Magnetic Perturbations

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I. MAGNETIC PERTURBATIONS IN FLUX COORDINATES

In the r, θ, ϕ flux coordinate system (where all lengths are normalized to R_0, and all magnetic field-strengths to B_0), the perturbed magnetic field is written

$$\mathbf{b} = b^r\,\mathcal{J}\,\nabla\theta\times\nabla\phi + b^\theta\,\mathcal{J}\,\nabla\phi\times\nabla r + b^\phi\,\mathcal{J}\,\nabla r\times\nabla\theta. \tag{1}$$

T7 Eqs. (25) and (49) yield

$$b^r = \frac{1}{r\,R^2}\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right)y, \tag{2}$$

$$b^\phi = \frac{x}{R^2}, \tag{3}$$

whereas TJ Eqs. (78), (79), (80), (98), and (99) imply that

$$y(r,\theta) = \sum_m \frac{\psi_m(r)}{m-n\,q(r)}\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{4}$$

$$x(r,\theta) = n\sum_m \frac{Z_m(r)+k_m(r)\,\psi_m(r)}{m-n\,q(r)}\,{\rm e}^{\,{\rm i}\,m\,\theta}. \tag{5}$$

It follows that

$$R^2\,b^r = \frac{\rm i}{r}\sum_m \psi_m\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{6}$$

$$R^2\,b^\phi = n\sum_m z_m\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{7}$$

where

$$z_m = \frac{Z_m + k_m\,\psi_m}{m-n\,q}. \tag{8}$$

Now, TJ Eqs. (2) and (A10) yield

$$\mathcal{J}\,\nabla\cdot\mathbf{b} = \frac{\partial}{\partial r}\left(r\,R^2\,b^r\right) + \frac{\partial}{\partial\theta}\left(r\,R^2\,b^\theta\right) - {\rm i}\,n\,r\,R^2\,b^\phi = 0, \tag{9}$$

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so

$$\frac{\partial}{\partial\theta}\,(r^2\,R^{\,2}\,b^\theta) = {\rm i}\sum_m\left[-r\,\frac{d\psi_m}{dr}+\frac{n^2\,r^2\,(Z_m+k_m\,\psi_m)}{m-n\,q}\right]{\rm e}^{\,{\rm i}\,m\,\theta}. \tag{10}$$

But, TJ Eq. (102) gives

$$r\,\frac{d\psi_m}{dr} = \sum_{m'}\frac{L_m^{m'}\,Z_{m'}+M_m^{m'}\,\psi_{m'}}{m'-n\,q}, \tag{11}$$

so

$$\frac{\partial}{\partial\theta}\,(r^2\,R^{\,2}\,b^\theta) = -{\rm i}\sum_m\chi_m\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{12}$$

where

$$\chi_m(r) = \sum_{m'}\frac{\chi_m^{m'}}{m'-n\,q}, \tag{13}$$

$$\chi_m^{m'}(r) = (L_m^{m'}-n^2\,r^2\,\delta_m^{m'})\,Z_{m'}+(M_m^{m'}-n^2\,r^2\,\delta_m^{m'}\,k_{m'})\,\psi_{m'} \tag{14}$$

Note from TJ Eqs. (100), (262), (263), (266), (267), (268), and (269) that $\chi_0^{m'}=0$ for all m', which implies that $\chi_0(r)=0$. Thus,

$$r\,R^{\,2}\,b^\theta = -\frac{1}{r}\sum_m^{m\neq 0}\hat{\chi}_m\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{15}$$

where

$$\hat{\chi}_m(r) = \frac{\chi_m}{m}. \tag{16}$$

According to Eqs. (4), (26), (79), and (98) of TJ,

$$\xi^r = \frac{q}{r\,g}\sum_m\hat{\psi}_m\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{17}$$

where

$$\hat{\psi}_m(r) = \frac{\psi_m}{m-n\,q}. \tag{18}$$

Thus,

$$\delta T_e = -\xi^r\,\frac{dT_e}{dr}. \tag{19}$$

But,

$$T_e(r) = \frac{B_0^{\,2}}{2\,\mu_0\,e\,n_0}\,\epsilon_a^{\,2}\,p_c\,(1-\hat{r}^{\,2})^{\mu-\alpha}, \tag{20}$$

giving

$$\frac{dT_e}{dr} = \frac{B_0^{\,2}}{\mu_0\,e\,n_0}\,\epsilon_a\,p_c\,(\mu-\alpha)\,\hat{r}\,(1-\hat{r}^{\,2})^{\mu-\alpha-1}. \tag{21}$$

II. REGULARIZATION

In evaluating the perturbed magnetic field associated with the unreconnected eigenfunction at the kth rational surface, we make the transformation

$$\frac{1}{m_k - n\,q} \rightarrow \frac{m_k - n\,q}{\delta_k^2 + (m_k - n\,q)^2}, \tag{22}$$

where m_k is the resonant poloidal mode number at the kth rational surface, and

$$\delta_k = \frac{m_k\,s(r_k)}{2\,\hat{r}_k}\,\frac{W_k}{\epsilon_a}. \tag{23}$$

Here, $r_k = \epsilon_a\,\hat{r}_k$ is the minor radius of the kth rational surface, $s(r)$ is the magnetic shear, and

$$W_k = 4\left(\frac{q}{g\,s}\right)_{r_k}^{1/2} \Psi_k^{1/2} \tag{24}$$

is the magnetic island width at the kth rational surface. Thus,

$$\Psi_k = \epsilon_a^2 \left(\frac{W_k}{4\,\epsilon_a}\right)^2 \left(\frac{g\,s}{q}\right)_{r_k}. \tag{25}$$

For the special case of an $m = 1$ mode,

$$\delta_k = \frac{m_k\,s(r_k)}{2\,\hat{r}_k}\,\frac{\xi_k}{\epsilon_a}, \tag{26}$$

$$\Psi_k = \epsilon_a^2 \left(\frac{\hat{r}\,g\,s}{q}\right)_{r_k} \frac{1}{|E_{kk}|} \left(\frac{\xi_k}{\epsilon_a}\right), \tag{27}$$

where ξ_k is the displacement of the plasma core.

III. MAGNETIC PERTURBATIONS IN CYLINDRICAL COORDINATES

We can write the perturbed magnetic field associated with the tearing mode that reconnects magnetic flux at the kth rational surface as

$$b^R \equiv \mathbf{b}\cdot\nabla R = b^r\,\frac{\partial R}{\partial r} + b^\theta\,\frac{\partial R}{\partial \theta}, \tag{28}$$

$$b^Z \equiv \mathbf{b}\cdot\nabla Z = b^r\,\frac{\partial Z}{\partial r} + b^\theta\,\frac{\partial Z}{\partial \theta}, \tag{29}$$

$$R\,b^\phi \equiv R\,\mathbf{b}\cdot\nabla\phi = R\,b^\phi. \tag{30}$$

Thus,

$$b^R(r,\theta,\phi) = b_C^R(r,\theta)\,\cos(n\,\phi) + b_S^R(r,\theta)\,\sin(n\,\phi), \tag{31}$$

$$b^Z(r,\theta,\phi) = b_C^Z(r,\theta)\,\cos(n\,\phi) + b_S^Z(r,\theta)\,\sin(n\,\phi), \tag{32}$$

$$R\,b^\phi(r,\theta,\phi) = b_C^\phi(r,\theta)\,\cos(n\,\phi) + b_S^\phi(r,\theta)\,\sin(n\,\phi), \tag{33}$$

$$\xi^r(r,\theta,\phi) = \xi_C^r(r,\theta)\,\cos(n\,\phi) + \xi_S^r\,\sin(n\,\phi), \tag{34}$$

where

$$b_C^R(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial R}{\partial \hat{r}}\sum_m\left[\mathrm{Re}(\psi_m)\,\sin(m\,\theta) + \mathrm{Im}(\psi_m)\,\cos(m\,\theta)\right]\right. \tag{35}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial R}{\partial \theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\cos(m\,\theta) - \mathrm{Im}(\hat{\chi}_m)\,\sin(m\,\theta)\right]\right\}, \tag{36}$$

$$b_S^R(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial R}{\partial \hat{r}}\sum_m\left[-\mathrm{Re}(\psi_m)\,\cos(m\,\theta) + \mathrm{Im}(\psi_m)\,\sin(m\,\theta)\right]\right. \tag{37}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial R}{\partial \theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\sin(m\,\theta) + \mathrm{Im}(\hat{\chi}_m)\,\cos(m\,\theta)\right]\right\}, \tag{38}$$

$$b_C^Z(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial Z}{\partial r}\sum_m\left[\mathrm{Re}(\psi_m)\,\sin(m\,\theta) + \mathrm{Im}(\psi_m)\,\cos(m\,\theta)\right]\right. \tag{39}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial Z}{\partial \theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\cos(m\,\theta) - \mathrm{Im}(\hat{\chi}_m)\,\sin(m\,\theta)\right]\right\}, \tag{40}$$

$$b_S^Z(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial Z}{\partial \hat{r}}\sum_m\left[-\mathrm{Re}(\psi_m)\,\cos(m\,\theta) + \mathrm{Im}(\psi_m)\,\sin(m\,\theta)\right]\right. \tag{41}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial Z}{\partial \theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\sin(m\,\theta) + \mathrm{Im}(\hat{\chi}_m)\,\cos(m\,\theta)\right]\right\}, \tag{42}$$

$$b_C^\phi(r,\theta) = \frac{n\,\Psi_k}{R}\sum_m\left[\mathrm{Re}(z_m)\,\cos(m\,\theta) - \mathrm{Im}(z_m)\,\sin(m\,\theta)\right], \tag{43}$$

$$b_S^\phi(r,\theta) = \frac{n\,\Psi_k}{R}\sum_m\left[\mathrm{Re}(z_m)\,\sin(m\,\theta) + \mathrm{Im}(z_m)\,\cos(m\,\theta)\right], \tag{44}$$

$$\xi_C^r(r,\theta) = \frac{\Psi_k\,q}{\epsilon_a\,\hat{r}\,g}\sum_m\left[\mathrm{Re}(\hat{\psi}_m)\,\cos(m\,\theta) - \mathrm{Im}(\hat{\psi}_m)\,\sin(m\,\theta)\right], \tag{45}$$

$$\xi_S^r(r,\theta) = \frac{\Psi_k\,q}{\epsilon_a\,\hat{r}\,g}\sum_m\left[\mathrm{Re}(\hat{\psi}_m)\,\sin(m\,\theta) + \mathrm{Im}(\hat{\psi}_m)\,\cos(m\,\theta)\right]. \tag{46}$$

IV. VALUE OF k_m

Now,

$$\begin{aligned}\tilde{k}_m = &-\frac{2-s}{m} - \frac{\epsilon_a^{2}}{m}\left(-\hat{r}\,p_2' + \frac{3\,\hat{r}^{\,2}}{2} - 2\,\hat{r}\,H_1' + S_2\right)\\ &+\epsilon_a^{2}\,\frac{(2-s)}{m}\left(-\frac{3\,\hat{r}^{\,2}}{4} + \frac{\hat{r}^{\,2}}{q^2} + H_1 + S_1\right)\\ &+\epsilon_a^{2}\,\frac{n\,\hat{r}}{m^2}\left[-q\,p_2' + \frac{\hat{r}}{m\,q}\,(2-s)\,(m-n\,q)\right].\end{aligned} \tag{47}$$

But, for the special case $m = 0$,

$$\begin{aligned}\tilde{k}_0 = &-\frac{q\,p_2'}{n\,\hat{r}} - \frac{2-s}{n\,q}\\ &-\frac{\epsilon_a^{2}}{n\,q}\left(\frac{3\,\hat{r}^{\,2}}{2} - 2\,\hat{r}\,H_1 + S_2\right)\\ &+\epsilon_a^{2}\,\frac{(2-s)}{n\,q}\left(-\frac{3\,\hat{r}^{\,2}}{4} + \frac{\hat{r}^{\,2}}{q^2} + H_1 + S_1\right)\\ &+\epsilon_a^{2}\,\frac{q\,p_2'}{n\,\hat{r}}\left(2\,g_2 + \frac{\hat{r}^{\,2}}{2} + \frac{\hat{r}^{\,2}}{q^2} - 2\,H_1 - 3\,\hat{r}\,H_1'\right).\end{aligned} \tag{48}$$

It turns out that

$$k_m = \tilde{k}_m - \tilde{k}_0 \tag{49}$$

for all m.