I. HIGHER ORDER EQUILIBRIUM

We have

$$R(\hat{r},\omega) = 1 - \epsilon\,\hat{r}\,\cos\omega + \epsilon^2\sum_{j>0} H_j(\hat{r})\,\cos[(j-1)\,\omega] + \epsilon^2\sum_{j>1} V_j(\hat{r})\,\sin[(j-1)\,\omega]$$

$$+\,\epsilon^3\,[L_3(\hat{r}) + \epsilon\,L_4(\hat{r})]\,\cos\omega, \tag{1}$$

$$Z(\hat{r},\omega) = \epsilon\,\hat{r}\,\sin\omega + \epsilon^2\sum_{j>1} H_j(\hat{r})\,\sin[(j-1)\,\omega] - \epsilon^2\sum_{j>1} V_j(\hat{r})\,\cos[(j-1)\,\omega]$$

$$-\,\epsilon^3\,[L_3(\hat{r}) + \epsilon\,L_4(\hat{r})]\,\sin\omega, \tag{2}$$

where

$$L_3(\hat{r}) = \frac{\hat{r}^{\,3}}{8} - \frac{\hat{r}\,H_1}{2} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{H_j^{\,2}}{\hat{r}} - \frac{1}{2}\sum_{j>1}(j-1)\,\frac{V_j^{\,2}}{\hat{r}}, \tag{3}$$

$$L_4(\hat{r}) = \frac{\hat{r}^{\,2}\,H_2}{4} - \frac{\hat{r}^{\,3}\,H_2'}{4} - \sum_{j>1}\frac{j-1}{2}\,\hat{r}\left(H_j'\,H_{j+1} + H_{j+1}'\,H_j\right) - \sum_{j>1}\frac{j-1}{2}\,H_j\,H_{j+1}$$

$$-\sum_{j>1}\frac{j-1}{2}\,\hat{r}\left(V_j'\,V_{j+1} + V_{j+1}'\,V_j\right) - \sum_{j>1}\frac{j-1}{2}\,V_j\,V_{j+1}. \tag{4}$$

Furthermore,

$$\theta = \omega + \epsilon\,F_1(\hat{r},\omega) + \epsilon^2\,F_2(\hat{r},\omega), \tag{5}$$

where

$$F_1(\hat{r},\omega) = \hat{r}\,\sin\omega - \sum_{j>0}\frac{1}{j}\left[H_j' - (j-1)\,\frac{H_j}{\hat{r}}\right]\sin(j\,\omega) + \sum_{j>1}\frac{1}{j}\left[V_j' - (j-1)\,\frac{V_j}{\hat{r}}\right]\cos(j\,\omega), \tag{6}$$

$$F_2(\hat{r},\omega) = \left(\frac{V_2}{2} + \frac{\hat{r}\,V_2'}{2} + \frac{H_1'\,V_2}{\hat{r}}\right)\cos\omega + \left(\frac{\hat{r}\,V_3'}{4} + \frac{H_1'\,V_3}{\hat{r}}\right)\cos(2\,\omega)$$

$$+\sum_{j>2}\left\{\frac{1}{2\,j}\left[-(j-2)\,(V_{j-1}+V_{j+1}) + \hat{r}\,(V_{j-1}'+V_{j+1}')\right] + \frac{H_1'\,V_{j+1}}{\hat{r}}\right\}\cos(j\,\omega)$$

$$-\left\{\frac{H_2}{2} + \frac{\hat{r}\,H_2'}{2} + \frac{H_1'\,H_2}{\hat{r}} + \sum_{j'>1} j'\left(\frac{H_{j'}'\,H_{j'+1}}{\hat{r}} + \frac{V_{j'}'\,V_{j'+1}}{\hat{r}}\right)\right.$$

$$\left.+\sum_{j'>1}(j'-1)\left(\frac{H_{j'+1}'\,H_{j'}}{\hat{r}} + \frac{V_{j'+1}'\,V_{j'}}{\hat{r}}\right)\right\}\sin\omega$$

$$
\begin{aligned}
&-\Bigg\{-\frac{\hat{r}^2}{4}+\frac{\hat{r}}{4}\left(H_1'+H_3'\right)+\frac{H_1'\,H_3}{\hat{r}}+\sum_{j'>1}\frac{j'+1}{2}\left(\frac{H_{j'}'\,H_{j'+2}}{\hat{r}}+\frac{V_{j'}'\,V_{j'+2}}{\hat{r}}\right)\\
&\quad+\sum_{j'>1}\frac{j'-1}{2}\left(\frac{H_{j'+2}'\,H_{j'}}{\hat{r}}+\frac{V_{j'+2}'\,V_{j'}}{\hat{r}}\right)\Bigg\}\sin(2\,\omega)\\
&\quad-\sum_{j>2}\Bigg\{-\frac{j-2}{2\,j}\left(H_{j-1}+H_{j+1}\right)+\frac{\hat{r}}{2\,j}\left(H_{j-1}'+H_{j+1}'\right)+\frac{H_1'\,H_{j+1}}{\hat{r}}\\
&\quad+\sum_{j'>1}\frac{j'+j-1}{j}\left(\frac{H_{j'}'\,H_{j'+j}}{\hat{r}}+\frac{V_{j'}'\,V_{j'+j}}{\hat{r}}\right)+\sum_{j'>1}\frac{j'-1}{j}\left(\frac{H_{j'+j}'\,H_{j'}}{\hat{r}}+\frac{V_{j'+j}'\,V_{j'}}{\hat{r}}\right)\Bigg\}\sin(j\,\omega).
\end{aligned}
\tag{7}
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