

# Calculation of Vertical Stability in an Inverse Aspect-Ratio Expanded Tokamak Plasma Equilibrium

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## I. INTRODUCTION

It is well known that an increase in the net toroidal plasma current flowing around a tokamak leads to an enhancement of both the maximum stable  $\beta$  value and the energy confinement time.<sup>1,2</sup> The conventional method of maximizing the total plasma current, without degrading the stability of the system to non-axisymmetric magnetohydrodynamical (MHD) modes, is to modify the plasma's poloidal cross-section such that it is both vertically elongated and triangular.<sup>3</sup> Unfortunately, tokamak plasmas possessing strong cross-sectional shaping are subject to severe axisymmetric instabilities.<sup>4,5</sup> Such instabilities involve bulk vertical motion of the plasma on an Alfvénic timescale (i.e.,  $10^{-7}$  s), which results in the sudden and violent termination of the discharge when it comes into contact with the first wall. It is possible to stabilize an axisymmetric mode by placing a perfectly conducting wall around the plasma. In reality, the mode remains unstable because the wall inevitably possesses finite electrical conductivity.<sup>6</sup> However, growth time of the mode is increased from the Alfvén time to the very much longer characteristic  $L/R$  time of the wall.<sup>7-9</sup> Usually, the vacuum vessel plays the role of the wall, and has an  $L/R$  time that is well in excess of  $10^{-3}$  s. Such a time is much shorter than the length of the plasma discharge, but is still long enough to allow active feedback stabilization of the axisymmetric mode with practical power supplies.<sup>10</sup>

Ref. 11 describes the TJ toroidal tearing mode code, which calculates the stability of an inverse aspect-ratio expanded tokamak plasma equilibrium to *non-axisymmetric* tearing modes via asymptotic matching techniques. Ref. 12 describes a generalization of the TJ code that permits it to calculate the stability of the plasma to non-axisymmetric ideal modes in the presence of a perfectly conducting wall surrounding the plasma. The aim of this paper is to describe a further generalization the TJ code that allows it to calculate the stability of the plasma to *axisymmetric* ideal modes in the presence of a resistive wall that surrounds the plasma. It might be hope that, in order to investigate axisymmetric modes, we could simply take the existing TJ code and set the toroidal mode number,  $n$ , to zero. Unfortunately, this simple scheme does not work, as evidenced by the large number of terms involving  $n^{-1}$  in the analysis of Ref. 11. Hence, as described in this paper, it is necessary to redo much of the

TJ analysis for the special case  $n = 0$ .

The simplified approach to calculating the vertical stability of tokamak plasmas described in this paper is quite different to that described in Refs. 13 and 14. In fact, Refs. 13 and 14 employ up-down symmetric, analytic Solov'ev plasma equilibria in which the equilibrium current jumps discontinuously to zero across the plasma-vacuum interface. Such equilibria can have arbitrary aspect-ratios, and possess the simplifying property that the perturbed plasma currents generated by vertical instabilities are surface currents that are localized on the interface. On the other hand, this paper employs realistic diffuse plasma equilibria in which the equilibrium current goes smoothly to zero as the plasma-vacuum interface is crossed, and in which the perturbed plasma currents associated with vertical instabilities are distributed throughout the plasma volume. In addition, our equilibria can be non up-down symmetric. Our main simplifying approximation is to restrict the inverse aspect-ratio of the plasma to only take small values.

This paper is organized as follows. In Sect. II, we introduce a general axisymmetric plasma equilibrium. In Sect. III, we examine axisymmetric perturbations to this equilibrium. Magnetic perturbations in the vacuum region surrounding the plasma are discussed in Sect. IV. The stability of the equilibrium to ideal axisymmetric modes is examined in Sect. V. The stability of the equilibrium to axisymmetric resistive wall modes is discussed in Sect. VI. The large aspect-ratio approximation is introduced in Sect. VII. The results of the calculations performed with the enhanced version of the TJ code are presented in Ref. VIII. Finally, Sect. IX contains a summary of the paper, as well as a discussion.

## II. GENERAL PLASMA EQUILIBRIUM

All lengths in this paper are normalized to the major radius of the plasma magnetic axis,  $R_0$ . All magnetic field-strengths are normalized to the toroidal field-strength at the magnetic axis,  $B_0$ . All current densities are normalized to  $B_0/(\mu_0 R_0)$ . All plasma pressures are normalized to  $B_0^2/\mu_0$ . All energies are normalized to  $B_0^2 R_0^3/\mu_0$ .

Let  $R, \phi, Z$  be right-handed cylindrical coordinates whose Jacobian is  $(\nabla R \times \nabla \phi \cdot \nabla Z)^{-1} =$

$R$ . Note that  $|\nabla\phi| = 1/R$ . Let  $r, \theta, \phi$  be right-handed flux-coordinates whose Jacobian is <sup>15,16</sup>

$$\mathcal{J}(r, \theta) \equiv (\nabla r \times \nabla \theta \cdot \nabla \phi)^{-1} \equiv R \left( \frac{\partial R}{\partial \theta} \frac{\partial Z}{\partial r} - \frac{\partial R}{\partial r} \frac{\partial Z}{\partial \theta} \right) = r R^2. \quad (1)$$

Note that  $r = r(R, Z)$  and  $\theta = \theta(R, Z)$ . The magnetic axis corresponds to  $r = 0$ . The plasma-vacuum interface corresponds to  $r = a$ . The inboard mid-plane corresponds to  $\theta = 0$ .

Consider an axisymmetric tokamak equilibrium whose magnetic field takes the form

$$\mathbf{B}(r, \theta) = f(r) \nabla \phi \times \nabla r + g(r) \nabla \phi = f \nabla(\phi - q \theta) \times \nabla r, \quad (2)$$

where

$$q(r) = \frac{r g}{f} \quad (3)$$

is the safety-factor profile. Note that  $\mathbf{B} \cdot \nabla r = 0$ , which implies that  $r$  is a magnetic flux-surface label. We require  $g = 1$  on the magnetic axis in order to ensure that the normalized toroidal magnetic field-strength at the axis is unity.

It is easily demonstrated that <sup>11</sup>

$$B^r = \mathbf{B} \cdot \nabla r = 0, \quad (4)$$

$$B^\theta = \mathbf{B} \cdot \nabla \theta = \frac{f}{r R^2}, \quad (5)$$

$$B^\phi = \mathbf{B} \cdot \nabla \phi = \frac{g}{R^2}, \quad (6)$$

$$B_r = \mathcal{J} \nabla \theta \times \nabla \phi \cdot \mathbf{B} = -r f \nabla r \cdot \nabla \theta, \quad (7)$$

$$B_\theta = \mathcal{J} \nabla \phi \times \nabla r \cdot \mathbf{B} = r f |\nabla r|^2, \quad (8)$$

$$B_\phi = \mathcal{J} \nabla r \times \nabla \theta \cdot \mathbf{B} = g. \quad (9)$$

The Maxwell equation (neglecting the displacement current, because the plasma velocity perturbations due to axisymmetric modes are far smaller than the velocity of light in vacuum)  $\mathbf{J} = \nabla \times \mathbf{B}$  yields

$$\mathcal{J} J^r = \frac{\partial B_\phi}{\partial \theta} = 0, \quad (10)$$

$$\mathcal{J} J^\theta = -\frac{\partial B_\phi}{\partial r} = -g', \quad (11)$$

$$\mathcal{J} J^\phi = \frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{\partial \theta} = \frac{\partial}{\partial r}(r f |\nabla r|^2) + \frac{\partial}{\partial \theta}(r f \nabla r \cdot \nabla \theta), \quad (12)$$

where  $\mathbf{J}$  is the equilibrium current density,  $' \equiv d/dr$ , and use has been made of Eqs. (7)–(9).

Equilibrium force balance requires that

$$\nabla P = \mathbf{J} \times \mathbf{B}, \quad (13)$$

where  $P(r)$  is the equilibrium scalar plasma pressure. Here, for the sake of simplicity, we have neglected the small centrifugal modifications to force balance due to subsonic plasma rotation.<sup>17,18</sup> It follows that

$$P' = \mathcal{J}(J^\theta B^\phi - J^\phi B^\theta) = -g' \frac{g}{R^2} - \frac{f}{r R^2} \left[ \frac{\partial}{\partial r}(r f |\nabla r|^2) + \frac{\partial}{\partial \theta}(r f \nabla r \cdot \nabla \theta) \right], \quad (14)$$

where use has been made of Eqs. (4)–(6), and (10)–(12). The other two components of Eq. (13) are identically zero.

Equation (14) yields the *inverse Grad-Shafranov equation*:<sup>16</sup>

$$\frac{f}{r} \frac{\partial}{\partial r}(r f |\nabla r|^2) + \frac{f}{r} \frac{\partial}{\partial \theta}(r f \nabla r \cdot \nabla \theta) + g g' + R^2 P' = 0. \quad (15)$$

It follows from Eqs. (3), (12), and (15) that

$$\mathcal{J} J^\phi = -q g' - \frac{r R^2 P'}{f}. \quad (16)$$

It is clear from Eqs. (11) and (16) that  $g' = P' = 0$  in the current-free “vacuum” region surrounding the plasma,  $r \geq a$ . We shall also assume that  $g' = P' = 0$  at the plasma-vacuum interface, so as to ensure that the equilibrium plasma current density is zero at the interface,  $r = a$ .

### III. GENERAL AXISYMMETRIC PLASMA PERTURBATION

#### A. Derivation of Axisymmetric Ideal-MHD P.D.E.s

Let us assume that all perturbed quantities have no dependence on the toroidal angle,  $\phi$ . The perturbed plasma equilibrium satisfies the linearized, marginally-stable, ideal-MHD equations<sup>16,19,27</sup>

$$\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \quad (17)$$

$$\nabla p = \mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b}, \quad (18)$$

$$\mathbf{j} = \nabla \times \mathbf{b}, \quad (19)$$

$$p = -\boldsymbol{\xi} \cdot \nabla P, \quad (20)$$

where  $\boldsymbol{\xi}(r, \theta)$  is the plasma displacement,  $\mathbf{b}(r, \theta)$  the perturbed magnetic field,  $\mathbf{j}(r, \theta)$  the perturbed current density, and  $p(r, \theta)$  the perturbed scalar pressure.

Now,<sup>11</sup>

$$(\boldsymbol{\xi} \times \mathbf{B})_\theta = \mathcal{J} (\xi^\phi B^r - \xi^r B^\phi) = -\mathcal{J} B^\phi \xi^r, \quad (21)$$

$$(\boldsymbol{\xi} \times \mathbf{B})_\phi = \mathcal{J} (\xi^r B^\theta - \xi^\theta B^r) = \mathcal{J} B^\theta \xi^r, \quad (22)$$

where use has been made of the fact that  $B^r = J^r = 0$ . [See Eqs. (4) and (10).] Combining Eqs. (17) and (22), we obtain

$$\mathcal{J} b^r = \frac{\partial}{\partial \theta} (\mathcal{J} B^\theta \xi^r). \quad (23)$$

Thus, Eqs. (1), (3), and (5) give<sup>11</sup>

$$r R^2 b^r = \frac{\partial y}{\partial \theta}, \quad (24)$$

where

$$y(r, \theta) = f \xi^r. \quad (25)$$

The constraint  $\nabla \cdot \mathbf{b} = 0$ , which follows from Eq. (17), immediately yields

$$r R^2 b^\theta = -\frac{\partial y}{\partial r}. \quad (26)$$

Note that the preceding expression is radically different from the expression, (54), for  $b^\theta$  given in Ref. 11. Thus, it is at this stage that our analysis starts to diverge from that of Ref. 11.

According to Eq. (20),

$$p = -P' \nabla r \cdot \boldsymbol{\xi} = -P' \xi^r. \quad (27)$$

So, the perturbed force balance equation, (18), yields

$$-\frac{\partial (P' \xi^r)}{\partial r} = (\mathbf{j} \times \mathbf{B})_r + (\mathbf{J} \times \mathbf{b})_r, \quad (28)$$

$$-\frac{\partial (P' \xi^r)}{\partial \theta} = (\mathbf{j} \times \mathbf{B})_\theta + (\mathbf{J} \times \mathbf{b})_\theta, \quad (29)$$

$$0 = (\mathbf{j} \times \mathbf{B})_\phi + (\mathbf{J} \times \mathbf{b})_\phi, \quad (30)$$

giving<sup>11</sup>

$$-\frac{\partial (P' \xi^r)}{\partial r} = r R^2 (j^\theta B^\phi - j^\phi B^\theta) + r R^2 (J^\theta b^\phi - J^\phi b^\theta), \quad (31)$$

$$-\frac{\partial (P' \xi^r)}{\partial \theta} = r R^2 (j^\phi B^r - j^r B^\phi) + r R^2 (J^\phi b^r - J^r b^\phi), \quad (32)$$

$$0 = r R^2 (j^r B^\theta - j^\theta B^r) + r R^2 (J^r b^\theta - J^\theta b^r), \quad (33)$$

where use has been made of Eq. (1). Thus, according to Eqs. (4)–(6), (10), (11), and (16),

$$-\frac{\partial (P' \xi^r)}{\partial r} = f (q j^\theta - j^\phi) - g' b^\phi + \left( q g' + \frac{r R^2 P'}{f} \right) b^\theta, \quad (34)$$

$$-\frac{\partial (P' \xi^r)}{\partial \theta} = -r g j^r - \left( q g' + \frac{r R^2 P'}{f} \right) b^r, \quad (35)$$

$$0 = f j^r + g' b^r. \quad (36)$$

It follows from Eqs. (24) and (36) that

$$r R^2 j^r = -\alpha_g \frac{\partial y}{\partial \theta}, \quad (37)$$

where

$$\alpha_g(r) = \frac{g'}{f}. \quad (38)$$

Note that Eq. (35) is trivially satisfied. Hence, of the three components of the perturbed force balance equation, only Eq. (34) remains to be solved.

Equation (19) yields<sup>11</sup>

$$r R^2 j^r = \frac{\partial b_\phi}{\partial \theta}, \quad (39)$$

$$r R^2 j^\theta = -\frac{\partial b_\phi}{\partial r}, \quad (40)$$

$$r R^2 j^\phi = \frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial \theta}, \quad (41)$$

where use has been made of Eq. (1). It follows from Eqs. (37), (39), and (40) that

$$b_\phi = -\alpha_g y, \quad (42)$$

$$r R^2 j^\theta = \frac{\partial(\alpha_g y)}{\partial r}. \quad (43)$$

Note that  $\nabla \cdot \mathbf{j} = 0$ , in accordance with Eq. (19).

Now,

$$\mathbf{b} = b_r \nabla r + b_\theta \nabla \theta + b_\phi \nabla \phi, \quad (44)$$

so

$$b^r = \mathbf{b} \cdot \nabla r = |\nabla r|^2 b_r + (\nabla r \cdot \nabla \theta) b_\theta, \quad (45)$$

$$b^\theta = \mathbf{b} \cdot \nabla \theta = (\nabla r \cdot \nabla \theta) b_r + |\nabla \theta|^2 b_\theta, \quad (46)$$

$$b^\phi = \mathbf{b} \cdot \nabla \phi = \frac{b_\phi}{R^2}. \quad (47)$$

Equations (1), (45), and (46) can be rearranged to give<sup>11</sup>

$$b_r = \left( \frac{1}{|\nabla r|^2} \right) b^r - \left( \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) b_\theta, \quad (48)$$

$$b^\theta = \left( \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \right) b^r + \left( \frac{1}{r^2 R^2 |\nabla r|^2} \right) b_\theta. \quad (49)$$

Let

$$\mathcal{Z}(r, \theta) = |\nabla r|^2 r \frac{\partial y}{\partial r} + r \nabla r \cdot \nabla \theta \frac{\partial y}{\partial \theta}. \quad (50)$$

Equations (24), (26), (42), (48) and (49) yield

$$b_r = \frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z}, \quad (51)$$

$$b_\theta = -\mathcal{Z}, \quad (52)$$

$$b^\phi = -\frac{\alpha_g}{R^2} y. \quad (53)$$

Equations (41), (51), and (52) give

$$r R^2 j^\phi = -\frac{\partial \mathcal{Z}}{\partial r} - \frac{\partial}{\partial \theta} \left[ \frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right]. \quad (54)$$



It follows from Eqs. (25), (26), (34), (43), (53), and (54) that

$$\begin{aligned}
-\frac{\partial}{\partial r} \left( \frac{P'}{f} y \right) &= \frac{f q}{r R^2} \frac{\partial(\alpha_g y)}{\partial r} + \frac{f}{r R^2} \frac{\partial \mathcal{Z}}{\partial r} \\
&+ \frac{f}{r R^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{r |\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{\nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right] \\
&+ \frac{g' \alpha_g}{R^2} y - \left( q g' + \frac{r R^2 P'}{f} \right) \frac{1}{r R^2} \frac{\partial y}{\partial r}.
\end{aligned} \tag{55}$$

Hence,

$$- \left[ (\alpha_f \alpha_p + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2 \right] y = r \frac{\partial \mathcal{Z}}{\partial r} + \frac{\partial}{\partial \theta} \left[ \frac{1}{|\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} + \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right], \tag{56}$$

$$R^2 j^\phi = \left[ (\alpha_f \alpha_p + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2 \right] y, \tag{57}$$

where

$$\alpha_p(r) = \frac{r P'}{f^2}, \tag{58}$$

$$\alpha_f(r) = \frac{r^2}{f} \frac{d}{dr} \left( \frac{f}{r} \right). \tag{59}$$

It is clear from Eqs. (37), (43), and (57) that  $j^r = j^\theta = j^\phi = 0$  in the vacuum region,  $r \geq a$ , which is characterized by  $\alpha_g = \alpha_p = 0$ . On the other hand, the contravariant components of the perturbed plasma current are clearly non-zero throughout the plasma.

Finally, Eqs. (50) and (56) yield the *axisymmetric ideal-MHD partial differential equations (p.d.e.s)*,

$$r \frac{\partial y}{\partial r} = \frac{\mathcal{Z}}{|\nabla r|^2} - \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \frac{\partial y}{\partial \theta}, \tag{60}$$

$$\begin{aligned}
r \frac{\partial \mathcal{Z}}{\partial r} &= - \left[ (\alpha_f \alpha_p + r \alpha'_p) R^2 + q r \alpha'_g + r^2 \alpha_g^2 \right] y - \frac{\partial}{\partial \theta} \left( \frac{1}{|\nabla r|^2 R^2} \frac{\partial y}{\partial \theta} \right) \\
&- \frac{\partial}{\partial \theta} \left( \frac{r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \mathcal{Z} \right),
\end{aligned} \tag{61}$$

which govern the perturbed equilibrium.

## B. Derivation of the Axisymmetric Ideal-MHD O.D.E.s

Let

$$y(r, \theta) = \sum_m y_m(r) e^{im\theta}, \quad (62)$$

$$\mathcal{Z}(r, \theta) = \sum_m Z_m(r) e^{im\theta}. \quad (63)$$

Here, the  $m$  are the set of (integer, but not necessarily positive) poloidal mode numbers included in the calculation. Equations (60) and (61) yield the *axisymmetric ideal-MHD ordinary differential equations (o.d.e.s)*:

$$r \frac{dy_m}{dr} = \sum_{m'} \left( A_m^{m'} Z_{m'} + B_m^{m'} y_{m'} \right), \quad (64)$$

$$r \frac{dZ_m}{dr} = \sum_{m'} \left( C_m^{m'} Z_{m'} + D_m^{m'} y_{m'} \right), \quad (65)$$

where

$$A_m^{m'} = c_m^{m'}, \quad (66)$$

$$B_m^{m'} = -m' f_m^{m'}, \quad (67)$$

$$C_m^{m'} = -m f_m^{m'}, \quad (68)$$

$$D_m^{m'} = -(\alpha_f \alpha_p + r \alpha_p') a_m^{m'} - (q r \alpha_g' + r^2 \alpha_g^2) \delta_m^{m'} + m m' b_m^{m'}, \quad (69)$$

and

$$a_m^{m'}(r) = \oint R^2 \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (70)$$

$$b_m^{m'}(r) = \oint |\nabla r|^{-2} R^{-2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (71)$$

$$c_m^{m'}(r) = \oint |\nabla r|^{-2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}, \quad (72)$$

$$f_m^{m'}(r) = \oint \frac{i r \nabla r \cdot \nabla \theta}{|\nabla r|^2} \exp[-i(m - m')\theta] \frac{d\theta}{2\pi}. \quad (73)$$

Here,  $\delta_m^{m'}$  is a Kronecker delta symbol. Note that  $Z_0$  is independent of  $r$  in the vacuum region,  $r \geq a$ , in which  $\alpha_g = \alpha_p = 0$ .

The axisymmetric ideal-MHD o.d.e.s play the same role for axisymmetric perturbations that the outer-region o.d.e.s, (102) and (103) of Ref. 11, play for non-axisymmetric perturbations. The main difference is that there are no singularities in the axisymmetric ideal-MHD o.d.e.s because an axisymmetric perturbation does not resonate with the plasma. In other words, there are no equilibrium flux-surfaces in the plasma at which  $\mathbf{k} \cdot \mathbf{B} = 0$ , where  $\mathbf{k}$  is the wavevector of the perturbation. This is because  $|B^\theta(r)| > 0$  throughout a conventional tokamak plasma equilibrium.

### C. Properties of Axisymmetric Ideal-MHD O.D.E.s

Note that  $a_{m'}^m = a_m^{m'*}$ ,  $b_{m'}^m = b_m^{m'*}$ ,  $c_{m'}^m = c_m^{m'*}$ , and  $f_{m'}^m = -f_m^{m'*}$ , which implies that

$$A_{m'}^m = A_{m'}^{m*}, \quad (74)$$

$$B_{m'}^m = -C_{m'}^{m*}, \quad (75)$$

$$C_{m'}^m = -B_{m'}^{m*}, \quad (76)$$

$$D_{m'}^m = D_{m'}^{m*}. \quad (77)$$

It follows from Eqs. (64), (65), and (74)–(77) that

$$r \frac{d}{dr} \left[ \sum_m (Z_m y_m^* - y_m Z_m^*) \right] = 0. \quad (78)$$

Thus, checking that  $\sum_m (Z_m y_m^* - y_m Z_m^*)$  is indeed constant throughout the plasma is a good numerical check on the accuracy of the TJ code.

### D. Perturbed Electric Field

Suppose that all perturbed quantities vary in time as  $e^{-i\omega t}$ . Let  $\mathbf{e}$  be the perturbed electric field, which satisfies

$$\nabla \times \mathbf{e} = i\omega \mathbf{b}. \quad (79)$$

Hence,

$$e_\phi = i\omega y, \quad (80)$$

and

$$\frac{\partial e_\theta}{\partial r} - \frac{\partial e_r}{\partial \theta} = -i\omega r \alpha_g y, \quad (81)$$

where use has been made of Eqs. (24), (26), and (53). We also expect  $\nabla \cdot \mathbf{e} = 0$ , which implies that  $e_r = e_\theta = 0$  in the vacuum region,  $r \geq a$ , in which  $\alpha_g = 0$ .

### E. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque exerted on the region of the plasma lying within the magnetic flux-surface whose label is  $r$  is <sup>11</sup>

$$T_\phi(r) = \oint \oint r R^2 b_\phi b^r d\theta d\phi. \quad (82)$$

It follows from Eqs. (24) and (42) that

$$T_\phi(r) = -\pi \alpha_g \oint \left( y^* \frac{\partial y}{\partial \theta} + y \frac{\partial y^*}{\partial \theta} \right) d\theta = -\pi \alpha_g \oint \frac{\partial |y|^2}{\partial \theta} d\theta = 0. \quad (83)$$

We conclude, not surprisingly, that an axisymmetric perturbation is incapable of exerting a net toroidal electromagnetic torque on the plasma.

In Ref. 12, the conservation of toroidal angular momentum is used to prove that the plasma and vacuum perturbed potential energy matrices are Hermitian. The fact that the toroidal electromagnetic torque is automatically zero in the axisymmetric case negates this proof. Hence, we need to find a new conserved quantity that does not automatically take the value zero. The electromagnetic energy flux is found to play this role.

### F. Electromagnetic Energy Flux

The net flux of electromagnetic energy across the plasma-vacuum interface is

$$\begin{aligned} \mathcal{E} &= \left[ \oint \oint (\mathbf{e} \times \mathbf{b}) \cdot \nabla r \mathcal{J} d\theta d\phi \right]_{r=a} = \left[ \oint \oint (e_\theta b_\phi - e_\phi b_\theta) d\theta d\phi \right]_{r=a} \\ &= i\pi\omega \oint (y \mathcal{Z}^* - y^* \mathcal{Z})_{r=a} d\theta = i\pi^2\omega \sum_m (Z_m^* y_m - y_m^* Z_m)_{r=a}. \end{aligned} \quad (84)$$

Here, use has been made of Eqs. (52), (62), (63), and (80), as well as the fact that  $e_\theta = 0$  for  $r \geq a$ .

## G. Perturbed Plasma Potential Energy

The perturbed plasma potential energy in the region of the plasma lying within the magnetic flux-surface whose label is  $r$  is<sup>12,19</sup>

$$\delta W_p = \frac{1}{2} \oint \oint r R^2 \xi^{r*} (-\mathbf{B} \cdot \mathbf{b} + \xi^r P') d\theta d\phi. \quad (85)$$

However,

$$\mathbf{B} \cdot \mathbf{b} - \xi^r P' = B^\theta b_\theta + B^\phi b_\phi - \xi^r P' = -\frac{f}{r R^2} (\mathcal{Z} + q \alpha_g y + \alpha_p R^2), \quad (86)$$

where use has been made of Eqs. (3)–(6), (25), (42), (52), and (58). Hence, we obtain

$$\delta W_p(r) = \frac{1}{2} \oint \oint y^* [\mathcal{Z} + (q \alpha_g + \alpha_p R^2) y] d\theta d\phi = \pi^2 \sum_m y_m^* \chi_m, \quad (87)$$

where

$$\chi_m(r) = Z_m + q \alpha_g y_m + \alpha_p \sum_{m'} a_m^{m'} y_{m'}. \quad (88)$$

## IV. GENERAL VACUUM SOLUTION

### A. Toroidal Coordinates

Let  $\mu, \eta, \phi$  be right-handed *orthogonal toroidal coordinates* defined such that<sup>11,12,20</sup>

$$R = \frac{\sinh \mu}{\cosh \mu - \cos \eta}, \quad (89)$$

$$Z = \frac{\sin \eta}{\cosh \mu - \cos \eta}. \quad (90)$$

The scale-factors of the toroidal coordinate system are

$$h_\mu = h_\eta = \frac{1}{\cosh \mu - \cos \eta} \equiv h, \quad (91)$$

$$h_\phi = \frac{\sinh \mu}{\cosh \mu - \cos \eta} = h \sinh \mu. \quad (92)$$

Moreover,

$$\mathcal{J}' \equiv (\nabla \mu \times \nabla \eta \cdot \nabla \phi)^{-1} = h^3 \sinh \mu. \quad (93)$$

## B. Perturbed Magnetic Field

The curl-free perturbed magnetic field in the vacuum region is written  $\mathbf{b} = \mathbf{i} \nabla V$ , where  $\nabla^2 V = 0$ . The most general axisymmetric solution to Laplace's equation is<sup>12,21</sup>

$$V(z, \eta) = \sum_m (z - \cos \eta)^{1/2} U_m(z) e^{-i m \eta}, \quad (94)$$

$$U_m(z) = p_m \hat{P}_{|m|-1/2}(z) + q_m \hat{Q}_{|m|-1/2}(z), \quad (95)$$

where  $z = \cosh \mu$ , the  $p_m$  and  $q_m$  are arbitrary complex coefficients, and

$$\hat{P}_{|m|-1/2}(z) = \cos(|m| \pi) \frac{\sqrt{\pi} \Gamma(|m| + 1/2) a^{|m|}}{2^{|m|-1/2} |m|!} P_{|m|-1/2}(z), \quad (96)$$

$$\hat{Q}_{|m|-1/2}(z) = \cos(|m| \pi) \frac{2^{|m|-1/2} |m|!}{\sqrt{\pi} \Gamma(|m| + 1/2) a^{|m|}} Q_{|m|-1/2}(z). \quad (97)$$

Here, the  $P_{|m|-1/2}(z)$  and  $Q_{|m|-1/2}(z)$  are toroidal functions,<sup>22</sup> and  $\Gamma(z)$  is a gamma function.<sup>23</sup>

## C. Toroidal Electromagnetic Angular Momentum Flux

The outward flux of toroidal angular momentum across a constant- $z$  surface is<sup>11,12</sup>

$$T_\phi(z) = - \oint \oint \mathcal{J}' b_\phi b^\mu d\eta d\phi = 0, \quad (98)$$

because  $b_\phi = \mathbf{i} \partial V / \partial \phi = 0$ . Of course, the flux has to be zero because the flux of angular momentum across the plasma-vacuum interface is zero, and there are no sources of angular momentum in the vacuum region surrounding the plasma. (See Sect. III E.)

## D. Electromagnetic Energy Flux

The outward flux of electromagnetic energy flux across a constant- $z$  surface is

$$\mathcal{E}(z) = - \oint \oint \mathcal{J}' \mathbf{e} \times \mathbf{b} \cdot \nabla \mu d\eta d\phi = -i \pi \oint \left( e_\phi \frac{\partial V^*}{\partial \eta} - e_\phi^* \frac{\partial V}{\partial \eta} \right) d\eta, \quad (99)$$

given that  $e_\mu = e_\eta = 0$  in the vacuum. However,  $\nabla \times \mathbf{e} = i \omega \mathbf{b}$  implies that

$$\frac{\partial e_\phi}{\partial \eta} = -\omega h \sinh \mu \frac{\partial V}{\partial \mu} = -\omega h \sinh^2 \mu \frac{\partial V}{\partial z}. \quad (100)$$

Thus,

$$\begin{aligned}\mathcal{E}(z) &= \mathrm{i} \pi \oint \left( \frac{\partial e_\phi}{\partial \eta} V^* - \frac{\partial e_\phi^*}{\partial \eta} V \right) d\eta = -\mathrm{i} \pi \omega \oint h \sinh^2 \mu \left( \frac{\partial V}{\partial z} V^* - \frac{\partial V^*}{\partial z} V \right) d\eta \\ &= \mathrm{i} \pi^2 \omega \sum_m (p_m q_m^* - q_m p_m^*) (z^2 - 1) \mathcal{W}(P_{|m|-1/2}, Q_{|m|-1/2}),\end{aligned}\quad (101)$$

where  $\mathcal{W}(f, g) = f dg/dz - g df/dz$ . However,<sup>24</sup>

$$\mathcal{W}(P_{|m|-1/2}, Q_{|m|-1/2}) = \frac{1}{1 - z^2}, \quad (102)$$

so

$$\mathcal{E}(z) = -\mathrm{i} \pi^2 \omega \sum_m (p_m q_m^* - q_m p_m^*). \quad (103)$$

Note that  $\mathcal{E}$  is independent of  $z$ , as must be the case because there are no energy sources in the vacuum region.

### E. Solution in Vacuum Region

In the large-aspect ratio limit,  $r \ll 1$ , it can be demonstrated that<sup>24</sup>

$$z \simeq \frac{1}{r}, \quad (104)$$

$$z^{1/2} \hat{P}_{-1/2}(z) \simeq \frac{1}{2} \ln(8z), \quad (105)$$

$$z^{1/2} \hat{P}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\pi) (az)^{|m|}}{|m|}, \quad (106)$$

$$z^{1/2} \hat{Q}_{|m|-1/2}(z) \simeq \frac{\cos(|m|\pi) (az)^{-|m|}}{2}. \quad (107)$$

Note that Eq. (106) only applies to  $|m| > 0$ .

According to Eq. (24) and (52),

$$\frac{\partial y}{\partial \theta} = \mathcal{J} \mathbf{b} \cdot \nabla r = \mathrm{i} \mathcal{J} \nabla V \cdot \nabla r, \quad (108)$$

$$\mathcal{Z} = -\mathcal{J} \nabla \phi \times \nabla r \cdot \mathbf{b} = -\mathrm{i} \frac{\partial V}{\partial \theta}. \quad (109)$$

Now, we have already seen that  $Z_0 \equiv \oint \mathcal{Z} d\theta / (2\pi) = 0$  is independent of  $r$  in the vacuum region. (See Sect. IIIB.) However, it is clear that Eq. (109) mandates that this constant

value is zero. This constraint implies that the axisymmetric ideal modes to which the plasma equilibrium is subject do not change the net toroidal current flowing in the plasma. [See Eqs. (52) and (219).] Note, further, that  $y_0 \equiv \oint y d\theta/(2\pi)$  does not influence the vacuum potential,  $V(r, \theta)$ .

The previous two equations yield<sup>12</sup>

$$\underline{V}(r) = \underline{\underline{P}}(r) \underline{p} + \underline{\underline{Q}}(r) \underline{q}, \quad (110)$$

$$\underline{\psi}(r) = \underline{\underline{R}}(r) \underline{p} + \underline{\underline{S}}(r) \underline{q}, \quad (111)$$

where  $V(r, \theta) = \sum_m V_m(r) e^{im\theta}$ ,  $Z_m(r) = m V_m(r)$ ,  $\psi_m(r) = m y_m(r)$ ,  $\underline{V}(r)$  is the vector of the  $V_m(r)$  values,  $\underline{\psi}(r)$  is the vector of the  $\psi_m(r)$  values,  $\underline{\underline{P}}(r)$  is the matrix of the

$$\mathcal{P}_{mm'}(r) = \oint_r (z - \cos \eta)^{1/2} \hat{P}_{|m'|-1/2}(z) \exp[-i(m\theta + m'\eta)] \frac{d\theta}{2\pi} \quad (112)$$

values,  $\underline{\underline{Q}}(r)$  is the matrix of the

$$\mathcal{Q}_{mm'}(r) = \oint_r (z - \cos \eta)^{1/2} \hat{Q}_{|m'|-1/2}(z) \exp[-i(m\theta + m'\eta)] \frac{d\theta}{2\pi} \quad (113)$$

values,  $\underline{\underline{R}}(r)$  is the matrix of the

$$\begin{aligned} \mathcal{R}_{mm'}(r) = & \oint_r \left\{ \left[ \frac{1}{2} (z - \cos \eta)^{-1/2} \hat{P}_{|m'|-1/2}(z) + (z - \cos \eta)^{1/2} \frac{d\hat{P}_{|m'|-1/2}}{dz} \right] \mathcal{J} \nabla r \cdot \nabla z \right. \\ & + \left. \left[ \frac{1}{2} (z - \cos \eta)^{-1/2} \sin \eta - i m' (z - \cos \eta)^{1/2} \right] \hat{P}_{|m'|-1/2}(z) \mathcal{J} \nabla r \cdot \nabla \eta \right\} \\ & \times \exp[-i(m\theta + m'\eta)] \frac{d\theta}{2\pi} \end{aligned} \quad (114)$$

values,  $\underline{\underline{S}}(r)$  is the matrix of the

$$\begin{aligned} \mathcal{S}_{mm'}(r) = & \oint_r \left\{ \left[ \frac{1}{2} (z - \cos \eta)^{-1/2} \hat{Q}_{|m'|-1/2}(z) + (z - \cos \eta)^{1/2} \frac{d\hat{Q}_{|m'|-1/2}}{dz} \right] \mathcal{J} \nabla r \cdot \nabla z \right. \\ & + \left. \left[ \frac{1}{2} (z - \cos \eta)^{-1/2} \sin \eta - i m' (z - \cos \eta)^{1/2} \right] \hat{Q}_{|m'|-1/2}(z) \mathcal{J} \nabla r \cdot \nabla \eta \right\} \\ & \times \exp[-i(m\theta + m'\eta)] \frac{d\theta}{2\pi} \end{aligned} \quad (115)$$

values,  $\underline{p}$  is the vector of the  $p_m$  coefficients, and  $\underline{q}$  is the vector of the  $q_m$  coefficients. Here, the subscript  $r$  on the integrals indicates that they are taken at constant  $r$ .



## F. Energy Conservation

According to Eq. (84), the net flux of electromagnetic energy across the plasma-vacuum interface is

$$\mathcal{E} = \mathrm{i} \pi^2 \omega (\underline{V}^\dagger \underline{\psi} - \underline{\psi}^\dagger \underline{V}). \quad (116)$$

However, energy conservation requires this flux to be equal to the energy flux through the vacuum region, so Eq. (103) gives

$$\mathcal{E} = -\mathrm{i} \pi^2 \omega (\underline{q}^\dagger \underline{p} - \underline{p}^\dagger \underline{q}). \quad (117)$$

Equations (110), (111), and the previous two equations, yield <sup>12</sup>

$$\underline{\underline{\mathcal{P}}}^\dagger \underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{R}}}^\dagger \underline{\underline{\mathcal{P}}}, \quad (118)$$

$$\underline{\underline{\mathcal{Q}}}^\dagger \underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{S}}}^\dagger \underline{\underline{\mathcal{Q}}}, \quad (119)$$

$$\underline{\underline{\mathcal{P}}}^\dagger \underline{\underline{\mathcal{S}}} - \underline{\underline{\mathcal{R}}}^\dagger \underline{\underline{\mathcal{Q}}} = \underline{\underline{1}}. \quad (120)$$

It can also be demonstrated that

$$\underline{\underline{\mathcal{Q}}} \underline{\underline{\mathcal{P}}}^\dagger = \underline{\underline{\mathcal{P}}} \underline{\underline{\mathcal{Q}}}^\dagger, \quad (121)$$

$$\underline{\underline{\mathcal{R}}} \underline{\underline{\mathcal{S}}}^\dagger = \underline{\underline{\mathcal{S}}} \underline{\underline{\mathcal{R}}}^\dagger. \quad (122)$$

The previous five equations hold throughout the vacuum region.

## G. No-Wall Matching Condition

Suppose that the plasma is surrounded by a vacuum region that extends to infinity. In this case,

$$\underline{q} = \underline{0}, \quad (123)$$

because the  $\underline{\underline{\mathcal{Q}}}(r)$  solutions blow up in an unphysical manner as  $r \rightarrow \infty$ . [See Eqs. (104) and (107)]. It immediately follows from Eq. (117) that

$$\mathcal{E} = 0. \quad (124)$$

In other words, as we would expect, there is zero net flux of electromagnetic energy out of a plasma surrounded by a vacuum region that extends to infinity (because there is nothing at infinity that could absorb the energy flux). Equations (110) and (111) imply that

$$\underline{V}(r = a_+) = \underline{\underline{H}} \underline{\psi}(r = a), \quad (125)$$

where

$$\underline{\underline{H}} = \underline{\underline{\mathcal{P}}}_a \underline{\underline{\mathcal{R}}}_a^{-1} \quad (126)$$

is termed the *no-wall vacuum matrix*. Here,  $\underline{\underline{\mathcal{P}}}_a = \underline{\underline{\mathcal{P}}}(r = a)$ , et cetera. Equation (118) ensures that  $\underline{\underline{H}}$  is Hermitian.

## H. Perfect-Wall Matching Condition

Suppose that the plasma is surrounded by a vacuum region that is bounded by a perfectly conducting wall that is conformal to the plasma-vacuum interface, and whose inner surface lies at  $r = b_w a$ , where  $b_w \geq 1$ . Because the wall is perfectly conducting,  $\underline{\psi}(r = b_w a) = 0$ .<sup>19</sup> In other words, the normal component of the perturbed magnetic field is zero at the inner surface of the wall. It follows from Eq. (111) that

$$\underline{p} = -\underline{\underline{I}}_b \underline{q}, \quad (127)$$

where

$$\underline{\underline{I}}_b = \underline{\underline{\mathcal{R}}}_b^{-1} \underline{\underline{\mathcal{S}}}_b \quad (128)$$

is termed the *wall matrix*. Here,  $\underline{\underline{\mathcal{R}}}_b = \underline{\underline{\mathcal{R}}}(r = b_w a)$ , et cetera. Equation (122) ensures that  $\underline{\underline{I}}_b$  is Hermitian. It immediately follows from Eq. (117) that

$$\mathcal{E} = 0. \quad (129)$$

In other words, as we would expect, there is zero net electromagnetic energy flux out of a plasma surrounded by a vacuum region that is bounded by perfectly conducting wall (because a perfect wall could not absorb the energy flux).

Making use of Eqs. (110) and (111), the matching condition at the plasma-vacuum interface for a perfectly-conducting wall becomes

$$\underline{V}(r = a_+) = \underline{\underline{G}} \underline{\psi}(r = a), \quad (130)$$

where

$$\underline{\underline{G}} = (\underline{\underline{Q}}_a - \underline{\underline{P}}_a \underline{\underline{I}}_b) (\underline{\underline{S}}_a - \underline{\underline{R}}_a \underline{\underline{I}}_b)^{-1} \quad (131)$$

is termed the *perfect-wall vacuum matrix*. It is easily demonstrated from Eqs. (118)–(120) that

$$\underline{\underline{G}} - \underline{\underline{G}}^\dagger = -[(\underline{\underline{S}}_a - \underline{\underline{R}}_a \underline{\underline{I}}_b)^{-1}]^\dagger (\underline{\underline{I}}_b - \underline{\underline{I}}_b^\dagger) (\underline{\underline{S}}_a + \underline{\underline{R}}_a \underline{\underline{I}}_b)^{-1}. \quad (132)$$

Thus, the vacuum matrix,  $\underline{\underline{G}}$ , is Hermitian because the wall matrix,  $\underline{\underline{I}}_b$ , is Hermitian.

## I. Perturbed Vacuum Potential Energy

Equation (24) implies that

$$\mathcal{J} \nabla r \cdot \nabla V = -i \frac{\partial y}{\partial \theta}. \quad (133)$$

The perturbed potential energy in the vacuum region is<sup>12,19</sup>

$$\begin{aligned} \delta W_v &= \frac{1}{2} \int_{a_+}^{\infty} \oint \oint \mathbf{b}^* \cdot \mathbf{b} \mathcal{J} dr d\theta d\phi = \frac{1}{2} \int_{a_+}^{\infty} \oint \oint \nabla V^* \cdot \nabla V \mathcal{J} dr d\theta d\phi \\ &= -\frac{1}{2} \left( \oint \oint \mathcal{J} \nabla r \cdot \nabla V^* V d\theta d\phi \right)_{a_+} = -\frac{1}{2} \left[ \oint \oint \left( -i \frac{\partial y}{\partial \theta} \right)^* V d\theta d\phi \right]_{a_+} \\ &= -\pi^2 \left( \sum_m m y_m^* V_m \right)_{a_+}, \end{aligned} \quad (134)$$

where use has been made of the facts that  $\nabla^2 V = 0$  and  $\nabla r \cdot \nabla V = 0$  at the ideal wall.

## V. GENERAL IDEAL STABILITY

### A. Perturbed Plasma Potential Energy

Suppose that the poloidal harmonics included in the calculation range from  $m = -m_{\max}$  to  $m = m_{\max}$ , where  $m_{\max} > 0$ . Let the  $y_{mm'}(r)$  and the  $Z_{mm'}(r)$  be linearly independent

solutions of the axisymmetric ideal-MHD o.d.e.s, (64) and (65), that are well-behaved at the magnetic axis. Here,  $m$  indexes the poloidal harmonic, whereas  $m'$  indexes the dominant poloidal harmonic close to the magnetic axis. (See Sect. VII H.) Let

$$\chi_{mm'}(r) = Z_{mm'} + q \alpha_g y_{mm'} + \alpha_p \sum_{m''} a_m^{m''} y_{m''m'}. \quad (135)$$

We can form  $J = 2m_{\max}$  linearly independent solutions that all have  $Z_0 = 0$  at  $r = a$ , as mandated by the matching condition (109). The most general solution to the axisymmetric ideal-MHD o.d.e.s that satisfied the constraint  $Z_0(a) = 0$  is written

$$y_m(r) = \sum_{m'=1,J} y_{mm'}(r) \alpha_{m'}, \quad (136)$$

$$\chi_m(r) = \sum_{m'=1,J} \chi_{mm'}(r) \alpha_{m'}, \quad (137)$$

where the  $\alpha_m$  are arbitrary complex coefficients. Here,  $m$  indexes the poloidal harmonic and  $m'$  indexes the solution. Note that  $\chi_m(a_-) = Z_m(a_-) = 0$  because  $\alpha_g(a) = \alpha_p(a) = 0$ . [See Eq. (88).]

According to Eq. (87), the net perturbed plasma potential energy is

$$\delta W_p = \pi^2 \underline{y}^\dagger \underline{\chi}, \quad (138)$$

where  $\underline{y}$  is the vector of the  $y_m(a)$  values, excluding the  $m = 0$  harmonic, and  $\underline{\chi}$  the vector of the  $\chi_m(a_-)$  values, excluding the  $m = 0$  harmonic. We can exclude the  $m = 0$  harmonic because, by construction our solutions are such that  $\chi_0(a_-) = 0$ . It follows that

$$\delta W_p = \pi^2 \underline{\alpha}^\dagger \underline{\underline{y}}^\dagger \underline{\underline{\chi}} \underline{\alpha}, \quad (139)$$

where  $\underline{\alpha}$  is the vector of the  $\alpha_m$  values,  $\underline{\underline{y}}$  the matrix of the  $y_{mm'}(a)$  values, excluding the  $m = 0$  harmonic, and  $\underline{\underline{\chi}}$  the matrix of the  $\chi_{mm'}(a_-)$  values, excluding the  $m = 0$  harmonic. If

$$\underline{\underline{\chi}} = \underline{\underline{W}}_p \underline{\underline{y}}, \quad (140)$$

then

$$\delta W_p = \pi^2 \underline{\alpha}^\dagger \underline{\underline{y}}^\dagger \underline{\underline{W}}_p \underline{\underline{y}} \underline{\alpha}. \quad (141)$$

Equations (84), (124), and (129) imply that  $\underline{\underline{W}}_p$  is Hermitian.

## B. Perturbed Vacuum Potential Energy

According to Eq. (134), the perturbed vacuum potential energy is

$$\delta W_v = -\pi^2 \sum_m m y_m^*(a) V_m(a_+), \quad (142)$$

excluding the  $m = 0$  harmonic, which obviously does not affect the vacuum energy. However, Eqs. (125) and (130) imply that

$$V_m(a_+) = \sum_{m'} H_{mm'} m' y_{m'}(a) \quad (143)$$

in the no-wall case, and

$$V_m(a_+) = \sum_{m'} G_{mm'} m' y_{m'}(a) \quad (144)$$

in the perfect-wall case. Here, we have excluded the  $m' = 0$  harmonic, which also obviously does not affect the vacuum energy. Hence, we can write

$$\delta W_v = \pi^2 \underline{y}^\dagger \underline{W}_v \underline{y}, \quad (145)$$

where  $\underline{W}_v$  is the matrix of the  $-m H_{mm'} m'$  values in the no-wall case, excluding the  $m = 0$  and  $m' = 0$  harmonics, and the  $-m G_{mm'} m'$  values in the perfect-wall case, likewise excluding the  $m = 0$  and  $m' = 0$  harmonics. Given that  $H_{mm'}$  and  $G_{mm'}$  are Hermitian, we deduce that  $\underline{W}_v$  is Hermitian. It follows that

$$\delta W_v = \pi^2 \underline{\alpha}^\dagger \underline{y}^\dagger \underline{W}_v \underline{y} \underline{\alpha}. \quad (146)$$

## C. Total Perturbed Potential Energy

The total perturbed potential energy is

$$\delta W = \delta W_p + \delta W_v = \pi^2 \underline{\alpha}^\dagger \underline{y}^\dagger \underline{W} \underline{y} \underline{\alpha}, \quad (147)$$

where

$$\underline{W} = \underline{W}_p + \underline{W}_v. \quad (148)$$

Given that  $\underline{W}_p$  and  $\underline{W}_v$  are both Hermitian, we deduce that  $\underline{W}$  is Hermitian.

### D. Ideal Stability

The fact that  $\underline{\underline{W}}$  is Hermitian allows us to write

$$\underline{\underline{W}}\underline{\underline{\beta}} = \underline{\underline{\beta}}\underline{\underline{A}}, \quad (149)$$

$$\underline{\underline{\beta}}^\dagger \underline{\underline{\beta}} = \underline{\underline{1}}, \quad (150)$$

where  $\underline{\underline{\beta}}$  is real, and  $\underline{\underline{A}}$  is the diagonal matrix of the real  $\lambda_m$  values. If  $\hat{\underline{\underline{\alpha}}} = \underline{\underline{\beta}}^\dagger \underline{\underline{y}} \underline{\underline{\alpha}}$  then

$$\delta W = \pi^2 \hat{\underline{\underline{\alpha}}}^\dagger \underline{\underline{A}} \hat{\underline{\underline{\alpha}}} = \pi^2 \sum_{m=1,J} |\hat{\alpha}_m|^2 \lambda_m. \quad (151)$$

Thus, if any of the  $\lambda_m$  are negative then solutions exist for which  $\delta W$  is negative, and the plasma is consequently unstable to an axisymmetric ideal mode.<sup>19</sup>

Suppose that  $\sum_m |\hat{\alpha}_m|^2 = 1$ . The ideal energy of the  $m$ th mode, for which  $\hat{\alpha}_{m'} = \delta_{mm'}$ , is

$$\delta W_m = \pi^2 \lambda_m. \quad (152)$$

However,

$$\underline{\underline{A}} = \underline{\underline{\beta}}^\dagger \underline{\underline{W}} \underline{\underline{\beta}}, \quad (153)$$

Thus, the diagonal components of  $\underline{\underline{\beta}}^\dagger \underline{\underline{W}}_p \underline{\underline{\beta}}$  and  $\underline{\underline{\beta}}^\dagger \underline{\underline{W}}_v \underline{\underline{\beta}}$  are the plasma and vacuum contributions to the  $\lambda_m$ , respectively. The eigenfunction of the  $m$ th mode is conveniently normalized such that  $y_{m'}(a) = \beta_{m'm}$ .

## VI. GENERAL RESISTIVE WALL MODE STABILITY

The unnormalized minor radius of the wall is  $b = b_w \bar{a}$ , where  $\bar{a} = \epsilon R_0$  is the unnormalized minor radius of the plasma. Suppose that the wall is resistive, and possesses an (unnormalized) electrical conductivity  $\sigma_w$ , as well as an (unnormalized) uniform radial thickness  $d$ . Consider a particular axisymmetric ideal mode for which the total perturbed potential energy in the absence of a wall,  $\delta W_{nw}$ , is negative, but the total perturbed potential energy in the presence of a perfectly conducting wall,  $\delta W_{pw}$ , is positive. In this case, the no-wall ideal mode is unstable (because  $\delta W_{nw} < 0$ ). In the absence of the wall, this mode would grow

very rapidly on an Alfvénic timescale. On the other hand, the mode is completely stabilized if the wall is perfectly conducting (because  $\delta W_{pw} > 0$ ). However, because the wall is not perfectly conducting, the unstable ideal mode is instead converted into a much more slowly growing *resistive wall mode*.

Let the (unnormalized) growth-rate of the resistive wall mode be

$$\gamma = \frac{\hat{\gamma}}{\tau_w}, \quad (154)$$

where

$$\tau_w = \mu_0 \sigma_w b d \quad (155)$$

is the (unnormalized) L/R time of the wall. The normalized growth-rate of the resistive wall mode is specified by<sup>25,26</sup>

$$\sqrt{\frac{\hat{\gamma}}{\delta_w}} \tanh \left( \sqrt{\delta_w \hat{\gamma}} \right) = -\frac{\delta W_{nw}}{\alpha_w \delta W_{pw}}, \quad (156)$$

where  $\delta_w = d/b$ , and

$$\alpha_w = \frac{(1/2) \int |\mathbf{A}_{nw} \times \mathbf{n}_w|^2 dS_w}{\epsilon b_w (\delta W_{vpw} - \delta W_{vnw})}. \quad (157)$$

Here,  $\delta W_{vnw}$  is the vacuum potential energy in the absence of a wall, whereas  $\delta W_{npw}$  is the vacuum energy in the presence of a perfectly conducting wall. Moreover,  $dS_w$  is an element of the inner surface of the wall, and the integral is over the whole inner surface. Furthermore, the perturbed magnetic field in the vacuum region, in the no-wall case, is written  $\nabla \times \mathbf{A}_{nw}$ . It is easily demonstrated from Eqs. (24) and (26) that  $\mathbf{A}_{nw} = \sum_m y_{mnw}(r) e^{im\theta} \nabla \phi$ , where the  $y_{mnw}(r)$  are the  $y$ -components of the no-wall eigenfunction of the mode in question. Finally,  $dS_w = \mathcal{J} d\theta d\phi$ . Hence, we deduce that

$$\alpha_w = \frac{\pi^2 \left( \sum_{m \neq 0} |y_{mnw}|^2 \right)_{\hat{r}=b_w}}{\delta W_{vpw} - \delta W_{vnw}}. \quad (158)$$

Here, we have neglected  $y_{0nw}$  because  $y_0$  does not affect the vacuum energy.

Given that  $\underline{q} = \underline{0}$  for a no-wall solution, Eq. (111) yields

$$\underline{\underline{m}} \underline{\underline{y}}_b = \underline{\underline{\mathcal{R}}}_b \underline{\underline{\mathcal{R}}}_a^{-1} \underline{\underline{m}} \underline{\underline{y}}_a, \quad (159)$$

which enables us to determine the  $y_{m \neq 0 \, nw}(\hat{r} = b_w)$  from the  $y_{m \neq 0 \, nw}(\hat{r} = 1)$ . Here,  $\underline{\underline{m}}$  is the diagonal matrix of the poloidal mode numbers.

Finally, in the *thin-wall* limit,  $\delta_w \ll 1$ , Eq. (156) simplifies to give

$$\hat{\gamma} = -\frac{\delta W_{nw}}{\alpha_w \delta W_{pw}}. \quad (160)$$

## VII. INVERSE ASPECT-RATIO EXPANDED TOKAMAK EQUILIBRIUM

### A. Introduction

Up to now, our analysis has been completely general. At this stage, we introduce the simplifying approximation of an inverse aspect-ratio expanded plasma equilibrium.

### B. Equilibrium Magnetic Flux-Surfaces

Let us assume that the inverse aspect-ratio of the plasma,  $\epsilon = a/R_0 = a$  (since  $R_0$  is normalized to unity), is such that  $0 < \epsilon \ll 1$ . Let  $r = \epsilon \hat{r}$ ,  $\nabla = \epsilon^{-1} \hat{\nabla}$ , and  $' \rightarrow \epsilon^{-1} '$ . Suppose that the loci of the equilibrium magnetic flux-surfaces can be written in the parametric form: <sup>11,12,16</sup>

$$\begin{aligned} R(\hat{r}, \omega) = & 1 - \epsilon \hat{r} \cos \omega + \epsilon^2 \sum_{j>0} H_j(\hat{r}) \cos[(j-1)\omega] + \epsilon^2 \sum_{j>1} V_j(\hat{r}) \sin[(j-1)\omega] \\ & + \epsilon^3 L(\hat{r}) \cos \omega, \end{aligned} \quad (161)$$

$$\begin{aligned} Z(\hat{r}, \omega) = & \epsilon \hat{r} \sin \omega + \epsilon^2 \sum_{j>1} H_j(\hat{r}) \sin[(j-1)\omega] - \epsilon^2 \sum_{j>1} V_j(\hat{r}) \cos[(j-1)\omega] \\ & - \epsilon^3 L(\hat{r}) \sin \omega, \end{aligned} \quad (162)$$

where  $j$  is a positive integer. Here,  $H_1(\hat{r})$  controls the relative horizontal locations of the flux-surface centroids,  $H_2(\hat{r})$  and  $V_2(\hat{r})$  control the magnitudes and vertical tilts of the flux-surface ellipticities,  $H_3(\hat{r})$  and  $V_3(\hat{r})$  control the magnitudes and vertical tilts of the flux-surface triangularities, et cetera, whereas  $L(\hat{r})$  is a flux-surface re-labelling parameter. Moreover,  $\omega(R, Z)$  is a poloidal angle that is distinct from  $\theta$ . Note that  $V_1$  does not appear in Eq. (162)



because such a factor merely gives rise to a rigid vertical shift of the plasma that can be eliminated by a suitable choice of the origin of the flux-coordinate system.

Let

$$J(\hat{r}, \omega) = \frac{1}{\epsilon^2} \left( \frac{\partial R}{\partial \omega} \frac{\partial Z}{\partial \hat{r}} - \frac{\partial R}{\partial \hat{r}} \frac{\partial Z}{\partial \omega} \right) \quad (163)$$

be the Jacobian of the  $\hat{r}, \omega$  coordinate system. We can transform to the  $\hat{r}, \theta$  coordinate system by writing

$$\theta(\hat{r}, \omega) = 2\pi \int_0^\omega \frac{J(\hat{r}, \tilde{\omega})}{R(\hat{r}, \tilde{\omega})} d\tilde{\omega} \Big/ \oint \frac{J(\hat{r}, \omega)}{R(\hat{r}, \omega)} d\omega, \quad (164)$$

$$\hat{r} = \frac{1}{2\pi} \oint \frac{J(\hat{r}, \omega)}{R(\hat{r}, \omega)} d\omega. \quad (165)$$

This transformation ensures that

$$\frac{\partial \theta}{\partial \omega} = \frac{J}{\hat{r} R}, \quad (166)$$

and, hence, that

$$\mathcal{J} \equiv \frac{R}{\epsilon} \left( \frac{\partial R}{\partial \theta} \frac{\partial Z}{\partial \hat{r}} - \frac{\partial R}{\partial \hat{r}} \frac{\partial Z}{\partial \theta} \right) = \epsilon R J \frac{\partial \omega}{\partial \theta} = r R^2, \quad (167)$$

in accordance with Eq. (1).

### C. Metric Elements

We can determine the metric elements of the flux-coordinate system by combining Eqs. (161)–(165). Evaluating the elements up to  $\mathcal{O}(\epsilon)$ , but retaining  $\mathcal{O}(\epsilon^2)$  contributions to terms that are independent of  $\omega$ , we obtain,<sup>11,12</sup>

$$L(\hat{r}) = \frac{\hat{r}^3}{8} - \frac{\hat{r} H_1}{2} - \frac{1}{2} \sum_{j>1} (j-1) \frac{H_j^2}{\hat{r}} - \frac{1}{2} \sum_{j>1} (j-1) \frac{V_j^2}{\hat{r}}, \quad (168)$$

$$\begin{aligned} \theta &= \omega + \epsilon \hat{r} \sin \omega - \epsilon \sum_{j>0} \frac{1}{j} \left[ H_j' - (j-1) \frac{H_j}{\hat{r}} \right] \sin(j \omega) \\ &+ \epsilon \sum_{j>1} \frac{1}{j} \left[ V_j' - (j-1) \frac{V_j}{\hat{r}} \right] \cos(j \omega), \end{aligned} \quad (169)$$

$$|\hat{\nabla} \hat{r}|^2 = 1 + 2\epsilon \sum_{j>0} H_j' \cos(j \theta) + 2\epsilon \sum_{j>1} V_j' \sin(j \theta)$$

$$\begin{aligned}
& + \epsilon^2 \left( \frac{3\hat{r}^2}{4} - H_1 + \frac{1}{2} \sum_{j>0} \left[ H_j'^2 + (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] \right. \\
& \left. + \frac{1}{2} \sum_{j>1} \left[ V_j'^2 + (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right] \right), \tag{170}
\end{aligned}$$

$$\begin{aligned}
\hat{\nabla}\hat{r} \cdot \hat{\nabla}\theta &= \epsilon \sin \theta - \epsilon \sum_{j>0} \frac{1}{j} \left[ H_j'' + \frac{H_j'}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \right] \sin(j\theta) \\
& + \epsilon \sum_{j>1} \frac{1}{j} \left[ V_j'' + \frac{V_j'}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \right] \cos(j\theta), \tag{171}
\end{aligned}$$

$$R^2 = 1 - 2\epsilon \hat{r} \cos \theta - \epsilon^2 \left( \frac{\hat{r}^2}{2} - \hat{r} H_1' - 2 H_1 \right). \tag{172}$$

Here,  $' \equiv d/d\hat{r}$ . Moreover, we have made use of the fact that  $V_j \propto H_j$ , for  $j > 1$ , because  $V_j$  and  $H_j$  satisfy the identical differential equations, (178) and (179).

#### D. Expansion of Inverse Grad-Shafranov Equation

Let us write<sup>11,12</sup>

$$f(\hat{r}) = \epsilon \frac{\hat{r} g}{q}, \tag{173}$$

$$g(\hat{r}) = 1 + \epsilon^2 g_2(\hat{r}) + \epsilon^4 g_4(\hat{r}), \tag{174}$$

$$P'(\hat{r}) = \epsilon^2 p_2'(\hat{r}), \tag{175}$$

where  $q$ ,  $g_2$ ,  $g_4$ , and  $p_2$  are all  $\mathcal{O}(1)$ . Here, the safety-factor,  $q(\hat{r})$ , and the second-order plasma pressure gradient,  $p_2'(\hat{r})$ , are the two free flux-surface functions that characterize the plasma equilibrium.

Expanding the inverse Grad-Shafranov equation, (15), order by order in the small parameter  $\epsilon$ , making use of Eqs. (170)–(175), we obtain<sup>11,12,27</sup>

$$g_2' = -p_2' - \frac{\hat{r}}{q^2} (2 - s), \tag{176}$$

$$H_1'' = -(3 - 2s) \frac{H_1'}{\hat{r}} - 1 + \frac{2p_2' q^2}{\hat{r}}, \tag{177}$$

$$H_j'' = -(3 - 2s) \frac{H_j'}{\hat{r}} + (j^2 - 1) \frac{H_j}{\hat{r}^2} \quad \text{for } j > 1, \tag{178}$$

$$V_j'' = -(3 - 2s) \frac{V_j'}{\hat{r}} + (j^2 - 1) \frac{V_j}{\hat{r}^2} \quad \text{for } j > 1, \quad (179)$$

$$g_4' = g_2 \left[ p_2' - \frac{\hat{r}}{q^2} (2 - s) \right] - \frac{\hat{r}}{q} \Sigma + p_2' \left( \frac{\hat{r}^2}{2} + \frac{\hat{r}^2}{q^2} - 2H_1 - 3\hat{r}H_1' \right), \quad (180)$$

where  $s = \hat{r}q'/q$  is the magnetic shear, and

$$\Sigma = \frac{S_2}{q} - \frac{2-s}{q} S_3 \quad (181)$$

$$S_1(\hat{r}) = \frac{1}{2} \sum_{j>0} \left[ 3H_j'^2 - (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] + \frac{1}{2} \sum_{j>1} \left[ 3V_j'^2 - (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right], \quad (182)$$

$$S_2(\hat{r}) = \frac{3\hat{r}^2}{2} - 2\hat{r}H_1' + \sum_{j>0} \left[ H_j'^2 + 2(j^2 - 1) \frac{H_j'H_j}{\hat{r}} - (j^2 - 1) \frac{H_j^2}{\hat{r}^2} \right] \\ + \sum_{j>1} \left[ V_j'^2 + 2(j^2 - 1) \frac{V_j'V_j}{\hat{r}} - (j^2 - 1) \frac{V_j^2}{\hat{r}^2} \right], \quad (183)$$

$$S_3(\hat{r}) = -\frac{3\hat{r}^2}{4} + \frac{\hat{r}^2}{q^2} + H_1 + S_1, \quad (184)$$

$$S_4(\hat{r}) = \frac{7\hat{r}^2}{4} - H_1 - 3\hat{r}H_1' + S_1. \quad (185)$$

Note that the relative horizontal shift of magnetic flux-surfaces,  $-H_1$ , otherwise known as the *Shafranov shift*, is driven by toroidicity [the second term on the right-hand side of Eq. (177)], and plasma pressure gradients (the third term). All of the other shaping terms (i.e., the  $H_j$ , for  $j > 1$ , and the  $V_j$ ) are driven by axisymmetric currents flowing in external magnetic field-coils.

Equations (38), (58), (59), and (173)–(175) yield<sup>11</sup>

$$\alpha_p(\hat{r}) = \frac{p_2' q^2}{\hat{r}} (1 - 2\epsilon^2 g_2), \quad (186)$$

$$\alpha_g(\hat{r}) = \frac{q}{\hat{r}} (g_2' - \epsilon^2 g_2 g_2' + \epsilon^2 g_4'), \quad (187)$$

$$\alpha_f(\hat{r}) = -s + \epsilon^2 \hat{r} g_2'. \quad (188)$$

Finally, it follows from Eqs. (171) and (177)–(179) that

$$\hat{\nabla} \hat{r} \cdot \hat{\nabla} \theta = 2\epsilon \left[ 1 - \frac{p_2' q^2}{\hat{r}} + (1 - s) \frac{H_1'}{\hat{r}} \right] \sin \theta$$

$$\begin{aligned}
& -2\epsilon \sum_{j>1} \frac{1}{j} \left[ -(1-s) \frac{H'_j}{\hat{r}} + (j^2-1) \frac{H_j}{\hat{r}^2} \right] \sin(j\theta) \\
& + 2\epsilon \sum_{j>1} \frac{1}{j} \left[ -(1-s) \frac{V'_j}{\hat{r}} + (j^2-1) \frac{V_j}{\hat{r}^2} \right] \cos(j\theta).
\end{aligned} \tag{189}$$

### E. Calculation of Coupling Coefficients

Equations (170) and (182) yield

$$|\hat{\nabla}\hat{r}|^{-2} = 1 - 2\epsilon \sum_{j>0} H'_j \cos(j\theta) - 2\epsilon \sum_{j>0} V'_j \sin(j\theta) + \epsilon^2 \left( -\frac{3\hat{r}^2}{4} + H_1 + S_1 \right), \tag{190}$$

Equation (172) gives

$$R^{-2} = 1 + 2\epsilon \hat{r} \cos\theta + \epsilon^2 \left( \frac{5\hat{r}^2}{2} - \hat{r} H'_1 - 2H_1 \right). \tag{191}$$

The previous two equations imply that

$$\begin{aligned}
|\hat{\nabla}\hat{r}|^{-2} R^{-2} &= 1 + 2\epsilon \hat{r} \cos\theta - 2\epsilon \sum_{j>0} H'_j \cos(j\theta) - 2\epsilon \sum_{j>1} V'_j \sin(j\theta) \\
&+ \epsilon^2 \left( \frac{7\hat{r}^2}{4} - H_1 - 3\hat{r} H'_1 + S_1 \right).
\end{aligned} \tag{192}$$

Finally, Eqs. (189) and (190) yield

$$\begin{aligned}
\hat{\nabla}\hat{r} \cdot \hat{\nabla}\theta |\hat{\nabla}\hat{r}|^{-2} &= 2\epsilon \left[ 1 - \frac{p'_2 q^2}{\hat{r}} + (1-s) \frac{H'_1}{\hat{r}} \right] \sin\theta \\
&- 2\epsilon \sum_{j>1} \frac{1}{j} \left[ -(1-s) \frac{H'_j}{\hat{r}} + (j^2-1) \frac{H_j}{\hat{r}^2} \right] \sin(j\theta) \\
&+ 2\epsilon \sum_{j>1} \frac{1}{j} \left[ -(1-s) \frac{V'_j}{\hat{r}} + (j^2-1) \frac{V_j}{\hat{r}^2} \right] \cos(j\theta),
\end{aligned} \tag{193}$$

where use has been made of the fact that  $V_j(\hat{r}) \propto H_j(\hat{r})$  for  $j > 1$ .

Equations (70)–(73), (172), (190), (192), and (193) imply that

$$\begin{aligned}
a_m^{m'} &= \delta_m^{m'} - \epsilon \hat{r} (\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon^2 \left( \frac{\hat{r}^2}{2} - \hat{r} H'_1 - 2H_1 \right) \delta_m^{m'}, \\
b_m^{m'} &= \delta_m^{m'} + \epsilon \hat{r} (\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon \sum_{j>0} H'_j (\delta_{m'-m-j} + \delta_{m'-m+j})
\end{aligned} \tag{194}$$

$$- \epsilon \sum_{j>1} i V_j' (\delta_{m'-m-j} - \delta_{m'-m+j}) + \epsilon^2 \left( \frac{7 \hat{r}^2}{4} - H_1 - 3 \hat{r} H_1' + S_1 \right) \delta_m^{m'}, \quad (195)$$

$$\begin{aligned} c_m^{m'} &= \delta_m^{m'} - \epsilon \sum_{j>0} H_j' (\delta_{m'-m-j} + \delta_{m'-m+j}) - \epsilon \sum_{j>1} i V_j' (\delta_{m'-m-j} - \delta_{m'-m+j}) \\ &\quad + \epsilon^2 \left( -\frac{3 \hat{r}^2}{4} + H_1 + S_1 \right) \delta_m^{m'}, \end{aligned} \quad (196)$$

$$\begin{aligned} f_m^{m'} &= -\epsilon [\hat{r} - p_2' q^2 + (1-s) H_1'] (\delta_{m'-m-1} - \delta_{m'-m+1}) \\ &\quad + \epsilon \sum_{j>1} \frac{1}{j} \left[ -(1-s) H_j' + (j^2 - 1) \frac{H_j}{\hat{r}} \right] (\delta_{m'-m-j} - \delta_{m'-m+j}) \\ &\quad + \epsilon \sum_{j>1} \frac{i}{j} \left[ -(1-s) V_j' + (j^2 - 1) \frac{V_j}{\hat{r}} \right] (\delta_{m'-m-j} + \delta_{m'-m+j}). \end{aligned} \quad (197)$$

If we write

$$\alpha_g = \alpha_g^{(0)} + \epsilon^2 \alpha_g^{(2)}, \quad (198)$$

$$\alpha_p = \alpha_p^{(0)} + \epsilon^2 \alpha_p^{(2)}, \quad (199)$$

$$\alpha_f = \alpha_f^{(0)} + \epsilon^2 \alpha_f^{(2)}, \quad (200)$$

$$a_m^{m'} = 1 + \epsilon a_m^{m'(1)} + \epsilon^2 a_m^{m'(2)}, \quad (201)$$

$$b_m^{m'} = 1 + \epsilon b_m^{m'(1)} + \epsilon^2 b_m^{m'(2)}, \quad (202)$$

$$D_m^{m'} = D_m^{m'(0)} + \epsilon D_m^{m'(1)} + \epsilon^2 D_m^{m'(2)}, \quad (203)$$

where  $\alpha_g^{(0)}$ ,  $\alpha_g^{(2)}$ , et cetera, are  $\mathcal{O}(1)$ , then it follows from Eq. (69) that

$$D_m^{m(0)} = -\alpha_f^{(0)} \alpha_p^{(0)} - \hat{r} \alpha_p'^{(0)} - q \hat{r} \alpha_g'^{(0)} + m^2, \quad (204)$$

$$D_m^{m'(1)} = -\left[ \alpha_f^{(0)} \alpha_p^{(0)} + \hat{r} \alpha_p'^{(0)} \right] a_m^{m'(1)} + m m' b_m^{m'(1)}, \quad (205)$$

$$\begin{aligned} D_m^{m(2)} &= -\left[ \alpha_f^{(0)} \alpha_p^{(0)} + \hat{r} \alpha_p'^{(0)} \right] a_m^{m'(2)} - \alpha_f^{(0)} \alpha_p^{(2)} - \alpha_f^{(2)} \alpha_p^{(0)} - \hat{r} \alpha_p'^{(2)} - q \hat{r} \alpha_g'^{(2)} \\ &\quad - \hat{r}^2 [\alpha_g^{(0)}]^2 + m^2 b_m^{m(2)}. \end{aligned} \quad (206)$$

Finally, Eqs. (70)–(73), (194)–(197), and (204)–(206) give the following expressions for the coupling coefficients appearing in the axisymmetric ideal-MHD o.d.e.s, (64) and (74):

$$A_m^m(\hat{r}) = 1 + \epsilon^2 \left( -\frac{3 \hat{r}^2}{4} + H_1 + S_1 \right), \quad (207)$$

$$A_m^{m\pm 1}(\hat{r}) = -\epsilon H'_1, \quad (208)$$

$$A_m^{m\pm j}(\hat{r}) = -\epsilon (H'_j \pm i V'_j) \quad \text{for } j > 1, \quad (209)$$

$$B_m^m(\hat{r}) = 0, \quad (210)$$

$$B_m^{m\pm 1}(\hat{r}) = \pm \epsilon (m \pm 1) [\hat{r} - p'_2 q^2 + (1-s) H'_1], \quad (211)$$

$$B_m^{m\pm j}(\hat{r}) = \pm \epsilon \frac{m \pm j}{j} \left[ (1-s) (H'_j \pm i V'_j) - (j^2 - 1) \frac{H_j \pm i V_j}{\hat{r}} \right] \quad \text{for } j > 1, \quad (212)$$

$$C_m^m(\hat{r}) = 0, \quad (213)$$

$$C_m^{m\pm 1}(\hat{r}) = \pm \epsilon m [\hat{r} - p'_2 q^2 + (1-s) H'_1], \quad (214)$$

$$C_m^{m\pm j}(\hat{r}) = \pm \epsilon \frac{m}{j} \left[ (1-s) (H'_j \pm i V'_j) - (j^2 - 1) \frac{H_j \pm i V_j}{\hat{r}} \right] \quad \text{for } j > 1, \quad (215)$$

$$\begin{aligned} D_m^m(\hat{r}) = & m^2 + q \hat{r} \frac{d}{d\hat{r}} \left( \frac{2-s}{q} \right) + \epsilon^2 m^2 S_4 \\ & + \epsilon^2 \left\{ -\hat{r}^2 \left( \frac{2-s}{q} \right)^2 + q \hat{r} \frac{d\Sigma}{d\hat{r}} - \hat{r} \frac{d}{d\hat{r}} (\hat{r} p'_2) - 2(1-s) \hat{r} p'_2 \right. \\ & \left. + 2 \hat{r} p'_2 q^2 \left( -2 + \frac{3 p'_2 q^2}{\hat{r}} \right) + 2 H'_1 q^2 \left[ \frac{d}{d\hat{r}} (\hat{r} p'_2) - 4(1-s) p'_2 \right] \right\}, \end{aligned} \quad (216)$$

$$D_m^{m\pm 1}(\hat{r}) = \epsilon q^2 \left[ \frac{d}{d\hat{r}} (\hat{r} p'_2) - (2-s) p'_2 \right] + \epsilon m (m \pm 1) (\hat{r} - H'_1), \quad (217)$$

$$D_m^{m\pm j}(\hat{r}) = -\epsilon m (m \pm j) (H'_j \pm i V'_j) \quad \text{for } j > 1. \quad (218)$$

Our system of equations is now complete.

## F. Toroidal Plasma Current

The net toroidal plasma current flowing in the plasma is given by

$$I_p = \oint_{r=a} \mathcal{J} \nabla \phi \times \nabla r \cdot \mathbf{B} d\theta = \oint_{r=a} B_\theta d\theta, \quad (219)$$

which yields

$$I_p = \gamma_{\text{shape}} I_{p\text{cly}}, \quad (220)$$

where

$$I_{p\text{ cyl}} = \frac{2\pi a^2 g(a)}{q(a)} \quad (221)$$

is the plasma current predicted by cylindrical theory, whereas

$$\gamma_{\text{shape}} = \oint_{r=a} |\hat{\nabla} \hat{r}|^2 \frac{d\theta}{2\pi} \quad (222)$$

is the factor by which the toroidal current is increased due to the shaping of the plasma's poloidal boundary.

### G. Plasma Self-Inductance

The dimensionless self-inductance of the plasma is defined as <sup>11</sup>

$$l_i = \frac{2 \int_0^1 \hat{r} f^2 \langle |\hat{\nabla} \hat{r}|^2 \rangle d\hat{r}}{(f \langle |\hat{\nabla} \hat{r}|^2 \rangle)_{\hat{r}=1}^2}. \quad (223)$$

### H. Behavior Close to Magnetic Axis

When  $\hat{r} \ll 1$ , the well-behaved solution of the axisymmetric ideal-MHD o.d.e.s, (64) and (65), that is dominated by the poloidal harmonic whose poloidal mode number is  $m$  is such that

$$y_m(\hat{r}) = \hat{r}^{|m|}, \quad (224)$$

$$Z_m(\hat{r}) = |m| \hat{r}^{|m|}, \quad (225)$$

with  $y_{m'}(\hat{r}) = Z_{m'}(\hat{r}) = 0$  for  $m' \neq 0$ .

## VIII. RESULTS

### IX. SUMMARY AND DISCUSSION

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## DATA AVAILABILITY STATEMENT

The digital data used in the figures in this paper can be obtained from the author upon reasonable request.

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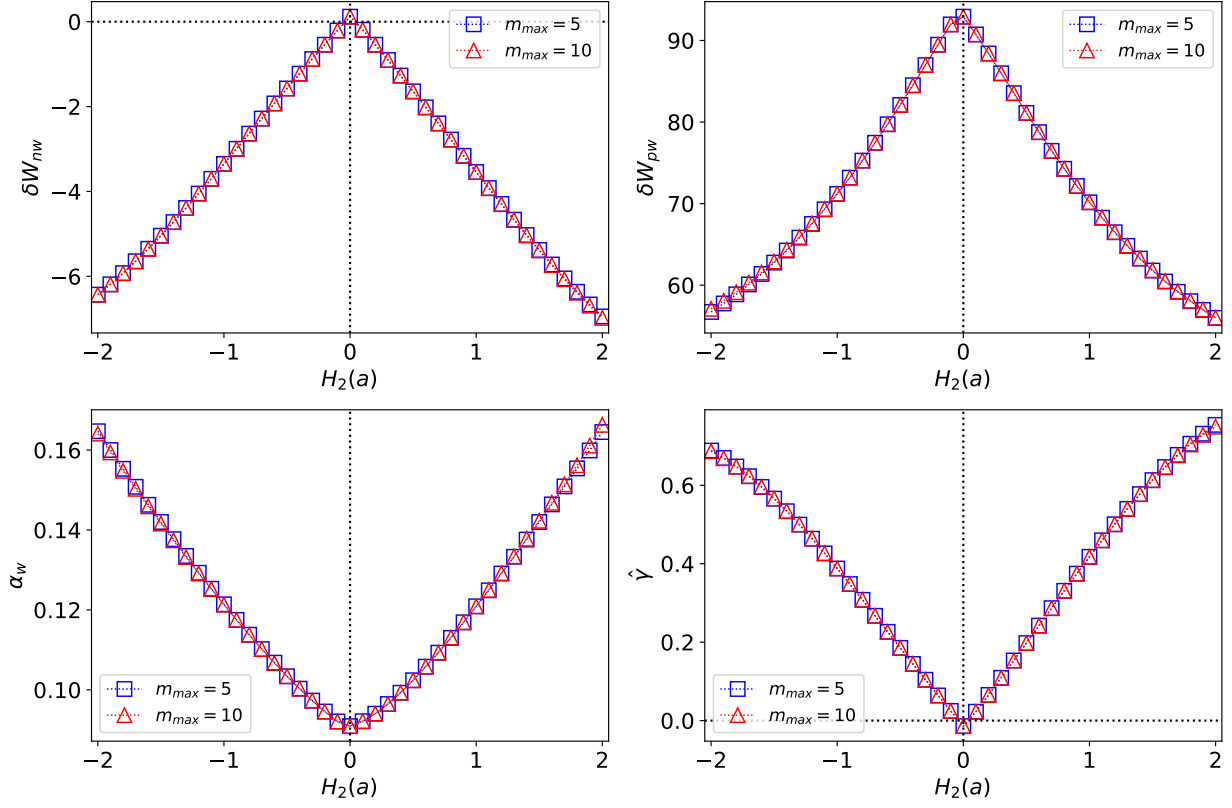


FIG. 1. The no-wall perturbed potential energy,  $\delta W_{nw}$ , the perfect-wall perturbed potential energy,  $\delta W_{pw}$ , the wall parameter,  $\alpha_w$ , and the normalized thin-wall growth-rate,  $\hat{\gamma}$ , of the unstable axisymmetric mode, plotted as a function of the plasma elongation,  $H_2(a)$ , for plasma equilibria characterized by  $\epsilon = 0.1$ ,  $V_2(a) = H_3(a) = 0$ ,  $q(0) = 1.01$ ,  $q(a) = 3.6$ ,  $p_2(0) = 0.05$ , and  $b_w = 1.1$ .

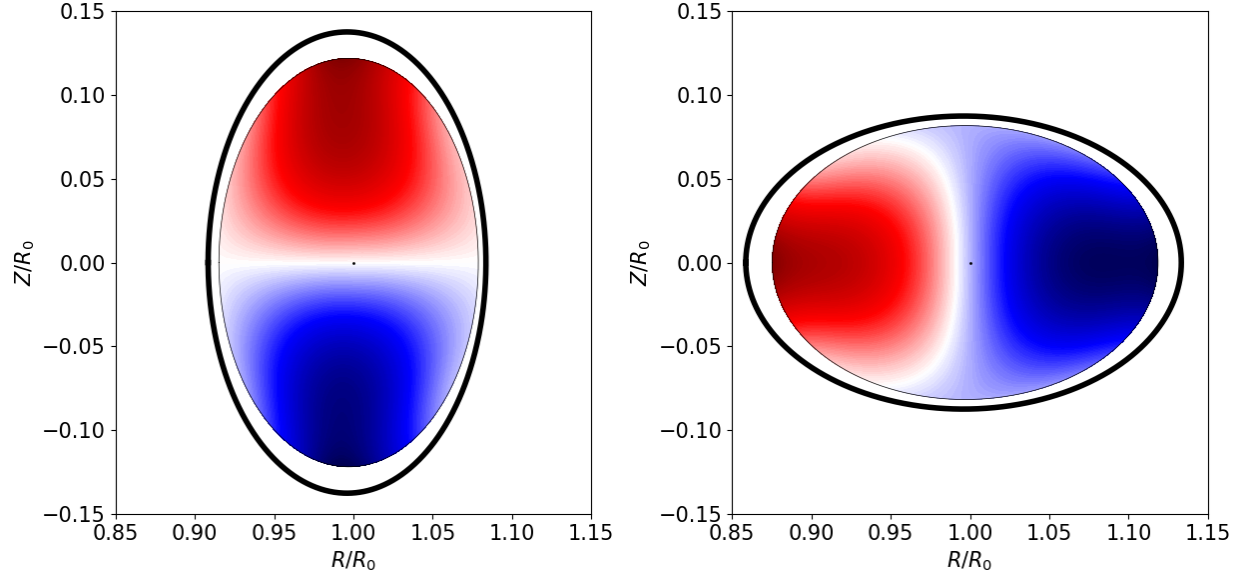


FIG. 2. The perfect-wall, ideal, radial plasma displacement function,  $y(r, \theta) = f \xi^r$ , for the cases  $H_2(a) = +2$  (left panel) and  $H_2(a) = -2$  (right-panel) that feature in Fig. 1. Here, red is positive and blue is negative. The thick black line indicates the wall.

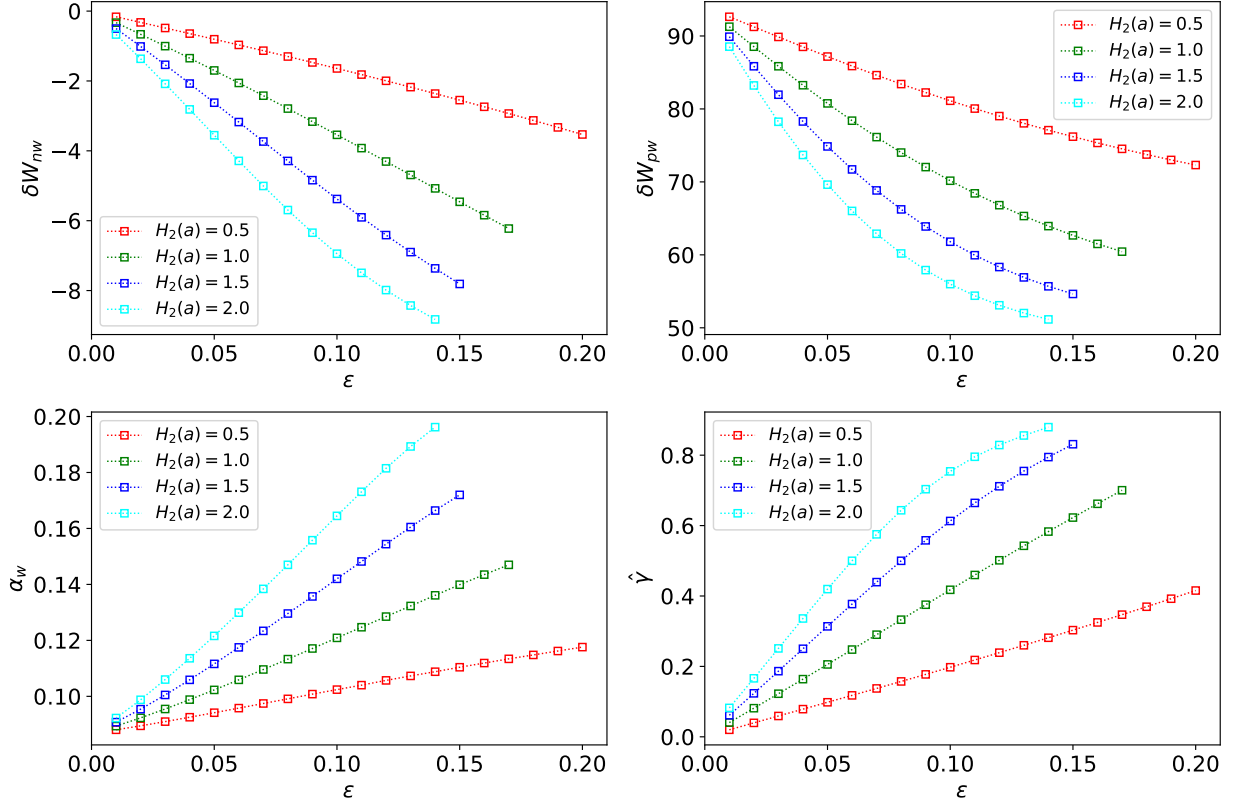


FIG. 3. The no-wall perturbed potential energy,  $\delta W_{nw}$ , the perfect-wall perturbed potential energy,  $\delta W_{pw}$ , the wall parameter,  $\alpha_w$ , and the normalized thin-wall growth-rate,  $\hat{\gamma}$ , of the unstable axisymmetric mode, plotted as a function of the plasma inverse aspect-ratio,  $\epsilon$ , for plasma equilibria characterized by  $V_2(a) = 0$ ,  $H_3(a) = 0$ ,  $q(0) = 1.01$ ,  $q(a) = 3.6$ ,  $p_2(0) = 0.05$ ,  $b_w = 1.1$ , and various different values of  $H_2(a)$ .

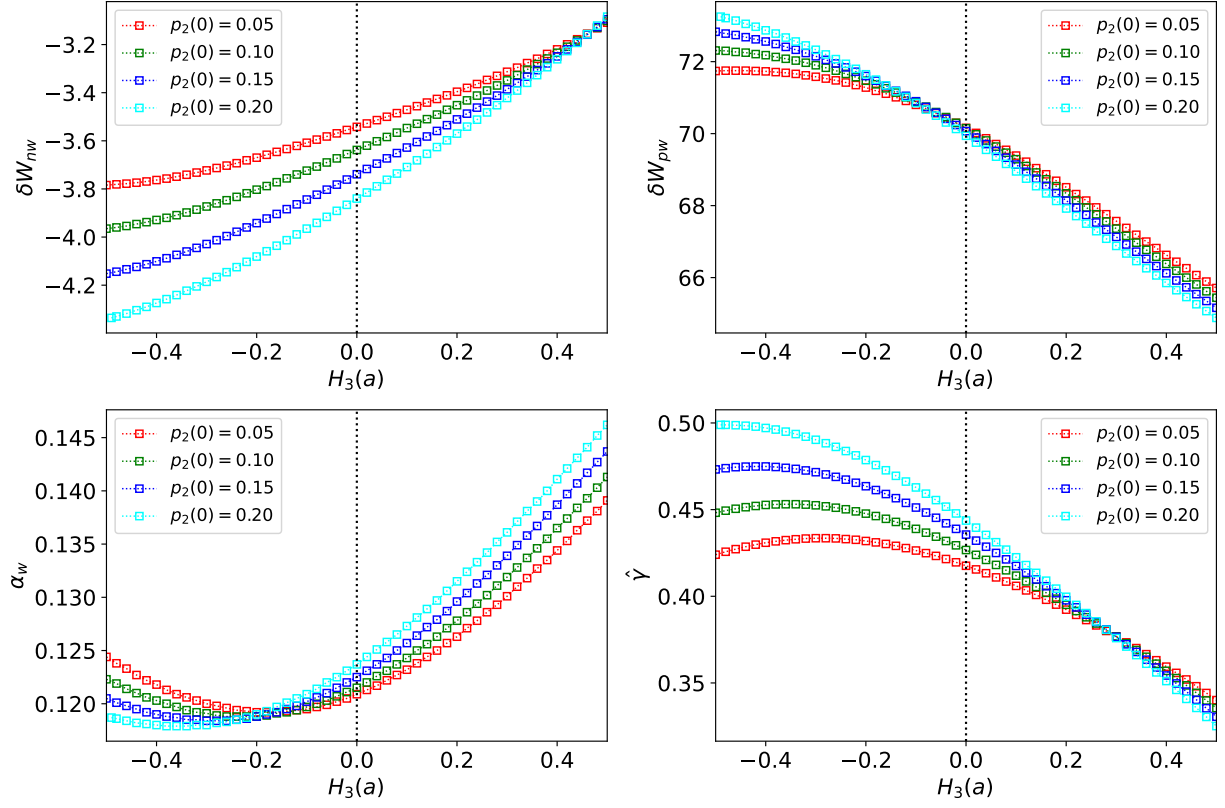


FIG. 4. The no-wall perturbed potential energy,  $\delta W_{nw}$ , the perfect-wall perturbed potential energy,  $\delta W_{pw}$ , the wall parameter,  $\alpha_w$ , and the normalized thin-wall growth-rate,  $\hat{\gamma}$ , of the unstable axisymmetric mode, plotted as a function of the plasma triangularity,  $H_3(a)$ , for plasma equilibria characterized by  $\epsilon = 0.1$ ,  $H_2(a) = 1$ ,  $V_2(a) = 0$ ,  $q(0) = 1.01$ ,  $q(a) = 3.6$ ,  $b_w = 1.1$ , and various different values of  $p_2(0)$ .

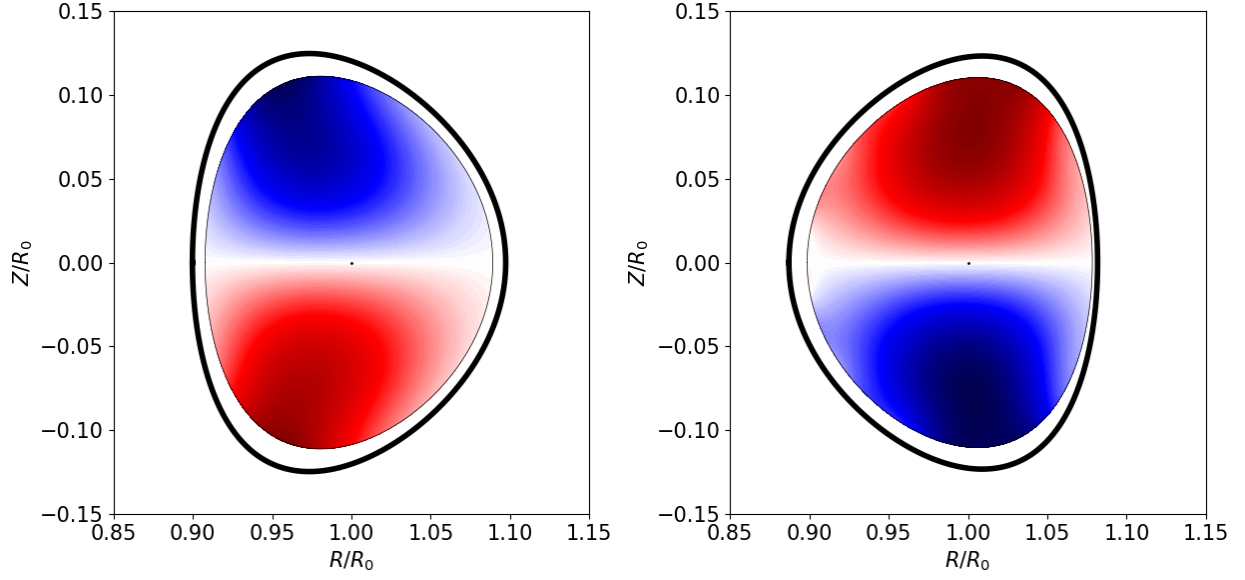


FIG. 5. The perfect-wall, ideal, radial plasma displacement function,  $y(r, \theta) = f \xi^r$ , for the cases  $H_3(a) = +0.5$ ,  $p_2(0) = 0.2$  (left panel) and  $H_3(a) = -0.5$ ,  $p_2(0) = 0.2$  (right-panel) that feature in Fig. 4. Here, red is positive and blue is negative. The thick black line indicates the wall.

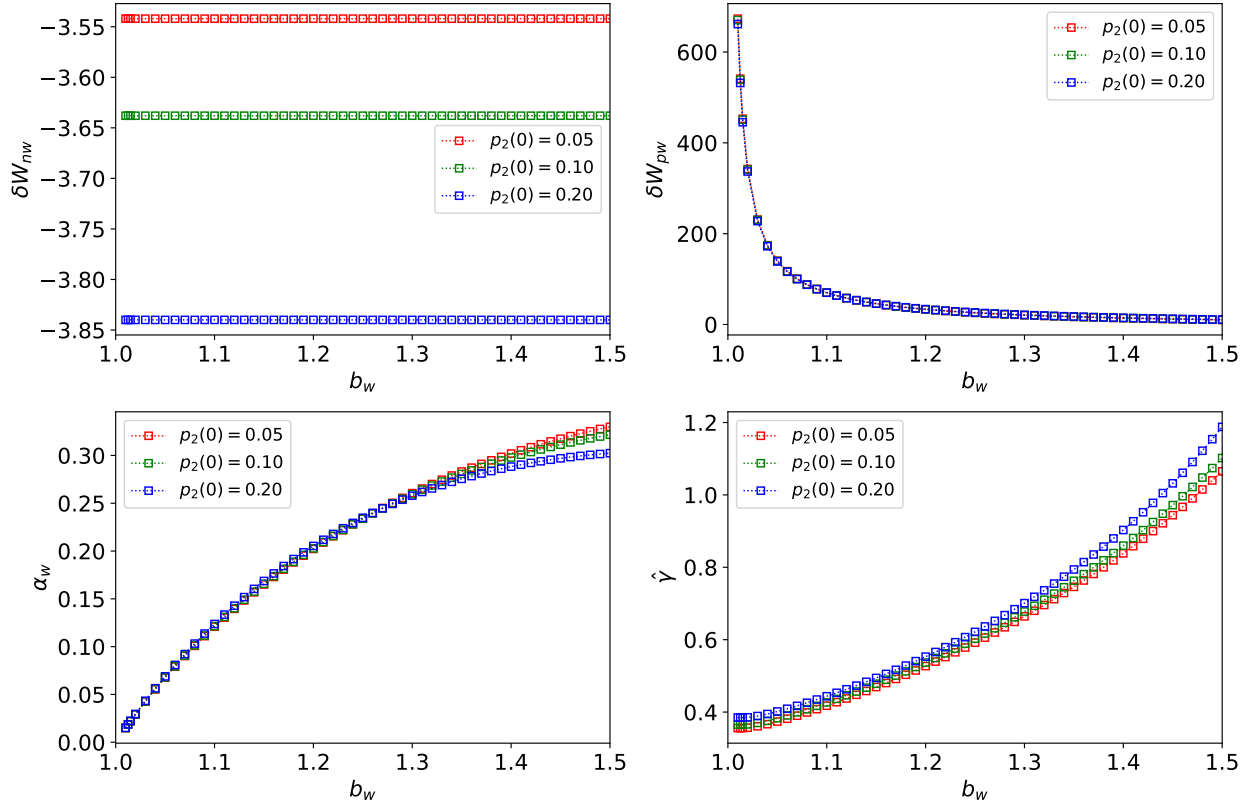


FIG. 6. The no-wall perturbed potential energy,  $\delta W_{nw}$ , the perfect-wall perturbed potential energy,  $\delta W_{pw}$ , the wall parameter,  $\alpha_w$ , and the normalized thin-wall growth-rate,  $\hat{\gamma}$ , of the unstable axisymmetric mode, plotted as a function of the relative wall radius  $b_w$ , for plasma equilibria characterized by  $\epsilon = 0.1$ ,  $H_2(a) = 1$ ,  $V_2(a) = H_3(a) = 0$ ,  $q(0) = 1.01$ ,  $q(a) = 3.6$ , and various different values of  $p_2(0)$ .

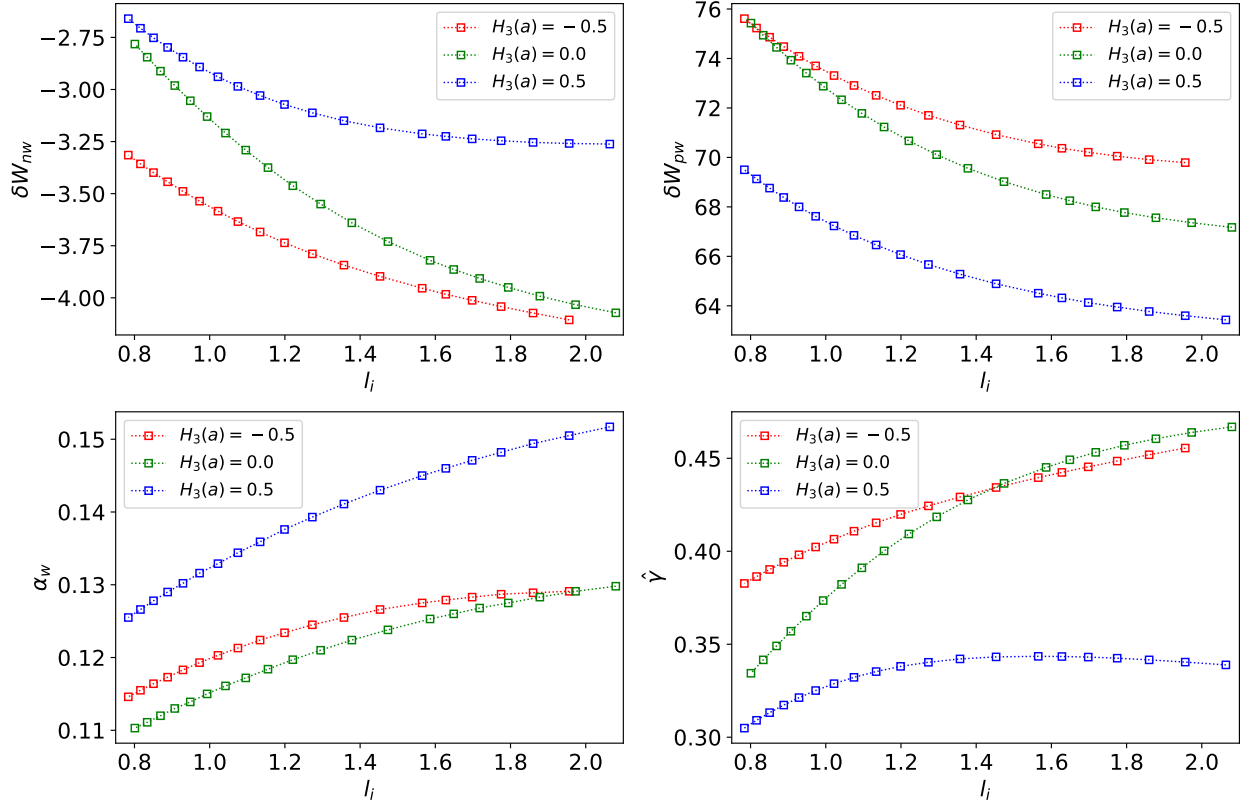


FIG. 7. The no-wall perturbed potential energy,  $\delta W_{nw}$ , the perfect-wall perturbed potential energy,  $\delta W_{pw}$ , the wall parameter,  $\alpha_w$ , and the normalized thin-wall growth-rate,  $\hat{\gamma}$ , of the unstable axisymmetric mode, plotted as a function of the plasma self-inductance,  $l_i$ , for plasma equilibria characterized by  $\epsilon = 0.1$ ,  $H_2(a) = 1$ ,  $V_2(a) = 0$ ,  $q(a) = 3.6$ ,  $p_2 = 0.05$ , and various values of  $H_3(a)$ .



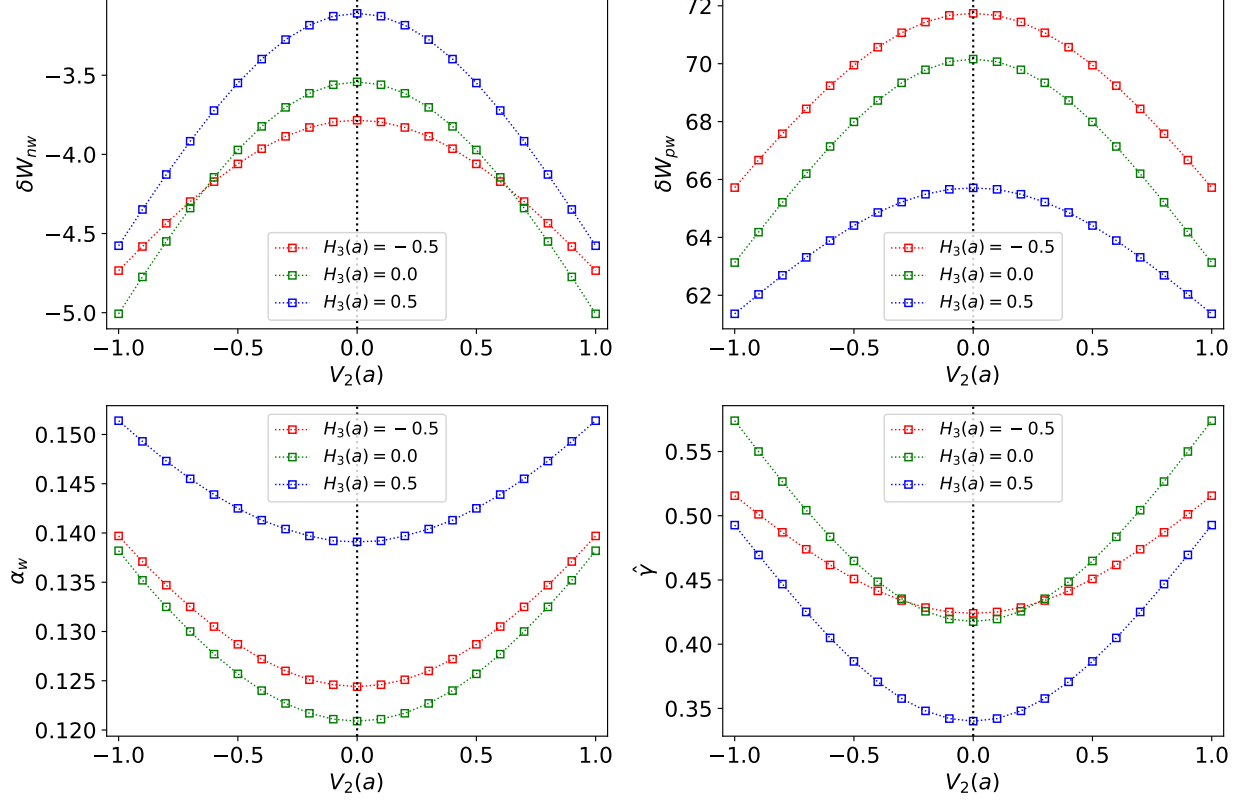


FIG. 8. The no-wall perturbed potential energy,  $\delta W_{nw}$ , the perfect-wall perturbed potential energy,  $\delta W_{pw}$ , the wall parameter,  $\alpha_w$ , and the normalized thin-wall growth-rate,  $\hat{\gamma}$ , of the unstable axisymmetric mode, plotted as a function of the plasma tilt,  $V_2(a)$ , for plasma equilibria characterized by  $\epsilon = 0.1$ ,  $H_2(a) = 1$ ,  $H_3(a) = 0$ ,  $q(0) = 1.01$ ,  $q(a) = 3.6$ ,  $p_2 = 0.05$ , and various different values of  $H_3(a)$ .

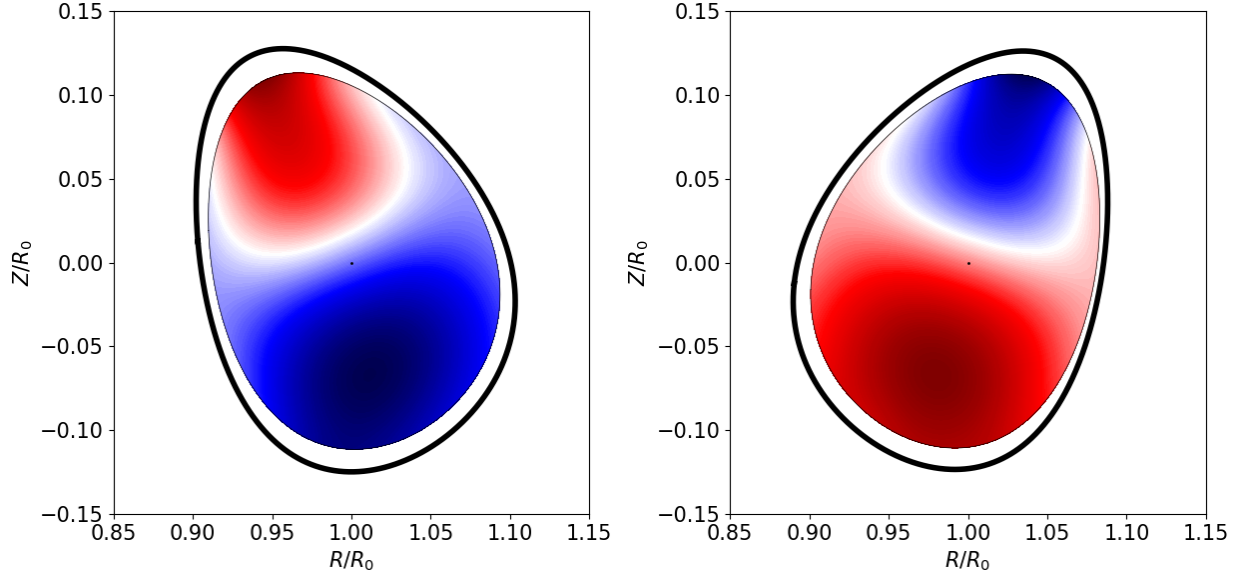


FIG. 9. The real part of the perfect-wall, ideal, radial plasma displacement function,  $\text{Re}[y(r, \theta)] = \text{Re}(f \xi^r)$ , for the cases  $V_2(a) = -1.0$ ,  $H_3(a) = 0.5$  (left panel) and  $V_2(a) = 1.0$ ,  $H_3(a) = -0.5$  (right-panel) that feature in Fig. 8. Here, red is positive and blue is negative. The thick black line indicates the wall.