Four field equations parallel flow

Retaining the V_z' term in the four field model while still neglecting electron viscosity results in the additional term shown in the ODE for $Y(p)$

$$
\begin{aligned}
\frac{d}{dp}\left(\frac{p^2}{i(Q-Q_e)+p^2}Y'\right) - G(p)p^2Y - \frac{c_\beta^2}{D^2}\frac{i(Q-Q_i)D^2p^2+(1+\tau)PD^2p^4}{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}\frac{dV_z}{dp} &= 0 \\
G(p) &= \frac{-Q(Q-Q_i)+i(Q-Q_i)(P+c_\beta^2)p^2+c_\beta^2Pp^4}{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}
\end{aligned}
$$

with the equation for V_z being

$$
\begin{aligned}
V_z &= \frac{\frac{dZ}{dp} - iQ_e\psi}{iQ+Pp^2} \\
&= \frac{\frac{d}{dp}\left(\frac{1-F(p)}{F(p)}Y + \frac{c_\beta^2}{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}\frac{dV_z}{dp}\right) - iQ_e\frac{Y'}{i(Q-Q_e)+p^2}}{iQ+Pp^2} \qquad (278) \\
F(p) &= \frac{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}{iQ+c_\beta^2p^2+\tau PD^2p^4}
\end{aligned}
$$

Rearranging the first equation yields

$$
\frac{dV_z}{dp} = \frac{\frac{d}{dp}\left(\frac{p^2}{i(Q-Q_e)+p^2}Y'\right) - G(p)p^2Y}{\frac{c_\beta^2}{D^2}\frac{i(Q-Q_i)D^2p^2+(1+\tau)PD^2p^4}{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}}
$$

which we can then substitute into the derivative of (278), yielding

$$
\begin{aligned}
&\frac{d}{dp}\left(\frac{p^2}{i(Q-Q_e)+p^2}Y'\right) - G(p)p^2Y - \frac{c_\beta^2}{D^2}\frac{i(Q-Q_i)D^2p^2+(1+\tau)PD^2p^4}{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}\frac{d}{dp}\Bigg(\\
&\frac{\frac{d}{dp}\left(\frac{1-F(p)}{F(p)}Y + \frac{c_\beta^2}{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}\left(\frac{\frac{d}{dp}\left(\frac{p^2}{i(Q-Q_e)+p^2}Y'\right)-G(p)p^2Y}{\frac{c_\beta^2}{D^2}\frac{i(Q-Q_i)D^2p^2+(1+\tau)PD^2p^4}{i(Q-Q_e)+(c_\beta^2+i(Q-Q_i)D^2)p^2+(1+\tau)PD^2p^4}}\right)\right) - iQ_e\frac{Y'}{i(Q-Q_e)+p^2}}{iQ+Pp^2}\Bigg) = 0
\end{aligned}
$$

where we have a fourth order ODE in $Y(p)$ in which V_z has been decoupled. In the limit $P \to 0, D \to 0$, this equation simplifies significantly to

$$
\begin{aligned}
\frac{d}{dp}\left(\frac{p^2Y'}{i(Q-Q_e)+p^2}\right) - G(p)p^2Y - \frac{c_\beta^2 i(Q-Q_i)p^2)}{iQ(i(Q-Q_e)+c_\beta^2p^2)}\frac{d}{dp}\Bigg[\frac{d}{dp}\left(\frac{iQ_eY}{i(Q-Q_e)+c_\beta^2p^2}\right) - \\
\frac{iQ_eY'}{i(Q-Q_e)+p^2} + \frac{d}{dp}\left(\frac{1}{i(Q-Q_i)p^2}\left(\frac{d}{dp}\left(\frac{p^2Y'}{i(Q-Q_e)+p^2}\right) - G(p)p^2Y\right)\right)\Bigg] &= 0.
\end{aligned}
$$

One complication in solving this equation for large c_β (~ 1) is that the large p layer dominant balance includes nearly all the terms, which is not analytically tractable. We can, however, manage this equation for smaller β and by setting convenient scalings for c_β, we can create more manageable dominant balances. This of course will create restrictions on our solutions' regions of validity in (c_β, Q) space.

$c_\beta^2 \sim (Q-Q_e)Q^2$ Scaling

Suppose $c_\beta^2 \sim (Q-Q_e)Q^2$ and $c_\beta^2 \ll 1$. In the small-p limit ($p^2 \sim Q-Q_e$), the fourth order term will dominate, leaving us with

$$\frac{d^2}{dp^2}\left(\frac{1}{p^2}\frac{d}{dp}\left(\frac{p^2 Y'(p)}{i(Q-Q_e)+p^2}\right)\right) \simeq 0 \tag{279}$$

which has the solution

$$Y(p) \simeq a_1\left(\frac{i(Q-Q_e)}{p}-p\right)+a_2+a_3\left(p^2+\frac{p^4}{2i(Q-Q_e)}\right)+a_4\left(p^3+\frac{3p^5}{5i(Q-Q_e)}\right). \tag{280}$$

Note that this agrees with our expected asymptotic behavior of $Y(p) \to Y_0(\frac{\hat{\Delta}}{\pi p}+1)$ as $p \to 0$. In what we will call the mid-p layer, when $p^2 \sim Q$, we have a balance between the fourth order term and the very first term from our full equation,

$$Y''(p) - \frac{c_\beta^2 p^2}{Q(Q_e-Q)}\frac{d^2}{dp^2}\left(\frac{Y''(p)}{p^2}\right) \simeq 0 \tag{281}$$

The solution to this equation is

$$Y_{p^2\sim Q}(p) \simeq b_1+b_2p+b_3e^{-p/\sqrt{\alpha}}(6\alpha+4\alpha^{1/2}p+p^2)+b_4e^{p/\sqrt{\alpha}}(6\alpha-4\alpha^{1/2}p+p^2), \tag{282}$$

where $\alpha = \frac{c_\beta^2}{Q(Q_e-Q)}$ and taken to be positive. Since we will need to match to a decaying solution in the large-p layer, we can drop the exponentially increasing solution setting $b_4 = 0$. In the large-p layer, we get a dominate balance when $p^2 \sim \frac{(Q-Q_e)^{1/2}}{Q}$, corresponding to a balance between the first two terms,

$$Y''(p) + \frac{iQ(Q-Q_i)}{(Q_e-Q)}p^2Y(p) \simeq 0. \tag{283}$$

The solution to this equation which decays to zero for $p \to \infty$ is

$$Y_{p^2\sim\frac{(Q-Q_e)^{1/2}}{Q}}(p) \simeq D_{-1/2}\left(-(-1)^{7/8}\sqrt{2}\left(\frac{Q(Q-Q_i)}{Q_e-Q}\right)^{1/4}p\right). \tag{284}$$

Now, we need to match between the layers in p-space. The small-p expansion for the large-p layer is

$$Y_{p^2\sim\frac{(Q-Q_e)^{1/2}}{Q}}(p) \to c_1\left(1+\frac{2(-1)^{7/8}\Gamma(3/4)}{\Gamma(1/4)}\left(\frac{Q(Q-Q_i)}{Q_e-Q}\right)^{1/4}p+O(p^4)\right). \tag{285}$$

The large-p expansion for the mid-p layer is

$$Y_{p^2\sim Q}(p) \to b_1+b_2p, \tag{286}$$

and its small-p expansion is

$$Y_{p^2\sim Q}(p) \to b_1+b_2p+b_3\left(6\alpha-2\alpha^{1/2}p+\frac{p^4}{12\alpha}-\frac{p^5}{20\alpha^{3/2}}+O(p^6)\right). \tag{287}$$

The large-p expansion for the small-p layer is

$$Y_{p^2\sim Q-Q_e} \quad \rightarrow \quad -a_1 p + a_2 + \frac{a_3 p^4}{2i(Q-Q_e)} + \frac{a_4 p^5}{5i(Q-Q_e)} \tag{288}$$

and its small-p expansion is

$$Y_{p^2\sim Q-Q_e} \quad \rightarrow \quad \frac{a_1 i(Q-Q_e)}{p} + a_2 + a_3 p^2 + a_4 p^3. \tag{289}$$

Matching between layers results in the following expression for the inner layer $\hat{\Delta}$,

$$\begin{aligned} \frac{\hat{\Delta}}{\pi} &= \frac{-i(Q-Q_e)(1-2\alpha^{1/2}\frac{b_3}{b_2})}{\frac{1}{\gamma}+6\alpha\frac{b_3}{b_2}}, \\ \gamma &= \frac{2(-1)^{7/8}\Gamma(3/4)}{\Gamma(1/4)}\left(\frac{Q(Q-Q_i)}{Q_e-Q}\right)^{1/4} \end{aligned} \tag{290}$$

$c_\beta^2 \sim (Q-Q_e)/Q$ Scaling

Suppose $c_\beta^2 \sim (Q-Q_e)/Q$. For the small-p layer ($p^2 \sim Q-Q_e$), the fourth derivative term will dominate, again leaving us with

$$Y_{p^2\sim Q-Q_e}(p) \simeq a_1\left(\frac{i(Q-Q_e)}{p} - p\right) + a_2 + a_3\left(p^2 + \frac{p^4}{2i(Q-Q_e)}\right) + a_4\left(p^3 + \frac{3p^5}{5i(Q-Q_e)}\right) \tag{291}$$

The next balance occurs when $p^6 \sim (Q-Q_e)/Q^4$, where part of the second term ($G(p)p^2Y$) balances with the fourth derivative term, resulting in the equation

$$Y - \frac{c_\beta^2}{iQ^2(Q-Q_i)}\frac{d^2}{dp^2}\left(\frac{Y''}{p^2}\right) \simeq 0, \tag{292}$$

which can be rewritten for simplicity as

$$Y + i\alpha_1 \frac{d^2}{dp^2}\left(\frac{Y''}{p^2}\right) \simeq 0 \tag{293}$$

where $\alpha_1 > 0$. The solution to this equation is

$$\begin{aligned} Y_{p^6\sim(Q-Q_e)/Q^4}(p) &= b_1\ {}_0F_3(1/6, 1/3, 5/6, ip^6/1296\alpha_1) + b_2\ p\ {}_0F_3(1/3, 1/2, 7/6, ip^6/1296\alpha_1) \\ &\quad + b_3\ p^4\ {}_0F_3(5/6, 3/2, 5/3, ip^6/1296\alpha_1) + b_4\ p^5\ {}_0F_3(7/6, 5/3, 11/6, ip^6/1296\alpha_1) \end{aligned} \tag{294}$$

The large-p expansion of the large-p layer solution is

$$\begin{aligned} Y_{p^6\sim(Q-Q_e)/Q^4}(p) &\rightarrow e^{O(p^{3/2})}(b_1\beta_1 + b_2\beta_2 + b_3\beta_3 + b_4\beta_4)p^{1/4}, \\ \beta_1 &= \frac{(\frac{i}{\alpha_1})^{1/24}\Gamma(1/3)}{2^{5/3}3^{1/6}\sqrt{\pi}} \\ \beta_2 &= \frac{\sqrt{3}\Gamma(1/3)\Gamma(7/6)}{4(\frac{i}{\alpha_1})^{1/8}\pi} \\ \beta_3 &= \frac{9\sqrt{3}\Gamma(5/6)\Gamma(5/3)}{2(\frac{i}{\alpha_1})^{5/8}\pi} \\ \beta_4 &= \frac{27(3^{1/6})\Gamma(11/3)}{8(2^{1/3})(\frac{i}{\alpha_1})^{19/24}\sqrt{\pi}}, \end{aligned}$$

and its small-p expansion is

$$Y_{p^6\sim(Q-Q_e)/Q^4}(p) \quad \rightarrow \quad b_1 + b_2 p + b_3 p^4 + b_4 p^5.$$

The large-p limit of the small-p layer solution is

$$Y_{p^2\sim Q-Q_e}(p) \quad \rightarrow \quad a_2 - a_1 p + \frac{a_3 p^4}{2i(Q-Q_e)} + \frac{a_4 3p^5}{5i(Q-Q_e)},$$

and its small-p limit is

$$Y_{p^2\sim Q-Q_e}(p) \quad \rightarrow \quad \frac{a_1 i(Q-Q_e)}{p} + a_2 + a_3 p^2 + a_4 p^3.$$

Matching between layers yields

$$\frac{\hat{\Delta}}{\pi} \quad = \quad \frac{i(Q-Q_e)(-b_2)}{b_1}. \tag{295}$$

We can eliminate one of the four 'b' constants by setting its large-p limit to zero, however we will still need more information to fully obtain an expression for $\hat{\Delta}$ just like in the last c_β scaling. This motivates the need for higher order matching.

Higher order asymptotic expansion

The lowest order asymptotic expansion for the four fields are

$$\begin{aligned}
\psi(X)/\Psi \rightarrow \left[1 + \frac{\hat{\Delta}}{2}|X|\right] + \frac{\psi_{-1}}{|X|} + \frac{\psi_{-2}}{X^2} + O(\frac{1}{|X|^3}) \\
Z(X)/\Psi \rightarrow \frac{Q_e}{X}\left[1 + \frac{\hat{\Delta}}{2}|X|\right] + \frac{Z_{-2}}{X|X|} + \frac{Z_{-3}}{X^3} + O(\frac{1}{X^3|X|}) \\
\phi(X)/\Psi \rightarrow \frac{Q}{X}\left[1 + \frac{\hat{\Delta}}{2}|X|\right] + \frac{\phi_{-2}}{X|X|} + \frac{\phi_{-3}}{X^3} + O(\frac{1}{X^3|X|}) \\
V_z(X)/\Psi \rightarrow \frac{V_{-3}}{|X|^3} + \frac{V_{-4}}{X^4}.
\end{aligned}$$

with the governing equations being

$$\begin{aligned}
i(Q-Q_e)\psi \ &= \ iX(\phi - Z) + \frac{d^2\psi}{dX^2} + O(\varepsilon^2) \\
iQZ \ &= \ iQ_e\phi + iD^2 X\frac{d^2\psi}{dX^2} + ic_\beta^2 X V_z + c_\beta^2\frac{d^2 Z}{dX^2} + O(\varepsilon^2) \\
i(Q-Q_i)\frac{d^2\phi}{dX^2} \ &= \ iX\frac{d^2\psi}{dX^2} + P\frac{d^4(\phi+\tau Z)}{dX^4} + O(\varepsilon^2) \\
iQV_z \ &= \ -iQ_e\psi + iXZ + P\frac{d^2 V_z}{dX^2} + O(\varepsilon^2).
\end{aligned}$$

Note that $\psi(X)$ has a tearing parity (even), which implies that V_z is also even, and ϕ and Z are odd. Plugging in the asymptotic expansions into the above equations and then converting to Fourier space results in

$$Y(p) \rightarrow Y_0\left[\frac{\hat{\Delta}}{\pi p} + 1 + i\psi_{-1}p - \frac{\psi_{-2}}{2}p^2 + O(p^3)\right].$$

Further analysis shows that $\psi_{-1} = 0$, $\psi_{-2} = \frac{1}{3}(Q - Q_i)Q\psi_0$, and $\psi_{-3} = 0$ so we can rewrite this as

$$Y(p) \to Y_0\left[\frac{\hat{\Delta}}{\pi p} + 1 + -\frac{Q(Q-Q_i)}{6}p^2 + O(p^3)\right].$$

$\hat{\Delta}$ calculations

Using this new asymptotic information, we can determine the ratio b_3/b_2 from equation 290 for the $c_\beta^2 \sim Q^2(Q - Q_e)$ scaling which yields

$$\hat{\Delta}_{c_\beta^2\sim Q^2(Q-Q_e)} = \frac{-i(Q-Q_e)\left(1 - 2\sqrt{\alpha}\frac{-Q(Q-Q_i)}{\gamma(\frac{i(Q-Q_e)}{\alpha}+6\alpha Q(Q-Q_i)}\right)}{\frac{1}{\gamma} + 6\alpha\frac{-Q(Q-Q_i)}{\gamma(\frac{i(Q-Q_e)}{\alpha}+6\alpha Q(Q-Q_i)}}, \tag{296}$$

where again,

$$\begin{aligned}
\alpha &= \frac{c_\beta^2}{Q(Q_e - Q)}, \\
\gamma &= \frac{2(-1)^{7/8}\Gamma(3/4)}{\Gamma(1/4)}\left(\frac{Q(Q-Q_i)}{Q_e - Q}\right)^{1/4}.
\end{aligned}$$

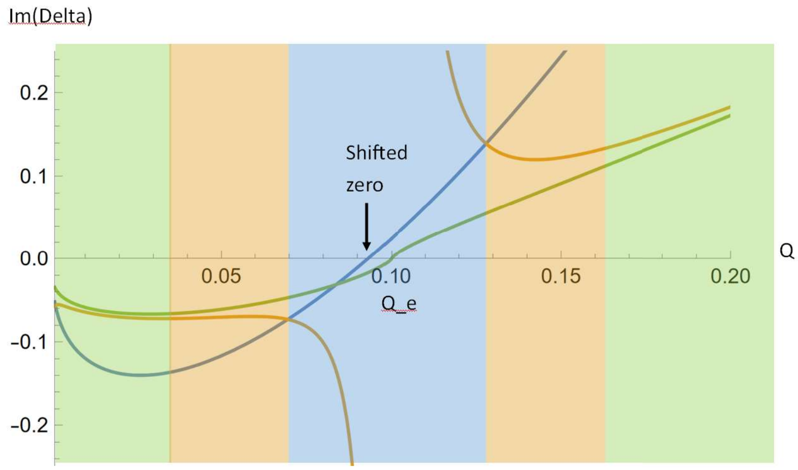
For the $c_\beta^2 \sim (Q - Q_e)/Q$ scaling,

$$\hat{\Delta}_{c_\beta^2\sim(Q-Q_e)/Q} = \frac{i\pi(Q-Q_e)\left(\beta_1 + \beta_3\frac{iQ(Q-Q_i)}{12(Q-Q_e)}\right)}{\beta_2}, \tag{297}$$

where again,

$$\begin{aligned}
\beta_1 &= \frac{(\frac{i}{\alpha_1})^{1/24}\Gamma(1/3)}{2^{5/3}3^{1/6}\sqrt{\pi}} \\
\beta_2 &= \frac{\sqrt{3}\Gamma(1/3)\Gamma(7/6)}{4(\frac{i}{\alpha_1})^{1/8}\pi} \\
\beta_3 &= \frac{9\sqrt{3}\Gamma(5/6)\Gamma(5/3)}{2(\frac{i}{\alpha_1})^{5/8}\pi}. \\
\alpha_1 &= \frac{c_\beta^2}{Q^2(Q-Q_i)}
\end{aligned} \tag{298}$$

Below is a plot of $Im(\hat{\Delta})$ for $c_\beta = 0.04$, $Q_e = 0.1 = -Q_i$. The green line is the $c_\beta^2/(Q - Q_e) = 0$ solution, the orange line is the $c_\beta^2 \sim Q^2(Q-Q_e)$ solution, and the blue line is the $c_\beta^2 \sim (Q-Q_e)/Q$ solution. The shaded regions indicate roughly where each solution is valid. Close to the electron resonance, we see a shifted zero point predicted by the blue region solution, a key prediction from Lee 2024. Also shown is the zero crossing predicted by the blue region solution as a function of c_β. These solutions should indicate the behavior of this shifted zero crossing for the small β limit.



Zero crossing of Im(Delta)