Pressure Flattening due to Asymmetric Magnetic Island

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I. MAGNETIC ISLAND

Let $x = r - r_s$, $X = x/W$, and $\zeta = m\,\theta - n\,\phi$, where W is the island width. The magnetic flux-surfaces of the magnetic island are contours of

$$\Omega(X,\zeta) = 8\,X^2 + \cos(\zeta - \delta^2\,\sin\zeta) - 2\sqrt{8}\,\delta\,X\,\cos\zeta + \delta^2\,\cos^2\zeta, \tag{1}$$

where $|\delta| < 1$. The X-points lie at $X = \delta/\sqrt{8}$ and $\zeta = 0,\ 2\pi$, whereas the O-point lies at $X = -\delta/\sqrt{8}$ and $\zeta = \pi$. The O-point corresponds to $\Omega = -1$, whereas the magnetic separatrix corresponds to $\Omega = 1$.

Let

$$Y = X - \frac{\delta}{\sqrt{8}}\,\cos\zeta, \tag{2}$$

$$\xi = \zeta - \delta^2\,\sin\zeta. \tag{3}$$

It follows that

$$\Omega(Y,\zeta) = 8\,Y^2 + \cos\xi, \tag{4}$$

The X-points lie at $Y = 0$ and $\xi = 0,\ 2\pi$, whereas the O-point lies at $Y = 0$ and $\zeta = \pi$. Moreover,

$$\zeta = \xi + 2\sum_{n=1,\infty}\left[\frac{J_n(n\,\delta^2)}{n}\right]\sin(n\,\xi), \tag{5}$$

$$\cos\zeta = -\frac{\delta^2}{2} + \sum_{n=1,\infty}\left[\frac{J_{n-1}(n\,\delta^2) - J_{n+1}(n\,\delta^2)}{n}\right]\cos(n\,\xi), \tag{6}$$

$$\sin\zeta = \frac{2}{\delta^2}\sum_{n=1,\infty}\left[\frac{J_n(n\,\delta^2)}{n}\right]\sin(n\,\xi), \tag{7}$$

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$$\cos(m\,\zeta) = m \sum_{n=1,\infty} \left[\frac{J_{n-m}(n\,\delta^2) - J_{n+m}(n\,\delta^2)}{n}\right] \cos(n\,\xi), \tag{8}$$

$$\sin(m\,\zeta) = m \sum_{n=1,\infty} \left[\frac{J_{n-m}(n\,\delta^2) + J_{n+m}(n\,\delta^2)}{n}\right] \sin(n\,\xi), \tag{9}$$

for $m > 1$.

II. PLASMA DISPLACEMENT

Outside the separatrix, we can write

$$\Omega(X,\zeta) = 8\,(X - \Xi)^2, \tag{10}$$

where $\Xi = \xi^r/W$. It follows that

$$\begin{aligned}\Xi(X,\zeta) &\simeq -\frac{[\Omega(X,\zeta) - 8\,X^2]}{16\,X}\\ &= -\frac{\cos(\zeta - \delta^2\,\sin\zeta) + \delta^2\,\cos^2\zeta}{16\,X} + \frac{\delta}{\sqrt{8}}\,\cos\zeta.\end{aligned} \tag{11}$$

Note that Ξ is an even function of ζ. Let us write

$$\Xi(X,\zeta) = \sum_{n=0,\infty} \Xi_n(X)\,\cos(n\,\zeta). \tag{12}$$

Thus,

$$\Xi_1(X) = 2\oint \Xi(X,\zeta)\,\cos(\zeta)\,\frac{d\zeta}{2\pi} = -\frac{1}{8\,X}\oint \cos(\zeta - \delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi} + \frac{\delta}{\sqrt{8}} \tag{13}$$

$$\begin{aligned}&= -\frac{1}{16\,X}\oint \cos(-\delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi}\\ &\quad - \frac{1}{16\,X}\oint \cos(2\,\zeta - \delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi} + \frac{\delta}{\sqrt{8}}.\end{aligned} \tag{14}$$

But,

$$J_n(\delta^2) = \oint \cos(n\,\zeta - \delta^2\,\sin\zeta)\,\frac{d\zeta}{2\pi}, \tag{15}$$

so

$$\Xi_1(X) = -\frac{J_0(\delta^2) + J_2(\delta^2)}{16\,X} + \frac{\delta}{\sqrt{8}}, \tag{16}$$

and

$$\xi_1^r(x) = -\frac{W^2}{16\,x}\,[J_0(\delta^2) + J_2(\delta^2)] + \frac{W\,\delta}{\sqrt{8}}. \tag{17}$$

Thus,

$$\psi_1(x) = \frac{r\,g}{q}\,(m - n\,q)\,\xi_1^r = -(n\,s\,g)_{r_s}\,x\,\xi_1^r$$
$$= (n\,s\,g)_{r_k}\,\frac{W^2}{16}\,[J_0(\delta^2) + J_2(\delta^2)] - (n\,s\,g)_{r_s}\,\frac{W\,\delta}{\sqrt{8}}\,x, \tag{18}$$

so

$$\frac{\psi_m(x)}{m} = J_0(\delta^2) + J_2(\delta^2) - 2\sqrt{8}\,\frac{\delta}{W}\,x. \tag{19}$$

It follows that

$$\delta = -\frac{W}{2\sqrt{8}\,m}\,\frac{d\psi_m(0)}{dr} \tag{20}$$
$$\simeq -\left[\frac{\psi_m(r_s+W) - \psi_m(r_s-W)}{4\sqrt{8}\,m}\right]. \tag{21}$$

III. FLUX-SURFACE AVERAGE OPERATOR

Now,

$$[A,B] \equiv \left.\frac{\partial A}{\partial X}\right|_\zeta \left.\frac{\partial B}{\partial \zeta}\right|_X - \left.\frac{\partial B}{\partial X}\right|_\zeta \left.\frac{\partial A}{\partial \zeta}\right|_X. \tag{22}$$

But,

$$\left.\frac{\partial}{\partial X}\right|_\zeta = \left.\frac{\partial \Omega}{\partial X}\right|_\zeta \left.\frac{\partial}{\partial \Omega}\right|_\xi + \left.\frac{\partial \xi}{\partial X}\right|_\zeta \left.\frac{\partial}{\partial \xi}\right|_\Omega = 16\,Y\,\left.\frac{\partial}{\partial \Omega}\right|_\xi, \tag{23}$$

and

$$\left.\frac{\partial}{\partial \zeta}\right|_X = \left.\frac{\partial \Omega}{\partial \zeta}\right|_X \left.\frac{\partial}{\partial \Omega}\right|_\xi + \left.\frac{\partial \xi}{\partial \zeta}\right|_X \left.\frac{\partial}{\partial \xi}\right|_\Omega, \tag{24}$$

so

$$[A,B] \equiv \frac{16\,Y}{\sigma}\left(\left.\frac{\partial A}{\partial \Omega}\right|_\xi \left.\frac{\partial B}{\partial \xi}\right|_\Omega - \left.\frac{\partial B}{\partial \Omega}\right|_\xi \left.\frac{\partial A}{\partial \xi}\right|_\Omega\right), \tag{25}$$

where

$$\sigma(\xi) \equiv \frac{d\zeta}{d\xi} = 1 + 2\sum_{n=1,\infty} J_n(n\,\delta^2)\,\cos(n\,\xi). \tag{26}$$

In particular,

$$[A,\Omega] = -\frac{16\,Y}{\sigma}\,\left.\frac{\partial A}{\partial \xi}\right|_\Omega. \tag{27}$$

The flux-surface average operator, $\langle\cdots\rangle$, is the annihilator of $[A,\Omega]$ for arbitrary $A(s,\Omega,\xi)$. Here, $s=+1$ for $Y>0$ and $s=-1$ for $Y<0$. It follows that

$$\langle A\rangle = \int_{\zeta_0}^{2\pi-\zeta_0} \frac{\sigma(\xi)\,A_+(\Omega,\xi)}{\sqrt{2\,(\Omega-\cos\xi)}}\,\frac{d\xi}{2\pi} \tag{28}$$

for $-1 \leq \Omega \leq 1$, and

$$\langle A\rangle = \int_0^{2\pi} \frac{\sigma(\xi)\,A(s,\Omega,\xi)}{\sqrt{2\,(\Omega-\cos\xi)}}\,\frac{d\xi}{2\pi} \tag{29}$$

for $\Omega > 1$. Here, $\xi_0 = \cos^{-1}(\Omega)$, and

$$A_+(\Omega,\xi) = \frac{1}{2}\left[A(+1,\Omega,\xi)+A(-1,\Omega,\xi)\right]. \tag{30}$$

IV. TEMPERATURE PERTURBATION

The electron temperature in the vicinity of the island can be written

$$T_e(X,\zeta) = T_{e\,s} + s\,W\,T_{e\,s}'\,\tilde{T}(\Omega). \tag{31}$$

Here, $\tilde{T}(\Omega)$ satisfies

$$\left\langle \left.\frac{\partial^2\tilde{T}}{\partial X^2}\right|_\zeta \right\rangle = 0, \tag{32}$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \rightarrow |X| \tag{33}$$

as $|X| \rightarrow \infty$. It follows that

$$\frac{d}{d\Omega}\left(\langle Y^2\rangle\,\frac{d\tilde{T}}{d\Omega}\right) = 0 \tag{34}$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \rightarrow \frac{\Omega^{1/2}}{\sqrt{8}} \tag{35}$$

as $\Omega \rightarrow \infty$.

Outside the separatrix,

$$\langle Y^2\rangle(\Omega) = \frac{1}{16}\int_0^{2\pi}\sigma(\xi)\sqrt{2\,(\Omega-\cos\xi)}\,\frac{d\xi}{2\pi}. \tag{36}$$

Let

$$k = \left(\frac{1+\Omega}{2}\right)^{1/2}. \tag{37}$$

Thus, the O-point corresponds to $k=0$ and the separatrix to $k=1$. It follows that

$$\langle Y^2\rangle(k) = \frac{k}{4\pi}\int_0^{\pi/2}\sigma(2\,\theta-\pi)\,(1-\sin^2\theta/k^2)^{1/2}\,d\theta. \tag{38}$$

Thus,

$$\langle Y^2\rangle(k) = \frac{k}{4\pi}\,G(1/k), \tag{39}$$

where

$$G(p) = E_0(p) + 2\,\cos(n\,\pi)\sum_{n=1,\infty} J_n(n\,\delta^2)\,E_n(p), \tag{40}$$

$$E_n(p) = \int_0^{\pi/2}\cos(2\,n\,\theta)\,(1-p^2\,\sin^2\theta)^{1/2}\,d\theta. \tag{41}$$

Equation (34) yields

$$\tilde{T}(k) = 0 \tag{42}$$

for $0\leq k\leq 1$, and

$$\frac{d}{dk}\left[G(1/k)\,\frac{d\tilde{T}}{dk}\right] = 0 \tag{43}$$

for $k>1$. Thus,

$$\frac{d\tilde{T}}{dk} = \frac{c}{G(1/k)} \tag{44}$$

for $k>1$, subject to the boundary condition that

$$\tilde{T}(k)\rightarrow\frac{k}{2} \tag{45}$$

as $k\rightarrow\infty$. In the limit that $p\rightarrow 0$,

$$E_0(p) = \frac{\pi}{2}, \tag{46}$$

$$E_{n>0}(p) = 0, \tag{47}$$

which implies that $c=\pi/4$. So

$$\frac{d\tilde{T}}{dk} = \frac{\pi}{4}\,\frac{1}{G(1/k)}, \tag{48}$$

$$\tilde{T}(k) = F(k), \tag{49}$$

$$F(k) = \frac{\pi}{4}\int_1^k\frac{dk'}{G(1/k')} \tag{50}$$

for $k>1$.

V. HARMONICS OF TEMPERATURE PERTURBATION

We can write

$$\tilde{T}(X,\zeta) = \sum_{\nu=0,\infty} \delta T_\nu(X)\,\cos(\nu\,\zeta). \tag{51}$$

Now,

$$\delta T_0(X) = \oint \tilde{T}(X,\zeta)\,\frac{d\zeta}{2\pi}, \tag{52}$$

where the integral is at constant X. It follows that

$$\delta T_0(X) = \int_0^{\xi_c} F(k)\,\sigma(\xi)\,\frac{d\xi}{\pi}, \tag{53}$$

where

$$\xi_c = \cos^{-1}(1 - 8\,Y^2) \tag{54}$$

for $|Y| < 1/2$, and $\xi_c = \pi$ for $|Y| \geq 1/2$. Furthermore,

$$k = \left[4\,Y^2 + \cos^2\left(\frac{\xi}{2}\right)\right]^{1/2}. \tag{55}$$

For $\nu > 0$, we have

$$\delta T_\nu(X) = 2\oint \tilde{T}(X,\zeta)\,\cos(\nu\,\zeta)\,\frac{d\zeta}{2\pi}. \tag{56}$$

Integrating by parts, we obtain

$$\delta T_\nu(X) = -\frac{2}{\nu}\oint \left.\frac{\partial\tilde{T}}{\partial\zeta}\right|_X \sin(\nu\,\zeta)\,\frac{d\zeta}{2\pi}. \tag{57}$$

But,

$$\left.\frac{\partial\tilde{T}}{\partial\zeta}\right|_X = \frac{d\tilde{T}}{dk}\left.\frac{\partial k}{\partial\zeta}\right|_X = \frac{1}{4\,k}\frac{d\tilde{T}}{dk}\left.\frac{\partial\Omega}{\partial\zeta}\right|_X = -\frac{1}{4\,k}\,\frac{d\tilde{T}}{dk}\,\kappa(\xi), \tag{58}$$

where

$$\kappa(\xi) = \sin\xi\,(1-\delta^2\,\cos\zeta) - 2\sqrt{8}\,\delta\,X\,\sin\zeta + \delta^2\,\sin(2\,\zeta). \tag{59}$$

Hence,

$$\delta T_\nu(X) = \frac{1}{8\,\nu}\int_0^{\xi_c} \frac{\sin(\nu\,\zeta)\,\kappa(\xi)\,\sigma(\xi)}{k\,G(1/k)}\,d\xi. \tag{60}$$