A Four-Field Resonant Response Model for Tokamak Plasmas

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1 Introduction

Externally generated, static, *resonant magnetic perturbations* (RMPs) can drive magnetic reconnection in tokamak plasmas that are intrinsically stable to tearing perturbations [1, 2, 3, 4, 5]. Driven reconnection leads to the formation of magnetic island chains at so-called *rational* magnetic flux-surfaces [6]. Such chains locally flatten the pressure profile, and, thereby, degrade the plasma confinement [7].

The analysis of the response of a tokamak plasma to an RMP is most efficiently formulated as an asymptotic matching problem in which the plasma is divided into two distinct regions [8]. In the so-called *outer region*, which comprises most of the plasma, the plasma perturbation is governed by the equations of linearized, marginally-stable, ideal-magnetohydrodynamics (MHD). However, these equations become singular on rational magnetic flux-surfaces at which the perturbed magnetic field resonates with the equilibrium field. In the *inner region*, which consists of a set of narrow layers centered on the various rational surfaces, non-ideal-MHD effects become important.

It is well known that single-fluid resistive-MHD offers a very poor description of the response of the inner region to the magnetic perturbation in the outer region. For instance, the strong diamagnetic flows present in tokamak plasmas imply that the electron and ion fluid velocities are significantly different from one another, necessitating a two-fluid treatment [9]. Moreover, resistive-MHD does not take into account the important ion sound radius lengthscale below which electron and ion dynamics become decoupled from one another [10, 11]. Previously, Cole & Fitzpatrick [12] used the four-field model of Fitzpatrick & Waelbroeck [13] (which is based on the original four-field model of Hazeltine, Kotschenreuther & Morrison [14]) to determine the linear two-fluid response of a resonant layer to the perturbation in the outer region. This treatment was extended in Ref. [15] to take into account the anomalously large perpendicular energy diffusivity present in tokamak plasmas.

In configuration space, the four-field models of Refs. [12] and [15] yield a set of resonant layer equations that can be expressed as *ten* coupled first-order linear differential equations [16]. However, in Fourier space, the resonant layer equations can be written as *four* first-order linear differential equatons [12]. It is clearly advantageous to solve the equations in Fourier space. In Refs. [12] and [15], an approximation is made by which one of the terms in the Fourier-transformed layer equations is neglected. This approximation, which is valid in low-β plasmas, is such that the layer equation that governs the parallel ion dynamics decouples from the other three equations, effectively converting a four-field resonant response model into a three-field model. Furthermore, the three remaining layer equations can be combined to give a single second-order linear ordinary differential equation. This second-order equation is most conveniently solved numerically by means of a Riccati transformation that converts it into a first-order nonlinear differential equation [17, 18]. The advantage of the Riccati approach is that it can deal with numerically problematic solutions that blow

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up as $\exp(p^2)$, or faster, at large p, where p is the Fourier-space variable.

Lee, Park & Na [16] recently demonstrated how to solve the full tenth-order four-field resonant layer equations in configuration space using a Riccati transformation. In the process, they discovered that, in a high-β plasma, the RMP frequency (as seen in the local $\mathbf{E}\times\mathbf{B}$ frame at the rational surface) at which driven magnetic reconnection is maximized is shifted in the ion diamagnetic direction from the electron diamagnetic frequency. This result is significant because there is some experimental evidence for such a shift [19]. In the present paper, we demonstrate how the calculation of Lee et alia can be reimplemented in Fourier space. The Fourier version of the calculation is more convenient, from a numerical point of view, because it involves the solution of a fourth-order, rather than a tenth-order, system of equations.

2 Asymptotic Matching

2.1 Plasma Equilibrium

Consider a large aspect-ratio tokamak plasma equilibrium whose magnetic flux-surfaces map out (almost) concentric circles in the poloidal plane. Such an equilibrium can be approximated as a periodic cylinder [4]. Let r, θ, z be right-handed cylindrical coordinates. The magnetic axis corresponds to $r=0$, and the plasma boundary to $r=a$, where a is the simulated minor radius of the plasma. The system is assumed to be periodic in the z-direction with periodicity length $2\pi\,R_0$, where R_0 is the simulated major radius of the plasma. The safety-factor profile takes the form $q(r) = r\,B_z/[R_0\,B_\theta(r)]$, where B_z is the constant "toroidal" magnetic field-strength, and $B_\theta(r)$ is the poloidal magnetic field-strength. The equilibrium poloidal and toroidal magnetic fluxes (divided by 2π) are written $\psi_p(r) = B_z\int_0^r r'/q(r')\,dr'$ and $\psi_t(r) = B_z\,r^2/2$, respectively. The standard large aspect-ratio orderings, $r/R_0 \ll 1$ and $B_\theta/B_z \ll 1$, are adopted.

2.2 Outer Region

Consider a static (in the laboratory frame) magnetic perturbation that has m periods in the poloidal direction, and n periods in the toroidal direction. The response of the plasma to the perturbation is governed by the linearized equations of marginally-stable ideal-MHD everywhere in the plasma, apart from a (radially) narrow layer centered on the rational surface whose minor radius, r_s, is such that $q(r_s) = m/n$ [8].

The perturbed magnetic field associated with the tearing mode is written $\delta\mathbf{B} \simeq \nabla\delta\psi\times\mathbf{e}_z$, where $\delta\psi(r,\theta,\varphi) = \delta\psi(r)\,\exp[\,{\rm i}\,(m\,\theta - n\,\varphi)]$, and $\varphi = z/R_0$ is a simulated toroidal angle. In the outer region (i.e., everywhere in the plasma apart from the resonant layer), the perturbed helical magnetic flux, $\delta\psi(r)$, satisfies the *cylindrical tearing mode equation* [20, 21],

$$\frac{d^2\delta\psi}{dr^2} + \frac{1}{r}\,\frac{d\delta\psi}{dr} - \frac{m^2}{r^2}\,\delta\psi - \frac{J_z'\,\delta\psi}{r\,(1/q - n/m)} = 0, \tag{1}$$

where $J_z(r) = R_0\,\mu_0\,j_z(r)/B_z$, and $j_z(r)$ is the equilibrium "toroidal" current density. Here, $' \equiv d/dr$.

In general, the solution of Eq. (1) that satisfies physical boundary conditions at the magnetic axis and the plasma boundary is such that $\delta\psi(r)$ is continuous across the rational surface, whereas $d\delta\psi/dr$ is discontinuous. The discontinuity of $d\delta\psi/dr$ across the rational surface is indicative of the presence of a helical current sheet at the surface. The complex quantity $\Psi_s = \delta\psi(r_s)$ determines the amplitude and phase of the reconnected helical magnetic flux at the rational surface, whereas the complex quantity

$$\Delta\Psi_s = \left[r\,\frac{d\delta\psi}{dr}\right]_{r_{s-}}^{r_{s+}} \tag{2}$$

parameterizes the amplitude and phase of the helical current sheet [4].

The solution of the cylindrical tearing mode equation in the outer region, in the presence of an externally generated RMP (with m periods in the poloidal direction and n periods in the toroidal direction), leads to the tearing mode dispersion relation [4, 8]

$$\Delta\Psi_s - E_{ss}\,\Psi_s = -E_{ss}\,\Psi_v, \tag{3}$$

where E_{ss} is a real dimensionless quantity known as the *tearing stability index* [8]. Moreover, Ψ_v is the so-called *vacuum flux*, and is defined as the reconnected magnetic flux that would be driven at the rational surface by the RMP were the plasma intrinsically tearing-stable (i.e., $E_{ss} < 0$), and were there no current sheet at the rational surface (i.e., $\Delta\Psi_s = 0$) [4].

2.3 Inner Region

The current sheet at the rational surface is resolved by employing a extended-MHD resonant response model in the inner region (i.e., the region of the plasma in the immediate vicinity of the rational surface) to determine the complex *layer response index*, Δ_s. Asymptotic matching between the solutions in the inner and the outer regions yields

$$\Delta_s = \frac{\Delta\Psi_s}{\Psi_s}. \tag{4}$$

The previous two equations lead to the *plasma response equation*,

$$\Psi_s = \frac{\Psi_v}{1 + \Delta_s/(-E_{ss})}. \tag{5}$$

Note that if $|\Delta_s|/(-E_{ss}) \ll 1$ then $\Psi_s \simeq \Psi_v$ and $\Delta\Psi_s \simeq 0$, which is termed a *vacuum response*. On the other hand, if $|\Delta_s|/(-E_{ss}) \gg 1$ then $|\Psi_s| \ll |\Psi_v|$, which is termed an *ideal response*. In the vacuum response regime, the current sheet that is excited in the resonant layer is too feeble to prevent driven magnetic reconnection, and the reconnected flux at the rational surface is the same as that which would be driven if there were no plasma present at the surface. On the other hand, in the ideal response regime, the current sheet excited in the layer is large enough to almost completely suppress driven magnetic reconnection, which implies that the response of the plasma is equivalent to that which would occur if the ideal-MHD flux freezing constraint, $\Psi_s = 0$, were imposed at the rational surface [22].

3 Four-Field Resonant Response Model

3.1 Description

Our four-field resonant response model is described in detail in Appendix A. The model is defined by the following eight dimensionless parameters: $S = \tau_R/\tau_H$, $Q_E = -S^{1/3}\, n\,\omega_E\,\tau_H$, $Q_e = -S^{1/3}\, n\,\omega_{\ast\,e}\,\tau_H$, $Q_i = -S^{1/3}\, n\,\omega_{\ast\,i}\,\tau_H$, $c_\beta = \sqrt{\beta/(1+\beta)}$, $D = S^{1/3}\,\iota_e^{1/2}\,\hat{d}_\beta$, $P_E = \tau_R/\tau_E$, and $P_\varphi = \tau_R/\tau_\varphi$. Here, τ_R is the resistive diffusion timescale, τ_H the hydrodynamic timescale, τ_E the energy confinement timescale, τ_φ the momentum confinement timescale, ω_E the $\mathbf{E}\times\mathbf{B}$ frequency, $\omega_{\ast\,e}$ the electron diamagnetic frequency, $\omega_{\ast\,i}$ the ion diamagnetic frequency, $\beta = (5/3)\,\mu_0\,p/B_z^2$ the square of the ratio of the sound speed to the Alfvén speed, p the plasma pressure, $d_\beta = c_\beta\,d_i$ the ion sound radius, and d_i the collisionless ion skin-depth. Moreover, $\iota_e = -\omega_{\ast\,e}/(\omega_{\ast\,i} - \omega_{\ast\,e})$, and $\hat{d}_\beta = d_\beta/r_s$. All quantities are evaluated at the rational surface.

After linearization and Fourier transformation (with respect to $x = r - r_s$), our four-field model reduces to the 2×2 Ricatti matrix differential equation specified in Eq. (85). Here, the elements of the 2×2 matrices $\underline{\underline{E}}$ and $\underline{\underline{F}}$ are given in Eqs. (88)–(91) and (92)–(95). The method of solution is as follows. A solution of the Ricatti equation is launched from a large value of the Fourier transform variable p, with the initial conditions specified in Eq. (152), and integrated to a small value of p. The layer response index, $\Delta_s = S^{1/3}\,\hat{\Delta}$, is then deduced from Eqs. (40) and (116).

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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A Four-Field Resonant Plasma Response Model

A.1 Fundamental Definitions

The plasma is assumed to consist of two species. First, electrons of mass m_e, electrical charge $-e$, number density n_e, and temperature T_e. Second, ions of mass m_i, electrical charge $+e$, number density n_e, and temperature T_i. Let $p = n_e\,(T_e + T_i)$ be the total plasma pressure.

Let r_s be the minor radius of the rational surface. It is helpful to define $n_0 = n_e(r_s)$, $p_0 = p(r_s)$,

$$\eta_e = \left.\frac{d\ln T_e}{d\ln n_e}\right|_{r=r_s}, \tag{6}$$

$$\eta_i = \left.\frac{d\ln T_i}{d\ln n_e}\right|_{r=r_s}, \tag{7}$$

$$\iota = \left(\frac{T_e}{T_i}\right)_{r=r_s}\left(\frac{1+\eta_e}{1+\eta_i}\right) = \left(\frac{dp_e}{dp_i}\right)_{r_s}, \tag{8}$$

where $n_e(r)$, $p(r)$, $p_e(r)$, $p_i(r)$, $T_e(r)$, and $T_i(r)$ refer to electron number density, total pressure, electron pressure, ion pressure, electron temperature, and ion temperature profiles, respectively, in the absence of the magnetic perturbation.

For the sake of simplicity, the perturbed electron and ion temperature profiles are assumed to be functions of the perturbed electron number density profile in the immediate vicinity of the rational surface. In other words, $T_e = T_e(n_e)$ and $T_i = T_i(n_e)$. This implies that $p = p(n_e)$. The "MHD velocity", which is the velocity of a fictional MHD fluid [23], is defined $\mathbf{V} = \mathbf{V}_E + V_{\parallel\,i}\,\mathbf{b}$, where $\mathbf{V}_E$ is the $\mathbf{E}\times\mathbf{B}$ drift velocity, $V_{\parallel\,i}$ is the parallel component of the ion fluid velocity, $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$, and $\mathbf{B}$ is the magnetic field-strength.

A.2 Fundamental Fields

The four fundamental fields in our four-field model—namely, ψ, N, ϕ, and V—have the following definitions:

$$\nabla\psi = \frac{\mathbf{e}_\parallel \times \mathbf{B}}{r_s\,B_z}, \tag{9}$$

$$N = -\hat{d}_i\left(\frac{p-p_0}{B_z^2/\mu_0}\right), \tag{10}$$

$$\nabla\phi = \frac{\mathbf{e}_\parallel \times \mathbf{V}}{r_s\,V_A}, \tag{11}$$

$$V = \hat{d}_i\left(\frac{\mathbf{e}_\parallel\cdot\mathbf{V}}{V_A}\right). \tag{12}$$

Here, $\mathbf{e}_\parallel = (0,\,\epsilon/q_s,\,1)$, $\epsilon = r/R_0$, $q_s = m/n$, $V_A = (B_z/\sqrt{\mu_0\,n_0\,m_i})_{r_s}$, $d_i = [m_i/(n_0\,e^{\,2}\,\mu_0)]_{r_s}^{1/2}$, and $\hat{d}_i = d_i/r_s$. Our model also employs the auxiliary field

$$J = -\frac{2\,\epsilon_s}{q_s} + \hat{\nabla}^2\psi, \tag{13}$$

where $\epsilon_s = r_s/R_0$, and $\hat{\nabla} = r_s\,\nabla$. Note that V_A is the *Alfvén speed* at the rational surface, whereas d_i is the *collisionless ion skin-depth*.

A.3 Fundamental Equations

For the case of a static (in the laboratory frame) magnetic perturbation, the four-field model takes the form [13, 12, 15]:

$$0 = [\phi,\psi] - \iota_e\,[N,\psi] + \hat{\eta}_\parallel\,J + \hat{E}_\parallel, \tag{14}$$

$$0 = [\phi,N] + \hat{d}_\beta^{\,2}\,[J,\psi] + c_\beta^2\,[V,\psi] + \hat{\chi}_E\,\hat{\nabla}^2 N, \tag{15}$$

$$0 = [\phi,\hat{\nabla}^2\phi] - \frac{\iota_i}{2}\left(\hat{\nabla}^2[\phi,N] + [\hat{\nabla}^2\phi,N] + [\hat{\nabla}^2 N,\phi]\right) + [J,\psi] + \hat{\chi}_\varphi\,\hat{\nabla}^4(\phi + \iota_i\,N)\,, \tag{16}$$

$$0 = [\phi,V] + [N,\psi] + \hat{\chi}_\varphi\,\hat{\nabla}^2 V. \tag{17}$$

Here, $[A,B] \equiv \hat{\nabla}A\times\hat{\nabla}B\cdot\mathbf{e}_\parallel$, $\iota_e = \iota/(1+\iota)$, $\iota_i = 1/(1+\iota)$, $\hat{t} = t/(r_s/V_A)$, $\hat{\eta}_\parallel = \eta_\parallel/(\mu_0\,r_s\,V_A)$, $\hat{E}_\parallel = E_\parallel/(B_z\,V_A)$, $\hat{\chi}_E = \chi_E/(r_s\,V_A)$, $\hat{\chi}_\varphi = \chi_\varphi/(r_s\,V_A)$, where $\eta_\parallel$ is the parallel plasma electrical resistivity at the rational surface, $E_\parallel$ the parallel inductive electric field that maintains the equilibrium toroidal plasma current in the vicinity of the rational surface, χ_E the anomalous perpendicular heat diffusivity at the rational surface, and χ_φ the anomalous perpendicular ion momentum diffusivity at the rational surface. Moreover, $d_\beta = c_\beta\,d_i$, and $\hat{d}_\beta = d_\beta/r_s$, where $c_\beta = [\beta/(1+\beta)]_{r_s}^{1/2}$, and $\beta = (5/3)\,\mu_0\,p_0/B_z^2$. Here, d_β is usually referred to as the *ion sound radius*.

A.4 Matching to Plasma Equilibrium

The unperturbed plasma equilibrium is such that $\mathbf{B} = (0,\,B_\theta(r),\,B_z)$, $p = p(r)$, $\mathbf{V} = (0,\,V_E(r),\,V_z(r))$, where $V_E(r) \simeq -E_r/B_z$ is the (dominant θ-component of the) $\mathbf{E}\times\mathbf{B}$ velocity. Now, the resonant layer is assumed to have a radial thickness that is much smaller than r_s. Hence, we only need to evaluate plasma equilibrium quantities in the immediate vicinity of the rational surface. Equations (9)–(12) suggest that

$$\psi(\hat{x}) = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s}, \tag{18}$$

$$N(\hat{x}) = -\hat{V}_*\,\hat{x}, \tag{19}$$

$$\phi(\hat{x}) = -\hat{V}_E\,\hat{x}, \tag{20}$$

$$V(\hat{x}) = \hat{V}_\parallel, \tag{21}$$

where $\hat{x} = (r - r_s)/r_s$, $\hat{L}_s = L_s/r_s$, $L_s = R_0\,q_s/s_s$, $\hat{V}_E = V_E(r_s)/V_A$, $\hat{V}_* = V_*(r_s)/V_A$, $V_*(r) = (dp/dr)/(e\,n_0\,B_z)$ is the (dominant θ-component of the) diamagnetic velocity, and $\hat{V}_\parallel = \hat{d}_i\,V_z(r_s)/V_A$. Here, $s_s = s(r_s)$ and $s(r) = d\ln q/d\ln r$. We also have

$$J(\hat{x}) = -\left(\frac{2}{s_s} - 1\right)\frac{1}{\hat{L}_s}, \tag{22}$$

and $\hat{E}_\parallel(\hat{x}) = (2/s_s - 1)\,(\hat{\eta}_\parallel/\hat{L}_s)$.

A.5 Derivation of Linear Layer Equations

In accordance with Eqs. (18)–(22), let us write

$$\psi(\hat{x},\zeta) = \frac{\hat{x}^{\,2}}{2\,\hat{L}_s} + \tilde{\psi}(\hat{x})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{23}$$

$$\phi(\hat{x},\zeta) = -\hat{V}_E\,\hat{x} + \tilde{\phi}(\hat{x})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{24}$$

$$N(\hat{x},\zeta) = -\hat{V}_*\,\hat{x} + \iota_e\,\tilde{N}(\hat{x})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{25}$$

$$V(\hat{x},\zeta) = \hat{V}_\parallel + \iota_e\,\tilde{V}(\hat{x})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{26}$$

$$J(\hat{x},\zeta) = -\left(\frac{2}{s_s} - 1\right)\frac{1}{\hat{L}_s} + \hat{\nabla}^2\tilde{\psi}(\hat{x})\,\mathrm{e}^{\,\mathrm{i}\,\zeta}, \tag{27}$$

where $\zeta = m\,\theta - n\,\varphi$. Substituting Eqs. (23)–(27) into Eqs. (13)–(17), and only retaining terms that are first order in perturbed quantities, we obtain the following set of linear equations:

$$-\mathrm{i}\,n\,(\omega_E + \omega_{*\,e})\,\tau_H\,\tilde{\psi} = -\mathrm{i}\,\hat{x}\,(\tilde{\phi} - \tilde{N}) + S^{-1}\,\hat{\nabla}^2\tilde{\psi}, \tag{28}$$

$$-\mathrm{i}\,n\,\omega_E\,\tau_H\,\tilde{N} = \mathrm{i}\,n\,\omega_{*\,e}\,\tau_H\,\tilde{\phi} - \mathrm{i}\,\iota_e\,\hat{d}_\beta^{\,2}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} - \mathrm{i}\,c_\beta^{\,2}\,\hat{x}\,\tilde{V} + S^{-1}\,P_E\,\hat{\nabla}^2\tilde{N}, \tag{29}$$

$$-\mathrm{i}\,n\,(\omega_E + \omega_{*\,i})\,\tau_H\,\hat{\nabla}^2\tilde{\phi} = -\mathrm{i}\,\hat{x}\,\hat{\nabla}^2\tilde{\psi} + S^{-1}\,P_\varphi\,\hat{\nabla}^4\left(\tilde{\phi} + \frac{\tilde{N}}{\iota}\right), \tag{30}$$

$$-\mathrm{i}\,n\,\omega_E\,\tau_H\,\tilde{V} = -\mathrm{i}\,n\,\omega_{*\,e}\,\tau_H\,\tilde{\psi} - \mathrm{i}\,\hat{x}\,\tilde{N} + S^{-1}\,P_\varphi\,\hat{\nabla}^2\tilde{V}. \tag{31}$$

Here, $\tau_H = L_s/(m\,V_A)$ is the hydromagnetic time, $\omega_E = -(q_s/r_s)\,V_E(r_s) = -(d\Phi/d\psi_p)_{r_s}$ the $\mathbf{E}\times\mathbf{B}$ frequency, $\omega_{*\,e} = \iota_e\,(q_s/r_s)V_*(r_s) = [(dp_e/d\psi_p)/(e\,n_e)]_{r_s}$ the electron diamagnetic frequency, $\omega_{*\,i} = -\iota_i\,(m/r_s)V_*(r_s) = -[(dp_i/d\psi_p)/(e\,n_e)]_{r_s}$ the ion diamagnetic frequency, $\Phi(r)$ the equilibrium electrostatic potential, $S = \tau_R/\tau_H$ the Lundquist number, $\tau_R = \mu_0\,r_s^{\,2}/\eta_\parallel$ the resistive diffusion time, $\tau_E = r_s^{\,2}/\chi_E$ the energy confinement time, and $\tau_\varphi = r_s^{\,2}/\chi_\varphi$ the toroidal momentum confinement time. Furthermore, $P_E = \tau_R/\tau_E$ and $P_\varphi = \tau_R/\tau_\varphi$ are magnetic Prandtl numbers.

Let us define the stretched radial variable $X = S^{\,1/3}\,\hat{x}$. Assuming that $X \sim \mathcal{O}(1)$ in the layer (i.e., assuming that the layer thickness is roughly of order $S^{-1/3}\,r_s$), and making use of the fact that $S \gg 1$ in conventional tokamak plasmas, Eqs. (28)–(31) reduce to the following set of linear layer equations [12, 15]:

$$\mathrm{i}\,(Q_E+Q_e)\,\tilde{\psi} = -\mathrm{i}\,X\left(\tilde{\phi}-\tilde{N}\right)+\frac{d^2\tilde{\psi}}{dX^2}, \tag{32}$$

$$\mathrm{i}\,Q_E\,\tilde{N} = -\mathrm{i}\,Q_e\,\tilde{\phi}-\mathrm{i}\,D^2\,X\,\frac{d^2\tilde{\psi}}{dX^2}-\mathrm{i}\,c_\beta^2\,X\,\tilde{V}+P_E\,\frac{d^2\tilde{N}}{dX^2}, \tag{33}$$

$$\mathrm{i}\,(Q_E+Q_i)\,\frac{d^2\tilde{\phi}}{dX^2} = -\mathrm{i}\,X\,\frac{d^2\tilde{\psi}}{dX^2}+P_\varphi\,\frac{d^4}{dX^4}\left(\tilde{\phi}+\frac{\tilde{N}}{\iota}\right), \tag{34}$$

$$\mathrm{i}\,Q_E\,\tilde{V} = \mathrm{i}\,Q_e\,\tilde{\psi}-\mathrm{i}\,X\,\tilde{N}+P_\varphi\,\frac{d^2\tilde{V}}{dX^2}. \tag{35}$$

Here, $Q_E = -S^{1/3}\,n\,\omega_E\,\tau_H$, $Q_{e,i} = -S^{1/3}\,n\,\omega_{*\,e,i}\,\tau_H$, and $D = S^{1/3}\,\iota_e^{1/2}\,\hat{d}_\beta$. If we write $P_E = c_\beta^2$ then Eqs. (32)–(35) become equivalent to the set of layer equations solved by Lee et alia. (To be slightly more exact, we get from our equations to those of Lee et alia by making the following transformation: $Q_E\to Q$, $Q_e\to -Q_{*\,e}$, $Q_i\to -Q_{*\,i}$, $\iota\to 1/\tau$, $P_E\to c_\beta^2$, $P_\varphi\to P$, $\tilde{\psi}\to-\tilde{\psi}$, $\tilde{N}\to\tilde{Z}$, $\tilde{\phi}\to\tilde{\phi}$, and $\tilde{V}\to-\tilde{V}_z$.)

The previously mentioned low-β approximation used in Refs. [12] and [15] involves neglecting the term containing c_β^2 in Eq. (33). This approximation decouples Eq. (35) from the three preceding equations, and effectively converts a four-field resonant response model into a three-field model. In the following, we shall not use this approximation.

A.6 Asymptotic Matching

The linear layer equations, (32)–(35), possess tearing parity solutions characterized by the symmetry $\tilde{\psi}(-X) = \tilde{\psi}(X)$, $\tilde{N}(-X) = -\tilde{N}(X)$, $\tilde{\phi}(-X) = -\tilde{\phi}(X)$, $\tilde{V}(-X) = \tilde{V}(X)$. As is easily demonstrated, the asymptotic behavior of the tearing parity solutions to Eqs. (32)–(35) are such that

$$\tilde{\psi}(X)\to\psi_0\left[\frac{\hat{\Delta}}{2}\,|X|+1+\mathcal{O}\left(\frac{1}{X^2}\right)\right], \tag{36}$$

$$\tilde{\phi}(X)\to-\psi_0\,Q_E\left[\frac{\hat{\Delta}}{2}\,\mathrm{sgn}(X)+\frac{1}{X}+\mathcal{O}\left(\frac{1}{X^2}\right)\right], \tag{37}$$

$$\tilde{N}(X)\to\psi_0\,Q_e\left[\frac{\hat{\Delta}}{2}\,\mathrm{sgn}(X)+\frac{1}{X}+\mathcal{O}\left(\frac{1}{X^2}\right)\right], \tag{38}$$

$$\tilde{V}(X)\to\mathcal{O}\left(\frac{1}{X^3}\right) \tag{39}$$

as $|X|\to\infty$, where ψ_0 is an arbitrary constant The layer response index is

$$\Delta_s = S^{1/3}\,\hat{\Delta}. \tag{40}$$

A.7 Fourier Transformation

Equations (32)–(35) are most conveniently solved in Fourier transform space [12]. Let

$$\bar{\phi}(p) = \int_{-\infty}^{\infty}\tilde{\phi}(X)\,\mathrm{e}^{-\mathrm{i}\,p\,X}\,dX, \tag{41}$$

et cetera. The Fourier transformed linear layer equations become

$$\mathrm{i}\,(Q_E+Q_e)\,\bar{\psi} = \frac{d}{dp}\big(\bar{\phi}-\bar{N}\big) - p^2\,\bar{\psi}, \tag{42}$$

$$\mathrm{i}\,Q_E\,\bar{N} = -\mathrm{i}\,Q_e\,\bar{\phi} - D^{\,2}\,\frac{d(p^2\,\bar{\psi})}{dp} + c_\beta^2\,\frac{d\bar{V}}{dp} - P_E\,p^2\,\bar{N}, \tag{43}$$

$$\mathrm{i}\,(Q_E+Q_i)\,p^2\,\bar{\phi} = \frac{d(p^2\,\bar{\psi})}{dp} - P_\varphi\,p^4\left(\bar{\phi}+\frac{\bar{N}}{\iota}\right), \tag{44}$$

$$\mathrm{i}\,Q_E\,\bar{V} = \mathrm{i}\,Q_e\,\bar{\psi} + \frac{d\bar{N}}{dp} - P_\varphi\,p^2\,\bar{V}, \tag{45}$$

where, for a tearing parity solution,

$$\bar{\phi}(p) - \bar{N}(p) \equiv \bar{Y}(p) \to \bar{Y}_0\left[\frac{\hat{\Delta}}{\pi\,p} + 1 + \mathcal{O}(p)\right] \tag{46}$$

as $p\to 0$, where Y_0 is an arbitrary constant.

Finally, if we define

$$\bar{J}(p) = p^2\,\bar{\psi}, \tag{47}$$

then Eqs. (42)–(45) can be converted into the following equivalent set of four coupled first-order differential equations:

$$\frac{d\bar{Y}}{dp} = \left[\frac{\mathrm{i}\,(Q_E+Q_e)+p^2}{p^2}\right]\bar{J}, \tag{48}$$

$$\frac{d\bar{N}}{dp} = -\left(\frac{\mathrm{i}\,Q_e}{p^2}\right)\bar{J} + (\,\mathrm{i}\,Q_E + P_\varphi\,p^2)\,\bar{V}, \tag{49}$$

$$\frac{d\bar{J}}{dp} = [\,\mathrm{i}\,(Q_E+Q_i)\,p^2 + P_\varphi\,p^4]\,\bar{Y} + [\,\mathrm{i}\,(Q_E+Q_i)\,p^2 + \iota_e^{-1}\,P_\varphi\,p^4]\,\bar{N}, \tag{50}$$

$$\begin{aligned} c_\beta^2\,\frac{d\bar{V}}{dp} &= [\,\mathrm{i}\,Q_e + \mathrm{i}\,(Q_E+Q_i)\,D^{\,2}\,p^2 + D^{\,2}\,P_\varphi\,p^4]\,\bar{Y} \\ &\quad + \{\,\mathrm{i}\,(Q_E+Q_e) + [P_E + \mathrm{i}\,(Q_E+Q_i)\,D^{\,2}]\,p^2 + \iota_e^{-1}\,D^{\,2}\,P_\varphi\,p^4\}\,\bar{N}, \end{aligned} \tag{51}$$

Note that $\iota_e = -Q_e/(Q_i-Q_e)$.

A.8 Small Argument Expansion

Let us search for power-law solutions of Eqs. (48)–(51) at small values of p. Given that we have four coupled first-order differential equations, we expect to find four independent power-law solutions. The first solution is such that

$$\bar{Y}(p) = \mathrm{i}\,(Q_E+Q_e)\,a_{-1}\,p^{-1} - \left[\frac{\mathrm{i}}{2}\,Q_E\,(Q_E+Q_e)\,(Q_E+Q_i)+1\right]a_{-1}\,p + \mathcal{O}(p^3), \tag{52}$$

$$\bar{N}(p) = -\mathrm{i}\,Q_e\,a_{-1}\,p^{-1} + \frac{\mathrm{i}}{2}\,Q_E\,Q_e\,(Q_E+Q_i)\,a_{-1}\,p + \mathcal{O}(p^3), \tag{53}$$

$$\bar{J}(p) = -a_{-1} - \frac{1}{2}\,Q_E\,(Q_E+Q_i)\,a_{-1}\,p^2 + \mathcal{O}(p^4), \tag{54}$$

$$\bar{V}(p) = -\frac{[\,\mathrm{i}\,Q_e\,(1+P_E) + Q_E\,(Q_E+Q_i)\,D^{\,2}]}{2\,c_\beta^2}\,a_{-1}\,p^2 + \mathcal{O}(p^4), \tag{55}$$

where a_{-1} is an arbitrary constant. The second solution is such that

$$\bar{Y}(p) = \mathrm{i}\,(Q_E + Q_e)\,a_0 - \frac{\mathrm{i}}{6}\,Q_E\,(Q_E+Q_e)\,(Q_E+Q_i)\,a_0\,p^2 + \mathcal{O}(p^4), \tag{56}$$

$$\bar{N}(p) = -\mathrm{i}\,Q_e\,a_0 + \frac{\mathrm{i}}{6}\,Q_E\,Q_e\,(Q_E+Q_i)\,a_0\,p^2 + \mathcal{O}(p^4), \tag{57}$$

$$\bar{J}(p) = -\frac{1}{3}\,Q_E\,(Q_E+Q_i)\,a_0\,p^3 + \mathcal{O}(p^5), \tag{58}$$

$$\bar{V}(p) = -\frac{1}{3}\,\frac{[\,\mathrm{i}\,Q_e\,P_E + Q_E\,(Q_E+Q_i)\,D^2]}{c_\beta^2}\,a_0\,p^3 + \mathcal{O}(p^5), \tag{59}$$

where a_0 is an arbitrary constant. The third solution is such that

$$\bar{Y}(p) = -\frac{1}{6}\,(Q_E+Q_e)\,(Q_E+Q_i)\,a_2\,p^2 + \mathcal{O}(p^4), \tag{60}$$

$$\bar{N}(p) = a_2 + \frac{\mathrm{i}}{2}\,(Q_E+Q_i)\left(-\frac{\mathrm{i}}{3}\,Q_e + \frac{g}{c_\beta^2}\right)a_2\,p^2 + \mathcal{O}(p^4), \tag{61}$$

$$\bar{J}(p) = \frac{\mathrm{i}}{3}\,(Q_E+Q_i)\,a_2\,p^3 + \mathcal{O}(p^5), \tag{62}$$

$$\bar{V}(p) = \mathrm{i}\,\frac{(Q_E+Q_e)}{c_\beta^2}\,a_2\,p + \mathcal{O}(p^3), \tag{63}$$

where a_2 is an arbitrary constant. The final solution is such that

$$\bar{Y}(p) = -\frac{\mathrm{i}}{12}\,Q_E\,(Q_E+Q_e)\,(Q_E+Q_i)\,a_3\,p^3 + \mathcal{O}(p^5), \tag{64}$$

$$\bar{N}(p) = -\mathrm{i}\,Q_E\,a_3\,p + \mathcal{O}(p^3), \tag{65}$$

$$\bar{J}(p) = -\frac{1}{4}\,Q_E\,(Q_E+Q_i)\,a_3\,p^4 + \mathcal{O}(p^6), \tag{66}$$

$$\bar{V}(p) = -\frac{Q_E\,(Q_E+Q_e)}{2\,c_\beta^2}\,a_3\,p^2 + \mathcal{O}(p^4), \tag{67}$$

where a_3 is an arbitrary constant. Note, however, that the third solution is not consistent with Equations (36)–(39), which mandate that $\bar{N}(p) \to -[Q_e/(Q_E+Q_e)]\,\bar{Y}(p) + \mathcal{O}(p)$ and $\bar{V}(p) \to \mathcal{O}(p^2)$ as $p \to 0$. Hence, we deduce that $a_2 = 0$.

We conclude that, at small values of p, the most general solution for $\bar{Y}(p)$ and $\bar{N}(p)$ takes the form

$$\bar{Y}(p) = \mathrm{i}\,(Q_E+Q_e)\,(a_{-1}\,p^{-1} + a_0) + \mathcal{O}(p), \tag{68}$$

$$\bar{N}(p) = -\mathrm{i}\,Q_e\,(a_{-1}\,p^{-1} + a_0) + \mathcal{O}(p), \tag{69}$$

which is consistent with Eqs. (36)–(39).

A.9 Ricatti Matrix Differential Equation

Let

$$\underline{u} = \begin{pmatrix} \bar{Y} \\ \bar{N} \end{pmatrix}, \tag{70}$$

$$\underline{v} = \begin{pmatrix} \bar{J} \\ c_\beta^2\,\bar{V} \end{pmatrix}. \tag{71}$$

Equations (48)–(51) can be written in the form

$$\frac{d\underline{u}}{dp} = \underline{\underline{A}}\,\underline{v}, \tag{72}$$

$$\frac{d\underline{v}}{dp} = \underline{\underline{B}}\,\underline{u}, \tag{73}$$

where

$$A_{11} = \frac{\mathrm{i}\,(Q_E + Q_e) + p^2}{p^2}, \tag{74}$$

$$A_{12} = 0, \tag{75}$$

$$A_{21} = -\frac{\mathrm{i}\,Q_e}{p^2}, \tag{76}$$

$$A_{22} = \frac{\mathrm{i}\,Q_E + P_\varphi\,p^2}{c_\beta^2}, \tag{77}$$

$$B_{11} = \mathrm{i}\,(Q_E + Q_i)\,p^2 + P_\varphi\,p^4, \tag{78}$$

$$B_{12} = \mathrm{i}\,(Q_E + Q_i)\,p^2 + \iota_e^{-1}\,P_\varphi\,p^4, \tag{79}$$

$$B_{21} = \mathrm{i}\,Q_e + \mathrm{i}\,(Q_E + Q_i)\,D^{\,2}\,p^2 + D^{\,2}\,P_\varphi\,p^4, \tag{80}$$

$$B_{22} = \mathrm{i}\,(Q_E + Q_e) + [P_E + \mathrm{i}\,(Q_E + Q_i)\,D^{\,2}]\,p^2 + \iota_e^{-1}\,D^{\,2}\,P_\varphi\,p^4. \tag{81}$$

Thus, we obtain the following matrix differential equation:

$$\frac{d}{dp}\left(\underline{\underline{A}}^{-1}\,\frac{d\underline{u}}{dp}\right) = \underline{\underline{B}}\,\underline{u}. \tag{82}$$

Let

$$p\,\frac{d\underline{u}}{dp} = \underline{\underline{W}}\,\underline{u}. \tag{83}$$

The previous two equations can be combined to give

$$\left(p\,\frac{d\underline{\underline{W}}}{dp} - \underline{\underline{W}} + \underline{\underline{W}}\,\underline{\underline{W}} + \underline{\underline{A}}\,p\,\frac{d\underline{\underline{A}}^{-1}}{dp}\,\underline{\underline{W}} - p^2\,\underline{\underline{A}}\,\underline{\underline{B}}\right)\underline{u} = \underline{0}, \tag{84}$$

which yields the Riccati matrix differential equation,

$$p\,\frac{d\underline{\underline{W}}}{dp} = \underline{\underline{W}} - \underline{\underline{W}}\,\underline{\underline{W}} - \underline{\underline{E}}\,\underline{\underline{W}} + \underline{\underline{F}}, \tag{85}$$

where

$$\underline{\underline{E}}(p) = \underline{\underline{A}}\,p\,\frac{d\underline{\underline{A}}^{-1}}{dp}, \tag{86}$$

$$\underline{\underline{F}}(p) = p^2\,\underline{\underline{A}}\,\underline{\underline{B}}. \tag{87}$$

In fact, it is easily demonstrated that

$$E_{11} = \frac{2\,\mathrm{i}\,(Q_E+Q_e)}{\mathrm{i}\,(Q_E+Q_e)+p^2}, \tag{88}$$

$$E_{12} = 0, \tag{89}$$

$$E_{21} = -\frac{2\,\mathrm{i}\,Q_e\,(\,\mathrm{i}\,Q_E+2\,P_\varphi\,p^2)}{[\,\mathrm{i}\,(Q_E+Q_e)+p^2]\,(\,\mathrm{i}\,Q_E+P_\varphi\,p^2)}, \tag{90}$$

$$E_{22} = -\frac{2\,P_\varphi\,p^2}{\mathrm{i}\,Q_E+P_\varphi\,p^2}, \tag{91}$$

and

$$F_{11} = p^2\,[\,\mathrm{i}\,(Q_E+Q_e)+p^2]\,[\,\mathrm{i}\,(Q_E+Q_i)+P_\varphi\,p^2], \tag{92}$$

$$F_{12} = p^2\,[\,\mathrm{i}\,(Q_E+Q_e)+p^2]\,[\,\mathrm{i}\,(Q_E+Q_i)+\iota_e^{-1}\,P_\varphi\,p^2], \tag{93}$$

$$F_{21} = -\mathrm{i}\,Q_e\,p^2\,[\,\mathrm{i}\,(Q_E+Q_i)+P_\varphi\,p^2] + c_\beta^{-2}\,p^2\,(\,\mathrm{i}\,Q_E+P_\varphi\,p^2)\,[\,\mathrm{i}\,Q_e+\mathrm{i}\,(Q_E+Q_i)\,D^{\,2}\,p^2+D^{\,2}\,P_\varphi\,p^4], \tag{94}$$

$$F_{22} = -\mathrm{i}\,Q_e\,p^2\,[\,\mathrm{i}\,(Q_E+Q_i)+\iota_e^{-1}\,P_\varphi\,p^2] + c_\beta^{-2}\,p^2\,(\,\mathrm{i}\,Q_E+P_\varphi\,p^2)\,\{\,\mathrm{i}\,(Q_E+Q_e)+[P_E+\mathrm{i}\,(Q_E+Q_i)\,D^{\,2}]\,p^2+\iota_e^{-1}\,D^{\,2}\,P_\varphi\,p^4\}. \tag{95}$$

Finally, if

$$\underline{\underline{W}}(p) = \begin{pmatrix} W_{11}, & W_{12}\\ W_{21}, & W_{22}\end{pmatrix} \tag{96}$$

then Eq. (85) yields

$$p\,\frac{dW_{11}}{dp} = W_{11} - W_{11}\,W_{11} - W_{12}\,W_{21} - E_{11}\,W_{11} + F_{11}, \tag{97}$$

$$p\,\frac{dW_{12}}{dp} = W_{12} - W_{11}\,W_{12} - W_{12}\,W_{22} - E_{11}\,W_{12} + F_{12}, \tag{98}$$

$$p\,\frac{dW_{21}}{dp} = W_{21} - W_{21}\,W_{11} - W_{22}\,W_{21} - E_{21}\,W_{11} - E_{22}\,W_{21} + F_{21}, \tag{99}$$

$$p\,\frac{dW_{22}}{dp} = W_{22} - W_{21}\,W_{12} - W_{22}\,W_{22} - E_{21}\,W_{12} - E_{22}\,W_{22} + F_{22}. \tag{100}$$

Thus, our final system of equations consists of a set of four coupled nonlinear differential equations.

A.10 Small Argument Behavior of Riccati Matrix Differential Equation

It follows from Eqs. (88)–(91) that $\underline{\underline{E}}(p) = \underline{\underline{E}}^{\,(0)} + \mathcal{O}(p^2)$ at small values of p, where

$$E_{11}^{(0)} = 2, \tag{101}$$

$$E_{12}^{(0)} = 0, \tag{102}$$

$$E_{21}^{(0)} = -\frac{2\,Q_e}{Q_E+Q_e}, \tag{103}$$

$$E_{22}^{(0)} = 0. \tag{104}$$

Likewise, Eqs. (92)–(95) imply that $\underline{\underline{F}}(p) = \mathcal{O}(p^2)$.

Suppose that $\underline{\underline{W}}(p) = \underline{\underline{W}}^{(0)} + \underline{\underline{W}}^{(1)}\, p$ at small values of p, where the elements of $\underline{\underline{W}}^{(0)}$ and $\underline{\underline{W}}^{(1)}$ are independent of p. Equation (85) gives

$$\underline{\underline{0}} = \underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)}\,\underline{\underline{W}}^{(0)} - \underline{\underline{E}}^{(0)}\,\underline{\underline{W}}^{(0)}, \tag{105}$$

$$\underline{\underline{0}} = -\underline{\underline{W}}^{(1)}\,\underline{\underline{W}}^{(0)} - \underline{\underline{W}}^{(0)}\,\underline{\underline{W}}^{(1)} - \underline{\underline{E}}^{(0)}\,\underline{\underline{W}}^{(1)}. \tag{106}$$

Suitable solutions are

$$\underline{\underline{W}}^{(0)} = \begin{pmatrix} -1, & 0 \\ -E_{21}^{(0)}/2, & 0 \end{pmatrix}, \tag{107}$$

$$W_{12}^{(1)} = 0, \tag{108}$$

$$W_{21}^{(1)} = \frac{E_{21}^{(0)}}{2}\left[W_{11}^{(1)} - W_{22}^{(1)}\right]. \tag{109}$$

At small values of p, let

$$\underline{u}(p) = \underline{u}_{-1}\, p^{-1} + \underline{u}_0, \tag{110}$$

where the elements of $\underline{u}_{-1}$ (which are y_{-1} and n_{-1}, respectively) and the elements of $\underline{u}_0$ (which are y_0 and n_0, respectively) are all constants. Equation (83) gives

$$\underline{\underline{W}}^{(0)}\,\underline{u}_{-1} = -\underline{u}_{-1}, \tag{111}$$

$$\underline{\underline{W}}^{(0)}\,\underline{u}_0 + \underline{\underline{W}}^{(1)}\,\underline{u}_{-1} = \underline{0}. \tag{112}$$

Thus, making use of Eq. (107), we get

$$\begin{pmatrix} -1, & 0 \\ -E_{21}^{(0)}/2, & 0 \end{pmatrix}\begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix} = -\begin{pmatrix} y_{-1} \\ n_{-1} \end{pmatrix}, \tag{113}$$

which implies that

$$\frac{E_{21}^{(0)}}{2}\, y_{-1} = -\frac{Q_e}{Q_E + Q_e}\, y_{-1} = n_{-1}, \tag{114}$$

in accordance with Eqs. (68) and (69), where use has been made of Eq. (103). Equations (107)–(109) and (112) yield

$$\frac{y_0}{y_{-1}} = W_{11}^{(1)}, \tag{115}$$

with n_0 undetermined. It follows from Equation (46) that

$$\frac{\pi}{\hat{\Delta}} \equiv \frac{y_0}{y_{-1}} = W_{11}^{(1)} = \frac{dW_{11}(0)}{dp}. \tag{116}$$

Note, incidentally, that the result $\pi/\hat{\Delta} = y_0/y_{-1}$ follows directly from Equation (32), and would hold even if a_2 were non-zero.

A.11 Large Argument Behavior of Riccati Matrix Differential Equation

At large values of p, it is clear from Eqs. (92)–(95) that $\underline{\underline{F}}(p) = \underline{\underline{F}}^{(6)}\, p^6 + \underline{\underline{F}}^{(8)}\, p^8$, where the elements of $\underline{\underline{F}}^{(6)}$ and $\underline{\underline{F}}^{(8)}$ are constants. On the other hand, Eqs. (88)–(91) imply that $\underline{\underline{E}}(p) = \underline{\underline{E}}^{(0)}$, where the elements of

$\underline{\underline{E}}^{(0)}$ are constants. Thus, if we write $\underline{\underline{W}}(p) = \underline{\underline{W}}^{(2)}\,p^2 + \underline{\underline{W}}^{(4)}\,p^4$, where the elements of $\underline{\underline{W}}^{(2)}$ and $\underline{\underline{W}}^{(4)}$ are constants, then Eq. (85) gives

$$\underline{\underline{W}}^{(4)}\,\underline{\underline{W}}^{(4)} = \underline{\underline{F}}^{(8)}, \tag{117}$$

$$\underline{\underline{W}}^{(2)}\,\underline{\underline{W}}^{(4)} + \underline{\underline{W}}^{(4)}\,\underline{\underline{W}}^{(2)} = \underline{\underline{F}}^{(6)}. \tag{118}$$

Now, according to Eqs. (92)–(95),

$$F_{11}^{(8)} = 0, \tag{119}$$

$$F_{12}^{(8)} = 0, \tag{120}$$

$$F_{21}^{(8)} = c_\beta^{-2}\,D^2\,P_\varphi^2, \tag{121}$$

$$F_{22}^{(8)} = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^2, \tag{122}$$

so Eq. (117) yields

$$W_{11}^{(4)} = 0, \tag{123}$$

$$W_{12}^{(4)} = 0, \tag{124}$$

$$W_{21}^{(4)} = -c_\beta^{-1}\,\iota_e^{1/2}\,D\,P_\varphi, \tag{125}$$

$$W_{22}^{(4)} = -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi, \tag{126}$$

where we have chosen the sign of the square root that is associated with well-behaved solutions at large values of p. Here, we are assuming that $\iota_e > 0$. Equations (92)–(95) also give

$$F_{11}^{(6)} = P_\varphi, \tag{127}$$

$$F_{12}^{(6)} = \iota_e^{-1}\,P_\varphi, \tag{128}$$

$$F_{21}^{(6)} = \mathrm{i}\,c_\beta^{-2}\,Q_E\,D^2\,P_\varphi + \mathrm{i}\,c_\beta^{-2}\,(Q_E+Q_i)\,D^2\,P_\varphi, \tag{129}$$

$$F_{22}^{(6)} = \mathrm{i}\,c_\beta^{-2}\,\iota_e^{-1}\,Q_E\,D^2\,P_\varphi + c_\beta^{-2}\,[P_E + \mathrm{i}\,(Q_E+Q_i)\,D^2]\,P_\varphi. \tag{130}$$

Thus, Eq. (118) yields

$$W_{12}^{(2)}\,W_{21}^{(4)} = F_{11}^{(6)}, \tag{131}$$

$$W_{12}^{(2)}\,W_{22}^{(4)} = F_{12}^{(6)}, \tag{132}$$

which gives

$$W_{12}^{(2)} = -c_\beta\,\iota_e^{-1/2}\,D^{-1}. \tag{133}$$

Now, if

$$\underline{\underline{W}}\,\underline{u} = \lambda(p)\,\underline{u} \tag{134}$$

then Eq. (83) yields

$$p\,\frac{d\underline{u}}{dp} = \lambda\,\underline{u}, \tag{135}$$

which implies that

$$\underline{u}(p) = \underline{u}(p_0)\,\exp\left[\int_{p_0}^{p}\frac{\lambda_r(p')}{p'}\,dp'\right]\exp\left[\mathrm{i}\int_{p_0}^{p}\frac{\lambda_i(p')}{p'}\,dp'\right], \tag{136}$$

where λ_r and λ_i are the real and imaginary parts of λ, respectively. Of course, a solution that is well behaved at large values of p is such that λ_r is negative. As we have seen, the large-p limit of Eq. (85) is

$$\underline{\underline{W}}\,\underline{\underline{W}} = \underline{\underline{F}}. \tag{137}$$

Hence, if

$$\underline{\underline{F}}\,\underline{u} = \Lambda\,\underline{u} \tag{138}$$

then Eqs. (134) and (138) imply that

$$\lambda^2 = \Lambda. \tag{139}$$

The eigenvalue problem for the F-matrix reduces to

$$\Lambda^2 - (F_{11} + F_{22})\,\Lambda + F_{11}\,F_{22} - F_{12}\,F_{21} = 0. \tag{140}$$

Now,

$$F_{11} + F_{22} \simeq F_{22}^{(8)}\,p^8 = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^2\,p^8, \tag{141}$$

$$F_{11}\,F_{22} - F_{12}\,F_{21} \simeq \left[F_{11}^{(6)}\,F_{22}^{(8)} - F_{12}^{(6)}\,F_{21}^{(8)}\right]p^{14} + \left[F_{11}^{(6)}\,F_{22}^{(6)} - F_{12}^{(6)}\,F_{21}^{(6)}\right]p^{12} = c_\beta^{-2}\,R\,P_\varphi^2\,p^{12}, \tag{142}$$

where

$$R = P_E + \mathrm{i}\,(1 - \iota_e^{-1})\,(Q_E + Q_i)\,D^2, \tag{143}$$

Hence, the two eigenvalues of the F-matrix are

$$\Lambda_1 \simeq F_{22}^{(8)}\,p^8 = c_\beta^{-2}\,\iota_e^{-1}\,D^2\,P_\varphi^2\,p^8, \tag{144}$$

$$\Lambda_2 \simeq \frac{[F_{11}^{(6)}\,F_{22}^{(6)} - F_{12}^{(6)}\,F_{21}^{(6)}]}{F_{22}^{(8)}}\,p^4 = \iota_e\,D^{-2}\,R\,p^4. \tag{145}$$

Thus, we deduce that the two eigenvalues of the W-matrix are

$$\lambda_1 = -\Lambda_1^{1/2} = -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi\,p^4, \tag{146}$$

$$\lambda_2 = -\Lambda_2^{1/2} = -\iota_e^{1/2}\,D^{-1}\,R^{1/2}\,p^2, \tag{147}$$

Here, the square root of R is taken such that the real part of λ_2 is negative. Now, the eigenvalue problem for the W-matrix reduces to

$$\lambda^2 - W_{22}^{(4)}\,p^4\,\lambda + \left[W_{11}^{(2)}\,W_{22}^{(4)} - W_{12}^{(2)}\,W_{21}^{(4)}\right]p^6 = 0. \tag{148}$$

which yields

$$\lambda_1 \simeq W_{22}^{(4)}\,p^4, \tag{149}$$

which is in agreement with Eq. (146), and

$$\lambda_2 \simeq \left[W_{11}^{(2)} - \frac{W_{12}^{(2)}\,W_{21}^{(4)}}{W_{22}^{(4)}}\right]p^2, \tag{150}$$

which implies that

$$W_{11}^{(2)} = -\iota_e^{1/2}\,D^{-1}\,R^{1/2} - c_\beta\,\iota_e^{1/2}\,D^{-1}. \tag{151}$$

Hence, the large-p boundary condition for the W-matrix is

$$\underline{\underline{W}}(p) = \begin{pmatrix} -\iota_e^{1/2}\,D^{-1}\,R^{1/2}\,p^2 - c_\beta\,\iota_e^{1/2}\,D^{-1}\,p^2, & -c_\beta\,\iota_e^{-1/2}\,D^{-1}\,p^2\\ -c_\beta^{-1}\,\iota_e^{1/2}\,D\,P_\varphi\,p^4, & -c_\beta^{-1}\,\iota_e^{-1/2}\,D\,P_\varphi\,p^4 \end{pmatrix}. \tag{152}$$

A.12 Method of Solution

The method of solution is to launch the well-behaved asymptotic solution (152) of Eqs. (97)–(100) from large p, and then integrate the equations backward to small p. The complex layer response index, Δ_s, is then determined from Eqs. (40) and (116). Note that, because there are no free parameters in expression (152), the layer response index is uniquely determined by this procedure.