

# Advanced Divertors

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## 1 Plasma Equilibrium

Let  $x, y, z$  be right-handed Cartesian coordinates. The system is assumed to be periodic in the  $z$ -direction with period  $2\pi R_0$ , where  $R_0$  is the simulated major radius of the plasma. Let  $\phi = z/R_0$  be a simulated toroidal angle. The equilibrium magnetic field is written

$$\mathbf{B} = \nabla\phi \times \nabla\psi_p + B_0 R_0 \nabla\phi, \quad (1)$$

where  $B_0$  is the toroidal magnetic field-strength, and  $\psi_p$  the poloidal magnetic flux (divided by  $2\pi$ ). Let  $a$  be the mean minor radius of the plasma, and let  $I_p$  be the toroidal plasma current. Let  $X = x/a$ ,  $Y = y/a$ , and

$$\psi_p(x, y) = \frac{\mu_0 I_p R_0}{2\pi} \psi(X, Y). \quad (2)$$

The safety-factor on a given magnetic flux-surface is

$$q = \frac{q_*}{2\pi} \oint \frac{dL}{|\hat{\nabla}\psi|}, \quad (3)$$

where

$$q_* = \frac{2\pi B_0 a^2}{\mu_0 I_p R_0}, \quad (4)$$

and  $\hat{\nabla} = a \nabla$ . Here,  $dL$  is an element of normalized length, in the  $X$ - $Y$  plane, along the flux-surface.

We require  $\hat{\nabla}^2\psi = 0$  everywhere that a current does not flow, where  $\hat{\nabla}^2 \equiv \partial^2/\partial X^2 + \partial^2/\partial Y^2$ . If we let  $z = X + iY$  then we can automatically ensure this by writing

$$F(z) = \psi(X, Y) + i\chi(X, Y), \quad (5)$$

where  $\psi(X, Y)$  and  $\chi(X, Y)$  are real functions. The Cauchy-Reiman relations yield

$$\frac{\partial\psi}{\partial X} = \frac{\partial\chi}{\partial Y}, \quad (6)$$

$$\frac{\partial\psi}{\partial Y} = -\frac{\partial\chi}{\partial X}, \quad (7)$$

from which it follows that  $\hat{\nabla}^2\psi = 0$ . The poloidal magnetic field is

$$\mathbf{B}_p \equiv \nabla\phi \times \nabla\psi_p = \frac{\mu_0 I_p}{2\pi a} \hat{\mathbf{B}}, \quad (8)$$

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where

$$\hat{B}_X = \frac{\partial \psi}{\partial Y}, \quad (9)$$

$$\hat{B}_Y = -\frac{\partial \psi}{\partial X}. \quad (10)$$

Now,

$$\frac{dF}{dz} = \frac{\partial \psi}{\partial X} + i \frac{\partial \chi}{\partial X} = \frac{\partial \psi}{\partial X} - i \frac{\partial \psi}{\partial Y}, \quad (11)$$

which implies that

$$\hat{B}_X = -\text{Im} \left( \frac{dF}{dz} \right), \quad (12)$$

$$\hat{B}_Y = -\text{Re} \left( \frac{dF}{dz} \right). \quad (13)$$

## 2 First-Order Magnetic Null

Let

$$F(z) = \ln z + \zeta \ln(z + i), \quad (14)$$

where  $\zeta$  is real and positive. The first term corresponds to the plasma current filament located at the origin, whereas the second corresponds to the divertor coil filament located at  $X = 0, Y = -1$ . The current in the plasma filament is  $I_p$ , whereas that in the divertor coil filament is  $\zeta I_p$ .

Now,

$$\frac{dF}{dz} = \frac{1}{z} + \frac{\zeta}{z + i}. \quad (15)$$

If  $z = z_0$  corresponds to a null in the poloidal magnetic field, at which  $|\hat{\mathbf{B}}| = 0$ , then we require  $(dF/dz)_{z_0} = 0$ . It follows that

$$z_0 = -\frac{i}{1 + \zeta}. \quad (16)$$

In other words, the magnetic null is located at  $X = 0, Y = Y_0$ , where  $Y_0 = -1/(1 + \zeta)$ .

Now,

$$\frac{d^2F}{dz^2} = -\frac{1}{z^2} - \frac{\zeta}{(z + i)^2}. \quad (17)$$

So, at the magnetic null,

$$\left( \frac{d^2F}{dz^2} \right)_{z_0} = \frac{(1 + \zeta)^3}{\zeta}. \quad (18)$$

Thus, in the vicinity of the magnetic null

$$F(z) \simeq \frac{1}{2} \frac{(1 + \zeta)^3}{\zeta} (z - z_0)^2. \quad (19)$$

Let

$$z - z_0 = u + i v. \quad (20)$$

It follows that

$$\psi(u, v) = \text{Re}(F) = \frac{1}{2} \frac{(1 + \zeta)^3}{\zeta} (u^2 - v^2). \quad (21)$$

The separatrix curves corresponds to  $\psi = 0$ . Thus, the separatrix curves are the two straight-lines  $u = v$  and  $u = -v$ , which subtend right-angles with respect to one another.

Consider the flux-surface

$$v^2 - u^2 = \epsilon^2, \quad (22)$$

whose distance of closest approach to the null point is  $\epsilon$ . Here,  $0 < \epsilon \ll 1$ . Note that

$$\frac{dv}{du} = \frac{u}{v}. \quad (23)$$

This flux-surface corresponds to the contour

$$\psi(u, v) = -\frac{1}{2} \frac{(1 + \zeta)^3}{\zeta} \epsilon^2. \quad (24)$$

On the flux-surface,

$$|\hat{\nabla}\psi| = \left[ \left( \frac{\partial\psi}{\partial u} \right)^2 + \left( \frac{\partial\psi}{\partial v} \right)^2 \right]^{1/2} = \frac{1}{2} \frac{(1 + \zeta)^3}{\zeta} [(2u)^2 + (2v)^2]^{1/2} = \frac{(1 + \zeta)^3}{\zeta} (u^2 + v^2)^{1/2}, \quad (25)$$

and

$$dL = \left[ 1 + \left( \frac{dv}{du} \right)^2 \right]^{1/2} du = \left( 1 + \frac{u^2}{v^2} \right)^{1/2} du = \frac{(u^2 + v^2)^{1/2} du}{v} = \frac{(u^2 + v^2)^{1/2} du}{(u^2 + \epsilon^2)^{1/2}}. \quad (26)$$

Thus,

$$\frac{q(\epsilon)}{q_*} = \frac{1}{\pi} \frac{\zeta}{(1 + \zeta)^3} \int_0^1 \frac{du}{(u^2 + \epsilon^2)^{1/2}} = \frac{1}{\pi} \frac{\zeta}{(1 + \zeta)^3} \left[ \ln \left\{ (u^2 + \epsilon^2)^{1/2} + u \right\} \right]_0^1 \simeq \frac{1}{\pi} \frac{\zeta}{(1 + \zeta)^3} \ln \left( \frac{2}{\epsilon} \right). \quad (27)$$

### 3 Second-Order Magnetic Null

Let

$$F(z) = \ln z + \frac{\zeta}{2} \ln(z + i + \Delta) + \frac{\zeta}{2} \ln(z + i - \Delta), \quad (28)$$

where  $\zeta$  and  $\Delta$  are real and positive. The first term corresponds to the plasma current filament located at the origin, whereas the second and third terms corresponds to two divertor coil filaments, carrying equal currents, located at  $X = \pm\Delta$ ,  $Y = -1$ .

Now,

$$\frac{dF}{dz} = \frac{1}{z} + \frac{\zeta}{2(z + i + \Delta)} + \frac{\zeta}{2(z + i - \Delta)}. \quad (29)$$

If  $z = z_0$  corresponds to a null in the poloidal magnetic field, at which  $|\hat{\mathbf{B}}| = 0$ , then we require  $(dF/dz)_{z_0} = 0$ . It follows that

$$(1 + \zeta) z_0^2 + i(2 + \zeta) z_0 - 1 - \Delta^2 = 0, \quad (30)$$

which gives

$$z_0 = \frac{-i(2 + \zeta) \pm i\sqrt{\zeta^2 - 4(1 + \zeta)\Delta^2}}{2(1 + \zeta)}. \quad (31)$$

Thus, in general, there are two magnetic null points.

Now,

$$\frac{d^2F}{dz^2} = -\frac{1}{z^2} - \frac{\zeta}{2(z + i + \Delta)^2} - \frac{\zeta}{2(z + i - \Delta)^2}. \quad (32)$$

If  $z = z_0$  corresponds to a second-order null point then we require  $(d^2F/dz^2)_{z_0} = 0$ . It follows that

$$\Delta = \frac{\zeta}{2\sqrt{1 + \zeta}}, \quad (33)$$

which is, of course, the condition for the two solutions in Eq. (31) to merge. The second-order null is located at  $X = 0$ ,  $Y = Y_0$ , where

$$Y_0 = -\frac{(2 + \zeta)}{2(1 + \zeta)}. \quad (34)$$

Now,

$$\frac{d^3F}{dz^3} = \frac{2}{z^3} + \frac{\zeta}{(z + i + \Delta)^3} + \frac{\zeta}{(z + i - \Delta)^3}. \quad (35)$$

So, at the magnetic null,

$$\left(\frac{d^3F}{dz^3}\right)_{z_0} = -\frac{16i(1 + \zeta)^4}{2 + \zeta)^2 \zeta^2}. \quad (36)$$

Thus, in the vicinity of the magnetic null

$$F(z) \simeq -\frac{1}{6} \frac{16i(1 + \zeta)^4}{(2 + \zeta)^2 \zeta^2} (z - z_0)^3. \quad (37)$$

Let

$$z - z_0 = u + iv. \quad (38)$$

It follows that

$$\psi(u, v) = \operatorname{Re}(F) = \frac{8}{3} \frac{(1 + \zeta)^4}{(2 + \zeta)^2 \zeta^2} (3u^2 - v^2)v \quad (39)$$

The separatrix curves corresponds to  $\psi = 0$ . Thus, the separatrix curves are the three straight-lines  $u = \pm v/\sqrt{3}$  and  $v = 0$  which subtend  $60^\circ$  with respect to one another.

Consider the he flux-surface

$$(v^2 - 3u^2)v = \epsilon^3 \quad (40)$$

whose distance of closest approach to the null point is  $\epsilon$ . Here,  $0 < \epsilon \ll 1$ . Note that

$$\frac{du}{dv} = \frac{v^2 - u^2}{2uv}. \quad (41)$$

This flux-surface corresponds to the contour

$$\psi(u, v) = -\frac{8}{3} \frac{(1 + \zeta)^4}{(2 + \zeta)^2 \zeta^2} \epsilon^3. \quad (42)$$

On the flux-surface,

$$\begin{aligned} |\hat{\nabla}\psi| &= \left[ \left( \frac{\partial\psi}{\partial u} \right)^2 + \left( \frac{\partial\psi}{\partial v} \right)^2 \right]^{1/2} = \frac{8}{3} \frac{(1 + \zeta)^4}{(2 + \zeta)^2 \zeta^2} [(6uv)^2 + (3u^2 - 3v^2)^2]^{1/2} \\ &= \frac{8(1 + \zeta)^4}{(2 + \zeta)^2 \zeta^2} (u^2 + v^2), \end{aligned} \quad (43)$$

and

$$dL = \left[ \left( \frac{du}{dv} \right)^2 + 1 \right]^{1/2} dv = \left[ \left( \frac{v^2 - u^2}{2uv} \right)^2 + 1 \right]^{1/2} dv = \frac{(u^2 + v^2)dv}{2uv}. \quad (44)$$

Thus,

$$\begin{aligned} \frac{q(\epsilon)}{q_*} &= \frac{1}{8\pi} \frac{(2 + \zeta)\zeta^2}{(1 + \zeta)^4} \int_{\epsilon}^{\infty} \frac{dv}{2uv} = \frac{\sqrt{3}}{16\pi} \frac{(2 + \zeta)\zeta^2}{(1 + \zeta)^4} \int_{\epsilon}^{\infty} \frac{dv}{v^{1/2}(v^3 - \epsilon^3)^{1/2}} \\ &= \frac{\sqrt{3}}{16\pi} \frac{(2 + \zeta)\zeta^2}{(1 + \zeta)^4} \frac{1}{\epsilon} \int_1^{\infty} \frac{dx}{x^{1/2}(x^3 - 1)^{1/2}} = \frac{\sqrt{3}\Gamma(4/3)}{16\pi^{1/2}\Gamma(5/6)} \frac{(2 + \zeta)\zeta^2}{(1 + \zeta)^4} \frac{1}{\epsilon}. \end{aligned} \quad (45)$$