Calculation of Vertical Stability in an Inverse Aspect-Ratio Expanded Tokamak Plasma Equilibrium

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I. PLASMA EQUILIBRIUM

All lengths are normalized to the major radius of the plasma magnetic axis, R_0. All magnetic field-strengths are normalized to the toroidal field-strength at the magnetic axis, B_0. All current densities are normalized to $B_0/(\mu_0\,R_0)$. All plasma pressures are normalized to B_0^2/μ_0.

Let R, ϕ, Z be right-handed cylindrical coordinates whose Jacobian is

$$(\nabla R\times\nabla\phi\cdot\nabla Z)^{-1} = R. \tag{1}$$

Note that $|\nabla\phi| = 1/R$.

Let r, θ, ϕ be right-handed flux-coordinates whose Jacobian is

$$\mathcal{J}(r,\theta)\equiv(\nabla r\times\nabla\theta\cdot\nabla\phi)^{-1}\equiv R\left(\frac{\partial R}{\partial\theta}\frac{\partial Z}{\partial r}-\frac{\partial R}{\partial r}\frac{\partial Z}{\partial\theta}\right) = r\,R^2. \tag{2}$$

Note that $r=r(R,Z)$ and $\theta=\theta(R,Z)$. The magnetic axis corresponds to $r=0$. The inboard mid-plane corresponds to $\theta=0$.

Consider an axisymmetric tokamak equilibrium whose magnetic field takes the form

$$\mathbf{B}(r,\theta) = f(r)\,\nabla\phi\times\nabla r + g(r)\,\nabla\phi = f\,\nabla(\phi-q\,\theta)\times\nabla r, \tag{3}$$

where

$$q(r) = \frac{r\,g}{f} \tag{4}$$

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is the safety-factor (i.e., the inverse of the rotational transform). Note that $\mathbf{B}\cdot\nabla r = 0$, which implies that r is a magnetic flux-surface label. We require $g=1$ on the magnetic axis in order to ensure that the normalized toroidal magnetic field-strength at the axis is unity.

It is easily demonstrated that

$$B^{\,r} = \mathbf{B}\cdot\nabla r = 0, \tag{5}$$

$$B^{\,\theta} = \mathbf{B}\cdot\nabla\theta = \frac{f}{r\,R^2}, \tag{6}$$

$$B^{\,\phi} = \mathbf{B}\cdot\nabla\phi = \frac{g}{R^2}, \tag{7}$$

$$B_r = \mathcal{J}\,\nabla\theta\times\nabla\phi\cdot\mathbf{B} = -r\,f\,\nabla r\cdot\nabla\theta, \tag{8}$$

$$B_\theta = \mathcal{J}\,\nabla\phi\times\nabla r\cdot\mathbf{B} = r\,f\,|\nabla r|^2, \tag{9}$$

$$B_\phi = \mathcal{J}\,\nabla r\times\nabla\theta\cdot\mathbf{B} = g. \tag{10}$$

The Maxwell equation (neglecting the displacement current, because the plasma velocity perturbations due to axisymmetric modes are far smaller than the velocity of light in vacuum) $\mathbf{J} = \nabla\times\mathbf{B}$ yields

$$\mathcal{J}\,J^{\,r} = \frac{\partial B_\phi}{\partial\theta} = 0, \tag{11}$$

$$\mathcal{J}\,J^{\,\theta} = -\frac{\partial B_\phi}{\partial r} = -g', \tag{12}$$

$$\mathcal{J}\,J^{\,\phi} = \frac{\partial B_\theta}{\partial r} - \frac{\partial B_r}{\partial\theta} = \frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\,, \tag{13}$$

where $\mathbf{J}$ is the equilibrium current density, $' \equiv d/dr$, and use has been made of Eqs. (8)–(10).

Equilibrium force balance requires that

$$\nabla P = \mathbf{J}\times\mathbf{B}, \tag{14}$$

where $P(r)$ is the equilibrium scalar plasma pressure. Here, for the sake of simplicity, we have neglected the small centrifugal modifications to force balance due to subsonic plasma rotation. It follows that

$$P' = \mathcal{J}(J^{\,\theta}\,B^{\,\phi} - J^{\,\phi}\,B^{\,\theta}) = -g'\,\frac{g}{R^2} - \frac{f}{r\,R^2}\left[\frac{\partial}{\partial r}\big(r\,f\,|\nabla r|^2\big) + \frac{\partial}{\partial\theta}(r\,f\,\nabla r\cdot\nabla\theta)\right], \tag{15}$$

where use has been made of Eqs. (5)–(7), and (11)–(13). The other two components of Eq. (14) are identically zero.

Equation (15) yields the *inverse Grad-Shafranov equation*:

$$\frac{f}{r}\frac{\partial}{\partial r}\left(r\, f\, |\nabla r|^2\right)+\frac{f}{r}\frac{\partial}{\partial\theta}(r\, f\,\nabla r\cdot\nabla\theta)+g\, g'+R^2\, P'=0. \tag{16}$$

It follows from Eqs. (4), (13), and (16) that

$$\mathcal{J}\, J^{\phi}=-q\, g'-\frac{r\, R^2\, P'}{f}. \tag{17}$$

It is clear from Eqs. (12) and (17) that $g'=P'=0$ in the current-free "vacuum" region surrounding the plasma. We shall also assume that $g'=P'=0$ at the plasma-vacuum interface, so as to ensure that the equilibrium plasma current density is zero at the interface.

II. AXISYMMETRIC PLASMA PERTURBATION

A. Derivation of Axisymmetric Ideal-MHD P.D.E.s

Let us assume that all perturbed quantities are independent of the toroidal angle, ϕ. The perturbed plasma equilibrium satisfies the marginally-stable ideal-MHD equations

$$\mathbf{b}=\nabla\times(\boldsymbol{\xi}\times\mathbf{B}), \tag{18}$$

$$\nabla p=\mathbf{j}\times\mathbf{B}+\mathbf{J}\times\mathbf{b}, \tag{19}$$

$$\mathbf{j}=\nabla\times\mathbf{b}, \tag{20}$$

$$p=-\boldsymbol{\xi}\cdot\nabla P, \tag{21}$$

where $\boldsymbol{\xi}(r,\theta)$ is the plasma displacement, $\mathbf{b}(r,\theta)$ the perturbed magnetic field, $\mathbf{j}(r,\theta)$ the perturbed current density, and $p(r,\theta)$ the perturbed scalar pressure.

Now,

$$(\boldsymbol{\xi}\times\mathbf{B})_\theta=\mathcal{J}\,(\xi^{\phi}\, B^{r}-\xi^{r}\, B^{\phi})=-\mathcal{J}\, B^{\phi}\,\xi^{r}, \tag{22}$$

$$(\boldsymbol{\xi}\times\mathbf{B})_\phi=\mathcal{J}\,(\xi^{r}\, B^{\theta}-\xi^{\theta}\, B^{r})=\mathcal{J}\, B^{\theta}\,\xi^{r}, \tag{23}$$

where use has been made of the fact that $B^r = J^r = 0$. [See Eqs. (5) and (11).] Combining Eqs. (18) and (23), we obtain

$$\mathcal{J}\, b^r = \frac{\partial}{\partial\theta}\left(\mathcal{J}\, B^\theta\, \xi^r\right). \tag{24}$$

Thus, Eqs. (2), (4), (6), and (7) give

$$r\, R^2\, b^r = \frac{\partial y}{\partial\theta}, \tag{25}$$

where

$$y(r,\theta) = f\, \xi^r. \tag{26}$$

The constraint $\nabla\cdot\mathbf{b} = 0$, which follows from Eq. (18), immediately yields

$$r\, R^2\, b^\theta = -\frac{\partial y}{\partial r}. \tag{27}$$

According to Eq. (21),

$$p = -P'\,\nabla r\cdot\boldsymbol{\xi} = -P'\,\xi^r. \tag{28}$$

So, the perturbed force balance equation, (19), yields

$$-\frac{\partial\,(P'\,\xi^r)}{\partial r} = (\mathbf{j}\times\mathbf{B})_r + (\mathbf{J}\times\mathbf{b})_r, \tag{29}$$

$$-\frac{\partial\,(P'\,\xi^r)}{\partial\theta} = (\mathbf{j}\times\mathbf{B})_\theta + (\mathbf{J}\times\mathbf{b})_\theta, \tag{30}$$

$$0 = (\mathbf{j}\times\mathbf{B})_\phi + (\mathbf{J}\times\mathbf{b})_\phi, \tag{31}$$

giving

$$-\frac{\partial\,(P'\,\xi^r)}{\partial r} = r\,R^2\,(j^\theta\,B^\phi - j^\phi\,B^\theta) + r\,R^2\,(J^\theta\,b^\phi - J^\phi\,b^\theta), \tag{32}$$

$$-\frac{\partial\,(P'\,\xi^r)}{\partial\theta} = r\,R^2\,(j^\phi\,B^r - j^r\,B^\phi) + r\,R^2\,(J^\phi\,b^r - J^r\,b^\phi), \tag{33}$$

$$0 = r\,R^2\,(j^r\,B^\theta - j^\theta\,B^r) + r\,R^2\,(J^r\,b^\theta - J^\theta\,b^r), \tag{34}$$

where use has been made of Eq. (2). Thus, according to Eqs. (5)–(7), (11), (12), and (17),

$$-\frac{\partial\,(P'\,\xi^r)}{\partial r} = f\,(q\,j^\theta - j^\phi) - g'\,b^\phi + \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)b^\theta, \tag{35}$$

$$-\frac{\partial\,(P'\,\xi^{\,r})}{\partial\theta} = -r\,g\,j^{\,r} - \left(q\,g' + \frac{r\,R^2\,P'}{f}\right) b^{\,r}, \tag{36}$$

$$0 = f\,j^{\,r} + g'\,b^{\,r}. \tag{37}$$

It follows from Eqs. (25) and (37) that

$$r\,R^2\,j^{\,r} = -\alpha_g\,\frac{\partial y}{\partial\theta}, \tag{38}$$

where

$$\alpha_g(r) = \frac{g'}{f}. \tag{39}$$

Note that Eq. (36) is trivially satisfied. Hence, of the three components of the perturbed force balance equation, only Eq. (35) remains to be solved.

Equation (20) yields

$$r\,R^2\,j^{\,r} = \frac{\partial b_\phi}{\partial\theta}, \tag{40}$$

$$r\,R^2\,j^{\,\theta} = -\frac{\partial b_\phi}{\partial r}, \tag{41}$$

$$r\,R^2\,j^{\,\phi} = \frac{\partial b_\theta}{\partial r} - \frac{\partial b_r}{\partial\theta}, \tag{42}$$

where use has been made of Eq. (2). It follows from Eqs. (38), (40), and (41) that

$$b_\phi = -\alpha_g\,y, \tag{43}$$

$$r\,R^2\,j^{\,\theta} = \frac{\partial(\alpha_g\,y)}{\partial r}. \tag{44}$$

Note that $\nabla\cdot\mathbf{j} = 0$, in accordance with Eq. (20).

Now,

$$\mathbf{b} = b_r\,\nabla r + b_\theta\,\nabla\theta + b_\phi\,\nabla\phi, \tag{45}$$

so

$$b^{\,r} = \mathbf{b}\cdot\nabla r = |\nabla r|^2\,b_r + (\nabla r\cdot\nabla\theta)\,b_\theta, \tag{46}$$

$$b^{\,\theta} = \mathbf{b}\cdot\nabla\theta = (\nabla r\cdot\nabla\theta)\,b_r + |\nabla\theta|^2\,b_\theta, \tag{47}$$

$$b^\phi = \mathbf{b}\cdot\nabla\phi = \frac{b_\phi}{R^2}. \tag{48}$$

Equations (2), (46), and (47) can be rearranged to give

$$b_r = \left(\frac{1}{|\nabla r|^2}\right) b^r - \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right) b_\theta, \tag{49}$$

$$b^\theta = \left(\frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\right) b^r + \left(\frac{1}{r^2\,R^2\,|\nabla r|^2}\right) b_\theta. \tag{50}$$

Let

$$z = |\nabla r|^2\,r\,\frac{\partial y}{\partial r} + r\,\nabla r\cdot\nabla\theta\,\frac{\partial y}{\partial\theta}. \tag{51}$$

Equations (25), (27), (43), (49) and (50) yield

$$b_r = \frac{1}{r\,|\nabla r|^2\,R^2}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z, \tag{52}$$

$$b_\theta = -z, \tag{53}$$

$$b^\phi = -\frac{\alpha_g}{R^2}\,y. \tag{54}$$

Equations (42), (52), and (53) give

$$r\,R^2\,j^\phi = -\frac{\partial z}{\partial r} - \frac{\partial}{\partial\theta}\left[\frac{1}{r\,|\nabla r|^2\,R^2}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right]. \tag{55}$$

It follows from Eqs. (26), (27), (35), (44), (54), and (55) that

$$\begin{aligned} -\frac{\partial}{\partial r}\left(\frac{P'}{f}\,y\right) = {}& \frac{f\,q}{r\,R^2}\,\frac{\partial(\alpha_g\,y)}{\partial r} + \frac{f}{r\,R^2}\,\frac{\partial z}{\partial r}\\ &+ \frac{f}{r\,R^2}\,\frac{\partial}{\partial\theta}\left[\frac{1}{r\,|\nabla r|^2\,R^2}\,\frac{\partial y}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right]\\ &+ \frac{g'\,\alpha_g}{R^2}\,y - \left(q\,g' + \frac{r\,R^2\,P'}{f}\right)\frac{1}{r\,R^2}\,\frac{\partial y}{\partial r}. \end{aligned} \tag{56}$$

Hence,

$$-\left[(\alpha_p\,\alpha_f + r\,\alpha_p')R^2 + q\,r\,\alpha_g' + r^2\,\alpha_g^2\right] y = r\,\frac{\partial z}{\partial r} + \frac{\partial}{\partial\theta}\left[\frac{1}{|\nabla r|^2\,R^2}\,\frac{\partial y}{\partial\theta} + \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right], \tag{57}$$

where

$$\alpha_p(r) = \frac{r\,P'}{f^2}, \tag{58}$$

$$\alpha_f(r) = \frac{r^2}{f}\,\frac{d}{dr}\!\left(\frac{f}{r}\right). \tag{59}$$

Finally, Eqs. (51) and (57) yield the *axisymmetric ideal-MHD p.d.e.s*:

$$r\,\frac{\partial y}{\partial r} = \frac{z}{|\nabla r|^2} - \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,\frac{\partial y}{\partial\theta}, \tag{60}$$

$$\begin{aligned} r\,\frac{\partial z}{\partial r} =& -\left[(\alpha_p\,\alpha_f + r\,\alpha_p')R^{\,2} + q\,r\,\alpha_g' + r^{\,2}\,\alpha_g^{\,2}\right] y - \frac{\partial}{\partial\theta}\!\left(\frac{1}{|\nabla r|^2\,R^{\,2}}\,\frac{\partial y}{\partial\theta}\right)\\ &- \frac{\partial}{\partial\theta}\!\left(\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,z\right). \end{aligned} \tag{61}$$

B. Derivation of the Axisymmetric Ideal-MHD O.D.E.s

Let

$$y(r,\theta) = \sum_m y_m(r)\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}, \tag{62}$$

$$z(r,\theta) = \sum_m z_m(r)\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}. \tag{63}$$

Equations (60) and (61) yield the *axisymmetric ideal-MHD o.d.e.s*:

$$r\,\frac{dy_m}{dr} = \sum_{m'}\left(A_m^{m'}\,z_{m'} + B_m^{m'}\,y_{m'}\right), \tag{64}$$

$$r\,\frac{dz_m}{dr} = \sum_{m'}\left(C_m^{m'}\,z_{m'} + D_m^{m'}\,y_{m'}\right), \tag{65}$$

where

$$A_m^{m'} = c_m^{m'}, \tag{66}$$

$$B_m^{m'} = -m'\,f_m^{m'}, \tag{67}$$

$$C_m^{m'} = -m\,f_m^{m'}, \tag{68}$$

$$D_m^{m'} = -(\alpha_f\,\alpha_p + r\,\alpha_p')\,a_m^{m'} - (q\,r\,\alpha_g' + r^{\,2}\,\alpha_g^{\,2})\,\delta_m^{m'} + m\,m'\,b_m^{m'}, \tag{69}$$

and

$$a_m^{m'}(r) = \oint R^{\,2}\,\exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{70}$$

$$b_m^{m'}(r) = \oint |\nabla r|^{-2}\, R^{-2}\, \exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{71}$$

$$c_m^{m'}(r) = \oint |\nabla r|^{-2}\, \exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}, \tag{72}$$

$$f_m^{m'}(r) = \oint \frac{\mathrm{i}\,r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\, \exp[-\mathrm{i}\,(m-m')\,\theta]\,\frac{d\theta}{2\pi}. \tag{73}$$

C. Properties of Axisymmetric Ideal-MHD O.D.E.s

Note that $a_{m'}^{m} = a_m^{m'\,*}$, $b_{m'}^{m} = b_m^{m'\,*}$, $c_{m'}^{m} = c_m^{m'\,*}$, and $f_{m'}^{m} = -f_m^{m'\,*}$, which implies that

$$A_{m'}^{m} = A_{m'}^{m\,*}, \tag{74}$$

$$B_{m'}^{m} = -C_{m'}^{m\,*}, \tag{75}$$

$$C_{m'}^{m} = -B_{m'}^{m\,*}, \tag{76}$$

$$D_{m'}^{m} = D_{m'}^{m\,*}. \tag{77}$$

It follows from Eqs. (64), (65), and (74)–(77) that

$$r\,\frac{d}{dr}\left[\sum_m (z_m\,y_m^{\,*} - y_m\,z_m^{\,*})\right] = 0. \tag{78}$$

D. Perturbed Electric Field

Let $\mathbf{e}$ be the perturbed electric field. It follows that

$$\nabla\times\mathbf{e} = \mathrm{i}\,\omega\,\mathbf{b}. \tag{79}$$

Hence,

$$e_\phi = \mathrm{i}\,\omega\,y, \tag{80}$$

and

$$\frac{\partial e_\theta}{\partial r} - \frac{\partial e_r}{\partial \theta} = -\mathrm{i}\,\omega\,r\,\alpha_g\,y, \tag{81}$$

where use has been made of Eqs. (25), (27), and (54). We also expect

$$\nabla\cdot\mathbf{e} = 0, \tag{82}$$

which implies that

$$r\,R^2\,e^r = \mathrm{i}\,\omega\,\frac{\partial u}{\partial\theta}, \tag{83}$$

$$r\,R^2\,e^\theta = -\mathrm{i}\,\omega\,\frac{\partial u}{\partial r}, \tag{84}$$

where $u = u(r,\theta)$. Let

$$v = |\nabla r|^2\,r\,\frac{\partial u}{\partial r} + r\,\nabla r\cdot\nabla\theta\,\frac{\partial u}{\partial\theta}. \tag{85}$$

It follows that

$$e_r = \mathrm{i}\,\omega\left[\frac{1}{r\,|\nabla r|^2\,R^2}\,\frac{\partial u}{\partial\theta} + \frac{\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,v\right], \tag{86}$$

$$e_\theta = -\mathrm{i}\,\omega\,v. \tag{87}$$

Hence, Eq. (81) gives

$$r^2\,\alpha_g\,y = r\,\frac{\partial v}{\partial r} + \frac{\partial}{\partial\theta}\left[\frac{1}{|\nabla r|^2\,R^2}\,\frac{\partial u}{\partial\theta} + \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,v\right]. \tag{88}$$

Finally, in the vacuum region, in which $\alpha_g = 0$, we have $u = v = 0$.

E. Electromagnetic Energy Flux

The net electromagnetic energy flux out of the plasma is

$$\begin{aligned}\mathcal{E} &= \left[\oint\oint(\mathbf{e}\times\mathbf{b})\cdot\nabla r\,\mathcal{J}\,d\theta\,d\phi\right]_{r=a} = \left[\oint\oint(e_\theta\,b_\phi - e_\phi\,b_\theta)\,d\theta\,d\phi\right]_{r=a}\\ &= \mathrm{i}\,2\,\pi\,\omega\left(\oint\left[\alpha_g\,(v\,y^* - v^*\,y) + (y\,z^* - y^*\,z)\right]\right)_{r=a},\\ &= \mathrm{i}\,\pi^2\,\omega\sum_m (z_m^*\,y_m - y_m^*\,z_m)_{r=a}.\end{aligned} \tag{89}$$

Here, use has been made of the fact that $v = 0$ for $r \geq a$.

F. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque exerted on the plasma lying within the magnetic flux-surface whose label is r is

$$T_\phi(r) = \oint\oint r\,R^2\,b_\phi\,b^r\,d\theta\,d\phi. \tag{90}$$

It follows from Eqs. (25) and (43) that

$$T_\phi(r) = -\pi\,\alpha_g \oint \left(y^\ast\,\frac{\partial y}{\partial\theta} + y\,\frac{\partial y^\ast}{\partial\theta}\right) d\theta = -\pi\,\alpha_g \oint \frac{\partial |y|^2}{\partial\theta}\,d\theta = 0. \tag{91}$$

We conclude that an axisymmetric perturbation is incapable of exerting a net toroidal electromagnetic torque on the plasma.

G. Perturbed Plasma Potential Energy

The perturbed plasma potential energy in the region of the plasma lying within the magnetic flux-surface whose label is r is

$$\delta W_p = \frac{1}{2}\oint\oint r\,R^2\,\xi^{\,r\ast}(-\mathbf{B}\cdot\mathbf{b} + \xi^{\,r}\,P')\,d\theta\,d\phi. \tag{92}$$

However,

$$\mathbf{B}\cdot\mathbf{b} - \xi^{\,r}\,P' = B^{\,\theta}\,b_\theta + B^{\,\phi}\,b_\phi - \xi^{\,r}\,P' = -\frac{f}{r\,R^2}\left(z + q\,\alpha_g\,y + \alpha_p\,R^2\right), \tag{93}$$

where use has been made of Eqs. (4)–(7), (26), (43), (53), and (58). Hence, we obtain

$$\delta W_p(r) = \frac{1}{2}\oint\oint y^\ast\left[z + (q\,\alpha_g + \alpha_p\,R^2)\,y\right] d\theta\,d\phi = \pi^2\sum_m y_m^\ast\,\chi_m, \tag{94}$$

where

$$\chi_m(r) = z_m + q\,\alpha_g\,y_m + \alpha_p\sum_{m'} a_m^{m'}\,y_{m'}. \tag{95}$$

III. INVERSE ASPECT-RATIO EXPANDED TOKAMAK EQUILIBRIUM

A. Equilibrium Magnetic Flux-Surfaces

Let us assume that the inverse aspect-ratio of the plasma, ϵ, is such that $0 < \epsilon \ll 1$. Let $r = \epsilon\,\hat{r}$, $\nabla = \epsilon^{-1}\,\hat{\nabla}$, and $' \to \epsilon^{-1}\,'$. Suppose that the loci of the equilibrium magnetic flux-surfaces can be written in the parametric form:

$$R(\hat{r},\omega) = 1 - \epsilon\,\hat{r}\,\cos\omega + \epsilon^2\sum_{j>0} H_j(\hat{r})\,\cos[(j-1)\,\omega] + \epsilon^2\sum_{j>1} V_j(\hat{r})\,\sin[(j-1)\,\omega]$$

$$+ \epsilon^3 L(\hat{r}) \cos\omega, \tag{96}$$

$$Z(\hat{r},\omega) = \epsilon\,\hat{r}\,\sin\omega + \epsilon^2 \sum_{j>1} H_j(\hat{r})\,\sin[(j-1)\,\omega] - \epsilon^2 \sum_{j>1} V_j(\hat{r})\,\cos[(j-1)\,\omega]$$

$$- \epsilon^3 L(\hat{r}) \sin\omega, \tag{97}$$

where j is a positive integer. Here, $H_1(\hat{r})$ controls the relative horizontal locations of the flux-surface centroids, $H_2(\hat{r})$ and $V_2(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface ellipticities, $H_3(\hat{r})$ and $V_3(\hat{r})$ control the magnitudes and vertical tilts of the flux-surface triangularities, et cetera, whereas $L(\hat{r})$ is a flux-surface re-labelling parameter. Moreover, $\omega(R,Z)$ is a poloidal angle that is distinct from θ. Note that V_1 does not appear in Eq. (97) because such a factor merely gives rise to a rigid vertical shift of the plasma that can be eliminated by a suitable choice of the origin of the flux-coordinate system.

Let

$$J(\hat{r},\omega) = \frac{1}{\epsilon^2}\left(\frac{\partial R}{\partial\omega}\,\frac{\partial Z}{\partial\hat{r}} - \frac{\partial R}{\partial\hat{r}}\,\frac{\partial Z}{\partial\omega}\right) \tag{98}$$

be the Jacobian of the $\hat{r}$, ω coordinate system. We can transform to the $\hat{r}$, θ coordinate system by writing

$$\theta(\hat{r},\omega) = 2\pi \int_0^\omega \frac{J(\hat{r},\tilde{\omega})}{R(\hat{r},\tilde{\omega})}\,d\tilde{\omega} \bigg/ \oint \frac{J(\hat{r},\omega)}{R(\hat{r},\omega)}\,d\omega, \tag{99}$$

$$\hat{r} = \frac{1}{2\pi}\oint \frac{J(\hat{r},\omega)}{R(\hat{r},\omega)}\,d\omega. \tag{100}$$

This transformation ensures that

$$\frac{\partial\theta}{\partial\omega} = \frac{J}{\hat{r}\,R}, \tag{101}$$

and, hence, that

$$\mathcal{J} \equiv \frac{R}{\epsilon}\left(\frac{\partial R}{\partial\theta}\,\frac{\partial Z}{\partial\hat{r}} - \frac{\partial R}{\partial\hat{r}}\,\frac{\partial Z}{\partial\hat{r}}\right) = \epsilon\,R\,J\,\frac{\partial\omega}{\partial\theta} = r\,R^2, \tag{102}$$

in accordance with Eq. (2).

B. Metric Elements

We can determine the metric elements of the flux-coordinate system by combining Eqs. (96)–(100). Evaluating the elements up to $\mathcal{O}(\epsilon)$, but retaining $\mathcal{O}(\epsilon^2)$ contributions to terms that are independent of ω, we obtain,

$$L(\hat{r}) = \frac{\hat{r}^{\,3}}{8} - \frac{\hat{r}\,H_1}{2} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{H_j^{\,2}}{\hat{r}} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{V_j^{\,2}}{\hat{r}}, \tag{103}$$

$$\theta = \omega + \epsilon\,\hat{r}\,\sin\omega - \epsilon\sum_{j>0}\frac{1}{j}\left[H_j' - (j-1)\,\frac{H_j}{\hat{r}}\right]\sin(j\,\omega)$$
$$+\,\epsilon\sum_{j>1}\frac{1}{j}\left[V_j' - (j-1)\,\frac{V_j}{\hat{r}}\right]\cos(j\,\omega), \tag{104}$$

$$|\hat{\nabla}\hat{r}|^2 = 1 + 2\,\epsilon\sum_{j>0}H_j'\,\cos(j\,\theta) + 2\,\epsilon\sum_{j>1}V_j'\,\sin(j\,\theta)$$
$$+\,\epsilon^{\,2}\left(\frac{3\,\hat{r}^{\,2}}{4} - H_1 + \frac{1}{2}\sum_{j>0}\left[H_j'^{\,2} + (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right]\right.$$
$$\left. +\frac{1}{2}\sum_{j>1}\left[V_j'^{\,2} + (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right]\right), \tag{105}$$

$$\hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta = \epsilon\,\sin\theta - \epsilon\sum_{j>0}\frac{1}{j}\left[H_j'' + \frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}}\right]\sin(j\,\theta)$$
$$+\,\epsilon\sum_{j>1}\frac{1}{j}\left[V_j'' + \frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}}\right]\cos(j\,\theta), \tag{106}$$

$$R^{\,2} = 1 - 2\,\epsilon\,\hat{r}\,\cos\theta - \epsilon^{\,2}\left(\frac{\hat{r}^{\,2}}{2} - \hat{r}\,H_1' - 2\,H_1\right). \tag{107}$$

Here, $' \equiv d/d\hat{r}$. Moreover, we have made use of the fact that $V_j \propto H_j$, for $j>1$, because V_j and H_j satisfy the identical differential equations, (113) and (114).

C. Expansion of Grad-Shafranov Equation

Let us write

$$f(\hat{r}) = \epsilon\,\frac{\hat{r}\,g}{q}, \tag{108}$$

$$g(\hat{r}) = 1 + \epsilon^{\,2}\,g_2(\hat{r}) + \epsilon^{\,4}\,g_4(\hat{r}), \tag{109}$$

$$P'(\hat{r}) = \epsilon^{\,2}\,p_2'(\hat{r}), \tag{110}$$

where q, g_2, g_4, and p_2 are all $\mathcal{O}(1)$. Here, the safety-factor, $q(\hat{r})$, and the second-order plasma pressure gradient, $p_2'(\hat{r})$, are the two free flux-surface functions that characterize the plasma equilibrium.

Expanding the Grad-Shafranov equation, (16), order by order in the small parameter ϵ, making use of Eqs. (105)–(110), we obtain

$$g_2' = -p_2' - \frac{\hat{r}}{q^2}\,(2-s), \tag{111}$$

$$H_1'' = -(3-2\,s)\,\frac{H_1'}{\hat{r}} - 1 + \frac{2\,p_2'\,q^2}{\hat{r}}, \tag{112}$$

$$H_j'' = -(3-2\,s)\,\frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}} \qquad \text{for } j>1, \tag{113}$$

$$V_j'' = -(3-2\,s)\,\frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}} \qquad \text{for } j>1, \tag{114}$$

$$g_4' = g_2\left[p_2' - \frac{\hat{r}}{q^2}\,(2-s)\right] - \frac{\hat{r}}{q}\,\Sigma + p_2'\left(\frac{\hat{r}^{\,2}}{2} + \frac{\hat{r}^{\,2}}{q^2} - 2\,H_1 - 3\,\hat{r}\,H_1'\right), \tag{115}$$

where $s = \hat{r}\,q'/q$,

$$\Sigma = \frac{1}{q}\left(\frac{3\,\hat{r}^{\,2}}{2} - 2\,\hat{r}\,H_1' + S_2\right) - \frac{2-s}{q}\left(-\frac{3\,\hat{r}^{\,2}}{4} + \frac{\hat{r}^{\,2}}{q^2} + H_1 + S_1\right), \tag{116}$$

$$S_1(\hat{r}) = \frac{1}{2}\sum_{j>0}\left[3\,H_j'^{\,2} - (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^2}\right] + \frac{1}{2}\sum_{j>1}\left[3\,V_j'^{\,2} - (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^2}\right], \tag{117}$$

$$\begin{aligned} S_2(\hat{r}) = {}& \sum_{j>0}\left[H_j'^{\,2} + 2\,(j^2-1)\,\frac{H_j'\,H_j}{\hat{r}} - (j^2-1)\,\frac{H_j^{\,2}}{\hat{r}^{\,2}}\right] \\ & + \sum_{j>1}\left[V_j'^{\,2} + 2\,(j^2-1)\,\frac{V_j'\,V_j}{\hat{r}} - (j^2-1)\,\frac{V_j^{\,2}}{\hat{r}^{\,2}}\right]. \end{aligned} \tag{118}$$

Note that the relative horizontal shift of magnetic flux-surfaces, H_1, otherwise known as the *Shafranov shift*, is driven by toroidicity [the second term on the right-hand side of Eq. (112)], and plasma pressure gradients (the third term). All of the other shaping terms (i.e., the H_j, for $j>1$, and the V_j) are driven by axisymmetric currents flowing in external magnetic field-coils.

Equations (39), (58), (59), and (108)–(110) yield

$$\alpha_p(\hat{r}) = \frac{p_2'\,q^2}{\hat{r}}\left(1 - 2\,\epsilon^{\,2}\,g_2\right), \tag{119}$$

$$\alpha_g(\hat{r}) = \frac{q}{\hat{r}}\left(g_2' - \epsilon^2\, g_2\, g_2' + \epsilon^2\, g_4'\right), \tag{120}$$

$$\alpha_f(\hat{r}) = -s + \epsilon^2\,\hat{r}\, g_2'. \tag{121}$$

Finally, it follows from Eqs. (106) and (112)–(114) that

$$\begin{aligned}\hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta = {} & 2\,\epsilon\left[1 - \frac{p_2'\,q^2}{\hat{r}} + (1-s)\,\frac{H_1'}{\hat{r}}\right]\sin\theta\\ & - 2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}}\right]\sin(j\,\theta)\\ & + 2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}}\right]\cos(j\,\theta).\end{aligned} \tag{122}$$

D. Calculation of Coupling Coefficients

Equations (105) and (117) yield

$$|\hat{\nabla}\hat{r}|^{-2} = 1 - 2\,\epsilon\sum_{j>0} H_j'\,\cos(j\,\theta) - 2\,\epsilon\sum_{j>0} V_j'\sin(j\,\theta) + \epsilon^2\left(-\frac{3\,\hat{r}^{\,2}}{4} + H_1 + S_1\right), \tag{123}$$

Equation (107) gives

$$R^{-2} = 1 + 2\,\epsilon\,\hat{r}\,\cos\theta + \epsilon^2\left(\frac{5\,\hat{r}^{\,2}}{2} - \hat{r}\,H_1' - 2\,H_1\right). \tag{124}$$

The previous two equations imply that

$$\begin{aligned}|\hat{\nabla}\hat{r}|^{-2}\,R^{-2} = {} & 1 + 2\,\epsilon\,\hat{r}\,\cos\theta - 2\,\epsilon\sum_{j>0} H_j'\,\cos(j\,\theta) - 2\,\epsilon\sum_{>1} V_j'\,\sin(j\,\theta)\\ & + \epsilon^2\left(\frac{7\,\hat{r}^{\,2}}{4} - H_1 - 3\,\hat{r}\,H_1' + S_1\right).\end{aligned} \tag{125}$$

Finally, Eqs. (122) and (123) give

$$\begin{aligned}\hat{\nabla}\hat{r}\cdot\hat{\nabla}\theta\,|\hat{\nabla}\hat{r}|^{-2} = {} & 2\,\epsilon\left[1 - \frac{p_2'\,q^2}{\hat{r}} + (1-s)\,\frac{H_1'}{\hat{r}}\right]\sin\theta\\ & - 2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{H_j'}{\hat{r}} + (j^2-1)\,\frac{H_j}{\hat{r}^{\,2}}\right]\sin(j\,\theta)\\ & + 2\,\epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,\frac{V_j'}{\hat{r}} + (j^2-1)\,\frac{V_j}{\hat{r}^{\,2}}\right]\cos(j\,\theta),\end{aligned} \tag{126}$$

where use has been made of the fact that $V_j' \propto H_j'$ for $j > 1$.

Equations (70)–(73), (107), (123), (125), and (126) imply that

$$a_m^{m'} = \delta_m^{m'} - \epsilon\,\hat{r}\,(\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon^2\left(\frac{\hat{r}^{\,2}}{2} - \hat{r}\,H_1' - 2\,H_1\right)\delta_m^{m'}, \tag{127}$$

$$b_m^{m'} = \delta_m^{m'} + \epsilon\,\hat{r}\,(\delta_{m'-m-1} + \delta_{m'-m+1}) - \epsilon\sum_{j>0} H_j'\,(\delta_{m'-m-j} + \delta_{m'-m+j})$$
$$- \epsilon\sum_{j>1} \mathrm{i}\,V_j'\,(\delta_{m'-m-j} - \delta_{m'-m+j}) + \epsilon^2\left(\frac{7\,\hat{r}^{\,2}}{4} - H_1 - 3\,\hat{r}\,H_1' + S_1\right)\delta_m^{m'}, \tag{128}$$

$$c_m^{m'} = \delta_m^{m'} - \epsilon\sum_{j>0} H_j'\,(\delta_{m'-m-j} + \delta_{m'-m+j}) - \epsilon\sum_{j>1} \mathrm{i}\,V_j'\,(\delta_{m'-m-j} - \delta_{m'-m+j})$$
$$+ \epsilon^2\left(-\frac{3\,\hat{r}^{\,2}}{4} + H_1 + S_1\right)\delta_m^{m'}, \tag{129}$$

$$f_m^{m'} = -\epsilon\left[\hat{r} - p_2'\,q^2 + (1-s)\,H_1'\right](\delta_{m'-m-1} - \delta_{m'-m+1})$$
$$+ \epsilon\sum_{j>1}\frac{1}{j}\left[-(1-s)\,H_j' + (j^2-1)\,\frac{H_j}{\hat{r}}\right](\delta_{m'-m-j} - \delta_{m'-m+j})$$
$$+ \epsilon\sum_{j>1}\frac{\mathrm{i}}{j}\left[-(1-s)\,V_j' + (j^2-1)\,\frac{V_j}{\hat{r}}\right](\delta_{m'-m-j} + \delta_{m'-m+j}). \tag{130}$$

If we write

$$\alpha_g = \alpha_g^{(0)} + \epsilon^2\,\alpha_g^{(2)}, \tag{131}$$

$$\alpha_p = \alpha_p^{(0)} + \epsilon^2\,\alpha_p^{(2)}, \tag{132}$$

$$\alpha_f = \alpha_f^{(0)} + \epsilon^2\,\alpha_f^{(2)}, \tag{133}$$

$$a_m^{m'} = 1 + \epsilon\,a_m^{m'\,(1)} + \epsilon^2\,a_m^{m'\,(2)}, \tag{134}$$

$$b_m^{m'} = 1 + \epsilon\,b_m^{m'\,(1)} + \epsilon^2\,b_m^{m'\,(2)}, \tag{135}$$

$$D_m^{m'} = D_m^{m'\,(0)} + \epsilon\,D_m^{m'\,(1)} + \epsilon^2\,D_m^{m'\,(2)}, \tag{136}$$

where $\alpha_g^{(0)}$, $\alpha_g^{(2)}$, et cetera, are $\mathcal{O}(\infty)$, then it follows from Eq. (69) that

$$D_m^{m\,(0)} = -\alpha_f^{(0)}\,\alpha_p^{(0)} - \hat{r}\,\alpha_p'^{(0)} - q\,\hat{r}\,\alpha_g'^{(0)} + m^2, \tag{137}$$

$$D_m^{m'\,(1)} = -\left[\alpha_f^{(0)}\,\alpha_p^{(0)} + \hat{r}\,\alpha_p'^{(0)}\right]a_m^{m'\,(1)} + m\,m'\,b_m^{m'\,(1)}, \tag{138}$$

$$
\begin{aligned}
D_m^{m\,(2)} = &- \left[\alpha_f^{(0)}\,\alpha_p^{(0)} + \hat{r}\,\alpha_p'^{(0)}\right] a_m^{m'\,(2)} - \alpha_f^{(0)}\,\alpha_p^{(2)} - \alpha_f^{(2)}\,\alpha_p^{(0)} - \hat{r}\,\alpha_p'^{(2)} - q\,\hat{r}\,\alpha_g'^{(2)}\\
&- \hat{r}^{\,2}\left[\alpha_g^{(0)}\right]^2 + m^2\,b_m^{m\,(2)}.
\end{aligned}
\tag{139}
$$

Finally, Eqs. (70)–(73), (127)–(130), and (137)–(139) give

$$
A_m^m(\hat{r}) = 1 + \epsilon^2\left(-\frac{3\,\hat{r}^{\,2}}{4} + H_1 + S_1\right), \tag{140}
$$

$$
A_m^{m\pm 1}(\hat{r}) = -\epsilon\,H_1', \tag{141}
$$

$$
A_m^{m\pm j}(\hat{r}) = -\epsilon\,(H_j' \pm \mathrm{i}\,V_j') \qquad \text{for } j>1, \tag{142}
$$

$$
B_m^m(\hat{r}) = 0, \tag{143}
$$

$$
B_m^{m\pm 1}(\hat{r}) = \pm\epsilon\,(m\pm 1)\left[\hat{r} - p_2'\,q^2 + (1-s)\,H_1'\right], \tag{144}
$$

$$
B_m^{m\pm j}(\hat{r}) = \pm\epsilon\,\frac{m\pm j}{j}\left[(1-s)\,(H_j' \pm \mathrm{i}\,V_j') - (j^2-1)\,\frac{H_j \pm \mathrm{i}\,V_j}{\hat{r}}\right] \qquad \text{for } j>1, \tag{145}
$$

$$
C_m^m(\hat{r}) = 0, \tag{146}
$$

$$
C_m^{m\pm 1}(\hat{r}) = \pm\epsilon\,m\left[\hat{r} - p_2'\,q^2 + (1-s)\,H_1'\right], \tag{147}
$$

$$
C_m^{m\pm j}(\hat{r}) = \pm\epsilon\,\frac{m}{j}\left[(1-s)\,(H_j' \pm \mathrm{i}\,V_j') - (j^2-1)\,\frac{H_j \pm \mathrm{i}\,V_j}{\hat{r}}\right] \qquad \text{for } j>1, \tag{148}
$$

$$
\begin{aligned}
D_m^m(\hat{r}) = &\, m^2 + q\,\hat{r}\,\frac{d}{d\hat{r}}\left(\frac{2-s}{q}\right) + \epsilon^2\,m^2\left(\frac{7\,\hat{r}^{\,2}}{4} - H_1 - 3\,\hat{r}\,H_1' + S_1\right)\\
&+ \epsilon^2\left\{-\hat{r}^{\,2}\,\frac{(2-s)^2}{q^2} + q\,\hat{r}\,\frac{d\Sigma}{d\hat{r}} - \hat{r}\,\frac{d}{d\hat{r}}(\hat{r}\,p_2') - 2\,(1-s)\,\hat{r}\,p_2'\right.\\
&\left. +2\,\hat{r}\,p_2'\,q^2\left(-2 + \frac{3\,p_2'\,q^2}{\hat{r}}\right) + 2\,H_1'\,q^2\left[\frac{d}{d\hat{r}}(\hat{r}\,p_2') - 4\,(1-s)\,p_2'\right]\right\},
\end{aligned}
\tag{149}
$$

$$
D_m^{m\pm 1}(\hat{r}) = \epsilon\left[\frac{d}{d\hat{r}}(\hat{r}\,p_2') - (2-s)\,p_2'\right] q^2 + \epsilon\,m\,(m\pm 1)\,(\hat{r} - H_1'), \tag{150}
$$

$$
D_m^{m\pm j}(\hat{r}) = -\epsilon\,m\,(m\pm j)\,(H_j' \pm \mathrm{i}\,V_j') \qquad \text{for } j>1. \tag{151}
$$

E. Behavior Close to Magnetic Axis

When $\hat{r} \ll 1$, the well-behaved solution of the axisymmetric ideal-MHD o.d.e.s, (64) and (65), that is dominated by the poloidal harmonic whose poloidal mode number is m is such that

$$y_m(\hat{r}) = \hat{r}^{\,|m|}, \tag{152}$$

$$z_m(\hat{r}) = |m|\,\hat{r}^{\,|m|}, \tag{153}$$

with $y_{m'}(\hat{r}) = z_{m'}(\hat{r}) = 0$ for $m' \neq 0$.