Resistive Wall Mode

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I. VACUUM SOLUTION

A. Normalization

Let all lengths be normalized to the major radius of the magnetic axis, R_0. Let all magnetic field-strengths be normalized to the toroidal magnetic field-strength at the magnetic axis, B_0.

B. Toroidal Coordinates

Let μ, η, ϕ be toroidal coordinates such that

$$R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}, \tag{1}$$

$$Z = \frac{\sin\eta}{\cosh\mu - \cos\eta}, \tag{2}$$

where R, ϕ, Z are cylindrical coordinates. The scale-factors of the toroidal coordinate system are

$$h_\mu = h_\eta = \frac{1}{\cosh\mu - \cos\eta} = h, \tag{3}$$

$$h_\phi = \frac{\sinh\mu}{\cosh\mu - \cos\eta} = h\,\sinh\mu. \tag{4}$$

C. Perturbed Magnetic Field

The perturbed magnetic field in the vacuum region is written

$$\mathbf{b} = \mathrm{i}\,\nabla\left[V(\mu,\eta)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi}\right], \tag{5}$$

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where

$$\nabla^2 V = (z-\cos\eta)^3\left\{\frac{\partial}{\partial z}\left[\frac{z^2-1}{z-\cos\eta}\,\frac{\partial V}{\partial z}\right]\right.$$
$$\left.+\frac{\partial}{\partial \eta}\left[\frac{1}{z-\cos\eta}\,\frac{\partial V}{\partial \eta}\right]-\frac{n^2\,V}{(z^2-1)\,(z-\cos\eta)}\right\}=0. \tag{6}$$

Here, $z=\cosh\mu$.

Let

$$f_z = z^2-1, \tag{7}$$
$$f_\eta = (z-\cos\eta)^{1/2}. \tag{8}$$

It follows that

$$\frac{df_z}{dz} = 2\,z, \tag{9}$$
$$\frac{\partial f_\eta}{\partial z} = \frac{1}{2\,f_\eta}, \tag{10}$$
$$\frac{\partial f_\eta}{\partial \eta} = \frac{\sin\eta}{2\,f_\eta}. \tag{11}$$

Let

$$V(z,\eta) = \sum_m f_\eta\,F_m(z)\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}. \tag{12}$$

Then, taking the sum and eikonal as read, and letting $' = d/dz$, we get

$$\frac{\partial V}{\partial z} = \frac{F_m}{2\,f_\eta} + f_\eta\,F_m', \tag{13}$$

$$\frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^2}\,\frac{\partial V}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{f_z\,F_m}{2\,f_\eta^3}+\frac{f_z\,F_m'}{f_\eta}\right) = \frac{z\,F_m}{f_\eta^3}-\frac{3\,f_z\,F_m}{4\,f_\eta^5}+\frac{f_z\,F_m'}{2\,f_\eta^3}+\frac{2\,z\,F_m'}{f_\eta}-\frac{f_z\,F_m'}{2\,f_\eta^3}+\frac{f_z\,F_m''}{f_\eta}$$
$$= \frac{z\,F_m}{f_\eta^3}-\frac{3\,(z^2-1)\,F_m}{4\,f_\eta^5}+\frac{2\,z\,F_m'}{f_\eta}+\frac{(z^2-1)\,F_m''}{f_\eta}, \tag{14}$$

$$\frac{\partial V}{\partial \eta} = \frac{\sin\eta\,F_m}{2\,f_\eta} - \mathrm{i}\,m f_\eta\,F_m, \tag{15}$$

$$\frac{\partial}{\partial \eta}\left(\frac{1}{f_\eta^2}\,\frac{\partial V}{\partial \eta}\right) = \frac{\partial}{\partial \eta}\left(\frac{\sin\eta\,F_m}{2\,f_\eta^3}-\frac{\mathrm{i}\,m\,F_m}{f_\eta}\right) = \frac{\cos\eta\,F_m}{2\,f_\eta^3}-\frac{3\,\sin^2\eta\,F_m}{4\,f_\eta^5}-\frac{\mathrm{i}\,m\,\sin\eta\,F_m}{2\,f_\eta^3}$$
$$+\frac{\mathrm{i}\,m\,\sin\eta\,F_m}{2\,f_\eta^3}-\frac{m^2\,F_m}{f_\eta}$$
$$= \frac{\cos\eta\,F_m}{2\,f_\eta^3}-\frac{3\,\sin^2\eta\,F_m}{4\,f_\eta^5}-\frac{m^2\,F_m}{f_\eta}, \tag{16}$$

$$-\frac{n^2\,V}{f_z\,f_\eta^{\,2}} = -\frac{n^2\,F_m}{(z^2-1)\,f_\eta}. \tag{17}$$

Thus,

$$\begin{aligned}
0 &= \frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^{\,2}}\,\frac{\partial V}{\partial z}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{f_\eta^{\,2}}\,\frac{\partial V}{\partial\eta}\right) - \frac{n^2\,V}{f_z\,f_\eta^{\,2}}\\
&= \frac{z\,F_m}{f_\eta^{\,3}} - \frac{3\,(z^2-1)\,F_m}{4\,f_\eta^{\,5}} + \frac{2\,z\,F_m'}{f_\eta} + \frac{(z^2-1)\,F_m''}{f_\eta}\\
&\quad + \frac{\cos\eta\,F_m}{2\,f_\eta^{\,3}} - \frac{3\,\sin^2\eta\,F_m}{4\,f_\eta^{\,5}} - \frac{m^2\,F_m}{f_\eta} - \frac{n^2\,F_m}{(z^2-1)\,f_\eta}\\
&= \frac{1}{f_\eta}\left[(z^2-1)\,F_m'' + 2\,z\,F_m' + \left(\frac{1}{4}-m^2\right)F_m - \frac{n^2\,F_m}{z^2-1}\right].
\end{aligned} \tag{18}$$

The most general solution of the previous equation is

$$F_m(z) = p_m\,P^{\,n}_{m-1/2}(z) + q_m\,Q^{\,n}_{m-1/2}(z), \tag{19}$$

where p_m and q_m are arbitrary complex coefficients. Note that

$$P^{\,n}_{-m-1/2}(z) = P^{\,n}_{m-1/2}(z), \tag{20}$$

$$Q^{\,n}_{-m-1/2}(z) = Q^{\,n}_{m-1/2}(z), \tag{21}$$

so

$$F_m(z) = p_m\,P^{\,n}_{|m|-1/2}(z) + q_m\,Q^{\,n}_{|m|-1/2}(z), \tag{22}$$

D. Toroidal Electromagnetic Angular Momentum Flux

The outward flux of toroidal angular momentum across a constant-z surface is

$$T_\phi(z) = -\oint\oint \mathcal{J}\,b_\phi\,b^\mu\,d\eta\,d\phi, \tag{23}$$

where

$$\mathcal{J} = (\nabla\mu\times\nabla\eta\cdot\nabla\phi)^{-1} = h^3\,\sinh\mu. \tag{24}$$

Now,

$$b^\mu = \mathbf{b}\cdot\nabla\mu = \mathrm{i}\,\frac{\partial V}{\partial\mu}\,|\nabla\mu|^2 = \mathrm{i}\,\frac{\sinh\mu}{h^2}\,\frac{\partial V}{\partial z}, \tag{25}$$

$$b^\phi = \mathcal{J}\,\nabla\mu\times\nabla\eta\cdot\nabla V = n\,V, \tag{26}$$

so

$$
\begin{aligned}
T_\phi(z) &= -\frac{\mathrm{i}\,n\,\pi}{2}\oint \frac{z^2-1}{z-\cos\eta}\left(\frac{\partial V}{\partial z}\,V^\ast - \frac{\partial V^\ast}{\partial z}\,V\right)d\eta\\
&= -\mathrm{i}\,n\,\pi^2\sum_m (z^2-1)\left(\frac{dF_m}{dz}\,F_m^\ast - \frac{dF_m^\ast}{dz}\,F_m\right)\\
&= -\mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast)\,(z^2-1)\left(\frac{dP^n_{|m|-1/2}}{dz}\,Q^n_{|m|-1/2} - \frac{dQ^n_{|m|-1/2}}{dz}\,P^n_{|m|-1/2}\right)\\
&= \mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast)\,(z^2-1)\,\mathcal{W}[P^n_{|m|-1/2},Q^n_{|m|-1/2}]. \qquad (27)
\end{aligned}
$$

But,

$$
\mathcal{W}[P^n_{|m|-1/2},Q^n_{|m|-1/2}] = \frac{(-1)^n}{1-z^2}\,\frac{\Gamma(1/2+n+|m|)}{\Gamma(1/2-n+m)}, \qquad (28)
$$

so

$$
T_\phi(z) = 2\pi^2\,n\,(-1)^n\,\frac{\Gamma(1/2+n+|m|)}{\Gamma(1/2-n+m)}\,\mathrm{Im}(p_m\,q_m^\ast). \qquad (29)
$$

II. WALL PHYSICS

A. Resistive Wall

Let the resistive wall extend from $z = z_w$ to $z = z_w - d_w$, where d_w is a constant. The physical wall thickness is

$$
d(\eta) = \frac{d_w}{|\nabla z|} = \frac{h_w(\eta)\,d_w}{(z_w^2-1)^{1/2}}, \qquad (30)
$$

where

$$
h_w(\eta) = \frac{1}{z_w-\cos\eta}. \qquad (31)
$$

Let the conductivity of the wall material vary as

$$
\sigma(\eta) = \frac{\sigma_w\,(z_w^2-1)}{h_w^2(\eta)}, \qquad (32)
$$

where σ_w is a constant. It follows that $\sigma\,d^2 = \sigma_w\,d_w^2$.

B. Wall Matching Conditions

If we write

$$
\mathbf{b} = \nabla\times\mathbf{A} \qquad (33)
$$

in the vacuum region then the boundary conditions at the wall are

$$\mathbf{n}_w \times \mathbf{A}\big|_{z_{w-}} = \frac{1}{\cosh\lambda}\,\mathbf{n}_w \times \mathbf{A}\big|_{z_{w+}} \tag{34}$$

$$\mathbf{n}_w \times (\nabla\times\mathbf{A})\big|_{z_{w+}} = -\frac{\lambda\,\tanh\lambda\,(z_w^2-1)^{1/2}}{h_w\,d_w}\,\mathbf{n}_w \times (\mathbf{n}_w\times\mathbf{A})\big|_{z_{w+}} + \frac{\mathbf{n}_w\times(\nabla\times\mathbf{A})\big|_{z_{w-}}}{\cosh\lambda}, \tag{35}$$

$$\lambda = \sqrt{\mu_0\,R_0^2\,\sigma_w\,d_w^2\,\gamma}, \tag{36}$$

where γ is the growth-rate. Here, $\mathbf{n}_w = -\mathbf{e}_\mu$ is an outward unit normal vector to the wall. Now,

$$\nabla\times\mathbf{A} = \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\phi}{\partial\eta} - \frac{\partial\hat{A}_\eta}{\partial\phi}\right)\mathbf{e}_\mu + \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)\mathbf{e}_\eta$$
$$+\frac{1}{h^2}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)\mathbf{e}_\phi, \tag{37}$$

where

$$\hat{A}_\mu = h\,A_\mu, \tag{38}$$

$$\hat{A}_\eta = h\,A_\eta, \tag{39}$$

$$\hat{A}_\phi = h\,\sinh\mu\,A_\phi. \tag{40}$$

Furthermore,

$$\mathbf{n}_w\times\mathbf{A} = -\mathbf{e}_\mu\times\mathbf{A} = A_\phi\,\mathbf{e}_\eta - A_\eta\,\mathbf{e}_\phi, \tag{41}$$

$$\mathbf{n}_w\times(\mathbf{n}_w\times\mathbf{A}) = -\mathbf{e}_\mu\times(\mathbf{n}_w\times\mathbf{A}) = -A_\eta\,\mathbf{e}_\eta - A_\phi\,\mathbf{e}_\phi, \tag{42}$$

$$\mathbf{n}_w\times(\nabla\times\mathbf{A}) = -\mathbf{e}_\mu\times(\nabla\times\mathbf{A}) = \frac{1}{h^2}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)\mathbf{e}_\eta - \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)\mathbf{e}_\phi. \tag{43}$$

Thus, the wall matching conditions become

$$\hat{A}_\eta\Big|_{z_{w-}} = \frac{1}{\cosh\lambda}\,\hat{A}_\eta\Big|_{z_{w+}}, \tag{44}$$

$$\hat{A}_\phi\Big|_{z_{w+}} = \frac{1}{\cosh\lambda}\,\hat{A}_\phi\Big|_{z_{w-}}, \tag{45}$$

$$\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)_{z_{w+}} = \frac{\lambda\,\tanh\lambda\,(z_w^2-1)^{1/2}}{d_w}\,\hat{A}_\eta\Big|_{z_{w+}} + \frac{1}{\cosh\lambda}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)_{z_{w-}}, \tag{46}$$

$$\left(\frac{\partial\hat{A}_\mu}{\partial\phi}-\frac{\partial\hat{A}_\phi}{\partial\mu}\right)_{z_{w+}} = -\frac{\lambda\,\tanh\lambda\,(z_w^2-1)^{1/2}}{d_w}\,\hat{A}_\phi\Big|_{\mu_{w+}}+\frac{1}{\cosh\lambda}\left(\frac{\partial\hat{A}_\mu}{\partial\phi}-\frac{\partial\hat{A}_\phi}{\partial\mu}\right)_{z_{w-}}. \tag{47}$$

Let

$$C(z,\eta,\phi)=\frac{\partial\hat{A}_\eta}{\partial\phi}-\frac{\partial\hat{A}_\phi}{\partial\eta}. \tag{48}$$

The wall matching conditions reduce to

$$C(z_{w-},\eta,\phi)=\frac{1}{\cosh\lambda}\,C(z_{w+},\eta,\phi), \tag{49}$$

$$\frac{\partial C(z_{w+},\eta,\phi)}{\partial z}=\frac{\lambda\,\tanh\lambda}{d_w}\,C(z_{w+},\eta,\phi)+\frac{1}{\cosh\lambda}\,\frac{\partial C(z_{w-},\eta,\phi)}{\partial z}. \tag{50}$$

However, if

$$\mathbf{b}=\mathrm{i}\,\nabla V=\nabla\times\mathbf{A} \tag{51}$$

then

$$C=-\mathrm{i}\,h\,\sinh\mu\,\frac{\partial V}{\partial\mu}=-\mathrm{i}\,h\,(z^2-1)\,\frac{\partial V}{\partial z}. \tag{52}$$

Thus,

$$C=-\mathrm{i}\,\frac{z^2-1}{z-\cos\eta}\sum_m\left[\frac{F_m}{2\,(z-\cos\eta)^{1/2}}+(z-\cos\eta)^{1/2}\,\frac{dF_m}{dz}\right]\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}, \tag{53}$$

$$\frac{\partial C}{\partial z}=-\mathrm{i}\sum_m\left[\frac{(3/4)\,\sin^2\eta}{(z-\cos\eta)^{5/4}}-\frac{(1/2)\,\cos\eta}{(z-\cos\eta)^{3/2}}+\frac{m^2+n^2/(z^2-1)}{(z-\cos\eta)^{1/2}}\right]F_m\,\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}. \tag{54}$$

It follows that

$$\sum_m\left[\frac{F_m}{2}+(z-\cos\eta)\,\frac{dF_m}{dz}\right]\mathrm{e}^{-\mathrm{i}\,m\,\eta} \tag{55}$$