

# Pressure Flattening at Rational Surface

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## I. FUNDAMENTAL EQUATION

Consider a rational surface whose resonant poloidal mode number is  $m$ , and that is located at the magnetic flux-surface  $r = r_s$ . Let  $x = r - r_s$ . The resonant harmonics of the perturbed magnetic field satisfy

$$x \frac{d\psi_m}{dx} = L_0 Z_m, \quad (1)$$

$$x \frac{dZ_m}{dx} = P_0 \psi_m + Z_m. \quad (2)$$

The previous equations can be combined to give

$$\frac{d^2\psi_m}{dx^2} = \frac{\nu(\nu+1)}{x^2} \psi_m, \quad (3)$$

where

$$\nu(\nu+1) = -D_I - \frac{1}{4} = L_0 P_0 = \left[ \frac{2r \mu_0 p' (1-q^2)}{B_0^2 s^2} \right]_{r_s}. \quad (4)$$

Let

$$\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} + L_0 P_0} = -\frac{1}{2} + \sqrt{-D_I} \geq -\frac{1}{2}. \quad (5)$$

It follows that  $\nu_L = -\nu$  and  $\nu_S = 1 + \nu$ . In the limit that  $|L_0 P_0| \ll 1$ ,

$$\nu \simeq L_0 P_0. \quad (6)$$

The most general tearing parity solution of Eq. (3) is

$$\psi(x) = A_L |x|^{-\nu} + A_S \operatorname{sgn}(x) |x|^{1+\nu}. \quad (7)$$

Making the definitions

$$\Psi = r_s^{-\nu} \left( \frac{1 + 2\nu}{L_m^m} \right)^{1/2} A_L, \quad (8)$$

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$$\Delta\Psi = r_s^{1+\nu} \left( \frac{1+2\nu}{L_m^m} \right)^{1/2} 2 A_S, \quad (9)$$

$$\Delta_{\text{out}} = \frac{\Delta\Psi}{\Psi}, \quad (10)$$

it is clear that

$$\Delta_{\text{out}} = r_s^{1+2\nu} \frac{2 A_S}{A_L}. \quad (11)$$

## II. PRESSURE FLATTENING

Suppose that the pressure gradient is locally flattened in the vicinity of the rational surface in such a manner that

$$p'(x) = p'_{\text{out}} \frac{x^2}{x^2 + \delta^2}. \quad (12)$$

Here, it is assumed that  $\delta \ll r_s$ . Equation (3) becomes

$$\frac{d^2\psi_m}{dx^2} = \frac{\nu(\nu+1)}{x^2 + \delta^2} \psi_m, \quad (13)$$

where

$$\nu(\nu+1) = \left[ \frac{2 r \mu_0 p'_{\text{out}} (1 - q^2)}{B_0^2 s^2} \right]_{r_s}. \quad (14)$$

Let  $X = x/\delta$ . Equation (13) becomes

$$(1 + X^2) \frac{d^2\psi_m}{dX^2} = \nu_k (1 + \nu_k) \psi_m \quad (15)$$

Consider the small- $|X|$  behavior of the solution to Eq. (15). If we write

$$\psi_m(X) = \sum_{m=0,2,4} a_m X^{\mu+m} \quad (16)$$

then we obtain

$$a_0 \mu(\mu-1) = 0, \quad (17)$$

$$a_2 = \frac{\nu(\nu+1) - \mu(\mu-1)}{(\mu+2)(\mu+1)} a_0. \quad (18)$$

The solutions are  $\mu = 0$  with

$$a_2 = \frac{\nu(1+\nu)}{2} a_0 \quad (19)$$

and  $\mu = 1$  with

$$a_2 = \frac{\nu(1+\nu)}{6} a_0. \quad (20)$$

It follows that the most general tearing parity small- $|X|$  solution takes the form

$$\psi_m(X) \simeq \hat{A}_{L\text{in}} \left[ 1 + \frac{\nu(1+\nu)}{2} X^2 \right] + \hat{A}_{S\text{in}} |X| \left[ 1 + \frac{\nu(1+\nu)}{6} X^2 \right]. \quad (21)$$

We can define

$$\Delta_{\text{in}} = \left( \frac{r_s}{\delta} \right) \frac{2 \hat{A}_{S\text{in}}}{\hat{A}_{L\text{in}}}. \quad (22)$$

Consider the large- $|X|$  behavior of the solution to Eq. (15). Let  $Y = 1/X$ . Equation (15) transforms into

$$(1 + Y^2) \frac{d}{dY} \left( Y^2 \frac{d\psi_m}{dY} \right) = \nu(\nu + 1) \psi_m. \quad (23)$$

If we write

$$\psi_m(Y) = \sum_{m=0,2,4} a_m Y^{\mu+m} \quad (24)$$

then we obtain

$$a_0 [\mu(\mu + 1) - \nu(\nu + 1)] = 0, \quad (25)$$

$$a_2 = - \frac{\mu(\mu + 1)}{(\mu + 2)(\mu + 3) - \nu(1 + \nu)} a_0. \quad (26)$$

The solutions are  $\mu = \nu$  with

$$a_2 = - \frac{\nu(1 + \nu)}{2(3 + 2\nu)} a_0 \quad (27)$$

and  $\mu = -1 - \nu$  with

$$a_2 = - \frac{\nu(1 + \nu)}{2(1 - 2\nu)} a_0. \quad (28)$$

It follows that the most general tearing parity large- $|X|$  solution takes the form

$$\begin{aligned} \psi_m(X) = & \hat{A}_{L\text{out}} |X|^{-\nu} \left[ 1 - \frac{\nu(1+\nu)}{2(3+2\nu)} \frac{1}{X^2} \right] \\ & + \hat{A}_{S\text{out}} \text{sgn}(X) |X|^{1+\nu} \left[ 1 - \frac{\nu(1+\nu)}{2(1-2\nu)} \frac{1}{X^2} \right] \end{aligned} \quad (29)$$

By analogy with Eq. (11), we can write

$$\Delta_{\text{out}} = \left( \frac{r_s}{\delta} \right)^{1+2\nu} \frac{2 \hat{A}_{S\text{out}}}{\hat{A}_{L\text{out}}}. \quad (30)$$

### III. CONNECTION FORMULAE

Suppose that we launch the ‘large’ solution

$$\psi_L(X) = X^{-\nu} \left[ 1 - \frac{\nu(1+\nu)}{2(3+2\nu)} \frac{1}{X^2} \right] \quad (31)$$

from large  $X$  and integrate to  $X = 0$ . Suppose that

$$\psi_L(0) = a_{LL}, \quad (32)$$

$$\frac{d\psi_L(0)}{dx} = a_{SL}. \quad (33)$$

Suppose that we launch the ‘small’ solution

$$\psi_S(X) = X^{1+\nu} \left[ 1 - \frac{\nu(1+\nu)}{2(1-2\nu)} \frac{1}{X^2} \right] \quad (34)$$

from large  $X$  and integrate to  $X = 0$ . Suppose that

$$\psi_L(0) = a_{LS}, \quad (35)$$

$$\frac{d\psi_L(0)}{dx} = a_{SS}. \quad (36)$$

The most general solution is

$$\psi_m(X) = \hat{A}_{L\text{out}} \psi_L(X) + \hat{A}_{S\text{out}} \psi_S(X). \quad (37)$$

It follows that

$$\hat{A}_{L\text{in}} = a_{LL} \hat{A}_{L\text{out}} + a_{LS} \hat{A}_{S\text{out}}, \quad (38)$$

$$\hat{A}_{S\text{in}} = a_{SL} \hat{A}_{L\text{out}} + a_{SS} \hat{A}_{S\text{out}}. \quad (39)$$

It follows from Eqs. (11) and (22) that

$$\left( \frac{\delta}{r_s} \right) \frac{\Delta_{\text{in}}}{2} = \frac{a_{SL} + a_{SS} (\delta/r_s)^{1+2\nu} (\Delta_{\text{out}}/2)}{a_{LL} + a_{LS} (\delta/r_s)^{1+2\nu} (\Delta_{\text{out}}/2)}. \quad (40)$$

#### IV. ANALYTIC SOLUTION

Let  $\psi_m = (1 + X^2) \phi$ . Equation (15) transforms to give

$$(1 + X^2) \frac{d^2 \phi}{dX^2} + 4X \frac{d\phi}{dX} + [2 - \nu(\nu + 1)] \phi = 0. \quad (41)$$

Let  $z = iX$ . The previous equation becomes

$$(1 - z^2) \frac{d^2 \phi}{dz^2} - 4z \frac{d\phi}{dz} - [2 - \nu(\nu + 1)] \phi = 0. \quad (42)$$

Now,  $Q_\nu(z)$  and  $Q_{-1-\nu}(z)$  satisfy

$$(1 - z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + \nu(\nu + 1) w = 0. \quad (43)$$

Let  $w' = dw/dz$ . Differentiation of the previous equation yields

$$(1 - z^2) \frac{d^2 w'}{dz^2} - 2z \frac{dw'}{dz} - [2 - \nu(\nu + 1)] w' = 0. \quad (44)$$

It is clear from a comparison of Eqs. (42) and (44) that the two independent solutions of Eq. (15) can be written  $(1 + X^2) Q'_\nu(iX)$  and  $(1 + X^2) Q'_{-1-\nu}(iX)$ , where ' denotes differentiation with respect to argument.

Now [Erdelyi, p. 134, (40)],

$$\begin{aligned} Q_\nu(z) = & \frac{\pi^{1/2} \Gamma(1/2 + \nu/2) e^{-i(\pi/2)(1+\nu)}}{2\Gamma(1 + \nu/2)} F\left(-\frac{\nu}{2}, \frac{1}{2} + \frac{\nu}{2}; \frac{1}{2}; z^2\right) \\ & + \frac{\pi^{1/2} \Gamma(1 + \nu/2) e^{-i(\pi/2)\nu}}{\Gamma(1/2 + \nu/2)} z F\left(\frac{1}{2} - \frac{\nu}{2}, 1 + \frac{\nu}{2}; \frac{3}{2}; z^2\right), \end{aligned} \quad (45)$$

which yields

$$\begin{aligned} Q_\nu(z) \simeq & \frac{\pi^{1/2} \Gamma(1/2 + \nu/2) e^{-i(\pi/2)(1+\nu)}}{2\Gamma(1 + \nu/2)} \left[1 - \frac{\nu(1 + \nu)z^2}{2}\right] \\ & + \frac{\pi^{1/2} \Gamma(1 + \nu/2) e^{-i(\pi/2)\nu}}{\Gamma(1/2 + \nu/2)} z, \end{aligned} \quad (46)$$

and

$$\begin{aligned} Q'_\nu(z) \simeq & -\frac{\pi^{1/2} \nu(1 + \nu) \Gamma(1/2 + \nu/2) e^{-i(\pi/2)(1+\nu)}}{2\Gamma(1 + \nu/2)} z \\ & + \frac{\pi^{1/2} \Gamma(1 + \nu/2) e^{-i(\pi/2)\nu}}{\Gamma(1/2 + \nu/2)}. \end{aligned} \quad (47)$$

So at small- $X$ ,

$$\begin{aligned} (1 + X^2) Q'_\nu(iX) = & e^{-i\nu\pi/2} \left[ \frac{\pi^{1/2} \Gamma(1 + \nu/2)}{\Gamma(1/2 + \nu/2)} \right. \\ & \left. - \frac{\pi^{1/2} \nu(1 + \nu) \Gamma(1/2 + \nu/2)}{2\Gamma(1 + \nu/2)} X \right] \end{aligned} \quad (48)$$

Furthermore, [Erdelyi, p. 134, (41)],

$$Q_\nu(z) = \frac{\pi^{1/2} \Gamma(1 + \nu)}{2^{1+\nu} \Gamma(3/2 + \nu)} z^{-1-\nu} F\left(1 + \frac{\nu}{2}, \frac{1}{2} + \frac{\nu}{2}; \frac{3}{2} + \nu; \frac{1}{z^2}\right), \quad (49)$$

which yields

$$Q'_\nu(z) \simeq -\frac{\pi^{1/2} (1 + \nu) \Gamma(1 + \nu)}{2^{1+\nu} \Gamma(3/2 + \nu)} z^{-2-\nu} \quad (50)$$

So, at large- $X$ ,

$$(1 + X^2) Q'_\nu(iX) \simeq e^{-i\nu\pi/2} \frac{\pi^{1/2} (1 + \nu) \Gamma(1 + \nu)}{2^{1+\nu} \Gamma(3/2 + \nu)} X^{-\nu}. \quad (51)$$

It is clear from a comparison of Eqs. (21), (29), (38), (39), (48), and (51) that

$$\hat{A}_{L\text{in}} = a_{LL} \hat{A}_{L\text{out}}, \quad (52)$$

$$\hat{A}_{S\text{in}} = a_{SL} \hat{A}_{L\text{out}}, \quad (53)$$

where

$$a_{LL} = \frac{2^{1+\nu} \Gamma(1 + \nu/2) \Gamma(3/2 + \nu)}{(1 + \nu) \Gamma(1/2 + \nu/2) \Gamma(1 + \nu)}, \quad (54)$$

$$a_{SL} = -\frac{2^\nu \nu \Gamma(1/2 + \nu/2) \Gamma(3/2 + \nu)}{\Gamma(1 + \nu/2) \Gamma(1 + \nu)}. \quad (55)$$

In the limit  $\nu \rightarrow 0$ ,

$$a_{LL} \rightarrow 1, \quad (56)$$

$$a_{SL} \rightarrow -\frac{\nu \pi}{2}. \quad (57)$$

Now, according to Eq. (48),

$$(1 + X^2) Q'_{-1-\nu}(iX) = e^{i(1+\nu)\pi/2} \left[ \frac{\pi^{1/2} \Gamma(1/2 - \nu/2)}{\Gamma(-\nu/2)} - \frac{\pi^{1/2} \nu (1 + \nu) \Gamma(-\nu/2)}{2 \Gamma(1/2 - \nu/2)} X \right]. \quad (58)$$

Furthermore, according to Eq. (51),

$$(1 + X^2) Q'_{-1-\nu}(iX) \simeq -e^{i(1+\nu)\pi/2} \frac{\pi^{1/2} \nu \Gamma(-\nu)}{2^{-\nu} \Gamma(1/2 - \nu)} X^{1+\nu}. \quad (59)$$

A comparison of Eqs. (21), (29), (38), (39), (58), and (59) yields

$$\hat{A}_{L\text{in}} = a_{LS} \hat{A}_{S\text{out}}, \quad (60)$$

$$\hat{A}_{S\text{in}} = a_{SS} \hat{A}_{S\text{out}}, \quad (61)$$

where

$$a_{LS} = -\frac{\Gamma(1/2 - \nu/2) \Gamma(1/2 - \nu)}{2^\nu \nu \Gamma(-\nu) \Gamma(-\nu/2)} = -\frac{\nu \Gamma(1/2 - \nu/2) \Gamma(1/2 - \nu)}{2^{1+\nu} \Gamma(1 - \nu) \Gamma(1 - \nu/2)}, \quad (62)$$

$$a_{SS} = \frac{(1+\nu) \Gamma(-\nu/2) \Gamma(1/2-\nu)}{2^{1+\nu} \Gamma(1/2-\nu/2) \Gamma(-\nu)} = \frac{(1+\nu) \Gamma(1-\nu/2) \Gamma(1/2-\nu)}{2^\nu \Gamma(1/2-\nu/2) \Gamma(1-\nu)}. \quad (63)$$

In the limit  $\nu \rightarrow 0$ ,

$$a_{LS} \rightarrow -\frac{\nu \pi}{2}, \quad (64)$$

$$a_{SS} \rightarrow 1. \quad (65)$$