Pressure Flattening due to Magnetic Island

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I.   MAGNETIC ISLAND

Let $x = r - r_s$, $X = x/W$, and $\zeta = m\,\theta - n\,\phi$, where W is the island width. The magnetic flux-surfaces of the magnetic island are contours of

$$\Omega(X, \zeta) = 8\,X^2 + \cos\zeta. \tag{1}$$

The X-points lie at $X = 0$ and $\zeta = 0$, 2π, whereas the X-point lies at $X = 0$ and $\zeta = \pi$. The O-point corresponds to $\Omega = -1$, whereas the magnetic separatrix corresponds to $\Omega = 1$. Note that $\Omega \simeq 8\,X^2$ in the limit $|X| \gg 1$.

II.   TEMPERATURE PERTURBATION IN INNER REGION

Let $T_{e\,0}(r)$ be the unperturbed electron temperature profile. Let

$$T_e(X, \zeta) = T_{e\,s} + \text{sgn}(X)\,W\,T'_{e\,s}\,\tilde{T}(\Omega) \tag{2}$$

be the temperature profile in the presence of the island, where $T_{e\,s} = T_{e\,0}(r_s)$, and $T'_{e\,s} = (dT_{e\,0}/dr)_{r=r_s}$. The normalized perturbed electron temperature profile, $\tilde{T}(\Omega)$, satisfies the energy conservation equation

$$\frac{d}{d\Omega}\left[\oint (\Omega - \cos\zeta)^{1/2}\,\frac{d\zeta}{2\pi}\,\frac{d\tilde{T}}{d\Omega}\right] = 0, \tag{3}$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \to |X| \tag{4}$$

as $|X| \to \infty$. Note that $T(X, \zeta) - T_{e\,s}$ is an odd function of X.

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Equations (2) and (3) imply that

$$\tilde{T}(\Omega) = 0 \tag{5}$$

for $-1 \leq \Omega < 1$, and

$$\frac{d\tilde{T}}{d\Omega} = \frac{c}{\oint (\Omega - \cos \zeta)^{1/2}\, d\zeta/2\pi} \tag{6}$$

for $\Omega \geq 1$, where c is a constant. Let

$$k = \left(\frac{1+\Omega}{2}\right)^{1/2}. \tag{7}$$

The island O-point corresponds to $k = 0$, whereas the magnetic separatrix corresponds to $k = 1$. Note that $k \to 2\,|X|$ as $|X| \to \infty$.

Equation (6) yields

$$\frac{d\tilde{T}}{dk} = \frac{\sqrt{2}\,\pi\,c}{E(1/k)}, \tag{8}$$

where

$$E(p) \equiv \int_0^{\pi/2} (1 - p^2 \sin^2 \theta)^{1/2}\, d\theta \tag{9}$$

is a complete elliptic integral. The boundary condition (4) implies that

$$c = \frac{1}{4\sqrt{2}}. \tag{10}$$

Hence, we conclude that

$$\frac{d\tilde{T}}{dk} = \frac{\pi}{4}\,\frac{1}{E(1/k)} \tag{11}$$

for $k \geq 1$. Thus,

$$\tilde{T}(k) = 0 \tag{12}$$

for $0 \leq k < 1$, and

$$\tilde{T}(k) = F(k) \tag{13}$$

for $k \geq 1$, where

$$F(k) = \frac{\pi}{4} \int_1^k \frac{dk'}{E(1/k')}. \tag{14}$$

III.   HARMONICS OF TEMPERATURE PERTURBATION

We can write

$$\tilde{T}(|X|, \zeta) = \sum_{\nu=0,\infty} \delta T_\nu(|X|)\, \cos(\nu\,\zeta). \tag{15}$$

Now,

$$\delta T_0(|X|) = \oint \tilde{T}(|X|, \zeta) \, \frac{d\zeta}{2\pi}, \tag{16}$$

where the integral is at constant $|X|$. It follows that

$$\delta T_0(|X|) = \int_0^{\zeta_c} F(k) \, \frac{d\zeta}{\pi}, \tag{17}$$

where

$$\zeta_c = \cos^{-1}\left(1 - 8\,X^2\right) \tag{18}$$

for $|X| < 1/2$, and $\zeta_c = \pi$ for $|X| \geq 1/2$. Furthermore,

$$k = \left[4\,|X|^2 + \cos^2\left(\frac{\zeta}{2}\right)\right]^{1/2}. \tag{19}$$

For $\nu > 0$, we have

$$\delta T_\nu(|X|) = 2 \oint \tilde{T}(|X|, \zeta) \, \cos(\nu\,\zeta) \, \frac{d\zeta}{2\pi}, \tag{20}$$

where the integral is at constant $|X|$. Integrating by parts, we obtain

$$\delta T_\nu(|X|) = -\frac{2}{\nu} \oint \frac{\partial \tilde{T}}{\partial \zeta} \, \sin(\nu\,\zeta) \, \frac{d\zeta}{2\pi}. \tag{21}$$

But,

$$\frac{\partial \tilde{T}}{\partial \zeta} = \frac{d\tilde{T}}{dk} \frac{\partial k}{\partial \zeta} = -\frac{d\tilde{T}}{dk} \frac{\sin\zeta}{4\,k} = -\frac{\pi}{16} \frac{\sin\zeta}{k\,E(1/k)}, \tag{22}$$

so

$$\delta T_\nu(X) = \frac{1}{16\,\nu} \int_0^{\zeta_c} \frac{\cos[(\nu - 1)\,\zeta] - \cos[(\nu + 1)\,\zeta]}{k\,E(1/k)} \, d\zeta. \tag{23}$$

IV. ASYMPTOTIC BEHAVIOR

In the limit $|X| \ll 1$, we have

$$\zeta_c \simeq 4\,|X|, \tag{24}$$

$$k \simeq 1 + \frac{\zeta_c^2 - \zeta^2}{8}, \tag{25}$$

$$E(1/k) \simeq 1, \tag{26}$$

$$F(k) \simeq \frac{\pi}{4}\,(k - 1). \tag{27}$$

It follows that

$$\delta T_0(|X|) \simeq \frac{4}{3}\,|X|^3, \tag{28}$$

$$\delta T_{\nu>0}(|X|) \simeq \frac{8}{3}\,|X|^3 \tag{29}$$

In the limit $|x|/W \gg 1$, we have

$$k \simeq 2\,|X|, \tag{30}$$

$$E(1/k) \simeq \frac{\pi}{2}. \tag{31}$$

It follows that

$$F(k) \simeq \frac{k}{2} - F_\infty, \tag{32}$$

$$\delta T_0(|X|) \simeq |X| - F_\infty, \tag{33}$$

$$\delta T_1(|X|) \simeq \frac{1}{16\,|X|}, \tag{34}$$

$$\delta T_{\nu>1}(|X|) \sim \mathcal{O}\left(\frac{1}{|X|^3}\right), \tag{35}$$

where

$$F_\infty = 0.3447. \tag{36}$$

V. ASYMPTOTIC MATCHING

Consider the kth rational surface whose radius is r_k and whose resonant poloidal mode number is m_k. Let $x = r - r_k$ and $\zeta_k = m_k\,\theta - n\,\phi$.

In the outer region, we write the total electron temperature as

$$\tilde{T}_e(r,\theta,\phi) = T_{e\,0}(r) - \Psi_k\,\frac{q(r)}{r\,g(r)}\,\frac{T'_{e\,0}(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\,e^{\,i\,\zeta_k}, \tag{37}$$

where $T'_{e\,0} = dT_{e\,0}/dr$, $T_{e\,0}(r)$ is the equilibrium electron temperature profile, Ψ_k is the reconnected flux, and

$$W_k = 4\left(\frac{q}{g\,s}\right)_{r_k}^{1/2}\,\Psi_k^{1/2} \tag{38}$$

is the island width. In the limit, $|x| \ll 1$, Eq. (37) yield

$$\tilde{T}_e(x,\theta,\phi) = T_{e\,k} + T'_{e\,k}\,x + \frac{T'_{e\,k}\,W_k^2}{16\,x}\,e^{\,i\,\zeta_k}, \tag{39}$$

Here, $T_{ek} = T_{e0}(r_k)$ and $T'_{ek} = (dT_{e0}/dr)_{r_k}$, and we have made use of the fact that $\psi_{m_k}(r_k) \simeq m_k$.

In the inner region, we write the total electron temperature as

$$\tilde{T}_e(x, \theta, \phi) = T_{ek} + \text{sgn}(x)\, T'_{ek}\, W_k \sum_{\nu=0,\infty} \delta T_\nu(|x|/W_k)\, e^{i\nu\zeta_k} + T'_{ek}\, W_k\, F_\infty, \tag{40}$$

In the limit $x \gg W_k$, the previous equation yields

$$\tilde{T}_e(x, \theta, \phi) \simeq T_{ek} + T'_{ek}\, x + \frac{T'_{ek}\, W_k^2}{16\, x}\, e^{i\zeta_k}, \tag{41}$$

On the other hand, in the limit $x \ll -W_k$, we get

$$\tilde{T}_e(x, \theta, \phi) \simeq T_{ek} + T'_{ek}\, x + 2\, T'_{ek}\, W_k\, F_\infty + \frac{T'_{ek}\, W_k^2}{16\, x}\, e^{i\zeta_k}. \tag{42}$$

The asymptotic matching process consists of writing

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e+} - \Psi_{k+}\, \frac{q(r)}{r\, g(r)}\, \frac{T'_{e0}(r)\, \psi_{m_k}(r)}{m_k - n\, q(r)}\, e^{i\zeta_k} \tag{43}$$

in the region $r > r_k + W_k$,

$$\tilde{T}_e(r, \theta, \phi) = T_{e0}(r) + \delta T_{e-} - \Psi_{k-}\, \frac{q(r)}{r\, g(r)}\, \frac{T'_{e0}(r)\, \psi_{m_k}(r)}{m_k - n\, q(r)}\, e^{i\zeta_k} \tag{44}$$

in the region $r < r_k - W_k$, and

$$\tilde{T}_e(r, \theta, \phi) = T_{ek} + \text{sgn}(x)\, T'_{ek}\, W_k \sum_{\nu=0,\infty} \delta T_\nu(|x|/W_k)\, e^{i\nu\zeta_k} + T'_{ek}\, W_k\, F_\infty \tag{45}$$

in the region $r_k - W_k \leq r \leq r_k + W_k$. Continuity of the solution at $r = r_k \pm W_k$ implies that

$$\delta T_{e+} = T'_{ek}\, W_k\, \delta T_{e0}(1) + T'_{ek}\, W_k\, F_\infty - T'_{ek}\, W_k, \tag{46}$$

$$\delta T_{e-} = -T'_{ek}\, W_k\, \delta T_{e0}(1) + T'_{ek}\, W_k\, F_\infty + T'_{ek}\, W_k, \tag{47}$$

$$\Psi_{k+} = -T'_{ek}\, W_k\, \delta T_1(1) \left(\frac{r\, g}{q}\, \frac{m_k - n\, q}{T'_{e0}\, \psi_{mk}} \right)_{r_k + W_k}, \tag{48}$$

$$\Psi_{k-} = T'_{ek}\, W_k\, \delta T_1(1) \left(\frac{r\, g}{q}\, \frac{m_k - n\, q}{T'_{e0}\, \psi_{mk}} \right)_{r_k - W_k}. \tag{49}$$

Finally, for the special case $m = 1$, we write

$$\tilde{T}_e(r, \theta, \phi) = -\xi^r(r, \theta, \phi)\, \frac{dT_{e0}}{dr}. \tag{50}$$

A.   Normalized Quantities

Let $\hat{r} = r/\epsilon_a$, $\hat{r}_k = r_k/\epsilon_a$, $\hat{x} = x/\epsilon_a$, $\hat{T}'_{e0} = \epsilon_a T'_{e0}$, $\hat{T}'_{ek} = \epsilon_a \hat{T}'_{ek}$, $\hat{W}_k = W_k/\epsilon_a$, and $\hat{\Psi}_k = \Psi_k/\epsilon_a^2$, etc., then

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e+} - \hat{\Psi}_{k+} \frac{q(\hat{r})}{\hat{r}\, g(\hat{r})} \frac{\hat{T}'_e(r)\, \psi_{m_k}(r)}{m_k - n\, q(\hat{r})}\, \mathrm{e}^{\mathrm{i}\zeta_k} \tag{51}$$

in the region $\hat{r} > \hat{r}_k + \hat{W}_k$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{e0}(\hat{r}) + \delta T_{e-} - \hat{\Psi}_{k-} \frac{q(\hat{r})}{\hat{r}\, g(\hat{r})} \frac{\hat{T}'_e(r)\, \psi_{m_k}(r)}{m_k - n\, q(\hat{r})}\, \mathrm{e}^{\mathrm{i}\zeta_k} \tag{52}$$

in the region $\hat{r} < \hat{r}_k - \hat{W}_k$, and

$$\tilde{T}_e(\hat{r}, \theta, \phi) = T_{ek} + \mathrm{sgn}(\hat{x})\, \hat{T}'_{ek}\, \hat{W}_k \sum_{\nu=0,\infty} \delta T_\nu(|\hat{x}|/\hat{W}_k)\, \mathrm{e}^{\mathrm{i}\nu\zeta_k} + \hat{T}'_{ek}\, \hat{W}_k\, F_\infty \tag{53}$$

in the region $\hat{r}_k - \hat{W}_k \leq \hat{r} \leq \hat{r}_k + \hat{W}_k$. Here,

$$\delta T_{e+} = \hat{T}'_{ek}\, \hat{W}_k\, \delta T_0(1) + \hat{T}'_{ek}\, \hat{W}_k\, F_\infty - \hat{T}'_{ek}\, \hat{W}_k, \tag{54}$$

$$\delta T_{e-} = -\hat{T}'_{ek}\, \hat{W}_k\, \delta T_0(1) + \hat{T}'_{ek}\, \hat{W}_k\, F_\infty + \hat{T}'_{ek}\, \hat{W}_k, \tag{55}$$

$$\hat{\Psi}_k = \left(\frac{\hat{W}_k}{4}\right)^2 \left(\frac{g\, s}{q}\right)_{\hat{r}_k}, \tag{56}$$

$$\hat{\Psi}_{k+} = -\hat{T}'_{ek}\, \hat{W}_k\, \delta T_1(1) \left(\frac{\hat{r}\, g}{q} \frac{m_k - n\, q}{\hat{T}'_{e0}\, \psi_{mk}}\right)_{\hat{r}_k + \hat{W}_k}, \tag{57}$$

$$\hat{\Psi}_{k-} = \hat{T}'_{ek}\, \hat{W}_k\, \delta T_1(1) \left(\frac{\hat{r}\, g}{q} \frac{m_k - n\, q}{\hat{T}'_{e0}\, \psi_{mk}}\right)_{\hat{r}_k - \hat{W}_k}. \tag{58}$$

For the special case $m = 1$,

$$\tilde{T}_e(\hat{r}, \theta, \phi) = -\frac{\xi^r(\hat{r}, \theta, \phi)}{\epsilon_a} \frac{dT_{e0}}{d\hat{r}}. \tag{59}$$