq W_{22}^{(4)} p^4, \quad (171)$$

which is satisfied, and

$$\lambda_2 \simeq \left[W_{11}^{(2)} - \frac{W_{12}^{(2)} W_{21}^{(4)}}{W_{22}^{(4)}} \right] p^2, \quad (172)$$

which implies that

$$W_{11}^{(2)} = -\iota_e^{1/2} D^{-1} R^{1/2} - c_\beta \iota_e^{1/2} D^{-1}. \quad (173)$$

Hence, the large- p boundary condition for the W -matrix is

$$\underline{\underline{W}}(p) = \begin{pmatrix} -\iota_e^{1/2} D^{-1} R^{1/2} p^2 - c_\beta \iota_e^{1/2} D^{-1} p^2, & -c_\beta \iota_e^{-1/2} D^{-1} p^2 \\ -c_\beta^{-1} \iota_e^{1/2} D P_\varphi p^4, & -c_\beta^{-1} \iota_e^{-1/2} D P_\varphi p^4 \end{pmatrix}. \quad (174)$$

VII. DETERMINATION OF P_{MAX}

At large values of p , we have

$$F_{11}^{(4)} = g + i Q_i + P_\varphi (g + i Q_e), \quad (175)$$

$$F_{12}^{(4)} = g + i Q_i + \iota_e^{-1} P_\varphi (g + i Q_e), \quad (176)$$

$$F_{21}^{(4)} = -i Q_e P_\varphi + c_\beta^{-2} [g D^2 (g + i Q_i) + P_\varphi i Q_e], \quad (177)$$

$$F_{22}^{(4)} = -i Q_e \iota_e^{-1} P_\varphi + c_\beta^{-2} \{g [P_\perp + D^2 (g + i Q_i) p^2] + P_\varphi (g + i Q_e)\}. \quad (178)$$

We require

$$p \gg \left| \frac{F_{21}^{(6)}}{F_{21}^{(8)}} \right|^{1/2}, \left| \frac{F_{22}^{(6)}}{F_{22}^{(8)}} \right|^{1/2}, \left| \frac{F_{11}^{(4)}}{F_{11}^{(6)}} \right|^{1/2}, \left| \frac{F_{12}^{(4)}}{F_{12}^{(6)}} \right|^{1/2}, \left| \frac{F_{21}^{(4)}}{F_{21}^{(6)}} \right|^{1/2}, \left| \frac{F_{22}^{(4)}}{F_{22}^{(6)}} \right|^{1/2}. \quad (179)$$