Calculation of Tearing Mode Stability in an Inverse Aspect-Ratio Expanded Tokamak Plasma Equilibrium

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I. INTRODUCTION

The calculation of tearing mode stability in a high temperature tokamak plasma is most efficiently formulated as an asymptotic matching problem.[1] In such a problem, the plasma is divided into two regions. In the "outer region", which comprises most of the plasma, the tearing perturbation is described by the equations of linearized, marginally-stable, ideal magnetohydrodynamics (ideal-MHD). [See Sect. III B.] However, these equations become singular on so-called "rational" magnetic flux-surfaces at which the perturbed magnetic field resonates with the equilibrium field. In the so-called "inner region", which consists of a set of narrow layers centered on the various rational surfaces, non-ideal-MHD effects such as plasma inertia, resistivity, and viscosity become important. The growth-rate and angular rotation frequency of the reconnected magnetic flux at a given rational surface (numbered k) are fixed by asymptotically matching the resistive layer solution in the associated segment of the inner region, which is characterized by a dimensionless complex quantity Δ_k, to the ideal-MHD solution in the outer region. In a realistic axisymmetric tokamak plasma equilibrium, tearing perturbations with different toroidal mode numbers are independent of one another, whereas perturbations with different poloidal mode numbers are coupled together via toroidicity and the non-circular shaping of magnetic flux-surfaces.[2] Consequently, for a tearing perturbation with a given toroidal mode number, the Δ_k values associated with the various rational surfaces in the plasma are interrelated via a matrix equation.[3–9]

In general, the numerical determination of the elements of the matrix equation that links the various Δ_k values from the ideal-MHD equations in the outer region is a exceptionally challenging task.[10–14] One way of greatly reducing the complexity of this task is to employ an inverse aspect-ratio expanded plasma equilibrium.[15] In such an equilibrium, the metric elements of the flux-coordinate system can be expressed analytically in terms of a relatively small number of flux-surface functions, which represents a major simplification.[2] Another significant advantage of an inverse aspect-ratio expanded equilibrium is that the magnetic perturbation in the plasma can be matched to a vacuum solution expressed as an expansion in toroidal functions.[5] The alternative approach of using a Green's function solution in the vacuum region is much more computationally intensive.[16] The inverse aspect-ratio expansion approach to determining tearing mode stability in tokamak plasmas was first introduced in Ref. 3 in a calculation that allowed for three poloidal harmonics coupled via toroidicity. The inverse aspect-ratio expansion approach was extended in Ref. 5 in a calculation that allowed for seven poloidal harmonics coupled via toroidicity, flux-surface elongation, and flux-surface triangularity. In this paper, we intend to generalize the inverse aspect-ratio expansion approach to allow for an arbitrary number of poloidal harmonics coupled by flux-surfaces of general shape. Furthermore, unlike Refs. 3 and 5, we shall not assume that the plasma is up-down symmetric.

II. PRELIMINARY ANALYSIS

A. Normalization

All lengths in this paper are normalized to the major radius of the plasma magnetic axis, R_0. All magnetic field-strengths are normalized to the toroidal field-strength at the magnetic axis, B_0. All current densities are normalized to $B_0/(\mu_0\, R_0)$. All plasma pressures are normalized to $B_0^{\,2}/\mu_0$. All toroidal electromagnetic torques are normalized to $B_0^{\,2}\, R_0^{\,3}/\mu_0$.

B. Axisymmetric Tokamak Plasma Equilibrium

Let R, ϕ, Z be right-handed cylindrical coordinates whose Jacobian is

$$(\nabla R\times\nabla\phi\cdot\nabla Z)^{-1} = R. \tag{1}$$

Note that $|\nabla\phi| = 1/R$.

Let r, θ, ϕ be right-handed flux-coordinates whose Jacobian is [3,17]

$$\mathcal{J}(r,\theta)\equiv(\nabla r\times\nabla\theta\cdot\nabla\phi)^{-1}\equiv R\left(\frac{\partial R}{\partial\theta}\,\frac{\partial Z}{\partial r}-\frac{\partial R}{\partial r}\,\frac{\partial Z}{\partial\theta}\right) = r\,R^{2}. \tag{2}$$

Note that $r = r(R,Z)$ and $\theta=\theta(R,Z)$. The magnetic axis corresponds to $r=0$. The inboard mid-plane corresponds to $\theta = 0$.

Consider an axisymmetric tokamak equilibrium [18] whose magnetic field takes the form [3,5]

$$\mathbf{B}(r,\theta) = f(r)\,\nabla\phi\times\nabla r + g(r)\,\nabla\phi = f\,\nabla(\phi - q\,\theta)\times\nabla r, \tag{3}$$

where

$$q(r) = \frac{r\,g}{f} \tag{4}$$

is the safety-factor (i.e., the inverse of the rotational transform). Note that $\mathbf{B}\cdot\nabla r = 0$, which implies that r is a magnetic flux-surface label. We require $g = 1$ on the magnetic axis in order to ensure that the normalized toroidal magnetic field-strength at the axis is unity.

It is easily demonstrated that

$$B^{\,r} = \mathbf{B}\cdot\nabla r = 0, \tag{5}$$

$$B^{\,\theta} = \mathbf{B}\cdot\nabla\theta = \frac{f}{r\,R^2}, \tag{6}$$

$$B^{\,\phi} = \mathbf{B}\cdot\nabla\phi = \frac{g}{R^2}, \tag{7}$$

$$B_r = \mathcal{J}\,\nabla\theta\times\nabla\phi\cdot\mathbf{B} = -r\,f\,\nabla r\cdot\nabla\theta, \tag{8}$$

$$B_\theta = \mathcal{J}\,\nabla\phi\times\nabla r\cdot\mathbf{B} = r\,f\,|\nabla r|^2, \tag{9}$$

$$B_\phi = \mathcal{J}\,\nabla r\times\nabla\theta\cdot\mathbf{B} = g, \tag{10}$$

where use has been made of Sect. A.

The Maxwell equation (neglecting the displacement current, because tearing modes are comparatively low-frequency phenomena) $\mathbf{J} = \nabla\times\mathbf{B}$ yields

$$\mathcal{J}\,J^{\,r} = \frac{\partial B_\phi}{\partial\theta} = 0, \tag{11}$$

$$\mathcal{J}\,J^{\,\theta} = -\frac{\partial B_\phi}{\partial r} = -g', \tag{12}$$

$$\mathcal{J}\,J^{\,\phi} = \frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{\partial\theta} = \frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\,, \tag{13}$$

where $\mathbf{J}$ is the equilibrium current density, $' \equiv d/dr$, and use has been made of Eqs. (8)–(10) and (A11)–(A13).

Equilibrium force balance requires that

$$\nabla P = \mathbf{J}\times\mathbf{B}, \tag{14}$$

where $P(r)$ is the equilibrium plasma pressure. Here, for the sake of simplicity, we have neglected the small centrifugal modifications to force balance due to plasma rotation.[19,20] It follows that

$$P' = \mathcal{J}(J^{\theta}\,B^{\phi} - J^{\phi}\,B^{\theta}) = -g'\,\frac{g}{R^2} - \frac{f}{r\,R^2}\left[\frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\right]. \tag{15}$$

where use has been made of Eqs. (5)–(7), (11)–(13), and (A4)–(A6). The other two components of Eq. (14) are identically zero.

Equation (15) yields the *Grad-Shafranov equation*,[18]

$$\frac{f}{r}\,\frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{f}{r}\,\frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta) + g\,g' + R^2\,P' = 0. \tag{16}$$

It follows from Eqs. (4), (13), and (16) that

$$\mathcal{J}\,J^{\phi} = -q\,g' - \frac{r\,R^2\,P'}{f}. \tag{17}$$

It is clear from Eqs. (12) and (17) that $g' = P' = 0$ in the current-free vacuum region surrounding the plasma. We shall also assume that $g' = P' = 0$ at the plasma/vacuum interface, so as to ensure that the equilibrium plasma current density is zero at the interface.

III. DERIVATION OF OUTER REGION PDES

A. Introduction

The outer region partial differential equations (PDEs) were first presented in Ref. 3 without an explicit derivation. However, the derivation is sufficiently non-obvious that it is worth outlining in this section.

B. Governing Equations

In the outer region, the perturbed plasma equilibrium satisfies the ideal-MHD equations[3,5,9,18]

$$\mathbf{b} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}), \tag{18}$$

$$\nabla p = \mathbf{j}\times\mathbf{B} + \mathbf{J}\times\mathbf{b}, \tag{19}$$

$$\mathbf{j} = \nabla \times \mathbf{b}, \tag{20}$$

$$p = -\boldsymbol{\xi} \cdot \nabla P, \tag{21}$$

where $\boldsymbol{\xi}(r, \theta, \phi)$ is the plasma displacement, $\mathbf{b}(r, \theta, \phi)$ the perturbed magnetic field, $\mathbf{j}(r, \theta, \phi)$ the perturbed current density, and $p(r, \theta, \phi)$ the perturbed pressure. Let us assume that all perturbed quantities vary with ϕ as $\exp(-{\rm i}\, n\, \phi)$, where the real positive integer n is the toroidal mode number of the tearing mode. For example, $p(r, \theta, \phi) = p(r, \theta)\, \exp(-{\rm i}\, n\, \phi)$.

C. Radial Plasma Displacement

Equations (A5) and (A6) yield

$$(\boldsymbol{\xi} \times \mathbf{B})_\theta = \mathcal{J}\, (\xi^{\,\phi}\, B^{\,r} - \xi^{\,r}\, B^{\,\phi}) = -\mathcal{J}\, B^{\,\phi}\, \xi^{\,r}, \tag{22}$$

$$(\boldsymbol{\xi} \times \mathbf{B})_\phi = \mathcal{J}\, (\xi^{\,r}\, B^{\,\theta} - \xi^{\,\theta}\, B^{\,r}) = \mathcal{J}\, B^{\,\theta}\, \xi^{\,r}, \tag{23}$$

where use has been made of the fact that $B^{\,r} = J^{\,r} = 0$. [See Eqs. (5) and (11).] Combining the previous two equations with Eqs. (18) and (A11), we obtain

$$\mathcal{J}\, b^r = \frac{\partial}{\partial\theta}\left(\mathcal{J}\, B^{\,\theta}\, \xi^{\,r}\right) - {\rm i}\, n\, \mathcal{J}\, B^{\,\phi}\, \xi^{\,r}. \tag{24}$$

Thus, Eqs. (2), (4), (6), and (7) give

$$r\, R^2\, b^r = \left(\frac{\partial}{\partial\theta} - {\rm i}\, n\, q\right) y, \tag{25}$$

where

$$y(r, \theta) = f\, \xi^{\,r}. \tag{26}$$

D. Perturbed Force Balance

According to Eq. (21),

$$p = -P'\, \nabla r \cdot \boldsymbol{\xi} = -P'\, \xi^{\,r}. \tag{27}$$

So, the perturbed force balance equation, (19), yields

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = (\mathbf{j}\times\mathbf{B})_r + (\mathbf{J}\times\mathbf{b})_r, \tag{28}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial \theta} = (\mathbf{j}\times\mathbf{B})_\theta + (\mathbf{J}\times\mathbf{b})_\theta, \tag{29}$$

$$\mathrm{i}\,n\,P'\,\xi^{\,r} = (\mathbf{j}\times\mathbf{B})_\phi + (\mathbf{J}\times\mathbf{b})_\phi, \tag{30}$$

giving

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = r\,R^2\,(j^{\,\theta}\,B^{\phi} - j^{\,\phi}\,B^{\theta}) + r\,R^2\,(J^{\,\theta}\,b^{\phi} - J^{\,\phi}\,b^{\theta}), \tag{31}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial \theta} = r\,R^2\,(j^{\,\phi}\,B^{r} - j^{\,r}\,B^{\phi}) + r\,R^2\,(J^{\,\phi}\,b^{r} - J^{\,r}\,b^{\phi}), \tag{32}$$

$$\mathrm{i}\,n\,P'\,\xi^{\,r} = r\,R^2\,(j^{\,r}\,B^{\theta} - j^{\,\theta}\,B^{r}) + r\,R^2\,(J^{\,r}\,b^{\theta} - J^{\,\theta}\,b^{r}), \tag{33}$$

where use has been made of Eqs. (2) and (A4)–(A6). Thus, according to Eqs. (5)–(7), (11), (12), and (17),

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial r} = f\,(q\,j^{\,\theta} - j^{\,\phi}) - g'\,b^{\phi} + \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)b^{\theta}, \tag{34}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial \theta} = -r\,g\,j^{\,r} - \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)b^{r}, \tag{35}$$

$$\mathrm{i}\,n\,P'\,\xi^{\,r} = f\,j^{\,r} + g'\,b^{r}. \tag{36}$$

It follows from Eqs. (26), (25), and (36) that

$$r\,j^{\,r} = \mathrm{i}\,n\,\alpha_p\,y - \frac{\alpha_g}{R^2}\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right)y, \tag{37}$$

where

$$\alpha_p(r) = \frac{r\,P'}{f^2}, \tag{38}$$

$$\alpha_g(r) = \frac{g'}{f}. \tag{39}$$

Note that Eq. (35) is trivially satisfied. Hence, only Eq. (34) remains to be solved.

E. Perturbed Plasma Current Density

Equation (20) yields

$$r\,R^2\,j^{\,r} = \frac{\partial b_\phi}{\partial\theta} + {\rm i}\,n\,b_\theta, \tag{40}$$

$$r\,R^2\,j^{\,\theta} = -{\rm i}\,n\,b_r - \frac{\partial b_\phi}{\partial r}, \tag{41}$$

$$r\,R^2\,j^{\,\phi} = \frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial\theta}, \tag{42}$$

where use has been made of Eqs. (2) and (A11)–(A13). Now, according to Sect. A,

$$\mathbf{b} = b_r\,\nabla r + b_\theta\,\nabla\theta + b_\phi\,\nabla\phi, \tag{43}$$

so

$$b^{\,r} = \mathbf{b}\cdot\nabla r = |\nabla r|^2\,b_r + (\nabla r\cdot\nabla\theta)\,b_\theta, \tag{44}$$

$$b^{\,\theta} = \mathbf{b}\cdot\nabla\theta = (\nabla r\cdot\nabla\theta)\,b_r + |\nabla\theta|^2\,b_\theta, \tag{45}$$

$$b^{\,\phi} = \mathbf{b}\cdot\nabla\phi = \frac{b_\phi}{R^2}. \tag{46}$$

Let us define

$$x(r,\theta) = b_\phi. \tag{47}$$

It follows from Eqs. (37), (40), (46), and (47) that

$$b_\theta = -\frac{\alpha_g}{{\rm i}\,n}\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right)y + \alpha_p\,R^2\,y - \frac{1}{{\rm i}\,n}\,\frac{\partial x}{\partial\theta}, \tag{48}$$

$$b^{\,\phi} = \frac{x}{R^2}. \tag{49}$$

Equations (44) and (45) can be rearranged to give

$$b_r = \left(\frac{1}{|\nabla r|^2}\right)b^{\,r} - \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right)b_\theta, \tag{50}$$

$$b^{\,\theta} = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right)b^{\,r} + \left[|\nabla\theta|^2 - \frac{(\nabla r\cdot\nabla\theta)^2}{|\nabla r|^2}\right]b_\theta. \tag{51}$$

But, from Eq. (2),

$$|\nabla r|^2\,|\nabla\theta|^2 - (\nabla r\cdot\nabla\theta)^2 = \frac{1}{r^2\,R^2}. \tag{52}$$

Thus, Eq. (51) reduces to

$$b^{\theta} = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right) b^r + \left(\frac{1}{r^2\,R^2\,|\nabla r|^2}\right) b_\theta. \tag{53}$$

Making use of Eqs. (25) and (48), we obtain

$$r^2\,R^2\,b^\theta = T\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right) y + U\,y - Q\,\frac{\partial x}{\partial\theta}, \tag{54}$$

where

$$Q(r,\theta) = \frac{1}{\mathrm{i}\,n\,|\nabla r|^2}, \tag{55}$$

$$U(r,\theta) = \frac{\alpha_p\,R^2}{|\nabla r|^2}, \tag{56}$$

$$T(r,\theta) = \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2} - \frac{\alpha_g}{\mathrm{i}\,n\,|\nabla r|^2}. \tag{57}$$

Equation (50) gives

$$b_r = A\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right) y - B\,y + C\,\frac{\partial x}{\partial\theta}, \tag{58}$$

where

$$A(r,\theta) = \frac{1}{r\,R^2\,|\nabla r|^2} + \frac{\alpha_g}{\mathrm{i}\,n}\,\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}, \tag{59}$$

$$B(r,\theta) = \alpha_p\,\frac{R^2\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}, \tag{60}$$

$$C(r,\theta) = \frac{1}{\mathrm{i}\,n}\,\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}, \tag{61}$$

and use has been made of Eqs. (25) and (48).

F. First Outer Region PDE

According to Eq. (18),

$$\nabla\cdot\mathbf{b} = 0, \tag{62}$$

which implies that

$$r\,\frac{\partial}{\partial r}\left[\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right) y\right] + \frac{\partial(r^2\,R^2\,b^\theta)}{\partial\theta} - S\,x = 0, \tag{63}$$

where

$$S(r) = \mathrm{i}\, n\, r^2, \tag{64}$$

and use has been made of Eqs. (2), (25), (49), and (A10). Thus, employing Eq. (54), we obtain the *first outer region partial differential equation* (PDE),[3]

$$r\,\frac{\partial}{\partial r}\left[\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y\right] = \frac{\partial}{\partial\theta}\left(Q\,\frac{\partial x}{\partial\theta}\right)+S\,x-\frac{\partial}{\partial\theta}\left[T\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y+U\,y\right]. \tag{65}$$

G. Second Outer Region PDE

According to Eqs. (41), (42), (47), (48), and (58),

$$r\,R^2\,j^{\,\theta} = -\mathrm{i}\,n\left[A\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y-B\,y+C\,\frac{\partial x}{\partial\theta}\right]-\frac{\partial x}{\partial r}, \tag{66}$$

$$\begin{aligned} r\,R^2\,j^{\,\phi} = {} & \frac{\partial}{\partial r}\left[-\frac{\alpha_g}{\mathrm{i}\,n}\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y+\alpha_p\,R^2\,y\right]-\frac{1}{\mathrm{i}\,n}\,\frac{\partial^2 x}{\partial r\,\partial\theta}\\ & -\frac{\partial}{\partial\theta}\left[A\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y-B\,y+C\,\frac{\partial x}{\partial\theta}\right]. \end{aligned} \tag{67}$$

So,

$$\begin{aligned} r\,R^2\,(q\,j^{\,\theta}-j^{\,\phi}) = {} & \left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)\left[A\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y-B\,y+C\,\frac{\partial x}{\partial\theta}\right]\\ & +\frac{1}{\mathrm{i}\,n}\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)\frac{\partial x}{\partial r}\\ & -\frac{\partial}{\partial r}\left[-\frac{\alpha_g}{\mathrm{i}\,n}\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y+\alpha_p\,R^2\,y\right]. \end{aligned} \tag{68}$$

Thus, Eq. (34) gives

$$\begin{aligned} -\frac{r\,R^2}{f}\,\frac{\partial}{\partial r}\left(\frac{f}{r}\,\alpha_p\,y\right) = {} & \left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)\left[A\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y-B\,y+C\,\frac{\partial x}{\partial\theta}\right]\\ & +\frac{1}{\mathrm{i}\,n}\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)\frac{\partial x}{\partial r}\\ & -\frac{\partial}{\partial r}\left[-\frac{\alpha_g}{\mathrm{i}\,n}\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y+\alpha_p\,R^2\,y\right]-r\,\alpha_g\,x\\ & +\frac{1}{r}\left(q\,\alpha_g+R^2\,\alpha_p\right)\left[T\left(\frac{\partial}{\partial\theta}-\mathrm{i}\,n\,q\right)y+U\,y-Q\,\frac{\partial x}{\partial\theta}\right], \end{aligned} \tag{69}$$

where use has been made of Eqs. (26), (38), (39), (49), (54). The previous equation reduces to

$$\begin{aligned}-{\rm i}\,n\,\alpha_p\,\alpha_f\,R^2\,y = {\rm i}\,n\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)&\left[r\,A\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y - r\,B\,y + r\,C\,\frac{\partial x}{\partial\theta}\right]\\
&+\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)r\,\frac{\partial x}{\partial r}+r\,\alpha_g'\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y\\
&+\alpha_g\,r\,\frac{\partial}{\partial r}\left[\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y\right]-{\rm i}\,n\,r\,\frac{\partial R^2}{\partial r}\,\alpha_p\,y-\alpha_g\,S\,x\\
&+{\rm i}\,n\left(q\,\alpha_g+R^2\,\alpha_p\right)\left[T\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y+U\,y-Q\,\frac{\partial x}{\partial\theta}\right],\end{aligned}\tag{70}$$

where

$$\alpha_f(r)=\frac{r^2}{f}\,\frac{d}{dr}\left(\frac{f}{r}\right),\tag{71}$$

and use has been made of Eq. (64). Employing Eq. (65), we obtain

$$\begin{aligned}-{\rm i}\,n\,\alpha_p\,\alpha_f\,R^2\,y = {\rm i}\,n\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)&\left[r\,A\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y - r\,B\,y + r\,C\,\frac{\partial x}{\partial\theta}\right]\\
&+\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)r\,\frac{\partial x}{\partial r}+r\,\alpha_g'\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y\\
&+\alpha_g\,\frac{\partial}{\partial\theta}\left(Q\,\frac{\partial x}{\partial\theta}\right)+\alpha_g\,S\,x-\alpha_g\,\frac{\partial}{\partial\theta}\left[T\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y+U\,y\right]\\
&-{\rm i}\,n\,r\,\frac{\partial R^2}{\partial r}\,\alpha_p\,y-\alpha_g\,S\,x\\
&+{\rm i}\,n\left(q\,\alpha_g+R^2\,\alpha_p\right)\left[T\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y+U\,y-Q\,\frac{\partial x}{\partial\theta}\right],\end{aligned}\tag{72}$$

which yields

$$\begin{aligned}-{\rm i}\,n\,\alpha_p\,\alpha_f\,R^2\,y = {\rm i}\,n\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)&\left[r\,A\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y - r\,B\,y + r\,C\,\frac{\partial x}{\partial\theta}\right]\\
&+\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)r\,\frac{\partial x}{\partial r}+r\,\alpha_g'\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y\\
&+\alpha_g\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)\left[Q\,\frac{\partial x}{\partial\theta}-T\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y-U\,y\right]\\
&-{\rm i}\,n\,r\,\frac{\partial R^2}{\partial r}\,\alpha_p\,y+{\rm i}\,n\,R^2\,\alpha_p\left[T\left(\frac{\partial}{\partial\theta}-{\rm i}\,n\,q\right)y+U\,y-Q\,\frac{\partial x}{\partial\theta}\right],\end{aligned}\tag{73}$$

which reduces to the *second outer region PDE*,[3]

$$\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right) r\,\frac{\partial x}{\partial r} = -\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right) T^\ast\,\frac{\partial x}{\partial\theta} + U\,\frac{\partial x}{\partial\theta} + X\,y$$
$$-\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right) V\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right) y + W\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right) y, \qquad (74)$$

where

$$V(r,\theta) = \frac{1}{|\nabla r|^2}\left(\frac{{\rm i}\,n}{R^2} + \frac{\alpha_g^2}{{\rm i}\,n}\right), \qquad (75)$$

$$W(r,\theta) = \frac{2\,\alpha_g\,\alpha_p\,R^2}{|\nabla r|^2} - r\,\alpha_g', \qquad (76)$$

$$X(r,\theta) = {\rm i}\,n\,\alpha_p\left[\frac{\partial}{\partial\theta}(T^\ast\,R^2) + r\,\frac{\partial R^2}{\partial r} - \alpha_f\,R^2 - U\,R^2\right], \qquad (77)$$

and * denotes a complex conjugate.

IV. OUTER REGION ODES

A. Primitive Outer Region ODEs

Let

$$x(r,\theta) = n\,z(r,\theta), \qquad (78)$$

and let us express $y(r,\theta)$ and $z(r,\theta)$ as Fourier series in the poloidal angle, θ:

$$y(r,\theta) = \sum_m y_m(r)\,\exp(\,{\rm i}\,m\,\theta), \qquad (79)$$

$$z(r,\theta) = \sum_m z_m(r)\,\exp(\,{\rm i}\,m\,\theta), \qquad (80)$$

Here, the (not necessarily positive) integers m are the poloidal mode numbers of the coupled Fourier harmonics included in the calculation. The outer region PDEs, (65) and (74), reduce to the *primitive outer region ordinary differential equations* (ODEs),[3,5,9]

$$r\,\frac{d}{dr}\left[(m - n\,q)\,y_m\right] = \sum_{m'}\left(A_m^{m'}\,z_{m'} + B_m^{m'}\,y_{m'}\right), \qquad (81)$$

$$(m - n\,q)\,r\,\frac{dz_m}{dr} = \sum_{m'}\left(C_m^{m'}\,z_{m'} + D_m^{m'}\,y_{m'}\right), \tag{82}$$

where

$$n^{-1}\,A_m^{m'}(r) = \frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m\,\theta}\left(\frac{\partial}{\partial\theta}\,Q\,\frac{\partial}{\partial\theta} + S\right)\mathrm{e}^{\,\mathrm{i}\,m'\,\theta}\,d\theta, \tag{83}$$

$$B_m^{m'}(r) = \frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m\,\theta}\left[-\frac{\partial}{\partial\theta}\,T\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right) - \frac{\partial U}{\partial\theta}\right]\mathrm{e}^{\,\mathrm{i}\,m'\,\theta}\,d\theta, \tag{84}$$

$$C_m^{m'}(r) = \frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m\,\theta}\left[-\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right)T^*\,\frac{\partial}{\partial\theta} + U\,\frac{\partial}{\partial\theta}\right]\mathrm{e}^{\,\mathrm{i}\,m'\,\theta}\,d\theta, \tag{85}$$

$$\begin{aligned} n\,D_m^{m'}(r) = \frac{1}{2\pi\,\mathrm{i}}\oint \mathrm{e}^{-\mathrm{i}\,m\,\theta}\Bigg[&-\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right)V\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right)\\ &+W\left(\frac{\partial}{\partial\theta} - \mathrm{i}\,n\,q\right) + X\Bigg]\mathrm{e}^{\,\mathrm{i}\,m'\,\theta}\,d\theta. \end{aligned} \tag{86}$$

Hence, it follows from Eqs. (55)–(57), (64), and (75)–(77) that [9]

$$A_m^{m'} = m\,m'\,c_m^{m'} + n^2\,r^2\,\delta_m^{m'} \tag{87}$$

$$B_m^{m'} = m\,(m' - n\,q)\left(-f_m^{m'} + n^{-1}\,\alpha_g\,c_m^{m'}\right) - m\,\alpha_p\,d_m^{m'}, \tag{88}$$

$$C_m^{m'} = -(m - n\,q)\,m'\left(f_m^{m'} + n^{-1}\,\alpha_g\,c_m^{m'}\right) + m'\,\alpha_p\,d_m^{m'}, \tag{89}$$

$$\begin{aligned} D_m^{m'} = {}& (m - n\,q)\,(m' - n\,q)\left(b_m^{m'} - n^{-2}\,\alpha_g^2\,c_m^{m'}\right) - (m - n\,q)\,n^{-1}\,r\,\alpha_g'\,\delta_m^{m'}\\ &+ \alpha_p\left[(m - m')\,g_m^{m'} + n^{-1}\,\alpha_g\,(m + m' - 2\,n\,q)\,d_m^{m'} + r\,\frac{da_m^{m'}}{dr} - \alpha_f\,a_m^{m'} - \alpha_p\,e_m^{m'}\right], \end{aligned} \tag{90}$$

where

$$a_m^{m'}(r) = \oint R^2\,\exp[-\mathrm{i}\,(m - m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{91}$$

$$b_m^{m'}(r) = \oint |\nabla r|^{-2}\,R^{-2}\,\exp[-\mathrm{i}\,(m - m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{92}$$

$$c_m^{m'}(r) = \oint |\nabla r|^{-2}\,\exp[-\mathrm{i}\,(m - m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{93}$$

$$d_m^{m'}(r) = \oint |\nabla r|^{-2}\,R^2\,\exp[-\mathrm{i}\,(m - m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{94}$$

$$e_m^{m'}(r) = \oint |\nabla r|^{-2}\,R^4\,\exp[-\mathrm{i}\,(m - m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{95}$$

$$f_m^{m'}(r) = \oint \frac{\mathrm{i}\, r\, \nabla r\cdot\nabla\theta}{|\nabla r|^2}\, \exp[-\mathrm{i}\,(m-m')\,\theta]\, \frac{d\theta}{2\pi}, \tag{96}$$

$$g_m^{m'}(r) = \oint \frac{\mathrm{i}\, r\, \nabla r\cdot\nabla\theta}{|\nabla r|^2}\, R^2\, \exp[-\mathrm{i}\,(m-m')\,\theta]\, \frac{d\theta}{2\pi}. \tag{97}$$

Here, we have extended the analysis of Ref. 9 to take into account the fact that the $A_m^{m'}$, $B_m^{m'}$, $a_m^{m'}$, $b_m^{m'}$, et cetera, are complex quantities in a realistic non-up-down-symmetric plasma equilibrium.

B. Outer Region ODEs

Let

$$y_m(r) = \frac{\psi_m(r)}{m-n\,q}, \tag{98}$$

$$z_m(r) = \frac{Z_m(r) + k_j\,\psi_m(r)}{m-n\,q}, \tag{99}$$

where

$$k_m(r) = -\mathrm{Re}\left(\frac{B_m^m}{A_m^m}\right) = -\left[\frac{m\,(m-n\,q)\,n^{-1}\,\alpha_g\,c_m^m - m\,\alpha_p\,d_m^m}{m^2\,c_m^m + n^2\,r^2}\right]. \tag{100}$$

Here, we have made use of the fact that f_m^m is imaginary. [See Eq. (96).] It follows from Eq. (25) that

$$b^r(r,\theta) = \mathrm{i}\sum_m \frac{\psi_m(r)}{r\,R^2}\, \exp(\,\mathrm{i}\,m\,\theta). \tag{101}$$

Furthermore, Eqs. (81) and (82) transform to give the *outer region ODEs*,[5,9]

$$r\,\frac{d\psi_m}{dr} = \sum_{m'} \frac{L_m^{m'}\,Z_{m'} + M_m^{m'}\,\psi_{m'}}{m'-n\,q}, \tag{102}$$

$$(m-n\,q)\,r\,\frac{d}{dr}\left(\frac{Z_m}{m-n\,q}\right) = \sum_{m'} \frac{N_m^{m'}\,Z_{m'} + P_m^{m'}\,\psi_{m'}}{m'-n\,q}, \tag{103}$$

where

$$L_m^{m'}(r) = A_m^{m'}, \tag{104}$$

$$M_m^{m'}(r) = B_m^{m'} + k_{m'}\,L_m^{m'}, \tag{105}$$

$$N_m^{m'}(r) = C_m^{m'} - k_m\,L_m^{m'}, \tag{106}$$

$$P_m^{m'}(r) = D_m^{m'} + k_{m'}\,C_m^{m'} - k_m\,M_m^{m'} - k_m\,n\,q\,s\,\delta_m^{m'} - (m-n\,q)\,r\,\frac{dk_m}{dr}\,\delta_m^{m'}, \tag{107}$$

with

$$s(r) = \frac{r\,q'}{q}. \tag{108}$$

Note that

$$M_m^m = N_m^m = -m\,(m-n\,q)\,f_m^m. \tag{109}$$

C. Symmetry Properties

Equations (91)–(97) imply that $a_{m'}^m = a_m^{m'*}$, $b_{m'}^m = b_m^{m'*}$, $c_{m'}^m = c_m^{m'*}$, $d_{m'}^m = d_m^{m'*}$, $e_{m'}^m = e_m^{m'*}$, $f_{m'}^m = -f_m^{m'*}$, $g_{m'}^m = -g_m^{m'*}$, for all m, m'. Hence, Eqs. (87)–(90), Eqs. (100), and (104)–(107) give

$$L_{m'}^m = L_m^{m'*}, \tag{110}$$

$$M_{m'}^m = -N_m^{m'*}, \tag{111}$$

$$N_{m'}^m = -M_m^{m'*}, \tag{112}$$

$$P_{m'}^m = P_m^{m'*}. \tag{113}$$

D. Toroidal Electromagnetic Torque

The volume integrated toroidal electromagnetic torque acting between the magnetic axis and a magnetic flux-surface whose label is r is given by

$$\begin{aligned} T_\phi(r) &= \int_0^r \oint\oint R^2\,\nabla\phi\cdot(\mathbf{J}+\mathbf{j})\times(\mathbf{B}\times\mathbf{b})\,\mathcal{J}\,d\tilde{r}\,d\theta\,d\phi \\ &= \int_0^r \oint\oint (\mathbf{j}\times\mathbf{b})_\phi\,\mathcal{J}\,d\tilde{r}\,d\theta\,d\phi \end{aligned} \tag{114}$$

Here, use has been made of Eq. (14), as well as the fact that $P = P(r)$. We have also taken into account that $\mathbf{b}$ and $\mathbf{j}$ vary with ϕ as $\exp(-{\rm i}\,n\,\phi)$, whereas $\mathbf{B}$, $\mathbf{J}$, $\mathcal{J}$, and $|\nabla\phi|$ are independent of ϕ. It is clear that the zeroth-order (in perturbed quantities) contribution to T_ϕ is identically zero, whereas the first-order contributions average to zero, leaving only second-order (i.e., nonlinear in perturbed quantities) contributions. Making use of Sect. A, as well as Eqs. (2), (20), (50), (53), and (62), we deduce that

$$\mathcal{J}\,(\mathbf{j}\times\mathbf{b})_\phi = \frac{\partial}{\partial r}(\mathcal{J}\,b_\phi\,b^r) + \frac{\partial}{\partial\theta}\left(\mathcal{J}\,b_\phi\,b^\theta\right) + \frac{\partial}{\partial\phi}\left(\mathcal{J}\,b_\phi\,b^\phi\right) - \frac{1}{2}\,\frac{\partial}{\partial\phi}\left[\mathcal{J}\left(b_r\,b^r + b_\theta\,b^\theta + b_\phi\,b^\phi\right)\right]. \tag{115}$$

Hence, we obtain

$$T_\phi(r) = \oint\oint \mathcal{J}\,b_\phi\,b^r\,d\theta\,d\phi = r\oint\oint R^2\,b_\phi\,b^r\,d\theta\,d\phi, \tag{116}$$

where the integral on the right-hand side is evaluated on the magnetic flux-surface whose label is r. We can reinterpret the previous expression as specifying the net outward flux of toroidal electromagnetic angular momentum across the magnetic flux-surface whose label is r. Finally, making use of Eqs. (47), (78), (80), and (99)–(101), the previous expression reduces to [5]

$$T_\phi(r) = \mathrm{i}\,\pi^2\,n\sum_m \frac{Z_m^\ast\,\psi_m - \psi_m^\ast\,Z_m}{m-n\,q}. \tag{117}$$

It follows from Eqs. (102), (103) and (110)–(113) that

$$r\,\frac{d}{dr}\left(\sum_m \frac{Z_m^\ast\,\psi_m - \psi_m^\ast\,Z_m}{m-n\,q}\right) = 0. \tag{118}$$

Hence, we deduce that [5]

$$\frac{dT_\phi}{dr} = 0 \tag{119}$$

in any region of the plasma that satisfies the outer region ODEs. Thus, the volume integrated toroidal electromagnetic torque is constant between rational magnetic flux-surfaces. As will become apparent in Sect. V D, the integrated torque can have discontinuous jumps across rational flux-surfaces. It follows that net electromagnetic torques can only develop in the plasma in the immediate vicinity of rational magnetic flux-surfaces, where the ideal-MHD equations become singular.[21]

V. BEHAVIOR IN VICINITY OF RATIONAL SURFACE

A. Introduction

The analysis of this section is a generalization of the analysis of Ref. 9 that takes into account the fact that the $L_m^{m'}$, $M_m^{m'}$, et cetera, are complex quantities in a realistic non-up-down-symmetric tokamak plasma equilibrium.

Let there be K rational magnetic flux-surfaces in the plasma. Suppose that the kth surface lies at $r = r_k$, and possesses the resonant poloidal mode number m_k, where $q(r_k) = m_k/n$.

B. General Case

Consider the solution of the outer region ODEs, (102) and (103), in the vicinity of the kth rational surface. Let $x = r - r_k$. The most general small-$|x|$ solution of the ODEs can be shown to take the form [5,9]

$$\begin{aligned}\psi_{m_k}(r_k+x) &= A_{L\,k}^{\pm}\,|x|^{\nu_{L\,k}}\,(1+\lambda_L\,x+\cdots)+A_{S\,k}^{\pm}\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(1+\cdots)\\ &\quad + A_C\,x\,(1+\cdots),\end{aligned} \tag{120}$$

$$\begin{aligned}Z_{m_k}(r_k+x) &= A_{L\,k}^{\pm}\,|x|^{\nu_{L\,k}}(b_L+\gamma_L\,x+\cdots)+A_{S\,k}^{\pm}\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(b_S+\cdots)\\ &\quad + B_C\,x\,(1+\cdots),\end{aligned} \tag{121}$$

and

$$\begin{aligned}\psi_{m_k+j}(r_k+x) &= A_{L\,k}^{\pm}\,|x|^{\nu_{L\,k}}\,(a_j+c_j\,x+\cdots)+A_{S\,k}^{\pm}\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(\tilde{a}_j+\cdots)\\ &\quad + (\bar{\psi}_{m_k+j}+\bar{\psi}'_{m_j+k}\,x+\cdots),\end{aligned} \tag{122}$$

$$\begin{aligned}Z_{m_k+j}(r_k+x) &= A_{L\,k}^{\pm}\,|x|^{\nu_{L\,k}}\,(b_j+d_j\,x+\cdots)+A_{S\,k}^{\pm}\,\mathrm{sgn}(x)\,|x|^{\nu_{S\,k}}\,(\tilde{b}_j+\cdots)\\ &\quad + (\bar{Z}_{m_k+j}+\bar{Z}'_{m_k+j}\,x+\cdots).\end{aligned} \tag{123}$$

The superscripts $^+$ and $^-$ correspond to $x>0$ and $x<0$, respectively. Here, $A_{L\,k}$ is known as the coefficient of the large solution, whereas $A_{S\,k}$ is termed the coefficient of the small solution.[5,9,22] Moreover,

$$\nu_{L\,k} = \frac{1}{2} - \sqrt{-D_{I\,k}}, \tag{124}$$

$$\nu_{S\,k} = \frac{1}{2} + \sqrt{-D_{I\,k}}, \tag{125}$$

$$D_{I\,k} = -L_0\,P_0 - \frac{1}{4}, \tag{126}$$

$$L_0 = -\left(\frac{L_{m_k}^{m_k}}{m_k\,s}\right)_{r_k}, \tag{127}$$

$$P_0 = -\left(\frac{P_{m_k}^{m_k}}{m_k\,s}\right)_{r_k}. \tag{128}$$

Note that, ordinarily, $\nu_{L\,k}$, $\nu_{S\,k}$, $D_{I\,k}$, L_0 and P_0 are all real quantities. Furthermore,

$$b_L = \frac{\nu_{L\,k}}{L_0}, \tag{129}$$

$$b_S = \frac{\nu_{S\,k}}{L_0}, \tag{130}$$

$$A_C = -\frac{1}{r_k\,P_0}\sum_{j\neq 0}\frac{1}{j}\left(N_{m_k}^{m_k+j}\,\bar{Z}_{m_k+j} + P_{m_k}^{m_k+j}\,\bar{\psi}_{m_k+j}\right)_{r_k}, \tag{131}$$

$$B_C = -\frac{1}{r_k\,L_0}\sum_{j\neq 0}\frac{1}{j}\left(L_{m_k}^{m_k+j}\,\bar{Z}_{m_k+j} + M_{m_k}^{m_k+j}\,\bar{\psi}_{m_k+j}\right)_{r_k} + \frac{A_C}{L_0}, \tag{132}$$

$$\begin{aligned}\lambda_L = {}& \frac{1}{2\,r_k}\left[\frac{P_1\,L_0}{\nu_{L\,k}} + T_1 + \nu_{L\,k}\left(\frac{L_1}{L_0} - 2\right) + 2\,M_1\right]_{r_k}\\ & - \frac{1}{2\,(m_k\,s)_{r_k}}\,\frac{1}{r_k\,\nu_{L\,k}}\sum_{j\neq 0}\frac{1}{j}\left[L_{m_k}^{m_k+j}\,P_{m_k+j}^{m_k} + P_{m_k}^{m_k+j}\,L_{m_k+j}^{m_k} + M_{m_k}^{m_k+j}\,M_{m_k+j}^{m_k} + N_{m_k}^{m_k+j}\,N_{m_k+j}^{m_k}\right.\\ & \left. + b_L\,(L_{m_k}^{m_k+j}\,N_{m_k+j}^{m_k} + M_{m_k}^{m_k+j}\,L_{m_k+j}^{m_k}) + \frac{1}{b_L}\,(N_{m_k}^{m_k+j}\,P_{m_k+j}^{m_k} + P_{m_k}^{m_k+j}\,M_{m_k+j}^{m_k})\right]_{r_k},\end{aligned} \tag{133}$$

$$\begin{aligned}\gamma_L = {}& \frac{1}{2\,r_k}\left[(1+\nu_{L\,k})\left(\frac{P_1}{\nu_{L\,k}} + \frac{T_1}{L_0} - \frac{\nu_{L\,k}}{L_0}\right) + P_0\left(\frac{L_1}{L_0} - 1\right) + 2\,b_L\,M_1\right]_{r_k}\\ & - \frac{1}{2\,(m_k\,s)_{r_k}}\,\frac{1}{r_k\,\nu_{L\,k}\,L_0}\sum_{j\neq 0}\frac{1}{j}\left[(\nu_{L\,k}+1)\,(P_{m_k}^{m_k+j}\,L_{m_k+j}^{m_k} + N_{m_k}^{m_k+j}\,N_{m_k+j}^{m_k})\right.\\ & + (\nu_{L\,k}-1)\,(L_{m_k}^{m_k+j}\,P_{m_k+j}^{m_k} + M_{m_k}^{m_k+j}\,M_{m_k+j}^{m_k}) + b_L\,(\nu_{L\,k}-1)\,(L_{m_k}^{m_k+j}\,N_{m_k+j}^{m_k} + M_{m_k}^{m_k+j}\,L_{m_k+j}^{m_k})\\ & \left. + \frac{1}{b_L}\,(\nu_{L\,k}+1)\,(N_{m_k}^{m_k+j}\,P_{m_k+j}^{m_k} + P_{m_k}^{m_k+j}\,M_{m_k+j}^{m_k})\right]_{r_k},\end{aligned} \tag{134}$$

$$a_j = -\frac{1}{(m_k\,s)_{r_k}} \left(\frac{L^{m_k}_{m_k+j}}{L_0} + \frac{M^{m_k}_{m_k+j}}{\nu_{L\,k}} \right)_{r_k}, \tag{135}$$

$$b_j = -\frac{1}{(m_k\,s)_{r_k}} \left(\frac{P^{m_k}_{m_k+j}}{\nu_{L\,k}} + \frac{N^{m_k}_{m_k+j}}{L_0} \right)_{r_k}, \tag{136}$$

$$\tilde{a}_j = -\frac{1}{(m_k\,s)_{r_k}} \left(\frac{L^{m_k}_{m_k+j}}{L_0} + \frac{M^{m_k}_{m_k+j}}{\nu_{S\,k}} \right)_{r_k}, \tag{137}$$

$$\tilde{b}_j = -\frac{1}{(m_k\,s)_{r_k}} \left(\frac{P^{m_k}_{m_k+j}}{\nu_{S\,k}} + \frac{N^{m_k}_{m_k+j}}{L_0} \right)_{r_k}, \tag{138}$$

$$\begin{aligned} c_j = \frac{1}{(1+\nu_{L\,k})\,r_k} \Bigg[& -\nu_{L\,k}\,a_j + L_{j\,1}\,b_L + M_{j\,1} \\ & - \frac{r_k}{m_k\,s} \left(L^{m_k}_{m_k+j}\,\gamma_L + M^{m_k}_{m_k+j}\,\lambda_L \right) + \sum_{j'\neq 0} \frac{1}{j'} \left(L^{m_k+j'}_{m_k+j}\,b_{j'} + M^{m_k+j'}_{m_k+j}\,a_{j'} \right) \Bigg]_{r_k}, \end{aligned} \tag{139}$$

$$\begin{aligned} d_j = \frac{1}{(1+\nu_{L\,k})\,r_k} \Bigg[& -\left(\nu_{L\,k} + \frac{m_k\,s}{j} \right) b_j + N_{j\,1}\,b_L + P_{j\,1} \\ & - \frac{r_k}{m_k\,s} \left(N^{m_k}_{m_k+j}\,\gamma_L + P^{m_k}_{m_k+j}\,\lambda_L \right) + \sum_{j'\neq 0} \frac{1}{j'} \left(N^{m_k+j'}_{m_k+j}\,b_{j'} + P^{m_k+j'}_{m_k+j}\,a_{j'} \right) \Bigg]_{r_k}, \end{aligned} \tag{140}$$

$$\begin{aligned} \bar{\psi}'_{m_k+j} = \frac{1}{r_k} \Bigg[& -\frac{r_k}{m_k\,s} \left(L^{m_k}_{m_k+j}\,B_C + M^{m_k}_{m_k+j}\,A_C \right) \\ & + \sum_{j'\neq 0} \frac{1}{j'} \left(L^{m_k+j'}_{m_k+j}\,\bar{Z}_{m_k+j'} + M^{m_k+j'}_{m_k+j}\,\bar{\psi}_{m_k+j'} \right) \Bigg]_{r_k}, \end{aligned} \tag{141}$$

$$\begin{aligned} \bar{Z}'_{m_k+j} = \frac{1}{r_k} \Bigg[& -\frac{m_k\,s}{j}\,\bar{Z}_{m_k+j} - \frac{r_k}{m_k\,s} \left(N^{m_k}_{m_k+j}\,B_C + P^{m_k}_{m_k+j}\,A_C \right) \\ & + \sum_{j'\neq 0} \frac{1}{j'} \left(N^{m_k+j'}_{m_k+j}\,\bar{Z}_{m_k+j'} + P^{m_k+j'}_{m_k+j}\,\bar{\psi}_{m_k+j'} \right) \Bigg]_{r_k}, \end{aligned} \tag{142}$$

and

$$L_1 = \lim_{x\to 0} \left(\frac{L^{m_k}_{m_k}}{m_k - n\,q} - \frac{r_k\,L_0}{x} \right), \tag{143}$$

$$P_1 = \lim_{x\to 0} \left(\frac{P^{m_k}_{m_k}}{m_k - n\,q} - \frac{r_k\,P_0}{x} \right), \tag{144}$$

$$T_1 = \lim_{x\to 0} \left(\frac{-n\,q\,s}{m_k - n\,q} - \frac{r_k}{x} \right), \tag{145}$$

$$M_1 = \lim_{x\to 0}\left(\frac{M_{m_k}^{m_k}}{m_k - n\,q}\right), \tag{146}$$

$$L_{j\,1} = \lim_{x\to 0}\left(\frac{L_{m_k+j}^{m_k}}{m_k - n\,q} + \frac{r_k}{m_k\,s}\,\frac{L_{m_k+j}^{m_k}}{x}\right), \tag{147}$$

$$M_{j\,1} = \lim_{x\to 0}\left(\frac{M_{m_k+j}^{m_k}}{m_k - n\,q} + \frac{r_k}{m_k\,s}\,\frac{M_{m_k+j}^{m_k}}{x}\right), \tag{148}$$

$$N_{j\,1} = \lim_{x\to 0}\left(\frac{N_{m_k+j}^{m_k}}{m_k - n\,q} + \frac{r_k}{m_k\,s}\,\frac{N_{m_k+j}^{m_k}}{x}\right), \tag{149}$$

$$P_{j\,1} = \lim_{x\to 0}\left(\frac{P_{m_k+j}^{m_k}}{m_k - n\,q} + \frac{r_k}{m_k\,s}\,\frac{P_{m_k+j}^{m_k}}{x}\right). \tag{150}$$

The coefficients of the large and the small solutions at the kth rational surface are evaluated as follows:

$$\begin{aligned}\bar{\psi}_{m_k+j} &= \psi_{m_k+j}(r_k+\delta) - (a_j + \delta\,c_j)\,A_{L\,k}\,|\delta|^{\,\nu_{L\,k}}\\ &\quad - \tilde{a}_j\,\mathrm{sgn}(\delta)\,|\delta|^{\nu_{S\,k}}\,A_{S\,k} - \bar{\psi}'_{m_k+j}\,\delta + \mathcal{O}(\delta^2),\end{aligned} \tag{151}$$

$$\begin{aligned}\bar{Z}_{m_k+j} &= Z_{m_k+j}(r_k+\delta) - (b_j + \delta\,d_j)\,A_{L\,k}\,|\delta|^{\,\nu_{L\,k}}\\ &\quad - \tilde{b}_j\,\mathrm{sgn}(\delta)\,|\delta|^{\nu_{S\,k}}\,A_{S\,k} - \bar{Z}'_{m_k+j}\,\delta + \mathcal{O}(\delta^2),\end{aligned} \tag{152}$$

$$\begin{aligned}A_{S\,k} &= \frac{Z_{m_k}(r_k+\delta) - b_L\,\psi_{m_k}(r_k+\delta) - \delta\,(B_C - b_L\,A_C) - \delta\,(\gamma_L - b_L\,\lambda_L)\,A_{L\,k}\,|\delta|^{\,\nu_{L\,k}}}{(b_S - b_L)\,\mathrm{sgn}(\delta)\,|\delta|^{\nu_{S\,k}}}\\ &\quad + \mathcal{O}(\delta),\end{aligned} \tag{153}$$

$$A_{L\,k} = \frac{\psi_{m_k}(r_k+\delta) - A_{S\,k}\,\mathrm{sgn}(\delta)\,|\delta|^{\nu_{S\,k}} - A_C\,\delta}{(1+\delta\,\lambda_L)\,|\delta|^{\nu_{L\,k}}} + \mathcal{O}(\delta^2) \tag{154}$$

for $j\neq 0$. The previous set of equations can be solved via iteration.

The previous analysis is based on the assumption that $D_{I\,k}<0$. If $D_{I\,k}>0$ then the indices $\nu_{L\,k}$ and $\nu_{S\,k}$ become complex, indicating that the plasma in the vicinity of the kth rational surface is unstable to localized ideal interchange modes.[23]

C. Special Case

In the limit $\nu_{L\,k}\to 0$, some of the previous expressions become singular, and a special treatment is required. The most general small-$|x|$ solution of the outer region ODEs takes the form

$$\psi_{m_k}(r_k+x) = A^{\pm}_{L\,k}\,[1+\nu_{L\,k}\,\ln|x| + \hat{\lambda}_L\,x\,(\ln|x|-1) + \mu_L\,x\,(\ln^2|x|-2\,\ln|x|-2) + \xi_L\,x + \cdots] + A^{\pm}_{S\,k}\,x\,(1+\cdots) + \hat{A}_C\,x\,(1+\cdots) + A_D\,x\,(\ln|x|-1+\cdots), \tag{155}$$

$$Z_{m_k}(r_k+x) = A^{\pm}_{L\,k}\left(b_L + \hat{\gamma}_L\,x\,\ln|x| + \delta_L\,x\,\ln^2|x| + \cdots\right) + A^{\pm}_{S\,k}\,x\,(b_S+\cdots) + B_D\,x\,(\ln|x|+\cdots), \tag{156}$$

and

$$\psi_{m_k+j}(r_k+x) = A^{\pm}_{L\,k}\,(\hat{a}_j\,\ln|x|+\cdots) + A^{\pm}_{S\,k}\,x\,(\tilde{a}_j+\cdots) + \bar{\psi}_{m_k+j}\,(1+\cdots), \tag{157}$$

$$Z_{m_k+j}(r_k+x) = A^{\pm}_{L\,k}\,(\hat{b}_j\,\ln|x|+\cdots) + A^{\pm}_{S\,k}\,x\,(\tilde{b}_j+\cdots) + \bar{Z}_{m_k+j}\,(1+\cdots). \tag{158}$$

Here,

$$\hat{A}_C = \frac{1}{r_k}\sum_{j\neq 0}\frac{1}{j}\left(L^{m_k+j}_{m_k}\,\bar{Z}_{m_k+j} + M^{m_k+j}_{m_k}\,\bar{\psi}_{m_k+j}\right)_{r_k}, \tag{159}$$

$$A_D = \frac{L_0}{r_k}\sum_{j\neq 0}\frac{1}{j}\left(N^{m_k+j}_{m_k}\,\bar{Z}_{m_k+j} + P^{m_k+j}_{m_k}\,\bar{\psi}_{m_k+j}\right)_{r_k} - \frac{\nu_{L\,k}}{r_k}\,\sum_{j\neq 0}\frac{1}{j}\left(L^{m_k+j}_{m_k}\,\bar{Z}_{m_k+j} + M^{m_k+j}_{m_k}\,\bar{\psi}_{m_k+j}\right)_{r_k}, \tag{160}$$

$$B_D = \frac{A_D}{L_0}, \tag{161}$$

$$\hat{\lambda}_L = \frac{P_1\,L_0\,(1+\nu_{L\,k})}{r_k} + \frac{\nu_{L\,k}\,T_1}{r_k} - \frac{1}{(m_k\,s)_{r_k}}\,\frac{1}{r_k}\sum_{j\neq 0}\frac{1}{j}\left(L^{m_k+j}_{m_k}\,P^{m_k}_{m_k+j} + M^{m_k+j}_{m_k}\,M^{m_k}_{m_k+j}\right)_{r_k} - \frac{1}{(m_k\,s)_{r_k}}\,\frac{\nu_{L\,k}}{r_k}\left(L^{m_k+j}_{m_k}\,N^{m_k}_{m_k+j} + M^{m_k+j}_{m_k}\,L^{m_k}_{m_k+j}\right)_{r_k}, \tag{162}$$

$$\mu_L = -\frac{1}{2\,(m_k\,s)_{r_k}}\frac{L_0}{r_k}\sum_{j\neq 0}\frac{1}{j}\,(N^{m_k+j}_{m_k}\,P^{m_k}_{m_k+j} + P^{m_k+j}_{m_k}\,M^{m_k}_{m_k+j})_{r_k}, \tag{163}$$

$$\xi_L = M_1 + \frac{\nu_{L\,k}}{r_k}\left(\frac{L_1}{L_0}-1\right), \tag{164}$$

$$\hat{\gamma}_L = \frac{P_1\,(1+\nu_{L\,k})}{L_0\,r_k} + \frac{\nu_{L\,k}\,T_1}{L_0\,r_k}, \tag{165}$$

$$\delta_L = \frac{\mu_L}{L_0}. \tag{166}$$

Here, $\hat{a}_j$, $\hat{b}_j$, $\tilde{a}_j$ and $\tilde{b}_j$ are again specified by Eqs. (135)–(138).

The coefficients of the large and the small solutions at the kth rational surface are evaluated as follows:

$$\bar{\psi}_{m_k+j} = \psi_{m_k+j}(r_k+\delta) - \hat{a}_j\, A_{L\,k}\, \ln|\delta| + {\cal O}(\delta), \tag{167}$$

$$\bar{Z}_{m_k+j} = Z_{m_k+j}(r_k+\delta) - \hat{b}_j\, A_{L\,k}\, \ln|\delta| + {\cal O}(\delta), \tag{168}$$

$$A_{S\,k} = \frac{Z_{m_k}(r_k+\delta) - b_L\, A_{L\,k} - \delta\, \ln|\delta|\,[B_D + (\hat{\gamma}_L + \delta_L\, \ln|\delta|)\, A_{L\,k}]}{b_S\,\delta} + {\cal O}(\delta), \tag{169}$$

$$A_{L\,k} = \frac{\psi_{m_k}(r_k+\delta) - \delta\,[A_{S\,k} + A_C + A_D\,(\ln|\delta|-1)]}{1+\nu_{L\,k}\,\ln|\delta| + \delta\,[\hat{\lambda}_L\,(\ln|\delta|-1) + \mu_L\,(\ln^2|\delta| - 2\,\ln|\delta| - 2) + \xi_L]} + {\cal O}(\delta^2) \tag{170}$$

for $j\neq 0$. As before, the previous equations can be solved via iteration.

D. Asymptotic Matching Across Rational Surfaces

Consider the resonant layer solution in the vicinity of the kth rational surface, whose resonant poloidal mode number is m_k. This solution can be separated into independent tearing and twisting parity components.[22] The tearing parity component is such that $\psi_{m_k}(r_k - x) = \psi_{m_k}(r_k + x)$ throughout the layer, whereas the twisting parity component is such that $\psi_{m_k}(r_k - x) = -\psi_{m_k}(r_k + x)$. It turns out, however, that the twisting parity response of a resonant layer to the solution in the outer region is generally negligible compared to the tearing parity response.[3,9,24] Hence, we shall neglect the twisting parity responses of the various resonant layers in the plasma all together.

The neglect of the twisting parity responses of the various resonant layers in the plasma implies that the coefficients of the large solution to the left and to the right of each rational surface in the plasma are equal to one another.[5] In other words,

$$A_{L\,k}^- = A_{L\,k}^+ = A_{L\,k}. \tag{171}$$

Note, however, that the coefficients of the small solution to the left and to the right of a given rational surface are not, in general, equal to one another.

Consider a solution that is completely continuous across the kth rational surface, so that $A_{S\,k}^- = A_{S\,k}^+$. According to the preceding analysis, the continuity condition for the resonant harmonic can be written as

$$\psi_{m_k}(r_k+|\delta|) = \psi_{m_k}(r_k-|\delta|) + 2\,|\delta|\,[A_{L\,k}\,|\delta|^{\nu_{L\,k}}\,\lambda_L + A_C] + 2\,A_{S\,k}^-\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}), \tag{172}$$

$$Z_{m_k}(r_k+|\delta|) = Z_{m_k}(r_k-|\delta|) + 2\,|\delta|\,[A_{L\,k}\,|\delta|^{\nu_{L\,k}}\,\gamma_L + B_C] + 2\,A_{S\,k}^-\,b_S\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}), \tag{173}$$

$$\psi_{m_k+j}(r_k+|\delta|) = \psi_{m_k+j}(r_k-|\delta|) + 2\,|\delta|\,[A_{L\,k}\,|\delta|^{\nu_{L\,k}}\,c_j + \bar{\psi}'_{m_k+j}] + 2\,A_{S\,k}^-\,a_j\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}), \tag{174}$$

$$Z_{m_k+j}(r_k+|\delta|) = Z_{m_k+j}(r_k-|\delta|) + 2\,|\delta|\,[A_{L\,k}\,|\delta|^{\nu_{L\,k}}\,d_j + \bar{Z}'_{m_k+j}] + 2\,A_{S\,k}^-\,b_j\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}) \tag{175}$$

in the general case, and

$$\psi_{m_k}(r_k+|\delta|) = \psi_{m_k}(r_k-|\delta|) + 2\,|\delta|\,\{A_{L\,k}\,[\hat{\lambda}_L\,(\ln|\delta|-1) + \hat{\mu}_L\,(\ln^2|\delta| - 2\,\ln|\delta| - 2) + \xi_L] + A_C + A_D\,(\ln|\delta|-1) + A_{S\,k}^-\} + {\cal O}(\delta^{\,2}), \tag{176}$$

$$Z_{m_k}(r_k+|\delta|) = Z_{m_k}(r_k-|\delta|) + 2\,|\delta|\,[A_{L\,k}\,\ln|\delta|\,(\hat{\gamma}_L + \delta_L\,\ln|\delta|) + B_D\,\ln|\delta| + A_{S\,k}^-\,b_S] + {\cal O}(\delta^{\,2}), \tag{177}$$

$$\psi_{m_k+j}(r_k+|\delta|) = \psi_{m_k+j}(r_k-|\delta|) + {\cal O}(\delta), \tag{178}$$

$$Z_{m_k+j}(r_k+|\delta|) = Z_{m_k+j}(r_k-|\delta|) + {\cal O}(\delta) \tag{179}$$

in the special case. In both cases, $j\neq 0$.

Consider a solution that is launched from the kth rational surface, so that $A_{L\,k}=A_{S\,k}^-=0$. It follows from the preceding analysis that

$$\psi_{m_k}(r_k+|\delta|) = A_{S\,k}^+\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}), \tag{180}$$

$$Z_{m_k}(r_k+|\delta|) = A_{S\,k}^+\,b_S\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}), \tag{181}$$

$$\psi_{m_k+j}(r_k+|\delta|) = A_{S\,k}^+\,\tilde{a}_j\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}), \tag{182}$$

$$Z_{m_k+j}(r_k+|\delta|) = A_{S\,k}^+\,\tilde{b}_j\,|\delta|^{\nu_{S\,k}} + {\cal O}(\delta^{\,2}) \tag{183}$$

for $j\neq 0$.

It is helpful to define the quantities [5]

$$\Psi_k = r_k^{\nu_{L\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{m_k}^{m_k}}\right)_{r_k}^{1/2} A_{L\,k}, \tag{184}$$

$$\Delta\Psi_k = r_k^{\nu_{S\,k}}\left(\frac{\nu_{S\,k}-\nu_{L\,k}}{L_{m_k}^{m_k}}\right)_{r_k}^{1/2} (A_{S\,k}^{+}-A_{S\,k}^{-}) \tag{185}$$

at each rational surface in the plasma. Here, the complex parameter Ψ_k is a measure of the reconnected helical magnetic flux at the kth rational surface, whereas the complex parameter $\Delta\Psi_k$ is a measure of the strength of a localized current sheet that flows parallel to the equilibrium magnetic field at the surface. It is evident from Eqs. (117), (119), (120), (121), (124), (125), (127), (129), (130), and (171)–(185) that [5,9]

$$T_\phi(r) = \int_0^r \sum_{k=1,K} \delta T_k\,\delta(\tilde{r}-r_k)\,d\tilde{r}, \tag{186}$$

where

$$\delta T_k = 2\,\pi^2\,n\,{\rm Im}(\Psi_k^\ast\,\Delta\Psi_k). \tag{187}$$

Here, δT_k is the net toroidal electromagnetic torque exerted on the plasma in the immediate vicinity of the kth rational surface.

VI. INVERSE ASPECT-RATIO EXPANDED TOKAMAK EQUILIBRIUM

A. Equilibrium Magnetic Flux-Surfaces

Let the plasma/vacuum boundary correspond to $r=\epsilon$, where $\epsilon\ll 1$ is the inverse aspect-ratio of the plasma. In other words, $\epsilon = a/R_0$, where $a\ll R_0$ is the effective minor radius of the plasma. Let $r=\epsilon\,\hat{r}$, $\nabla = \epsilon^{-1}\,\hat{\nabla}$, and $'\rightarrow \epsilon^{-1}\,'$. Suppose that the locii of the equilibrium magnetic flux-surfaces can be written in the parametric form: [2,5,25,26]

$$R(\hat{r},\omega) = 1-\epsilon\,\hat{r}\,\cos\omega + \epsilon^2\sum_{j>0} H_j(\hat{r})\,\cos[(j-1)\,\omega] + \epsilon^2\sum_{j>1} V_j(\hat{r})\,\sin[(j-1)\,\omega]$$
$$+\,\epsilon^3\,L(\hat{r})\,\cos\omega, \tag{188}$$

$$Z(\hat{r},\omega) = \epsilon\,\hat{r}\,\sin\omega + \epsilon^2 \sum_{j>1} H_j(\hat{r})\,\sin[(j-1)\,\omega] - \epsilon^2 \sum_{j>1} V_j(\hat{r})\,\cos[(j-1)\,\omega]$$
$$- \epsilon^3\,L(\hat{r})\,\sin\omega, \tag{189}$$

where j is a positive integer. Here, $H_1(\hat{r})$ controls the relative horizontal locations of the flux-surface centroids, $H_2(\hat{r})$ and $V_2(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface ellipticities, $H_3(\hat{r})$ and $V_3(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface triangularities, et cetera, whereas $L(\hat{r})$ is a re-labelling parameter. Moreover, $\omega(R, Z)$ is a poloidal angle that is distinct from θ. Note that V_1 does not appear in Eq. (189) because such a factor merely gives rise to a rigid vertical shift of the plasma that can be eliminated by a suitable choice of the origin of the flux-coordinate system.[26]

Let

$$J(\hat{r},\omega) = \frac{1}{\epsilon^2}\left(\frac{\partial R}{\partial\omega}\,\frac{\partial Z}{\partial\hat{r}} - \frac{\partial R}{\partial\hat{r}}\,\frac{\partial Z}{\partial\omega}\right) \tag{190}$$

be the Jacobian of the $\hat{r}$, ω coordinate system. We can transform to the $\hat{r}$, θ coordinate system by writing

$$\theta(\hat{r},\omega) = 2\pi\int_0^{\omega}\frac{J(\hat{r},\tilde{\omega})}{R(\hat{r},\tilde{\omega})}\,d\tilde{\omega}\bigg/\oint\frac{J(\hat{r},\omega)}{R(\hat{r},\omega)}\,d\omega, \tag{191}$$
$$\hat{r} = \frac{1}{2\pi}\oint\frac{J(\hat{r},\omega)}{R(\hat{r},\omega)}\,d\omega. \tag{192}$$

This transformation ensures that

$$\frac{\partial\theta}{\partial\omega} = \frac{J}{\hat{r}\,R}, \tag{193}$$

and, hence, that

$$\mathcal{J} \equiv \frac{R}{\epsilon}\left(\frac{\partial R}{\partial\theta}\,\frac{\partial Z}{\partial\hat{r}} - \frac{\partial R}{\partial\hat{r}}\,\frac{\partial Z}{\partial\hat{r}}\right) = \epsilon\,R\,J\,\frac{\partial\omega}{\partial\theta} = r\,R^2, \tag{194}$$

in accordance with Eq. (2).

B. Metric Elements

We can determine the metric elements of the flux-coordinate system by combining Eqs. (188)–(192). Evaluating the elements up to $\mathcal{O}(\epsilon)$, but retaining $\mathcal{O}(\epsilon^2)$ contributions to terms that are independent of ω, we obtain,[5,25,26]

$$L(\hat{r}) = \frac{\hat{r}^{\,3}}{8} - \frac{\hat{r}\,H_1}{2} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{H_j^{\,2}}{\hat{r}} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{V_j^{\,2}}{\hat{r}}, \tag{195}$$

$$\theta = \omega + \epsilon\,\hat{r}\,\sin\omega - \epsilon\sum_{j>0}\frac{1}{j}\left[H_j' - (j-1)\,\frac{H_j}{\hat{r}}\right]\sin(j\,\omega)$$
$$+\,\epsilon\sum_{j>1}\frac{1}{j}\left[V_j' - (j-1)\,\frac{V_j}{\hat{r}}\right]\cos(j\,\omega), \tag{196}$$

$$|\hat{\nabla}\hat{r}|^2 = 1 + 2\,\epsilon\sum_{j>0}H_j'\,\cos(j\,\theta) + 2\,\epsilon\sum_{j>1}V_j'\,\sin(j\,\theta)$$
$$+\,\epsilon^{\,2}\left(\frac{3\,\hat{r}^2}{4} - H_1 + \frac{1}{2}\sum_{j>0}\left[H_j'^{\,2} + (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right]\right.$$
$$\left.+\frac{1}{2}\sum_{j>1}\left[V_j'^{\,2} + (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right]\right), \tag{197}$$

$$\hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta = \epsilon\,\sin\theta - \epsilon\sum_{j>0}\frac{1}{j}\left[H_j'' + \frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}}\right]\sin(j\,\theta)$$
$$+\,\epsilon\sum_{j>1}\frac{1}{j}\left[V_j'' + \frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}}\right]\cos(j\,\theta), \tag{198}$$

$$R^{\,2} = 1 - 2\,\epsilon\,\hat{r}\,\cos\theta - \epsilon^{\,2}\left(\frac{\hat{r}^{\,2}}{2} - \hat{r}\,H_1' - 2\,H_1\right). \tag{199}$$

Here, $' \equiv d/d\hat{r}$. Moreover, we have made use of the fact that $V_j \propto H_j$, for $j > 1$, because V_j and H_j satisfy the identical differential equations, (205) and (206).

C. Expansion of Grad-Shafranov Equation

Let us write

$$f(\hat{r}) = \epsilon\,\frac{\hat{r}\,g}{q}, \tag{200}$$

$$g(\hat{r}) = 1 + \epsilon^{\,2}\,g_2(\hat{r}) + \epsilon^{\,4}\,g_4(\hat{r}), \tag{201}$$

$$P'(\hat{r}) = \epsilon^{\,2}\,p_2'(\hat{r}), \tag{202}$$

where q, g_2, g_4, and p_2 are all $\mathcal{O}(1)$. Here, the safety-factor, $q(\hat{r})$, and the second-order plasma pressure gradient, $p_2'(\hat{r})$, are the two free flux-surface functions that characterize the plasma equilibrium.[18]

Expanding the Grad-Shafranov equation, (16), order by order in the small parameter ϵ, making use of Eqs. (197)–(202), we obtain [2,3,25,26]

$$g_2' = -p_2' - \frac{\hat{r}}{q^2}\,(2-s), \tag{203}$$

$$H_1'' = -(3-2\,s)\,\frac{H_1'}{\hat{r}} - 1 + \frac{2\,p_2'\,q^2}{\hat{r}}, \tag{204}$$

$$H_j'' = -(3-2\,s)\,\frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}} \qquad \text{for } j>1, \tag{205}$$

$$V_j'' = -(3-2\,s)\,\frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}} \qquad \text{for } j>1, \tag{206}$$

$$\begin{aligned} g_4' = &-\frac{\hat{r}}{q^2}\left(\frac{3\,\hat{r}^{\,2}}{2} - 2\,\hat{r}\,H_1' \right.\\ &\quad + \sum_{j>0}\left[H_j'^{\,2} + 2\,(j^2-1)\,\frac{H_j'\,H_j}{\hat{r}} - (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right]\\ &\quad \left. + \sum_{j>1}\left[V_j'^{\,2} + 2\,(j^2-1)\,\frac{V_j'\,V_j}{\hat{r}} - (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right]\right)\\ &+\frac{\hat{r}}{q^2}\,(2-s)\left(-g_2 - \frac{3\,\hat{r}^{\,2}}{4} + \frac{\hat{r}^{\,2}}{q^2} + H_1 + \frac{1}{2}\sum_{j>0}\left[3\,H_j'^{\,2} - (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right]\right.\\ &\quad \left. + \frac{1}{2}\sum_{j>1}\left[3\,V_j'^{\,2} - (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right]\right)\\ &+ p_2'\left(g_2 + \frac{\hat{r}^{\,2}}{2} + \frac{\hat{r}^{\,2}}{q^2} - 2\,H_1 - 3\,\hat{r}\,H_1'\right). \end{aligned} \tag{207}$$

Note that the relative horizontal shift of magnetic flux-surfaces, H_1, otherwise known as the *Shafranov shift*,[27] is driven by toroidicity [the second term on the right-hand side of Eq. (204)], and plasma pressure gradients (the third term). All of the other shaping terms (i.e., the H_j, for $j>1$, and the V_j) are driven by currents flowing in axisymmetric external magnetic field-coils.[26]

Finally, it follows from Eqs. (38), (39), (71), and (200)–(202) that

$$\alpha_p(\hat{r}) = \frac{p_2'\,q^2}{\hat{r}}\left(1 - 2\,\epsilon^2\,g_2\right), \tag{208}$$

$$\alpha_g(\hat{r}) = \frac{q}{\hat{r}}\left(g_2' - \epsilon^2\,g_2\,g_2' + \epsilon^2\,g_4'\right), \tag{209}$$

$$\alpha_f(\hat{r}) = -s + \epsilon^2\,\hat{r}\,g_2'. \tag{210}$$

D. Self-Inductance and β values

The conventionally defined normalized self-inductance, toroidal beta, poloidal beta, and normalized beta values of the plasma equilibrium can be written [18]

$$l_i = \frac{2\int_0^1 \hat{r}\,f^2\,\langle|\nabla r|^2\rangle\,d\hat{r}}{(f^2\,\langle|\nabla r|^2\rangle^2)_{\hat{r}=1}}, \tag{211}$$

$$\beta_t = \frac{2\,\epsilon^2\int_0^1 \hat{r}\,\langle R^2\rangle\,p_2\,d\hat{r}}{\int_0^1 \hat{r}\,\langle R^2\rangle\,d\hat{r}}, \tag{212}$$

$$\beta_p = \frac{2\,\epsilon^2\int_0^1 \hat{r}\,\langle R^2\rangle\,p_2\,d\hat{r}}{(f^2\,\langle|\nabla r|^2/R^2\rangle)_{\hat{r}=1}\int_0^1 \hat{r}\,\langle R^2\rangle\,d\hat{r}}, \tag{213}$$

$$\beta_N = \frac{20\,\epsilon^2\int_0^1 \hat{r}\,\langle R^2\rangle\,p_2\,d\hat{r}}{(f\,\langle|\nabla r|^2\rangle)_{\hat{r}=1}\int_0^1 \hat{r}\,\langle R^2\rangle\,d\hat{r}}, \tag{214}$$

respectively. Here, $\langle\cdots\rangle \equiv \oint(\cdots)\,d\theta/2\pi$.

E. Coupling Coefficients

Let

$$S_1(\hat{r}) = \frac{1}{2}\sum_{j>0}\left[3\,(H_j'^{\,2} + V_j'^{\,2}) - (j^2-1)\,\frac{H_j^{\,2}+V_j^{\,2}}{\hat{r}^{\,2}}\right], \tag{215}$$

$$\begin{aligned} S_2(\hat{r}) = {} & \sum_{j>1} 2\,(j^2-1)\left(H_j'^{\,2} + V_j'^{\,2} - \frac{11}{3}\,\frac{H_j'\,H_j + V_j'\,V_j}{\hat{r}} + j^2\,\frac{H_j^{\,2}+V_j^{\,2}}{\hat{r}^{\,2}}\right) \\ & - \sum_{j>0}(1-s)\left(\frac{H_j'\,H_j + V_j'\,V_j}{\hat{r}} + \frac{1}{3}\,\frac{H_j^{\,2}+V_j^{\,2}}{\hat{r}^{\,2}}\right). \end{aligned} \tag{216}$$

The analysis of Sects. IV A, IV B, VI B, and VI C can be combined to give the following expressions for the coupling coefficients appearing in the outer region ODES, (102) and (103):[5]

$$L_m^m(\hat{r}) = m^2 + \epsilon^2\, m^2 \left(-\frac{3\,\hat{r}^{\,2}}{4} + H_1 + S_1\right) + \epsilon^2\, n^2\, \hat{r}^{\,2}, \tag{217}$$

$$M_m^m(\hat{r}) = 0, \tag{218}$$

$$N_m^m(\hat{r}) = 0, \tag{219}$$

$$\begin{aligned} P_m^m(\hat{r}) &= (m-n\,q)^2 + \frac{m-n\,q}{m}\,q\,\hat{r}\,\frac{d}{d\hat{r}}\left(\frac{2-s}{q}\right)\\ &\quad+\epsilon^2\,(m-n\,q)^2\left\{\frac{7\,\hat{r}^{\,2}}{4} - H_1 - 3\,\hat{r}\,H_1' + S_1\right.\\ &\quad\left.+\frac{1}{m^2}\left[\frac{n}{m}\,\hat{r}\,\frac{d}{d\hat{r}}\left(\hat{r}^{\,2}\,\frac{2-s}{q}\right) - \hat{r}^{\,2}\,\frac{(2-s)^2}{q^2} - \hat{r}\,\frac{d}{d\hat{r}}(\hat{r}\,p_2')\right]\right\}\\ &\quad-\epsilon^2\,\frac{m-n\,q}{m}\left\{2\,\hat{r}\,p_2'\,(2-s) + q\,\hat{r}\,\frac{d}{d\hat{r}}\left[\hat{r}^{\,2}\,\frac{2-s}{q^3} + \frac{s}{q}\left(\frac{3\,\hat{r}^{\,2}}{4} - H_1 - S_1\right)\right.\right.\\ &\quad\left.\left.-\frac{2}{q}\left(\frac{3\,\hat{r}^{\,2}}{2} - H_1 - \hat{r}\,H_1' - \frac{2}{3}\,S_1\right)\right] - S_2\right\} + \epsilon^2\,2\,\hat{r}\,p_2'\,(1-q^2), \end{aligned} \tag{220}$$

$$L_m^{m\pm 1}(\hat{r}) = -\epsilon\, m\,(m\pm 1)\,H_1', \tag{221}$$

$$L_m^{m\pm j}(\hat{r}) = -\epsilon\, m\,(m\pm j)\,(H_j' \pm {\rm i}\,V_j') \qquad \text{for } j>1, \tag{222}$$

$$M_m^{m\pm 1}(\hat{r}) = \mp\epsilon\, m\,(m-n\,q)\,p_2'\,q^2 \pm \epsilon\, m\,(m\pm 1-n\,q)\,[\hat{r} + (1-s)\,H_1'], \tag{223}$$

$$\begin{aligned} M_m^{m\pm j}(\hat{r}) &= \pm\epsilon\,\frac{m}{j}\,(m\pm j-n\,q)\left[(1-s)\,(H_j' \pm {\rm i}\,V_j') - (j^2-1)\,\frac{H_j \pm {\rm i}\,V_j}{\hat{r}}\right]\\ &\qquad \text{for } j>1, \end{aligned} \tag{224}$$

$$N_m^{m\pm 1}(\hat{r}) = \mp\epsilon\,(m\pm 1)\,(m\pm 1-n\,q)\,p_2'\,q^2 \pm \epsilon\,(m\pm 1)\,(m-n\,q)\,[\hat{r} + (1-s)\,H_1'], \tag{225}$$

$$\begin{aligned} N_m^{m\pm j}(\hat{r}) &= \pm\epsilon\,\frac{(m\pm j)}{j}\,(m-n\,q)\left[(1-s)\,(H_j' \pm {\rm i}\,V_j') - (j^2-1)\,\frac{H_j \pm {\rm i}\,V_j}{\hat{r}}\right]\\ &\qquad \text{for } j>1, \end{aligned} \tag{226}$$

$$P_m^{m\pm 1}(\hat{r}) = -\epsilon\,(1+s)\,p_2'\,q^2 + \epsilon\,(m-n\,q)\,(m\pm 1-n\,q)\,(\hat{r} - H_1'), \tag{227}$$

$$P_m^{m\pm j}(\hat{r}) = -\epsilon\,(m-n\,q)\,(m\pm j-n\,q)\,(H_j' \pm {\rm i}\,V_j')] \qquad \text{for } j>1. \tag{228}$$

When $m = 0$, some of the coupling coefficient take on special values:

$$P_0^{\,0}(\hat{r}) = n^2\,q^2 - \frac{q^2}{\hat{r}}\,\frac{d}{d\hat{r}}\left(\hat{r}^{\,2}\,\frac{2-s}{q^2}\right) - q^2\,\hat{r}\,\frac{d}{d\hat{r}}\left(\frac{p_2'}{\hat{r}}\right), \tag{229}$$

$$M_{\pm 1}^{\,0}(\hat{r}) = \lim_{m\to 0}\,M_{m\pm 1}^{\,m} \mp \epsilon\,(2-s)\,H_1', \tag{230}$$

$$M_{\pm j}^{\,0}(\hat{r}) = \lim_{m\to 0}\,M_{m\pm j}^{\,m} \mp \epsilon\,(2-s)\,j\,(H_j' \mp \mathrm{i}\,V_j') \qquad \text{for } j>1, \tag{231}$$

$$N_0^{\,\pm 1}(\hat{r}) = \lim_{m\to 0}\,N_m^{\,m\pm 1} \pm \epsilon\,(2-s)\,H_1', \tag{232}$$

$$N_0^{\,\pm j}(\hat{r}) = \lim_{m\to 0}\,N_m^{\,m\pm j} \pm \epsilon\,(2-s)\,j\,(H_j' \pm \mathrm{i}\,V_j') \qquad \text{for } j>1, \tag{233}$$

$$P_{\pm 1}^{\,0}(\hat{r}) = \lim_{m\to 0}\,P_{m\pm 1}^{\,m} - \epsilon\,(2-s)\left\{\pm n\,q^3\,p_2' + (1\mp n\,q)\,[\hat{r} + (1-s)\,H_1']\right\} \tag{234}$$

$$P_{\pm j}^{\,0}(\hat{r}) = \lim_{m\to 0}\,P_{m\pm j}^{\,m} - \epsilon\,(2-s)\,\frac{(j\mp n\,q)}{j}\left[(1-s)\,(H_j' \mp \mathrm{i}\,V_j') - (j^2-1)\left(\frac{H_j \mp \mathrm{i}\,V_j}{\hat{r}}\right)\right]$$

$$\text{for } j>1, \tag{235}$$

$$P_0^{\,\pm 1}(\hat{r}) = \lim_{m\to 0}\,P_{m\pm 1}^{\,m} - \epsilon\,(2-s)\left\{\pm n\,q^3\,p_2' + (1\mp n\,q)\,[\hat{r} + (1-s)\,H_1']\right\}, \tag{236}$$

$$P_0^{\,\pm j}(\hat{r}) = \lim_{m\to 0}\,P_m^{\,m\pm j} - \epsilon\,(2-s)\,\frac{(j\mp n\,q)}{j}\left[(1-s)\,(H_j' \pm \mathrm{i}\,V_j') - (j^2-1)\left(\frac{H_j \pm \mathrm{i}\,V_j}{\hat{r}}\right)\right]$$

$$\text{for } j>1. \tag{237}$$

Note that the coupling coefficients satisfy the symmetry requirements (110)–(113).

F. Behavior Close to Magnetic Axis

In the limit $\hat{r} \ll 1$, a well-behaved solution of the outer region ODEs, (102) and (103), with a dominant poloidal mode number $m>0$ is such that [5]

$$Z_m(\hat{r}) \simeq \frac{m-n\,q}{m}\,\psi_m(\hat{r}), \tag{238}$$

$$\psi_{m+1}(\hat{r}) \simeq -\epsilon\,\frac{\hat{r}\,[(m-n\,q) - 2\,p_2''\,q^2]}{2\,(m-n\,q)}\,\psi_m(\hat{r}), \tag{239}$$

$$\psi_{m+j}(\hat{r}) \simeq \epsilon\,\frac{2\,\hat{r}^{\,2}\,q''\,(H_j' - \mathrm{i}\,V_j')}{(m-n\,q)\,(m+1)\,q}\,\psi_m(\hat{r}) \qquad \text{for } j>1 \tag{240}$$

to lowest order, with all of the other Z_m and ψ_m approximately zero. A well-behaved solution with a dominant poloidal mode number $m<0$ is such that

$$Z_m(\hat{r}) \simeq \frac{m-n\,q}{|m|}\,\psi_m(\hat{r}), \tag{241}$$

$$\psi_{m-1}(\hat{r}) \simeq -\epsilon\,\frac{\hat{r}\,[(m-n\,q)+2\,p_2''\,q^2]}{2\,(m-n\,q)}\,\psi_m(\hat{r}), \tag{242}$$

$$\psi_{m-j}(\hat{r}) \simeq -\epsilon\,\frac{2\,\hat{r}^{\,2}\,q''\,(H_j'+\mathrm{i}\,V_j')}{(m-n\,q)\,(|m|+1)\,q}\,\psi_m(\hat{r}) \qquad \text{for } j>1 \tag{243}$$

to lowest order, with all of the other Z_m and ψ_m approximately zero. For the special case in which the dominant poloidal mode number is zero, the well-behaved solution is

$$Z_0(\hat{r}) \simeq \text{constant} \tag{244}$$

to lowest order, with all of the other Z_m and ψ_m approximately zero.

Note that the solutions (238)–(243) only exhibit "outward" coupling of different poloidal harmonics (i.e., coupling in the direction away from the $m=0$ harmonic). However, the solutions whose central poloidal mode numbers are $m=\pm 1$ are special cases, and also exhibit inward coupling. Thus, in addition, to the couplings described in Eqs. (238)–(240), an $m=1$ solution drives the harmonics

$$\psi_{1-j}(\hat{r}) \simeq 2\,\epsilon\,(H_j'-\mathrm{i}\,V_j')\,\psi_1(\hat{r}) \qquad \text{for } j>1. \tag{245}$$

Likewise, in addition to the couplings described in Eqs. (241)–(243), an $m=-1$ solution drives the harmonics

$$\psi_{-1+j}(\hat{r}) \simeq 2\,\epsilon\,(H_j'+\mathrm{i}\,V_j')\,\psi_{-1}(\hat{r}) \qquad \text{for } j>1. \tag{246}$$

G. Plasma/Vacuum Interface

We require the equilibrium plasma current to be zero at the plasma/vacuum interface, $\hat{r}=1$, which implies that $g'(1)=P'(1)=0$. (See Sect. II B.) It follows from Eqs. (201)–(203) and (207) that we need [5]

$$p_2'(1) = 0, \tag{247}$$

$$
\begin{aligned}
s(1) = 2 + \epsilon^{2} \Bigg(&\frac{3\,\hat{r}^{2}}{2} - 2\,\hat{r}\,H_1' \\
&+ \sum_{j>0}\left[H_j'^{\,2} + 2\,(j^2-1)\,\frac{H_j'\,H_j}{\hat{r}} - (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right] \\
&+ \sum_{j>1}\left[V_j'^{\,2} + 2\,(j^2-1)\,\frac{V_j'\,V_j}{\hat{r}} - (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right]\Bigg)_{\hat{r}=1} + {\cal O}(\epsilon^4).
\end{aligned}
\tag{248}
$$

H. Perturbed Vacuum Magnetic Field

In the vacuum region surrounding the plasma, the perturbed magnetic field can be written in the form [5]

$$
\mathbf{b} = {\rm i}\,\nabla V, \tag{249}
$$

where the scalar magnetic potential, $V(\hat{r},\theta,\phi) = V(\hat{r},\theta)\,\exp(-{\rm i}\,n\,\phi)$, is expanded as

$$
V(\hat{r},\theta) = \sum_m V_m(\hat{r})\,\exp({\rm i}\,m\,\theta). \tag{250}
$$

It follows from Eqs. (49), (78), (80), and (99)–(101) that

$$
Z_m(\hat{r}) = (m-n\,q)\,V_m(\hat{r}), \tag{251}
$$

$$
\psi_m(\hat{r}) = \sum_{m'}\left[h_m^{m'}\,\hat{r}\,\frac{dV_{m'}}{d\hat{r}} + i_m^{m'}\,m'\,V_m'(\hat{r})\right], \tag{252}
$$

where

$$
h_m^{m'}(\hat{r}) = \oint |\hat{\nabla}\hat{r}|^2\,R^2\,\exp[-{\rm i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{253}
$$

$$
i_m^{m'}(\hat{r}) = \oint {\rm i}\,\hat{r}\,\hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta\,R^2\,\exp[-{\rm i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}. \tag{254}
$$

In the vacuum region, the scalar magnetic potential satisfies Laplace's equation

$$
\nabla^2 V = 0. \tag{255}
$$

It is necessary to obtain solution of the previous equation that extend beyond the region of validity of the inverse aspect-ratio expansion. This goal can be achieved using *orthogonal toroidal coordinates*, μ, η, ϕ, [28]

$$
R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}, \tag{256}
$$

$$Z = \frac{\sin\eta}{\cosh\mu - \cos\eta}. \tag{257}$$

Here, $\mu(R,Z)\rightarrow 0$ corresponds to either $R\rightarrow 0$ or $(R^{2}+Z^{2})^{1/2}\rightarrow\infty$ (i.e., an approach to the toroidal symmetry axis or to infinity), whereas $\mu(R,Z)\rightarrow\infty$ corresponds to $(R,Z)\rightarrow(1,0)$ (i.e., an approach to the magnetic axis). Furthermore, $\eta(R,Z)$ is an angular variable in the poloidal plane. The most general solution of Laplace's equation in toroidal coordinates for a potential that has n periods around the toroidal symmetry axis is [29]

$$V(\mu,\eta) = \sum_m (z-\cos\eta)^{1/2}\,[A_m\,P^{\,n}_{m-1/2}(z) + B_m\,Q^{\,n}_{m-1/2}(z)]\,\exp(-{\rm i}\,m\,\eta), \tag{258}$$

where $z=\cosh\mu$, $P^{\,n}_{m-1/2}(z)$ and $Q^{\,n}_{m-1/2}(z)$ are *toroidal functions*,[30] and the A_m and B_m are arbitrary constants. Note that $P^{\,n}_{m-1/2}(z)$ is well behaved in the limit $z\rightarrow 1$, whereas $Q^{\,n}_{m-1/2}(z)$ is badly behaved. Conversely, $P^{\,n}_{m-1/2}(z)$ is badly behaved in the limit $z\rightarrow\infty$, whereas $Q^{\,n}_{m-1/2}(z)$ is well behaved.

In the immediate vicinity of the plasma/vacuum boundary, Eqs. (188), (189), (195), (256), and (257) can be combined to give [26]

$$\eta = \pi - \theta - \epsilon\,F_\theta(\hat{r},\theta) + {\cal O}(\epsilon^{2}), \tag{259}$$

where

$$F_\theta(\hat{r},\theta) = -\frac{\hat{r}}{2}\,\sin\theta + \sum_{j>0}\frac{1}{j}\left(H_j' + \frac{H_j}{\hat{r}}\right)\sin(j\,\theta) - \sum_{j>1}\frac{1}{j}\left(V_j' + \frac{V_j}{\hat{r}}\right)\cos(j\,\theta). \tag{260}$$

Furthermore,

$$z = \frac{\xi(\hat{r},\theta)}{\epsilon\,\hat{r}}, \tag{261}$$

where

$$\begin{aligned}\xi(\hat{r},\theta) = 1 &+ \epsilon\left[-\frac{\hat{r}}{2}\,\cos\theta + \sum_{j>0}\frac{H_j}{\hat{r}}\,\cos(j\,\theta) + \sum_{j>1}\frac{V_j}{\hat{r}}\,\sin(j\,\theta)\right] \\ &+ \epsilon^{2}\left[\frac{3\,\hat{r}^{2}}{16} + \frac{H_1}{4} - \frac{\hat{r}\,H_1'}{4} - \frac{1}{2}\sum_{j>0}\left(\frac{H_j'\,H_j}{\hat{r}} - \frac{H_j^{2}}{2\,\hat{r}^{2}}\right) - \frac{1}{2}\sum_{j>1}\left(\frac{V_j'\,V_j}{\hat{r}} - \frac{V_j^{2}}{2\,\hat{r}^{2}}\right)\right].\end{aligned} \tag{262}$$

Note that $\hat{r}\sim{\cal O}(1)$ in the near-vacuum region, which implies that $z\gg 1$.

It is easily demonstrated that $P^n_{m-1/2}(z) = P^n_{|m|-1/2}(z)$ and $Q^n_{m-1/2}(z) = Q^n_{|m|-1/2}(z)$.[31] In the limit $z \gg 1$, the toroidal functions $P^n_{|m|-1/2}(z)$ and $Q^n_{|m|-1/2}(z)$ have the following asymptotic behaviors:[32,33]

$$\frac{z^{1/2}\,P^n_{-1/2}(z)}{\tanh^n z} = \frac{2}{\sqrt{2\pi}\,\Gamma(1/2-n)}\left\{\ln\left(\frac{8\,z}{\zeta_n}\right)\right.$$
$$\left. +\frac{1}{4}\left[(n^2-5/4)+(n+1/2)\,(n+3/2)\,\ln\left(\frac{8\,z}{\zeta_n}\right)\right]\frac{1}{z^2}+{\cal O}\left(\frac{1}{z^4}\right)\right\}, \quad (263)$$

$$\frac{z^{1/2}\,P^n_{1/2}(z)}{\tanh^n z} = \frac{2\,z}{\sqrt{2\pi}\,\Gamma(3/2-n)}\left\{1\right.$$
$$\left. -\left[\frac{n^2-1/4}{2}\,\ln\left(\frac{8\,z}{\zeta_n}\right)+\frac{(n-1/2)\,(n-3/2)}{4}\right]\frac{1}{z^2}+{\cal O}\left(\frac{1}{z^4}\right)\right\}, \quad (264)$$

$$\frac{z^{1/2}\,P^n_{|m|-1/2}(z)}{\tanh^n z} = \frac{(|m|-1)!\,2^{\,|m|}\,z^{\,|m|}}{\sqrt{2\pi}\,\Gamma(|m|-n+1/2)}\left[1-\frac{(n-|m|+1/2)\,(n-|m|+3/2)}{4\,(|m|-1)}\,\frac{1}{z^2}+{\cal O}\left(\frac{1}{z^4}\right)\right], \quad (265)$$

$$\frac{z^{1/2}\,Q^n_{|m|-1/2}(z)}{\tanh^{-n} z} = \frac{\sqrt{\pi}\,\Gamma(|m|+n+1/2)\,z^{-|m|}}{\sqrt{2}\,2^{\,|m|}\,|m|!}\left[1\right.$$
$$\left. +\frac{(|m|-n+1/2)\,(|m|-n+3/2)}{4\,(|m|+1)}\,\frac{1}{z^2}+{\cal O}\left(\frac{1}{z^4}\right)\right], \quad (266)$$

where

$$\zeta_n = \exp\left(\sum_{i=1,n}\frac{2}{2\,i-1}\right). \quad (267)$$

Here, Eqs. (265) only applies to $|m|>1$. Note that there is a factor ${\rm i}^{\,n}$ difference between the definition of the $P^n_{m-1/2}(z)$ used in this paper and that employed in Ref. 29. Moreover, $\Gamma(z)$ is a Gamma function.[34]

Let

$$a_0 = \frac{2}{\sqrt{2\pi}\,\Gamma(1/2-n)}\,A_0, \quad (268)$$

$$a_m = \frac{\cos(|m|\,\pi)\,(|m|-1)!\,2^{\,|m|}\,\epsilon^{-|m|}}{\sqrt{2\pi}\,\Gamma(|m|-n+1/2)}\,A_m, \quad (269)$$

$$b_m = \frac{\cos(|m|\,\pi)\,\sqrt{\pi}\,\Gamma(|m|+n+1/2)\,\epsilon^{\,|m|}}{2^{\,|m|+1/2}\,|m|!}\,B_m, \quad (270)$$

where Eq. (269) only applies to $|m|>0$. Equations (197)–(199), (204)–(206), (247)–(248), (250)–(254), (258), and (260)–(266) can be combined to give

$$\frac{Z_m(1)}{m-n\,q(1)} = \sum_{m'}\left(a_{m'}\,\mathcal{P}_m^{m'} + b_{m'}\,\mathcal{Q}_m^{m'}\right), \tag{271}$$

$$\psi_m(1) = \sum_{m'}\left(a_{m'}\,\mathcal{R}_m^{m'} + b_{m'}\,\mathcal{S}_m^{m'}\right), \tag{272}$$

where [5]

$$\mathcal{P}_{m+\sigma\,j}^{m} = \epsilon\,\frac{|m|}{2\,j}\left[(H_j'-{\rm i}\,\sigma\,V_j') + (j+1)\,(H_j-{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{273}$$

$$\mathcal{P}_{m+\sigma}^{m} = \epsilon\left[\left(\frac{1}{4}-\frac{|m|}{2}\right)+|m|\left(H_1+\frac{H_1'}{2}\right)\right], \tag{274}$$

$$\mathcal{P}_{m}^{m} = 1+\epsilon^{\,2}\,[G_0(|m|)-|m|\,G_1-|m|^2\,G_2], \tag{275}$$

$$\mathcal{P}_{m-\sigma}^{m} = \epsilon\left(\frac{1}{4}-|m|\,\frac{H_1'}{2}\right), \tag{276}$$

$$\mathcal{P}_{m-\sigma\,j}^{m} = \epsilon\,\frac{|m|}{2\,j}\,\left[-\left(H_j'+{\rm i}\,\sigma\,V_j'\right) + (j-1)\,(H_j+{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{277}$$

$$\mathcal{Q}_{m+\sigma\,j}^{m} = \epsilon\,\frac{|m|}{2\,j}\,\left[\left(H_j'-{\rm i}\,\sigma\,V_j'\right) - (j-1)\,(H_j-{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{278}$$

$$\mathcal{Q}_{m+\sigma}^{m} = \epsilon\left(\frac{1}{4}+|m|\,\frac{H_1'}{2}\right) \tag{279}$$

$$\mathcal{Q}_{m}^{m} = 1+\epsilon^{\,2}\,[G_0(-|m|)+|m|\,G_1-|m|^2\,G_2], \tag{280}$$

$$\mathcal{Q}_{m-\sigma}^{m} = \epsilon\left[\left(\frac{1}{4}+\frac{|m|}{2}\right)-|m|\left(H_1+\frac{H_1'}{2}\right)\right], \tag{281}$$

$$\mathcal{Q}_{m-\sigma\,j}^{m} = \epsilon\,\frac{|m|}{2\,j}\,\left[-(H_j'+{\rm i}\,\sigma\,V_j') - (j+1)\,(H_j+{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{282}$$

$$\mathcal{R}_{m+\sigma\,j}^{m} = \epsilon\,\frac{|m|}{2}\left(1+\frac{|m|}{j}\right)\left[-(H_j'-{\rm i}\,\sigma\,V_j') - (j+1)\,(H_j-{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{283}$$

$$\mathcal{R}_{m+\sigma}^{m} = \epsilon\,(1+|m|)\left[\left(\frac{1}{4}+\frac{|m|}{2}\right)-|m|\left(H_1+\frac{H_1'}{2}\right)\right], \tag{284}$$

$$\mathcal{R}_{m}^{m} = |m|\,\left(-1-\epsilon^{\,2}\left[G_3(|m|)-|m|\left(G_1+\frac{H_1}{2}\right)-|m|^2\,G_2\right]\right), \tag{285}$$

$$\mathcal{R}_{m-\sigma}^{m} = \epsilon\left[\left(\frac{1}{4}+\frac{|m|}{4}\right)-|m|\,(1-|m|)\,\frac{H_1'}{2}\right], \tag{286}$$

$$\mathcal{R}^m_{m-\sigma\,j} = \epsilon\,\frac{|m|}{2}\left(1-\frac{|m|}{j}\right)\left[-\left(H_j'+{\rm i}\,\sigma\,V_j'\right)+(j-1)\,(H_j+{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{287}$$

$$\mathcal{S}^m_{m+\sigma\,j} = \epsilon\,\frac{|m|}{2}\left(1+\frac{|m|}{j}\right)\left[\left(H_j'-{\rm i}\,\sigma\,V_j'\right)-(j-1)\,(H_j-{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{288}$$

$$\mathcal{S}^m_{m+\sigma} = \epsilon\left[\left(\frac{1}{4}-\frac{|m|}{4}\right)+|m|\,(1+|m|)\,\frac{H_1'}{2}\right], \tag{289}$$

$$\mathcal{S}^m_m = |m|\,\left(1+\epsilon^2\left[G_3(-|m|)+|m|\left(G_1+\frac{H_1}{2}\right)-|m|^2\,G_2\right]\right), \tag{290}$$

$$\mathcal{S}^m_{m-\sigma} = \epsilon\,(1-|m|)\left[\left(\frac{1}{4}-\frac{|m|}{2}\right)+|m|\left(H_1+\frac{H_1'}{2}\right)\right], \tag{291}$$

$$\mathcal{S}^m_{m-\sigma\,j} = \epsilon\,\frac{|m|}{2}\left(1-\frac{|m|}{j}\right)\left[(H_j'+{\rm i}\,\sigma\,V_j')+(j+1)\,(H_j+{\rm i}\,\sigma\,V_j)\right] \quad \text{for } j>1, \tag{292}$$

and

$$G_0(|m|) = -\left[\frac{n^2+(|m|-2)\,(|m|-3/4)}{4\,(|m|-1)}\right]-\frac{H_1}{2}-\frac{H_1'}{4} \quad \text{for } |m|\neq 1, \tag{293}$$

$$G_0(1) = -\frac{n^2}{4}-\left(\frac{n^2}{2}-\frac{1}{8}\right)\ln\left(\frac{8}{\zeta_n\,\epsilon}\right)-\frac{H_1}{2}-\frac{H_1'}{4}, \tag{294}$$

$$G_1 = -\frac{3\,H_1}{4}-\frac{H_1'}{4}+\frac{1}{2}\sum_{j>0}H_j'\,H_j+\frac{1}{2}\sum_{j>1}V_j'\,V_j, \tag{295}$$

$$\begin{aligned} G_2 = {}& -\frac{H_1'}{4}+\frac{1}{4}\sum_{j>0}\frac{1}{j^2}\left[H_j'^{\,2}+2\,H_j'\,H_j-(j^2-1)\,H_j^{\,2}\right] \\ & +\frac{1}{4}\sum_{j>1}\frac{1}{j^2}\left[V_j'^{\,2}+2\,V_j'\,V_j-(j^2-1)\,V_j^{\,2}\right], \end{aligned} \tag{296}$$

$$G_3(|m|) = -\left[\frac{(|m|-2)\,n^2+|m|^2/4}{4\,(|m|-1)\,|m|}\right] \quad \text{for } |m|\neq 1, \tag{297}$$

$$G_3(1) = -\frac{n^2}{4}+\frac{1}{8}+\left(\frac{n^2}{2}-\frac{1}{8}\right)\ln\left(\frac{8}{\zeta_n\,\epsilon}\right). \tag{298}$$

Here $\sigma = {\rm sgn}(m)$. Moreover, the shaping functions and their derivatives are all evaluated at $\hat{r}=1$. There are a number of special cases:

$$\mathcal{P}^0_j = \epsilon\left(\frac{H_j-{\rm i}\,V_j}{2}\right) \quad \text{for } j>1, \tag{299}$$

$$\mathcal{P}^0_1 = \epsilon\left[\frac{1}{4}\ln\left(\frac{8}{\zeta_n\,\epsilon}\right)-\frac{1}{4}+\frac{H_1}{2}\right], \tag{300}$$

$$\mathcal{P}_0^0 = \ln\left(\frac{8}{\zeta_n\,\epsilon}\right) + \epsilon^2\left[\frac{n^2}{4} - \frac{5}{16} + \left(\frac{n^2}{4}+\frac{3}{8}\right)\ln\left(\frac{8}{\zeta_n\,\epsilon}\right)\right]$$
$$-\left(\frac{H_1}{2}+\frac{H_1'}{4}\right)\ln\left(\frac{8}{\zeta_n\,\epsilon}\right) - G_1, \tag{301}$$

$$\mathcal{P}_{-1}^0 = \epsilon\left[\frac{1}{4}\ln\left(\frac{8}{\zeta_n\,\epsilon}\right) - \frac{1}{4} + \frac{H_1}{2}\right], \tag{302}$$

$$\mathcal{P}_{-j}^0 = \epsilon\left(\frac{H_j + \mathrm{i}\,V_j}{2}\right) \quad \text{for } j>1, \tag{303}$$

$$\mathcal{R}_j^0 = \epsilon\left(-\frac{H_j' - \mathrm{i}\,V_j'}{2} - \frac{H_j - \mathrm{i}\,V_j}{2}\right) \quad \text{for } j>1, \tag{304}$$

$$\mathcal{R}_1^0 = \epsilon\left[\frac{1}{4}\ln\left(\frac{8}{\zeta_n\,\epsilon}\right) + \frac{1}{2} - \frac{H_1}{2} - \frac{H_1'}{2}\right], \tag{305}$$

$$\mathcal{R}_0^0 = -1 + \epsilon^2\,n^2\,\frac{1}{2}\left[\ln\left(\frac{8}{\zeta_n\,\epsilon}\right) + \frac{1}{2}\right], \tag{306}$$

$$\mathcal{R}_{-1}^0 = \epsilon\left[\frac{1}{4}\ln\left(\frac{8}{\zeta_n\,\epsilon}\right) + \frac{1}{2} - \frac{H_1}{2} - \frac{H_1'}{2}\right], \tag{307}$$

$$\mathcal{R}_{-j}^0 = \epsilon\left(-\frac{H_j' + \mathrm{i}\,V_j'}{2} - \frac{H_j + \mathrm{i}\,V_j}{2}\right) \quad \text{for } j>1, \tag{308}$$

$$\mathcal{S}_0^0 = \epsilon^2\,\frac{n^2}{2}. \tag{309}$$

It can be demonstrated that [5]

$$\mathcal{A}^{mm'} \equiv \sum_{m''}\left(\mathcal{P}_{m''}^{m\,*}\,\mathcal{R}_{m''}^{m'} - \mathcal{R}_{m''}^{m\,*}\,\mathcal{P}_{m''}^{m'}\right) = \delta^{\,mm'}\,\mathcal{O}(\epsilon^4) + (1-\delta^{\,mm'})\,\mathcal{O}(\epsilon^2), \tag{310}$$

$$\mathcal{B}^{mm'} \equiv \sum_{m''}\left(\mathcal{Q}_{m''}^{m\,*}\,\mathcal{S}_{m''}^{m'} - \mathcal{S}_{m''}^{m\,*}\,\mathcal{Q}_{m''}^{m'}\right) = \delta^{\,mm'}\,\mathcal{O}(\epsilon^4) + (1-\delta^{\,mm'})\,\mathcal{O}(\epsilon^2), \tag{311}$$

$$\mathcal{C}^{mm'} \equiv \sum_{m''}\left(\mathcal{P}_{m''}^{m\,*}\,\mathcal{S}_{m''}^{m'} - \mathcal{R}_{m''}^{m\,*}\,\mathcal{Q}_{m''}^{m'}\right) = \delta^{\,mm'}\,[h_m + \mathcal{O}(\epsilon^4)] + (1-\delta^{\,mm'})\,\mathcal{O}(\epsilon^2), \tag{312}$$

where

$$h_m = \begin{cases} 2\,|m| & |m|>0 \\ 1 & |m|=0 \end{cases}. \tag{313}$$

Thus, we can determine the parameters in the expansions (271) and (272) as follows:

$$a_m = \frac{1}{h_m}\sum_{m'}\left[-\mathcal{Q}_{m'}^{m\,*}\,\psi_{m'}(1) + \frac{\mathcal{S}_{m'}^{m\,*}\,Z_{m'}(1)}{m' - n\,q(1)}\right] + \mathcal{O}(\epsilon^2), \tag{314}$$

$$b_m = \frac{1}{h_m}\sum_{m'}\left[\mathcal{P}_{m'}^{m\,*}\,\psi_{m'}(1) - \frac{\mathcal{R}_{m'}^{m\,*}\,Z_{m'}(1)}{m'-n\,q(1)}\right] + \mathcal{O}(\epsilon^2), \tag{315}$$

where use has been made of Eq. (251).

I. Toroidal Electromagnetic Angular Momentum Flux

By analogy with Eq. (116), the outward flux of toroidal electromagnetic angular momentum across the plasma boundary, which is equal to the flux of toroidal electromagnetic angular momentum across a surface of constant μ (in the direction of decreasing μ) in the vacuum region, is given by

$$\begin{aligned} T_\phi(1) &= -\oint\oint(\nabla\mu\times\nabla\eta\cdot\nabla\phi)^{-1}\,b_\phi\,b^\mu\,d\eta\,d\phi \\ &= -\frac{\mathrm{i}\,\pi\,n}{2}\oint\frac{z^2-1}{z-\cos\eta}\left(\frac{\partial V}{\partial z}\,V^{\,*} - \frac{\partial V^{\,*}}{\partial z}\,V\right)d\eta, \end{aligned} \tag{316}$$

where $z = \cosh\mu$. From Eq. (258), we get

$$T_\phi(1) = -\mathrm{i}\,\pi^2\,n\sum_m(z^2-1)\left[\frac{dP_{m-1}^n}{dz}\,Q_{m-1/2}^n - P_{m-1/2}^n\,\frac{dQ_{m-1/2}^n}{dz}\right](A_m\,B_m^* - A_m^*\,B_m)\,. \tag{317}$$

However,[35]

$$\frac{dP_{m-1}^n}{dz}\,Q_{m-1/2}^n - P_{m-1/2}^n\,\frac{dQ_{m-1/2}^n}{dz} = \frac{\Gamma(|m|+n+1/2)}{\Gamma(|m|-n-1/2)\,(z^2-1)}. \tag{318}$$

Thus, making use of Eqs. (268)–(270), we obtain

$$T_\phi(1) = -\mathrm{i}\,\pi^2\,n\sum_m h_m\,(a_m\,b_m^* - a_m^*\,b_m). \tag{319}$$

J. Homogeneous Boundary Condition at Plasma/Vacuum Interface

In the absence of currents driven in non-axisymmetric external magnetic field-coils and eddy currents flowing in external conductors, the vacuum expansion (258) must contain none of the terms involving the $Q_{m-1/2}^n(z)$, because these terms are badly behaved a long way from the plasma, and could not, therefore, be generated by currents flowing within the plasma (which are the only type of currents present in the problem). It follows that the B_m coefficients must all be zero. Hence, according to Eq. (270),

$$b_m = 0 \tag{320}$$

for all m. Thus, Eq. (315) yields the following boundary condition at the plasma/vacuum interface:

$$\frac{Z_m(1)}{m - n\,q(1)} = \sum_{m'} H_{mm'}\,\psi_{m'}(1), \tag{321}$$

where

$$\sum_{m''} \mathcal{R}^{m\,*}_{m''}\,H_{m''m'} = \mathcal{P}^{m\,*}_{m'}. \tag{322}$$

Now, from Eq. (117), (271), (272), and (310)–(312),

$$\begin{aligned} T_\phi(1) &= {\rm i}\,\pi^2\,n\sum_m \frac{Z_m^\ast(1)\,\psi_m(1) - \psi_m^\ast(1)\,Z_m(1)}{m - n\,q(1)} \\ &= {\rm i}\,\pi^2\,n\sum_{m,m'}\left(a_m^\ast\,\mathcal{A}^{mm'}\,a_{m'} + a_m^\ast\,\mathcal{C}^{mm'}\,b_{m'} - b_m^\ast\,\mathcal{C}^{m'm\,*}\,a_{m'} + b_m^\ast\,\mathcal{B}^{mm'}\,b_{m'}\right) \\ &= -{\rm i}\,\pi^2\,n\sum_m h_m\,(a_m\,b_m^\ast - a_m^\ast\,b_m) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{323}$$

Given that Eq. (319) is an exact result (i.e., it is independent of our inverse aspect-ratio expansion), it is clear that the $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^4)$ residuals on the right-hand sides of Eqs. (310)–(312) would all be zero in an exact calculation.

Equations (319) and (320) imply that

$$T_\phi(1) = 0. \tag{324}$$

In other words, the flux of toroidal electromagnetic angular momentum across the plasma boundary is exactly zero, as must be the case for an isolated plasma. According to Eqs. (321) and (323), this constraint can only be satisfied if $H_{mm'}$ is an Hermitian matrix. Now, Eqs. (322) yields

$$\sum_{m'',m'''} \mathcal{R}^{m\,*}_{m''}\,(H_{m''m'''} - H^\ast_{m'''m''})\,\mathcal{R}^{m'}_{m'''} = \sum_{m''}\left(\mathcal{P}^{m\,*}_{m''}\,\mathcal{R}^{m'}_{m''} - \mathcal{R}^{m\,*}_{m''}\,\mathcal{P}^{m'}_{m''}\right) = \mathcal{O}(\epsilon^2), \tag{325}$$

where use has been made of Eq. (310). Thus, it is clear that $H_{mm'}$ is Hermitian to $\mathcal{O}(\epsilon^2)$. Moreover, we can enforce the physical constraint (324) by neglecting the $\mathcal{O}(\epsilon^2)$ non-Hermitian component of $H_{mm'}$.

VII. CALCULATION OF HOMOGENEOUS TEARING MODE DISPERSION RELATION

A. Introduction

Let the m_j, for $j = 1, J$, be the poloidal mode numbers included in the calculation. Here, it is assumed that $m_{j+1} = m_j + 1$ for $j = 1, J-1$. Let there be K rational surfaces in the plasma, and let the kth surface lie at radius $\hat{r}_k$, for $k = 1, K$. Here, it is assumed that $\hat{r}_{k+1} > \hat{r}_k$ for $k = 1, K-1$.

B. Well-Behaved Solutions Launched from Magnetic Axis

Let us launch J linearly independent, well-behaved solutions of the outer region ODEs, (102) and (103), from the magnetic axis, as described in Sect. VI F. Let us then numerically integrate these solutions to the plasma/vacuum interface. The poloidal harmonics of the solutions are denoted $\psi^a_{m_{j'}\,m_j}(\hat{r})$ and $Z^a_{m_{j'}\,m_j}(\hat{r})$, for $j, j' = 1, J$. Here, $m_{j'}$ is the poloidal mode number of the harmonic, whereas m_j is the dominant poloidal mode number of the solution close to the magnetic axis. The asymptotic matching conditions imposed at the rational surfaces are

$$A^-_{L\,k} = A^+_{L\,k}, \tag{326}$$

$$\varDelta\Psi_k = 0, \tag{327}$$

for $k = 1, K$. (See Sect. V D). Let $\varPi^a_{kj}$, for $k = 1, K$ and $j = 1, J$, be the value of Ψ_k at the kth rational surface associated with a solution launched from the magnetic axis with dominant poloidal mode number m_j.

A scheme similar to A.H. Glasser's "fixups"[8] is employed to periodically re-orthogonalize the set of solutions. These re-orthogonalizations are implemented at user-defined locations between the magnetic axis and the plasma boundary. At each re-orthogonalization location, the matrix of solutions is forced to become upper triangular, via a process similar to Gaussian elimination, such that only one solution has a non-zero amount of the highest poloidal harmonic, two solutions have non-zero amounts of the next highest harmonic, and so on. The solutions are then renormalized to their largest component. The re-orthogonalizations are necessary to prevent the solutions for becoming colinear as a result of rounding errors, given the significantly different rates at which poloidal harmonics with different poloidal mode numbers grow with increasing $\hat{r}$ close to the magnetic axis. (In fact, a harmonic with mode number m grows as $\hat{r}^{\,|m|}$. Hence, an $m = 10$ harmonic grows far faster than an $m = 1$ harmonic. Unchecked, each solution would quickly become dominated by its component with the largest mode number.)

C. Small Solutions Launched from Rational Surfaces

Let us launch a "small" solution of the outer region ODEs from each rational surface in the plasma, as described in Sects. V B and V C, and numerically integrate it to the plasma vacuum boundary. The poloidal harmonics of the solutions are denoted $\psi^s_{m_j\,k}(\hat{r})$ and $Z^s_{m_j\,k}(\hat{r})$, for $j = 1, J$ and $k = 1, K$. Here, m_j is the poloidal mode number of the harmonic, whereas k is the index of the rational surface from which the solution is launched. The launch conditions are

$$A^+_{L\,k} = 0, \tag{328}$$

$$\Delta\Psi_k = 1. \tag{329}$$

The asymptotic matching conditions imposed at the other rational surfaces are

$$A^-_{L\,k'} = A^+_{L\,k'}, \tag{330}$$

$$\Delta\Psi_{k'} = 0, \tag{331}$$

for $k' = k+1, K$. Let $\Pi^s_{k'k}$, for $k' = 1, K$ and $k = 1, K$, be the value of $\Psi_{k'}$ at the k'th rational surface associated with a small solution launched from the kth rational surface. Note that $\Pi^s_{k'k} = 0$ for $k' \leq k$.

D. Homogeneous Tearing Mode Dispersion Relation

The most general expression for the solution of the outer region ODEs at the plasma/vacuum interface is

$$\psi_{m_j}(1) = \sum_{j'=1,J} \psi^a_{m_j\,m_{j'}}(1)\,\alpha_{j'} + \sum_{k=1,K} \psi^s_{m_j\,k}(1)\,\Delta\Psi_k, \tag{332}$$

$$Z_{m_j}(1) = \sum_{j'=1,J} Z^a_{m_j\,m_{j'}}(1)\,\alpha_{j'} + \sum_{k=1,K} Z^s_{m_j\,k}(1)\,\Delta\Psi_k, \tag{333}$$

for $j = 1, J$, where the α_j are complex coefficients. However, in the absence of currents driven in non-axisymmetric externa;l magnetic field-coils, the solution must satisfy the homogenous boundary condition (321). It follows that

$$\sum_{j'=1,J} X_{jj'}\,\alpha_{j'} = \sum_{k=1,K} Y_{jk}\,\Delta\Psi_k \tag{334}$$

for $j = 1, J$, where

$$X_{jj'} = \frac{Z^a_{m_j\,m_{j'}}(1)}{m_j - n\,q(1)} - \sum_{j''=1,J} H_{m_j\,m_{j''}}\,\psi^a_{m''_j\,m_{j'}}(1), \tag{335}$$

$$Y_{jk} = \sum_{j''=1,J} H_{m_j\,m_{j''}}\,\psi^s_{m_{j''}\,k}(1) - \frac{Z^s_{m_j\,k}(1)}{m_j - n\,q(1)} \tag{336}$$

for $j, j' = 1, J$ and $k = 1, K$. Thus, we can write

$$\alpha_j = \sum_{k=1,K} \Omega_{jk}\,\Delta\Psi_k \tag{337}$$

for $j = 1, J$, where

$$\sum_{j'=1,J} X_{jj'}\,\Omega_{j'k} = Y_{jk} \tag{338}$$

for $j = 1, J$ and $k = 1, K$. It follows that our general solution is now free of arbitrary parameters. Finally, we obtain the *tearing mode dispersion relation*,[3–5,11]

$$\Psi_k = \sum_{k'=1,K} F_{kk'}\,\Delta\Psi_{k'} \tag{339}$$

for $k, k' = 1, K$, where

$$F_{kk'} = \sum_{j=1,J} \Pi^a_{kj}\,\Omega_{jk'} + \Pi^s_{kk'}. \tag{340}$$

Equation (339) specifies the reconnected magnetic flux, Ψ_k, driven at each rational surface in the plasma as a consequence of the current sheets, $\Delta\Psi_k$, flowing at the surfaces. It follows that $F_{kk'}$ is a dimensionless inductance matrix.[36]

We can construct the "fully reconnected" tearing eigenfunction [5] associated with the kth rational surface, which is defined to have the following properties,

$$\Psi_{k'} = F_{k'k}, \tag{341}$$

$$\Delta\Psi_{k'} = \delta_{k'k} \tag{342}$$

for $k' = 1, K$, as follows:

$$\psi^f_{m_j\,k}(\hat{r}) = \psi^s_{m_j\,k}(\hat{r}) + \sum_{j'=1,J} \psi^a_{m_j\,m_{j'}}(\hat{r})\,\Omega_{j'k}, \tag{343}$$

$$Z^f_{m_j\,k}(\hat{r}) = Z^s_{m_j\,k}(\hat{r}) + \sum_{j'=1,J} Z^a_{m_j\,m_{j'}}(\hat{r})\,\Omega_{j'k} \tag{344}$$

for $j = 1, J$. Note that the fully reconnected solution associated with the kth rational surface only has a current sheet at that surface.

The tearing mode dispersion relation can be written in the alternative form [4,5]

$$\Delta\Psi_k = \sum_{k'=1,K} E_{kk'}\,\Psi_{k'} \tag{345}$$

for $k = 1, K$, where $E_{kk'}$ is the inverse of $F_{kk'}$. The previous equation specifies the current sheets driven at each rational surface in the plasma as a consequence of the reconnected fluxes at the surfaces.

We can construct the "unreconnected" tearing eigenfunction [5] associated with the kth rational surface, which is defined to have the following properties,

$$\Psi_{k'} = \delta_{k'k}, \tag{346}$$

$$\Delta\Psi_{k'} = E_{k'k} \tag{347}$$

for $k' = 1, K$, as follows:

$$\psi^u_{m_j\,k}(\hat{r}) = \sum_{k'=1,K} \psi^f_{m_j\,k'}(\hat{r})\,E_{k'k}, \tag{348}$$

$$Z^u_{m_j\,k}(\hat{r}) = \sum_{k'=1,K} Z^f_{m_j\,k'}(\hat{r})\,E_{k'k}, \tag{349}$$

for $j = 1, J$. Note that the unreconnected solution associated with the kth rational surface only has reconnected flux at that surface.

Let

$$\varDelta_k = \frac{\varDelta\varPsi_k}{\varPsi_k} \tag{350}$$

be the complex quantity that characterizes the tearing response of the resonant layer at the kth rational surface to the ideal-MHD solution in the outer region.[1] In general, $\varDelta_k$ is a function of the growth-rate and phase-velocity of the reconnected magnetic flux at the surface.[5,37,38] The previous two equations can be combined to give the ultimate form of the tearing mode dispersion relation,

$$\sum_{k'=1,k} (\varDelta_k\,\delta_{kk'} - E_{kk'})\,\varPsi_{k'} = 0 \tag{351}$$

for $k = 1, K$. It is clear that E_{kk} is the tearing stability index[1] at the kth rational surface when magnetic reconnection takes place at this surface, but is suppressed at the other surfaces (as is likely to occur in the presence of sheared plasma rotation[5]).

E. Toroidal Electromagnetic Torques

According to Eqs. (187) and (345), the net toroidal electromagnetic torque exerted on the plasma in the immediate vicinity of the kth rational surface is

$$\delta T_k = 2\,\pi^2\,n \sum_{k'=1,K} \mathrm{Im}(\varPsi_k^\ast\,E_{kk'}\,\varPsi_{k'}). \tag{352}$$

The total electromagnetic torque exerted on the plasma is

$$T_\phi(1) = \sum_{k=1,K} \delta T_k = 2\,\pi^2\,n \sum_{k,k'=1,K} \mathrm{Im}(\varPsi_k^\ast\,E_{kk'}\,\varPsi_{k'}). \tag{353}$$

However, we have already established that this total torque is zero, irrespective of the values of the $\varPsi_k$. [See Eq. (324)]. Thus, it follows that

$$E_{kk'} = E_{k'k}^\ast. \tag{354}$$

In other words, the matrix $E_{kk'}$ is Hermitian,[5] which implies that $F_{kk'}$ is also Hermitian (as must ought to be the case if $F_{kk'}$ can be interpreted as a dimensionless inductance matrix).

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DATA AVAILABILITY STATEMENT

The digital data used in the figures in this paper can be obtained from the author upon reasonable request.

Appendix A: Nonorthogonal Curvilinear Coordinates

Consider the nonorthogonal curvilinear coordinate system, r, θ, ϕ, introduced in Sect. II A, and let $\mathcal{J} = (\nabla r \times \nabla\theta \cdot \nabla\phi)^{-1}$.

Let

$$\mathbf{A} = A^r\,\mathcal{J}\,\nabla\theta \times \nabla\phi + A^\theta\,\mathcal{J}\,\nabla\phi \times \nabla r + A^\phi\,\mathcal{J}\,\nabla r \times \nabla\theta, \tag{A1}$$

$$\mathbf{A} = A_r\,\nabla r + A_\theta\,\nabla\theta + A_\phi\,\nabla\phi, \tag{A2}$$

where $\mathbf{A}$ is a general vector. It is easily demonstrated that

$$\mathbf{A}\cdot\mathbf{B} = A_r\,B^r + A_\theta\,B^\theta + A_\phi\,B^\phi = A^r\,B_r + A^\theta\,B_\theta + A^\phi\,B_\phi, \tag{A3}$$

$$(\mathbf{A}\times\mathbf{B})_r = \mathcal{J}\,(A^\theta\,B^\phi - A^\phi\,B^\theta), \tag{A4}$$

$$(\mathbf{A}\times\mathbf{B})_\theta = \mathcal{J}\,(A^\phi\,B^r - A^r\,B^\phi), \tag{A5}$$

$$(\mathbf{A}\times\mathbf{B})_\phi = \mathcal{J}\,(A^r\,B^\theta - A^\theta\,B^r), \tag{A6}$$

$$\mathcal{J}\,(\mathbf{A}\times\mathbf{B})^r = A_\theta\,B_\phi - A_\phi\,B_\theta, \tag{A7}$$

$$\mathcal{J}\,(\mathbf{A}\times\mathbf{B})^{\,\theta} = A_\phi\,B_r - A_r\,B_\phi, \tag{A8}$$

$$\mathcal{J}\,(\mathbf{A}\times\mathbf{B})^{\,\phi} = A_r\,B_\theta - A_\theta\,B_r, \tag{A9}$$

where $\mathbf{B}$ is another general vector. Furthermore,

$$\mathcal{J}\,\nabla\cdot\mathbf{C} = \frac{\partial\,(\mathcal{J}\,C^{\,r})}{\partial r} + \frac{\partial\,(\mathcal{J}\,C^{\,\theta})}{\partial\theta} + \frac{\partial\,(\mathcal{J}\,C^{\,\phi})}{\partial\phi}, \tag{A10}$$

$$\mathcal{J}\,(\nabla\times\mathbf{C})^{\,r} = \frac{\partial C_\phi}{\partial\theta} - \frac{\partial C_\theta}{\partial\phi}, \tag{A11}$$

$$\mathcal{J}\,(\nabla\times\mathbf{C})^{\,\theta} = \frac{\partial C_r}{\partial\phi} - \frac{\partial C_\phi}{\partial r}, \tag{A12}$$

$$\mathcal{J}\,(\nabla\times\mathbf{C})^{\,\phi} = \frac{\partial C_\theta}{\partial r} - \frac{\partial C_r}{\partial\theta}, \tag{A13}$$

where $\mathbf{C}(\mathbf{r})$ is a general vector field.

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