Program FLUX

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1 Construction of Flux Coordinate System

Let us adopt a normalization scheme in which all lengths are normalized to R_0, all magnetic field-strengths to B_0, and all pressures to B_0^2/μ_0. In the following, all quantities are assumed to be normalized.

Let R, ϕ, Z be conventional right-handed cylindrical coordinates, such that

$$\nabla R\times\nabla\phi\cdot\nabla Z = \frac{1}{R}, \tag{1}$$

where $|\nabla\phi| = 1/R$.

We can write the equilibrium magnetic field in the form

$$\mathbf{B} = \nabla\phi\times\nabla\psi_p + g(\psi_p)\,\nabla\phi = \nabla(\phi - q\,\theta)\times\nabla\psi_p, \tag{2}$$

where the poloidal flux, $\psi_p(R,Z)$, and the toroidal flux function, $g(\psi_p)$, are both given. Furthermore,

$$\nabla\psi_p\times\nabla\theta\cdot\nabla\phi = \frac{g}{q\,R^2}, \tag{3}$$

where $q = q(\psi_p)$. Here, θ is a so-called "straight" poloidal angle, and $q(\psi_p)$ is the safety-factor.

Let $\Psi = \psi_p/\psi_c = 1-\Psi_N$, where ψ_c is the value of ψ_p on the magnetic axis. (It is assumed that $\psi_p = 0$ on the magnetic separatrix.) Thus, $\Psi = 1$ on the magnetic axis, and $\Psi = 0$ on the magnetic separatrix. The previous equation implies that

$$\frac{d\theta}{dl} = \frac{g}{q}\,\frac{1}{|\psi_c|\,R\,\sqrt{\Psi_R^2+\Psi_Z^2}}, \tag{4}$$

where dl is an element of poloidal path-length along a constant-Ψ surface, and $\Psi_R\equiv\partial\Psi/\partial R$, et cetera. Furthermore,

$$dR = -\frac{\Psi_Z\,dl}{\sqrt{\Psi_R^2+\Psi_Z^2}}, \tag{5}$$

$$dZ = \frac{\Psi_R\,dl}{\sqrt{\Psi_R^2+\Psi_Z^2}}. \tag{6}$$

It follows that

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{2\pi\,|\psi_c|}\oint\frac{dl}{R\,\sqrt{\Psi_R^2+\Psi_Z^2}}. \tag{7}$$

If we define

$$\tan\zeta = \frac{Z-Z_{\rm axis}}{R_{\rm axis}-R} \tag{8}$$

then

$$\frac{dR}{d\zeta} = -\Psi_Z\,F, \tag{9}$$

$$\frac{dZ}{d\zeta} = \Psi_R\,F, \tag{10}$$

$$\frac{q(\Psi)}{g(\Psi)} = \frac{1}{|\psi_c|}\oint\frac{F}{R}\,\frac{d\zeta}{2\pi}, \tag{11}$$

where

$$F = \frac{(R_{\rm axis}-R)^{\,2}+(Z-Z_{\rm axis})^{\,2}}{-(Z-Z_{\rm axis})\,\Psi_Z+(R_{\rm axis}-R)\,\Psi_R}. \tag{12}$$

Note that $\zeta=0$ on the inboard mid-plane.

It is helpful to define the length-like flux-surface label $r(\Psi)$, such that

$$\nabla r\times\nabla\theta\cdot\nabla\phi = \frac{1}{r\,R^{2}}. \tag{13}$$

It follows that

$$r(\Psi) = \left[2\,|\psi_c|\int_{\Psi}^{1}\frac{q(\Psi')}{g(\Psi')}\,d\Psi'\right]^{1/2}. \tag{14}$$

Let

$$a = r(\Psi_v), \tag{15}$$

where Ψ_v is the value of Ψ at the plasma/vacuum boundary. Note that $\Psi_v>0$ and $f(r) = d\psi_p/dr$.

We can calculate $R(r,\theta)$ and $Z(r,\theta)$ by integrating

$$\frac{dR}{d\theta} = -|\psi_c|\,\frac{q}{g}\,R\,\Psi_Z, \tag{16}$$

$$\frac{dZ}{d\theta} = |\psi_c|\,\frac{q}{g}\,R\,\Psi_R, \tag{17}$$

along constant-r surfaces. Here, $\theta=0$ on the inboard mid-plane.

2 Profile Functions

Program TJ also needs the following profile functions:

$$s(r) = r\,\frac{d\ln q}{dr}, \tag{18}$$

$$\alpha_g(r) = \frac{dg}{d\psi_p}, \tag{19}$$

$$\alpha_p(r) = \frac{q}{g}\,\frac{dP}{d\psi_p}, \tag{20}$$

$$\alpha_f(r) = -r\,\frac{d\ln(q/g)}{dr}, \tag{21}$$

$$\alpha_g'(r) = \frac{d\alpha_g}{dr}, \tag{22}$$

$$\alpha_p'(r) = \frac{d\alpha_p}{dr}. \tag{23}$$

where $P(r)$ is the equilibrium pressure profile.

3 Metric Elements

Let $R_r = \partial R/\partial r|_\theta$, et cetera. It follows that

$$R_\theta = -|\psi_c|\,\frac{q}{g}\,R\,\Psi_Z, \tag{24}$$

$$Z_\theta = |\psi_c|\,\frac{q}{g}\,R\,\Psi_R, \tag{25}$$

$$r\,R = R_\theta\,Z_r - R_r\,Z_\theta, \tag{26}$$

$$|\nabla r|^{-2} = \frac{r^2\,R^2}{R_\theta^2 + Z_\theta^2}, \tag{27}$$

$$\frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2} = -\frac{r\,(R_r\,R_\theta + Z_r\,Z_\theta)}{R_\theta^2 + Z_\theta^2}. \tag{28}$$

Let

$$a(r,\theta) = R^2, \tag{29}$$

$$b(r,\theta) = |\nabla r|^{-2}\,R^{-2}, \tag{30}$$

$$c(r,\theta) = |\nabla r|^{-2}, \tag{31}$$

$$d(r,\theta) = |\nabla r|^{-2}\,R^2, \tag{32}$$

$$e(r,\theta) = |\nabla r|^{-2}\,R^4, \tag{33}$$

$$f(r,\theta) = \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}, \tag{34}$$

$$g(r,\theta) = \frac{r\,\nabla r\cdot\nabla\theta}{|\nabla r|^2}\,R^2. \tag{35}$$

Program TJ requires the following metric functions:

$$a_j^c(r) = \oint a(r,\theta)\,\cos(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{36}$$

$$b_j^c(r) = \oint b(r,\theta)\,\cos(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{37}$$

$$c_j^c(r) = \oint c(r,\theta)\,\cos(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{38}$$

$$d_j^c(r) = \oint d(r,\theta)\,\cos(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{39}$$

$$e_j^c(r) = \oint e(r,\theta)\,\cos(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{40}$$

$$f_j^c(r) = \oint f(r,\theta)\,\cos(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{41}$$

$$g_j^c(r) = \oint g(r,\theta)\,\cos(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{42}$$

for $j = 0, J$,

$$a_j^s(r) = \oint a(r,\theta)\,\sin(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{43}$$

$$b_j^s(r) = \oint b(r,\theta)\,\sin(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{44}$$

$$c_j^s(r) = \oint c(r,\theta)\,\sin(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{45}$$

$$d_j^s(r) = \oint d(r,\theta)\,\sin(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{46}$$

$$e_j^s(r) = \oint e(r,\theta)\,\sin(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{47}$$

$$f_j^s(r) = \oint f(r,\theta)\,\sin(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{48}$$

$$g_j^s(r) = \oint g(r,\theta)\,\sin(j\,\theta)\,\frac{d\theta}{2\pi}, \tag{49}$$

for $j = 1, J$. The program also requires da_j^c/dr, for $j = 0, J$, da_j^s/dr, for $j = 1, J$, dc_0^c/dr, c_0^s/dr, dd_0^c/dr, and dd_0^s/dr.

4 Neoclassical Coordinate System

The neoclassical poloidal angle is defined

$$\mathbf{b}\cdot\nabla\Theta = \frac{1}{\gamma(\Psi)}. \tag{50}$$

It follows that

$$\frac{d\Theta}{dl} = \frac{B\,R}{|\psi_c|\,\gamma\,\sqrt{\Psi_R^{\,2}+\Psi_Z^{\,2}}}, \tag{51}$$

where

$$B\,R = \left[g^{\,2} + |\psi_c|^{\,2}\,(\Psi_R^{\,2}+\Psi_Z^{\,2})\right]^{1/2}. \tag{52}$$

Hence,

$$\gamma(\Psi) = \frac{1}{2\pi\,|\psi_c|}\oint\frac{B\,R\,dl}{\sqrt{\Psi_R^{\,2}+\Psi_Z^{\,2}}} = \frac{1}{|\psi_c|}\oint B\,R\,F\,\frac{d\zeta}{2\pi}. \tag{53}$$

We can calculate $R(r,\Theta)$ and $Z(r,\Theta)$ from

$$\frac{dR}{d\Theta} = -|\psi_c|\,\frac{\gamma\,\Psi_Z}{B\,R}, \tag{54}$$

$$\frac{dZ}{d\Theta} = |\psi_c|\,\frac{\gamma\,\Psi_R}{B\,R}. \tag{55}$$

Note that

$$\frac{d\theta}{d\Theta} = \left(\frac{\gamma\,g}{q}\right)\frac{1}{B\,R^2}. \tag{56}$$

5 Vacuum Solution

The plasma/vacuum interface lie at $\Psi=\Psi_v$. Let us solve

$$\nabla^2\Phi = 0 \tag{57}$$

subject to the boundary conditions $\Phi=0$ on the plasma/vacuum boundary, and $\Phi=1$ on the bounding box. Let

$$\nabla\Phi\times\nabla\hat{\theta}\cdot\nabla\phi = \frac{1}{\Gamma(\Phi)\,R^2}. \tag{58}$$

It follows that

$$\frac{d\hat{\theta}}{dl} = \frac{1}{\Gamma\,R\,\sqrt{\Phi_R^{\,2}+\Phi_Z^{\,2}}}. \tag{59}$$

$$\frac{dR}{d\zeta} = \Phi_Z\,\hat{F}, \tag{60}$$

$$\frac{dZ}{d\zeta} = -\Phi_R\,\hat{F}, \tag{61}$$

$$\varGamma = \oint \frac{\hat{F}}{R}\,\frac{d\zeta}{2\pi}, \tag{62}$$

where

$$\hat{F} = \frac{(R_{\rm axis}-R)^{\,2}+(Z-Z_{\rm axis})^{\,2}}{(Z-Z_{\rm axis})\,\varPhi_Z-(R_{\rm axis}-R)\,\varPhi_R}. \tag{63}$$

We wish to define the length-like flux surface label $\hat{r}(\varPhi)$ such that

$$\nabla\hat{r}\times\nabla\hat{\theta}\,\cdot\nabla\phi = \frac{1}{r\,R^{\,2}}, \tag{64}$$

and $\hat{r}=a$ on the plasma/vacuum boundary. It follows that

$$\hat{r}^{\,2} = a^2 + 2\int_0^{\varPhi}\varGamma(\varPhi')\,d\varPhi'. \tag{65}$$

We can calculate $R(\hat{r},\hat{\theta})$ and $Z(\hat{r},\hat{\theta})$ by integrating

$$\frac{dR}{d\hat{\theta}} = \varGamma\,R\,\varPhi_Z, \tag{66}$$

$$\frac{dZ}{d\hat{\theta}} = -\varGamma\,R\,\varPhi_R, \tag{67}$$

along constant-$\hat{r}$ surfaces. Here, $\hat{\theta}=0$ on the inboard mid-plane.