Solution of Layer Equations

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I. DERIVATION OF LAYER EQUATIONS

A. Outer Region

Let r, θ, φ be right-handed cylindrical coordinates. Throughout the bulk of the plasma, the perturbed magnetic field can be written [1,2]

$$\delta B_r = \mathrm{i}\,\frac{m}{r}\,\delta\psi, \tag{1}$$

$$\delta B_\theta = -\frac{\partial\delta\psi}{\partial r}, \tag{2}$$

$$\delta B_\varphi \simeq 0, \tag{3}$$

where

$$\delta\psi(r,\zeta,t) = \psi_1(r,t)\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{4}$$

and $\zeta = m\,\theta - n\,\varphi$. Here, m and n are poloidal and toroidal mode numbers, respectively.

B. Inner Region

The *rational surface* lies at radius r_s, where $q(r_s) = m/n$, and $q(r)$ is the *safety-factor* profile. The inner region is a thin (in r) layer centered on the rational surface. The inner region is assumed to be governed by the four-field model described in Ref. 3. This model is an extension of a model introduced in Ref. 4, and is ultimately based on the four-field model of Hazeltine, et alia.[5]

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C. Useful Definitions

The plasma is assumed to consist of two species. First, electrons of mass m_e, electrical charge $-e$, number density n_e, and temperature T_e. Second, ions of mass m_i, electrical charge $+e$, number density n_e, and temperature T_i. Let $p = n_e\,(T_e + T_i)$ be the total plasma pressure.

It is helpful to define $n_0 = n_e(r_s)$, $p_0 = p(r_s)$,

$$\eta_e = \left.\frac{d\ln T_e}{d\ln n_e}\right|_{r=r_s}, \tag{5}$$

$$\eta_i = \left.\frac{d\ln T_i}{d\ln n_e}\right|_{r=r_s}, \tag{6}$$

$$\iota = \left(\frac{T_e}{T_i}\right)_{r=r_s}\left(\frac{1+\eta_e}{1+\eta_i}\right), \tag{7}$$

where $n_e(r)$, $p(r)$, $T_e(r)$, and $T_i(r)$ refer to density, pressure, and temperature profiles that are unperturbed by the magnetic perturbation. Note that ι is the ratio of the electron to the ion pressure gradient at the rational surface.

For the sake of simplicity, the perturbed electron and ion temperature profiles are assumed to be functions of the perturbed electron number density profile in the immediate vicinity of the rational surface. In other words, $T_e = T_e(n_e)$ and $T_i = T_i(n_e)$. This implies that $p = p(n_e)$. The MHD velocity, which is the velocity of a fictional MHD fluid, is defined as $\mathbf{V} = \mathbf{V}_E + V_{\parallel\,i}\,\mathbf{b}$, where $\mathbf{V}_E$ is the E-cross-B drift velocity, $V_{\parallel\,i}$ is the parallel component of the ion fluid velocity, and $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$. Here, $\mathbf{B} = [0,\,B_\theta(r),\,B_\varphi]$ is the equilibrium magnetic field. Note that $q(r) = r\,B_\varphi/[R_0\,B_\theta(r)]$, where R_0 is the plasma minor radius.

D. Dimensionless Fields

The four dimensionless fields in the four-field model—namely, ψ, ϕ, N, and V—have the following definitions:

$$\nabla\psi = \frac{\mathbf{n}\times\mathbf{B}}{r_s\,B_\varphi}, \tag{8}$$

$$\nabla\phi = \frac{\mathbf{n}\times\mathbf{V}}{r_s\,V_A}, \tag{9}$$

$$N = -\hat{d}_i\left(\frac{p-p_0}{B_\varphi^2/\mu_0}\right), \tag{10}$$

$$V = \hat{d}_i \left(\frac{\mathbf{n}\cdot\mathbf{V}}{V_A}\right). \tag{11}$$

Here, $\mathbf{n} = (0,\, \epsilon/q_s,\, 1)$, $\epsilon = r/R_0$, $q_s = m/n$, $V_A = B_\varphi/\sqrt{\mu_0\, n_0\, m_i}$ is the *Alfvén speed*, $d_i = \sqrt{m_i/(n_0\, e^2\, \mu_0)}$ is the *collisionless ion skin-depth*, and $\hat{d}_i = d_i/r_s$. Our model also employs the auxiliary dimensionless field

$$J = -\frac{2\,\epsilon_s}{q_s} + \hat{\nabla}^2\psi, \tag{12}$$

where $\epsilon_s = r_s/R_0$, and $\hat{\nabla} = r_s\,\nabla$. Note that

$$\psi(r,\zeta,t) = \frac{\psi_0(r) + \psi_1(r,t)\,\mathrm{e}^{\mathrm{i}\,\zeta}}{r_s\, B_\varphi}, \tag{13}$$

where

$$\psi_0(r) = \frac{B_\varphi}{R_0}\int_{r_s}^{r} r' \left[\frac{n}{m} - \frac{1}{q(r')}\right] dr', \tag{14}$$

and $\psi_1(r,t)$ is defined in Eqs. (1)–(4).

E. Four-Field Model

Our four-field model takes the form:[2–4,6]

$$\frac{\partial\psi}{\partial\hat{t}} = [\phi,\psi] - \iota_e\,[N,\psi] + \hat{\eta}_\parallel\, J + \hat{E}_\parallel, \tag{15}$$

$$\frac{\partial\hat{\nabla}^2\phi}{\partial\hat{t}} = [\phi,\hat{\nabla}^2\phi] - \frac{\iota_i}{2}\left(\hat{\nabla}^2[\phi,N] + [\hat{\nabla}^2\phi,N] + [\hat{\nabla}^2 N,\phi]\right) + [J,\psi] + \hat{\chi}_\varphi\,\hat{\nabla}^4(\phi + \iota_i\,N)\,, \tag{16}$$

$$\frac{\partial N}{\partial\hat{t}} = [\phi,N] + c_\beta^2\,[V,\psi] + \hat{d}_\beta^2\,[J,\psi] + \hat{\chi}_\perp\,\hat{\nabla}^2 N, \tag{17}$$

$$\frac{\partial V}{\partial\hat{t}} = [\phi,V] + [N,\psi] + \hat{\chi}_\varphi\,\hat{\nabla}^2 V. \tag{18}$$

Here, $[A,B] \equiv \hat{\nabla}A\times\hat{\nabla}B\cdot\mathbf{n}$, $\iota_e = \iota/(1+\iota)$, $\iota_i = 1/(1+\iota)$, $\hat{t} = t/(r_s/V_A)$, $\hat{\eta}_\parallel = \eta_\parallel/(\mu_0\, r_s\, V_A)$, $\hat{E}_\parallel = E_\parallel/(B_\varphi\, V_A)$, $\hat{\chi}_\varphi = \chi_\varphi/(r_s\, V_A)$, $\hat{\chi}_\perp = \chi_\perp/(r_s\, V_A)$, where $\eta_\parallel$ is the *parallel plasma electrical resistivity* at the rational surface, $E_\parallel$ the parallel inductive electric field that maintains the equilibrium toroidal plasma current in the vicinity of the rational surface, χ_φ the *anomalous perpendicular ion momentum diffusivity* at the rational surface, and $\chi_\perp$ the *anomalous perpendicular energy diffusivity* at the rational surface. Moreover, $d_\beta = c_\beta\, d_i$, and $\hat{d}_\beta = d_\beta/r_s$, where $c_\beta = \sqrt{\beta/(1+\beta)}$, and $\beta = (5/3)\,\mu_0\, p_0/B_\varphi^2$. Here, d_β is usually referred to as the *ion* *sound radius*. Note that we have neglected parallel energy transport in Eq. (17) because we expect the radial width of the inner region to be less than the critical width above which parallel transport forces the temperature and density profiles to be magnetic flux-surface functions.[7]

F. Matching to Plasma Equilibrium

The unperturbed plasma equilibrium is such that $\mathbf{B} = [0,\, B_\theta(r),\, B_\varphi]$, $p = p(r)$, $\mathbf{V} = [0,\, V_E(r),\, V_\varphi(r)]$, where $V_E(r) \simeq E_r/B_\varphi$ is the (dominant θ-component of the) E-cross-B drift velocity. Now, the inner region is assumed to have a radial thickness that is much smaller than r_s. Hence, we only need to evaluate plasma equilibrium quantities in the immediate vicinity of the rational surface. Equations (8)–(11) and (13)–(14) suggest that

$$\psi(\hat{x}) = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s}, \tag{19}$$

$$\phi(\hat{x}) = -\hat{V}_E\,\hat{x}, \tag{20}$$

$$N(\hat{x}) = -\hat{V}_\ast\,\hat{x}, \tag{21}$$

$$V = \hat{V}_\parallel, \tag{22}$$

where $\hat{L}_s = L_s/r_s$, $L_s = R_0\,q_s/s_s$, $\hat{V}_E = V_E(r_s)/V_A$, $\hat{V}_\ast = V_\ast(r_s)/V_A$, $V_\ast(r) = (dp/dr)/(e\,n_0\,B_\varphi)$ is the (dominant θ-component of the) diamagnetic velocity, and $\hat{V}_\parallel = \hat{d}_i\,V_\varphi(r_s)/V_A$. Here, $s_s = s(r_s)$, and $s(r) = d\ln q/\ln r$ is the *magnetic shear*. We also deduce that

$$J = -\left(\frac{2}{s_s} - 1\right)\frac{1}{\hat{L}_s}, \tag{23}$$

and $\hat{E}_\parallel = (2/s_s - 1)(\hat{\eta}_\parallel/\hat{L}_s)$.

G. Linearized Layer Equations

In accordance with Eqs. (12), (13), and (19)–(23), let us write

$$\psi(\hat{x},\zeta,\hat{t}) = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s} + \tilde{\psi}(\hat{x},\hat{t})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{24}$$

$$\phi(\hat{x},\zeta,\hat{t}) = -\hat{V}_E\,\hat{x} + \tilde{\phi}(\hat{x},\hat{t})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{25}$$

$$N(\hat{x},\zeta,\hat{t}) = -\hat{V}_\ast\,\hat{x} + \iota_e\,\tilde{N}(\hat{x},\hat{t})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{26}$$

$$V(\hat{x},\zeta,\hat{t}) = \hat{V}_\parallel + \iota_e\,\tilde{V}(\hat{x},\hat{t})\,\mathrm{e}^{\mathrm{i}\,\zeta}, \tag{27}$$

$$J(\hat{x},\zeta,\hat{t}) = -\left(\frac{2}{s_s}-1\right)\frac{1}{\hat{L}_s} + \hat{\nabla}^2\tilde{\psi}(\hat{x},\hat{t})\,\mathrm{e}^{\mathrm{i}\,\zeta}. \tag{28}$$

Substituting Eqs. (24)–(28) into Eqs. (15)–(18), and only retaining terms that are first order in perturbed quantities, we obtain the following set of linearized layer equations:

$$\tau_H\,\frac{\partial\tilde{\psi}}{\partial t} + \mathrm{i}\,(\omega_E+\omega_{\ast\,e})\,\tau_H\,\tilde{\psi} = -\mathrm{i}\,\hat{x}\,(\tilde{\phi}-\tilde{N}) + S^{-1}\,\hat{\nabla}^2\tilde{\psi}, \tag{29}$$

$$\tau_H\,\frac{\partial\hat{\nabla}^2\tilde{\phi}}{\partial t} + \mathrm{i}\,(\omega_E+\omega_{\ast\,i})\,\tau_H\,\hat{\nabla}^2\tilde{\phi} = -\mathrm{i}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\varphi\,\hat{\nabla}^4\left(\tilde{\phi}+\frac{\tilde{N}}{\iota}\right), \tag{30}$$

$$\begin{aligned}\tau_H\,\frac{\partial\tilde{N}}{\partial t} + \mathrm{i}\,\omega_E\,\tau_H\,\tilde{N} = {}&-\mathrm{i}\,\omega_{\ast\,e}\,\tau_H\,\tilde{\phi} - \mathrm{i}\,c_\beta^2\,\hat{x}\,\tilde{V} - \mathrm{i}\,\iota_e\,\hat{d}_\beta^2\,\hat{x}\,\hat{\nabla}^2\tilde{\psi}\\ &+ S^{-1}\,P_\perp\,\hat{\nabla}^2\tilde{N},\end{aligned} \tag{31}$$

$$\tau_H\,\frac{\partial\tilde{V}}{\partial t} + \mathrm{i}\,\omega_E\,\tau_H\,\tilde{V} = \mathrm{i}\,\omega_{\ast\,e}\,\tau_H\,\tilde{\psi} - \mathrm{i}\,\hat{x}\,\tilde{N} + S^{-1}\,P_\varphi\,\hat{\nabla}^2\tilde{V}. \tag{32}$$

Here, $\tau_H = L_s/(m\,V_A)$ is the *hydromagnetic time*, $\omega_E = (m/r_s)\,V_E(r_s)$ the *E-cross-B frequency*, $\omega_{\ast\,e} = -\iota_e\,(m/r_s)\,V_\ast(r_s)$ the *electron diamagnetic frequency*, $\omega_{\ast\,i} = \iota_i\,(m/r_s)\,V_\ast(r_s)$ the *ion diamagnetic frequency*, $S = \tau_R/\tau_H$ the *Lundquist number*, $\tau_R = \mu_0\,r_s^2/\eta_\parallel$ the *resistive diffusion time*, $\tau_\varphi = r_s^2/\chi_\varphi$ the *toroidal momentum confinement time*, and $\tau_\perp = r_s^2/\chi_\perp$ the *energy confinement time*. Furthermore, $P_\varphi = \tau_R/\tau_\varphi$ and $P_\perp = \tau_R/\tau_\perp$ are *magnetic Prandtl numbers*. Note that $\iota = -\omega_{\ast\,e}/\omega_{\ast\,i}$ and $\iota_e = \omega_{\ast\,e}/(\omega_{\ast\,e}-\omega_{\ast\,i})$.

In the following, in accordance with the analysis of Refs. 4 and 3, we shall neglect the term $-\mathrm{i}\,c_\beta^2\,\hat{x}\,\tilde{V}$ in Eq. (31). This approximation can be justified *a posteriori*. It is equivalent to the neglect of ion parallel dynamics in the inner region, and effectively converts our four-field model into the following three-field model:

$$\tau_H\,\frac{\partial\tilde{\psi}}{\partial t} + \mathrm{i}\,(\omega_E+\omega_{\ast\,e})\,\tau_H\,\tilde{\psi} = -\mathrm{i}\,\hat{x}\,(\tilde{\phi}-\tilde{N}) + S^{-1}\,\hat{\nabla}^2\tilde{\psi}, \tag{33}$$

$$\tau_H\,\frac{\partial\hat{\nabla}^2\tilde{\phi}}{\partial t} + \mathrm{i}\,(\omega_E+\omega_{\ast\,i})\,\tau_H\,\hat{\nabla}^2\tilde{\phi} = -\mathrm{i}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\varphi\,\hat{\nabla}^4\left(\tilde{\phi}+\frac{\tilde{N}}{\iota}\right), \tag{34}$$

$$\tau_H\,\frac{\partial\tilde{N}}{\partial t} + \mathrm{i}\,\omega_E\,\tau_H\,\tilde{N} = -\mathrm{i}\,\omega_{\ast\,e}\,\tau_H\,\tilde{\phi} - \mathrm{i}\,\iota_e\,\hat{d}_\beta^2\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\perp\,\hat{\nabla}^2\tilde{N}. \tag{35}$$

Finally, if we write

$$\frac{\partial}{\partial t} = g - \mathrm{i}\,\omega_E, \tag{36}$$

where g is the complex growth-rate in a frame of reference that co-rotates with the MHD fluid at the rational surface, then we get

$$(g + \mathrm{i}\,\omega_{*\,e})\,\tau_H\,\tilde{\psi} = -\mathrm{i}\,\hat{x}\,(\tilde{\phi} - \tilde{N}) + S^{-1}\,\hat{\nabla}^2\tilde{\psi}, \tag{37}$$

$$(g + \mathrm{i}\,\omega_{*\,i})\,\tau_H\,\hat{\nabla}^2\tilde{\phi} = -\mathrm{i}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\varphi\,\hat{\nabla}^4\left(\tilde{\phi} + \frac{\tilde{N}}{\iota}\right), \tag{38}$$

$$g\,\tau_H\,\tilde{N} = -\mathrm{i}\,\omega_{*\,e}\,\tau_H\,\tilde{\phi} - \mathrm{i}\,\iota_e\,\hat{d}_\beta^{\,2}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\perp\,\hat{\nabla}^2\tilde{N}. \tag{39}$$

H. Asymptotic Matching

The layer equations, (37)–(39) possess a trivial twisting parity solution, and a non-trivial tearing parity solution.[8] The latter solution is such that $\tilde{\psi}(-\hat{x}) = \tilde{\psi}(\hat{x})$, $\tilde{\phi}(-\hat{x}) = -\tilde{\phi}(\hat{x})$, and $\tilde{N}(-\hat{x}) = -\tilde{N}(\hat{x})$. When this solution is asymptotically matched to the outer solution, we find that

$$\tilde{\psi}(\hat{x}) \rightarrow \left(\frac{\Psi_s}{r_s\,B_\varphi}\right)\left(1 + \Delta_s\,\frac{|\hat{x}|}{2}\right) \tag{40}$$

at the interface between the inner and the outer regions. Here, Ψ_s is the (complex) asymptotic reconnected helical magnetic flux at the rational surface, whereas Δ_s is the (real, dimensionless) tearing stability index.

II. FOURIER TRANSFORMED LAYER EQUATIONS

A. Stretched Radial Coordinate

Let us define the stretched radial coordinate $X = S^{1/3}\,\hat{x}$. Assuming that $X \sim \mathcal{O}(1)$ in the resonant layer that constitutes the inner region (i.e., assuming that the thickness of the layer is roughly of order $S^{-1/3}\,r_s$), and making use of the fact that $S \gg 1$ in a conventional tokamak plasma, Eqs. (37)–(39) reduce to the following set of linear layer equations:

$$(\hat{g} + \mathrm{i}\,Q_e)\,\tilde{\psi} = -\mathrm{i}\,X\left(\tilde{\phi} - \tilde{N}\right) + \frac{d^2\tilde{\psi}}{dX^2}, \tag{41}$$

$$(\hat{g} + \mathrm{i}\,Q_i)\,\frac{d^2\tilde{\phi}}{dX^2} = -\mathrm{i}\,X\,\frac{d^2\tilde{\psi}}{dX^2} + P_\varphi\,\frac{d^4}{dX^4}\left(\tilde{\phi} + \frac{\tilde{N}}{\iota}\right), \tag{42}$$

$$\hat{g}\,\tilde{N} = -\mathrm{i}\,Q_e\,\tilde{\phi} - \mathrm{i}\,D^2\,X\,\frac{d^2\tilde{\psi}}{dX^2} + P_\perp\,\frac{d^2\tilde{N}}{dX^2}, \tag{43}$$

where

$$\hat{g} = S^{\,1/3}\, g\, \tau_H, \tag{44}$$

$$Q_{e,i} = S^{\,1/3}\, \omega_{\ast\, e,i}\, \tau_H, \tag{45}$$

$$D = S^{\,1/3}\, \iota_e^{1/2}\, \hat{d}_\beta. \tag{46}$$

According to Eq. (40), the asymptotic behavior of the tearing-parity solutions of Eqs. (41)–(43) is such that

$$\tilde{\psi}(X) \rightarrow \left(\frac{\Psi_s}{r_s\, B_\varphi}\right)\left[1 + \frac{\hat{\Delta}_s}{2}\,|X| + {\cal O}(X^2)\right] \tag{47}$$

as $|X|\rightarrow\infty$, where

$$\hat{\Delta}_s = S^{-1/3}\,\Delta_s. \tag{48}$$

It follows from Eqs. (41)–(43) that

$$\bar{\phi}(X) \rightarrow {\rm i}\,\hat{g}\left(\frac{\Psi_s}{r_s\, B_\varphi}\right)\left[\frac{1}{X} + \frac{\hat{\Delta}_s}{2}\,{\rm sgn}(X) + {\cal O}(X)\right] \tag{49}$$

as $|X|\rightarrow\infty$.

B. Fourier Transformation

Equations (41)–(43) are most conveniently solved in Fourier transform space.[4] Let

$$\hat{\phi}(p) = \int_{-\infty}^{\infty}\tilde{\phi}(X)\,{\rm e}^{-{\rm i}\,p\,X}\,dX, \tag{50}$$

et cetera. The Fourier transformed linear layer equations become

$$(\hat{g}+{\rm i}\,Q_e)\,\hat{\psi} = \frac{d}{dp}\left(\hat{\phi} - \hat{N}\right) - p^2\,\hat{\psi}, \tag{51}$$

$$(\hat{g}+{\rm i}\,Q_i)\,p^2\,\hat{\phi} = \frac{d(p^2\,\hat{\psi})}{dp} - P_\varphi\,p^4\left(\hat{\phi} + \frac{\hat{N}}{\iota}\right), \tag{52}$$

$$\hat{g}\,\hat{N} = -{\rm i}\,Q_e\,\hat{\phi} - D^2\,\frac{d(p^2\,\hat{\psi})}{dp} - P_\perp\,p^2\,\hat{N}, \tag{53}$$

where, for a tearing parity solution, Eq. (49) yields[9]

$$\hat{\phi}(p) \rightarrow= \pi\,\hat{g}\left(\frac{\Psi_s}{r_s\, B_\varphi}\right)\left[\frac{\hat{\Delta}_s}{\pi\,p} + 1 + {\cal O}(p)\right]. \tag{54}$$

as $p\to 0$.

Let

$$Y_e(p)\equiv \hat{\phi}(p)-\hat{N}(p)=\pi\,(\hat{g}+{\rm i}\,Q_e)\left(\frac{\Psi_s}{r_s\,B_\varphi}\right)\,\hat{Y}_e(p). \tag{55}$$

Equations (51)–(53) can be combined to give

$$\frac{d}{dp}\left[A(p)\,\frac{d\hat{Y}_e}{dp}\right]-\frac{B(p)}{C(p)}\,p^2\,\hat{Y}_e=0, \tag{56}$$

where

$$A(p)=\frac{p^2}{\hat{g}+{\rm i}\,Q_e+p^2}, \tag{57}$$

$$B(p)=\hat{g}\,(\hat{g}+{\rm i}\,Q_i)+(\hat{g}+{\rm i}\,Q_i)\,(P_\varphi+P_\perp)\,p^2+P_\varphi\,P_\perp\,p^4, \tag{58}$$

$$C(p)=\hat{g}+{\rm i}\,Q_e+[P_\perp+(\hat{g}+{\rm i}\,Q_i)\,D^2]\,p^2+\iota_e^{-1}\,P_\varphi\,D^2\,p^4. \tag{59}$$

Note that

$$\iota_e=\frac{Q_e}{Q_e-Q_i}. \tag{60}$$

Furthermore, because

$$\hat{Y}_e(p)=\left(\frac{\hat{g}+{\rm i}\,Q_e}{\hat{g}}\right)\hat{\phi}(p) \tag{61}$$

as $p\to 0$, Eqs. (54) and (55) yield the following small-p boundary condition that Eq. (56) must satisfy:[9]

$$\hat{Y}_e(p)\to\frac{\hat{\Delta}_s}{\pi\,p}+1+{\cal O}(p) \tag{62}$$

as $p\to 0$. Equation (56) must also satisfy the physical boundary condition

$$\hat{Y}_e(p)\to 0 \tag{63}$$

as $p\to\infty$.

III. TECHNICAL DETAILS

A. Large-p Limit

In the large-p limit, if we write

$$A=1+\frac{\alpha}{p^2}, \tag{64}$$

$$\frac{B}{C} = \beta + \frac{\gamma}{p^2}, \tag{65}$$

and look for a solution of Eq. (56) of the form

$$\hat{Y}_e(p) \propto p^x \exp\left(\frac{-\sqrt{\beta}\,p^2}{2}\right) \tag{66}$$

then we find that

$$x = \frac{\gamma - \sqrt{\beta}\,(1-\sqrt{\beta}\,\alpha)}{2\sqrt{\beta}}. \tag{67}$$

It is easily seen that

$$\alpha = -(\hat{g} + {\rm i}\,Q_e), \tag{68}$$

$$\beta = \frac{P_\varphi\,P_\perp}{\iota_e^{-1}\,P_\varphi\,D^2}, \tag{69}$$

$$\gamma = \frac{P_\varphi\,P_\perp}{\iota_e^{-1}\,P_\varphi\,D^2}\left[1 + (\hat{g}+{\rm i}\,Q_i)\,\frac{P_\varphi + P_\perp}{P_\varphi\,P_\perp} - (P_\perp + [\hat{g}+{\rm i}\,Q_i]\,D^2)\,\frac{1}{\iota_e^{-1}\,P_\varphi\,D^2}\right]. \tag{70}$$

In order to be in the large-p limit, we require

$$p \gg |\hat{g} + {\rm i}\,Q_e|^{1/2}, \tag{71}$$

$$p \gg \left|\frac{(\hat{g}+{\rm i}\,Q_i)\,(P_\varphi + P_\perp)}{P_\varphi\,P_\perp}\right|^{1/2}, \tag{72}$$

$$p \gg \left|\frac{\hat{g}\,(\hat{g}+{\rm i}\,Q_i)}{P_\varphi\,P_\perp}\right|^{1/4}, \tag{73}$$

$$p \gg \left|\frac{P_\perp + (\hat{g}+{\rm i}\,Q_i)\,D^2}{\iota_e^{-1}\,P_\varphi\,D^2}\right|^{1/2}, \tag{74}$$

$$p \gg \left|\frac{\hat{g}+{\rm i}\,Q_e}{\iota_e^{-1}\,P_\varphi\,D^2}\right|^{1/4}, \tag{75}$$

$$p \gg \left(\frac{\iota_e^{-1}\,P_\varphi\,D^2}{P_\varphi\,P_\perp}\right)^{1/4}. \tag{76}$$

B. Low-D Limit

If

$$D^2 \ll \left|\frac{P_\perp}{\hat{g}+{\rm i}\,Q_e}\right|,\ \frac{\iota_e\,P_\perp}{P_\varphi^{2/3}} \tag{77}$$

then the terms in the layer equations involving D^2 are negligible. In this case, in the large-p limit, we can write

$$A = 1 + \frac{\alpha}{p^2}, \tag{78}$$

$$\frac{B}{C} = \beta\, p^2 + \gamma, \tag{79}$$

where

$$\alpha = -(\hat{g} + \mathrm{i}\, Q_e), \tag{80}$$

$$\beta = P_\varphi, \tag{81}$$

$$\gamma = -\mathrm{i}\,(Q_e - Q_i)\,\frac{P_\varphi}{P_\perp} + \hat{g} + \mathrm{i}\, Q_i. \tag{82}$$

The solution of Eq. (56) becomes

$$\hat{Y}_e \propto p^{-1} \exp\left(x\, p - \frac{\sqrt{\beta}\, p^3}{3}\right), \tag{83}$$

where

$$x = \frac{\alpha\,\beta - \gamma}{2\sqrt{\beta}}. \tag{84}$$

In order to be in the large-p limit, we require

$$p \gg |\hat{g} + \mathrm{i}\, Q_e|^{1/2}, \tag{85}$$

$$p \gg \left|\frac{(\hat{g} + \mathrm{i}\, Q_i)\,(P_\varphi + P_\perp)}{P_\varphi\, P_\perp}\right|^{1/2}, \tag{86}$$

$$p \gg \left|\frac{\hat{g}\,(\hat{g} + \mathrm{i}\, Q_i)}{P_\varphi\, P_\perp}\right|^{1/4}, \tag{87}$$

$$p \gg \left|\frac{\hat{g} + \mathrm{i}\, Q_e}{P_\perp}\right|^{1/2}, \tag{88}$$

$$p \gg P_\varphi^{-1/6}. \tag{89}$$

C. Ricatti Transformation

Let

$$W = \frac{p}{\hat{Y}_e}\,\frac{d\hat{Y}_e}{dp}. \tag{90}$$

Equation (56) transforms to give

$$\frac{dW}{dp} = -\frac{A'}{p}\,W - \frac{W^{\,2}}{p} + \frac{B}{A\,C}\,p^3, \tag{91}$$

where

$$A' = \frac{\hat{g} + {\rm i}\,Q_e - p^2}{\hat{g} + {\rm i}\,Q_e + p^2}. \tag{92}$$

According to Sects. II B and III A, this equation must be solved subject to the boundary condition that

$$W(p) = x - \sqrt{\beta}\,p^2 \tag{93}$$

at large-p, and

$$W(p) = -1 + \frac{\pi\,p}{\hat{\Delta}_s} \tag{94}$$

at small-p. However, according to Sect. III B, in the low-D limit,

$$W(p) = -1 + x\,p - \sqrt{\beta}\,p^3 \tag{95}$$

at large-p.

D. Method of Solution

We launch a solution of Eqs (91) from large p, subject to the boundary condition (93) or (95), as appropriate, and integrate it to small p. We can then deduce the value of $\hat{\Delta}_s(\hat{g}, Q_e, Q_i, D, P_\varphi, P_\perp)$ from Eq. (94).

IV. INCORPORATION INTO TJ CODE

A. TJ Data

For each of the K rational surface in the plasma, TJ calculates

$$S_k^{1/3}, \tag{96}$$

$$\tau_k \equiv S_k^{1/3}\,\tau_{H\,k}, \tag{97}$$

$$Q_{E\,k}, \tag{98}$$

$$Q_{e\,k}, \tag{99}$$

$$Q_{i\,k}, \tag{100}$$

$$D_k, \tag{101}$$

$$P_{\varphi\,k}, \tag{102}$$

$$P_{\perp\,k}, \tag{103}$$

$$E_{kk}, \tag{104}$$

$$\Delta_{c\,k}, \tag{105}$$

$$|\chi_k|. \tag{106}$$

B. Calculation of Growth-Rate and Real Frequency

Solve

$$E_{kk} = S_k^{1/3}\,\hat{\Delta}_s(\hat{g}, Q_{e\,k}, Q_{i\,k}, D_k, P_{\varphi\,k}, P_{\perp\,k}) + \Delta_{c\,k} \tag{107}$$

for $\hat{g}$ starting from a suitable marginal stability point. The growth-rate is

$$\gamma = \frac{\mathrm{Re}(\hat{g})}{\tau_k}, \tag{108}$$

and the real frequency is

$$\omega = \frac{Q_E - \mathrm{Im}(\hat{g})}{\tau_k}. \tag{109}$$

C. Calculation of Shielding Factor and Locking Torque

The shielding factor at the rational surface is

$$\Xi_k(\omega) = \left|\frac{\Delta_{c\,k} - E_{kk}}{S_k^{1/3}\,\hat{\Delta}_s(\hat{g}, Q_{e\,k}, Q_{i\,k}, D_k, P_{\varphi\,k}, P_{\perp\,k}) + \Delta_{c\,k} - E_{kk}}\right|, \tag{110}$$

where

$$\hat{g} = \mathrm{i}\,(Q_E - \omega\,\tau_k). \tag{111}$$

Likewise, the locking torque takes the form

$$\delta T_k(\omega) = 2\pi^2\,n\,|\chi_k|^2\,\frac{\mathrm{Im}[S_k^{1/3}\,\hat{\Delta}_s(\hat{g}, Q_{e\,k}, Q_{i\,k}, D_k, P_{\varphi\,k}, P_{\perp\,k})]}{|S_k^{1/3}\,\hat{\Delta}_s(\hat{g}, Q_{e\,k}, Q_{i\,k}, D_k, P_{\varphi\,k}, P_{\perp\,k}) + \Delta_{c\,k} - E_{kk}|^2}. \tag{112}$$

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