Four-Field Resonant Layer Model

R. Fitzpatrick [a]

Institute for Fusion Studies, Department of Physics,

University of Texas at Austin, Austin TX 78712, USA

I. FOURIER-TRANSFORMED FOUR-FIELD EQUATIONS

In real space, the four-field resonant layer equations can be reduced to a set of ten coupled first-order differential equations. As we shall demonstrate, the equations can be reduced to a set of four coupled first-order differential equations in Fourier space. Clearly, it is advantageous to solve the equations in Fourier space.

The Fourier-transformed four-field layer equations take the form:

$$(g+\mathrm{i}\,Q_e)\,\bar{\psi} = \frac{d(\bar{\phi}-\bar{N})}{dp} - p^2\,\bar{\psi}, \tag{1}$$

$$g\,\bar{N} = -\mathrm{i}\,Q_e\,\bar{\phi} - D^2\,\frac{d(p^2\,\bar{\psi})}{dp} + c_\beta^2\,\frac{d\bar{V}}{dp} - P_\perp\,p^2\,\bar{N}, \tag{2}$$

$$(g+\mathrm{i}\,Q_i)\,p^2\,\bar{\phi} = \frac{d(p^2\,\bar{\psi})}{dp} - P_\varphi\,p^4\left(\bar{\phi}+\frac{\bar{N}}{\iota}\right), \tag{3}$$

$$g\,\bar{V} = \mathrm{i}\,Q_e\,\bar{\psi} + \frac{d\bar{N}}{dp} - P_\varphi\,p^2\,\bar{V}. \tag{4}$$

It follows that

$$\bar{\psi} = \frac{1}{g+\mathrm{i}\,Q_e+p^2}\,\frac{d(\bar{\phi}-\bar{N})}{dp}, \tag{5}$$

and

$$\frac{d(p^2\,\bar{\psi})}{dp} = [(g+\mathrm{i}\,Q_i)\,p^2 + P_\varphi\,p^4]\,\bar{\phi} + \frac{P_\varphi}{\iota}\,p^4\,\bar{N}, \tag{6}$$

and

$$c_\beta^2\,\frac{d\bar{V}}{dp} = (g+P_\perp\,p^2+\iota^{-1}\,D^2\,P_\varphi\,p^4)\,\bar{N} + [\mathrm{i}\,Q_e + D^2\,(g+\mathrm{i}\,Q_i)\,p^2 + D^2\,P_\varphi\,p^4]\,\bar{\phi}. \tag{7}$$

If we define

$$\bar{J} = p^2\,\bar{\psi}, \tag{8}$$

[a] rfitzp@utexas.edu

$$\bar{Y} = \bar{\phi} - \bar{N}, \tag{9}$$

then we can transform Eqs. (4)–(7) into the the following set of four coupled first-order differential equations:

$$\frac{d\bar{Y}}{dp} = \left(\frac{g + \mathrm{i}\,Q_e + p^2}{p^2}\right)\bar{J}, \tag{10}$$

$$\frac{d\bar{N}}{dp} = \left(\frac{-\mathrm{i}\,Q_e}{p^2}\right)\bar{J} + (g + P_\varphi\,p^2)\,\bar{V}, \tag{11}$$

$$\frac{d\bar{J}}{dp} = [(g + \mathrm{i}\,Q_i)\,p^2 + P_\varphi\,p^4]\,\bar{Y} + [(g + \mathrm{i}\,Q_i)\,p^2 + \iota_e^{-1}\,P_\varphi\,p^4]\,\bar{N}, \tag{12}$$

$$c_\beta^2\,\frac{d\bar{V}}{dp} = [\mathrm{i}\,Q_e + D^{\,2}\,(g + \mathrm{i}\,Q_i)\,p^2 + D^{\,2}\,P_\varphi\,p^4]\,\bar{Y}$$

$$+ (g + \mathrm{i}\,Q_e + [P_\perp + D^{\,2}\,(g + \mathrm{i}\,Q_i)]\,p^2 + \iota_e^{-1}\,D^{\,2}\,P_\varphi\,p^4)\,\bar{N}, \tag{13}$$

where $\iota_e = \iota/(1+\iota)$.

II. SMALL-p BEHAVIOR OF FOURIER-TRANSFORMED FOUR-FIELD EQUATIONS

A. Introduction

Let us search for power-law solutions of Eqs. (10)–(13) at small values of p. Given that we have four coupled first-order differential equations, we expect to find four independent power-law solutions.

B. First Solution

Suppose that

$$\bar{Y}(p) = y_{-1}\,p^{-1} + y_1\,p + \mathcal{O}(p^3), \tag{14}$$

$$\bar{N}(p) = n_{-1}\,p^{-1} + n_1\,p + \mathcal{O}(p^3), \tag{15}$$

$$\bar{J}(p) = j_0 + j_2\,p^2 + \mathcal{O}(p^4), \tag{16}$$

$$\bar{V}(p) = v_2\,p^2 + \mathcal{O}(p^4). \tag{17}$$

Equations (10)–(13) yield

$$-y_{-1}\,p^{-2}+y_1=(g+\mathrm{i}\,Q_e)\,(j_0\,p^{-2}+j_2)+j_0+\mathcal{O}(p^2), \tag{18}$$

$$-n_{-1}\,p^{-2}+n_1=-\mathrm{i}\,Q_e\,(j_0\,p^{-2}+j_2)+\mathcal{O}(p^2), \tag{19}$$

$$2\,j_2\,p=(g+\mathrm{i}\,Q_i)\,(y_{-1}+n_{-1})\,p+\mathcal{O}(p^3), \tag{20}$$

$$\begin{aligned}2\,c_\beta^2\,v_2\,p&=\mathrm{i}\,Q_e\,(y_{-1}\,p^{-1}+y_1\,p)+(g+\mathrm{i}\,Q_e)\,(n_{-1}\,p^{-1}+n_1\,p)\\&\quad+D^{\,2}\,(g+\mathrm{i}\,Q_i)\,y_{-1}\,p+[P_\perp+D^{\,2}\,(g+\mathrm{i}\,Q_i)]\,n_{-1}\,p+\mathcal{O}(p^3).\end{aligned} \tag{21}$$

It follows that

$$-y_{-1}=(g+\mathrm{i}\,Q_e)\,j_0, \tag{22}$$

$$y_1=(g+\mathrm{i}\,Q_e)\,j_2+j_0, \tag{23}$$

$$-n_{-1}=-\mathrm{i}\,Q_e\,j_0, \tag{24}$$

$$n_1=-\mathrm{i}\,Q_e\,j_2, \tag{25}$$

$$2\,j_2=(g+\mathrm{i}\,Q_i)\,(y_{-1}+n_{-1}), \tag{26}$$

$$0=\mathrm{i}\,Q_e\,y_{-1}+(g+\mathrm{i}\,Q_e)\,n_{-1}, \tag{27}$$

$$\begin{aligned}2\,c_\beta^2\,v_2&=\mathrm{i}\,Q_e\,y_1+(g+\mathrm{i}\,Q_e)\,n_1\\&\quad+D^{\,2}\,(g+\mathrm{i}\,Q_i)\,y_{-1}+[P_\perp+D^{\,2}\,(g+\mathrm{i}\,Q_i)]\,n_{-1},\end{aligned} \tag{28}$$

which gives

$$y_{-1}=(g+\mathrm{i}\,Q_e)\,a_{-1}, \tag{29}$$

$$y_1=\left[\frac{1}{2}\,g\,(g+\mathrm{i}\,Q_e)\,(g+\mathrm{i}\,Q_i)-1\right]a_{-1}, \tag{30}$$

$$n_{-1}=-\mathrm{i}\,Q_e\,a_{-1}, \tag{31}$$

$$n_1=-\frac{1}{2}\,g\,(\mathrm{i}\,Q_e)\,(g+\mathrm{i}\,Q_i)\,a_{-1}, \tag{32}$$

$$j_0=-a_{-1}, \tag{33}$$

$$j_2=\frac{1}{2}\,g\,(g+\mathrm{i}\,Q_i)\,a_{-1}, \tag{34}$$

$$v_2=\frac{[-\mathrm{i}\,Q_e\,(1+P_\perp)+g\,(g+\mathrm{i}\,Q_i)\,D^{\,2}]}{2\,c_\beta^2}\,a_{-1}, \tag{35}$$

where a_{-1} is an arbitrary constant.

C. Second Solution

Suppose that

$$\bar{Y}(p) = y_0 + y_2\,p^2 + \mathcal{O}(p^4), \tag{36}$$

$$\bar{N}(p) = n_0 + n_2\,p^2 + \mathcal{O}(p^4), \tag{37}$$

$$\bar{J}(p) = j_3\,p^3 + \mathcal{O}(p^5), \tag{38}$$

$$\bar{V}(p) = v_3\,p^3 + \mathcal{O}(p^5). \tag{39}$$

Equations (10)–(13) give

$$2\,y_2\,p = (g + \mathrm{i}\,Q_e)\,j_3\,p + \mathcal{O}(p^3), \tag{40}$$

$$2\,n_2\,p = -\mathrm{i}\,Q_e\,j_3\,p + \mathcal{O}(p^3), \tag{41}$$

$$3\,j_3\,p^2 = (g + \mathrm{i}\,Q_i)\,(y_0 + n_0)\,p^2 + \mathcal{O}(p^4), \tag{42}$$

$$\begin{aligned} 3\,c_\beta^2\,v_3\,p^2 &= \mathrm{i}\,Q_e\,(y_0 + y_2\,p^2) + (g + \mathrm{i}\,Q_e)\,(n_0 + n_2\,p^2) \\ &\quad + D^{\,2}\,(g + \mathrm{i}\,Q_i)\,y_0\,p^2 + [P_\perp + D^{\,2}\,(g + \mathrm{i}\,Q_i)]\,n_0\,p^2 + \mathcal{O}(p^4). \end{aligned} \tag{43}$$

It follows that

$$2\,y_2 = (g + \mathrm{i}\,Q_e)\,j_3, \tag{44}$$

$$2\,n_2 = -\mathrm{i}\,Q_e\,j_3, \tag{45}$$

$$3\,j_3 = (g + \mathrm{i}\,Q_i)\,y_0 + (g + \mathrm{i}\,Q_i)\,n_0, \tag{46}$$

$$0 = \mathrm{i}\,Q_e\,y_0 + (g + \mathrm{i}\,Q_e)\,n_0, \tag{47}$$

$$\begin{aligned} 3\,c_\beta^2\,v_3 &= \mathrm{i}\,Q_e\,y_2 + (g + \mathrm{i}\,Q_e)\,n_2 \\ &\quad + D^{\,2}\,(g + \mathrm{i}\,Q_e)\,n_0 + [P_\perp + D^{\,2}\,(g + \mathrm{i}\,Q_i)]\,n_0. \end{aligned} \tag{48}$$

which gives

$$y_0 = (g + \mathrm{i}\,Q_e)\,a_0, \tag{49}$$

$$y_2 = \frac{1}{6}\,g\,(g + \mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i)\,a_0, \tag{50}$$

$$n_0 = -\mathrm{i}\,Q_e\,a_0, \tag{51}$$

$$n_2 = -\frac{1}{6}\,g\,(\mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i)\,a_0, \tag{52}$$

$$j_3 = \frac{1}{3}\, g\,(g + \mathrm{i}\, Q_i)\, a_0, \tag{53}$$

$$v_3 = \frac{1}{3}\, \frac{[-\mathrm{i}\, Q_e\, P_\perp + g\,(g + \mathrm{i}\, Q_i)\, D^{\,2}]}{c_\beta^2}\, a_0, \tag{54}$$

where a_0 is an arbitrary constant.

D. Third Solution

Suppose that

$$\bar{Y}(p) = y_2\, p^2 + \mathcal{O}(p^4), \tag{55}$$

$$\bar{N}(p) = n_0 + n_2\, p^2 + \mathcal{O}(p^4), \tag{56}$$

$$\bar{J}(p) = j_3\, p^3 + \mathcal{O}(p^5), \tag{57}$$

$$\bar{V}(p) = v_1\, p + \mathcal{O}(p^3). \tag{58}$$

Equations (10)–(13) give

$$2\, y_2\, p = (g + \mathrm{i}\, Q_e)\, j_3\, p + \mathcal{O}(p^3), \tag{59}$$

$$2\, n_2\, p = -\mathrm{i}\, Q_e\, j_3\, p + g\, v_1\, p + \mathcal{O}(p^3), \tag{60}$$

$$3\, j_3\, p^2 = (g + \mathrm{i}\, Q_i)\, n_0\, p^2 + \mathcal{O}(p^4), \tag{61}$$

$$c_\beta^2\, v_1 = (g + \mathrm{i}\, Q_e)\, n_0 + \mathcal{O}(p^2). \tag{62}$$

It follows that

$$y_2 = \frac{1}{6}\, (g + \mathrm{i}\, Q_e)\, (g + \mathrm{i}\, Q_i)\, a_2, \tag{63}$$

$$n_0 = a_2, \tag{64}$$

$$n_2 = \frac{1}{2}\, (g + \mathrm{i}\, Q_i) \left(-\frac{1}{3}\, \mathrm{i}\, Q_e\, + \frac{g}{c_\beta^2}\right) a_2, \tag{65}$$

$$j_3 = \frac{1}{3}\, (g + \mathrm{i}\, Q_i)\, a_2, \tag{66}$$

$$v_1 = \frac{(g + \mathrm{i}\, Q_e)}{c_\beta^2}\, a_2, \tag{67}$$

where a_2 is an arbitrary constant.

E. Fourth Solution

Suppose that

$$\bar{Y}(p) = y_3\,p^3 + \mathcal{O}(p^5), \tag{68}$$

$$\bar{N}(p) = n_1\,p + \mathcal{O}(p^3), \tag{69}$$

$$\bar{J}(p) = j_4\,p^4 + \mathcal{O}(p^6), \tag{70}$$

$$\bar{V}(p) = v_0 + v_2\,p^2 + \mathcal{O}(p^4). \tag{71}$$

Equations (10)–(13) give

$$3\,y_3\,p^2 = (g + \mathrm{i}\,Q_e)\,j_4\,p^2 + \mathcal{O}(p^4), \tag{72}$$

$$n_1 = g\,v_0 + \mathcal{O}(p^2), \tag{73}$$

$$4\,j_4\,p^3 = (g + \mathrm{i}\,Q_i)\,n_1\,p^3 + \mathcal{O}(p^5), \tag{74}$$

$$2\,c_\beta^2\,v_2\,p = (g + \mathrm{i}\,Q_e)\,n_1\,p + \mathcal{O}(p^3). \tag{75}$$

It follows that

$$y_3 = \frac{1}{12}\,g\,(g + \mathrm{i}\,Q_e)\,(g + \mathrm{i}\,Q_i)\,a_3, \tag{76}$$

$$n_1 = g\,a_3, \tag{77}$$

$$j_4 = \frac{1}{4}\,g\,(g + \mathrm{i}\,Q_i)\,a_3, \tag{78}$$

$$v_2 = \frac{g\,(g + \mathrm{i}\,Q_e)}{2\,c_\beta^2}\,a_3, \tag{79}$$

where a_3 is an arbitrary constant.

F. General Solution

We conclude that, at small values of p, the most general solution for $\bar{Y}(p)$ and $\bar{N}(p)$ takes the form

$$\bar{Y}(p) = (g + \mathrm{i}\,Q_e)\,p^{-1}\,a_{-1} + (g + \mathrm{i}\,Q_e)\,a_0 + \mathcal{O}(p), \tag{80}$$

$$\bar{N}(p) = (-\mathrm{i}\,Q_e)\,p^{-1}\,a_{-1} + (-\mathrm{i}\,Q_e)\,a_0 + a_2 + \mathcal{O}(p). \tag{81}$$

III. MATRIX DIFFERENTIAL EQUATION

Let

$$\underline{u} = \begin{pmatrix} \bar{Y} \\ \bar{N} \end{pmatrix}, \tag{82}$$

$$\underline{v} = \begin{pmatrix} \bar{J} \\ c_\beta^2\,\bar{V} \end{pmatrix}. \tag{83}$$

Equations (10)–(13) can be written in the form

$$\frac{d\underline{u}}{dp} = \underline{\underline{A}}\,\underline{v}, \tag{84}$$

$$\frac{d\underline{v}}{dp} = \underline{\underline{B}}\,\underline{u}, \tag{85}$$

where

$$A_{11} = \frac{g + \mathrm{i}\,Q_e + p^2}{p^2}, \tag{86}$$

$$A_{21} = \frac{-\mathrm{i}\,Q_e}{p^2}, \tag{87}$$

$$A_{22} = \frac{g + P_\varphi\,p^2}{c_\beta^2}, \tag{88}$$

$$B_{11} = (g + \mathrm{i}\,Q_i)\,p^2 + P_\varphi\,p^4, \tag{89}$$

$$B_{12} = (g + \mathrm{i}\,Q_i)\,p^2 + \iota_e^{-1}\,P_\varphi\,p^4, \tag{90}$$

$$B_{21} = \mathrm{i}\,Q_e + D^2\,(g + \mathrm{i}\,Q_i)\,p^2 + D^2\,P_\varphi\,p^4, \tag{91}$$

$$B_{22} = g + \mathrm{i}\,Q_e + [P_\perp + D^2\,(g + \mathrm{i}\,Q_i)]\,p^2 + \iota_e^{-1}\,D^2\,P_\varphi\,p^4. \tag{92}$$

Thus, we obtain the following matrix differential equation:

$$\frac{d}{dp}\left(\underline{\underline{A}}^{-1}\,\frac{d\underline{u}}{dp}\right) = \underline{\underline{B}}\,\underline{u}. \tag{93}$$

IV. RICCATI MATRIX DIFFERENTIAL EQUATION

Let

$$p\,\frac{d\underline{u}}{dp} = \underline{\underline{W}}\,\underline{u}. \tag{94}$$

The previous equation can be combined with Eq. (93) to give

$$\left(p\,\frac{d\underline{\underline{W}}}{dp} - \underline{\underline{W}} + \underline{\underline{W}}\,\underline{\underline{W}} + \underline{\underline{A}}\,p\,\frac{d\underline{\underline{A}}^{-1}}{dp}\,\underline{\underline{W}} - p^2\,\underline{\underline{A}}\,\underline{\underline{B}}\right)\underline{u} = \underline{0}, \tag{95}$$

which implies that

$$p\,\frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}}\,\underline{\underline{W}} - \underline{\underline{A}}\,p\,\frac{d\underline{\underline{A}}^{-1}}{dp}\,\underline{\underline{W}} + p^2\,\underline{\underline{A}}\,\underline{\underline{B}}. \tag{96}$$

Now,

$$\underline{\underline{A}}^{-1} = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}, \tag{97}$$

where

$$C_{11} = \frac{p^2}{g + \mathrm{i}\,Q_e + p^2}, \tag{98}$$

$$C_{21} = \frac{\mathrm{i}\,c_\beta^2\,Q_e}{(g + \mathrm{i}\,Q_e + p^2)\,(g + P_\varphi\,p^2)}, \tag{99}$$

$$C_{22} = \frac{c_\beta^2}{g + P_\varphi\,p^2}. \tag{100}$$

So, if

$$p\,\frac{d\underline{\underline{A}}^{-1}}{dp} = \begin{pmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{pmatrix} \tag{101}$$

then

$$D_{11} = \frac{2\,p^2\,(g + \mathrm{i}\,Q_e)}{(g + \mathrm{i}\,Q_e + p^2)^2}, \tag{102}$$

$$D_{21} = -\frac{2\,\mathrm{i}\,c_\beta^2\,Q_e\,p^2\,[g + P_\varphi\,(g + \mathrm{i}\,Q_e) + 2\,P_\varphi\,p^2]}{(g + \mathrm{i}\,Q_e + p^2)^2\,(g + P_\varphi\,p^2)^2}, \tag{103}$$

$$D_{22} = -\frac{2\,c_\beta^2\,P_\varphi\,p^2}{(g + P_\varphi\,p^2)^2}. \tag{104}$$

Furthermore, if

$$\underline{\underline{A}}\,p\,\frac{d\underline{\underline{A}}^{-1}}{dp} = \begin{pmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{pmatrix} \tag{105}$$

then

$$E_{11} = \frac{2\,(g + \mathrm{i}\,Q_e)}{g + \mathrm{i}\,Q_e + p^2}, \tag{106}$$

$$E_{21} = -\frac{2\,\mathrm{i}\,Q_e\,(g + 2\,P_\varphi\,p^2)}{(g + \mathrm{i}\,Q_e + p^2)\,(g + P_\varphi\,p^2)}, \tag{107}$$

$$E_{22} = -\frac{2\,P_\varphi\,p^2}{g+P_\varphi\,p^2}. \tag{108}$$

Finally, if

$$p^2\,\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix} \tag{109}$$

then

$$F_{11} = p^2\,(g+\mathrm{i}\,Q_e+p^2)\,(g+\mathrm{i}\,Q_i+P_\varphi\,p^2), \tag{110}$$

$$F_{12} = p^2\,(g+\mathrm{i}\,Q_e+p^2)\,(g+\mathrm{i}\,Q_i+\iota_e^{-1}\,P_\varphi\,p^2), \tag{111}$$

$$\begin{aligned}F_{21} &= -\mathrm{i}\,Q_e\,p^2\,(g+\mathrm{i}\,Q_i+P_\varphi\,p^2)\\ &\quad + c_\beta^{-2}\,p^2\,(g+P_\varphi\,p^2)\,[\mathrm{i}\,Q_e+D^{\,2}\,(g+\mathrm{i}\,Q_i)\,p^2+D^{\,2}\,P_\varphi\,p^4],\end{aligned} \tag{112}$$

$$\begin{aligned}F_{22} &= -\mathrm{i}\,Q_e\,p^2\,(g+\mathrm{i}\,Q_i+\iota_e^{-1}\,P_\varphi\,p^2)\\ &\quad + c_\beta^{-2}\,p^2\,(g+P_\varphi\,p^2)\,[g+\mathrm{i}\,Q_e+[P_\perp+D^{\,2}\,(g+\mathrm{i}\,Q_i)]\,p^2+\iota_e^{-1}\,D^{\,2}\,P_\varphi\,p^4].\end{aligned} \tag{113}$$

Hence, Eq. (96) can be written as the following Riccati matrix differential equation:

$$p\,\frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}}\,\underline{\underline{W}} - \underline{\underline{E}}\,\underline{\underline{W}} + \underline{\underline{F}}. \tag{114}$$

Furthermore, if

$$\underline{\underline{W}} = \begin{pmatrix} W_{11} & W_{12}\\ W_{21} & W_{22}\end{pmatrix} \tag{115}$$

then

$$p\,\frac{dW_{11}}{dp} = W_{11} - W_{11}\,W_{11} - W_{12}\,W_{21} - E_{11}\,W_{11} + F_{11}, \tag{116}$$

$$p\,\frac{dW_{12}}{dp} = W_{12} - W_{11}\,W_{12} - W_{12}\,W_{22} - E_{11}\,W_{12} + F_{12}, \tag{117}$$

$$p\,\frac{dW_{21}}{dp} = W_{21} - W_{21}\,W_{11} - W_{22}\,W_{21} - E_{21}\,W_{11} - E_{22}\,W_{21} + F_{21}, \tag{118}$$

$$p\,\frac{dW_{22}}{dp} = W_{22} - W_{21}\,W_{12} - W_{22}\,W_{22} - E_{21}\,W_{12} - E_{22}\,W_{22} + F_{22}. \tag{119}$$

V. SMALL-p BEHAVIOR OF RICCATI MATRIX DIFFERENTIAL EQUATION

Let $\underline{\underline{E}} = \underline{\underline{E}}^{(0)}$ at $p=0$. It follows from Eqs. (105)–(108) that

$$E_{11}^{(0)} = 2, \tag{120}$$

$$E^{(0)}_{12} = 0, \tag{121}$$

$$E^{(0)}_{21} = -\frac{2\,\mathrm{i}\,Q_e}{g+\mathrm{i}\,Q_e}, \tag{122}$$

$$E^{(0)}_{22} = 0. \tag{123}$$

Likewise, at small values of p, we can write $\underline{\underline{F}} = p^2\,\underline{\underline{F}}^{(2)}$, where the elements of $\underline{\underline{F}}^{(2)}$ are constants, and where use has been made of Eqs. (110)–(113).

Suppose that $\underline{\underline{W}} = \underline{\underline{W}}^{(0)}$ at $p=0$. Equation (114) gives

$$\underline{\underline{0}} = \underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)}\,\underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)}\,\underline{\underline{W}}^{(0)}, \tag{124}$$

at $p=0$, which yields

$$(\underline{\underline{1}} - \underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)})\,\underline{\underline{W}}^{(0)} = \underline{\underline{0}}. \tag{125}$$

Hence, we deduce that

$$\underline{\underline{W}}^{(0)} = \underline{\underline{1}} - \underline{\underline{E}}^{(0)} = \left(\begin{array}{cc} -1 & 0\\ -E^{(0)}_{21} & 1\end{array}\right). \tag{126}$$

At small values of p, let

$$\underline{u}(p) = \underline{u}_{-1}\,p^{-1} + \underline{u}_0, \tag{127}$$

$$\underline{\underline{W}}(p) = \underline{\underline{W}}^{(0)} + p\,\underline{\underline{W}}^{(1)}, \tag{128}$$

where the elements of $\underline{u}_{-1}$ (which are y_{-1} and n_{-1}, respectively), the elements of $\underline{u}_0$ (which are y_0 and n_0, respectively), and the elements of $\underline{\underline{W}}^{(1)}$, are all constants. Equation (94) gives

$$\underline{\underline{W}}^{(0)}\,\underline{u}_{-1} = -\underline{u}_{-1}, \tag{129}$$

$$\underline{\underline{W}}^{(0)}\,\underline{u}_0 + \underline{\underline{W}}^{(1)}\,\underline{u}_{-1} = \underline{0}. \tag{130}$$

Thus, making use of Eq. (126), we get

$$\left(\begin{array}{cc} -1 & 0\\ -E^{(0)}_{21} & 1\end{array}\right)\left(\begin{array}{c} y_{-1}\\ n_{-1}\end{array}\right) = -\left(\begin{array}{c} y_{-1}\\ n_{-1}\end{array}\right), \tag{131}$$

which implies that

$$E^{(0)}_{21}\,y_{-1} = -\frac{2\,\mathrm{i}\,Q_e}{g+\mathrm{i}\,Q_e}\,y_{-1} = 2\,n_{-1}, \tag{132}$$

in accordance with Eqs. (29) and (31), where use has been made of Eq. (122). Thus, if we write

$$y_{-1} = (g + \mathrm{i}\, Q_e)\, a_{-1}, \tag{133}$$

$$n_{-1} = -\mathrm{i}\, Q_e\, a_{-1}, \tag{134}$$

$$y_0 = (g + \mathrm{i}\, Q_e)\, a_0, \tag{135}$$

$$n_0 = -\mathrm{i}\, Q_e\, a_0 + a_2, \tag{136}$$

in accordance with Eqs. (80) and (81), then we deduce from Eqs. (126) and (130) that

$$\frac{\pi}{\hat{\Delta}_s} \equiv \frac{a_0}{a_{-1}} = W_{11}^{(1)} - W_{12}^{(1)}\,\frac{(\mathrm{i}\, Q_e)}{g + \mathrm{i}\, Q_e}, \tag{137}$$

and

$$\frac{a_2}{a_{-1}} = \mathrm{i}\, Q_e\left[W_{22}^{(1)} - W_{11}^{(1)}\right] + \frac{(\mathrm{i}\, Q_e)^2}{g + \mathrm{i}\, Q_e}\, W_{12}^{(1)} - (g + \mathrm{i}\, Q_e)\, W_{21}^{(1)}. \tag{138}$$

VI. LARGE-p BEHAVIOR OF RICCATI MATRIX DIFFERENTIAL EQUATION

At large values of p, it is clear from Eqs. (110)–(113) that $\underline{\underline{F}}(p) = p^6\, \underline{\underline{F}}^{(6)} + p^8\, \underline{\underline{F}}^{(8)}$, where the elements of $\underline{\underline{F}}^{(6)}$ and $\underline{\underline{F}}^{(8)}$ are constants. On the other hand, Eqs. (105)–(108) imply that $\underline{\underline{E}}(p) = \underline{\underline{E}}^{(0)}$, where the elements of $\underline{\underline{E}}^{(0)}$ are constants. Thus, if we write $\underline{\underline{W}}(p) = p^2\, \underline{\underline{W}}^{(2)} + p^4\, \underline{\underline{W}}^{(4)}$, where the elements of $\underline{\underline{W}}^{(2)}$ and $\underline{\underline{W}}^{(4)}$ are constants, then Eq. (114) gives

$$\underline{\underline{W}}^{(4)}\, \underline{\underline{W}}^{(4)} = \underline{\underline{F}}^{(8)}, \tag{139}$$

$$\underline{\underline{W}}^{(2)}\, \underline{\underline{W}}^{(4)} + \underline{\underline{W}}^{(4)}\, \underline{\underline{W}}^{(2)} = \underline{\underline{F}}^{(6)}. \tag{140}$$

Now, according to Eqs. (110)–(113),

$$F_{11}^{(8)} = 0, \tag{141}$$

$$F_{12}^{(8)} = 0, \tag{142}$$

$$F_{21}^{(8)} = c_\beta^{-2}\, D^2\, P_\varphi^2, \tag{143}$$

$$F_{22}^{(8)} = c_\beta^{-2}\, \iota_e^{-1}\, D^2\, P_\varphi^2, \tag{144}$$

so Eq. (139) yields

$$W_{11}^{(4)} = 0, \tag{145}$$

$$W_{12}^{(4)} = 0, \tag{146}$$

$$W_{21}^{(4)} = -c_\beta^{-1}\,\iota_e^{1/2}\,D\,P_\varphi, \tag{147}$$

$$W_{22}^{(4)} = -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi, \tag{148}$$

where we have chosen the sign of the square root that is associated with well-behaved solutions at large values of p. Here, we are assuming that $\iota_e > 0$. Equations (110)–(113) also give

$$F_{11}^{(6)} = P_\varphi, \tag{149}$$

$$F_{12}^{(6)} = \iota_e^{-1}\,P_\varphi, \tag{150}$$

$$F_{21}^{(6)} = c_\beta^{-2}\,D^{\,2}\,g\,P_\varphi + c_\beta^{-2}\,D^{\,2}\,(g+{\rm i}\,Q_i)\,P_\varphi, \tag{151}$$

$$F_{22}^{(6)} = c_\beta^{-2}\,\iota_e^{-1}\,D^{\,2}\,g\,P_\varphi + c_\beta^{-2}\,[P_\perp + D^{\,2}\,(g+{\rm i}\,Q_i)]\,P_\varphi. \tag{152}$$

Thus, Eq. (140) yields

$$W_{12}^{(2)}\,W_{21}^{(4)} = F_{11}^{(6)}, \tag{153}$$

$$W_{12}^{(2)}\,W_{22}^{(4)} = F_{12}^{(6)}, \tag{154}$$

which gives

$$W_{12}^{(2)} = -c_\beta\,\iota_e^{-1/2}\,D^{-1}. \tag{155}$$

Now, if

$$\underline{\underline{W}}\,\underline{u} = \lambda(p)\,\underline{u} \tag{156}$$

then Eq. (94) yields

$$p\,\frac{d\underline{u}}{dp} = \lambda\,\underline{u}, \tag{157}$$

which implies that

$$\underline{u}(p) = \underline{u}(p_0)\,\exp\left[\int_{p_0}^{p}\frac{\lambda_r(p')}{p'}\,dp'\right]\exp\left[{\rm i}\int_{p_0}^{p}\frac{\lambda_i(p')}{p'}\,dp'\right], \tag{158}$$

where λ_r and λ_i are the real and imaginary parts of λ, respectively. Of course, a solution that is well behaved at large values of p is such that λ_r is negative. As we have seen, the large-p limit of Eq. (114) is

$$\underline{\underline{W}}\,\underline{\underline{W}} = \underline{\underline{F}}. \tag{159}$$

Hence, if

$$\underline{\underline{F}}\,\underline{u} = \Lambda\,\underline{u} \tag{160}$$

then Eqs. (156) and (160) imply that

$$\lambda^2 = \Lambda. \tag{161}$$

The eigenvalue problem for the F-matrix reduces to

$$\Lambda^2 - (F_{11} + F_{22})\,\Lambda + F_{11}\,F_{22} - F_{12}\,F_{21} = 0. \tag{162}$$

Now,

$$F_{11} + F_{22} \simeq F_{22}^{(8)}\,p^8 = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^2\,p^8, \tag{163}$$

$$\begin{aligned} F_{11}\,F_{22} - F_{12}\,F_{21} \simeq{}& \left[F_{11}^{(6)}\,F_{22}^{(8)} - F_{12}^{(6)}\,F_{21}^{(8)}\right]p^{14}\\ &+ \left[F_{11}^{(6)}\,F_{22}^{(6)} - F_{12}^{(6)}\,F_{21}^{(6)}\right]p^{12} = c_\beta^{-2}\,R\,P_\varphi^2\,p^{12}, \end{aligned} \tag{164}$$

where

$$R = P_\perp + (1 - \iota_e^{-1})\,D^2\,(g + \mathrm{i}\,Q_i), \tag{165}$$

Hence, the two eigenvalues of the F-matrix are

$$\Lambda_1 \simeq F_{22}^{(8)}\,p^8 = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^2\,p^8, \tag{166}$$

$$\Lambda_2 \simeq \frac{[F_{11}^{(6)}\,F_{22}^{(6)} - F_{12}^{(6)}\,F_{21}^{(6)}]}{F_{22}^{(8)}}\,p^4 = \iota_e\,D^{-2}\,R\,p^4. \tag{167}$$

Thus, we deduce that the two eigenvalues of the W-matrix are

$$\lambda_1 = -\Lambda_1^{1/2} = -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi\,p^4, \tag{168}$$

$$\lambda_2 = -\Lambda_2^{1/2} = -\iota_e^{1/2}\,D^{-1}\,R^{1/2}\,p^2, \tag{169}$$

Here, the square root of R is taken such that the real part of λ_2 is negative. Now, the eigenvalue problem for the W-matrix reduces to

$$\lambda^2 - W_{22}^{(4)}\,p^4\,\lambda + \left[W_{11}^{(2)}\,W_{22}^{(4)} - W_{12}^{(2)}\,W_{21}^{(4)}\right]p^6 = 0. \tag{170}$$

which yields

$$\lambda_1 \simeq W_{22}^{(4)}\,p^4, \tag{171}$$

which is satisfied, and

$$\lambda_2 \simeq \left[W_{11}^{(2)} - \frac{W_{12}^{(2)}\,W_{21}^{(4)}}{W_{22}^{(4)}}\right] p^2, \tag{172}$$

which implies that

$$W_{11}^{(2)} = -\iota_e^{1/2}\,D^{-1}\,R^{1/2} - c_\beta\,\iota_e^{1/2}\,D^{-1}. \tag{173}$$

Hence, the large-p boundary condition for the W-matrix is

$$\underline{\underline{W}}(p) = \begin{pmatrix} -\iota_e^{1/2}\,D^{-1}\,R^{1/2}\,p^2 - c_\beta\,\iota_e^{1/2}\,D^{-1}\,p^2, & -c_\beta\,\iota_e^{-1/2}\,D^{-1}\,p^2 \\ -c_\beta^{-1}\,\iota_e^{1/2}\,D\,P_\varphi\,p^4, & -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi\,p^4 \end{pmatrix}. \tag{174}$$