

# Separatrix

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## I. SIMPLE MODEL

Suppose that we have two current filaments running parallel to the  $z$ -axis. Let the upper filament carry the current  $I_0$  and pierce the  $x$ - $y$  plane at  $x = y = 0$ . Let the lower filament carry the current  $I_1$  and pierce the  $x$ - $y$  plane at  $x = 0$ ,  $y = -a$ . Suppose that there is a uniform  $z$ -directed magnetic field of strength  $B_0$ . Let the system be periodic in the  $z$  direction with period  $2\pi R_0$ . Let  $\phi = z/R_0$ . We can write

$$\mathbf{B} = \nabla\phi \times \nabla\psi + B_0 R_0 \nabla\phi, \quad (1)$$

where

$$\psi(x, y) = \frac{\mu_0 I_0 R_0}{4\pi} \ln(x^2 + y^2) + \frac{\mu_0 I_1 R_0}{4\pi} \ln[x^2 + (y + a)^2]. \quad (2)$$

Let us write

$$\mathbf{B} = \nabla(\phi - q\theta) \times \nabla\psi, \quad (3)$$

where  $q = q(\psi)$ . It follows that

$$\nabla\psi \times \nabla\theta \cdot \nabla\phi = \frac{B_0}{R_0 q}. \quad (4)$$

Note that  $\mathbf{B} \cdot \nabla\phi = 0$ , so  $\psi$  is a magnetic flux-surface label. Also,  $\mathbf{B} \cdot \nabla(\phi - q\theta) = 0$ . So, magnetic field-lines within a given flux-surface appear as straight lines when plotted in  $\theta$ - $\phi$  space. Thus,

$$\frac{d\theta}{dl} = \frac{B_0}{q |\nabla\psi|}, \quad (5)$$

where  $dl$  is an element of length around a flux-surface. Thus,

$$dx = -\frac{\psi_y dl}{\sqrt{\psi_x^2 + \psi_y^2}}, \quad (6)$$

$$dy = \frac{\psi_x dl}{\sqrt{\psi_x^2 + \psi_y^2}}, \quad (7)$$

so that  $d\mathbf{r} \cdot \nabla\psi = 0$ . It follows that

$$q(\psi) = \frac{B_0}{2\pi} \oint \frac{dl}{\sqrt{\psi_x^2 + \psi_y^2}} \quad (8)$$

Let  $\psi = \mu_0 I_0 R_0 \Psi / (2\pi)$ ,  $x = a X$ ,  $y = a Y$ ,  $l = a L$ , and  $r = a \hat{r}$ . It follows that

$$dX = -\frac{\Psi_Y dL}{\sqrt{\Psi_X^2 + \Psi_Y^2}}, \quad (9)$$

$$dY = \frac{\Psi_X dL}{\sqrt{\Psi_X^2 + \Psi_Y^2}}, \quad (10)$$

$$q(\psi) = \frac{1}{2\pi} \oint \frac{dL}{\sqrt{\Psi_X^2 + \Psi_Y^2}}. \quad (11)$$

Here, we have assumed that

$$\frac{B_0 a^2}{\mu_0 I_0 R_0} = \frac{1}{2\pi}, \quad (12)$$

which ensures that

$$q(\Psi) = \exp(2\Psi) \quad (13)$$

as  $X^2 + Y^2 \rightarrow 0$ . Moreover,

$$\Psi = \frac{1}{2} \ln(X^2 + Y^2) + \frac{\beta}{2} \ln[X^2 + (Y + 1)^2], \quad (14)$$

$$\Psi_X = \frac{X}{X^2 + Y^2} + \frac{\beta X}{X^2 + (Y + 1)^2}, \quad (15)$$

$$\Psi_Y = \frac{Y}{X^2 + Y^2} + \frac{\beta(Y + 1)}{X^2 + (Y + 1)^2}, \quad (16)$$

where  $\beta = I_1/I_0$ .

## II. SAFETY-FACTOR

Let  $\hat{r} = r/a$ . Suppose that the last closed magnetic flux surface is at  $\hat{r} = 1$ . Suppose that the safety-factor profile takes the form

$$q(\hat{r}) = q_0 + (q_{95} - q_0) \frac{\ln(1 - \hat{r}^2)}{\ln(1 - \hat{r}_{95}^2)}. \quad (17)$$

[Logarithmic increase in  $q$  at separatrix: N. Pomphrey and A. Reiman, Phys. Fluids B **4**, 938 (1992).]

### III. POLOIDAL MAGNETIC FLUX

Now,

$$\frac{d\Psi_p}{dr} = B_0 R_0 f = B_0 a \frac{\hat{r}}{q}, \quad (18)$$

assuming that  $g(\hat{r}) = 1$ . Let  $\psi_p = \Psi_p/(B_0 a^2)$ . It follows that

$$\frac{d\psi_p}{d\hat{r}} = \frac{\hat{r}}{q} = \frac{\hat{r}}{q_0 - \alpha \ln(1 - \hat{r}^2)}, \quad (19)$$

where

$$\alpha = -\frac{q_{95} - q_0}{\ln(1 - \hat{r}_{95}^2)}. \quad (20)$$

So,

$$\begin{aligned} \psi_p(\hat{r}) &= \int_0^{\hat{r}} \frac{d\psi_p}{d\hat{r}} d\hat{r} = \frac{1}{2} \int_0^{\hat{r}^2} \frac{dx}{q_0 - \alpha \ln(1 - x)} = \frac{1}{2} \int_{1-\hat{r}^2}^1 \frac{dy}{q_0 - \alpha \ln y} \\ &= \frac{e^{q_0/\alpha}}{2\alpha} \int_{q_0/\alpha}^{q_0/\alpha - \ln(1-\hat{r}^2)} \frac{e^{-z} dz}{z} = \frac{e^{q_0/\alpha}}{2\alpha} \left[ \int_{q_0/\alpha}^{\infty} \frac{e^{-z} dz}{z} - \int_{q_0/\alpha - \ln(1-\hat{r}^2)}^{\infty} \frac{e^{-z} dz}{z} \right] \\ &= \frac{e^{q_0/\alpha}}{2\alpha} \left[ \int_1^{\infty} \frac{e^{-(q_0/\alpha)t} dt}{t} - \int_1^{\infty} \frac{e^{-[q_0/\alpha - \ln(1-\hat{r}^2)]t} dt}{t} \right], \end{aligned} \quad (21)$$

giving

$$\psi_p(\hat{r}) = \frac{e^{q_0/\alpha}}{2\alpha} \{E_1(q_0/\alpha) - E_1[q_0/\alpha - \ln(1 - \hat{r}^2)]\}. \quad (22)$$

Thus,

$$\psi_p(1) = \frac{e^{q_0/\alpha}}{2\alpha} E_1(q_0/\alpha). \quad (23)$$

Let  $\hat{\psi}_p(\hat{r}) = \psi_p(\hat{r})/\psi_p(1)$ . It follows that

$$\hat{\psi}_p(\hat{r}) = 1 - \frac{E_1[q_0/\alpha - \ln(1 - \hat{r}^2)]}{E_1(q_0/\alpha)}. \quad (24)$$

By definition,  $\hat{\psi}_p(\hat{r}_{95}) = 0.95$ , so

$$0.95 = 1 - \frac{E_1[q_0/\alpha - \ln(1 - \hat{r}_{95}^2)]}{E_1(q_0/\alpha)} = 1 - \frac{E_1(q_{95}/\alpha)}{E_1(q_0/\alpha)}, \quad (25)$$

giving

$$0.05 = \frac{E_1(q_{95}/\alpha)}{E_1(q_0/\alpha)}. \quad (26)$$

Assuming that  $q_0$  and  $q_{95}$  are specified, the previous equation can be solved to give  $\alpha$ , which then determines  $\hat{r}_{95}$ .

#### IV. RATIONAL SURFACES

Suppose that  $n$  is the toroidal mode number. The  $m, n$  rational surface lies at  $\hat{r} = \hat{r}_m$ , where  $q(\hat{r}_m) = q_m$  and  $q_m = m/n$ . Thus,

$$\hat{r}_m = \left[ 1 - \exp \left( \frac{q_0 - q_m}{\alpha} \right) \right]^{1/2}. \quad (27)$$

Hence, if  $\hat{\psi}_m = \hat{\psi}_p(\hat{r}_m)$  then

$$\hat{\psi}_m = 1 - \frac{E_1(q_m/\alpha)}{E_1(q_0/\alpha)}. \quad (28)$$

The spacing (in  $\hat{r}$ ) between successive rational surfaces is

$$\hat{\delta}_{\text{rational}} = \frac{d\hat{r}_m}{dm} = \frac{1 - \hat{r}_m^2}{2 \hat{r}_m n \alpha}. \quad (29)$$

In the continuum limit, assuming that  $\hat{r}_m \simeq 1$ , we get

$$\hat{\delta}_{\text{rational}}(\hat{r}) \simeq \frac{1 - \hat{r}}{n \alpha} = \frac{x}{n \alpha}, \quad (30)$$

where  $x = 1 - \hat{r}$ .

#### V. MAGNETIC SHEAR

The magnetic shear is

$$s(\hat{r}) = \frac{\hat{r}}{q} \frac{dq}{d\hat{r}} = \frac{2 \alpha \hat{r}^2}{(1 - \hat{r}^2) q} \simeq \frac{1}{f(x)}, \quad (31)$$

where

$$f(x) = -x \ln(2x). \quad (32)$$

#### VI. PROFILES

Let

$$n_e(\hat{\psi}_p) = 20 n_{e95} (1 - \hat{\psi}_p) + n_{e100}, \quad (33)$$

$$T_e(\hat{\psi}_p) = 20 T_{e95} (1 - \hat{\psi}_p) + T_{e100}, \quad (34)$$

$$T_i(\hat{\psi}_p) = 20 T_{i95} (1 - \hat{\psi}_p) + T_{i100}. \quad (35)$$

Thus,

$$\begin{aligned} \omega_{*e}(\hat{\psi}_p) &= \frac{T_e}{e} \left( \frac{d \ln n_e}{d \Psi_p} + \frac{d \ln T_e}{d \Psi_p} \right) = \frac{T_e}{e B_0 a^2 \psi_p(1)} \left( \frac{d \ln n_e}{d \hat{\psi}_p} + \frac{d \ln T_e}{d \hat{\psi}_p} \right) \\ &\simeq -\frac{20 T_{e100}}{e B_0 a^2 \psi_p(1)} \left( \frac{n_{e95}}{n_{e100}} + \frac{T_{e95}}{T_{e100}} \right), \end{aligned} \quad (36)$$

$$\begin{aligned} \omega_{*i}(\hat{\psi}_p) &= -\frac{T_i}{e} \left( \frac{d \ln n_e}{d \Psi_p} + \frac{d \ln T_i}{d \Psi_p} \right) = -\frac{T_i}{e B_0 a^2 \psi_p(1)} \left( \frac{d \ln n_e}{d \hat{\psi}_p} + \frac{d \ln T_i}{d \hat{\psi}_p} \right) \\ &\simeq -\frac{20 T_{i100}}{e B_0 a^2 \psi_p(1)} \left( \frac{n_{e95}}{n_{e100}} + \frac{T_{i95}}{T_{i100}} \right). \end{aligned} \quad (37)$$

## VII. LAYER QUANTITIES

Now,

$$\tau_{ee}(\hat{\psi}_p) = \frac{6\sqrt{2} \pi^{3/2} \epsilon_0^2 m_e^{1/2} T_e^{3/2}}{\ln \Lambda e^4 n_e} \simeq \frac{6\sqrt{2} \pi^{3/2} \epsilon_0^2 m_e^{1/2} T_{e100}^{3/2}}{\ln \Lambda e^4 n_{e100}}, \quad (38)$$

$$\sigma_{\parallel} = 1.96 \frac{n_e e^2 \tau_{ee}}{m_e}, \quad (39)$$

$$\tau_R = \mu_0 a^2 \sigma_{\parallel} \hat{r}^2 \simeq \mu_0 a^2 \sigma_{\parallel}, \quad (40)$$

$$\tau_H = \frac{R_0}{B_0} \frac{\sqrt{\mu_0 m_i n_e}}{n s} \simeq \tau_A f, \quad (41)$$

$$\tau_A = \frac{R_0}{B_0} \frac{\sqrt{\mu_0 m_i n_{e100}}}{n}, \quad (42)$$

$$\tau_{\perp} = \frac{a^2 \hat{r}^2}{D_{\perp}} \simeq \frac{a^2}{D_{\perp}}, \quad (43)$$

$$\tau_{\varphi} = \frac{a^2 \hat{r}^2}{\chi_{\varphi}} \simeq \frac{a^2}{\chi_{\varphi}}, \quad (44)$$

$$\tau = -\frac{\omega_{*e}}{\omega_{*i}}, \quad (45)$$

$$d_{\beta} = \frac{\sqrt{(5/3) m_i (T_e + T_i)}}{e B_0} \simeq \frac{\sqrt{(5/3) m_i (T_{e100} + T_{i100})}}{e B_0}, \quad (46)$$

$$S = \frac{\tau_R}{\tau_H} = \mathcal{S} f^{-1} \quad (47)$$

$$\mathcal{S} = \frac{\tau_R}{\tau_A}, \quad (48)$$

$$P_\varphi = \frac{\tau_R}{\tau_\varphi}, \quad (49)$$

$$P_\perp = \frac{\tau_R}{\tau_\perp}, \quad (50)$$

$$D = S^{1/3} \left( \frac{\tau}{1+\tau} \right)^{1/2} \frac{d_\beta}{a \hat{r}} \simeq \mathcal{D} f^{-1/3}, \quad (51)$$

$$\mathcal{D} = S^{1/3} \left( \frac{\tau}{1+\tau} \right)^{1/2} \frac{d_\beta}{a}, \quad (52)$$

$$Q_E = -S^{1/3} n \omega_E \tau_H \simeq \mathcal{Q}_E f^{2/3}, \quad (53)$$

$$\mathcal{Q}_E = -\mathcal{S}^{1/3} n \omega_E \tau_A, \quad (54)$$

$$\mathcal{Q}_{e,i} = -S^{1/3} n \omega_{*e,i} \tau_H \simeq \mathcal{Q}_{*e,i} f^{2/3}, \quad (55)$$

$$\mathcal{Q}_{ei} = -\mathcal{S}^{1/3} n \omega_{*ei} \tau_A, \quad (56)$$

$$Q = S^{1/3} \omega \tau_H = \mathcal{Q} f^{2/3}, \quad (57)$$

$$\mathcal{Q} = \mathcal{S}^{1/3} \omega \tau_A. \quad (58)$$

### VIII. LAYER EQUATION

The layer equation is written

$$\frac{d}{dp} \left[ A(p) \frac{dY_e}{dp} \right] - \frac{B(p)}{C(p)} p^2 Y_e = 0, \quad (59)$$

where

$$A = \frac{p^2}{-\mathrm{i}(Q - Q_E - Q_e) + p^2}, \quad (60)$$

$$B = -\mathrm{i}(Q - Q_E)(Q - Q_E - Q_i) - \mathrm{i}(Q - Q_E - Q_i)(P_\varphi + P_\perp)p^2 + P_\varphi P_\perp p^4, \quad (61)$$

$$C = -\mathrm{i}(Q - Q_E - Q_e) + [P_\perp - \mathrm{i}(Q - Q_E - Q_i)D^2]p^2 + (1 + 1/\tau)P_\varphi D^2 p^4. \quad (62)$$

Let

$$p = f^{-1/6} \hat{p}. \quad (63)$$

It follows that

$$\frac{d}{d\hat{p}} \left[ \mathcal{A} \frac{dY_e}{d\hat{p}} \right] - \frac{\mathcal{B}}{\mathcal{C}} \hat{p}^2 Y_e = 0, \quad (64)$$

where

$$\mathcal{A} = \frac{\hat{p}^2}{-i(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_e) f + \hat{p}^2}, \quad (65)$$

$$\mathcal{B} = -i(\mathcal{Q} - \mathcal{Q}_E)(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_i) f^2 - i(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_i)(P_\varphi + P_\perp) \hat{p}^2 f + P_\varphi P_\perp \hat{p}^4, \quad (66)$$

$$\mathcal{C} = -i(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_e) f^2 + [P_\perp - i(\mathcal{Q} - \mathcal{Q}_E - \mathcal{Q}_i) \mathcal{D}^2] \hat{p}^2 f + (1 + 1/\tau) P_\varphi \mathcal{D}^2 \hat{p}^4. \quad (67)$$

Thus, Eq. (64) reduces to

$$\frac{d^2 Y_e}{d\hat{p}^2} - \frac{P_\perp}{(1 + 1/\tau) \mathcal{D}^2} \hat{p}^2 Y_e \simeq 0. \quad (68)$$

The characteristic layer width in  $\hat{p}$  space is

$$\hat{p}_* = \left[ \frac{\mathcal{D}^2 (1 + 1/\tau)}{P_\perp} \right]^{1/4}. \quad (69)$$

Thus, the characteristic layer width in  $\hat{r}$  space is

$$\hat{\delta}_{\text{layer}} = \frac{S^{-1/3}}{f^{-1/6} \hat{p}_*} = \mathcal{S}^{-1/3} \left[ \frac{P_\perp}{\mathcal{D}^2 (1 + 1/\tau)} \right]^{1/4} f^{1/2} = n^{-1/2} \Delta_{\text{layer}} f^{1/2}, \quad (70)$$

where

$$\Delta_{\text{layer}} = \frac{\tau_A^{1/2}}{\tau_R^{1/4} \tau_\perp^{1/4} \hat{d}_\beta^{1/2}} \quad (71)$$

$$\hat{\tau}_A = \frac{R_0}{B_0} \sqrt{\mu_0 m_i n_{e100}}, \quad (72)$$

$$\hat{d}_\beta = \frac{\sqrt{(5/3) m_i (T_{e100} + T_{i100})}}{e B_0 a}. \quad (73)$$

## IX. OVERLAP CRITERION

The resistive layer width exceeds the spacing between rational surfaces when

$$\hat{\delta}_{\text{layer}} > \hat{\delta}_{\text{rational}}, \quad (74)$$

or

$$x < x_c \tag{75}$$

where

$$\frac{x_c}{-\ln(2x_c)} = n(\alpha\Delta_{\text{layer}})^2. \tag{76}$$