Pressure Flattening due to Asymmetric Magnetic Island

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I. MAGNETIC ISLAND

Let $x = r - r_s$, $X = x/W$, and $\zeta = m\,\theta - n\,\phi$, where W is the island width. The magnetic flux-surfaces of the magnetic island are contours of

$$\Omega(X,\zeta) = 8\,X^2 + \cos(\zeta - \delta^2\,\sin\zeta) - 2\sqrt{8}\,\delta\,X\,\cos\zeta + \delta^2\,\cos^2\zeta, \tag{1}$$

where $|\delta| < 1$. The X-points lie at $X = \delta/\sqrt{8}$ and $\zeta = 0,\ 2\pi$, whereas the O-point lies at $X = -\delta/\sqrt{8}$ and $\zeta = \pi$. The O-point corresponds to $\Omega = -1$, whereas the magnetic separatrix corresponds to $\Omega = 1$. The maximum width of the separatrix (in x) is W.

Let

$$Y = X - \frac{\delta}{\sqrt{8}}\,\cos\zeta, \tag{2}$$

$$\xi = \zeta - \delta^2\,\sin\zeta. \tag{3}$$

It follows that

$$\Omega(Y,\zeta) = 8\,Y^2 + \cos\xi, \tag{4}$$

The X-points lie at $Y = 0$ and $\xi = 0,\ 2\pi$, whereas the O-point lies at $Y = 0$ and $\zeta = \pi$. Moreover,

$$\zeta = \xi + 2\sum_{n=1,\infty}\left[\frac{J_n(n\,\delta^2)}{n}\right]\sin(n\,\xi), \tag{5}$$

$$\cos\zeta = -\frac{\delta^2}{2} + \sum_{n=1,\infty}\left[\frac{J_{n-1}(n\,\delta^2) - J_{n+1}(n\,\delta^2)}{n}\right]\cos(n\,\xi), \tag{6}$$

$$\sin\zeta = \frac{2}{\delta^2}\sum_{n=1,\infty}\left[\frac{J_n(n\,\delta^2)}{n}\right]\sin(n\,\xi), \tag{7}$$

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$$\cos(m\,\zeta) = m\sum_{n=1,\infty}\left[\frac{J_{n-m}(n\,\delta^2)-J_{n+m}(n\,\delta^2)}{n}\right]\cos(n\,\xi), \tag{8}$$

$$\sin(m\,\zeta) = m\sum_{n=1,\infty}\left[\frac{J_{n-m}(n\,\delta^2)+J_{n+m}(n\,\delta^2)}{n}\right]\sin(n\,\xi), \tag{9}$$

for $m > 1$.

II. PLASMA DISPLACEMENT

Outside the separatrix, we can write

$$\Omega(X,\zeta) = 8\,(X-\Xi)^2, \tag{10}$$

where $\Xi = \xi^r/W$. It follows that

$$\begin{aligned}\Xi(X,\zeta) &\simeq -\frac{[\Omega(X,\zeta)-8\,X^2-8\,\Xi^2]}{16\,X}\\ &= -\frac{\cos(\zeta-\delta^2\,\sin\zeta)+\delta^2\,\cos^2\zeta}{16\,X}+\frac{\delta}{\sqrt{8}}\,\cos\zeta+\frac{\Xi^2}{2\,X}\\ &\simeq -\frac{\cos(\zeta-\delta^2\,\sin\zeta)}{16\,X}+\frac{\delta}{\sqrt{8}}\,\cos\zeta\end{aligned} \tag{11}$$

Note that $\Xi(X,\zeta)$ is an even function of ζ. Let us write

$$\Xi(X,\zeta) = \sum_{n=0,\infty}\Xi_n(X)\,\cos(n\,\zeta). \tag{12}$$

Thus,

$$\begin{aligned}\Xi_1(X) = 2\oint\Xi(X,\zeta)\,\cos(\zeta)\,\frac{d\zeta}{2\pi} &= -\frac{1}{8\,X}\oint\cos(\zeta-\delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi}+\frac{\delta}{\sqrt{8}}\\ &= -\frac{1}{16\,X}\oint\cos(-\delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi}\\ &\quad-\frac{1}{16\,X}\oint\cos(2\,\zeta-\delta^2\,\sin\zeta)\,\cos\zeta\,\frac{d\zeta}{2\pi}+\frac{\delta}{\sqrt{8}}.\end{aligned} \tag{13}$$

But,

$$J_n(\delta^2) = \oint\cos(n\,\zeta-\delta^2\,\sin\zeta)\,\frac{d\zeta}{2\pi}, \tag{14}$$

so

$$\Xi_1(X) = -\frac{J_0(\delta^2)+J_2(\delta^2)}{16\,X}+\frac{\delta}{\sqrt{8}}, \tag{15}$$

and

$$\xi_1^r(x) = -\frac{W^2}{16\,x}\left[J_0(\delta^2)+J_2(\delta^2)\right] + \frac{W\,\delta}{\sqrt{8}}. \tag{16}$$

Thus,

$$\begin{aligned}\psi_1(x) &= \frac{r\,g}{q}\,(m-n\,q)\,\xi_1^r = -(n\,s\,g)_{r_s}\,x\,\xi_1^r\\ &= (n\,s\,g)_{r_k}\,\frac{W^2}{16}\left[J_0(\delta^2)+J_2(\delta^2)\right] - (n\,s\,g)_{r_s}\,\frac{W\,\delta}{\sqrt{8}}\,x.\end{aligned} \tag{17}$$

Now,

$$\Psi = \frac{\psi_m(0)}{(L_m^m)_{r_s}^{1/2}}, \tag{18}$$

which yields

$$\Psi = \left(\frac{W}{4}\right)^2\left(\frac{s\,g}{h\,q}\right)_{r_s}\left[J_0(\delta^2)+J_2(\delta^2)\right], \tag{19}$$

where

$$h = \frac{(L_m^m)^{1/2}}{m}. \tag{20}$$

Thus, assuming that $\psi_m(x)$ is normalized such that $\Psi = 1$,

$$\frac{\psi_m(x)}{(L_m^m)_{r_s}^{1/2}} = 1 - 2\sqrt{8}\,\frac{\delta}{J_0(\delta^2)+J_2(\delta^2)}\,\frac{x}{W}. \tag{21}$$

It follows that

$$\begin{aligned}\frac{\delta}{J_0(\delta^2)+J_2(\delta^2)} &= -\frac{W}{2\sqrt{8}\,(L_m^m)_{r_s}^{1/2}}\,\frac{d\psi_m(0)}{dr}\\ &\simeq -\left[\frac{\psi_m(r_s+W)-\psi_m(r_s-W)}{4\sqrt{8}\,(L_m^m)_{r_s}^{1/2}}\right].\end{aligned} \tag{22}$$

III. FLUX-SURFACE AVERAGE OPERATOR

Now,

$$[A,B] \equiv \left.\frac{\partial A}{\partial X}\right|_\zeta \left.\frac{\partial B}{\partial \zeta}\right|_X - \left.\frac{\partial B}{\partial X}\right|_\zeta \left.\frac{\partial A}{\partial \zeta}\right|_X. \tag{23}$$

But,

$$\left.\frac{\partial}{\partial X}\right|_\zeta = \left.\frac{\partial \Omega}{\partial X}\right|_\zeta \left.\frac{\partial}{\partial \Omega}\right|_\xi + \left.\frac{\partial \xi}{\partial X}\right|_\zeta \left.\frac{\partial}{\partial \xi}\right|_\Omega = 16\,Y \left.\frac{\partial}{\partial \Omega}\right|_\xi, \tag{24}$$

and

$$\left.\frac{\partial}{\partial \zeta}\right|_X = \left.\frac{\partial \Omega}{\partial \zeta}\right|_X \left.\frac{\partial}{\partial \Omega}\right|_\xi + \left.\frac{\partial \xi}{\partial \zeta}\right|_X \left.\frac{\partial}{\partial \xi}\right|_\Omega, \tag{25}$$

so

$$[A,B] \equiv \frac{16\,Y}{\sigma}\left(\left.\frac{\partial A}{\partial \mathit{\Omega}}\right|_{\xi}\left.\frac{\partial B}{\partial \xi}\right|_{\mathit{\Omega}} - \left.\frac{\partial B}{\partial \mathit{\Omega}}\right|_{\xi}\left.\frac{\partial A}{\partial \xi}\right|_{\mathit{\Omega}}\right), \tag{26}$$

where

$$\sigma(\xi) \equiv \frac{d\zeta}{d\xi} = 1 + 2\sum_{n=1,\infty} J_n(n\,\delta^2)\,\cos(n\,\xi). \tag{27}$$

In particular,

$$[A,\mathit{\Omega}] = -\frac{16\,Y}{\sigma}\left.\frac{\partial A}{\partial \xi}\right|_{\mathit{\Omega}}. \tag{28}$$

The flux-surface average operator, $\langle\cdots\rangle$, is the annihilator of $[A,\mathit{\Omega}]$ for arbitrary $A(s,\mathit{\Omega},\xi)$. Here, $s=+1$ for $Y>0$ and $s=-1$ for $Y<0$. It follows that

$$\langle A\rangle = \int_{\zeta_0}^{2\pi-\zeta_0}\frac{\sigma(\xi)\,A_+(\mathit{\Omega},\xi)}{\sqrt{2\,(\mathit{\Omega}-\cos\xi)}}\,\frac{d\xi}{2\pi} \tag{29}$$

for $-1\leq\mathit{\Omega}\leq 1$, and

$$\langle A\rangle = \int_{0}^{2\pi}\frac{\sigma(\xi)\,A(s,\mathit{\Omega},\xi)}{\sqrt{2\,(\mathit{\Omega}-\cos\xi)}}\,\frac{d\xi}{2\pi} \tag{30}$$

for $\mathit{\Omega}>1$. Here, $\xi_0=\cos^{-1}(\mathit{\Omega})$, and

$$A_+(\mathit{\Omega},\xi) = \frac{1}{2}\left[A(+1,\mathit{\Omega},\xi)+A(-1,\mathit{\Omega},\xi)\right]. \tag{31}$$

IV. TEMPERATURE PERTURBATION

The electron temperature in the vicinity of the island can be written

$$T_e(X,\zeta) = T_{e\,s} + s\,W\,T_{e\,s}'\,\tilde{T}(\mathit{\Omega}). \tag{32}$$

Here, $\tilde{T}(\mathit{\Omega})$ satisfies

$$\left\langle\left.\frac{\partial^2\tilde{T}}{\partial X^2}\right|_{\zeta}\right\rangle = 0, \tag{33}$$

subject to the boundary condition that

$$\tilde{T}(\mathit{\Omega}) \rightarrow |X| \tag{34}$$

as $|X|\rightarrow\infty$. It follows that

$$\frac{d}{d\mathit{\Omega}}\left(\langle Y^2\rangle\,\frac{d\tilde{T}}{d\mathit{\Omega}}\right) = 0 \tag{35}$$

subject to the boundary condition that

$$\tilde{T}(\Omega) \to \frac{\Omega^{1/2}}{\sqrt{8}} \tag{36}$$

as $\Omega \to \infty$.

Outside the separatrix,

$$\langle Y^2\rangle(\Omega) = \frac{1}{16}\int_0^{2\pi} \sigma(\xi)\sqrt{2\,(\Omega-\cos\xi)}\,\frac{d\xi}{2\pi}. \tag{37}$$

Let

$$k = \left(\frac{1+\Omega}{2}\right)^{1/2}. \tag{38}$$

Thus, the O-point corresponds to $k=0$ and the separatrix to $k=1$. It follows that

$$\langle Y^2\rangle(k) = \frac{k}{4\pi}\int_0^{\pi/2} \sigma(2\,\theta-\pi)\,(1-\sin^2\theta/k^2)^{1/2}\,d\theta. \tag{39}$$

Thus,

$$\langle Y^2\rangle(k) = \frac{k}{4\pi}\,G(1/k), \tag{40}$$

where

$$G(p) = E_0(p) + 2\,\cos(n\,\pi)\sum_{n=1,\infty} J_n(n\,\delta^2)\,E_n(p), \tag{41}$$

$$E_n(p) = \int_0^{\pi/2} \cos(2\,n\,\theta)\,(1-p^2\,\sin^2\theta)^{1/2}\,d\theta. \tag{42}$$

Equation (35) yields

$$\tilde{T}(k) = 0 \tag{43}$$

for $0\leq k\leq 1$, and

$$\frac{d}{dk}\left[G(1/k)\,\frac{d\tilde{T}}{dk}\right] = 0 \tag{44}$$

for $k>1$. Thus,

$$\frac{d\tilde{T}}{dk} = \frac{c}{G(1/k)} \tag{45}$$

for $k>1$, subject to the boundary condition that

$$\tilde{T}(k) \to \frac{k}{2} \tag{46}$$

as $k\to\infty$. In the limit that $p\to 0$,

$$E_0(p) = \frac{\pi}{2}, \tag{47}$$

$$E_{n>0}(p) = 0, \tag{48}$$

which implies that $c = \pi/4$. So

$$\frac{d\tilde{T}}{dk} = \frac{\pi}{4}\,\frac{1}{G(1/k)}, \tag{49}$$

$$\tilde{T}(k) = F(k), \tag{50}$$

$$F(k) = \frac{\pi}{4}\int_1^k \frac{dk'}{G(1/k')} \tag{51}$$

for $k > 1$.

V. HARMONICS OF TEMPERATURE PERTURBATION

We can write

$$\tilde{T}(X,\zeta) = \sum_{\nu=0,\infty} \delta T_\nu(X)\,\cos(\nu\,\zeta). \tag{52}$$

Now,

$$\delta T_0(X) = \oint \tilde{T}(X,\zeta)\,\frac{d\zeta}{2\pi}, \tag{53}$$

where the integral is at constant X. It follows that

$$\delta T_0(X) = \int_0^{\xi_c} F(k)\,\sigma(\xi)\,\frac{d\xi}{\pi}, \tag{54}$$

where

$$\xi_c = \cos^{-1}(1 - 8\,Y^2) \tag{55}$$

for $|Y| < 1/2$, and $\xi_c = \pi$ for $|Y| \geq 1/2$. Furthermore,

$$k = \left[4\,Y^2 + \cos^2\left(\frac{\xi}{2}\right)\right]^{1/2}. \tag{56}$$

Let

$$\delta T_{0\,\infty} = \lim_{X\rightarrow\infty}\left[X - \delta T_0(X)\right]. \tag{57}$$

For $\nu > 0$, we have

$$\delta T_\nu(X) = 2\oint \tilde{T}(X,\zeta)\,\cos(\nu\,\zeta)\,\frac{d\zeta}{2\pi}. \tag{58}$$

Integrating by parts, we obtain

$$\delta T_\nu(X) = -\frac{2}{\nu}\oint \left.\frac{\partial\tilde{T}}{\partial\zeta}\right|_X \sin(\nu\,\zeta)\,\frac{d\zeta}{2\pi}. \tag{59}$$

But,

$$\left.\frac{\partial\tilde{T}}{\partial\zeta}\right|_X = \frac{d\tilde{T}}{dk}\left.\frac{\partial k}{\partial\zeta}\right|_X = \frac{1}{4\,k}\frac{d\tilde{T}}{dk}\left.\frac{\partial\Omega}{\partial\zeta}\right|_X = -\frac{1}{4\,k}\frac{d\tilde{T}}{dk}\,\kappa(\xi), \tag{60}$$

where

$$\kappa(\xi) = \sin\xi\,(1-\delta^2\,\cos\zeta) - 2\sqrt{8}\,\delta\,X\,\sin\zeta + \delta^2\,\sin(2\,\zeta). \tag{61}$$

Hence,

$$\delta T_\nu(X) = \frac{1}{8\,\nu}\int_0^{\xi_c}\frac{\sin(\nu\,\zeta)\,\kappa(\xi)\,\sigma(\xi)}{k\,G(1/k)}\,d\xi. \tag{62}$$

VI. ASYMPTOTIC MATCHING

Consider the kth rational surface whose radius is r_k and whose resonant poloidal mode number is m_k. Let $x = r - r_k$ and $\zeta_k = m_k\,\theta - n\,\phi$.

In the outer region, we write the total electron temperature as

$$\tilde{T}_e(r,\theta,\phi) = T_{e\,0}(r) - \Psi_k\,\frac{q(r)}{r\,g(r)}\,\frac{T_{e\,0}'(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k}, \tag{63}$$

where $T_{e\,0}' = dT_{e\,0}/dr$, $T_{e\,0}(r)$ is the equilibrium electron temperature profile, Ψ_k is the reconnected flux, and

$$\Psi_k = \left(\frac{W_k}{4}\right)^2\left(\frac{s\,g}{h\,q}\right)_{r_k}\left[J_0(\delta_k^2) + J_2(\delta_k^2)\right], \tag{64}$$

where W_k is the island width. In the limit, $|x| \ll 1$, Eq. (63) yields

$$\tilde{T}_e(x,\theta,\phi) = T_{e\,k} + T_{e\,k}'\,x + T_{e\,k}'\,W_k\left([J_0(\delta_k^2) + J_2(\delta_k^2)]\,\frac{W_k}{16\,x} - \frac{\delta_k}{\sqrt{8}}\right)\mathrm{e}^{\,\mathrm{i}\,\zeta_k}, \tag{65}$$

Here, $T_{e\,k} = T_{e\,0}(r_k)$ and $T_{e\,k}' = (dT_{e\,0}/dr)_{r_k}$,

$$\frac{\delta_k}{J_0(\delta_k^2) + J_2(\delta_k^2)} = -\frac{W_k}{2\sqrt{8}\,(L_{m_k}^{m_k})_{r_k}^{1/2}}\,\frac{d\psi_{m_k}(r_k)}{dr}, \tag{66}$$

and we have made use of the fact that $\psi_{m_k}(r_k) = (L_{m_k}^{m_k})_{r_k}^{1/2}$.

In the inner region, we write the total electron temperature as

$$\tilde{T}_e(x,\theta,\phi) = T_{e\,k} + T_{e\,k}'\,W_k\sum_{\nu=0,\infty}\delta T_\nu(x/W_k)\,\mathrm{e}^{\,\mathrm{i}\,\nu\,\zeta_k} + T_{e\,k}'\,W_k\,\delta T_{0\,\infty}, \tag{67}$$

The asymptotic matching process consists of writing

$$\tilde{T}_e(r,\theta,\phi) = T_{e\,0}(r) + \delta T_{e\,+} - \Psi_{k+}\,\frac{q(r)}{r\,g(r)}\,\frac{T_{e\,0}'(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\,\mathrm{e}^{\,\mathrm{i}\,\zeta_k} \tag{68}$$

in the region $r > r_k + W_k$,

$$\tilde{T}_e(r,\theta,\phi) = T_{e\,0}(r) + \delta T_{e\,-} - \Psi_{k-}\,\frac{q(r)}{r\,g(r)}\,\frac{T'_{e\,0}(r)\,\psi_{m_k}(r)}{m_k - n\,q(r)}\,{\rm e}^{\,{\rm i}\,\zeta_k} \tag{69}$$

in the region $r < r_k - W_k$, and

$$\tilde{T}_e(r,\theta,\phi) = T_{e\,k} + T'_{e\,k}\,W_k \sum_{\nu=0,\infty} \delta T_\nu(x/W_k)\,{\rm e}^{\,{\rm i}\,\nu\,\zeta_k} + T'_{e\,k}\,W_k\,\delta T_{0\,\infty} \tag{70}$$

in the region $r_k - W_k \leq r \leq r_k + W_k$. Continuity of the solution at $r = r_k \pm W_k$ implies that

$$\delta T_{e\,+} = T'_{e\,k}\,W_k\,\delta T_0(1) + T'_{e\,k}\,W_k\,\delta T_{0\,\infty} - T'_{e\,k}\,W_k, \tag{71}$$

$$\delta T_{e\,-} = T'_{e\,k}\,W_k\,\delta T_0(-1) + T'_{e\,k}\,W_k\,\delta T_{0\,\infty} + T'_{e\,k}\,W_k, \tag{72}$$

$$\Psi_{k\,+} = -T'_{e\,k}\,W_k\,\delta T_1(1)\left(\frac{r\,g}{q}\,\frac{m_k - n\,q}{T'_{e\,0}\,\psi_{m\,k}}\right)_{r_k+W_k}, \tag{73}$$

$$\Psi_{k\,-} = -T'_{e\,k}\,W_k\,\delta T_1(-1)\left(\frac{r\,g}{q}\,\frac{m_k - n\,q}{T'_{e\,0}\,\psi_{m\,k}}\right)_{r_k-W_k}. \tag{74}$$

Finally, for the special case $m = 1$, we write

$$\tilde{T}_e(r,\theta,\phi) = -\xi^r(r,\theta,\phi)\,\frac{dT_{e\,0}}{dr}. \tag{75}$$

VII. NORMALIZED QUANTITIES

Let $\hat{r} = r/\epsilon_a$, $\hat{r}_k = r_k/\epsilon_a$, $\hat{x} = x/\epsilon_a$, $\hat{T}'_{e\,0} = \epsilon_a\,T'_{e\,0}$, $\hat{T}'_{e\,k} = \epsilon_a\,\hat{T}'_{e\,k}$, $\hat{W}_k = W_k/\epsilon_a$, and $\hat{\Psi}_k = \Psi_k/\epsilon_a^{\,2}$, etc., then

$$\tilde{T}_e(\hat{r},\theta,\phi) = T_{e\,0}(\hat{r}) + \delta T_{e\,+} - \hat{\Psi}_{k+}\,\frac{q(\hat{r})}{\hat{r}\,g(\hat{r})}\,\frac{\hat{T}'_e(r)\,\psi_{m_k}(r)}{m_k - n\,q(\hat{r})}\,{\rm e}^{\,{\rm i}\,\zeta_k} \tag{76}$$

in the region $\hat{r} > \hat{r}_k + \hat{W}_k$,

$$\tilde{T}_e(\hat{r},\theta,\phi) = T_{e\,0}(\hat{r}) + \delta T_{e\,-} - \hat{\Psi}_{k-}\,\frac{q(\hat{r})}{\hat{r}\,g(\hat{r})}\,\frac{\hat{T}'_e(r)\,\psi_{m_k}(r)}{m_k - n\,q(\hat{r})}\,{\rm e}^{\,{\rm i}\,\zeta_k} \tag{77}$$

in the region $\hat{r} < \hat{r}_k - \hat{W}_k$, and

$$\tilde{T}_e(\hat{r},\theta,\phi) = T_{e\,k} + \hat{T}'_{e\,k}\,\hat{W}_k \sum_{\nu=0,\infty} \delta T_\nu(\hat{x}/\hat{W}_k)\,{\rm e}^{\,{\rm i}\,\nu\,\zeta_k} + \hat{T}'_{e\,k}\,\hat{W}_k\,\delta T_{0\,\infty} \tag{78}$$

in the region $\hat{r}_k - \hat{W}_k \leq \hat{r} \leq \hat{r}_k + \hat{W}_k$. Here,

$$\delta T_{e\,+} = \hat{T}'_{e\,k}\,\hat{W}_k\,\delta T_0(1) + \hat{T}'_{e\,k}\,\hat{W}_k\,\delta T_{0\,\infty} - \hat{T}'_{e\,k}\,\hat{W}_k, \tag{79}$$

$$\delta T_{e\,-} = \hat{T}'_{e\,k}\,\hat{W}_k\,\delta T_0(-1) + \hat{T}'_{e\,k}\,\hat{W}_k\,\delta T_{0\,\infty} + \hat{T}'_{e\,k}\,\hat{W}_k, \tag{80}$$

$$\hat{\Psi}_k = \left(\frac{\hat{W}_k}{4}\right)^2 \left(\frac{g\,s}{h\,q}\right)_{\hat{r}_k} [J_0(\delta_k^2) + J_2(\delta_k^2)], \tag{81}$$

$$\hat{\Psi}_{k\,+} = -\hat{T}'_{e\,k}\,\hat{W}_k\,\delta T_1(1)\left(\frac{\hat{r}\,g}{q}\,\frac{m_k - n\,q}{\hat{T}'_{e\,0}\,\psi_{m\,k}}\right)_{\hat{r}_k+\hat{W}_k}, \tag{82}$$

$$\hat{\Psi}_{k\,-} = -\hat{T}'_{e\,k}\,\hat{W}_k\,\delta T_1(-1)\left(\frac{\hat{r}\,g}{q}\,\frac{m_k - n\,q}{\hat{T}'_{e\,0}\,\psi_{m\,k}}\right)_{\hat{r}_k-\hat{W}_k}. \tag{83}$$

For the special case $m = 1$,

$$\tilde{T}_e(\hat{r},\theta,\phi) = -\frac{\xi^r(\hat{r},\theta,\phi)}{\epsilon_a}\,\frac{dT_{e\,0}}{d\hat{r}}. \tag{84}$$

VIII. RELATIVISTIC DOWNSHIFTING AND BROADENING

Neglecting doppler broadening, the angular frequency of an nth harmonic electron cyclotron emission (ece) signal is

$$\omega = \frac{n\,\Omega_0\,R_0}{R}\left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2}, \tag{85}$$

where v is the electron speed, and $\Omega_0 = e\,B_0/m_e$. Here, we are neglecting the poloidal magnetic field-strength, and assuming that the toroidal field-strength falls off like $1/R$. Let

$$R_\omega(\omega) = \frac{n\,\Omega_0\,R_0}{\omega} \tag{86}$$

be the major radius from which the ece of frequency ω is emitted in the absence of relativistic downshifting and broadening. We can write

$$\frac{v}{c} = \begin{cases} \left[1 - \left(\frac{R}{R_\omega}\right)^2\right]^{1/2} & R \leq R_\omega \\ 0 & R > R_\omega \end{cases}. \tag{87}$$

Now, the distribution of electron speeds is

$$f(v) = A\,v^2\left(-\frac{1}{\theta_\omega}\left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}\right), \tag{88}$$

where

$$\theta_\omega(\omega) = \frac{T_e(R_\omega)}{m_e\,c^2}. \tag{89}$$

Thus, we can define

$$F(R,R_\omega) = \left[1-\left(\frac{R}{R_\omega}\right)^2\right]\exp\!\left(-\frac{1}{\theta_\omega}\,\frac{R_\omega}{R}\right). \tag{90}$$

The electron temperature measured by the ece diagnostic is

$$T_e(R_\omega) = \frac{\int_{R_{\rm min}}^{R_\omega} T_e(R)\,F(R,R_\omega)\,dR}{\int_{R_{\rm min}}^{R_\omega} F(R,R_\omega)\,dR}. \tag{91}$$

IX. ELECTRON CYCLOTRON CURRENT DRIVE

The stabilizing electron cyclotron current drive contribution to the Rutherford equation is

$$\varDelta_{eccd}\,W = -\frac{16}{W}\int_{-1}^{\infty} J_+(\varOmega)\,\langle\cos\zeta\rangle\,d\varOmega, \tag{92}$$

where $J(x,\zeta)$ is the driven current density. Suppose that

$$J(x,\zeta) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\exp\!\left[-\frac{(x-d)^2}{2\,\sigma^2}\right]\left[\frac{1-\cos(\zeta-\varDelta\zeta)}{2}\right]. \tag{93}$$

However, only the component of $J(x,\zeta)$ that is even in ζ contributes to $\varDelta_{eccd}$, so

$$J(x,\zeta) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\exp\!\left[-\frac{(x-d)^2}{2\,\sigma^2}\right]\left(\frac{1-\cos\zeta\,\cos\varDelta\zeta}{2}\right). \tag{94}$$

Let $\hat{\sigma}=\sigma/W$ and $\hat{d}=d/W$. It follows that

$$J(s,Y,\zeta) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\exp\!\left[-\frac{(s\,Y+\delta\,\cos\zeta/\sqrt{8}-\hat{d})^2}{2\,\hat{\sigma}^2}\right]\left(\frac{1-\cos\zeta\,\cos\varDelta\zeta}{2}\right). \tag{95}$$

Note that

$$Y = \frac{\sqrt{k^2-\cos^2(\xi/2)}}{2}. \tag{96}$$

Thus,

$$J_+(Y,\zeta) = \frac{J(1,Y,\zeta)+J(-1,Y,\zeta)}{2}. \tag{97}$$

Moreover,

$$J_+(\varOmega) = \frac{\langle J_+(Y,\zeta)\rangle}{\langle 1\rangle}. \tag{98}$$

Thus,

$$\varDelta_{eccd}\,W = -64\int_0^\infty \frac{\langle J_+\rangle\,\langle\cos\zeta\rangle}{\langle 1\rangle}\,k\,dk. \tag{99}$$

Finally,

$$\langle A(k,\xi)\rangle(k) = \frac{1}{\pi}\int_0^{\pi/2} \frac{\sigma(\xi)\,A(k,\xi)}{\sqrt{1-k^2\,\sin^2\theta}}\,d\theta \tag{100}$$

for $0 \leq k \leq 1$, where $\xi = 2\,\cos^{-1}(k\,\sin\theta)$. Likewise,

$$\langle A(k,\xi)\rangle(k) = \frac{1}{\pi}\int_0^{\pi/2} \frac{\sigma(\xi)\,A(k,\xi)}{\sqrt{k^2-\sin^2\theta}}\,d\theta \tag{101}$$

for $k > 1$, where $\xi = \pi - 2\,\theta$.