# Magnetic Perturbations

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## I. MAGNETIC PERTURBATIONS IN FLUX COORDINATES

In the $r$, $\theta$, $\phi$ flux coordinate system (where all lengths are normalized to $R_0$, and all magnetic field-strengths to $B_0$), the perturbed magnetic field is written

$$\mathbf{b} = b^r\,\mathcal{J}\,\nabla\theta\times\nabla\phi + b^\theta\,\mathcal{J}\,\nabla\phi\times\nabla r + b^\phi\,\mathcal{J}\,\nabla r\times\nabla\theta. \tag{1}$$

T7 Eqs. (25) and (49) yield

$$b^r = \frac{1}{r\,R^2}\left(\frac{\partial}{\partial\theta} - {\rm i}\,n\,q\right)y, \tag{2}$$

$$b^\phi = \frac{x}{R^2}, \tag{3}$$

whereas TJ Eqs. (78), (79), (80), (98), and (99) imply that

$$y(r,\theta) = \sum_m \frac{\psi_m(r)}{m - n\,q(r)}\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{4}$$

$$x(r,\theta) = n\sum_m \frac{Z_m(r) + k_m(r)\,\psi_m(r)}{m - n\,q(r)}\,{\rm e}^{\,{\rm i}\,m\,\theta}. \tag{5}$$

It follows that

$$R^2\,b^r = \frac{\rm i}{r}\sum_m \psi_m\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{6}$$

$$R^2\,b^\phi = n\sum_m z_m\,{\rm e}^{\,{\rm i}\,m\,\theta}, \tag{7}$$

where

$$z_m = \frac{Z_m + k_m\,\psi_m}{m - n\,q}. \tag{8}$$

Now, TJ Eqs. (2) and (A10) yield

$$\mathcal{J}\,\nabla\cdot\mathbf{b} = \frac{\partial}{\partial r}\left(r\,R^2\,b^r\right) + \frac{\partial}{\partial\theta}\left(r\,R^2\,b^\theta\right) - {\rm i}\,n\,r\,R^2\,b^\phi = 0, \tag{9}$$

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so

$$\frac{\partial}{\partial\theta}\,(r^2\,R^2\,b^\theta) = \mathrm{i}\sum_m\left[-r\,\frac{d\psi_m}{dr}+\frac{n^2\,r^2\,(Z_m+k_m\,\psi_m)}{m-n\,q}\right]\mathrm{e}^{\,\mathrm{i}\,m\,\theta}. \tag{10}$$

But, TJ Eq. (102) gives

$$r\,\frac{d\psi_m}{dr} = \sum_{m'}\frac{L_m^{m'}\,Z_{m'}+M_m^{m'}\,\psi_{m'}}{m'-n\,q}, \tag{11}$$

so

$$\frac{\partial}{\partial\theta}\,(r^2\,R^2\,b^\theta) = -\mathrm{i}\sum_m\chi_m\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}, \tag{12}$$

where

$$\chi_m(r) = \sum_{m'}\frac{\chi_m^{m'}}{m'-n\,q}, \tag{13}$$

$$\chi_m^{m'}(r) = (L_m^{m'}-n^2\,r^2\,\delta_m^{m'})\,Z_{m'}+(M_m^{m'}-n^2\,r^2\,\delta_m^{m'}\,k_{m'})\,\psi_{m'} \tag{14}$$

Note from TJ Eqs. (100), (262), (263), (266), (267), (268), and (269) that $\chi_0^{m'}=0$ for all $m'$, which implies that $\chi_0(r)=0$. Thus,

$$r\,R^2\,b^\theta = -\frac{1}{r}\sum_m^{m\neq 0}\hat{\chi}_m\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}, \tag{15}$$

where

$$\hat{\chi}_m(r) = \frac{\chi_m}{m}. \tag{16}$$

According to Eqs. (4), (26), (79), and (98) of TJ,

$$\xi^r = \frac{q}{r\,g}\sum_m\hat{\psi}_m\,\mathrm{e}^{\,\mathrm{i}\,m\,\theta}, \tag{17}$$

where

$$\hat{\psi}_m(r) = \frac{\psi_m}{m-n\,q}. \tag{18}$$

## II. REGULARIZATION

In the vicinity of the $k$th rational surface, whose resonant poloidal mode number is $m_k$, the resonant harmonic of the displacement varies as

$$\xi^r_{m_k}(x) = \frac{q}{r\,g}\,\frac{\psi_{m_k}\,\Psi_k}{m_k-n\,q}\simeq-\left(\frac{q}{g\,s}\right)_{r_k}\frac{\Psi_k}{x} = -\frac{W_k^{\,2}}{16\,x}, \tag{19}$$

where $x = r - r_k$ , $\Psi_k$ is the reconnected magnetic flux at the $k$th rational surface, and

$$W_k = 4\left(\frac{q}{g\,s}\right)_{r_k}^{1/2} \Psi_k^{1/2} \tag{20}$$

is the corresponding magnetic island width. Here, we have made use of the fact that $\psi_{m_k}(r_k) \simeq m_k$. We wish $\xi^r_{m_k}(x)$ to be zero at $x = 0$, and to reach a maximum value of $-W_k/\sqrt{8}$ at $x = W_k/\sqrt{8}$. The function

$$F(y) = \frac{y^3}{3/4 - (5/4)\,y^2 + y^4} \tag{21}$$

is zero at $y = 0$, attains its maximum value 2 at $x = 1$, and is approximately $1/y$ for $y \gg 1$. Thus, we can write

$$\xi^r_{m_k}(r) = \frac{q}{r\,g}\,\frac{\psi_{m_k}\,\Psi_k\,(m_k - n\,q)^3}{(3/4)\,\delta_k^4 - (5/4)\,\delta_k^2\,(m_k - n\,q)^2 + (m_k - n\,q)^4}, \tag{22}$$

where

$$\delta_k = \frac{m_k\,s(r_k)}{\sqrt{8}\,\hat{r}_k}\,\frac{W_k}{\epsilon_a}. \tag{23}$$

Note that

$$\frac{\Psi_k}{\epsilon_a} = \epsilon_a \left(\frac{g\,s}{q}\right)_{r_k} \left(\frac{W_k}{4\,\epsilon_a}\right)^2. \tag{24}$$

For the special case of $m_k = 1$, we expect

$$\Delta\Psi_k = E_{kk}\,\Psi_k = -\frac{r_k}{m_k}\,A_{S\,k}, \tag{25}$$

so that

$$\Psi_k\,\psi_{m_k}(x) \simeq \frac{m_k\,E_{kk}\,\Psi_k}{r_k}\,x \tag{26}$$

for $x < 0$. Thus,

$$\xi^r_{m_k} \simeq -\left(\frac{q}{g\,s}\right)_{r_k} \frac{E_{kk}\,\Psi_k}{r_k} \tag{27}$$

for $x < 0$. Hence, we can write

$$\frac{\Psi_k}{\epsilon_a} = -\epsilon_a \left(\frac{\hat{r}\,g\,s}{q}\right)_{r_k} \frac{1}{E_{kk}} \left(\frac{\xi_c}{\epsilon_a}\right), \tag{28}$$

where $\xi_c$ is the plasma displacement in the core. We can also write

$$\xi^r_{m_k}(r) = \frac{q}{r\,g}\,\frac{\psi_{m_k}\,\Psi_k\,(m_k - n\,q)}{\delta_k^2 + (m_k - n\,q)^2}, \tag{29}$$

where

$$\delta_k = \frac{m_k\,s(r_k)}{2\,\hat{r}_k}\,\frac{W_k}{\epsilon_a}. \tag{30}$$

## III. MAGNETIC PERTURBATIONS IN CYLINDRICAL COORDINATES

We can write the perturbed magnetic field associated with the tearing mode that reconnects magnetic flux at the $k$th rational surface as

$$b^R \equiv \mathbf{b}\cdot\nabla R = b^r\,\frac{\partial R}{\partial r} + b^\theta\,\frac{\partial R}{\partial\theta}, \tag{31}$$

$$b^Z \equiv \mathbf{b}\cdot\nabla Z = b^r\,\frac{\partial Z}{\partial r} + b^\theta\,\frac{\partial Z}{\partial\theta}, \tag{32}$$

$$R\,b^\phi \equiv R\,\mathbf{b}\cdot\nabla\phi = R\,b^\phi. \tag{33}$$

It follows that

$$b^R = \frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\sum_m \mathrm{Re}\left[\left(\frac{1}{\epsilon_a}\,\frac{\partial R}{\partial\hat{r}}\,\mathrm{i}\,\psi_m - \frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial R}{\partial\theta}\,\hat{\chi}_m\right)\mathrm{e}^{\,\mathrm{i}\,(m\,\theta-n\,\phi)}\right], \tag{34}$$

$$b^Z = \frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\sum_m \mathrm{Re}\left[\left(\frac{1}{\epsilon_a}\,\frac{\partial Z}{\partial\hat{r}}\,\mathrm{i}\,\psi_m - \frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial Z}{\partial\theta}\,\hat{\chi}_m\right)\mathrm{e}^{\,\mathrm{i}\,(m\,\theta-n\,\phi)}\right], \tag{35}$$

$$R\,b^\phi = \frac{n\,\Psi_k}{R}\sum_m \mathrm{Re}\left[z_m\,\mathrm{e}^{\,\mathrm{i}\,(m\,\theta-n\,\phi)}\right]. \tag{36}$$

Thus,

$$b^R(r,\theta,\phi) = b^R_C(r,\theta)\,\cos(n\,\phi) + b^R_S(r,\theta)\,\sin(n\,\phi), \tag{37}$$

$$b^Z(r,\theta,\phi) = b^Z_C(r,\theta)\,\cos(n\,\phi) + b^Z_S(r,\theta)\,\sin(n\,\phi), \tag{38}$$

$$R\,b^\phi(r,\theta,\phi) = b^\phi_C(r,\theta)\,\cos(n\,\phi) + b^\phi_S(r,\theta)\,\sin(n\,\phi), \tag{39}$$

$$\xi^r(r,\theta,\phi) = \xi^r_C(r,\theta)\,\cos(n\,\phi) + \xi^r_S\,\sin(n\,\phi), \tag{40}$$

where

$$b^R_C(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial R}{\partial\hat{r}}\sum_m\left[\mathrm{Re}(\psi_m)\,\sin(m\,\theta) + \mathrm{Im}(\psi_m)\,\cos(m\,\theta)\right]\right. \tag{41}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial R}{\partial\theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\cos(m\,\theta) - \mathrm{Im}(\hat{\chi}_m)\,\sin(m\,\theta)\right]\right\}, \tag{42}$$

$$b^R_S(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial R}{\partial\hat{r}}\sum_m\left[-\mathrm{Re}(\psi_m)\,\cos(m\,\theta) + \mathrm{Im}(\psi_m)\,\sin(m\,\theta)\right]\right. \tag{43}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial R}{\partial\theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\sin(m\,\theta) + \mathrm{Im}(\hat{\chi}_m)\,\cos(m\,\theta)\right]\right\}, \tag{44}$$

$$b_C^Z(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial Z}{\partial r}\sum_m\left[\mathrm{Re}(\psi_m)\,\sin(m\,\theta)+\mathrm{Im}(\psi_m)\,\cos(m\,\theta)\right]\right. \tag{45}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial Z}{\partial\theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\cos(m\,\theta)-\mathrm{Im}(\hat{\chi}_m)\,\sin(m\,\theta)\right]\right\}, \tag{46}$$

$$b_S^Z(r,\theta) = -\frac{\Psi_k}{\epsilon_a\,\hat{r}\,R^2}\left\{\frac{1}{\epsilon_a}\frac{\partial Z}{\partial \hat{r}}\sum_m\left[-\mathrm{Re}(\psi_m)\,\cos(m\,\theta)+\mathrm{Im}(\psi_m)\,\sin(m\,\theta)\right]\right. \tag{47}$$

$$\left.+\frac{1}{\epsilon_a\,\hat{r}}\,\frac{\partial Z}{\partial\theta}\sum_{m\neq 0}\left[\mathrm{Re}(\hat{\chi}_m)\,\sin(m\,\theta)+\mathrm{Im}(\hat{\chi}_m)\,\cos(m\,\theta)\right]\right\}, \tag{48}$$

$$b_C^\phi(r,\theta) = \frac{n\,\Psi_k}{R}\sum_m\left[\mathrm{Re}(z_m)\,\cos(m\,\theta)-\mathrm{Im}(z_m)\,\sin(m\,\theta)\right], \tag{49}$$

$$b_S^\phi(r,\theta) = \frac{n\,\Psi_k}{R}\sum_m\left[\mathrm{Re}(z_m)\,\sin(m\,\theta)+\mathrm{Im}(z_m)\,\cos(m\,\theta)\right], \tag{50}$$

$$\xi_C^r(r,\theta) = \frac{\Psi_k\,q}{\epsilon_a\,\hat{r}\,g}\sum_m\left[\mathrm{Re}(\hat{\psi}_m)\,\cos(m\,\theta)-\mathrm{Im}(\hat{\psi}_m)\,\sin(m\,\theta)\right], \tag{51}$$

$$\xi_S^r(r,\theta) = \frac{\Psi_k\,q}{\epsilon_a\,\hat{r}\,g}\sum_m\left[\mathrm{Re}(\hat{\psi}_m)\,\sin(m\,\theta)+\mathrm{Im}(\hat{\psi}_m)\,\cos(m\,\theta)\right]. \tag{52}$$

## IV. VALUE OF $k_m$

Now,

$$\begin{aligned}\tilde{k}_m = &-\frac{2-s}{m}-\frac{\epsilon_a^2}{m}\left(-\hat{r}\,p_2'+\frac{3\,\hat{r}^{\,2}}{2}-2\,\hat{r}\,H_1'+S_2\right)\\ &+\epsilon_a^2\,\frac{(2-s)}{m}\left(-\frac{3\,\hat{r}^{\,2}}{4}+\frac{\hat{r}^{\,2}}{q^2}+H_1+S_1\right)\\ &+\epsilon_a^2\,\frac{n\,\hat{r}}{m^2}\left[-q\,p_2'+\frac{\hat{r}}{m\,q}\,(2-s)\,(m-n\,q)\right].\end{aligned} \tag{53}$$

But, for the special case $m=0$,

$$\begin{aligned}\tilde{k}_0 = &-\frac{q\,p_2'}{n\,\hat{r}}-\frac{2-s}{n\,q}\\ &-\frac{\epsilon_a^2}{n\,q}\left(\frac{3\,\hat{r}^{\,2}}{2}-2\,\hat{r}\,H_1+S_2\right)\\ &+\epsilon_a^2\,\frac{(2-s)}{n\,q}\left(-\frac{3\,\hat{r}^{\,2}}{4}+\frac{\hat{r}^{\,2}}{q^2}+H_1+S_1\right)\\ &+\epsilon_a^2\,\frac{q\,p_2'}{n\,\hat{r}}\left(2\,g_2+\frac{\hat{r}^{\,2}}{2}+\frac{\hat{r}^{\,2}}{q^2}-2\,H_1-3\,\hat{r}\,H_1'\right).\end{aligned} \tag{54}$$

It turns out that

$$k_m = \tilde{k}_m - \tilde{k}_0 \tag{55}$$

for all $m$.