Calculation of Vertical Stability in an Inverse Aspect-Ratio Expanded Tokamak Plasma Equilibrium

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I. PLASMA EQUILIBRIUM

All lengths are normalized to the major radius of the plasma magnetic axis, R_0. All magnetic field-strengths are normalized to the toroidal field-strength at the magnetic axis, B_0. All current densities are normalized to $B_0/(\mu_0\, R_0)$. All plasma pressures are normalized to B_0^2/μ_0.

Let R, ϕ, Z be right-handed cylindrical coordinates whose Jacobian is

$$(\nabla R\times\nabla\phi\cdot\nabla Z)^{-1} = R. \tag{1}$$

Note that $|\nabla\phi| = 1/R$.

Let r, θ, ϕ be right-handed flux-coordinates whose Jacobian is

$$\mathcal{J}(r,\theta)\equiv(\nabla r\times\nabla\theta\cdot\nabla\phi)^{-1}\equiv R\left(\frac{\partial R}{\partial\theta}\,\frac{\partial Z}{\partial r}-\frac{\partial R}{\partial r}\,\frac{\partial Z}{\partial\theta}\right) = r\,R^2. \tag{2}$$

Note that $r = r(R,Z)$ and $\theta=\theta(R,Z)$. The magnetic axis corresponds to $r=0$. The plasma-vacuum interface corresponds to $r=a$. The inboard mid-plane corresponds to $\theta=0$.

Consider an axisymmetric tokamak equilibrium whose magnetic field takes the form

$$\mathbf{B}(r,\theta) = f(r)\,\nabla\phi\times\nabla r + g(r)\,\nabla\phi = f\,\nabla(\phi-q\,\theta)\times\nabla r, \tag{3}$$

where

$$q(r) = \frac{r\,g}{f} \tag{4}$$

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is the safety-factor (i.e., the inverse of the rotational transform). Note that $\mathbf{B}\cdot\nabla r = 0$, which implies that r is a magnetic flux-surface label. We require $g = 1$ on the magnetic axis in order to ensure that the normalized toroidal magnetic field-strength at the axis is unity.

It is easily demonstrated that

$$B^r = \mathbf{B}\cdot\nabla r = 0, \tag{5}$$

$$B^\theta = \mathbf{B}\cdot\nabla\theta = \frac{f}{r\,R^2}, \tag{6}$$

$$B^\phi = \mathbf{B}\cdot\nabla\phi = \frac{g}{R^2}, \tag{7}$$

$$B_r = \mathcal{J}\,\nabla\theta\times\nabla\phi\cdot\mathbf{B} = -r\,f\,\nabla r\cdot\nabla\theta, \tag{8}$$

$$B_\theta = \mathcal{J}\,\nabla\phi\times\nabla r\cdot\mathbf{B} = r\,f\,|\nabla r|^2, \tag{9}$$

$$B_\phi = \mathcal{J}\,\nabla r\times\nabla\theta\cdot\mathbf{B} = g. \tag{10}$$

The Maxwell equation (neglecting the displacement current, because the plasma velocity perturbations due to axisymmetric modes are far smaller than the velocity of light in vacuum) $\mathbf{J} = \nabla\times\mathbf{B}$ yields

$$\mathcal{J}\,J^r = \frac{\partial B_\phi}{\partial\theta} = 0, \tag{11}$$

$$\mathcal{J}\,J^\theta = -\frac{\partial B_\phi}{\partial r} = -g', \tag{12}$$

$$\mathcal{J}\,J^\phi = \frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{\partial\theta} = \frac{\partial}{\partial r}\left(r\,f\,|\nabla r|^2\right) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\,, \tag{13}$$

where $\mathbf{J}$ is the equilibrium current density, $'\equiv d/dr$, and use has been made of Eqs. (8)–(10).

Equilibrium force balance requires that

$$\nabla P = \mathbf{J}\times\mathbf{B}, \tag{14}$$

where $P(r)$ is the equilibrium scalar plasma pressure. Here, for the sake of simplicity, we have neglected the small centrifugal modifications to force balance due to subsonic plasma rotation. It follows that

$$P' = \mathcal{J}(J^\theta\,B^\phi - J^\phi\,B^\theta) = -g'\,\frac{g}{R^2} - \frac{f}{r\,R^2}\left[\frac{\partial}{\partial r}\left(r\,f\,|\nabla r|^2\right) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\right], \tag{15}$$

where use has been made of Eqs. (5)–(7), and (11)–(13). The other two components of Eq. (14) are identically zero.

Equation (15) yields the *inverse Grad-Shafranov equation*:

$$\frac{f}{r}\frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{f}{r}\frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta) + g\,g' + R^2\,P' = 0. \tag{16}$$

It follows from Eqs. (4), (13), and (16) that

$$\mathcal{J}\,J^{\phi} = -q\,g' - \frac{r\,R^2\,P'}{f}. \tag{17}$$

It is clear from Eqs. (12) and (17) that $g' = P' = 0$ in the current-free "vacuum" region surrounding the plasma, $r > a$. We shall also assume that $g' = P' = 0$ at the plasma-vacuum interface, so as to ensure that the equilibrium plasma current density is zero at the interface, $r = a$.

II. AXISYMMETRIC PLASMA PERTURBATION

A. Derivation of Axisymmetric Ideal-MHD P.D.E.s

Let us assume that all perturbed quantities are independent of the toroidal angle, ϕ. The perturbed plasma equilibrium satisfies the linearized, marginally-stable, ideal-MHD equations

$$\mathbf{b} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}), \tag{18}$$

$$\nabla p = \mathbf{j}\times\mathbf{B} + \mathbf{J}\times\mathbf{b}, \tag{19}$$

$$\mathbf{j} = \nabla\times\mathbf{b}, \tag{20}$$

$$p = -\boldsymbol{\xi}\cdot\nabla P, \tag{21}$$

where $\boldsymbol{\xi}(r,\theta)$ is the plasma displacement, $\mathbf{b}(r,\theta)$ the perturbed magnetic field, $\mathbf{j}(r,\theta)$ the perturbed current density, and $p(r,\theta)$ the perturbed scalar pressure.

Now,

$$(\boldsymbol{\xi}\times\mathbf{B})_\theta = \mathcal{J}\,(\xi^{\phi}\,B^{r} - \xi^{r}\,B^{\phi}) = -\mathcal{J}\,B^{\phi}\,\xi^{r}, \tag{22}$$

$$(\boldsymbol{\xi}\times\mathbf{B})_\phi = \mathcal{J}\,(\xi^{\,r}\,B^{\,\theta}-\xi^{\,\theta}\,B^{\,r}) = \mathcal{J}\,B^{\,\theta}\,\xi^{\,r}, \tag{23}$$

where use has been made of the fact that $B^{\,r} = J^{\,r} = 0$. [See Eqs. (5) and (11).] Combining Eqs. (18) and (23), we obtain

$$\mathcal{J}\,b^{\,r} = \frac{\partial}{\partial\theta}\left(\mathcal{J}\,B^{\,\theta}\,\xi^{\,r}\right). \tag{24}$$

Thus, Eqs. (2), (4), and (6) give

$$r\,R^2\,b^{\,r} = \frac{\partial y}{\partial\theta}, \tag{25}$$

where

$$y(r,\theta) = f\,\xi^{\,r}. \tag{26}$$

The constraint $\nabla\cdot\mathbf{b} = 0$, which follows from Eq. (18), immediately yields

$$r\,R^2\,b^{\,\theta} = -\frac{\partial y}{\partial r}. \tag{27}$$

According to Eq. (21),

$$p = -P'\,\nabla r\cdot\boldsymbol{\xi} = -P'\,\xi^{\,r}. \tag{28}$$

So, the perturbed force balance equation, (19), yields

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = (\mathbf{j}\times\mathbf{B})_r + (\mathbf{J}\times\mathbf{b})_r, \tag{29}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial\theta} = (\mathbf{j}\times\mathbf{B})_\theta + (\mathbf{J}\times\mathbf{b})_\theta, \tag{30}$$

$$0 = (\mathbf{j}\times\mathbf{B})_\phi + (\mathbf{J}\times\mathbf{b})_\phi, \tag{31}$$

giving

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = r\,R^2\,(j^{\,\theta}\,B^{\,\phi} - j^{\,\phi}\,B^{\,\theta}) + r\,R^2\,(J^{\,\theta}\,b^{\,\phi} - J^{\,\phi}\,b^{\,\theta}), \tag{32}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial\theta} = r\,R^2\,(j^{\,\phi}\,B^{\,r} - j^{\,r}\,B^{\,\phi}) + r\,R^2\,(J^{\,\phi}\,b^{\,r} - J^{\,r}\,b^{\,\phi}), \tag{33}$$

$$0 = r\,R^2\,(j^{\,r}\,B^{\,\theta} - j^{\,\theta}\,B^{\,r}) + r\,R^2\,(J^{\,r}\,b^{\,\theta} - J^{\,\theta}\,b^{\,r}), \tag{34}$$

where use has been made of Eq. (2). Thus, according to Eqs. (5)–(7), (11), (12), and (17),

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = f\,(q\,j^{\,\theta} - j^{\,\phi}) - g'\,b^{\phi} + \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)b^{\,\theta}, \tag{35}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial \theta} = -r\,g\,j^{\,r} - \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)b^{\,r}, \tag{36}$$

$$0 = f\,j^{\,r} + g'\,b^{\,r}. \tag{37}$$

It follows from Eqs. (25) and (37) that

$$r\,R^2\,j^{\,r} = -\alpha_g\,\frac{\partial y}{\partial\theta}, \tag{38}$$

where

$$\alpha_g(r) = \frac{g'}{f}. \tag{39}$$

Note that Eq. (36) is trivially satisfied. Hence, of the three components of the perturbed force balance equation, only Eq. (35) remains to be solved.

Equation (20) yields

$$r\,R^2\,j^{\,r} = \frac{\partial b_\phi}{\partial\theta}, \tag{40}$$

$$r\,R^2\,j^{\,\theta} = -\frac{\partial b_\phi}{\partial r}, \tag{41}$$

$$r\,R^2\,j^{\,\phi} = \frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial\theta}, \tag{42}$$

where use has been made of Eq. (2). It follows from Eqs. (38), (40), and (41) that

$$b_\phi = -\alpha_g\,y, \tag{43}$$

$$r\,R^2\,j^{\,\theta} = \frac{\partial(\alpha_g\,y)}{\partial r}. \tag{44}$$

Note that $\nabla\cdot\mathbf{j} = 0$, in accordance with Eq. (20).

Now,

$$\mathbf{b} = b_r\,\nabla r + b_\theta\,\nabla\theta + b_\phi\,\nabla\phi, \tag{45}$$

so

$$b^{\,r} = \mathbf{b}\cdot\nabla r = |\nabla r|^2\,b_r + (\nabla r\cdot\nabla\theta)\,b_\theta, \tag{46}$$

$$b^{\theta} = \mathbf{b}\cdot\nabla\theta = (\nabla r\cdot\nabla\theta)\, b_r + |\nabla\theta|^2\, b_\theta, \tag{47}$$

$$b^{\phi} = \mathbf{b}\cdot\nabla\phi = \frac{b_\phi}{R^2}. \tag{48}$$

Equations (2), (46), and (47) can be rearranged to give

$$b_r = \left(\frac{1}{|\nabla r|^2}\right) b^r - \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right) b_\theta, \tag{49}$$

$$b^\theta = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right) b^r + \left(\frac{1}{r^2\, R^2\, |\nabla r|^2}\right) b_\theta. \tag{50}$$

Let

$$\mathcal{Z}(r,\theta) = |\nabla r|^2\, r\,\frac{\partial y}{\partial r} + r\,\nabla r\cdot\nabla\theta\,\frac{\partial y}{\partial\theta}. \tag{51}$$

Equations (25), (27), (43), (49) and (50) yield

$$b_r = \frac{1}{r\,|\nabla r|^2\, R^2}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\, z, \tag{52}$$

$$b_\theta = -\mathcal{Z}, \tag{53}$$

$$b^\phi = -\frac{\alpha_g}{R^2}\, y. \tag{54}$$

Equations (42), (52), and (53) give

$$r\, R^2\, j^{\phi} = -\frac{\partial\mathcal{Z}}{\partial r} - \frac{\partial}{\partial\theta}\left[\frac{1}{r\,|\nabla r|^2\, R^2}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\mathcal{Z}\right]. \tag{55}$$

It follows from Eqs. (26), (27), (35), (44), (54), and (55) that

$$\begin{aligned}
-\frac{\partial}{\partial r}\left(\frac{P'}{f}\, y\right) &= \frac{f\, q}{r\, R^2}\,\frac{\partial(\alpha_g\, y)}{\partial r} + \frac{f}{r\, R^2}\,\frac{\partial\mathcal{Z}}{\partial r}\\
&\quad + \frac{f}{r\, R^2}\frac{\partial}{\partial\theta}\left[\frac{1}{r\,|\nabla r|^2\, R^2}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\mathcal{Z}\right]\\
&\quad + \frac{g'\,\alpha_g}{R^2}\, y - \left(q\, g' + \frac{r\, R^2\, P'}{f}\right)\frac{1}{r\, R^2}\,\frac{\partial y}{\partial r}.
\end{aligned} \tag{56}$$

Hence,

$$-\left[(\alpha_f\,\alpha_p + r\,\alpha_p')R^2 + q\, r\,\alpha_g' + r^2\,\alpha_g^2\right] y = r\,\frac{\partial\mathcal{Z}}{\partial r} + \frac{\partial}{\partial\theta}\left[\frac{1}{|\nabla r|^2\, R^2}\,\frac{\partial y}{\partial\theta} + \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\mathcal{Z}\right], \tag{57}$$

where

$$\alpha_p(r) = \frac{r\, P'}{f^2}, \tag{58}$$

$$\alpha_f(r) = \frac{r^2}{f}\,\frac{d}{dr}\!\left(\frac{f}{r}\right). \tag{59}$$

Finally, Eqs. (51) and (57) yield the *axisymmetric ideal-MHD p.d.e.s*:

$$r\,\frac{\partial y}{\partial r} = \frac{\mathcal{Z}}{|\nabla r|^2} - \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\frac{\partial y}{\partial\theta}, \tag{60}$$

$$\begin{aligned} r\,\frac{\partial \mathcal{Z}}{\partial r} = &-\left[(\alpha_f\,\alpha_p + r\,\alpha_p')R^2 + q\,r\,\alpha_g' + r^2\,\alpha_g^2\right] y - \frac{\partial}{\partial\theta}\!\left(\frac{1}{|\nabla r|^2\,R^2}\,\frac{\partial y}{\partial\theta}\right)\\ &- \frac{\partial}{\partial\theta}\!\left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\mathcal{Z}\right). \end{aligned} \tag{61}$$

B. Derivation of the Axisymmetric Ideal-MHD O.D.E.s

Let

$$y(r,\theta) = \sum_m y_m(r)\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}, \tag{62}$$

$$\mathcal{Z}(r,\theta) = \sum_m Z_m(r)\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}. \tag{63}$$

Equations (60) and (61) yield the *axisymmetric ideal-MHD o.d.e.s*:

$$r\,\frac{dy_m}{dr} = \sum_{m'}\left(A_m^{m'}\,Z_{m'} + B_m^{m'}\,y_{m'}\right), \tag{64}$$

$$r\,\frac{dZ_m}{dr} = \sum_{m'}\left(C_m^{m'}\,Z_{m'} + D_m^{m'}\,y_{m'}\right), \tag{65}$$

where

$$A_m^{m'} = c_m^{m'}, \tag{66}$$

$$B_m^{m'} = -m'\,f_m^{m'}, \tag{67}$$

$$C_m^{m'} = -m\,f_m^{m'}, \tag{68}$$

$$D_m^{m'} = -(\alpha_f\,\alpha_p + r\,\alpha_p')\,a_m^{m'} - (q\,r\,\alpha_g' + r^2\,\alpha_g^2)\,\delta_m^{m'} + m\,m'\,b_m^{m'}, \tag{69}$$

and

$$a_m^{m'}(r) = \oint R^2\,\exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{70}$$

$$b_m^{m'}(r) = \oint |\nabla r|^{-2}\,R^{-2}\,\exp[-{\rm i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{71}$$

$$c_m^{m'}(r) = \oint |\nabla r|^{-2}\,\exp[-{\rm i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{72}$$

$$f_m^{m'}(r) = \oint \frac{{\rm i}\,r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\exp[-{\rm i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}. \tag{73}$$

C. Properties of Axisymmetric Ideal-MHD O.D.E.s

Note, from Eq. (25) that $\oint \mathcal{J}\,\mathbf{b}\cdot\nabla r\,d\theta = 0$. Note, from Eqs. (64)–(69), that $dZ_0/dr = 0$ in the vacuum region surrounding the plasma. Note also that $a_{m'}^m = a_m^{m'*}$, $b_{m'}^m = b_m^{m'*}$, $c_{m'}^m = c_m^{m'*}$, and $f_{m'}^m = -f_m^{m'*}$, which implies that

$$A_{m'}^m = A_{m'}^{m\,*}, \tag{74}$$

$$B_{m'}^m = -C_{m'}^{m\,*}, \tag{75}$$

$$C_{m'}^m = -B_{m'}^{m\,*}, \tag{76}$$

$$D_{m'}^m = D_{m'}^{m\,*}. \tag{77}$$

It follows from Eqs. (64), (65), and (74)–(77) that

$$r\,\frac{d}{dr}\left[\sum_m (Z_m\,y_m^* - y_m\,Z_m^*)\right] = 0. \tag{78}$$

D. Perturbed Electric Field

Let $\mathbf{e}$ be the perturbed electric field, which satisfies

$$\nabla\times\mathbf{e} = {\rm i}\,\omega\,\mathbf{b}. \tag{79}$$

Hence,

$$e_\phi = {\rm i}\,\omega\,y, \tag{80}$$

and

$$\frac{\partial e_\theta}{\partial r} - \frac{\partial e_r}{\partial\theta} = -{\rm i}\,\omega\,r\,\alpha_g\,y, \tag{81}$$

where use has been made of Eqs. (25), (27), and (54). We also expect $\nabla\cdot\mathbf{e} = 0$, which implies that $e_r = e_\theta = 0$ in the vacuum region, $r \geq a$, in which $\alpha_g = 0$.

E. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque exerted on the plasma lying within the magnetic flux-surface whose label is r is

$$T_\phi(r) = \oint\oint r\,R^2\,b_\phi\,b^r\,d\theta\,d\phi. \tag{82}$$

It follows from Eqs. (25) and (43) that

$$T_\phi(r) = -\pi\,\alpha_g \oint\left(y^\ast\,\frac{\partial y}{\partial\theta} + y\,\frac{\partial y^\ast}{\partial\theta}\right)d\theta = -\pi\,\alpha_g\oint\frac{\partial|y|^2}{\partial\theta}\,d\theta = 0. \tag{83}$$

We conclude that an axisymmetric perturbation is incapable of exerting a net toroidal electromagnetic torque on the plasma.

F. Electromagnetic Energy Flux

The net electromagnetic energy flux across the plasma-vacuum interface is

$$\begin{aligned}\mathcal{E} &= \left[\oint\oint(\mathbf{e}\times\mathbf{b})\cdot\nabla r\,\mathcal{J}\,d\theta\,d\phi\right]_{r=a} = \left[\oint\oint(e_\theta\,b_\phi - e_\phi\,b_\theta)\,d\theta\,d\phi\right]_{r=a}\\ &= \mathrm{i}\,\pi\,\omega\oint(y\,\mathcal{Z}^\ast - y^\ast\,\mathcal{Z})_{r=a}\,d\theta,\\ &= \mathrm{i}\,\pi^2\,\omega\sum_m\left(Z_m^\ast\,y_m - y_m^\ast\,Z_m\right)_{r=a}.\end{aligned} \tag{84}$$

Here, use has been made of Eqs. (53 and (80), as well as the fact that $e_\theta = 0$ for $r\geq a$.

G. Perturbed Plasma Potential Energy

The perturbed plasma potential energy in the region of the plasma lying within the magnetic flux-surface whose label is r is

$$\delta W_p = \frac{1}{2}\oint\oint r\,R^2\,\xi^{\,r\ast}(-\mathbf{B}\cdot\mathbf{b} + \xi^{\,r}\,P')\,d\theta\,d\phi. \tag{85}$$

However,

$$\mathbf{B}\cdot\mathbf{b} - \xi^{\,r}\,P' = B^{\,\theta}\,b_\theta + B^{\,\phi}\,b_\phi - \xi^{\,r}\,P' = -\frac{f}{r\,R^2}\left(\mathcal{Z} + q\,\alpha_g\,y + \alpha_p\,R^2\right), \tag{86}$$

where use has been made of Eqs. (4)–(7), (26), (43), (53), and (58). Hence, we obtain

$$\delta W_p(r) = \frac{1}{2}\oint\oint y^*\left[\mathcal{Z} + (q\,\alpha_g + \alpha_p\,R^2)\,y\right]d\theta\,d\phi = \pi^2\sum_m y_m^*\,\chi_m, \tag{87}$$

where

$$\chi_m(r) = Z_m + q\,\alpha_g\,y_m + \alpha_p\sum_{m'} a_m^{m'}\,y_{m'}. \tag{88}$$

III. INVERSE ASPECT-RATIO EXPANDED TOKAMAK EQUILIBRIUM

A. Equilibrium Magnetic Flux-Surfaces

Let us assume that the inverse aspect-ratio of the plasma, $\epsilon = a/R_0 = a$ (since R_0 is normalized to unity), is such that $0 < \epsilon \ll 1$. Let $r = \epsilon\,\hat{r}$, $\nabla = \epsilon^{-1}\,\hat{\nabla}$, and $' \to \epsilon^{-1}\,'$. Suppose that the loci of the equilibrium magnetic flux-surfaces can be written in the parametric form:

$$R(\hat{r},\omega) = 1 - \epsilon\,\hat{r}\,\cos\omega + \epsilon^2\sum_{j>0} H_j(\hat{r})\,\cos[(j-1)\,\omega] + \epsilon^2\sum_{j>1} V_j(\hat{r})\,\sin[(j-1)\,\omega]$$
$$+\,\epsilon^3\,L(\hat{r})\,\cos\omega, \tag{89}$$
$$Z(\hat{r},\omega) = \epsilon\,\hat{r}\,\sin\omega + \epsilon^2\sum_{j>1} H_j(\hat{r})\,\sin[(j-1)\,\omega] - \epsilon^2\sum_{j>1} V_j(\hat{r})\,\cos[(j-1)\,\omega]$$
$$-\,\epsilon^3\,L(\hat{r})\,\sin\omega, \tag{90}$$

where j is a positive integer. Here, $H_1(\hat{r})$ controls the relative horizontal locations of the flux-surface centroids, $H_2(\hat{r})$ and $V_2(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface ellipticities, $H_3(\hat{r})$ and $V_3(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface triangularities, et cetera, whereas $L(\hat{r})$ is a flux-surface re-labelling parameter. Moreover, $\omega(R,Z)$ is a poloidal angle that is distinct from θ. Note that V_1 does not appear in Eq. (90) because such a factor merely gives rise to a rigid vertical shift of the plasma that can be eliminated by a suitable choice of the origin of the flux-coordinate system.

Let

$$J(\hat{r},\omega) = \frac{1}{\epsilon^2}\left(\frac{\partial R}{\partial\omega}\frac{\partial Z}{\partial\hat{r}} - \frac{\partial R}{\partial\hat{r}}\frac{\partial Z}{\partial\omega}\right) \tag{91}$$

be the Jacobian of the $\hat{r}$, ω coordinate system. We can transform to the $\hat{r}$, θ coordinate system by writing

$$\theta(\hat{r},\omega) = 2\pi \int_0^\omega \frac{J(\hat{r},\tilde{\omega})}{R(\hat{r},\tilde{\omega})}\,d\tilde{\omega} \Big/ \oint \frac{J(\hat{r},\omega)}{R(\hat{r},\omega)}\,d\omega, \tag{92}$$

$$\hat{r} = \frac{1}{2\pi}\oint \frac{J(\hat{r},\omega)}{R(\hat{r},\omega)}\,d\omega. \tag{93}$$

This transformation ensures that

$$\frac{\partial\theta}{\partial\omega} = \frac{J}{\hat{r}\,R}, \tag{94}$$

and, hence, that

$$\mathcal{J} \equiv \frac{R}{\epsilon}\left(\frac{\partial R}{\partial\theta}\,\frac{\partial Z}{\partial\hat{r}} - \frac{\partial R}{\partial\hat{r}}\,\frac{\partial Z}{\partial\hat{r}}\right) = \epsilon\,R\,J\,\frac{\partial\omega}{\partial\theta} = r\,R^2, \tag{95}$$

in accordance with Eq. (2).

B. Metric Elements

We can determine the metric elements of the flux-coordinate system by combining Eqs. (89)–(93). Evaluating the elements up to $\mathcal{O}(\epsilon)$, but retaining $\mathcal{O}(\epsilon^2)$ contributions to terms that are independent of ω, we obtain,

$$L(\hat{r}) = \frac{\hat{r}^{\,3}}{8} - \frac{\hat{r}\,H_1}{2} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{H_j^{\,2}}{\hat{r}} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{V_j^{\,2}}{\hat{r}}, \tag{96}$$

$$\begin{aligned}\theta &= \omega + \epsilon\,\hat{r}\,\sin\omega - \epsilon\sum_{j>0}\frac{1}{j}\left[H_j' - (j-1)\,\frac{H_j}{\hat{r}}\right]\sin(j\,\omega)\\ &\quad + \epsilon\sum_{j>1}\frac{1}{j}\left[V_j' - (j-1)\,\frac{V_j}{\hat{r}}\right]\cos(j\,\omega),\end{aligned} \tag{97}$$

$$\begin{aligned}|\hat{\nabla}\hat{r}|^2 &= 1 + 2\,\epsilon\sum_{j>0}H_j'\,\cos(j\,\theta) + 2\,\epsilon\sum_{j>1}V_j'\,\sin(j\,\theta)\\ &\quad + \epsilon^2\left(\frac{3\,\hat{r}^{\,2}}{4} - H_1 + \frac{1}{2}\sum_{j>0}\left[H_j'^{\,2} + (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right]\right.\\ &\quad \left. + \frac{1}{2}\sum_{j>1}\left[V_j'^{\,2} + (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right]\right),\end{aligned} \tag{98}$$

$$\hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta = \epsilon\,\sin\theta - \epsilon\sum_{j>0}\frac{1}{j}\left[H_j'' + \frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}}\right]\sin(j\,\theta)$$

$$+\,\epsilon\sum_{j>1}\frac{1}{j}\left[V_j'' + \frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}}\right]\cos(j\,\theta), \tag{99}$$

$$R^{\,2} = 1 - 2\,\epsilon\,\hat{r}\,\cos\theta - \epsilon^{\,2}\left(\frac{\hat{r}^{\,2}}{2} - \hat{r}\,H_1' - 2\,H_1\right). \tag{100}$$

Here, $' \equiv d/d\hat{r}$. Moreover, we have made use of the fact that $V_j \propto H_j$, for $j>1$, because V_j and H_j satisfy the identical differential equations, (106) and (107).

C. Expansion of Inverse Grad-Shafranov Equation

Let us write

$$f(\hat{r}) = \epsilon\,\frac{\hat{r}\,g}{q}, \tag{101}$$

$$g(\hat{r}) = 1 + \epsilon^{\,2}\,g_2(\hat{r}) + \epsilon^{\,4}\,g_4(\hat{r}), \tag{102}$$

$$P'(\hat{r}) = \epsilon^{\,2}\,p_2'(\hat{r}), \tag{103}$$

where q, g_2, g_4, and p_2 are all $\mathcal{O}(1)$. Here, the safety-factor, $q(\hat{r})$, and the second-order plasma pressure gradient, $p_2'(\hat{r})$, are the two free flux-surface functions that characterize the plasma equilibrium.

Expanding the inverse Grad-Shafranov equation, (16), order by order in the small parameter ϵ, making use of Eqs. (98)–(103), we obtain

$$g_2' = -p_2' - \frac{\hat{r}}{q^2}\,(2-s), \tag{104}$$

$$H_1'' = -(3-2\,s)\,\frac{H_1'}{\hat{r}} - 1 + \frac{2\,p_2'\,q^2}{\hat{r}}, \tag{105}$$

$$H_j'' = -(3-2\,s)\,\frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}} \qquad \text{for } j>1, \tag{106}$$

$$V_j'' = -(3-2\,s)\,\frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}} \qquad \text{for } j>1, \tag{107}$$

$$g_4' = g_2\left[p_2' - \frac{\hat{r}}{q^2}\,(2-s)\right] - \frac{\hat{r}}{q}\,\Sigma + p_2'\left(\frac{\hat{r}^{\,2}}{2} + \frac{\hat{r}^{\,2}}{q^2} - 2\,H_1 - 3\,\hat{r}\,H_1'\right), \tag{108}$$

where $s = \hat{r}\,q'/q$ is the magnetic shear, and

$$\Sigma = \frac{S_2}{q} - \frac{2-s}{q}\,S_3 \tag{109}$$

$$S_1(\hat{r}) = \frac{1}{2}\sum_{j>0}\left[3\,H_j'^{\,2} - (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^2}\right] + \frac{1}{2}\sum_{j>1}\left[3\,V_j'^{\,2} - (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^2}\right], \tag{110}$$

$$\begin{aligned} S_2(\hat{r}) &= \frac{3\,\hat{r}^{\,2}}{2} - 2\,\hat{r}\,H_1' + \sum_{j>0}\left[H_j'^{\,2} + 2\,(j^2-1)\,\frac{H_j'\,H_j}{\hat{r}} - (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right] \\ &\quad + \sum_{j>1}\left[V_j'^{\,2} + 2\,(j^2-1)\,\frac{V_j'\,V_j}{\hat{r}} - (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right], \end{aligned} \tag{111}$$

$$S_3(\hat{r}) = -\,\frac{3\,\hat{r}^{\,2}}{4} + \frac{\hat{r}^{\,2}}{q^2} + H_1 + S_1, \tag{112}$$

$$S_5(\hat{r}) = \frac{7\,\hat{r}^{\,2}}{4} - H_1 - 3\,\hat{r}\,H_1' + S_1. \tag{113}$$

Note that the relative horizontal shift of magnetic flux-surfaces, $-H_1$, otherwise known as the *Shafranov shift*, is driven by toroidicity [the second term on the right-hand side of Eq. (105)], and plasma pressure gradients (the third term). All of the other shaping terms (i.e., the H_j, for $j > 1$, and the V_j) are driven by axisymmetric currents flowing in external magnetic field-coils.

Equations (39), (58), (59), and (101)–(103) yield

$$\alpha_p(\hat{r}) = \frac{p_2'\,q^2}{\hat{r}}\left(1 - 2\,\epsilon^{\,2}\,g_2\right), \tag{114}$$

$$\alpha_g(\hat{r}) = \frac{q}{\hat{r}}\left(g_2' - \epsilon^{\,2}\,g_2\,g_2' + \epsilon^{\,2}\,g_4'\right), \tag{115}$$

$$\alpha_f(\hat{r}) = -s + \epsilon^{\,2}\,\hat{r}\,g_2'. \tag{116}$$

Finally, it follows from Eqs. (99) and (105)–(107) that

$$\begin{aligned} \hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta &= 2\,\epsilon\left[1 - \frac{p_2'\,q^2}{\hat{r}} + (1-s)\,\frac{H_1'}{\hat{r}}\right]\sin\theta \\ &\quad - 2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}}\right]\sin(j\,\theta) \\ &\quad + 2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}}\right]\cos(j\,\theta). \end{aligned} \tag{117}$$

D. Calculation of Coupling Coefficients

Equations (98) and (110) yield

$$|\hat{\nabla}\hat{r}|^{-2} = 1 - 2\,\epsilon\sum_{j>0} H_j'\,\cos(j\,\theta) - 2\,\epsilon\sum_{j>0} V_j'\sin(j\,\theta) + \epsilon^2\left(-\frac{3\,\hat{r}^{\,2}}{4} + H_1 + S_1\right), \tag{118}$$

Equation (100) gives

$$R^{-2} = 1 + 2\,\epsilon\,\hat{r}\,\cos\theta + \epsilon^2\left(\frac{5\,\hat{r}^{\,2}}{2} - \hat{r}\,H_1' - 2\,H_1\right). \tag{119}$$

The previous two equations imply that

$$|\hat{\nabla}\hat{r}|^{-2}\,R^{-2} = 1 + 2\,\epsilon\,\hat{r}\,\cos\theta - 2\,\epsilon\sum_{j>0} H_j'\,\cos(j\,\theta) - 2\,\epsilon\sum_{>1} V_j'\,\sin(j\,\theta)$$
$$+\,\epsilon^2\left(\frac{7\,\hat{r}^{\,2}}{4} - H_1 - 3\,\hat{r}\,H_1' + S_1\right). \tag{120}$$

Finally, Eqs. (117) and (118) give

$$\hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta\,|\hat{\nabla}\hat{r}|^{-2} = 2\,\epsilon\left[1 - \frac{p_2'\,q^2}{\hat{r}} + (1-s)\,\frac{H_1'}{\hat{r}}\right]\sin\theta$$
$$-\,2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}}\right]\sin(j\,\theta)$$
$$+\,2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}}\right]\cos(j\,\theta), \tag{121}$$

where use has been made of the fact that $V_j' \propto H_j'$ for $j>1$.

Equations (70)–(73), (100), (118), (120), and (121) imply that

$$a_m^{m'} = \delta_m^{m'} - \epsilon\,\hat{r}\,(\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon^2\left(\frac{\hat{r}^{\,2}}{2} - \hat{r}\,H_1' - 2\,H_1\right)\delta_m^{m'}, \tag{122}$$

$$b_m^{m'} = \delta_m^{m'} + \epsilon\,\hat{r}\,(\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon\sum_{j>0} H_j'\,(\delta_{m'-m-j} + \delta_{m'-m+j})$$
$$-\,\epsilon\sum_{j>1}\mathrm{i}\,V_j'\,(\delta_{m'-m-j} - \delta_{m'-m+j}) + \epsilon^2\left(\frac{7\,\hat{r}^{\,2}}{4} - H_1 - 3\,\hat{r}\,H_1' + S_1\right)\delta_m^{m'}, \tag{123}$$

$$c_m^{m'} = \delta_m^{m'} - \epsilon\sum_{j>0} H_j'\,(\delta_{m'-m-j} + \delta_{m'-m+j}) - \epsilon\sum_{j>1}\mathrm{i}\,V_j'\,(\delta_{m'-m-j} - \delta_{m'-m+j})$$

$$+\,\epsilon^2\left(-\frac{3\,\hat{r}^{\,2}}{4}+H_1+S_1\right)\delta_m^{\,m'}, \tag{124}$$

$$\begin{aligned} f_m^{\,m'} &= -\epsilon\left[\hat{r}-p_2'\,q^2+(1-s)\,H_1'\right](\delta_{m'-m-1}-\delta_{m'-m+1})\\ &\quad+\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,H_j'+(j^2-1)\,\frac{H_j}{\hat{r}}\right](\delta_{m'-m-j}-\delta_{m'-m+j})\\ &\quad+\epsilon\sum_{j>1}\frac{\mathrm{i}}{j}\left[-(1-s)\,V_j'+(j^2-1)\,\frac{V_j}{\hat{r}}\right](\delta_{m'-m-j}+\delta_{m'-m+j})\,. \end{aligned} \tag{125}$$

If we write

$$\alpha_g = \alpha_g^{(0)}+\epsilon^2\,\alpha_g^{(2)}, \tag{126}$$

$$\alpha_p = \alpha_p^{(0)}+\epsilon^2\,\alpha_p^{(2)}, \tag{127}$$

$$\alpha_f = \alpha_f^{(0)}+\epsilon^2\,\alpha_f^{(2)}, \tag{128}$$

$$a_m^{m'} = 1+\epsilon\,a_m^{m'\,(1)}+\epsilon^2\,a_m^{m'\,(2)}, \tag{129}$$

$$b_m^{m'} = 1+\epsilon\,b_m^{m'\,(1)}+\epsilon^2\,b_m^{m'\,(2)}, \tag{130}$$

$$D_m^{m'} = D_m^{m'\,(0)}+\epsilon\,D_m^{m'\,(1)}+\epsilon^2\,D_m^{m'\,(2)}, \tag{131}$$

where $\alpha_g^{(0)}$, $\alpha_g^{(2)}$, et cetera, are $\mathcal{O}(1)$, then it follows from Eq. (69) that

$$D_m^{m\,(0)} = -\alpha_f^{(0)}\,\alpha_p^{(0)}-\hat{r}\,\alpha_p'^{(0)}-q\,\hat{r}\,\alpha_g'^{(0)}+m^2, \tag{132}$$

$$D_m^{m'\,(1)} = -\left[\alpha_f^{(0)}\,\alpha_p^{(0)}+\hat{r}\,\alpha_p'^{(0)}\right]a_m^{m'\,(1)}+m\,m'\,b_m^{m'\,(1)}, \tag{133}$$

$$\begin{aligned} D_m^{m\,(2)} &= -\left[\alpha_f^{(0)}\,\alpha_p^{(0)}+\hat{r}\,\alpha_p'^{(0)}\right]a_m^{m'\,(2)}-\alpha_f^{(0)}\,\alpha_p^{(2)}-\alpha_f^{(2)}\,\alpha_p^{(0)}-\hat{r}\,\alpha_p'^{(2)}-q\,\hat{r}\,\alpha_g'^{(2)}\\ &\quad-\hat{r}^{\,2}\left[\alpha_g^{(0)}\right]^2+m^2\,b_m^{m\,(2)}. \end{aligned} \tag{134}$$

Finally, Eqs. (70)–(73), (122)–(125), and (132)–(134) give

$$A_m^m(\hat{r}) = 1+\epsilon^2\left(-\frac{3\,\hat{r}^{\,2}}{4}+H_1+S_1\right), \tag{135}$$

$$A_m^{m\pm 1}(\hat{r}) = -\epsilon\,H_1', \tag{136}$$

$$A_m^{m\pm j}(\hat{r}) = -\epsilon\,(H_j'\pm\mathrm{i}\,V_j')\qquad\text{for } j>1, \tag{137}$$

$$B_m^m(\hat{r}) = 0, \tag{138}$$

$$B_m^{m\pm 1}(\hat{r}) = \pm\epsilon\,(m\pm 1)\left[\hat{r} - p_2'\,q^2 + (1-s)\,H_1'\right], \tag{139}$$

$$B_m^{m\pm j}(\hat{r}) = \pm\epsilon\,\frac{m\pm j}{j}\,\left[(1-s)\,(H_j'\pm \mathrm{i}\,V_j') - (j^2-1)\,\frac{H_j \pm \mathrm{i}\,V_j}{\hat{r}}\right] \qquad \text{for } j>1, \tag{140}$$

$$C_m^{m}(\hat{r}) = 0, \tag{141}$$

$$C_m^{m\pm 1}(\hat{r}) = \pm\epsilon\,m\left[\hat{r} - p_2'\,q^2 + (1-s)\,H_1'\right], \tag{142}$$

$$C_m^{m\pm j}(\hat{r}) = \pm\epsilon\,\frac{m}{j}\,\left[(1-s)\,(H_j'\pm \mathrm{i}\,V_j') - (j^2-1)\,\frac{H_j \pm \mathrm{i}\,V_j}{\hat{r}}\right] \qquad \text{for } j>1, \tag{143}$$

$$\begin{aligned} D_m^{m}(\hat{r}) &= m^2 + q\,\hat{r}\,\frac{d}{d\hat{r}}\left(\frac{2-s}{q}\right) + \epsilon^2\,m^2\,S_5 \\ &\quad + \epsilon^2\left\{-\hat{r}^{\,2}\,\frac{(2-s)^2}{q^2} + q\,\hat{r}\,\frac{d\Sigma}{d\hat{r}} - \hat{r}\,\frac{d}{d\hat{r}}(\hat{r}\,p_2') - 2\,(1-s)\,\hat{r}\,p_2'\right. \\ &\quad \left. +2\,\hat{r}\,p_2'\,q^2\left(-2+\frac{3\,p_2'\,q^2}{\hat{r}}\right) + 2\,H_1'\,q^2\left[\frac{d}{d\hat{r}}(\hat{r}\,p_2') - 4\,(1-s)\,p_2'\right]\right\}, \end{aligned} \tag{144}$$

$$D_m^{m\pm 1}(\hat{r}) = \epsilon\left[\frac{d}{d\hat{r}}(\hat{r}\,p_2') - (2-s)\,p_2'\right]q^2 + \epsilon\,m\,(m\pm 1)\,(\hat{r} - H_1'), \tag{145}$$

$$D_m^{m\pm j}(\hat{r}) = -\epsilon\,m\,(m\pm j)\,(H_j'\pm \mathrm{i}\,V_j') \qquad \text{for } j>1. \tag{146}$$

E. Behavior Close to Magnetic Axis

When $\hat{r} \ll 1$, the well-behaved solution of the axisymmetric ideal-MHD o.d.e.s, (64) and (65), that is dominated by the poloidal harmonic whose poloidal mode number is m is such that

$$y_m(\hat{r}) = \hat{r}^{\,|m|}, \tag{147}$$

$$z_m(\hat{r}) = |m|\,\hat{r}^{\,|m|}, \tag{148}$$

with $y_{m'}(\hat{r}) = z_{m'}(\hat{r}) = 0$ for $m' \neq 0$.

IV. VACUUM SOLUTION

A. Toroidal Coordinates

Let μ, η, ϕ be right-handed toroidal coordinates defined such that

$$R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}, \tag{149}$$

$$Z = \frac{\sin\eta}{\cosh\mu - \cos\eta}. \tag{150}$$

The scale-factors of the toroidal coordinate system are

$$h_\mu = h_\eta = \frac{1}{\cosh\mu - \cos\eta} \equiv h, \tag{151}$$

$$h_\phi = \frac{\sinh\mu}{\cosh\mu - \cos\eta} = h\,\sinh\mu. \tag{152}$$

Moreover,

$$\mathcal{J}' \equiv (\nabla\mu\times\nabla\eta\cdot\nabla\phi)^{-1} = h^3\,\sinh\mu. \tag{153}$$

B. Perturbed Magnetic Field

The curl-free perturbed magnetic field in the vacuum region is written $\mathbf{b} = \mathrm{i}\,\nabla V$, where $\nabla^2 V = 0$. The most general axisymmetric solution to Laplace's equation is

$$V(z,\eta) = \sum_m (z-\cos\eta)^{1/2}\,U_m(z)\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}, \tag{154}$$

$$U_m(z) = p_m\,\hat{P}_{|m|-1/2}(z) + q_m\,\hat{Q}_{m-1/2}(z), \tag{155}$$

where $z = \cosh\mu$, the p_m and q_m are arbitrary complex coefficients, and

$$\hat{P}_{|m|-1/2}(z) = \cos(|m|\,\pi)\,\frac{\sqrt{\pi}\,\Gamma(|m|+1/2)\,a^{\,|m|}}{2^{\,|m|-1/2}\,|m|!}\,P_{|m|-1/2}(z), \tag{156}$$

$$\hat{Q}_{|m|-1/2}(z) = \cos(|m|\,\pi)\,\frac{2^{\,|m|-1/2}\,|m|!}{\sqrt{\pi}\,\Gamma(|m|+1/2)\,a^{\,|m|}}\,Q_{|m|-1/2}(z). \tag{157}$$

Here, the $P_{|m|-1/2}(z)$ and $Q_{|m|-1/2}(z)$ are toroidal functions, and $\Gamma(z)$ is a gamma function.

C. Toroidal Electromagnetic Angular Momentum Flux

The outward flux of toroidal angular momentum across a constant-z surface is

$$T_\phi(z) = -\oint\oint \mathcal{J}'\, b_\phi\, b^\mu\, d\eta\, d\phi = 0, \tag{158}$$

because $b_\phi = \mathrm{i}\,\partial V/\partial\phi = 0$. Of course, this has to be the case because the flux of angular momentum across the plasma-vacuum interface is zero. (See Sect. II E.)

D. Electromagnetic Energy Flux

The outward flux of electromagnetic energy flux across a constant-z surface is

$$\mathcal{E}(z) = -\oint\oint \mathcal{J}'\,\mathbf{e}\times\mathbf{b}\cdot\nabla\mu\, d\eta\, d\phi = -\mathrm{i}\,\pi\oint\left(e_\phi\,\frac{\partial V^\ast}{\partial\eta} - e_\phi^\ast\,\frac{\partial V}{\partial\eta}\right)d\eta, \tag{159}$$

given that $e_\mu = e_\eta = 0$ in the vacuum. However, $\nabla\times\mathbf{e} = \mathrm{i}\,\omega\,\mathbf{b}$ implies that

$$\frac{\partial e_\phi}{\partial\eta} = -\omega\, h\,\sinh\mu\,\frac{\partial V}{\partial\mu} = -\omega\, h\,\sinh^2\mu\,\frac{\partial V}{\partial z}. \tag{160}$$

Thus,

$$\begin{aligned}\mathcal{E}(z) &= \mathrm{i}\,\pi\oint\left(\frac{\partial e_\phi}{\partial\eta}\,V^\ast - \frac{\partial e_\phi^\ast}{\partial\eta}\,V\right)d\eta = -\mathrm{i}\,\pi\,\omega\oint h\,\sinh^2\mu\left(\frac{\partial V}{\partial z}\,V^\ast - \frac{\partial V^\ast}{\partial z}\,V\right)d\eta\\ &= \mathrm{i}\,\pi^2\,\omega\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast)\,(z^2-1)\,\mathcal{W}(P_{|m|-1/2}, Q_{|m|-1/2}),\end{aligned} \tag{161}$$

where $\mathcal{W}(f,g) = f\,dg/dz - g\,df/dz$. However,

$$\mathcal{W}(P_{|m|-1/2}, Q_{|m|-1/2}) = \frac{1}{1-z^2}, \tag{162}$$

so

$$\mathcal{E}(z) = -\mathrm{i}\,\pi^2\,\omega\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast). \tag{163}$$

Note that $\mathcal{E}$ is independent of z, as must be the case because there are no energy sources in the vacuum region.

E. Solution in Vacuum Region

In the large-aspect ratio limit, $r \ll 1$, it can be demonstrated that

$$z \simeq \frac{1}{r}, \tag{164}$$

$$z^{\,1/2}\,\hat{P}_{-1/2}(z) \simeq \frac{1}{2}\ln(8\,z), \tag{165}$$

$$z^{1/2}\,\hat{P}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\,\pi)\,(a\,z)^{|m|}}{|m|}, \tag{166}$$

$$z^{1/2}\,\hat{Q}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\,\pi)\,(a\,z)^{-|m|}}{2}. \tag{167}$$

Note that Eq. (166) only applies to $|m| > 0$.

Now, according to Eqs. (25) and (38)

$$\frac{\partial y}{\partial\theta} = \mathcal{J}\,\mathbf{b}\cdot\nabla r = \mathrm{i}\,\mathcal{J}\,\nabla V\cdot\nabla r, \tag{168}$$

$$\mathcal{Z} = -\mathcal{J}\,\nabla\phi\times\nabla r\cdot\mathbf{b} = -\mathrm{i}\,\frac{\partial V}{\partial\theta}. \tag{169}$$

Thus,

$$\underline{V}(r) = \underline{\underline{\mathcal{P}}}(r)\,\underline{p} + \underline{\underline{\mathcal{Q}}}(r)\,\underline{q}, \tag{170}$$

$$\underline{\psi}(r) = \underline{\underline{\mathcal{R}}}(r)\,\underline{p} + \underline{\underline{\mathcal{S}}}(r)\,\underline{q}, \tag{171}$$

where $Z_m(r) = m\,V_m(r)$, $\psi_m(r) = m\,y_m(r)$, $\underline{V}(r)$ is the vector of the $V_m(r)$ values, $\underline{\psi}(r)$ is the vector of the $\psi_m(r)$ values, $\underline{\underline{\mathcal{P}}}(r)$ is the matrix of the

$$\mathcal{P}_{mm'}(r) = \oint_r (z-\cos\eta)^{1/2}\,\hat{P}_{|m'|-1/2}(z)\,\exp[-\mathrm{i}\,(m\,\theta + m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{172}$$

values, $\underline{\underline{\mathcal{Q}}}(r)$ is the matrix of the

$$\mathcal{Q}_{mm'}(r) = \oint_r (z-\cos\eta)^{1/2}\,\hat{Q}_{|m'|-1/2}(z)\,\exp[-\mathrm{i}\,(m\,\theta + m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{173}$$

values, $\underline{\underline{\mathcal{R}}}(r)$ is the matrix of the

$$\mathcal{R}_{mm'}(r) = \oint_r \Bigg\{\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\hat{P}_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{d\hat{P}_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z$$

$$+\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta - {\rm i}\,m'\,(z-\cos\eta)^{1/2}\right]\hat{P}_{|m'|-1/2}z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\Big\}$$

$$\times\exp[-{\rm i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{174}$$

values, $\underline{\underline{\mathcal{S}}}(r)$ is the matrix of the

$$\mathcal{S}_{mm'}(r) = \oint_r \Big\{\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\hat{Q}_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{d\hat{Q}_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z$$

$$+\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta - {\rm i}\,m'\,(z-\cos\eta)^{1/2}\right]\hat{Q}_{|m'|-1/2}(z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\Big\}$$

$$\times\exp[-{\rm i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{175}$$

values, $\underline{p}$ is the vector of the p_m coefficients, and $\underline{q}$ is the vector of the q_m coefficients. Here, the subscript r on the integrals indicates that they are taken at constant r. Note from Eqs. (168) and (169), $\oint \mathcal{J}\,\nabla V\cdot\nabla r\,d\theta = Z_0 = 0$.

F. Energy Conservation

According to Eq. (84), the net flux of electromagnetic energy across the plasma-vacuum interface is

$$\mathcal{E} = {\rm i}\,\pi^2\,\omega\,(\underline{V}^\dagger\,\underline{\psi} - \underline{\psi}^\dagger\,\underline{V}). \tag{176}$$

However, this flux must be equal the energy flux through the vacuum region, so Eq. (163) gives

$$\mathcal{E} = -{\rm i}\,\pi^2\,\omega\,(\underline{q}^\dagger\,\underline{p} - \underline{p}^\dagger\,\underline{q}). \tag{177}$$

Equations (170), (171), and the previous two equations, yield

$$\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{P}}}, \tag{178}$$

$$\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{S}}}^\dagger\,\underline{\underline{\mathcal{Q}}}, \tag{179}$$

$$\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{S}}} - \underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{Q}}} = \underline{\underline{1}}. \tag{180}$$

It can also be demonstrated that

$$\underline{\underline{\mathcal{Q}}}\,\underline{\underline{\mathcal{P}}}^\dagger = \underline{\underline{\mathcal{P}}}\,\underline{\underline{\mathcal{Q}}}^\dagger, \tag{181}$$

$$\underline{\underline{\mathcal{R}}}\,\underline{\underline{\mathcal{S}}}^\dagger = \underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{R}}}^\dagger. \tag{182}$$

The previous five equations hold throughout the vacuum region.

G. Ideal-Wall Matching Condition

Suppose that the plasma is surrounded by a wall that lies at $r = b_w\,a$, where $b_w \geq 1$. If the wall is perfectly conducting then $\underline{\psi}(b_w\,a) = 0$. It follows from Eq. (171) that

$$\underline{q} = \underline{\underline{I}}_b\,\underline{p}, \tag{183}$$

where

$$\underline{\underline{I}}_b = -\underline{\underline{\mathcal{S}}}_b^{-1}\,\underline{\underline{\mathcal{R}}}_b \tag{184}$$

is termed the *wall matrix*. Here, $\underline{\underline{\mathcal{S}}}_b = \underline{\underline{\mathcal{S}}}(r = b_w\,a)$, et cetera. Equation (182) ensures that $\underline{\underline{I}}_b$ is Hermitian. It immediately follows from Eq. (177) that $\mathcal{E} = 0$. In other words, there is zero net electromagnetic energy flux out of a plasma surrounded by a perfectly conducting wall.

Making use of Eqs. (170) and (171), the matching condition at the plasma-vacuum interface for a perfectly-conducting wall becomes

$$\underline{V}(r = a_+) = \underline{\underline{H}}\,\underline{\psi}(r = a), \tag{185}$$

where

$$\underline{\underline{H}} = (\underline{\underline{\mathcal{P}}}_a + \underline{\underline{\mathcal{Q}}}_a\,\underline{\underline{I}}_b)\,(\underline{\underline{\mathcal{R}}}_a + \underline{\underline{\mathcal{S}}}_a\,\underline{\underline{I}}_b)^{-1} \tag{186}$$

is termed the *vacuum matrix*. Here, $\underline{\underline{\mathcal{P}}}_a = \underline{\underline{\mathcal{P}}}(r = a_+)$, et cetera. Making use of Eqs. (178)–(180), it is easily demonstrated that

$$\underline{\underline{H}} - \underline{\underline{H}}^\dagger = -[(\underline{\underline{\mathcal{R}}}_a + \underline{\underline{\mathcal{S}}}_a\,\underline{\underline{I}}_b)^{-1}]^\dagger\,(\underline{\underline{I}}_b - \underline{\underline{I}}_b^\dagger)\,(\underline{\underline{\mathcal{R}}}_a + \underline{\underline{\mathcal{S}}}_a\,\underline{\underline{I}}_b)^{-1}. \tag{187}$$

Thus, the vacuum matrix, $\underline{\underline{H}}$, is Hermitian because the wall matrix, $\underline{\underline{I}}_b$, is Hermitian.

H. Model Wall Matrix

Equations (164)–(167), (174), and (175) suggest that

$$\underline{\underline{\mathcal{R}}}_b = \underline{\underline{\mathcal{R}}}_a\,\underline{\underline{\rho}}^{-1}, \tag{188}$$

$$\underline{\underline{\mathcal{S}}}_b = \underline{\underline{\mathcal{S}}}_a\,\underline{\underline{\rho}}, \tag{189}$$

where

$$\rho_{mm'} = \delta_{mm'}\,\rho_m, \tag{190}$$

$$\rho_0 = 1 + \ln b_w, \tag{191}$$

$$\rho_{m\neq 0} = b_w^{|m|}. \tag{192}$$

Hence,

$$\underline{\underline{I}}_b = -\underline{\underline{\rho}}^{-1\,\dagger}\,\underline{\underline{\mathcal{S}}}_a^{-1}\,\underline{\underline{\mathcal{R}}}_a\,\underline{\underline{\rho}}^{-1}. \tag{193}$$

Note that our model wall matrix, $\underline{\underline{I}}_b$, is Hermitian given that $\underline{\underline{\mathcal{S}}}_a^{-1}\,\underline{\underline{\mathcal{R}}}_a$ is Hermitian. [See Eq. (182).] Our model wall matrix allows us to smoothly interpolate between a plasma with no wall (which corresponds to $b_w \rightarrow \infty$ and $\underline{\underline{H}} = \underline{\underline{\mathcal{P}}}_a\,\underline{\underline{\mathcal{R}}}_a^{-1}$), and a fixed boundary plasma (which corresponds to $b_w = 1$ and $\underline{\underline{H}}^{-1} = \underline{\underline{0}}$).