Ideal-MHD Energy Principle Analysis

I. IDEAL STABILITY ANALAYIS

A. Fundamental Equations

The fundamental equations of ideal magnetohydrodynamics (ideal-MHD) are

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0, \tag{1}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] + \nabla p - \mathbf{j}\times\mathbf{B} = 0, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma\,p\,\nabla\cdot\mathbf{v} = 0, \tag{3}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \tag{4}$$

$$\mu_0\,\mathbf{j} = \nabla\times\mathbf{B}, \tag{5}$$

where ρ is the plasma mass density, $\mathbf{v}$ the plasma velocity, p the (scalar) plasma pressure, $\gamma = 5/3$ the ratio of specific heats, $\mathbf{B}$ the magnetic field-strength, and $\mathbf{j}$ the electric current density. Note that Eq. (4) ensures that the magnetic field remains divergence-free, provided that this is the case initially.

B. Plasma Equilibrium

The plasma equilibrium is such that

$$\rho(\mathbf{r},t) = \rho_0(\mathbf{r}), \tag{6}$$

$$\mathbf{v}(\mathbf{r},t) = \mathbf{0}, \tag{7}$$

$$p(\mathbf{r},t) = p_0(\mathbf{r}), \tag{8}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r}), \tag{9}$$

$$\mathbf{j}(\mathbf{r},t) = \mathbf{j}_0(\mathbf{r}), \tag{10}$$

where

$$\nabla\cdot\mathbf{B}_0 = 0, \tag{11}$$

$$\nabla p_0 = \mathbf{j}_0 \times \mathbf{B}_0, \tag{12}$$

$$\mu_0\, \mathbf{j}_0 = \nabla \times \mathbf{B}_0. \tag{13}$$

Note that we are neglecting equilibrium plasma flows, as these are generally unimportant provided that they remain sub-sonic and sub-Alfvénic.

C. Perturbed Quantities

Let us formulate the linear stability problem as a normal mode problem. This goal is achieved by letting all perturbed quantities vary in time as $\exp(-{\rm i}\,\omega\, t)$. Thus, the perturbation to the plasma equilibrium is written

$$\rho_1(\mathbf{r}, t) = \rho_1(\mathbf{r})\, {\rm e}^{-{\rm i}\,\omega\, t}, \tag{14}$$

$$\mathbf{v}_1(\mathbf{r}, t) = \mathbf{v}_1(\mathbf{r})\, {\rm e}^{-{\rm i}\,\omega\, t} = -{\rm i}\,\omega\, \boldsymbol{\xi}(\mathbf{r})\, {\rm e}^{-{\rm i}\,\omega\, t}, \tag{15}$$

$$p(\mathbf{r}, t) = p_1(\mathbf{r})\, {\rm e}^{-{\rm i}\,\omega\, t}, \tag{16}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_1(\mathbf{r})\, {\rm e}^{-{\rm i}\,\omega\, t}, \tag{17}$$

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_1(\mathbf{r})\, {\rm e}^{-{\rm i}\,\omega\, t}. \tag{18}$$

Note that $\boldsymbol{\xi}(\mathbf{r})$ represents the displacement of the plasma from its equilibrium position.

The linearized perturbed versions of Eqs. (1)–(5) are

$$\rho_1 = -\nabla\cdot(\rho_0\, \boldsymbol{\xi}), \tag{19}$$

$$-\omega^2\, \rho_0\, \boldsymbol{\xi} = \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 - \nabla p_1, \tag{20}$$

$$p_1 = -\boldsymbol{\xi}\cdot\nabla p_0 - \gamma\, p_0\, \nabla\cdot\boldsymbol{\xi}, \tag{21}$$

$$\mathbf{B}_1 = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0), \tag{22}$$

$$\mu_0\, \mathbf{j}_1 = \nabla\times\mathbf{B}_1. \tag{23}$$

Note that Eq. (19) is not coupled to the other linearized equations. Combining the previous four equations, we obtain the perturbed plasma equation of motion,

$$-\omega^2\, \rho_0\, \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{24}$$

where

$$\mathbf{F}(\boldsymbol{\xi}) = \mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0 + \gamma\, p_0\, \nabla\cdot\boldsymbol{\xi}) \tag{25}$$

is known as the *force operator*, and

$$\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \tag{26}$$

is the perturbed magnetic field. The force operator clearly specifies the perturbed force density that develops in the plasma in response to the displacement $\boldsymbol{\xi}$.

D. Perfectly Conducting Wall

Suppose that the plasma is surrounded by a perfectly conducting wall whose inner surface has the outward-directed unit normal $\mathbf{n}$. Both the electric field and the magnetic field are zero within the wall. Standard electromagnetic boundary conditions require that

$$\mathbf{n} \times \mathbf{E} = 0, \tag{27}$$

$$\mathbf{n} \cdot \mathbf{B} = 0, \tag{28}$$

at the wall. Here, $\mathbf{E}$ is the electric field.

The plasma velocity is assumed to be dominated by the $\mathbf{E} \times \mathbf{B}$ drift velocity. In other words,

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \tag{29}$$

If we write $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) \exp(-\mathrm{i}\,\omega\,t)$ then it is clear from Eq. (7) that $\mathbf{E}_0 = \mathbf{0}$. It follows from the previous equation that

$$\mathbf{v}_1 = \frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2}. \tag{30}$$

Now, Eqs. (27) and (28) imply that

$$\mathbf{n} \cdot \mathbf{B}_0 = 0, \tag{31}$$

and

$$\mathbf{n} \times \mathbf{E}_1 = 0, \tag{32}$$

$$\mathbf{n} \cdot \mathbf{B}_1 = 0 \tag{33}$$

at the wall. Equations (30) and (32) yield

$$\mathbf{n} \cdot \mathbf{v}_1 = 0 \tag{34}$$

at the wall. Hence, it follows from Eq. (15) that

$$\mathbf{n}\cdot\boldsymbol{\xi} = 0 \tag{35}$$

at the wall.

E. Self-Adjoint Property of Force Operator

We wish to demonstrate that the force operator is *self-adjoint*. In other words, we wish to prove that

$$\int \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r} = \int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})\,d\mathbf{r}, \tag{36}$$

where the integrals are taken over the whole plasma volume, and $\boldsymbol{\xi}(\mathbf{r})$ and $\boldsymbol{\eta}(\mathbf{r})$ are two arbitrary vector fields that satisfy the physical boundary conditions

$$\mathbf{n}\cdot\boldsymbol{\xi} = \mathbf{n}\cdot\boldsymbol{\eta} = 0 \tag{37}$$

at the wall.

According to Eq. (25), the integrand of the left-hand side of Eq. (36) takes the form

$$\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = \boldsymbol{\eta}\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0) + \nabla(\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi})\right]. \tag{38}$$

The final term can be written

$$\boldsymbol{\eta}\cdot\nabla(\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi}) = \nabla\cdot(\boldsymbol{\eta}\,\gamma\,p_0\,\nabla\cdot\boldsymbol{\xi}) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}). \tag{39}$$

However, according to Eq. (37), the divergence term integrates to zero in Eq. (36). Hence, we can safely neglect this term, to give

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\xi) = \boldsymbol{\eta}\cdot&\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0)\right]\\ &-\gamma\,p_0\,(\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}).\end{aligned} \tag{40}$$

Let us write

$$\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel\,\mathbf{b}, \tag{41}$$

$$\boldsymbol{\eta} = \boldsymbol{\eta}_\perp + \eta_\parallel\,\mathbf{b}, \tag{42}$$

where

$$\mathbf{b} = \frac{\mathbf{B}_0}{B_0}, \tag{43}$$

$$\mathbf{b}\cdot\boldsymbol{\xi}_\perp = \mathbf{b}\cdot\boldsymbol{\eta}_\perp = 0. \tag{44}$$

Thus, $\xi_\parallel\,\mathbf{b}$ and $\boldsymbol{\xi}_\perp$ are the component of $\boldsymbol{\xi}$ that are parallel to and perpendicular to the equilibrium magnetic field, et cetera. According to Eqs. (31) and (37),

$$\mathbf{n}\cdot\boldsymbol{\xi}_\perp = \mathbf{n}\cdot\boldsymbol{\eta}_\perp \tag{45}$$

at the wall. It follows from Eq. (26) that

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0). \tag{46}$$

Moreover, Eq. (12) implies that

$$\boldsymbol{\xi}\cdot\nabla p_0 = \boldsymbol{\xi}\cdot\mathbf{j}_0\times\mathbf{B}_0 = \boldsymbol{\xi}_\perp\cdot\mathbf{j}_0\times\mathbf{B}_0 = \boldsymbol{\xi}_\perp\cdot\nabla p_0. \tag{47}$$

Now,

$$\mathbf{B}_0\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0\right] = 0, \tag{48}$$

and

$$\mathbf{B}_0\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q}\right] = \mathbf{B}_0\cdot\mathbf{j}_0\times\mathbf{Q} = -\mathbf{j}_0\times\mathbf{B}_0\cdot\mathbf{Q} = -\nabla p_0\cdot\mathbf{Q}, \tag{49}$$

where use has been made of Eqs. (12) and (13). However, according to Eq. (46),

$$\begin{aligned}-\nabla p_0\cdot\mathbf{Q} &= -\nabla p_0\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0) = \nabla\cdot[\nabla p_0\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0)] = -\nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,\mathbf{B}_0]\\ &= -\mathbf{B}_0\cdot\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0),\end{aligned} \tag{50}$$

where use has been made of Eqs. (11) and (12). The previous two equations, combined with Eq. (47), imply that

$$\mathbf{B}_0\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p_0)\right] = 0. \tag{51}$$

Thus, Eqs. (40), (42), (47), (48), and (51) yield

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) &= \boldsymbol{\eta}_\perp\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 + \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} + \nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\right]\\ &\quad-\gamma\,p_0\,(\nabla\cdot\boldsymbol{\eta})\,(\nabla\cdot\boldsymbol{\xi}).\end{aligned} \tag{52}$$

Note that $\xi_\parallel$ and $\eta_\parallel$ only occur on the right-hand side of the previous equation in the final term.

Let

$$I = \boldsymbol{\eta}_\perp\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0+\mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q}+\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\right]. \tag{53}$$

Now,

$$\boldsymbol{\eta}_\perp\cdot\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0) = \nabla\cdot[\boldsymbol{\eta}_\perp\,(\boldsymbol{\xi}_\perp\cdot\nabla p_0)]-(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,(\nabla\cdot\boldsymbol{\eta}_\perp) = -(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,(\nabla\cdot\boldsymbol{\eta}_\perp), \tag{54}$$

where the divergence term has integrated to zero because of Eq. (45). Furthermore,

$$(\nabla\times\mathbf{Q})\times\mathbf{B}_0+\mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{Q} = \mathbf{Q}\cdot\nabla\mathbf{B}_0+\mathbf{B}_0\cdot\nabla\mathbf{Q}-\nabla(\mathbf{B}_0\cdot\mathbf{Q}). \tag{55}$$

Hence,

$$I = \mu_0^{-1}\,\boldsymbol{\eta}_\perp\cdot[\mathbf{Q}\cdot\nabla\mathbf{B}_0+\mathbf{B}_0\cdot\nabla\mathbf{Q}-\nabla(\mathbf{B}_0\cdot\mathbf{Q})]-(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,(\nabla\cdot\boldsymbol{\eta}_\perp). \tag{56}$$

Note that

$$Q_i = \frac{\partial}{\partial x_j}\,(\xi_{\perp\,i}\,B_{0\,j}-\xi_{\perp\,j}\,B_{0\,i}). \tag{57}$$

It follows that

$$\begin{aligned}(\mathbf{Q}\cdot\nabla\mathbf{B}_0)_i &= Q_j\,\frac{\partial B_{0\,i}}{\partial x_j} = \frac{\partial}{\partial x_k}\,(\xi_{\perp\,j}\,B_{0\,k}-\xi_{\perp\,k}\,B_{0\,j})\,\frac{\partial B_{0\,i}}{\partial x_j}\\ &= B_{0\,k}\,\frac{\partial \xi_{\perp\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j}-\xi_{\perp\,k}\,\frac{\partial B_{0\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j}-B_{0\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,\frac{\partial \xi_{\perp\,k}}{\partial x_k}\\ &= [(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}_0-(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot\nabla\mathbf{B}_0-(\mathbf{B}_0\cdot\nabla\mathbf{B}_0)\,(\nabla\cdot\boldsymbol{\xi}_\perp)]_i\,,\end{aligned} \tag{58}$$

where use has been made of Eq. (11). However,

$$(\mathbf{B}_0\cdot\nabla\mathbf{B}_0) = B_0^{\,2}\,\boldsymbol{\kappa}+(\mathbf{B}_0\cdot\nabla B_0)\,\mathbf{b}, \tag{59}$$

where

$$\boldsymbol{\kappa} = \mathbf{b}\cdot\nabla\mathbf{b} = \frac{\mathbf{R}_c}{R_c^{\,2}} \tag{60}$$

is the curvature vector of the equilibrium magnetic field (i.e., $\mathbf{R}_c$ is the local radius of curvature of equilibrium magnetic field-lines). Hence, we deduce that

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{Q}\cdot\nabla\mathbf{B}_0) = \boldsymbol{\eta}_\perp\cdot\left[(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}_0-(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot\nabla\mathbf{B}_0-B_0^{\,2}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,\boldsymbol{\kappa}\right]. \tag{61}$$

Now,

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{Q}) = \eta_{\perp\,i}\,B_{0\,j}\,\frac{\partial Q_i}{\partial x_j} = \frac{\partial}{\partial x_j}\,(\eta_{\perp i}\,Q_i\,B_{0\,j}) - Q_i\,B_{0\,j}\,\frac{\partial\eta_{\perp\,i}}{\partial x_j}$$
$$= \nabla\cdot[(\boldsymbol{\eta}_\perp\cdot\mathbf{Q})\,\mathbf{B}_0] - \mathbf{Q}\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) = -\mathbf{Q}\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp), \tag{62}$$

where we have used Eq. (11), and the divergence term has integrated to zero because of Eq. (31). However, from Eqs. (11) and (57),

$$\mathbf{Q} = \mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0 - (\nabla\cdot\boldsymbol{\xi}_\perp)\,\mathbf{B}_0. \tag{63}$$

Thus,

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{Q}) = -(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) + (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp)$$
$$+ (\nabla\cdot\boldsymbol{\xi}_\perp)\,\mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp). \tag{64}$$

But,

$$\mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) = B_{0\,i}\,B_{0\,j}\,\frac{\partial\eta_{\perp\,i}}{\partial x_j} = \frac{\partial}{\partial x_j}\,(B_{0\,i}\,B_{0\,j}\,\eta_{\perp\,i}) - B_{0\,i}\,\frac{\partial B_{0\,j}}{\partial x_j}\,\eta_{\perp\,i}$$
$$= \nabla\cdot[(\mathbf{B}_0\cdot\boldsymbol{\eta}_\perp)\,\mathbf{B}_0] - B_0^{\,2}\,\boldsymbol{\kappa}\cdot\boldsymbol{\eta}_\perp = -B_0^{\,2}\,\boldsymbol{\kappa}\cdot\boldsymbol{\eta}_\perp, \tag{65}$$

where we have made use of Eq. (11), as well as Eq. (59), and the divergence term has integrated to zero because of Eq. (31). Thus, we arrive at

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{Q}) = -(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) + (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - B_0^{\,2}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp). \tag{66}$$

Now,

$$-\boldsymbol{\eta}_\perp\cdot\nabla(\mathbf{B}_0\cdot\mathbf{Q}) = -\eta_{\perp i}\,\frac{\partial}{\partial x_i}\,(B_{0\,j}\,Q_j) = -\frac{\partial}{\partial x_i}\,(\eta_{\perp\,i}\,B_{0\,j}\,Q_j) + B_{0\,j}\,Q_j\,\frac{\partial\eta_{\perp i}}{\partial x_i}$$
$$= -\nabla\cdot[(\mathbf{B}_0\cdot\mathbf{Q})\,\boldsymbol{\eta}_\perp] + (\mathbf{B}_0\cdot\mathbf{Q})\,(\nabla\cdot\boldsymbol{\eta}_\perp) = (\mathbf{B}_0\cdot\mathbf{Q})\,(\nabla\cdot\boldsymbol{\eta}_\perp), \tag{67}$$

where the divergence term has integrated to zero because of Eq. (45). But, from Eq. (63),

$$\mathbf{B}_0\cdot\mathbf{Q} = -B_0^{\,2}\,\nabla\cdot\boldsymbol{\xi}_\perp + \mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp) - \mathbf{B}_0\cdot(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)$$
$$= -B_0^{\,2}\,\nabla\cdot\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\cdot\nabla(B_0^{\,2}/2) + \mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp). \tag{68}$$

However,

$$\mathbf{B}_0\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp) = B_{0\,i}\,B_{0\,j}\,\frac{\partial\xi_{\perp\,i}}{\partial x_j} = \frac{\partial}{\partial x_j}\,(B_{0\,i}\,B_{0\,j}\,\xi_{\perp\,i}) - B_{0\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,\xi_{\perp\,i}$$

$$= \nabla\cdot[(\mathbf{B}_0\cdot\boldsymbol{\xi}_\perp)\,\mathbf{B}_0] - \boldsymbol{\xi}_\perp\cdot(\mathbf{B}_0\cdot\nabla\mathbf{B}_0) = -B_0^{\,2}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}), \tag{69}$$

where we have used Eqs. (11) and (59), and the divergence term has integrated to zero because of Eq. (31). Thus, we deduce that

$$-\boldsymbol{\eta}_\perp\cdot\nabla(\mathbf{B}_0\cdot\mathbf{Q}) = -B_0^{\,2}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp) - [\boldsymbol{\xi}_\perp\cdot\nabla(B_0^{\,2}/2) + B_0^{\,2}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})]\,(\nabla\cdot\boldsymbol{\eta}_\perp). \tag{70}$$

Hence, it follows from Eqs. (40), (53), (56), (61), (66), and (70) that

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) &= \frac{B_0^{\,2}}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{1}{\mu_0}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta})\\ &\quad - \left[\boldsymbol{\xi}_\perp\cdot\nabla\left(p_0 + \frac{B_0^{\,2}}{2\,\mu_0}\right) + \frac{B_0^{\,2}}{\mu_0}\,\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right]\nabla\cdot\boldsymbol{\eta}_\perp\\ &\quad - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp) + R,\end{aligned} \tag{71}$$

where

$$\mu_0\,R = \boldsymbol{\eta}_\perp\cdot[(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}_0 - (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot\nabla\mathbf{B}_0] + (\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}_0)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp). \tag{72}$$

However, from Eqs. (12) and (13),

$$\nabla p_0 = \mu_0^{-1}\,(\nabla\times\mathbf{B}_0)\times\mathbf{B}_0 = \mu_0^{-1}\left[(\mathbf{B}_0\cdot\nabla)\,\mathbf{B}_0 - \nabla(B_0^{\,2}/2)\right]. \tag{73}$$

Thus, Eq. (59) yields

$$\boldsymbol{\xi}_\perp\cdot\nabla\left(p_0 + \frac{B_0^{\,2}}{2}\right) = \frac{B_0^{\,2}}{\mu_0}\,\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}. \tag{74}$$

Hence, Eq. (71) simplifies to give

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) &= \frac{B_0^{\,2}}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{1}{\mu_0}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi})\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta})\\ &\quad - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{2\,B_0^{\,2}}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp) + R.\end{aligned} \tag{75}$$

Now,

$$\begin{aligned}\mu_0\,R &= \eta_{\perp\,i}\,B_{0\,k}\,\frac{\partial\xi_{\perp\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} - \eta_{\perp\,i}\,\xi_{\perp\,k}\,\frac{\partial B_{0\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} + \xi_{\perp\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,B_{0\,k}\,\frac{\partial\eta_{\perp\,i}}{\partial x_k}\\ &= \frac{\partial}{\partial x_k}\left(\eta_{\perp\,i}\,B_{0\,k}\,\xi_{\perp\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\right) - \xi_{\perp\,j}\,\frac{\partial}{\partial x_k}\left(\eta_{\perp\,i}\,B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_j}\right)\\ &\quad - \eta_{\perp\,i}\,\xi_{\perp\,k}\,\frac{\partial B_{0\,j}}{\partial x_k}\,\frac{\partial B_{0\,i}}{\partial x_j} + \xi_{\perp\,j}\,\frac{\partial B_{0\,i}}{\partial x_j}\,B_{0\,k}\,\frac{\partial\eta_{\perp\,i}}{\partial x_k}\end{aligned}$$

$$= -\eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial}{\partial x_k}\left(B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_j}\right) - \eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial B_{0\,k}}{\partial x_j}\,\frac{\partial B_{0\,i}}{\partial x_k}$$

$$= -\eta_{\perp\,i}\,\xi_{\perp\,j}\,B_{0\,k}\,\frac{\partial^2 B_{0\,i}}{\partial x_j\,\partial x_k} - \eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial B_{0\,k}}{\partial x_j}\,\frac{\partial B_{0\,i}}{\partial x_k}$$

$$= -\eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial}{\partial x_j}\left(B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_k}\right), \tag{76}$$

where the divergence term has integrated to zero because of Eq. (31), and use has been made of Eq. (11). However, from Eq. (73),

$$\mu_0^{-1}\,B_{0\,k}\,\frac{\partial B_{0\,i}}{\partial x_k} = \frac{\partial}{\partial x_i}\left(p_0 + \frac{B_0^2}{2\,\mu_0}\right). \tag{77}$$

Thus,

$$R = -\eta_{\perp\,i}\,\xi_{\perp\,j}\,\frac{\partial^2}{\partial x_i\,\partial x_j}\left(p_0 + \frac{B_0^2}{2\,\mu_0}\right) = -(\boldsymbol{\eta}_\perp\,\boldsymbol{\xi}_\perp : \nabla\nabla)\left(p_0 + \frac{B_0^2}{2\,\mu_0}\right). \tag{78}$$

Thus, it follows from Eq. (75) that

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = &-\frac{1}{\mu_0}\,(\mathbf{B}_0\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}_0\cdot\nabla\boldsymbol{\eta}_\perp) - \gamma\,p_0\,(\nabla\cdot\boldsymbol{\xi})\,(\nabla\cdot\boldsymbol{\eta}) + \frac{B_0^2}{\mu_0}\,(\nabla\cdot\boldsymbol{\xi}_\perp)\,(\nabla\cdot\boldsymbol{\eta}_\perp)\\ &-\frac{2\,B_0^2}{\mu_0}\,(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{2\,B_0^2}{\mu_0}\,(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\,(\nabla\cdot\boldsymbol{\xi}_\perp)\\ &-(\boldsymbol{\eta}_\perp\,\boldsymbol{\xi}_\perp : \nabla\nabla)\left(p_0 + \frac{B_0^2}{2\,\mu_0}\right).\end{aligned} \tag{79}$$

The self-adjointness property (36) is now obviously satisfied.

F. Boundary Conditions at Perfectly Conducting Wall

Equations (31), (35), and (41) can be combined to give

$$\mathbf{n}\cdot\boldsymbol{\xi}_\perp = 0 \tag{80}$$

at the wall. Making use of Eqs. (22), (33) and (41), we also require

$$\mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}_0) = 0 \tag{81}$$

at the wall. Now, Eqs. (31) and (80) imply that

$$\boldsymbol{\xi}_\perp\times\mathbf{B}_0 = f\,\mathbf{n} \tag{82}$$

at the wall, where f is some scalar. Thus, the boundary condition (81) becomes

$$\mathbf{n}\cdot\nabla\times(f\,\mathbf{n}) = \mathbf{n}\cdot[\nabla f\times\mathbf{n} + f\,\nabla\times\mathbf{n}) = f\,\mathbf{n}\cdot\nabla\times\mathbf{n} = 0. \tag{83}$$

Now, according to Eq. (31), the inner surface of the perfectly conducting wall must correspond to a contour of the equilibrium poloidal magnetic flux, $\psi(\mathbf{r})$. It follows that

$$\mathbf{n} = \frac{\nabla\psi}{|\nabla\psi|}. \tag{84}$$

Thus,

$$\mathbf{n}\cdot\nabla\times\mathbf{n} = \frac{\nabla\psi}{|\nabla\psi|}\cdot\left[\nabla\left(\frac{1}{|\nabla\psi|}\right)\times\nabla\psi\right] = 0. \tag{85}$$

Hence, we deduce that the boundary condition (81) is satisfied provided that the boundary condition (80) is satisfied.

G. Reality of ω^2

Consider a discrete normal mode with frequency ω and displacment $\boldsymbol{\xi}(\mathbf{r})$. Taking the scalar product of Eq. (25) with $\boldsymbol{\xi}^*(\mathbf{r})$, and integrating over the plasma volume, we obtain

$$\omega^2\int\rho_0\,|\boldsymbol{\xi}|^2\,d\mathbf{r} = -\int\boldsymbol{\xi}^*\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r}. \tag{86}$$

Likewise, taking the scalar product of the complex conjugate of Eq. (25) with $\boldsymbol{\xi}(\mathbf{r})$, and integrating over the whole plasma volume, we get

$$(\omega^2)^*\int\rho_0\,|\boldsymbol{\xi}|^2\,d\mathbf{r} = -\int\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}^*)\,d\mathbf{r}. \tag{87}$$

Here, we have made use of the fact that $[\mathbf{F}(\boldsymbol{\xi})]^* = \mathbf{F}(\boldsymbol{\xi}^*)$. Taking the difference between the previous two equations, we obtain

$$\left[\omega^2-(\omega^2)^*\right]\int\rho_0\,|\boldsymbol{\xi}|^2\,d\mathbf{r} = -\int\boldsymbol{\xi}^*\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r}+\int\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}^*)\,d\mathbf{r}. \tag{88}$$

However, the self-adjoint property of the force operator, (36), which is validated by the physical boundary condition (80), yields

$$\left[\omega^2-(\omega^2)^*\right]\int\rho_0\,|\boldsymbol{\xi}|^2\,d\mathbf{r} = 0. \tag{89}$$

Given that the integral in the previous expression is positive-definite, we deduce that

$$\omega^2 = (\omega^2)^*. \tag{90}$$

In other words, ω^2 is a real quantity. Furthermore, the fact that all of the differential operators appearing on the right-hand side of Eq. (24) are real implies that $\boldsymbol{\xi}(\mathbf{r})$ is real. More generally, $\boldsymbol{\xi}(\mathbf{r})$ is a real function multiplied by a spatially uniform complex number.

In terms of the usual definition of exponential stability, a discrete normal mode with $\omega^2 > 0$ corresponds to a pure oscillation, and would, therefore, be considered stable. Conversely, a discrete mode with $\omega^2 < 0$ has one branch that grows exponentially in time, and would, therefore, be considered unstable. Clearly, the transition from stability to instability occurs when $\omega^2 = 0$.

H. Orthogonality of Normal Modes

Consider two discrete normal modes. Let the first have the frequency ω_a and the displacement $\boldsymbol{\xi}_a(\mathbf{r})$. Let the second have the frequency ω_b and the displacement $\boldsymbol{\xi}_b(\mathbf{r})$. Let the modes satisfy the physical boundary conditions

$$\mathbf{n}\cdot\boldsymbol{\xi}_{a\,\perp} = \mathbf{n}\cdot\boldsymbol{\xi}_{b\,\perp} = 0 \tag{91}$$

at the wall. It follows from Eq. (24) that

$$\omega_a^{\,2}\,\rho_0\,\boldsymbol{\xi}_a = -\mathbf{F}(\boldsymbol{\xi}_a), \tag{92}$$

$$\omega_b^{\,2}\,\rho_0\,\boldsymbol{\xi}_b = -\mathbf{F}(\boldsymbol{\xi}_b). \tag{93}$$

Forming the scalar product of the first equation with $\boldsymbol{\xi}_b$, and the second with $\boldsymbol{\xi}_a$, taking the difference, and integrating over the plasma volume, we obtain

$$(\omega_a^{\,2}-\omega_b^{\,2})\int\rho_0\,\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}_b\,d\mathbf{r} = -\int[\boldsymbol{\xi}_b\cdot\mathbf{F}(\boldsymbol{\xi}_a)-\boldsymbol{\xi}_a\cdot\mathbf{F}(\boldsymbol{\xi}_b)]\,d\mathbf{r}. \tag{94}$$

However, the self-adjoint property of the force operator, (36), which is validated by the physical boundary conditions (91), yields

$$(\omega_a^{\,2}-\omega_b^{\,2})\int\rho_0\,\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}_b\,d\mathbf{r} = 0. \tag{95}$$

Hence, we deduce that two discrete normal modes with different frequencies are orthogonal, in the sense that

$$\int\rho_0\,\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}_b\,d\mathbf{r} = 0. \tag{96}$$

Of course, the previous proof fails if we encounter multiple distinct normal modes that share the same frequency, ω. However, any linear combination of such degenerate modes is also a valid normal mode with frequency ω, and it is always possible to form linear combinations that are mutually orthogonal. With this caveat, we can state that discrete normal modes are mutually orthogonal.

I. Energy Conservation

Taking the scalar product of the perturbed plasma equation of motion, (24), with $(1/2)\,\boldsymbol{\xi}^*$, and integrating over the plasma volume, we obtain

$$\delta K + \delta W = 0, \tag{97}$$

where

$$\delta K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\omega^2\, K(\boldsymbol{\xi}^*, \boldsymbol{\xi}), \tag{98}$$

$$K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \frac{1}{2}\int \rho_0\, |\boldsymbol{\xi}|^2\, d\mathbf{r}, \tag{99}$$

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2}\int \boldsymbol{\xi}^*\cdot\mathbf{F}(\boldsymbol{\xi})\, d\mathbf{r}. \tag{100}$$

Equation (97) is clearly an energy conservation equation. We recognize δK as the perturbed kinetic energy associated with the normal mode. (Actually, for purely growing modes it is the perturbed *kinetic energy* at $t = 0$, and for purely oscillatory modes it is the peak perturbed kinetic energy.) It follows that δW is the perturbed *potential energy* associated with the mode. (With the same caveats as for the kinetic energy.) Of course, energy is conserved because the ideal-MHD equations, (1)–(5), contain no dissipative terms.

J. Variational Formulation

Equations (97)–(99) can be rearranged to give

$$\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi})} \tag{101}$$

We wish to demonstrate that any allowable $\boldsymbol{\xi}(\mathbf{r})$ function for which ω^2 becomes an extremum satisfies the perturbed plasma equation of motion, (24). The proof follows by letting $\boldsymbol{\xi}\rightarrow \boldsymbol{\xi}+\delta\boldsymbol{\xi}$ and $\omega^2\rightarrow\omega^2+\delta\omega^2$, and setting $\delta\omega^2 = 0$ (corresponding to ω^2 being an extremum). Neglecting terms that are quadratic in small quantities, we obtain

$$\omega^2 + \delta\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \delta W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \delta W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + K(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) + K(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})}. \tag{102}$$

Rearranging the previous equation, making use of Eq. (101), and again neglecting terms that are quadratic in small quantities, we get

$$\delta\omega^2 = \frac{\delta W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \delta W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi}) - \omega^2\,[K(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) + K(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})]}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi})}. \tag{103}$$

Setting $\delta\omega^2$ to zero yields

$$\omega^2 K(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) - \delta W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \omega^2 K(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi}) + \delta W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi}) = 0. \tag{104}$$

Making use of Eqs. (99) and (100), as well as the self-adjoint property of the force operator, (36), we get

$$\int \left\{\delta\boldsymbol{\xi}^* \cdot [\omega^2 \rho_0 \,\boldsymbol{\xi} - \mathbf{F}(\boldsymbol{\xi})] + \delta\boldsymbol{\xi} \cdot [\omega^2 \rho_0 \,\boldsymbol{\xi}^* + \mathbf{F}(\boldsymbol{\xi}^*)]\right\} d\mathbf{r} = 0. \tag{105}$$

However, in order for ω^2 to be an extremum, the previous equation must hold for arbitrary $\delta\boldsymbol{\xi}$. Hence, we obtain

$$-\omega^2 \rho_0 \,\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{106}$$

which is identical to Eq. (24).

K. Energy Principle

The ideal-MHD energy principle states that if

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \geq 0 \tag{107}$$

for all allowable plasma displacements (i.e., bounded in energy and satisfying appropriate boundary conditions) then the plasma is ideally stable. In other words, there exist no normal modes with $\omega^2 < 0$. Conversely, if δW is negative for any allowable displacement then the plasma is ideally unstable. In other words, there exists at least one normal mode with $\omega^2 < 0$.

The proof of the energy principle is straightforward if one assumes that the normal modes are discrete, and form a complete set of basis functions, $\boldsymbol{\xi}_n(\mathbf{r})$, each satisfying

$$-\omega_n^2 \rho_0 \,\boldsymbol{\xi}_n = \mathbf{F}(\boldsymbol{\xi}_n). \tag{108}$$

In this case, any arbitrary trial function, $\boldsymbol{\xi}(\mathbf{r})$, can be represented as

$$\boldsymbol{\xi}(\mathbf{r}) = \sum_n a_n \,\boldsymbol{\xi}_n(\mathbf{r}). \tag{109}$$

Now, we demonstrated in Sect. I H that the normal modes are orthogonal with respect to the weight function ρ_0. Let us normalize them such that they are orthonormal with respect to this weight function:

$$\int \rho_0 \,\boldsymbol{\xi}_n^* \cdot \boldsymbol{\xi}_m \, d\mathbf{r} = \delta_{nm}. \tag{110}$$

It follows that

$$\begin{aligned}\delta W(\boldsymbol{\xi}^\ast,\boldsymbol{\xi}) &= -\frac{1}{2}\int \boldsymbol{\xi}^\ast\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r}\\ &= -\frac{1}{2}\sum_{n,m} a_n^\ast\,a_m\int \boldsymbol{\xi}_n^\ast\cdot\mathbf{F}(\boldsymbol{\xi}_m)\,d\mathbf{r} = \frac{1}{2}\sum_{n,m} a_n^\ast\,a_m\,\omega_m^{\,2}\int \rho_0\,\boldsymbol{\xi}_n^\ast\cdot\boldsymbol{\xi}_m\,d\mathbf{r}\\ &= \frac{1}{2}\sum_n |a_n|^2\,\omega_n^{\,2}. \end{aligned} \tag{111}$$

The previous equation implies that if a $\boldsymbol{\xi}(\mathbf{r})$ can be found for which $\delta W<0$ then at least one of the $\omega_n^{\,2}$ is negative, indicating instability. Conversely, if $\delta W\geq 0$ for all $\boldsymbol{\xi}(\mathbf{r})$ then all of the $\omega_n^{\,2}$ are non-negative, indicating stability.

L. Perturbed Potential Energy

According to Eqs. (13), (52) and (100), the perturbed plasma potential energy can be written

$$\delta W = \frac{1}{2}\int\left\{\boldsymbol{\xi}^\ast\cdot\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\times\mathbf{B}_0 - \mathbf{j}_0\times\mathbf{Q} - \nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\right] + \gamma\,p_0\,|\nabla\cdot\boldsymbol{\xi}|^2\right\}d\mathbf{r}. \tag{112}$$

However, Eqs. (13), (47), and (51) imply that

$$\mathbf{b}\cdot[\mathbf{j}_0\times\mathbf{Q}+\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0)] = 0. \tag{113}$$

Hence, Eq. (112) becomes

$$\delta W = \frac{1}{2}\int\left[\mu_0^{-1}\,(\nabla\times\mathbf{Q})\cdot(\boldsymbol{\xi}_\perp^\ast\times\mathbf{B}_0) - \boldsymbol{\xi}_\perp^\ast\cdot\mathbf{j}_0\times\mathbf{Q} - \boldsymbol{\xi}_\perp^\ast\cdot\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0) + \gamma\,p_0\,|\nabla\cdot\boldsymbol{\xi}|^2\right]d\mathbf{r} \tag{114}$$

Now,

$$\begin{aligned}\int(\nabla\times\mathbf{Q})\cdot(\boldsymbol{\xi}_\perp^\ast\times\mathbf{B}_0)\,d\mathbf{r} &= \int\{\nabla\cdot[\mathbf{Q}\times(\boldsymbol{\xi}_\perp^\ast\times\mathbf{B}_0)] + \mathbf{Q}\cdot\nabla\times(\boldsymbol{\xi}_\perp^\ast\times\mathbf{B}_0)\}\,d\mathbf{r}\\ &= \int|\mathbf{Q}|^2\,d\mathbf{r}, \end{aligned} \tag{115}$$

where use has been made of Eq. (46). Here, the divergence term integrates to zero because

$$\mathbf{n}\cdot\mathbf{Q}\times(\boldsymbol{\xi}_\perp^\ast\times\mathbf{B}_0) = (\mathbf{Q}\cdot\mathbf{B}_0)\,(\mathbf{n}\cdot\boldsymbol{\xi}_\perp^\ast) - (\mathbf{Q}\cdot\boldsymbol{\xi}_\perp^\ast)\,(\mathbf{n}\cdot\mathbf{B}_0) = 0 \tag{116}$$

at the wall, where use has been made of Eqs. (31) and (80). Furthermore,

$$\int\boldsymbol{\xi}_\perp^\ast\cdot\nabla(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,d\mathbf{r} = \int\{\nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,\boldsymbol{\xi}_\perp^\ast] - (\boldsymbol{\xi}_\perp\cdot\nabla p_0)\,(\nabla\cdot\boldsymbol{\xi}_\perp^\ast)\}\,d\mathbf{r}$$

$$= -\int (\boldsymbol{\xi}_\perp \cdot \nabla p_0)\,(\nabla \cdot \boldsymbol{\xi}_\perp^*)\,d\mathbf{r}. \tag{117}$$

Here, the divergence term has integrated to zero because of Eq. (80). Combining Eqs. (114), (115), and (117), we obtain the following standard expression for the perturbed potential energy

$$\delta W = \frac{1}{2}\int \left[\mu_0^{-1}\,|\mathbf{Q}|^2 - \boldsymbol{\xi}_\perp^* \cdot \mathbf{j}_0 \times \mathbf{Q} + (\boldsymbol{\xi}_\perp \cdot \nabla p_0)\,(\nabla \cdot \boldsymbol{\xi}_\perp^*) + \gamma\,p_0\,|\nabla \cdot \boldsymbol{\xi}|^2\right] d\mathbf{r}. \tag{118}$$