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PAPER

Calculation of Tearing Mode Stability in a Magnetically Diverted Tokamak Plasma

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1 Two-Filament Model of a Magnetically Diverted Plasma

1.1 Introduction

The aim of this section is to construct a very simple model of a magnetically diverted tokamak plasma.

1.2 Equilibrium Magnetic Field

Suppose that two current filaments run parallel to the z-axis [1]. Let the first filament carry the current I_p, and pierce the x-y plane at $x = y = 0$. Let the second filament carry the current I_c, and pierce the x-y plane at $x = 0$, $y = -a$. Here, a is the effective minor radius of the plasma. The first filament represents the "toroidal" (i.e., z-directed) plasma current, whereas the second represents the current flowing in the magnetic divertor coil. Suppose that there is a uniform externally generated "toroidal" (i.e., z-directed) magnetic field of strength B_0. Let the system be periodic in the z direction with period $2\pi\,R_0$, where R_0 is the simulated major radius of the plasma. It is helpful to define the simulated toroidal angle, $\phi = z/R_0$.

The equilibrium magnetic field can be written in the divergence-free form

$$\mathbf{B} = \nabla\phi\times\nabla\psi_p + B_0\,R_0\,\nabla\phi, \tag{1}$$

where

$$\psi_p(x,y) = \frac{\mu_0\,I_0\,R_0}{4\pi}\,\ln(x^2+y^2) + \frac{\mu_0\,I_1\,R_0}{4\pi}\,\ln[x^2+(y+a)^2] \tag{2}$$

is the "poloidal" magnetic flux (divided by 2π) generated by the two current filaments.

1.3 Flux Coordinates

It is convenient to re-express the magnetic field as

$$\mathbf{B} = \nabla(\phi - q\,\theta)\times\nabla\psi_p, \tag{3}$$

where θ is a poloidal (i.e., in the x-y plane) angle, and $q = q(\psi_p)$ is the safety-factor [2]. Equations (1) and (3) can be reconciled provided

$$\nabla\psi_p\times\nabla\theta\cdot\nabla\phi = \frac{B_0}{R_0\,q}. \tag{4}$$

Note, from Eq. (3), that $\mathbf{B}\cdot\nabla\psi_p = 0$, which implies that ψ_p is a magnetic flux-surface label. Furthermore, $\mathbf{B}\cdot\nabla(\phi - q\,\theta) = 0$, which implies that magnetic field-lines within a given flux-surface appear as straight lines, with gradient $d\phi/d\theta = q$, when plotted in the θ-ϕ plane. In fact, ψ_p, θ, ϕ are known as *flux-coordinates*, and θ is termed a "straight" poloidal angle [2].

1.4 Non-Diverted Edge Safety-Factor

In the absence of the divertor current, the plasma would have a circular cross-section of minor radius a, and an edge safety-factor of

$$q_\ast = \frac{2\pi\,B_0\,a^2}{\mu_0\,I_p\,R_0}. \tag{5}$$

1.5 Normalization Scheme

Let $x = a\,X$, $y = a\,Y$, $\nabla = a^{-1}\,\hat{\nabla}$, and $\psi_p = \mu_0\,I_p\,R_0\,\psi/(2\pi)$. It follows that

$$\hat{\nabla}\psi\times\hat{\nabla}\theta\cdot\hat{\nabla}\phi = \frac{a}{R_0}\,\frac{q_\ast}{q}, \tag{6}$$

$$\psi = \frac{1}{2}\,\ln\left(X^2+Y^2\right)+\frac{\beta}{2}\,\ln\left[X^2+(Y+1)^2\right], \tag{7}$$

$$\psi_X = \frac{X}{X^2+Y^2}+\frac{\beta\,X}{X^2+(Y+1)^2}, \tag{8}$$

$$\psi_Y = \frac{Y}{X^2+Y^2}+\frac{\beta\,(Y+1)}{X^2+(Y+1)^2}, \tag{9}$$

where $\beta = I_c/I_p$. Here, $\psi_X\equiv\partial\psi/\partial X$, et cetera.

1.6 Magnetic X-point

The magnetic X-point forms at the point in the X-Y plane at which $\psi_X=\psi_Y=0$. As is easily demonstrated, the coordinates of this point are $(X_c,\,Y_c)$, where $X_c=0$ and $Y_c=-1/(1+\beta)$. The magnetic separatrix corresponds to the curve $\psi(X,Y)=\psi_c$, where

$$\psi_c\equiv\psi(X_c,Y_c) = \ln\left[\frac{\beta^{\,\beta}}{(1+\beta)^{1+\beta}}\right]. \tag{10}$$

It is helpful to define the normalized poloidal flux $\Psi = \psi_c/\psi$.

1.7 Construction of Flux Coordinate System

Equation (6) yields

$$\frac{d\theta}{dL} = \frac{q_\ast}{q\,|\hat{\nabla}\psi|}. \tag{11}$$

where $\hat{\nabla} = a\,\nabla$, and dL is an element of normalized length around a magnetic flux-surface. It follows that

$$q(\psi) = \frac{q_\ast}{2\pi}\oint\frac{dL}{|\hat{\nabla}\psi|}, \tag{12}$$

where $\oint$ implies a complete circuit in θ at constant ψ. It is easily demonstrated that, on such a circuit,

$$\frac{dX}{dL} = -\frac{\psi_Y}{\sqrt{\psi_X^2+\psi_Y^2}}, \tag{13}$$

$$\frac{dY}{dL} = \frac{\psi_X}{\sqrt{\psi_X^2+\psi_Y^2}}, \tag{14}$$

$$\frac{d\phi}{dL} = \frac{q_\ast}{\sqrt{\psi_X^2+\psi_Y^2}}, \tag{15}$$

$$\frac{d\omega}{dL} = \frac{X\,\psi_X+Y\,\psi_Y}{(X^2+Y^2)\sqrt{\psi_X^2+\psi_Y^2}}, \tag{16}$$

where

$$\omega = \tan^{-1}\left(\frac{Y}{X}\right) \tag{17}$$

is a geometric poloidal angle. Here, ϕ is calculated on the assumption that we are following a magnetic field-line within the flux-surface (i.e., $d\phi/d\theta = q$). We need to integrate Eqs. (13)–(16) from $\omega=0$ to $\omega=2\pi$, subject to the initial condition $\phi(\omega=0)=0$, and then set $q(\psi)=\phi(\omega=2\pi)/(2\pi)$. We can then compute θ using

$$\frac{d\theta}{dL} = \frac{1}{q\sqrt{\psi_X^2+\psi_Y^2}}. \tag{18}$$

1.8 Results

Let $q_* = 12$ and $\beta = 0.2$. Figure 1 shows the magnetic flux-surfaces $\Psi = 0.9$, $\Psi = 1.0$, and $\Psi = 1.1$, plotted in the X-Y plane. Flux-surfaces characterized by $\Psi < 1$ do not enclose the divertor current filament, whereas those characterized by $\Psi < 1$ do enclose the filament. The magnetic separatrix, $\Psi = 1$, separates flux-surfaces that do and do not enclose the divertor current filament, and crosses itself at the magnetic X-point.

Figure 2 shows the safety-factor profile $q(\Psi)$. It is clear that the safety-factor generally increases with increasing Ψ. However, $q \to \infty$ as $\Psi \to 1$. In other words, the safety-factor tends to infinity as the magnetic separatrix is approached. It is apparent from the bottom panel that q approaches infinity logarithmically as $\Psi \to 1$ (because the plot of q versus $\log_{10}(|\Psi - 1|)$ asymptotes to a straight line as $|\Psi - 1| \to 0$). In other words, close to the separatrix we can write

$$q(\Psi) \simeq -\alpha_- \log(1 - \Psi) \tag{19}$$

for $\Psi < 1$, and

$$q(\Psi) \simeq -\alpha_+ \log(\Psi - 1) \tag{20}$$

for $\Psi > 1$. Moreover, it is clear from the figure that $\alpha_+ > \alpha_-$.

Figures 3 and 4 show the flux coordinate system inside and outside the magnetic separatrix, respectively. It can be seen that, as the magnetic separatrix is crossed, all the contours of θ converge onto the X-point, and then diverge away from it. This singular behavior occurs because the Jacobian of the coordinate system, $\mathcal{J} \equiv (\hat{\nabla}\psi \times \hat{\nabla}\theta \cdot \hat{\nabla}\phi)^{-1} = (R_0/a)\,(q/q_*)$, is infinite at the X-point.

1.9 Significance of Flux Coordinates

To understand the significance of the flux coordinate system, suppose that the plasma is subject to a magnetic perturbation that varies with θ, ϕ and time, t, as $\exp[\,{\rm i}\,(m\,\theta - n\,\phi) + \gamma\,t]$. Here, m and n are integers. In other words, the perturbation, which is, of course, single-valued in the angular coordinates θ and ϕ, possesses m periods in the poloidal angle and n periods in the toroidal angle. The scalar product of the curl of the linearized MHD Ohm's law yields

$$-\gamma\, b^{\psi_p} + {\rm i}\,\gamma\,\frac{B_0}{R_0\,q}\,(m - n\,q)\,\xi^{\psi_p} \simeq \frac{B_0}{R_0\,q}\,{\rm i}\,m\,\eta_\parallel\,j_\phi, \tag{21}$$

where subscripts and superscripts denote covariant and contravariant components in the ψ_p, θ, ϕ coordinate system. Here, $\mathbf{b}$ is the perturbed magnetic field, $\boldsymbol{\xi}$ the plasma displacement, $\mathbf{j}$ the perturbed current density, and $\eta_\parallel$ the parallel plasma electrical resistivity.

The previous equation described how the inductive electric field generated by a time-varying magnetic field drives a current parallel to magnetic field-lines. On a general magnetic flux-surface, the first two terms on the left-hand side of the previous equation cancel one another out, and there is no driven current. In other words, the plasma displaces rather than allowing a parallel inductive current to flow. However, it is clear from the equation that there exist special magnetic flux-surfaces, termed *rational* flux-surfaces, at which the safety-factor takes the rational value $q = m/n$. On a rational flux-surface, the two terms on the left-hand side of the previous equation cannot cancel one another out, because the second term is zero. Hence, in general, a parallel inductive current flows on a rational flux-surface. The current is a shielding current that acts to suppress magnetic reconnection on the flux-surface, or, at least, to slow down reconnection such that it takes place on the comparatively long resistive timescale.

We can now appreciate that by employing a flux coordinate system we can distinguish rational magnetic flux-surfaces from irrational flux-surfaces. We can also calculate the angular variation of the particular magnetic perturbation that drives a shielding current at a particular rational surface. Note, finally, that it is clear from Figs. 2–4 that rational flux-surfaces exist both inside and outside the magnetic separatrix.

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Data availability

The digital data used in the figures in this paper can be obtained from the author upon reasonable request.

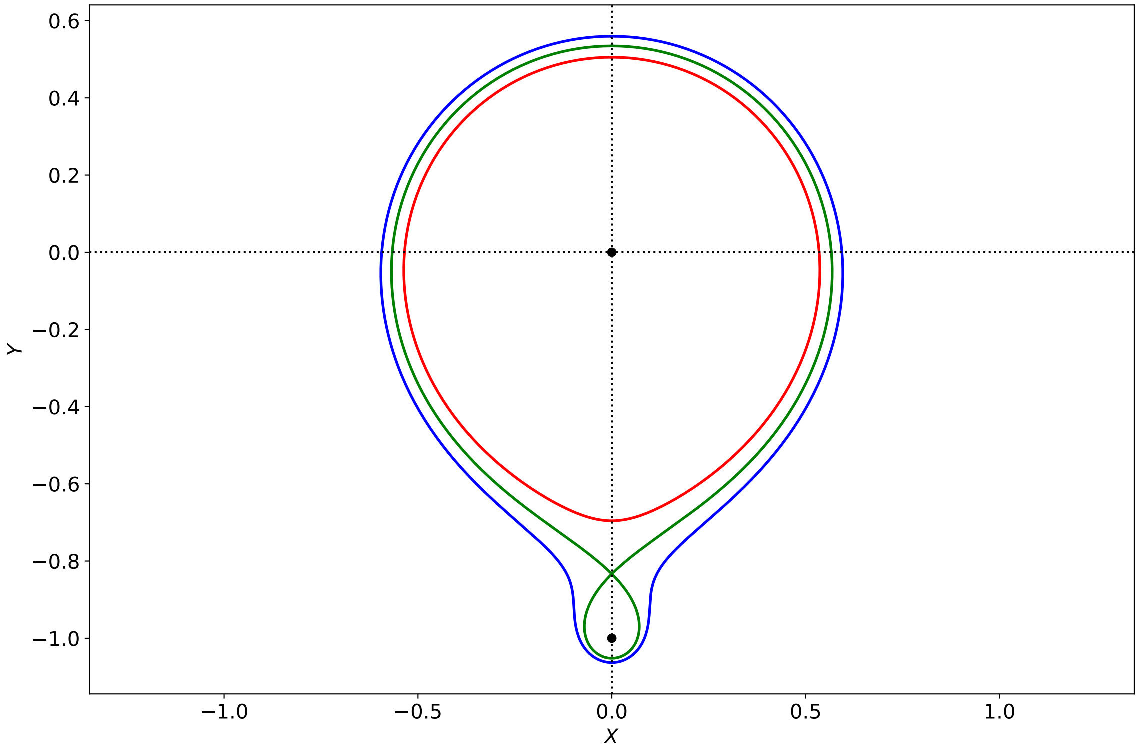

Figure 1. The magnetic flux-surfaces $\Psi = 0.9$ (red), $\Psi = 1.0$ (green), and $\Psi = 1.1$ (blue), plotted in the X-Y plane. The black dots shows the locations of the two current filaments. Here, $\beta = 0.2$.

References

[1] N. Pomphrey and A. Reiman, Phys. Fluids B **4**, 938 (1992).

[2] A.H. Boozer, Rev. Mod. Phys. **76**, 1071 (2004).

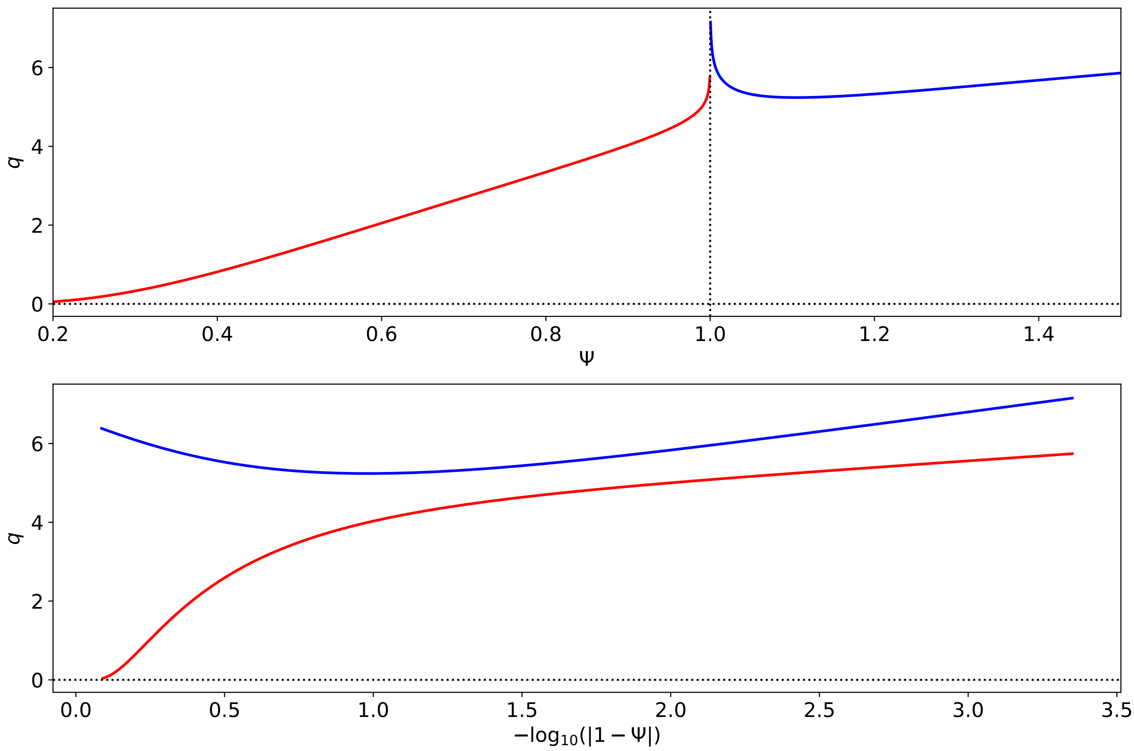

Figure 2. The safety-factor $q(\Psi)$ calculated for $q_* = 12$ and $\beta = 0.2$.

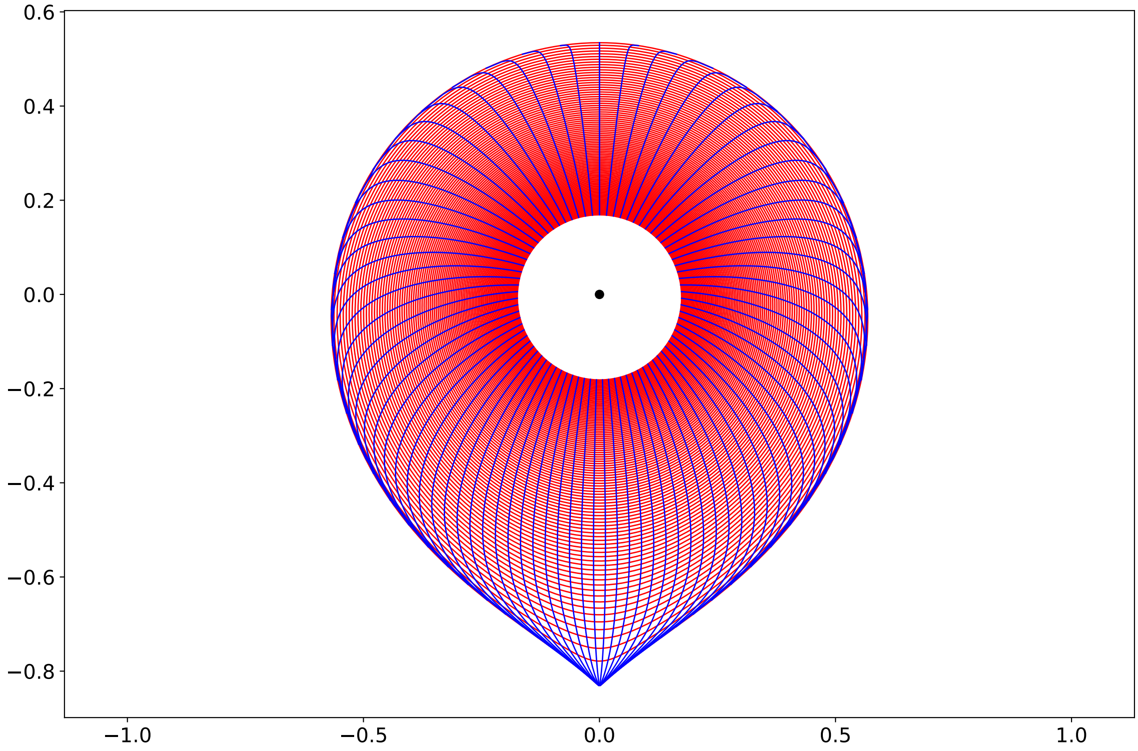

Figure 3. The flux coordinate system inside the magnetic separatrix calculated for $q_* = 12$ and $\beta = 0.2$. The red curves are surfaces of constant $\hat{\psi}$, whereas the blue curves are surfaces of constant θ. The black dots shows the locations of the plasma current filaments.

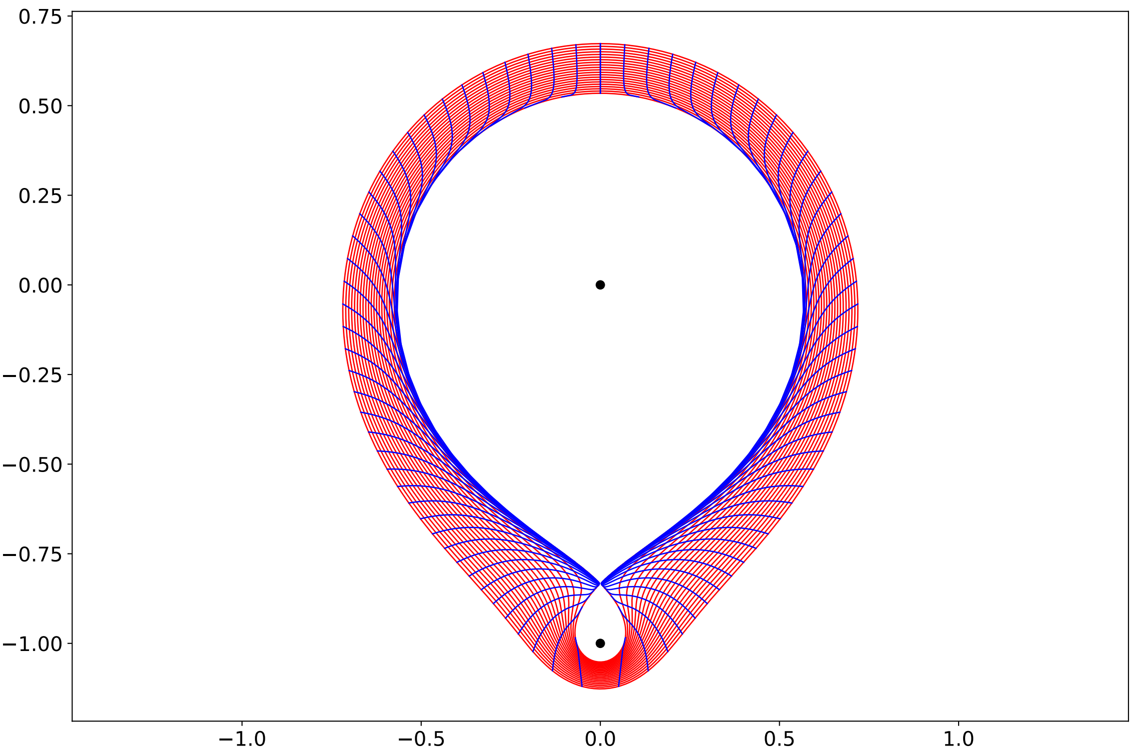

Figure 4. The flux coordinate system outside the magnetic separatrix calculated for $q_* = 12$ and $\beta = 0.2$. The red curves are surfaces of constant $\hat{\psi}$, whereas the blue curves are surfaces of constant θ. The black dots shows the locations of the two current filaments.

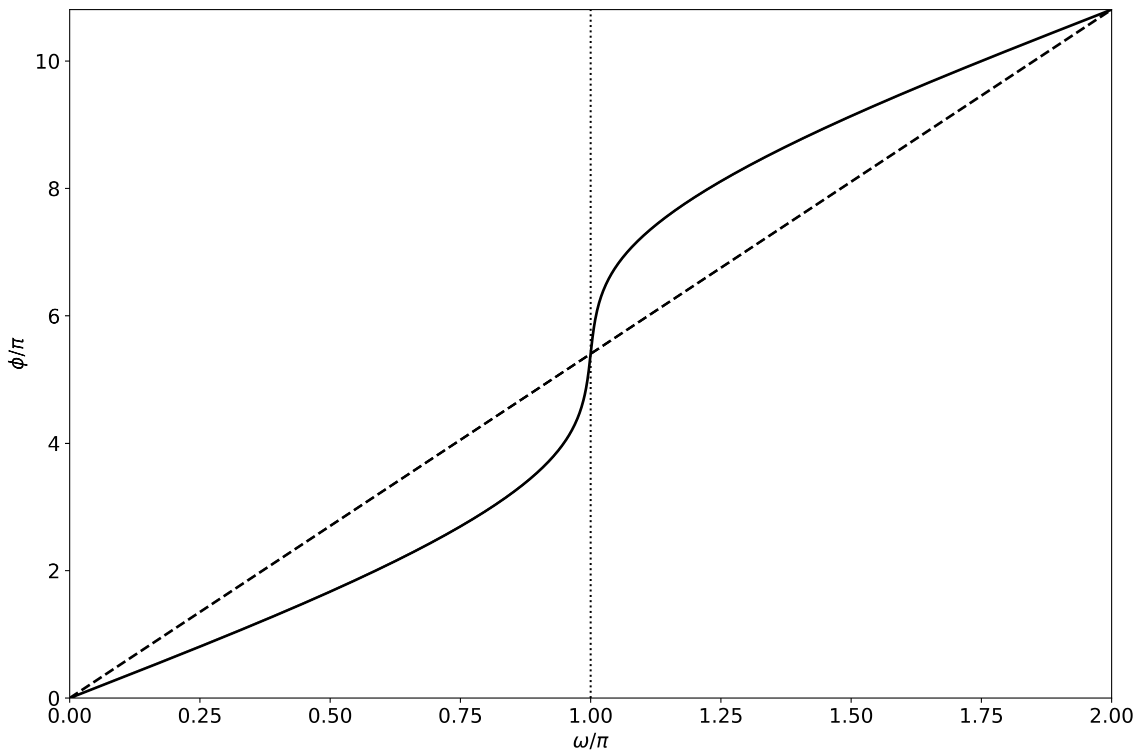

Figure 5. The variation of the toroidal angle, ϕ, with the geometric poloidal angle, ω, on a magnetic field-line lying within a magnetic flux-surface characterized by $\Psi = 0.999$, calculated with $q_* = 12$ and $\beta = 0.2$. The dashed line shows the mean gradient of the field-line.