Resistive Wall

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I. VACUUM SOLUTION

A. Normalization

Let all lengths be normalized to the major radius of the axisymmetric plasma equilibrium's magnetic axis, R_0. Let all magnetic field-strengths be normalized to the toroidal magnetic field-strength at the magnetic axis, B_0.

B. Toroidal Coordinates

Let μ, η, ϕ be right-handed toroidal coordinates defined such that

$$R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}, \tag{1}$$

$$Z = \frac{\sin\eta}{\cosh\mu - \cos\eta}, \tag{2}$$

where R, ϕ, Z are right-handed cylindrical coordinates whose symmetry axis corresponds to that of the plasma equilibrium. Note that $(\nabla R\times\nabla\phi\cdot\nabla Z)^{-1} = R$. The scale-factors of the toroidal coordinate system are

$$h_\mu = h_\eta = \frac{1}{\cosh\mu - \cos\eta} \equiv h, \tag{3}$$

$$h_\phi = \frac{\sinh\mu}{\cosh\mu - \cos\eta} = h\,\sinh\mu. \tag{4}$$

Moreover,

$$\mathcal{J}' \equiv (\nabla\mu\times\nabla\eta\cdot\nabla\phi) = h^3\,\sinh\mu. \tag{5}$$

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C. Perturbed Magnetic Field

The curl-free perturbed magnetic field in the vacuum region is written

$$\mathbf{b} = \mathrm{i}\,\nabla\left[V(\mu,\eta)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi}\right], \tag{6}$$

where where the toroidal mode number, n, is a positive integer. Given that $\nabla\cdot\mathbf{b} = 0$, we deduce that

$$\begin{aligned}\nabla^2 V \equiv (z-\cos\eta)^3 \Bigg\{ &\frac{\partial}{\partial z}\left[\frac{z^2-1}{z-\cos\eta}\,\frac{\partial V}{\partial z}\right] \\ &+\frac{\partial}{\partial\eta}\left[\frac{1}{z-\cos\eta}\,\frac{\partial V}{\partial\eta}\right] - \frac{n^2\,V}{(z^2-1)\,(z-\cos\eta)}\Bigg\} = 0.\end{aligned} \tag{7}$$

Here, $z = \cosh\mu$.

Let

$$f_z = z^2 - 1, \tag{8}$$

$$f_\eta = (z-\cos\eta)^{1/2}, \tag{9}$$

which implies that

$$\frac{df_z}{dz} = 2\,z, \tag{10}$$

$$\frac{\partial f_\eta}{\partial z} = \frac{1}{2\,f_\eta}, \tag{11}$$

$$\frac{\partial f_\eta}{\partial \eta} = \frac{\sin\eta}{2\,f_\eta} \tag{12}$$

Suppose that

$$V(z,\eta) = \sum_m (z-\cos\eta)^{1/2}\,U_m(z)\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}. \tag{13}$$

Taking the sum and eikonal as read, and letting $' = d/dz$, we get

$$\frac{\partial V}{\partial z} = \frac{U_m}{2\,f_\eta} + f_\eta\,U_m', \tag{14}$$

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^2}\,\frac{\partial V}{\partial z}\right) &= \frac{\partial}{\partial z}\left(\frac{f_z\,U_m}{2\,f_\eta^3} + \frac{f_z\,U_m'}{f_\eta}\right) = \frac{z\,U_m}{f_\eta^3} - \frac{3\,f_z\,U_m}{4\,f_\eta^5} + \frac{f_z\,U_m'}{2\,f_\eta^3} + \frac{2\,z\,U_m'}{f_\eta} - \frac{f_z\,U_m'}{2\,f_\eta^3} + \frac{f_z\,U_m''}{f_\eta} \\ &= \frac{z\,U_m}{f_\eta^3} - \frac{3\,(z^2-1)\,U_m}{4\,f_\eta^5} + \frac{2\,z\,U_m'}{f_\eta} + \frac{(z^2-1)\,U_m''}{f_\eta},\end{aligned} \tag{15}$$

$$\frac{\partial V}{\partial \eta} = \frac{\sin\eta\,U_m}{2\,f_\eta} - \mathrm{i}\,m f_\eta\,U_m, \tag{16}$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{f_\eta^2}\,\frac{\partial V}{\partial\eta}\right) = \frac{\partial}{\partial\eta}\left(\frac{\sin\eta\,U_m}{2\,f_\eta^3} - \frac{{\rm i}\,m\,U_m}{f_\eta}\right) = \frac{\cos\eta\,U_m}{2\,f_\eta^3} - \frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5} - \frac{{\rm i}\,m\,\sin\eta\,U_m}{2\,f_\eta^3}$$
$$+\frac{{\rm i}\,m\,\sin\eta\,U_m}{2\,f_\eta^3} - \frac{m^2\,U_m}{f_\eta}$$
$$= \frac{\cos\eta\,U_m}{2\,f_\eta^3} - \frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5} - \frac{m^2\,U_m}{f_\eta}, \tag{17}$$
$$-\frac{n^2\,V}{f_z\,f_\eta^2} = -\frac{n^2\,U_m}{(z^2-1)\,f_\eta}. \tag{18}$$

Thus, Eq. (7) becomes

$$0 = \frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^2}\,\frac{\partial V}{\partial z}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{f_\eta^2}\,\frac{\partial V}{\partial\eta}\right) - \frac{n^2\,V}{f_z\,f_\eta^2}$$
$$= \frac{z\,U_m}{f_\eta^3} - \frac{3\,(z^2-1)\,U_m}{4\,f_\eta^5} + \frac{2\,z\,U_m'}{f_\eta} + \frac{(z^2-1)\,U_m''}{f_\eta}$$
$$+\frac{\cos\eta\,U_m}{2\,f_\eta^3} - \frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5} - \frac{m^2\,U_m}{f_\eta} - \frac{n^2\,U_m}{(z^2-1)\,f_\eta}$$
$$= \frac{1}{f_\eta}\left[(z^2-1)\,U_m'' + 2\,z\,U_m' + \left(\frac{1}{4} - m^2\right)U_m - \frac{n^2\,U_m}{z^2-1}\right]. \tag{19}$$

The most general solution of the previous equation is

$$U_m(z) = p_m\,\hat{P}^{\,n}_{|m|-1/2}(z) + q_m\,\hat{Q}^{\,n}_{m-1/2}(z), \tag{20}$$

where

$$\hat{P}^{\,n}_{|m|-1/2}(z) = \cos(|m|\,\pi)\,\frac{\sqrt{\pi}\,\Gamma(|m|+1/2-n)\,\epsilon^{\,|m|}}{2^{\,|m|-1/2}\,|m|!}\,P^{\,n}_{|m|-1/2}(z), \tag{21}$$

$$\hat{Q}^{\,n}_{|m|-1/2}(z) = \cos(n\,\pi)\,\cos(|m|\,\pi)\,\frac{2^{\,|m|-1/2}\,|m|!\,\epsilon^{-|m|}}{\sqrt{\pi}\,\Gamma(|m|+1/2+n)}\,Q^{\,n}_{|m|-1/2}. \tag{22}$$

Here, ϵ is the inverse-aspect ratio of the plasma equilibrium, and p_m and q_m are arbitrary complex coefficients. Moreover, we have made use of the fact that

$$P^{\,n}_{-m-1/2}(z) = P^{\,n}_{m-1/2}(z), \tag{23}$$

$$Q^{\,n}_{-m-1/2}(z) = Q^{\,n}_{m-1/2}(z). \tag{24}$$

D. Toroidal Electromagnetic Angular Momentum Flux

The outward flux of toroidal angular momentum across a constant-z surface is

$$T_\phi(z) = -\oint\oint \mathcal{J}'\,b_\phi\,b^\mu\,d\eta\,d\phi. \tag{25}$$

Now,

$$b^\mu \equiv \mathbf{b}\cdot\nabla\mu = \mathrm{i}\,\frac{\partial V}{\partial\mu}\,|\nabla\mu|^2 = \mathrm{i}\,\frac{\sinh\mu}{h^2}\,\frac{\partial V}{\partial z}, \tag{26}$$

$$b^\phi \equiv \mathcal{J}'\,\nabla\mu\times\nabla\eta\cdot\nabla V = n\,V, \tag{27}$$

so

$$\begin{aligned} T_\phi(z) &= -\frac{\mathrm{i}\,n\,\pi}{2}\oint\frac{z^2-1}{z-\cos\eta}\left(\frac{\partial V}{\partial z}\,V^\ast - \frac{\partial V^\ast}{\partial z}\,V\right)d\eta \\ &= -\mathrm{i}\,n\,\pi^2\sum_m (z^2-1)\left(\frac{dU_m}{dz}\,U_m^\ast - \frac{dU_m^\ast}{dz}\,U_m\right) \\ &= -\mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast)\,(z^2-1)\left(\frac{d\hat{P}|^n_{|m|-1/2}}{dz}\,\hat{Q}^n_{|m|-1/2} - \frac{d\hat{Q}^n_{|m|-1/2}}{dz}\,\hat{P}^n_{|m|-1/2}\right) \\ &= \mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast)\,(z^2-1)\,\mathcal{W}(\hat{P}^n_{|m|-1/2},\hat{Q}^n_{|m|-1/2}). \end{aligned} \tag{28}$$

But,

$$\begin{aligned} \mathcal{W}(\hat{P}^n_{|m|-1/2},\hat{Q}^n_{|m|-1/2}) &= \cos(n\,\pi)\,\frac{\Gamma(|m|+1/2-n)}{\Gamma(|m|+1/2+n)}\,\mathcal{W}(P^n_{|m|-1/2},Q^n_{|m|-1/2}) \\ &= \cos(n\,\pi)\,\frac{\Gamma(|m|+1/2-n)}{\Gamma(|m|+1/2+n)}\,\frac{\cos(n\,\pi)}{1-z^2}\,\frac{\Gamma(|m|+1/2+n)}{\Gamma(|m|+1/2-n)} \\ &= \frac{1}{1-z^2}, \end{aligned} \tag{29}$$

so

$$T_\phi(z) = 2\pi^2\,n\sum_m \mathrm{Im}(q_m^\ast\,p_m). \tag{30}$$

II. RESISTIVE WALL PHYSICS

A. Resistive Wall

Let the inner surface of the resistive wall surrounding the plasma lie at $\mu=\mu_w$, and let the outer surface lie at $\mu=\mu_w-\bar{d}_w\,\sinh\mu_w$, where $\bar{d}_w\ll 1$ is a positive constant. The physical wall thickness is

$$d(\eta) = \frac{\bar{d}_w\,\sinh\mu_w}{|\nabla\mu|} = h_w(\eta)\,\sinh\mu_w\,\bar{d}_w, \tag{31}$$

where

$$h_w(\eta) = \frac{1}{z_w - \cos\eta}, \tag{32}$$

and $z_w = \cosh\mu_w$. Let the electrical conductivity of the wall material vary as

$$\sigma(\eta) = \frac{\bar{\sigma}_w}{h_w^2(\eta)\,\sinh^2\mu_w}, \tag{33}$$

where $\bar{\sigma}_w$ is a positive constant. It follows that $\sigma\,d^2 = \bar{\sigma}_w\,\bar{d}_w^2$.

B. Wall Matching Conditions

If we write

$$\mathbf{b} = \nabla\times\mathbf{A} \tag{34}$$

in the vacuum region then the boundary conditions at the wall are

$$\mathbf{n}_w\times\mathbf{A}|_{z_{w-}} = \frac{1}{\cosh\lambda}\,\mathbf{n}_w\times\mathbf{A}|_{z_{w+}} \tag{35}$$

$$\mathbf{n}_w\times(\nabla\times\mathbf{A})|_{z_{w+}} = -\frac{\lambda\,\tanh\lambda}{\bar{d}_w\,h_w\,\sinh\mu_w}\;\mathbf{n}_w\times(\mathbf{n}_w\times\mathbf{A})|_{z_{w+}} + \frac{\mathbf{n}_w\times(\nabla\times\mathbf{A})|_{z_{w-}}}{\cosh\lambda}, \tag{36}$$

$$\lambda = \sqrt{\hat{\gamma}\,\bar{d}_w}, \tag{37}$$

$$\hat{\gamma} = \gamma\,\bar{\tau}_w, \tag{38}$$

$$\bar{\tau}_w = \mu_0\,R_0^2\,\bar{\sigma}_w\,\bar{d}_w, \tag{39}$$

where γ is the growth-rate of the magnetic perturbation, and $\bar{\tau}_w$ is the effective L/R time of the wall. Here, $\mathbf{n}_w = -\mathbf{e}_\mu$ is an outward unit normal vector to the wall. Now,

$$\nabla\times\mathbf{A} = \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\phi}{\partial\eta} - \frac{\partial\hat{A}_\eta}{\partial\phi}\right)\mathbf{e}_\mu + \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)\mathbf{e}_\eta + \frac{1}{h^2}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)\mathbf{e}_\phi, \tag{40}$$

where

$$\hat{A}_\mu = h\,A_\mu, \tag{41}$$

$$\hat{A}_\eta = h\,A_\eta, \tag{42}$$

$$\hat{A}_\phi = h\,\sinh\mu\,A_\phi. \tag{43}$$

Furthermore,

$$\mathbf{n}_w \times \mathbf{A} = -\mathbf{e}_\mu \times \mathbf{A} = A_\phi\,\mathbf{e}_\eta - A_\eta\,\mathbf{e}_\phi, \tag{44}$$

$$\mathbf{n}_w \times (\mathbf{n}_w \times \mathbf{A}) = -\mathbf{e}_\mu \times (\mathbf{n}_w \times \mathbf{A}) = -A_\eta\,\mathbf{e}_\eta - A_\phi\,\mathbf{e}_\phi, \tag{45}$$

$$\mathbf{n}_w \times (\nabla \times \mathbf{A}) = -\mathbf{e}_\mu \times (\nabla \times \mathbf{A}) = \frac{1}{h^2}\left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta}\right)\mathbf{e}_\eta - \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu}\right)\mathbf{e}_\phi. \tag{46}$$

Thus, the wall matching conditions become

$$\hat{A}_\eta\Big|_{z_{w-}} = \frac{1}{\cosh\lambda}\,\hat{A}_\eta\Big|_{z_{w+}}, \tag{47}$$

$$\hat{A}_\phi\Big|_{z_{w+}} = \frac{1}{\cosh\lambda}\,\hat{A}_\phi\Big|_{z_{w-}}, \tag{48}$$

$$\left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta}\right)_{z_{w+}} = \frac{\lambda\,\tanh\lambda}{\bar{d}_w\,\sinh\mu_w}\,\hat{A}_\eta\Big|_{z_{w+}} + \frac{1}{\cosh\lambda}\left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta}\right)_{z_{w-}}, \tag{49}$$

$$\left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu}\right)_{z_{w+}} = -\frac{\lambda\,\tanh\lambda}{\bar{d}_w\,\sinh\mu_w}\,\hat{A}_\phi\Big|_{\mu_{z+}} + \frac{1}{\cosh\lambda}\left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu}\right)_{z_{w-}}. \tag{50}$$

Let

$$C(z,\eta,\phi) = \frac{\partial \hat{A}_\eta}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \eta}. \tag{51}$$

The wall matching conditions reduce to

$$C(z_{w-},\eta,\phi) = \frac{1}{\cosh\lambda}\,C(z_{w+},\eta,\phi), \tag{52}$$

$$\frac{\partial C(z_{w+},\eta,\phi)}{\partial z} = \frac{\lambda\,\tanh\lambda}{\bar{d}_w\,\sinh^2\mu_w}\,C(z_{w+},\eta,\phi) + \frac{1}{\cosh\lambda}\,\frac{\partial C(z_{w-},\eta,\phi)}{\partial z}. \tag{53}$$

However, if

$$\mathbf{b} = \mathrm{i}\,\nabla V = \nabla \times \mathbf{A} \tag{54}$$

then

$$C = -\mathrm{i}\,h\,\sinh\mu\,\frac{\partial V}{\partial \mu} = -\mathrm{i}\,h\,(z^2-1)\,\frac{\partial V}{\partial z}. \tag{55}$$

Thus,

$$C = -\mathrm{i}\,\frac{z^2-1}{z-\cos\eta}\sum_m\left[\frac{U_m}{2\,(z-\cos\eta)^{1/2}} + (z-\cos\eta)^{1/2}\,\frac{dU_m}{dz}\right]\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}, \tag{56}$$

$$\frac{\partial C}{\partial z} = -\mathrm{i}\sum_m\left[\frac{(3/4)\,\sin^2\eta}{(z-\cos\eta)^{5/2}} - \frac{(1/2)\,\cos\eta}{(z-\cos\eta)^{3/2}} + \frac{m^2+n^2/(z^2-1)}{(z-\cos\eta)^{1/2}}\right]U_m\,\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}. \tag{57}$$

It follows that

$$\sum_m \left[\frac{U_m}{2} + (z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w-}} \mathrm{e}^{-\mathrm{i}\,m\,\eta} = \frac{1}{\cosh\lambda}\sum_m \left[\frac{U_m}{2} + (z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w+}} \mathrm{e}^{-\mathrm{i}\,m\,\eta}, \tag{58}$$

$$\sum_m \left[\frac{3}{4}\,\sin^2\eta - \frac{1}{2}\,(z-\cos\eta)\,\cos\eta + (z-\cos\eta)^2\left(m^2+\frac{n^2}{z^2-1}\right)\right] U_m\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}\Big|_{z_{w+}} =$$
$$f_w\sum_m (z-\cos\eta)\left[\frac{U_m}{2} + (z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w+}} \mathrm{e}^{-\mathrm{i}\,m\,\eta}$$
$$+\frac{1}{\cosh\lambda}\sum_m \left[\frac{3}{4}\,\sin^2\eta - \frac{1}{2}\,(z-\cos\eta)\,\cos\eta + (z-\cos\eta)^2\left(m^2+\frac{n^2}{z^2-1}\right)\right] U_m\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}\Big|_{z_{w-}}, \tag{59}$$

where

$$f_w = \frac{\lambda\,\tanh\lambda}{\bar{d}_w}. \tag{60}$$

Thus, we can write

$$\sum_{m'} I_{mm'}\,U_{m'}(z_{w-}) = \frac{1}{\cosh\lambda}\sum_{m'} I_{mm'}\,U_{m'}(z_{w+}), \tag{61}$$

$$\sum_{m'} J_{mm'}\,U_{m'}(z_{w+}) = f_w\sum_{m',m''} k_{mm''}\,I_{m''m'}\,U_{m'}(z_{w+}) + \frac{1}{\cosh\lambda}\sum_{m'} J_{mm'}\,U_{m'}(z_{w-}), \tag{62}$$

where

$$I_{mm'} = \left(\frac{1}{2}+z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\,(\delta_{m\,m'+1}+\delta_{m\,m'-1}), \tag{63}$$

$$J_{mm'} = \left[\frac{5}{8}+\left(\frac{1}{2}+z^2\right)\left(m^2+\frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4}+\left(m^2+\frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+1}+\delta_{m\,m'-1})$$
$$+\left[-\frac{1}{16}+\frac{1}{4}\left(m^2+\frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+2}+\delta_{m\,m'-2}), \tag{64}$$

$$k_{mm'} = z\,\delta_{mm'} - \frac{1}{2}\,(\delta_{m\,m'+1}+\delta_{m\,m'-1}). \tag{65}$$

C. Vacuum Solution

Now,

$$U_m(z) = p_{m-}\,\hat{P}^{\,n}_{|m|-1/2}(z) \tag{66}$$

in the region $z < z_w$, whereas

$$U_m(z) = p_{m+}\,\hat{P}^n_{|m|-1/2}(z) + q_{m+}\,\hat{Q}^n_{|m|-1/2}(z) \tag{67}$$

in the region $z > z_w$. Let $\underline{\underline{I}}_p$ be the matrix of the

$$\left\{\left[\left(\frac{1}{2}+z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\left(\delta_{m\,m'+1}+\delta_{m\,m'-1}\right)\right]\hat{P}^n_{|m'|-1/2}(z)\right\}_{z_w} \tag{68}$$

values. Let $\underline{\underline{I}}_q$ be the matrix of the

$$\left\{\left[\left(\frac{1}{2}+z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\left(\delta_{m\,m'+1}+\delta_{m\,m'-1}\right)\right]\hat{Q}^n_{|m'|-1/2}(z)\right\}_{z_w} \tag{69}$$

values. Let $\underline{\underline{J}}_p$ be the matrix of the

$$\begin{aligned}&\left\{\left[\frac{5}{8}+\left(\frac{1}{2}+z^2\right)\left(m^2+\frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4}+\left(m^2+\frac{n^2}{z^2-1}\right)\right]\left(\delta_{m\,m'+1}+\delta_{m\,m'-1}\right)\right.\\ &\quad\left.+\left[-\frac{1}{16}+\frac{1}{4}\left(m^2+\frac{n^2}{z^2-1}\right)\right]\left(\delta_{m\,m'+2}+\delta_{m\,m'-2}\right)\right\}\hat{P}^n_{|m'|-1/2}(z_w)\end{aligned} \tag{70}$$

values. Let $\underline{\underline{J}}_q$ be the matrix of the

$$\begin{aligned}&\left\{\left[\frac{5}{8}+\left(\frac{1}{2}+z^2\right)\left(m^2+\frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4}+\left(m^2+\frac{n^2}{z^2-1}\right)\right]\left(\delta_{m\,m'+1}+\delta_{m\,m'-1}\right)\right.\\ &\quad\left.+\left[-\frac{1}{16}+\frac{1}{4}\left(m^2+\frac{n^2}{z^2-1}\right)\right]\left(\delta_{m\,m'+2}+\delta_{m\,m'-2}\right)\right\}\hat{Q}^n_{|m'|-1/2}(z_w)\end{aligned} \tag{71}$$

values. Let $\underline{\underline{k}}$ be the matrix of the $k_{mm'}$ values. Finally, let $\underline{p}_+$ be the vector of the p_{m+} values, et cetera. Thus, we obtain

$$\underline{\underline{I}}_p\,\underline{p}_- = \frac{1}{\cosh\lambda}\left(\underline{\underline{I}}_p\,\underline{p}_+ + \underline{\underline{I}}_q\,\underline{q}_+\right), \tag{72}$$

$$\underline{\underline{J}}_p\,\underline{p}_+ + \underline{\underline{J}}_q\,\underline{q}_+ = f_w\,\underline{\underline{k}}\left(\underline{\underline{I}}_p\,\underline{p}_+ + \underline{\underline{I}}_q\,\underline{q}_+\right) + \frac{1}{\cosh\lambda}\,\underline{\underline{J}}_p\,\underline{p}_-, \tag{73}$$

which can be rearranged to give

$$\left(\tanh^2\lambda\,\underline{\underline{J}}_p - f_w\,\hat{\underline{\underline{I}}}_p\right)\underline{p}_+ + \left(\underline{\underline{J}}_{pq} + \tanh^2\lambda\,\underline{\underline{J}}_{qp} - f_w\,\hat{\underline{\underline{I}}}_q\right)\underline{q}_+, \tag{74}$$

where

$$\hat{\underline{\underline{I}}}_p = \underline{\underline{k}}\,\underline{\underline{I}}_p, \tag{75}$$

$$\hat{\underline{\underline{I}}}_q = \underline{\underline{k}}\,\underline{\underline{I}}_q, \tag{76}$$

$$\underline{\underline{J}}_{pq} = \underline{\underline{J}}_{q} - \underline{\underline{J}}_{p}\,\hat{\underline{\underline{I}}}_{p}^{-1}\,\hat{\underline{\underline{I}}}_{q}, \tag{77}$$

$$\underline{\underline{J}}_{qp} = \underline{\underline{J}}_{p}\,\hat{\underline{\underline{I}}}_{p}^{-1}\,\hat{\underline{\underline{I}}}_{q}. \tag{78}$$

Now, $z_w \sim 1/\bar{b}_w$, where $\bar{b}_w$ is the mean wall minor radius. In the large aspect-ratio limit, $b_w \ll 1$, we have $\underline{\underline{I}}_p \sim \mathcal{O}(1)$, $\underline{\underline{I}}_q \sim \mathcal{O}(1)$, $\underline{\underline{J}}_p \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{\underline{J}}_q \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{\underline{K}}_p \sim \mathcal{O}(1/\bar{b}_w)$, $\underline{\underline{K}}_q \sim \mathcal{O}(1/\bar{b}_w)$, and $\underline{\underline{k}} \sim \mathcal{O}(1/\bar{b}_w)$ It follows that $\hat{\underline{\underline{I}}}_p \sim \mathcal{O}(1/\bar{b}_w)$, $\hat{\underline{\underline{I}}}_q \sim \mathcal{O}(1/\bar{b}_w)$, $\underline{\underline{J}}_{pq} \sim \mathcal{O}(1/\bar{b}_w^2)$ and $\underline{\underline{J}}_{qp} \sim \mathcal{O}(1/\bar{b}_w^2)$. Thus, the ratio of the first to the second term multiplying $\underline{p}_+$ in Eq. (74) is

$$\tanh\lambda\,\frac{\bar{d}_w}{\lambda\,\bar{b}_w}. \tag{79}$$

However, the wall analysis is premised on the assumption that

$$\frac{\bar{d}_w}{\lambda\,\bar{b}_w} \ll 1. \tag{80}$$

Hence, the first term is negligible with respect to the second, irrespective of the value of λ. The ratios of the three terms multiplying $\underline{q}_+$ in Eq. (74) are

$$\frac{\bar{d}_w}{\lambda\,\bar{b}_w},\;\; \tanh^2\lambda\,\frac{\bar{d}_w}{\lambda\,\bar{b}_w},\;\; \tanh\lambda. \tag{81}$$

Thus, in the thin-shell limit, $\lambda \ll 1$, the second term is negligible with respect to the first. In the thick-shell limit, $\lambda \gg 1$, the third term is dominant. Thus, we can neglect the second term. Hence, we deduce that

$$\underline{q}_+ = \underline{\underline{\mathcal{F}}}\,\underline{p}_+, \tag{82}$$

where

$$\underline{\underline{\mathcal{F}}} = f_w\,\underline{\underline{I}}\,(\underline{\underline{J}} + f_w\,\underline{\underline{1}})^{-1}, \tag{83}$$

$$\underline{\underline{I}} = -\hat{\underline{\underline{I}}}_{q}^{-1}\,\hat{\underline{\underline{I}}}_{p}, \tag{84}$$

$$\underline{\underline{J}} = \hat{\underline{\underline{I}}}_{p}^{-1}\,(\underline{\underline{J}}_{q}\,\underline{\underline{I}} + \underline{\underline{J}}_{p}). \tag{85}$$

Note that $\underline{\underline{I}} \sim \mathcal{O}(1)$ and $\underline{\underline{J}} \sim \mathcal{O}(1/\bar{b}_w)$.

D. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque acting on the plasma is

$$T_\phi = -2\pi^2\,n\,\mathrm{Im}(\underline{p}_+^\dagger\,\underline{q}_+) = -2\pi^2\,n\,\mathrm{Im}(\underline{p}_+^\dagger\,\underline{\underline{\mathcal{F}}}\,\underline{p}_+) = -\pi^2\,n\,\mathrm{Im}[\underline{p}_+^\dagger\,(\underline{\underline{\mathcal{F}}} - \underline{\underline{\mathcal{F}}}^\dagger)\,\underline{p}_+]. \tag{86}$$

However, we expect this torque to be zero if f_w is real, which implies that $\underline{\underline{\mathcal{F}}} = \underline{\underline{\mathcal{F}}}^\dagger$ when f_w is real. In other words,

$$f_w\,\underline{\underline{I}}\,(\underline{\underline{J}} + f_w\,\underline{\underline{1}})^{-1} = f_w\,(\underline{\underline{J}}^\dagger + f_w\,\underline{\underline{1}})^{-1}\,\underline{\underline{I}}^\dagger, \tag{87}$$

which implies that

$$f_w\,(\underline{\underline{J}}^\dagger + f_w\,\underline{\underline{1}})\,\underline{\underline{I}} = f_w\,\underline{\underline{I}}^\dagger\,(\underline{\underline{J}} + f_w\,\underline{\underline{1}}). \tag{88}$$

However, the previous equation holds for arbitrary real f_w, so we can separately equate the coefficients of f_w and f_w^2 to give

$$\underline{\underline{J}}^\dagger\,\underline{\underline{I}} = \underline{\underline{I}}^\dagger\,\underline{\underline{J}} \tag{89}$$

$$\underline{\underline{I}} = \underline{\underline{I}}^\dagger. \tag{90}$$

It follows that $\underline{\underline{I}}$ and

$$\underline{\underline{K}} = \underline{\underline{I}}\,\underline{\underline{J}} \tag{91}$$

are both real symmetric matrices. In general,

$$\underline{\underline{\mathcal{F}}} - \underline{\underline{\mathcal{F}}}^\dagger = (f_w - f_w^\ast)\,[(\underline{\underline{J}} + f_w\,\underline{\underline{1}})^{-1}]^\dagger\,\underline{\underline{K}}\,(\underline{\underline{J}} + f_w\,\underline{\underline{1}})^{-1}, \tag{92}$$

$$T_\phi = -2\pi^2\,n\,\mathrm{Im}(f_w)\,[(\underline{\underline{J}} + f_w\,\underline{\underline{1}})^{-1}\,\underline{p}_+]^\dagger\,\underline{\underline{K}}\,[(\underline{\underline{J}} + f_w\,\underline{\underline{1}})^{-1}\,\underline{p}_+]. \tag{93}$$

Thus, $\underline{\underline{\mathcal{F}}}$ is clearly Hermitian if f_w is real.

III. MATCHING AT PLASMA/VACUUM INTERFACE

A. Matching Condition

Let r, θ, ϕ be right-handed flux coordinates, where r is a flux-surface label, θ is a poloidal angle that is zero on the inboard mid-plane, and

$$\mathcal{J} \equiv (\nabla r \times \nabla\theta \cdot \nabla\phi)^{-1} = r\,R^2. \tag{94}$$

The plasma/vacuum interface lies at $r = \epsilon$. In the vacuum region between the interface and the wall,

$$V(z,\eta) = \sum_m (z - \cos\eta)^{1/2}\left[p_{m+}\,\hat{P}^n_{|m|-1/2}(z) + q_{m+}\,\hat{Q}^n_{|m|-1/2}(z)\right]\mathrm{e}^{-\mathrm{i}\,m\,\eta}. \tag{95}$$

Thus, if we write

$$V(r,\theta) = \sum_m V_m(r)\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}, \tag{96}$$

$$\psi(r,\theta) = \sum_m \psi_m(r)\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta} \tag{97}$$

in the same region, where

$$\psi(r,\theta) = \mathcal{J}\,\nabla V\cdot\nabla r, \tag{98}$$

then

$$\underline{V} = \underline{\underline{\mathcal{P}}}\;\underline{p}_{\,+} + \underline{\underline{\mathcal{Q}}}\;\underline{q}_{\,+}, \tag{99}$$

$$\underline{\psi} = \underline{\underline{\mathcal{R}}}\;\underline{p}_{\,+} + \underline{\underline{\mathcal{S}}}\;\underline{q}_{\,+}, \tag{100}$$

where $\underline{V}$ is the vector of the $V_m(\epsilon)$ values, $\underline{\psi}$ is the vector of the $\psi_m(\epsilon)$ values, $\underline{\underline{\mathcal{P}}}$ is the matrix of the

$$\mathcal{P}_{mm'} = \oint_{r=\epsilon}(z-\cos\eta)^{1/2}\,\hat{P}^{\,n}_{|m'|-1/2}(z)\,\exp[-\mathrm{i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{101}$$

values, $\underline{\underline{\mathcal{Q}}}$ is the matrix of the

$$\mathcal{Q}_{mm'} = \oint_{r=\epsilon}(z-\cos\eta)^{1/2}\,\hat{Q}^{\,n}_{|m'|-1/2}(z)\,\exp[-\mathrm{i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{102}$$

values, $\underline{\underline{\mathcal{R}}}$ is the matrix of the

$$\begin{aligned}\mathcal{R}_{mm'} = \oint_{r=\epsilon}&\left\{\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\hat{P}^{\,n}_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{d\hat{P}^{\,n}_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z\right.\\ &+\left.\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta - \mathrm{i}\,m'\,(z-\cos\eta)^{1/2}\right]\hat{P}^{\,n}_{|m'|-1/2}(z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\right\}\\ &\times\exp[-\mathrm{i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi}\end{aligned} \tag{103}$$

values, and $\underline{\underline{\mathcal{S}}}$ is the matrix of the

$$\begin{aligned}\mathcal{S}_{mm'} = \oint_{r=\epsilon}&\left\{\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\hat{Q}^{\,n}_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{d\hat{Q}^{\,n}_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z\right.\\ &+\left.\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta - \mathrm{i}\,m'\,(z-\cos\eta)^{1/2}\right]\hat{Q}^{\,n}_{|m'|-1/2}(z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\right\}\\ &\times\exp[-\mathrm{i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi}\end{aligned} \tag{104}$$

Equations (82), (96), and (97) imply that

$$\underline{V} = (\underline{\underline{\mathcal{P}}} + \underline{\underline{\mathcal{Q}}}\,\underline{\underline{\mathcal{F}}})\,\underline{p}_{+}, \tag{105}$$

$$\underline{\psi} = (\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}})\,\underline{p}_{+}, \tag{106}$$

which yields

$$\underline{V} = \underline{\underline{H}}\;\underline{\psi}, \tag{107}$$

where

$$\underline{\underline{H}} = (\underline{\underline{\mathcal{P}}} + \underline{\underline{\mathcal{Q}}}\,\underline{\underline{\mathcal{F}}})\,(\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}})^{-1}. \tag{108}$$

B. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque acting on the plasma is

$$\begin{aligned} T_\phi &= -2\pi^2\,n\,\mathrm{Im}(\underline{V}^\dagger\,\underline{\psi}) \\ &= -2\pi^2\,n\,\mathrm{Im}[\underline{p}_{+}^\dagger\,(\underline{\underline{\mathcal{P}}}^\dagger + \underline{\underline{\mathcal{F}}}^\dagger\,\underline{\underline{\mathcal{Q}}}^\dagger)\,(\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}})\,\underline{p}_{+}] \\ &= -2\pi^2\,n\,\mathrm{Im}[\underline{p}_{+}^\dagger\,(\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{F}}}^\dagger\,\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}} + \underline{\underline{\mathcal{F}}}^\dagger\,\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}})\,\underline{p}_{+}] \\ &= -\pi^2\,n\,\mathrm{Im}[\underline{p}_{+}^\dagger\,(\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{R}}} - \underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{P}}})\,\underline{p}_{+}] - \pi^2\,n\,\mathrm{Im}[\underline{p}_{+}^\dagger\,(\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{S}}} - \underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{Q}}})\,\underline{\underline{\mathcal{F}}}\,\underline{p}_{+}] \\ &\quad + \pi^2\,n\,\mathrm{Im}[\underline{p}_{+}^\dagger\,\underline{\underline{\mathcal{F}}}^\dagger\,(\underline{\underline{\mathcal{S}}}^\dagger\,\underline{\underline{\mathcal{P}}} - \underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{R}}})\,\underline{p}_{+}] - \pi^2\,n\,\mathrm{Im}[\underline{p}_{+}^\dagger\,\underline{\underline{\mathcal{F}}}^\dagger\,(\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{S}}} - \underline{\underline{\mathcal{S}}}^\dagger\,\underline{\underline{\mathcal{Q}}})\,\underline{\underline{\mathcal{F}}}\,\underline{p}_{+}]. \end{aligned} \tag{109}$$

The previous equation is consistent with Eq. (86) provided that

$$\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{R}}} = \underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{P}}}, \tag{110}$$

$$\underline{\underline{\mathcal{Q}}}^\dagger\,\underline{\underline{\mathcal{S}}} = \underline{\underline{\mathcal{S}}}^\dagger\,\underline{\underline{\mathcal{Q}}}, \tag{111}$$

$$\underline{\underline{\mathcal{P}}}^\dagger\,\underline{\underline{\mathcal{S}}} - \underline{\underline{\mathcal{R}}}^\dagger\,\underline{\underline{\mathcal{Q}}} = \underline{\underline{1}}. \tag{112}$$

Making use of the previous three equations, we can show that

$$\underline{\underline{H}} - \underline{\underline{H}}^\dagger = -[(\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}})^{-1}]^\dagger\,(\underline{\underline{\mathcal{F}}} - \underline{\underline{\mathcal{F}}}^\dagger)\,(\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}})^{-1}. \tag{113}$$

Thus, $\underline{\underline{H}}$ is Hermitian if $\underline{\underline{\mathcal{F}}}$ is Hermitian, which implies that $\underline{\underline{H}}$ is Hermitian if f_w is real.

IV. VACUUM MATRIX

A. No-Wall and Perfect-Wall Boundary Conditions

In the no-wall limit, $f_w = 0$, and $\underline{\underline{\mathcal{F}}} = \underline{\underline{0}}$. Hence, the boundary condition at the plasma/vacuum interface becomes

$$\underline{V} = \underline{\underline{H}}_{nw}\,\underline{\psi}, \tag{114}$$

where

$$\underline{\underline{H}}_{nw} = \underline{\underline{\mathcal{P}}}\,\underline{\underline{\mathcal{R}}}^{-1}. \tag{115}$$

Equation (110) implies that $\underline{\underline{H}}_{nw}$ is Hermitian.

In the perfect-wall limit, $f(\lambda) \to \infty$, and $\underline{\underline{\mathcal{F}}} = \underline{\underline{I}}$. Hence, the boundary condition at the interface becomes

$$\underline{V} = \underline{\underline{H}}_{pw}\,\underline{\psi}, \tag{116}$$

where

$$\underline{\underline{H}}_{pw} = (\underline{\underline{\mathcal{P}}} + \underline{\underline{\mathcal{Q}}}\,\underline{\underline{I}})\,(\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{S}}}\,\underline{\underline{I}})^{-1}. \tag{117}$$

Comparison to Eqs. (108) and (113) reveals that $\underline{\underline{H}}_{pw}$ is Hermitian, given that that $\underline{\underline{I}}$ is Hermitian.

B. Symmetric form of Vacuum Matrix

It is possible to demonstrate that

$$\underline{\underline{H}} = \underline{\underline{H}}_{nw} + f_w\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1}, \tag{118}$$

where

$$\underline{\underline{B}} = \underline{\underline{\mathcal{R}}}\,\underline{\underline{J}}\,(\underline{\underline{\mathcal{R}}} + \underline{\underline{\mathcal{S}}}\,\underline{\underline{I}})^{-1}. \tag{119}$$

Note that $\underline{\underline{B}} \sim \mathcal{O}(1/\bar{b}_w)$. Suppose that f_w is real. It follows that

$$\underline{\underline{H}} - \underline{\underline{H}}^{\dagger} = f_w\left[(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1} - (\underline{\underline{B}}^{\dagger} + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\right]. \tag{120}$$

However, we know that $\underline{\underline{H}}$ is Hermitian when f_w is real. Hence, we deduce that

$$(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1} = (\underline{\underline{B}}^{\dagger} + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw} - \underline{\underline{H}}_{nw}), \tag{121}$$

which implies that

$$\underline{\underline{B}}^{\dagger}\,(\underline{\underline{H}}_{pw}-\underline{\underline{H}}_{nw}) = (\underline{\underline{H}}_{pw}-\underline{\underline{H}}_{nw})\,\underline{\underline{B}}. \tag{122}$$

The previous equation yields

$$\begin{aligned}(\underline{\underline{\mathcal{R}}}^{\dagger}+\underline{\underline{I}}\,\underline{\underline{\mathcal{S}}}^{\dagger})\,\underline{\underline{J}}^{\dagger}\,\underline{\underline{\mathcal{R}}}^{\dagger}&\left[(\underline{\underline{\mathcal{P}}}+\underline{\underline{\mathcal{Q}}}\,\underline{\underline{I}})\,(\underline{\underline{\mathcal{R}}}+\underline{\underline{\mathcal{S}}}\,\underline{\underline{I}})^{-1}-\underline{\underline{\mathcal{R}}}^{-1\,\dagger}\,\underline{\underline{P}}^{\dagger}\right]\\
&=\left[(\underline{\underline{\mathcal{R}}}^{\dagger}+\underline{\underline{I}}\,\underline{\underline{\mathcal{S}}}^{\dagger})^{-1}\,(\underline{\underline{\mathcal{P}}}^{\dagger}+\underline{\underline{I}}\,\underline{\underline{\mathcal{Q}}}^{\dagger})-\underline{\underline{\mathcal{P}}}\,\underline{\underline{\mathcal{R}}}^{-1}\right]\underline{\underline{\mathcal{R}}}\,\underline{\underline{J}}\,(\underline{\underline{\mathcal{R}}}+\underline{\underline{\mathcal{S}}}\,\underline{\underline{I}})^{-1}.\end{aligned} \tag{123}$$

where use has been made of the fact that $\underline{\underline{H}}_{pw}$ and $\underline{\underline{H}}_{nw}$ are Hermitian. Making use of Eqs. (110)–(112), the previous equation reduces to

$$-\underline{\underline{J}}^{\dagger}\,\underline{\underline{I}} = -\underline{\underline{I}}\,\underline{\underline{J}}, \tag{124}$$

which Eqs. (89) and (90) ensure is satisfied. Equations (118) and (121) can be combined to give

$$\begin{aligned}\underline{\underline{H}} &= \underline{\underline{H}}_{nw}+\frac{f_w}{2}\left[(\underline{\underline{B}}^{\dagger}+f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw}-\underline{\underline{H}}_{nw})+(\underline{\underline{H}}_{pw}-\underline{\underline{H}}_{nw})\,(\underline{\underline{B}}+f_w\,\underline{\underline{1}})^{-1}\right]\\
&= \underline{\underline{H}}_{nw}+f_w\,(\underline{\underline{B}}^{\dagger}+f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{H}}_{pw}-\underline{\underline{H}}_{nw}),\\
&= \underline{\underline{H}}_{nw}+f_w\,(\underline{\underline{H}}_{pw}-\underline{\underline{H}}_{nw})\,(\underline{\underline{B}}+f_w\,\underline{\underline{1}})^{-1}.\end{aligned} \tag{125}$$

Note that $\underline{\underline{H}}$ is clearly Hermitian if f_w is real.

V. WALL TORQUE

Consider an unreconnected tearing mode, resonant at the kth rational surface, and rotating at the angular phase velocity ω_k. It follows that $\gamma = -\mathrm{i}\,\omega_k$. Thus,

$$f_w = \frac{\zeta}{2\,\bar{d}_w}\left[\left(\frac{\sinh\zeta-\sin\zeta}{\cosh\zeta+\cos\zeta}\right)-\mathrm{i}\left(\frac{\sinh\zeta+\sin\zeta}{\cosh\zeta+\cos\zeta}\right)\right], \tag{126}$$

where

$$\zeta = (2\,\hat{\omega}_k\,\bar{d}_w)^{1/2}, \tag{127}$$

and $\hat{\omega}_k = \omega_k\,\bar{\tau}_w$. The matching condition at the plasma vacuum interface becomes

$$\underline{V} = \underline{\underline{H}}(\zeta)\;\underline{\psi}. \tag{128}$$

Note that $\underline{\underline{H}}(\zeta)$ is not generally Hermitian, because f_w is complex, which implies that the E-matrix is not Hermitian. The toroidal electromagnetic torque acting at the kth rational surface is

$$\delta T_k = 2\pi^2\,n\,\mathrm{Im}(E_{kk})\,|\Psi_k|^2. \tag{129}$$

VI. RESISTIVE WALL MODE

We can write

$$\underline{V} = \underline{\underline{V}}_i\,\underline{\alpha}, \tag{130}$$

$$\underline{\psi} = \underline{\underline{\psi}}_i\,\underline{\alpha}, \tag{131}$$

where the $\underline{\underline{V}}_i$ and $\underline{\underline{\psi}}_i$ are ideal solutions. The net toroidal electromagnetic torque acting on the plasma is

$$T_\phi = -2\pi^2\,n\,\mathrm{Im}(\underline{V}^\dagger\,\underline{\psi}) = -2\pi^2\,n\,\mathrm{Im}(\underline{\alpha}^\dagger\,\underline{\underline{V}}_i^\dagger\,\underline{\underline{\psi}}_i\,\underline{\alpha}). \tag{132}$$

However, the net torque acting on an ideal plasma is zero, so

$$\underline{\underline{V}}_i^\dagger\,\underline{\underline{\psi}}_i = \underline{\underline{\psi}}_i^\dagger\,\underline{\underline{V}}_i. \tag{133}$$

Equation (107) implies that

$$\underline{\underline{V}}_i\,\underline{\alpha} = \underline{\underline{H}}\,\underline{\underline{\psi}}_i\,\underline{\alpha}. \tag{134}$$

Writing

$$\underline{\underline{\psi}}_i\,\underline{\alpha} = \underline{x}, \tag{135}$$

we obtain

$$\underline{\underline{W}}_p\,\underline{x} = \underline{\underline{H}}\,\underline{x}, \tag{136}$$

where

$$\underline{\underline{W}}_p = \underline{\underline{V}}_i\,\underline{\underline{\psi}}_i^{-1}. \tag{137}$$

Equation (133) ensures that $\underline{\underline{W}}_p$ is Hermitian.

Equation (125) can be combined with the previous equation to give

$$\underline{\underline{W}}_{nw}\,\underline{\beta} = -\frac{f_w}{2}\left[(\underline{\underline{W}}_{pw} - \underline{\underline{W}}_{nw})\,(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1} + (\underline{\underline{B}}^\dagger + f_w\,\underline{\underline{1}})^{-1}\,(\underline{\underline{W}}_{pw} - \underline{\underline{W}}_{nw})\right]\underline{\beta}, \tag{138}$$

where

$$\underline{\underline{W}}_{nw} = \underline{\underline{W}}_p - \underline{\underline{H}}_{nw}, \tag{139}$$

$$\underline{\underline{W}}_{pw} = \underline{\underline{W}}_p - \underline{\underline{H}}_{pw}. \tag{140}$$

Note that $\underline{\underline{W}}_{nw}$ and $\underline{\underline{W}}_{pw}$ are Hermitian. Writing

$$(\underline{\underline{B}} + f_w\,\underline{\underline{1}})^{-1}\,\underline{x} = \underline{y}, \tag{141}$$

we obtain

$$\left(f_w\,\underline{\underline{W}}_{pw}+\underline{\underline{B}}^{\dagger}\,\underline{\underline{W}}_{nw}\right)\underline{x}=\underline{0},\tag{142}$$

$$\left(f_w\,\underline{\underline{W}}_{pw}+\underline{\underline{W}}_{nw}\,\underline{\underline{B}}\right)\underline{y}=\underline{0}.\tag{143}$$

Note that the $\underline{x}$ are the left-eigenvectors of Eq. (156), whereas the $\underline{y}$ are the left-eigenvectors of Eq. (142). The previous two equations can be combined to give

$$(f_w-f_w^{\ast})\,\underline{y}^{\dagger}\,\underline{\underline{W}}_{pw}\,\underline{x}=0.\tag{144}$$

Hence, we deduce that the eigenvalues, f_w, are real, and that the eigenvectors, $\underline{x}$ and $\underline{y}$, are orthonormal, in the sense that

$$\underline{y}_i^{\dagger}\,\underline{\underline{W}}_{pw}\,\underline{x}_j=\delta_{ij}.\tag{145}$$

Once we have determined the eigenvectors then

$$\underline{\psi}=\underline{x},\tag{146}$$

$$\underline{V}=\underline{\underline{W}}_{p}\,\underline{x}.\tag{147}$$

We can write

$$\underline{\psi}=\underline{\underline{Q}}\,\underline{\Xi},\tag{148}$$

$$\underline{Z}=\underline{\underline{Q}}\,\underline{V},\tag{149}$$

where $\underline{\underline{Q}}$ is the diagonal matrix of the $m-n\,q(\epsilon)$ values. Let

$$\underline{\underline{\widetilde{W}}}_{nw}=\underline{\underline{Q}}\,\underline{\underline{W}}_{nw}\,\underline{\underline{Q}},\tag{150}$$

$$\underline{\underline{\widetilde{W}}}_{pw}=\underline{\underline{Q}}\,\underline{\underline{W}}_{pw}\,\underline{\underline{Q}},\tag{151}$$

$$\underline{\underline{\widetilde{B}}}=\underline{\underline{Q}}^{-1}\,\underline{\underline{B}}\,\underline{\underline{Q}},\tag{152}$$

$$\underline{\widetilde{x}}=\underline{\underline{Q}}^{-1}\,\underline{x},\tag{153}$$

$$\underline{\widetilde{y}}=\underline{\underline{Q}}^{-1}\,\underline{x}.\tag{154}$$

It follows that

$$\left(f_w\,\underline{\underline{\widetilde{W}}}_{pw}+\underline{\underline{\widetilde{B}}}^{\dagger}\,\underline{\underline{\widetilde{W}}}_{nw}\right)\underline{\widetilde{x}}=\underline{0},\tag{155}$$

$$\left(f_w \, \underline{\underline{\widetilde{W}}}_{pw} + \underline{\underline{\widetilde{W}}}_{nw} \, \underline{\underline{\widetilde{B}}}\right) \underline{\widetilde{y}} = \underline{0}. \tag{156}$$

Hence, we again deduce that

$$(f_w - f_w^*) \, \underline{\widetilde{y}}^\dagger \, \underline{\underline{\widetilde{W}}}_{pw} \, \underline{\widetilde{x}} = 0. \tag{157}$$

Furthermore,

$$\underline{\Xi} = \underline{\widetilde{x}}, \tag{158}$$

$$\underline{Z} = \underline{\underline{\widetilde{W}}}_p \, \underline{\widetilde{x}}. \tag{159}$$