Resistive Wall

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I. RESISTIVE WALL PHYSICS

A. Resistive Wall

Let the inner surface of the resistive wall surrounding the plasma lie at $\mu = \mu_w$, and let the outer surface lie at $\mu = \mu_w - \bar{d}_w \sinh\mu_w$, where $\bar{d}_w \ll 1$ is a positive constant. The physical wall thickness is

$$d(\eta) = \frac{\bar{d}_w \sinh\mu_w}{|\nabla\mu|} = h_w(\eta) \sinh\mu_w\, \bar{d}_w, \tag{1}$$

where

$$h_w(\eta) = \frac{1}{z_w - \cos\eta}, \tag{2}$$

and $z_w = \cosh\mu_w$. Let the electrical conductivity of the wall material vary as

$$\sigma(\eta) = \frac{\bar{\sigma}_w}{h_w^2(\eta) \sinh^2\mu_w}, \tag{3}$$

where $\bar{\sigma}_w$ is a positive constant. It follows that $\sigma\, d^2 = \bar{\sigma}_w\, \bar{d}_w^2$.

B. Wall Matching Conditions

If we write

$$\mathbf{b} = \nabla\times\mathbf{A} \tag{4}$$

in the vacuum region then the boundary conditions at the wall are

$$\mathbf{n}_w \times \mathbf{A}|_{z_{w-}} = \frac{1}{\cosh\lambda}\, \mathbf{n}_w \times \mathbf{A}|_{z_{w+}} \tag{5}$$

$$\mathbf{n}_w \times (\nabla\times\mathbf{A})|_{z_{w+}} = -\frac{\lambda\tanh\lambda}{\bar{d}_w\, h_w \sinh\mu_w}\, \mathbf{n}_w \times (\mathbf{n}_w \times \mathbf{A})|_{z_{w+}} + \frac{\mathbf{n}_w \times (\nabla\times\mathbf{A})|_{z_{w-}}}{\cosh\lambda}, \tag{6}$$

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$$\lambda = \sqrt{\hat{\gamma}\,\bar{d}_w}, \tag{7}$$

$$\hat{\gamma} = \gamma\,\bar{\tau}_w, \tag{8}$$

$$\bar{\tau}_w = \mu_0\,R_0^2\,\bar{\sigma}_w\,\bar{d}_w, \tag{9}$$

where γ is the growth-rate of the magnetic perturbation, and $\bar{\tau}_w$ is the effective L/R time of the wall. Here, $\mathbf{n}_w = -\mathbf{e}_\mu$ is an outward unit normal vector to the wall. Now,

$$\nabla\times\mathbf{A} = \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\phi}{\partial\eta} - \frac{\partial\hat{A}_\eta}{\partial\phi}\right)\mathbf{e}_\mu + \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)\mathbf{e}_\eta + \frac{1}{h^2}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)\mathbf{e}_\phi, \tag{10}$$

where

$$\hat{A}_\mu = h\,A_\mu, \tag{11}$$

$$\hat{A}_\eta = h\,A_\eta, \tag{12}$$

$$\hat{A}_\phi = h\,\sinh\mu\,A_\phi. \tag{13}$$

Furthermore,

$$\mathbf{n}_w\times\mathbf{A} = -\mathbf{e}_\mu\times\mathbf{A} = A_\phi\,\mathbf{e}_\eta - A_\eta\,\mathbf{e}_\phi, \tag{14}$$

$$\mathbf{n}_w\times(\mathbf{n}_w\times\mathbf{A}) = -\mathbf{e}_\mu\times(\mathbf{n}_w\times\mathbf{A}) = -A_\eta\,\mathbf{e}_\eta - A_\phi\,\mathbf{e}_\phi, \tag{15}$$

$$\mathbf{n}_w\times(\nabla\times\mathbf{A}) = -\mathbf{e}_\mu\times(\nabla\times\mathbf{A}) = \frac{1}{h^2}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)\mathbf{e}_\eta - \frac{1}{h^2\,\sinh\mu}\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)\mathbf{e}_\phi. \tag{16}$$

Thus, the wall matching conditions become

$$\hat{A}_\eta\Big|_{z_{w-}} = \frac{1}{\cosh\lambda}\,\hat{A}_\eta\Big|_{z_{w+}}, \tag{17}$$

$$\hat{A}_\phi\Big|_{z_{w+}} = \frac{1}{\cosh\lambda}\,\hat{A}_\phi\Big|_{z_{w-}}, \tag{18}$$

$$\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)_{z_{w+}} = \frac{\lambda\,\tanh\lambda}{\bar{d}_w\,\sinh\mu_w}\,\hat{A}_\eta\Big|_{z_{w+}} + \frac{1}{\cosh\lambda}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)_{z_{w-}}, \tag{19}$$

$$\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)_{z_{w+}} = -\frac{\lambda\,\tanh\lambda}{\bar{d}_w\,\sinh\mu_w}\,\hat{A}_\phi\Big|_{\mu_{z+}} + \frac{1}{\cosh\lambda}\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)_{z_{w-}}. \tag{20}$$

Let

$$C(z,\eta,\phi) = \frac{\partial \hat{A}_\eta}{\partial\phi} - \frac{\partial \hat{A}_\phi}{\partial\eta}. \tag{21}$$

The wall matching conditions reduce to

$$C(z_{w-},\eta,\phi) = \frac{1}{\cosh\lambda}\,C(z_{w+},\eta,\phi), \tag{22}$$

$$\frac{\partial C(z_{w+},\eta,\phi)}{\partial z} = \frac{\lambda\,\tanh\lambda}{\bar{d}_w\,\sinh^2\mu_w}\,C(z_{w+},\eta,\phi) + \frac{1}{\cosh\lambda}\,\frac{\partial C(z_{w-},\eta,\phi)}{\partial z}. \tag{23}$$

However, if

$$\mathbf{b} = \mathrm{i}\,\nabla V = \nabla\times\mathbf{A} \tag{24}$$

then

$$C = -\mathrm{i}\,h\,\sinh\mu\,\frac{\partial V}{\partial\mu} = -\mathrm{i}\,h\,(z^2-1)\,\frac{\partial V}{\partial z}. \tag{25}$$

Thus,

$$C = -\mathrm{i}\,\frac{z^2-1}{z-\cos\eta}\sum_m\left[\frac{U_m}{2\,(z-\cos\eta)^{1/2}} + (z-\cos\eta)^{1/2}\,\frac{dU_m}{dz}\right]\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}, \tag{26}$$

$$\frac{\partial C}{\partial z} = -\mathrm{i}\sum_m\left[\frac{(3/4)\,\sin^2\eta}{(z-\cos\eta)^{5/2}} - \frac{(1/2)\,\cos\eta}{(z-\cos\eta)^{3/2}} + \frac{m^2+n^2/(z^2-1)}{(z-\cos\eta)^{1/2}}\right]U_m\,\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}. \tag{27}$$

It follows that

$$\sum_m\left[\frac{U_m}{2} + (z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w-}}\mathrm{e}^{-\mathrm{i}\,m\,\eta} = \frac{1}{\cosh\lambda}\sum_m\left[\frac{U_m}{2} + (z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w+}}\mathrm{e}^{-\mathrm{i}\,m\,\eta}, \tag{28}$$

$$\begin{aligned}\sum_m\left[\frac{3}{4}\,\sin^2\eta - \frac{1}{2}\,(z-\cos\eta)\,\cos\eta + (z-\cos\eta)^2\left(m^2+\frac{n^2}{z^2-1}\right)\right]U_m\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}\bigg|_{z_{w+}} &= \\ f_w\sum_m(z-\cos\eta)\left[\frac{U_m}{2} + (z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w+}}&\mathrm{e}^{-\mathrm{i}\,m\,\eta} \\ +\frac{1}{\cosh\lambda}\sum_m\left[\frac{3}{4}\,\sin^2\eta - \frac{1}{2}\,(z-\cos\eta)\,\cos\eta + (z-\cos\eta)^2\left(m^2+\frac{n^2}{z^2-1}\right)\right]U_m\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}\bigg|_{z_{w-}}&,\end{aligned} \tag{29}$$

where

$$f_w = \frac{\lambda\,\tanh\lambda}{\bar{d}_w}. \tag{30}$$

Thus, we can write

$$\sum_{m'} I_{mm'}\,U_{m'}(z_{w-}) = \frac{1}{\cosh\lambda}\sum_{m'} I_{mm'}\,U_{m'}(z_{w+}), \tag{31}$$

$$\sum_{m'} J_{mm'}\,U_{m'}(z_{w+}) = f_w \sum_{m',m''} k_{mm''}\,I_{m''m'}\,U_{m'}(z_{w+}) + \frac{1}{\cosh\lambda}\sum_{m'} J_{mm'}\,U_{m'}(z_{w-}), \tag{32}$$

where

$$I_{mm'} = \left(\frac{1}{2} + z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\,(\delta_{m+1\,m'} + \delta_{m-1\,m'}), \tag{33}$$

$$J_{mm'} = \left[\frac{5}{8} + \left(\frac{1}{2} + z^2\right)\left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4} + \left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right](\delta_{m+1\,m'} + \delta_{m-1\,m'})$$
$$+ \left[-\frac{1}{16} + \frac{1}{4}\left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right](\delta_{m+2\,m'} + \delta_{m-2\,m'}), \tag{34}$$

$$k_{mm'} = z\,\delta_{mm'} - \frac{1}{2}\,(\delta_{m+1\,m'} + \delta_{m-1\,m'}). \tag{35}$$

C. Vacuum Solution

Now,

$$U_m(z) = p_{m-}\,\hat{P}^{\,n}_{|m|-1/2}(z) \tag{36}$$

in the region $z < z_w$, whereas

$$U_m(z) = p_{m+}\,\hat{P}^{\,n}_{|m|-1/2}(z) + q_{m+}\,\hat{Q}^{\,n}_{|m|-1/2}(z) \tag{37}$$

in the region $z > z_w$. Let $\underline{\underline{I}}_p$ be the matrix of the

$$\left\{\left[\left(\frac{1}{2} + z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\,(\delta_{m+1\,m'} + \delta_{m-1\,m'})\right]\hat{P}^{\,n}_{|m'|-1/2}(z)\right\}_{z_w} \tag{38}$$

values. Let $\underline{\underline{I}}_q$ be the matrix of the

$$\left\{\left[\left(\frac{1}{2} + z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\,(\delta_{m+1\,m'} + \delta_{m-1\,m'})\right]\hat{Q}^{\,n}_{|m'|-1/2}(z)\right\}_{z_w} \tag{39}$$

values. Let $\underline{\underline{J}}_p$ be the matrix of the

$$\Bigg\{\left[\frac{5}{8} + \left(\frac{1}{2} + z^2\right)\left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4} + \left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right](\delta_{m+1\,m'} + \delta_{m-1\,m'})$$
$$+ \left[-\frac{1}{16} + \frac{1}{4}\left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right](\delta_{m+2\,m'} + \delta_{m-2\,m'})\Bigg\}\,\hat{P}^{\,n}_{|m'|-1/2}(z_w) \tag{40}$$

values. Let $\underline{\underline{J}}_q$ be the matrix of the

$$\Bigg\{\left[\frac{5}{8} + \left(\frac{1}{2} + z^2\right)\left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4} + \left(m'^{\,2} + \frac{n^2}{z^2-1}\right)\right](\delta_{m+1\,m'} + \delta_{m-1\,m'})$$

$$+\left[-\frac{1}{16}+\frac{1}{4}\left(m'^{\,2}+\frac{n^2}{z^2-1}\right)\right](\delta_{m+2\,m'}+\delta_{m-2\,m'})\Bigg\}\,\hat{Q}^{\,n}_{|m'|-1/2}(z_w) \tag{41}$$

values. Let $\underline{\underline{k}}$ be the matrix of the $k_{mm'}$ values. Finally, let $\underline{p}_{\,+}$ be the vector of the p_{m+} values, et cetera. Thus, we obtain

$$\underline{\underline{I}}_{\,p}\,\underline{p}_{\,-} = \frac{1}{\cosh\lambda}\left(\underline{\underline{I}}_{\,p}\,\underline{p}_{\,+}+\underline{\underline{I}}_{\,q}\,\underline{q}_{\,+}\right), \tag{42}$$

$$\underline{\underline{J}}_{\,p}\,\underline{p}_{\,+}+\underline{\underline{J}}_{\,q}\,\underline{q}_{\,+} = f_w\,\underline{\underline{k}}\left(\underline{\underline{I}}_{\,p}\,\underline{p}_{\,+}+\underline{\underline{I}}_{\,q}\,\underline{q}_{\,+}\right)+\frac{1}{\cosh\lambda}\,\underline{\underline{J}}_{\,p}\,\underline{p}_{\,-}, \tag{43}$$

which can be rearranged to give

$$\left(\tanh^2\lambda\,\underline{\underline{J}}_{\,p}-f_w\,\hat{\underline{\underline{I}}}_{\,p}\right)\underline{p}_{\,+}+\left(\underline{\underline{J}}_{\,pq}+\tanh^2\lambda\,\underline{\underline{J}}_{\,qp}-f_w\,\hat{\underline{\underline{I}}}_{\,q}\right)\underline{q}_{\,+}, \tag{44}$$

where

$$\hat{\underline{\underline{I}}}_{\,p} = \underline{\underline{k}}\,\underline{\underline{I}}_{\,p}, \tag{45}$$

$$\hat{\underline{\underline{I}}}_{\,q} = \underline{\underline{k}}\,\underline{\underline{I}}_{\,q}, \tag{46}$$

$$\underline{\underline{J}}_{\,pq} = \underline{\underline{J}}_{\,q}-\underline{\underline{J}}_{\,p}\,\hat{\underline{\underline{I}}}_{\,p}^{\,-1}\,\hat{\underline{\underline{I}}}_{\,q}, \tag{47}$$

$$\underline{\underline{J}}_{\,qp} = \underline{\underline{J}}_{\,p}\,\hat{\underline{\underline{I}}}_{\,p}^{\,-1}\,\hat{\underline{\underline{I}}}_{\,q}. \tag{48}$$

Now, $z_w \sim 1/\bar{b}_w$, where $\bar{b}_w$ is the mean wall minor radius. In the large aspect-ratio limit, $\bar{b}_w \ll 1$, we have $\underline{\underline{I}}_{\,p} \sim \mathcal{O}(1)$, $\underline{\underline{I}}_{\,q} \sim \mathcal{O}(1)$, $\underline{\underline{J}}_{\,p} \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{\underline{J}}_{\,q} \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{\underline{K}}_{\,p} \sim \mathcal{O}(1/\bar{b}_w)$, $\underline{\underline{K}}_{\,q} \sim \mathcal{O}(1/\bar{b}_w)$, and $\underline{\underline{k}} \sim \mathcal{O}(1/\bar{b}_w)$ It follows that $\hat{\underline{\underline{I}}}_{\,p} \sim \mathcal{O}(1/\bar{b}_w)$, $\hat{\underline{\underline{I}}}_{\,q} \sim \mathcal{O}(1/\bar{b}_w)$, $\underline{\underline{J}}_{\,pq} \sim \mathcal{O}(1/\bar{b}_w^2)$ and $\underline{\underline{J}}_{\,qp} \sim \mathcal{O}(1/\bar{b}_w^2)$. Thus, the ratio of the first to the second term multiplying $\underline{p}_{\,+}$ in Eq. (44) is

$$\tanh\lambda\,\frac{\bar{d}_w}{\lambda\,\bar{b}_w}. \tag{49}$$

However, the wall analysis is premised on the assumption that

$$\frac{\bar{d}_w}{\lambda\,\bar{b}_w} \ll 1. \tag{50}$$

Hence, the first term is negligible with respect to the second, irrespective of the value of λ. The ratios of the three terms multiplying $\underline{q}_{\,+}$ in Eq. (44) are

$$\frac{\bar{d}_w}{\lambda\,\bar{b}_w},\;\;\tanh^2\lambda\,\frac{\bar{d}_w}{\lambda\,\bar{b}_w},\;\;\tanh\lambda. \tag{51}$$

Thus, in the thin-shell limit, $\lambda \ll 1$, the second term is negligible with respect to the first. In the thick-shell limit, $\lambda \gg 1$, the third term is dominant. Thus, we can neglect the second term. Hence, we deduce that

$$\underline{q}_{+} = \underline{\underline{\mathcal{F}}}\, \underline{p}_{+}, \tag{52}$$

where

$$\underline{\underline{\mathcal{F}}} = f_w\, \underline{\underline{I}}\, (\underline{\underline{J}} + f_w\, \underline{\underline{1}})^{-1}, \tag{53}$$

$$\underline{\underline{I}} = -\hat{\underline{\underline{I}}}_q^{\,-1}\, \hat{\underline{\underline{I}}}_p, \tag{54}$$

$$\underline{\underline{J}} = \hat{\underline{\underline{I}}}_p^{\,-1}\, (\underline{\underline{J}}_q\, \underline{\underline{I}} + \underline{\underline{J}}_p). \tag{55}$$

Note that $\underline{\underline{I}} \sim \mathcal{O}(1)$ and $\underline{\underline{J}} \sim \mathcal{O}(1/\bar{b}_w)$.

D. Toroidal Electromagnetic Torque

The net toroidal electromagnetic torque acting on the plasma is

$$T_\phi = -2\pi^2\, n\, \mathrm{Im}(\underline{p}_{+}^{\dagger}\, \underline{q}_{+}) = -2\pi^2\, n\, \mathrm{Im}(\underline{p}_{+}^{\dagger}\, \underline{\underline{\mathcal{F}}}\, \underline{p}_{+}) = -\pi^2\, n\, \mathrm{Im}[\underline{p}_{+}^{\dagger}\, (\underline{\underline{\mathcal{F}}} - \underline{\underline{\mathcal{F}}}^{\dagger})\, \underline{p}_{+}]. \tag{56}$$

However, we expect this torque to be zero if f_w is real, which implies that $\underline{\underline{\mathcal{F}}} = \underline{\underline{\mathcal{F}}}^{\dagger}$ when f_w is real. In other words,

$$f_w\, \underline{\underline{I}}\, (\underline{\underline{J}} + f_w\, \underline{\underline{1}})^{-1} = f_w\, (\underline{\underline{J}}^{\dagger} + f_w\, \underline{\underline{1}})^{-1}\, \underline{\underline{I}}^{\dagger}, \tag{57}$$

which implies that

$$f_w\, (\underline{\underline{J}}^{\dagger} + f_w\, \underline{\underline{1}})\, \underline{\underline{I}} = f_w\, \underline{\underline{I}}^{\dagger}\, (\underline{\underline{J}} + f_w\, \underline{\underline{1}}). \tag{58}$$

However, the previous equation holds for arbitrary real f_w, so we can separately equate the coefficients of f_w and f_w^2 to give

$$\underline{\underline{J}}^{\dagger}\, \underline{\underline{I}} = \underline{\underline{I}}^{\dagger}\, \underline{\underline{J}}, \tag{59}$$

$$\underline{\underline{I}} = \underline{\underline{I}}^{\dagger}. \tag{60}$$

It follows that $\underline{\underline{I}}$ and

$$\underline{\underline{K}} = \underline{\underline{I}}\, \underline{\underline{J}} \tag{61}$$

are both real symmetric matrices. In general,

$$\underline{\underline{\mathcal{F}}} - \underline{\underline{\mathcal{F}}}^{\dagger} = (f_w - f_w^{*})\, [(\underline{\underline{J}} + f_w\, \underline{\underline{1}})^{-1}]^{\dagger}\, \underline{\underline{K}}\, (\underline{\underline{J}} + f_w\, \underline{\underline{1}})^{-1}, \tag{62}$$

$$T_\phi = -2\pi^2\, n\, \mathrm{Im}(f_w)\, [(\underline{\underline{J}} + f_w\, \underline{\underline{1}})^{-1}\, \underline{p}_+]^\dagger\, \underline{\underline{K}}\, [(\underline{\underline{J}} + f_w\, \underline{\underline{1}})^{-1}\, \underline{p}_+]. \tag{63}$$

Thus, $\underline{\underline{\mathcal{F}}}$ is clearly Hermitian, and T_ϕ is zero, if f_w is real.