Neoclassical Toroidal Viscosity

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I. CURVILINEAR COORDINATES

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a position vector, where the x_i are Cartesian coordinates, and i runs from 1 to 3. Let the $q_i(x_1, x_2, x_3)$ be curvilinear coordinates. We can define the contravariant basis vectors,

$$\mathbf{e}^i = \frac{\partial q^i}{\partial \mathbf{x}}, \tag{1}$$

and the covariant basis vectors,

$$\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial q^i}. \tag{2}$$

Note that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \frac{\partial q^i}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial q^j} = \frac{\partial q^i}{\partial x_k} \frac{\partial x_k}{\partial q^j} = \frac{\partial q^i}{\partial q^j} = \delta^i_j. \tag{3}$$

Here, use has been made of the Einstein summation convention, as well as the chain rule.

The Jacobian, $\mathcal{J}$, is defined as

$$\mathcal{J}^{-1} = \frac{\partial q^1}{\partial \mathbf{x}} \cdot \frac{\partial q^2}{\partial \mathbf{x}} \times \frac{\partial q^3}{\partial \mathbf{x}}. \tag{4}$$

It is easily seen that

$$\mathbf{e}_i = \mathcal{J} \frac{\partial q^j}{\partial \mathbf{x}} \times \frac{\partial q^k}{\partial \mathbf{x}}, \tag{5}$$

where i, j, k are cyclic. To demonstrate this we need to show that $\mathbf{e}_i \cdot \mathbf{e}^l = \delta^l_i$, which implies that

$$\mathcal{J} \frac{\partial q^l}{\partial \mathbf{x}} \cdot \frac{\partial q^j}{\partial \mathbf{x}} \times \frac{\partial q^k}{\partial \mathbf{x}} = \delta^l_i, \tag{6}$$

which is obviously satisfied.

The contravariant components, a^i, of a vector $\mathbf{a}$ are defined via

$$\mathbf{a} = a^i \, \mathbf{e}_i. \tag{7}$$

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The covariant components, a_i, are defined via

$$\mathbf{a} = a_i\,\mathbf{e}^i. \tag{8}$$

Thus,

$$\mathbf{a}\cdot\mathbf{b} = a^i\,b_j\,\mathbf{e}_i\cdot\mathbf{e}^j = a^i\,b_j\,\delta_i^j = a^i\,b_i, \tag{9}$$

where use has been made of Eq. (3). Similarly,

$$\mathbf{a}\cdot\mathbf{b} = a_i\,b^j\,\mathbf{e}^i\cdot\mathbf{e}_j = a_i\,b^j\,\delta_j^i = a_i\,b^i, \tag{10}$$

Note that

$$a^i = \mathbf{a}\cdot\mathbf{e}^i, \tag{11}$$

$$a_i = \mathbf{a}\cdot\mathbf{e}_i. \tag{12}$$

The contravariant components of the metric tensor, $\overleftrightarrow{\mathbf{g}}$, are defined

$$g^{ij} = \mathbf{e}^i\cdot\mathbf{e}^j. \tag{13}$$

Note that $g^{ij} = g^{ji}$. Likewise, the covariant components of the metric tensor are

$$g_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j. \tag{14}$$

Note that $g_{ij} = g_{ji}$. Now,

$$\mathbf{e}^i = (\mathbf{e}^i\cdot\mathbf{e}^j)\,\mathbf{e}_j, \tag{15}$$

so

$$\mathbf{a}\cdot\mathbf{e}^i = \mathbf{a}\cdot[(\mathbf{e}^i\cdot\mathbf{e}^j)\,\mathbf{e}_j] = (\mathbf{e}^i\cdot\mathbf{e}^j)\,\mathbf{a}\cdot\mathbf{e}_j, \tag{16}$$

which yields

$$a^i = g^{ij}\,a_j. \tag{17}$$

Likewise,

$$\mathbf{e}_i = (\mathbf{e}_i\cdot\mathbf{e}_j)\,\mathbf{e}^j, \tag{18}$$

so

$$\mathbf{a}\cdot\mathbf{e}_i = \mathbf{a}\cdot[(\mathbf{e}_i\cdot\mathbf{e}_j)\,\mathbf{e}^j] = (\mathbf{e}_i\cdot\mathbf{e}_j)\,\mathbf{a}\cdot\mathbf{e}^j, \tag{19}$$

which yields

$$a_i = g_{ij}\,a^j. \tag{20}$$

Combining Eqs. (17) and (20), we get

$$a^i = g^{ij}\, g_{jk}\, a^k = \delta^i_k\, a^k, \tag{21}$$

which implies that

$$g^{ij}\, g_{jk} = \delta^i_k. \tag{22}$$

Likewise,

$$a_i = g_{ij}\, g^{\,jk}\, a_k = \delta_i^k\, a_k, \tag{23}$$

which implies that

$$g_{ij}\, g^{\,jk} = \delta_i^k. \tag{24}$$

The Christoffel symbol, $\Gamma_{k,ij}$ is defined such that

$$\frac{\partial \mathbf{e}_i}{\partial q^{\,j}} = \Gamma_{k,ij}\, \mathbf{e}^k. \tag{25}$$

It follows that

$$\Gamma_{k,ij} = \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_i}{\partial q^j}. \tag{26}$$

Thus,

$$\frac{\partial g_{ij}}{\partial q^k} = \frac{\partial(\mathbf{e}_i\cdot\mathbf{e}_j)}{\partial q^k} = \frac{\partial \mathbf{e}_i}{\partial q^k}\cdot\mathbf{e}_j + \mathbf{e}_i\cdot\frac{\partial \mathbf{e}_j}{\partial q^k} = \Gamma_{j,ik} + \Gamma_{i,jk}, \tag{27}$$

$$\frac{\partial g_{ik}}{\partial q^{\,j}} = \frac{\partial(\mathbf{e}_i\cdot\mathbf{e}_k)}{\partial q^{\,j}} = \frac{\partial \mathbf{e}_i}{\partial q^{\,j}}\cdot\mathbf{e}_k + \mathbf{e}_i\cdot\frac{\partial \mathbf{e}_k}{\partial q^{\,j}} = \Gamma_{k,ij} + \Gamma_{i,kj}, \tag{28}$$

$$\frac{\partial g_{jk}}{\partial q^i} = \frac{\partial(\mathbf{e}_j\cdot\mathbf{e}_k)}{\partial q^i} = \frac{\partial \mathbf{e}_j}{\partial q^i}\cdot\mathbf{e}_k + \mathbf{e}_j\cdot\frac{\partial \mathbf{e}_k}{\partial q^i} = \Gamma_{k,ji} + \Gamma_{j,ki}. \tag{29}$$

However,

$$\frac{\partial \mathbf{e}_i}{\partial q^{\,j}} = \frac{\partial^2 \mathbf{x}}{\partial q^i\,\partial q^{\,j}} = \frac{\partial^2 \mathbf{x}}{\partial q^{\,j}\,\partial q^i} = \frac{\partial \mathbf{e}_j}{\partial q^i}, \tag{30}$$

where use has been made of Eq. (2). Hence, we deduce that

$$\Gamma_{k,ij} = \Gamma_{k,ji}. \tag{31}$$

It follows that

$$\begin{aligned}\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^{\,j}} &= \Gamma_{j,ik} + \Gamma_{i,jk} + \Gamma_{k,ji} + \Gamma_{j,ki} - \Gamma_{k,ij} - \Gamma_{i,kj}\\ &= \Gamma_{j,ik} + \cancel{\Gamma_{i,jk}} + \cancel{\Gamma_{k,ji}} + \Gamma_{j,ik} - \cancel{\Gamma_{k,ji}} - \cancel{\Gamma_{i,jk}},\end{aligned} \tag{32}$$

which implies that

$$\Gamma_{j,ik} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^j}\right), \tag{33}$$

or

$$\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right). \tag{34}$$

The previous two equations yield

$$\frac{\partial g_{ij}}{\partial q^k} = \Gamma_{i,jk} + \Gamma_{j,ik}. \tag{35}$$

Let us define the Jacobian matrix

$$J_{ij} = \frac{\partial x_j}{\partial q^i}. \tag{36}$$

The Jacobian is the determinant of the Jacobian matrix

$$\mathcal{J} = ||J_{ij}|| = \frac{\partial \mathbf{x}}{\partial q^1}\cdot\frac{\partial \mathbf{x}}{\partial q^2}\times\frac{\partial \mathbf{x}}{\partial q^3}. \tag{37}$$

Now,

$$g_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j = \frac{\partial \mathbf{x}}{\partial q^i}\cdot\frac{\partial \mathbf{x}}{\partial q^j} = \frac{\partial x_k}{\partial q^i}\frac{\partial x_k}{\partial q^j} = J_{ik}\,J_{jk} = J_{ij}\,J^T_{kj}, \tag{38}$$

where $J^T_{ij} = J_{ji}$. However, $||A\,B|| = ||A||\,||B||$ and $||A^T|| = ||A||$. Hence, we deduce that

$$||g_{ij}|| = \mathcal{J}^{\,2}. \tag{39}$$

Jacobi's formula states that

$$\frac{\partial ||A||}{\partial q^k} = ||A||\,\mathrm{tr}\left(A^{-1}\,\frac{\partial A}{\partial q^k}\right) \tag{40}$$

for any square matrix, $A_{ij}(q_1, q_2, q_3)$. Let $A_{ij} = g_{ij}$. Now, the inverse of g_{ij} is g^{ij}. Hence, making use of Eq. (39), we deduce that

$$\frac{\partial \mathcal{J}^{\,2}}{\partial q^k} = \mathcal{J}^{\,2}\,g^{ij}\,\frac{\partial g_{ji}}{\partial q^k}, \tag{41}$$

which reduces to

$$\frac{\partial g_{ij}}{\partial q^k}\,g^{ij} = \frac{2}{\mathcal{J}}\,\frac{\partial \mathcal{J}}{\partial q^k}. \tag{42}$$

II. EQUILIBRIUM MAGNETIC FIELD

Let ψ, θ, α be magnetic coordinates, where ψ is a flux-surface label, θ a poloidal angle, and α a helical angle. Let $\mathcal{J}^{-1} = \nabla\psi\cdot\nabla\theta\times\nabla\alpha$. Suppose that the geometric toroidal angle is $\varphi = \alpha + q(\psi)\,\theta$. Let the equilibrium magnetic field take the form

$$\mathbf{B} = \nabla\alpha\times\nabla\psi = \mathcal{J}^{-1}\,\mathbf{e}_\theta. \tag{43}$$

It follows that

$$\mathbf{b} \equiv \frac{\mathbf{B}}{B} = (\mathcal{J}\,B)^{-1}\,\mathbf{e}_\theta. \tag{44}$$

Thus,

$$(\mathbf{b}\,\mathbf{b})^{ij} = (\mathcal{J}\,B)^{-2}\,(\mathbf{e}_\theta\cdot\mathbf{e}^i)\,(\mathbf{e}_\theta\cdot\mathbf{e}^j) = (\mathcal{J}\,B)^{-2}\,\delta^i_\theta\,\delta^j_\theta. \tag{45}$$

III. PLASMA VISCOSITY TENSOR

Let $\mathbf{1}$ be the identity tensor. It follows that

$$1^{ij} = \mathbf{e}^i\cdot\mathbf{e}^j = g^{ij}. \tag{46}$$

Thus, if

$$\overleftrightarrow{\mathbf{\Pi}} = (\delta p_\parallel - \delta p_\perp)\,\mathbf{b}\,\mathbf{b} + \delta p_\perp\,\overleftrightarrow{\mathbf{1}} \tag{47}$$

is the plasma viscosity tensor then

$$\mathit{\Pi}^{\,ij} = (\delta p_\parallel - \delta p_\perp)\,(\mathcal{J}\,B)^{-2}\,\delta^i_\theta\,\delta^j_\theta + \delta p_\perp\,g^{ij}. \tag{48}$$

Consider

$$\nabla\cdot\left(\overleftrightarrow{\mathbf{\Pi}}\cdot\mathbf{e}_k\right) = \nabla\cdot\overleftrightarrow{\mathbf{\Pi}}\cdot\mathbf{e}_k + \overleftrightarrow{\mathbf{\Pi}} : \frac{\partial\mathbf{e}_k}{\partial\mathbf{x}}. \tag{49}$$

Now,

$$\frac{\partial\mathbf{e}_k}{\partial\mathbf{x}} = \frac{\partial\mathbf{e}_k}{\partial q^j}\,\frac{\partial q^j}{\partial\mathbf{x}} = \mathit{\Gamma}_{i,kj}\,\mathbf{e}^i\,\mathbf{e}^j = \mathit{\Gamma}_{i,jk}\,\mathbf{e}^i\,\mathbf{e}^j, \tag{50}$$

where use has been made of Eqs. (1) and (25). Thus,

$$\overleftrightarrow{\mathbf{\Pi}} : \frac{\partial\mathbf{e}_k}{\partial\mathbf{x}} = \mathit{\Pi}^{\,ij}\,\mathit{\Gamma}_{i,jk}. \tag{51}$$

Now, $\mathit{\Pi}^{\,ij} = \mathit{\Pi}^{\,ji}$, so

$$\overleftrightarrow{\mathbf{\Pi}} : \frac{\partial\mathbf{e}_k}{\partial\mathbf{x}} = \frac{1}{2}\,\mathit{\Pi}^{\,ij}\,(\mathit{\Gamma}_{i,jk} + \mathit{\Gamma}_{j,ik}) = \frac{1}{2}\,\mathit{\Pi}^{\,ij}\,\frac{\partial g_{ij}}{\partial q^k}, \tag{52}$$

where use has been made of Eq. (35).

If we integrate Eq. (49) over all space, and neglect surface terms, then we find that

$$\int \nabla\cdot\overleftrightarrow{\mathbf{\Pi}}\cdot\mathbf{e}_k\,d^3\mathbf{x} = -\frac{1}{2}\int \Pi^{\,ij}\,\frac{\partial g_{ij}}{\partial q^k}\,d^3\mathbf{x}, \tag{53}$$

where use has been made of Eq. (52). It follows from Eq. (48) that

$$\int \nabla\cdot\overleftrightarrow{\mathbf{\Pi}}\cdot\mathbf{e}_k\,d^3\mathbf{x} = -\frac{1}{2}\int\left[\frac{(\delta p_\parallel-\delta p_\perp)}{(\mathcal{J}\,B)^2}\,\frac{\partial g_{\theta\theta}}{\partial q^k}+\delta p_\perp\,g^{ij}\,\frac{\partial g_{ij}}{\partial q^k}\right]d^3\mathbf{x}. \tag{54}$$

However, Eq. (43) implies that

$$g_{\theta\theta} = \mathbf{e}_\theta\cdot\mathbf{e}_\theta = (\mathcal{J}\,B)^2. \tag{55}$$

Finally, making use of Eq. (42), we get

$$\begin{aligned}\int \nabla\cdot\overleftrightarrow{\mathbf{\Pi}}\cdot\mathbf{e}_k\,d^3\mathbf{x} &= -\frac{1}{2}\int\left[\frac{(\delta p_\parallel-\delta p_\perp)}{(\mathcal{J}\,B)^2}\,\frac{\partial(\mathcal{J}\,B)^2}{\partial q^k}+\delta p_\perp\,\frac{2}{\mathcal{J}}\,\frac{\partial\mathcal{J}}{\partial q^k}\right]d^3\mathbf{x}\\ &= -\int\left[(\delta p_\parallel-\delta p_\perp)\,\frac{1}{B}\,\frac{\partial B}{\partial q^k}+\delta p_\parallel\,\frac{1}{\mathcal{J}}\,\frac{\partial\mathcal{J}}{\partial q^k}\right]d^3\mathbf{x}. \end{aligned}\tag{56}$$