Pressure Flattening due to Asymmetric Magnetic Island

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I.   MAGNETIC ISLAND

Let $x = r - r_s$, $X = x/W$, and $\zeta = m\,\theta - n\,\phi$, where W is the island width. The magnetic flux-surfaces of the magnetic island are contours of

$$\Omega(X, \zeta) = 8\,X^2 + \cos(\zeta - \delta^2\,\sin\zeta) - 2\sqrt{8}\,\delta\,X\,\cos\zeta + \delta^2\,\cos^2\zeta, \tag{1}$$

where $|\delta| < 1$. The X-points lie at $X = \delta/\sqrt{8}$ and $\zeta = 0,\, 2\pi$, whereas the O-point lies at $X = -\delta/\sqrt{8}$ and $\zeta = \pi$. The O-point corresponds to $\Omega = -1$, whereas the magnetic separatrix corresponds to $\Omega = 1$.

Let

$$Y = X - \frac{\delta}{\sqrt{8}}\,\cos\zeta, \tag{2}$$

$$\xi = \zeta - \delta^2\,\sin\zeta. \tag{3}$$

It follows that

$$\Omega(Y, \zeta) = 8\,Y^2 + \cos\xi, \tag{4}$$

The X-points lie at $Y = 0$ and $\xi = 0,\, 2\pi$, whereas the O-point lies at $Y = 0$ and $\zeta = \pi$. Moreover,

$$\zeta = \xi + 2\sum_{n=1,\infty}\left[\frac{J_n(n\,\delta^2)}{n}\right]\sin(n\,\xi), \tag{5}$$

$$\cos\zeta = -\frac{\delta^2}{2} + \sum_{n=1,\infty}\left[\frac{J_{n-1}(n\,\delta^2) - J_{n+1}(n\,\delta^2)}{n}\right]\cos(n\,\xi), \tag{6}$$

$$\sin\zeta = \frac{2}{\delta^2}\sum_{n=1,\infty}\left[\frac{J_n(n\,\delta^2)}{n}\right]\sin(n\,\xi), \tag{7}$$

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$$\cos(m\,\zeta) = m \sum_{n=1,\infty} \left[\frac{J_{n-m}(n\,\delta^2) - J_{n+m}(n\,\delta^2)}{n} \right] \cos(n\,\xi), \tag{8}$$

$$\sin(m\,\zeta) = m \sum_{n=1,\infty} \left[\frac{J_{n-m}(n\,\delta^2) + J_{n+m}(n\,\delta^2)}{n} \right] \sin(n\,\xi), \tag{9}$$

for $m > 1$.

II.   PLASMA DISPLACEMENT

Outside the separatrix, we can write

$$\Omega(X,\zeta) = 8\,(X - \Xi)^2, \tag{10}$$

where $\Xi = \xi^r/W$. It follows that

$$\begin{aligned}
\Xi(X,\zeta) &\simeq -\frac{[\Omega(X,\zeta) - 8\,X^2]}{16\,X} \\
&= -\frac{\cos(\zeta - \delta^2\,\sin\zeta) + \delta^2\,\cos^2\zeta}{16\,X} + \frac{\delta}{\sqrt{8}}\,\cos\zeta.
\end{aligned} \tag{11}$$

Note that Ξ is an even function of ζ. Let us write

$$\Xi(X,\zeta) = \sum_{n=0,\infty} \Xi_n(X)\,\cos(n\,\zeta). \tag{12}$$

Thus,

$$\Xi_1(X) = 2 \oint \Xi(X,\zeta)\,\cos(\zeta)\,\frac{d\zeta}{2\pi} = -\frac{1}{8\,X} \oint \cos(\zeta - \delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi} + \frac{\delta}{\sqrt{8}} \tag{13}$$

$$= -\frac{1}{16\,X} \oint \cos(-\delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi}$$

$$\qquad - \frac{1}{16\,X} \oint \cos(2\,\zeta - \delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi} + \frac{\delta}{\sqrt{8}}. \tag{14}$$

But,

$$J_n(\delta^2) = \oint \cos(n\,\zeta - \delta^2\,\sin\zeta)\,\frac{d\zeta}{2\pi}, \tag{15}$$

so

$$\Xi_1(X) = -\frac{J_0(\delta^2) + J_2(\delta^2)}{16\,X} + \frac{\delta}{\sqrt{8}}, \tag{16}$$

and

$$\xi_1^r(x) = -\frac{W^2}{16\,x}\,[J_0(\delta^2) + J_2(\delta^2)] + \frac{W\,\delta}{\sqrt{8}}. \tag{17}$$

Thus,

$$\psi_1(x) = \frac{r\,g}{q}\,(m - n\,q)\,\xi_1^r = -(n\,s\,g)_{r_s}\,x\,\xi_1^r$$

$$= (n\,s\,g)_{r_k}\,\frac{W^2}{16}\,[J_0(\delta^2) + J_2(\delta^2)] - (n\,s\,g)_{r_s}\,\frac{W\,\delta}{\sqrt{8}}\,x, \tag{18}$$

so

$$\frac{\psi_m(x)}{m} = J_0(\delta^2) + J_2(\delta^2) - 2\sqrt{8}\,\frac{\delta}{W}\,x. \tag{19}$$

It follows that

$$\delta = -\frac{W}{2\sqrt{8}\,m}\,\frac{d\psi_m(0)}{dr} \tag{20}$$

$$\simeq -\left[\frac{\psi_m(r_s + W) - \psi_m(r_s - W)}{4\sqrt{8}\,m}\right]. \tag{21}$$

III.   FLUX-SURFACE AVERAGE OPERATOR

Now,

$$[A, B] \equiv \left.\frac{\partial A}{\partial X}\right|_\zeta \left.\frac{\partial B}{\partial \zeta}\right|_X - \left.\frac{\partial B}{\partial X}\right|_\zeta \left.\frac{\partial A}{\partial \zeta}\right|_X. \tag{22}$$

But,

$$\left.\frac{\partial}{\partial X}\right|_\zeta = \left.\frac{\partial \Omega}{\partial X}\right|_\zeta \left.\frac{\partial}{\partial \Omega}\right|_\xi + \left.\frac{\partial \xi}{\partial X}\right|_\zeta \left.\frac{\partial}{\partial \xi}\right|_\Omega = 16\,Y\,\left.\frac{\partial}{\partial \Omega}\right|_\xi, \tag{23}$$

and

$$\left.\frac{\partial}{\partial \zeta}\right|_X = \left.\frac{\partial \Omega}{\partial \zeta}\right|_X \left.\frac{\partial}{\partial \Omega}\right|_\xi + \left.\frac{\partial \xi}{\partial \zeta}\right|_X \left.\frac{\partial}{\partial \xi}\right|_\Omega, \tag{24}$$

so

$$[A, B] \equiv \frac{16\,Y}{\sigma}\left(\left.\frac{\partial A}{\partial \Omega}\right|_\xi \left.\frac{\partial B}{\partial \xi}\right|_\Omega - \left.\frac{\partial B}{\partial \Omega}\right|_\xi \left.\frac{\partial A}{\partial \xi}\right|_\Omega\right), \tag{25}$$

where

$$\sigma(\xi) \equiv \frac{d\zeta}{d\xi} = 1 + 2\sum_{n=1,\infty} J_n(n\,\delta^2)\,\cos(n\,\xi). \tag{26}$$

In particular,

$$[A, \Omega] = -\frac{16\,Y}{\sigma}\left.\frac{\partial A}{\partial \xi}\right|_\Omega. \tag{27}$$

The flux-surface average operator, $\langle\cdots\rangle$, is the annihilator of $[A, \Omega]$ for arbitrary $A(s, \Omega, \xi)$. Here, $s = +1$ for $Y > 0$ and $s = -1$ for $Y < 0$. It follows that

$$\langle A\rangle = \int_{\zeta_0}^{2\pi-\zeta_0} \frac{\sigma(\xi)\,A_+(\Omega, \xi)}{\sqrt{2\,(\Omega - \cos\xi)}}\,\frac{d\xi}{2\pi} \tag{28}$$

for $-1 \leq \Omega \leq 1$, and

$$\langle A \rangle = \int_0^{2\pi} \frac{\sigma(\xi)\, A(s, \Omega, \xi)}{\sqrt{2\,(\Omega - \cos\xi)}}\, \frac{d\xi}{2\pi} \tag{29}$$

for $\Omega > 1$. Here, $\xi_0 = \cos^{-1}(\Omega)$, and

$$A_+(\Omega, \xi) = \frac{1}{2}\left[A(+1, \Omega, \xi) + A(-1, \Omega, \xi)\right]. \tag{30}$$

IV.   TEMPERATURE PERTURBATION

The electron temperature in the vicinity of the island can be written

$$T_e(X, \zeta) = T_{es} + s\, W\, T_{es}'\, \tilde{T}(\Omega). \tag{31}$$

Here, $\tilde{T}(\Omega)$ satisfies

$$\left\langle \left.\frac{\partial^2 \tilde{T}}{\partial X^2}\right|_\zeta \right\rangle = 0, \tag{32}$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \to |X| \tag{33}$$

as $|X| \to \infty$. It follows that

$$\frac{d}{d\Omega}\left(\langle Y^2 \rangle \frac{d\tilde{T}}{d\Omega}\right) = 0 \tag{34}$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \to \frac{\Omega^{1/2}}{\sqrt{8}} \tag{35}$$

as $\Omega \to \infty$.

Outside the separatrix,

$$\langle Y^2 \rangle(\Omega) = \frac{1}{16} \int_0^{2\pi} \sigma(\xi) \sqrt{2\,(\Omega - \cos\xi)}\, \frac{d\xi}{2\pi}. \tag{36}$$

Let

$$k = \left(\frac{1 + \Omega}{2}\right)^{1/2}. \tag{37}$$

Thus, the O-point corresponds to $k = 0$ and the separatrix to $k = 1$. It follows that

$$\langle Y^2 \rangle(k) = \frac{k}{4\pi} \int_0^{\pi/2} \sigma(2\,\theta - \pi)\,(1 - \sin^2\theta/k^2)^{1/2}\, d\theta. \tag{38}$$

Thus,

$$\langle Y^2 \rangle(k) = \frac{k}{4\pi} G(1/k), \tag{39}$$

where

$$G(p) = E_0(p) + 2 \cos(n\,\pi) \sum_{n=1,\infty} J_n(n\,\delta^2)\, E_n(p), \tag{40}$$

$$E_n(p) = \int_0^{\pi/2} \cos(2\,n\,\theta)\,(1 - p^2\,\sin^2\theta)^{1/2}\,d\theta. \tag{41}$$

Equation (34) yields

$$\tilde{T}(k) = 0 \tag{42}$$

for $0 \le k \le 1$, and

$$\frac{d}{dk}\left[G(1/k)\,\frac{d\tilde{T}}{dk} \right] = 0 \tag{43}$$

for $k > 1$. Thus,

$$\frac{d\tilde{T}}{dk} = \frac{c}{G(1/k)} \tag{44}$$

for $k > 1$, subject to the boundary condition that

$$\tilde{T}(k) \to \frac{k}{2} \tag{45}$$

as $k \to \infty$. In the limit that $p \to 0$,

$$E_0(p) = \frac{\pi}{2}, \tag{46}$$

$$E_{n>0}(p) = 0, \tag{47}$$

which implies that $c = \pi/4$. So

$$\frac{d\tilde{T}}{dk} = \frac{\pi}{4}\,\frac{1}{G(1/k)}, \tag{48}$$

$$\tilde{T}(k) = F(k), \tag{49}$$

$$F(k) = \frac{\pi}{4} \int_1^k \frac{dk'}{G(1/k')} \tag{50}$$

for $k > 1$.

V. HARMONICS OF TEMPERATURE PERTURBATION

We can write

$$\tilde{T}(X,\zeta) = \sum_{\nu=0,\infty} \delta T_\nu(X)\,\cos(\nu\,\zeta). \tag{51}$$

Now,

$$\delta T_0(X) = \oint \tilde{T}(X,\zeta)\,\frac{d\zeta}{2\pi}, \tag{52}$$

where the integral is at constant X. It follows that

$$\delta T_0(X) = \int_0^{\xi_c} F(k)\,\sigma(\xi)\,\frac{d\xi}{\pi}, \tag{53}$$

where

$$\xi_c = \cos^{-1}(1 - 8\,Y^2) \tag{54}$$

for $|Y| < 1/2$, and $\xi_c = \pi$ for $|Y| \geq 1/2$. Furthermore,

$$k = \left[4\,Y^2 + \cos^2\left(\frac{\xi}{2}\right)\right]^{1/2}. \tag{55}$$

Let

$$\delta T_{0\,\infty} = \lim_{X\to\infty}\left[X - \delta T_0(X)\right]. \tag{56}$$

For $\nu > 0$, we have

$$\delta T_\nu(X) = 2 \oint \tilde{T}(X,\zeta)\,\cos(\nu\,\zeta)\,\frac{d\zeta}{2\pi}. \tag{57}$$

Integrating by parts, we obtain

$$\delta T_\nu(X) = -\frac{2}{\nu} \oint \left.\frac{\partial\tilde{T}}{\partial\zeta}\right|_X \sin(\nu\,\zeta)\,\frac{d\zeta}{2\pi}. \tag{58}$$

But,

$$\left.\frac{\partial\tilde{T}}{\partial\zeta}\right|_X = \frac{d\tilde{T}}{dk}\left.\frac{\partial k}{\partial\zeta}\right|_X = \frac{1}{4\,k}\frac{d\tilde{T}}{dk}\left.\frac{\partial\Omega}{\partial\zeta}\right|_X = -\frac{1}{4\,k}\frac{d\tilde{T}}{dk}\,\kappa(\xi), \tag{59}$$

where

$$\kappa(\xi) = \sin\xi\,(1 - \delta^2\,\cos\zeta) - 2\sqrt{8}\,\delta\,X\,\sin\zeta + \delta^2\,\sin(2\,\zeta). \tag{60}$$

Hence,

$$\delta T_\nu(X) = \frac{1}{8\,\nu}\int_0^{\xi_c}\frac{\sin(\nu\,\zeta)\,\kappa(\xi)\,\sigma(\xi)}{k\,G(1/k)}\,d\xi. \tag{61}$$

VI.   ASYMPTOTIC MATCHING

Consider the kth rational surface whose radius is r_k and whose resonant poloidal mode number is m_k. Let $x = r - r_k$ and $\zeta_k = m_k \theta - n \phi$.

In the outer region, we write the total electron temperature as

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) - \Psi_k \frac{q(r)}{r\,g(r)} \frac{T'_{e0}(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\, e^{\,i\,\zeta_k}, \tag{62}$$

where $T'_{e0} = dT_{e0}/dr$, $T_{e0}(r)$ is the equilibrium electron temperature profile, Ψ_k is the reconnected flux, and

$$W_k = 4 \left(\frac{q}{g\,s} \right)_{r_k}^{1/2} \Psi_k^{1/2} \delta T_{0\infty} \tag{63}$$

is the island width. In the limit, $|x| \ll 1$, Eq. (62) yield

$$\tilde{T}_e(x, \theta, \phi) = T_{ek} + T'_{ek}\,x + T'_{ek}\,W_k \left(\frac{W_k}{16\,x} - \frac{\delta_k}{\sqrt{8}} \right) e^{\,i\,\zeta_k}, \tag{64}$$

Here, $T_{ek} = T_{e0}(r_k)$ and $T'_{ek} = (dT_{e0}/dr)_{r_k}$,

$$\delta_k = -\frac{W_k}{2\sqrt{8}\,m} \frac{d\psi_{m_k}(r_k)}{dr}, \tag{65}$$

and we have made use of the fact that $\psi_{m_k}(r_k) \simeq m_k$.

In the inner region, we write the total electron temperature as

$$\tilde{T}_e(x, \theta, \phi) = T_{ek} + T'_{ek}\,W_k \sum_{\nu=0,\infty} \delta T_\nu(x/W_k)\,e^{\,i\,\nu\,\zeta_k} + T'_{ek}\,W_k\,\delta T_{0\infty}, \tag{66}$$

The asymptotic matching process consists of writing

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e+} - \Psi_{k+} \frac{q(r)}{r\,g(r)} \frac{T'_{e0}(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\, e^{\,i\,\zeta_k} \tag{67}$$

in the region $r > r_k + W_k$,

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e-} - \Psi_{k-} \frac{q(r)}{r\,g(r)} \frac{T'_{e0}(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\, e^{\,i\,\zeta_k} \tag{68}$$

in the region $r < r_k - W_k$, and

$$\tilde{T}_e(r, \theta, \phi) = T_{ek} + T'_{ek}\,W_k \sum_{\nu=0,\infty} \delta T_\nu(x/W_k)\,e^{\,i\,\nu\,\zeta_k} + T'_{ek}\,W_k\,\delta T_{0\infty} \tag{69}$$

in the region $r_k - W_k \leq r \leq r_k + W_k$. Continuity of the solution at $r = r_k \pm W_k$ implies that

$$\delta T_{e+} = T'_{ek}\,W_k\,\delta T_{e0}(1) + T'_{ek}\,W_k\,\delta T_{0\infty} - T'_{ek}\,W_k, \tag{70}$$

$$\delta T_{e-} = T'_{ek}\, W_k\, \delta T_{e0}(-1) + T'_{ek}\, W_k\, \delta T_{0\infty} + T'_{ek}\, W_k, \tag{71}$$

$$\Psi_{k+} = -T'_{ek}\, W_k\, \delta T_1(1) \left(\frac{r\, g}{q}\, \frac{m_k - n\, q}{T'_{e0}\, \psi_{mk}} \right)_{r_k + W_k}, \tag{72}$$

$$\Psi_{k-} = -T'_{ek}\, W_k\, \delta T_1(-1) \left(\frac{r\, g}{q}\, \frac{m_k - n\, q}{T'_{e0}\, \psi_{mk}} \right)_{r_k - W_k}. \tag{73}$$

Finally, for the special case $m = 1$, we write

$$\tilde{T}_e(r, \theta, \phi) = -\xi^r(r, \theta, \phi)\, \frac{dT_{e0}}{dr}. \tag{74}$$

A.   Normalized Quantities

Let $\hat{r} = r/\epsilon_a$, $\hat{r}_k = r_k/\epsilon_a$, $\hat{x} = x/\epsilon_a$, $\hat{T}'_{e0} = \epsilon_a T'_{e0}$, $\hat{T}'_{ek} = \epsilon_a \hat{T}'_{ek}$, $\hat{W}_k = W_k/\epsilon_a$, and $\hat{\Psi}_k = \Psi_k/\epsilon_a^2$, etc., then

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e+} - \hat{\Psi}_{k+}\, \frac{q(\hat{r})}{\hat{r}\, g(\hat{r})}\, \frac{\hat{T}'_e(r)\, \psi_{m_k}(r)}{m_k - n\, q(\hat{r})}\, e^{i\zeta_k} \tag{75}$$

in the region $\hat{r} > \hat{r}_k + \hat{W}_k$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e-} - \hat{\Psi}_{k-}\, \frac{q(\hat{r})}{\hat{r}\, g(\hat{r})}\, \frac{\hat{T}'_e(r)\, \psi_{m_k}(r)}{m_k - n\, q(\hat{r})}\, e^{i\zeta_k} \tag{76}$$

in the region $\hat{r} < \hat{r}_k - \hat{W}_k$, and

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{ek} + \hat{T}'_{ek}\, \hat{W}_k \sum_{\nu = 0, \infty} \delta T_\nu(\hat{x}/\hat{W}_k)\, e^{i\nu\zeta_k} + \hat{T}'_{ek}\, \hat{W}_k\, \delta T_{0\infty} \tag{77}$$

in the region $\hat{r}_k - \hat{W}_k \leq \hat{r} \leq \hat{r}_k + \hat{W}_k$. Here,

$$\delta T_{e+} = \hat{T}'_{ek}\, \hat{W}_k\, \delta T_0(1) + \hat{T}'_{ek}\, \hat{W}_k\, \delta T_{0\infty} - \hat{T}'_{ek}\, \hat{W}_k, \tag{78}$$

$$\delta T_{e-} = \hat{T}'_{ek}\, \hat{W}_k\, \delta T_0(-1) + \hat{T}'_{ek}\, \hat{W}_k\, \delta T_{0\infty} + \hat{T}'_{ek}\, \hat{W}_k, \tag{79}$$

$$\hat{\Psi}_k = \left(\frac{\hat{W}_k}{4} \right)^2 \left(\frac{g\, s}{q} \right)_{\hat{r}_k}, \tag{80}$$

$$\hat{\Psi}_{k+} = -\hat{T}'_{ek}\, \hat{W}_k\, \delta T_1(1) \left(\frac{\hat{r}\, g}{q}\, \frac{m_k - n\, q}{\hat{T}'_{e0}\, \psi_{mk}} \right)_{\hat{r}_k + \hat{W}_k}, \tag{81}$$

$$\hat{\Psi}_{k-} = -\hat{T}'_{ek}\, \hat{W}_k\, \delta T_1(-1) \left(\frac{\hat{r}\, g}{q}\, \frac{m_k - n\, q}{\hat{T}'_{e0}\, \psi_{mk}} \right)_{\hat{r}_k - \hat{W}_k}. \tag{82}$$

For the special case $m = 1$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = -\frac{\xi^r(\hat{r}, \theta, \phi)}{\epsilon_a}\, \frac{dT_{e0}}{d\hat{r}}. \tag{83}$$

VII.   RELATIVISTIC DOWNSHIFTING AND BROADENING

Neglecting doppler broadening, the angular frequency of an nth harmonic electron cyclotron emission (ece) signal is

$$\omega = \frac{n\,\Omega_0\,R_0}{R} \left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2},$$ (84)

where v is the electron speed, and $\Omega_0 = e\,B_0/R_0$. Here, we are neglecting the poloidal magnetic field-strength, and assuming that the toroidal field-strength falls off like $1/R$. Let

$$R_\omega(\omega) = \frac{n\,\Omega_0\,R_0}{\omega}$$ (85)

be the major radius from which the ece of frequency ω is emitted in the absence of relativistic downshifting and broadening. We can write

$$\frac{v}{c} = \begin{cases} \left[1 - \left(\frac{R}{R_\omega}\right)^2\right]^{1/2} & R \leq R_\omega \\ 0 & R > R_\omega \end{cases}.$$ (86)

Now, the distribution of electron speeds is

$$f(v) = A\,v^2 \left(-\frac{1}{\theta_\omega}\left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}\right),$$ (87)

where

$$\theta_\omega(\omega) = \frac{T_e(R_\omega)}{m_e\,c^2}.$$ (88)

Thus, we can define

$$F(R, R_\omega) = \left[1 - \left(\frac{R}{R_\omega}\right)^2\right] \exp\left(-\frac{1}{\theta_\omega}\frac{R_\omega}{R}\right).$$ (89)

The electron temperature measured by the ece diagnostic is

$$T_e(R_\omega) = \frac{\int_{R_{\min}}^{R_\omega} T_e(R)\,F(R, R_\omega)\,dR}{\int_{R_{\min}}^{R_\omega} F(R, R_\omega)\,dR}.$$ (90)