Resistive Wall Mode

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I. VACUUM SOLUTION

A. Normalization

Let all lengths be normalized to the major radius of the magnetic axis, R_0. Let all magnetic field-strengths be normalized to the toroidal magnetic field-strength at the magnetic axis, B_0.

B. Toroidal Coordinates

Let μ, η, ϕ be toroidal coordinates such that

$$R = \frac{\sinh\mu}{\cosh\mu - \cos\eta}, \tag{1}$$

$$Z = \frac{\sin\eta}{\cosh\mu - \cos\eta}, \tag{2}$$

where R, ϕ, Z are cylindrical coordinates. The scale-factors of the toroidal coordinate system are

$$h_\mu = h_\eta = \frac{1}{\cosh\mu - \cos\eta} = h, \tag{3}$$

$$h_\phi = \frac{\sinh\mu}{\cosh\mu - \cos\eta} = h\,\sinh\mu. \tag{4}$$

C. Perturbed Magnetic Field

The perturbed magnetic field in the vacuum region is written

$$\mathbf{b} = \mathrm{i}\,\nabla\left[V(\mu,\eta)\,\mathrm{e}^{-\mathrm{i}\,n\,\phi}\right], \tag{5}$$

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where

$$\nabla^2 V = (z-\cos\eta)^3 \left\{ \frac{\partial}{\partial z}\left[\frac{z^2-1}{z-\cos\eta}\,\frac{\partial V}{\partial z}\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{z-\cos\eta}\,\frac{\partial V}{\partial\eta}\right] - \frac{n^2\,V}{(z^2-1)\,(z-\cos\eta)}\right\} = 0. \tag{6}$$

Here, $z = \cosh\mu$.

Let

$$f_z = z^2 - 1, \tag{7}$$

$$f_\eta = (z-\cos\eta)^{1/2}. \tag{8}$$

It follows that

$$\frac{df_z}{dz} = 2\,z, \tag{9}$$

$$\frac{\partial f_\eta}{\partial z} = \frac{1}{2\,f_\eta}, \tag{10}$$

$$\frac{\partial f_\eta}{\partial \eta} = \frac{\sin\eta}{2\,f_\eta}. \tag{11}$$

Let

$$V(z,\eta) = \sum_m f_\eta\,U_m(z)\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}. \tag{12}$$

Then, taking the sum and eikonal as read, and letting $' = d/dz$, we get

$$\frac{\partial V}{\partial z} = \frac{U_m}{2\,f_\eta} + f_\eta\,U_m', \tag{13}$$

$$\frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^2}\,\frac{\partial V}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{f_z\,U_m}{2\,f_\eta^3} + \frac{f_z\,U_m'}{f_\eta}\right) = \frac{z\,U_m}{f_\eta^3} - \frac{3\,f_z\,U_m}{4\,f_\eta^5} + \frac{f_z\,U_m'}{2\,f_\eta^3} + \frac{2\,z\,U_m'}{f_\eta} - \frac{f_z\,U_m'}{2\,f_\eta^3} + \frac{f_z\,U_m''}{f_\eta}$$
$$= \frac{z\,U_m}{f_\eta^3} - \frac{3\,(z^2-1)\,U_m}{4\,f_\eta^5} + \frac{2\,z\,U_m'}{f_\eta} + \frac{(z^2-1)\,U_m''}{f_\eta}, \tag{14}$$

$$\frac{\partial V}{\partial\eta} = \frac{\sin\eta\,U_m}{2\,f_\eta} - \mathrm{i}\,m f_\eta\,U_m, \tag{15}$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{f_\eta^2}\,\frac{\partial V}{\partial\eta}\right) = \frac{\partial}{\partial\eta}\left(\frac{\sin\eta\,U_m}{2\,f_\eta^3} - \frac{\mathrm{i}\,m\,U_m}{f_\eta}\right) = \frac{\cos\eta\,U_m}{2\,f_\eta^3} - \frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5} - \frac{\mathrm{i}\,m\,\sin\eta\,U_m}{2\,f_\eta^3} + \frac{\mathrm{i}\,m\,\sin\eta\,U_m}{2\,f_\eta^3} - \frac{m^2\,U_m}{f_\eta}$$
$$= \frac{\cos\eta\,U_m}{2\,f_\eta^3} - \frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5} - \frac{m^2\,U_m}{f_\eta}, \tag{16}$$

$$-\frac{n^2\,V}{f_z\,f_\eta^2} = -\frac{n^2\,U_m}{(z^2-1)\,f_\eta}. \tag{17}$$

Thus,

$$\begin{aligned}
0 &= \frac{\partial}{\partial z}\left(\frac{f_z}{f_\eta^2}\,\frac{\partial V}{\partial z}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{f_\eta^2}\,\frac{\partial V}{\partial \eta}\right) - \frac{n^2\,V}{f_z\,f_\eta^2}\\
&= \frac{z\,U_m}{f_\eta^3} - \frac{3\,(z^2-1)\,U_m}{4\,f_\eta^5} + \frac{2\,z\,U_m'}{f_\eta} + \frac{(z^2-1)\,U_m''}{f_\eta}\\
&\quad + \frac{\cos\eta\,U_m}{2\,f_\eta^3} - \frac{3\,\sin^2\eta\,U_m}{4\,f_\eta^5} - \frac{m^2\,U_m}{f_\eta} - \frac{n^2\,U_m}{(z^2-1)\,f_\eta}\\
&= \frac{1}{f_\eta}\left[(z^2-1)\,U_m'' + 2\,z\,U_m' + \left(\frac{1}{4}-m^2\right)U_m - \frac{n^2\,U_m}{z^2-1}\right].
\end{aligned} \tag{18}$$

The most general solution of the previous equation is

$$U_m(z) = p_m\,P^n_{m-1/2}(z) + q_m\,Q^n_{m-1/2}(z), \tag{19}$$

where p_m and q_m are arbitrary complex coefficients. Note that

$$P^n_{-m-1/2}(z) = P^n_{m-1/2}(z), \tag{20}$$

$$Q^n_{-m-1/2}(z) = Q^n_{m-1/2}(z), \tag{21}$$

so

$$U_m(z) = p_m\,P^n_{|m|-1/2}(z) + q_m\,Q^n_{|m|-1/2}(z), \tag{22}$$

Let

$$p_m = \bar{p}_m\,\hat{p}_m, \tag{23}$$

$$q_m = \bar{q}_m\,\hat{q}_m, \tag{24}$$

where

$$\bar{p}_{m\neq 0} = \cos(|m|\,\pi)\,\frac{\sqrt{\pi}\,\Gamma(|m|+1/2-n)\,\epsilon^{\,|m|}}{2^{|m|-1/2}\,|m|!}, \tag{25}$$

$$\bar{q}_{m\neq 0} = \cos(n\,\pi)\,\cos(|m|\,\pi)\,\frac{2^{|m|+1/2}\,(|m|-1)!\,\epsilon^{-|m|}}{\sqrt{\pi}\,\Gamma(|m|+1/2+n)}, \tag{26}$$

$$\bar{p}_0 = \frac{\sqrt{\pi}\,\Gamma(1/2-n)}{\sqrt{2}}, \tag{27}$$

$$\bar{q}_0 = \cos(n\,\pi)\,\frac{\sqrt{2}}{\sqrt{\pi}\,\Gamma(n+1/2)}. \tag{28}$$

D. Toroidal Electromagnetic Angular Momentum Flux

The outward flux of toroidal angular momentum across a constant-z surface is

$$T_\phi(z) = -\oint\oint \mathcal{J}\, b_\phi\, b^\mu\, d\eta\, d\phi, \tag{29}$$

where

$$\mathcal{J} = (\nabla\mu\times\nabla\eta\cdot\nabla\phi)^{-1} = h^3\,\sinh\mu. \tag{30}$$

Now,

$$b^\mu = \mathbf{b}\cdot\nabla\mu = \mathrm{i}\,\frac{\partial V}{\partial \mu}\,|\nabla\mu|^2 = \mathrm{i}\,\frac{\sinh\mu}{h^2}\,\frac{\partial V}{\partial z}, \tag{31}$$

$$b^\phi = \mathcal{J}\,\nabla\mu\times\nabla\eta\cdot\nabla V = n\,V, \tag{32}$$

so

$$\begin{aligned} T_\phi(z) &= -\frac{\mathrm{i}\,n\,\pi}{2}\oint \frac{z^2-1}{z-\cos\eta}\left(\frac{\partial V}{\partial z}\,V^\ast - \frac{\partial V^\ast}{\partial z}\,V\right) d\eta \\ &= -\mathrm{i}\,n\,\pi^2\sum_m (z^2-1)\left(\frac{dU_m}{dz}\,F_m^\ast - \frac{dU_m^\ast}{dz}\,F_m\right) \\ &= -\mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast)\,(z^2-1)\left(\frac{dP^n_{|m|-1/2}}{dz}\,Q^n_{|m|-1/2} - \frac{dQ^n_{|m|-1/2}}{dz}\,P^n_{|m|-1/2}\right) \\ &= \mathrm{i}\,n\,\pi^2\sum_m (p_m\,q_m^\ast - q_m\,p_m^\ast)\,(z^2-1)\,\mathcal{W}[P^n_{|m|-1/2}, Q^n_{|m|-1/2}]. \end{aligned} \tag{33}$$

But,

$$\mathcal{W}[P^n_{|m|-1/2}, Q^n_{|m|-1/2}] = \frac{\cos(n\,\pi)}{1-z^2}\,\frac{\Gamma(|m|+1/2+n)}{\Gamma(|m|+1/2-n)}, \tag{34}$$

so

$$T_\phi(z) = 2\pi^2\,n\,\cos(n\,\pi)\sum_m \frac{\Gamma(|m|+1/2+n)}{\Gamma(|m|+1/2-n)}\,\mathrm{Im}(p_m\,q_m^\ast), \tag{35}$$

or

$$T_\phi(z) = 2\pi^2\,n\sum_m \mathrm{Im}(\hat{p}_m\,\hat{q}_m^\ast)\,h_m, \tag{36}$$

where

$$h_{m\neq 0} = \frac{2}{|m|}, \tag{37}$$

$$h_0 = 1. \tag{38}$$

II. RESISTIVE WALL PHYSICS

A. Resistive Wall

Let the resistive wall extend from $\mu = \mu_w$ to $\mu = \mu_w - \bar{d}_w \sinh\mu_w$, where $\bar{d}_w \ll 1$ is a positive constant. In other words, $\mu = \mu_w$ is the inner surface of the wall, and $\mu = \mu_w - \bar{d}_w \sinh\mu_w$ is the outer surface. The physical wall thickness is

$$d(\eta) = \frac{\bar{d}_w \sinh\mu_w}{|\nabla\mu|} = g_w(\eta)\,\bar{d}_w, \tag{39}$$

where

$$g_w(\eta) = \frac{(z_w^2-1)^{1/2}}{z_w - \cos\eta}, \tag{40}$$

and $z_w = \cosh\mu_w$. Let the electrical conductivity of the wall material vary as

$$\sigma(\eta) = \frac{\bar{\sigma}_w}{g_w^2(\eta)}, \tag{41}$$

where $\bar{\sigma}_w$ is a positive constant. It follows that $\sigma\, d^2 = \bar{\sigma}_w\, \bar{d}_w^2$.

B. Wall Matching Conditions

If we write

$$\mathbf{b} = \nabla\times\mathbf{A} \tag{42}$$

in the vacuum region then the boundary conditions at the wall are

$$\mathbf{n}_w \times \mathbf{A}|_{z_{w-}} = \frac{1}{\cosh\lambda}\,\mathbf{n}_w\times\mathbf{A}|_{z_{w+}} \tag{43}$$

$$\mathbf{n}_w\times(\nabla\times\mathbf{A})|_{z_{w+}} = -\frac{\lambda\,\tanh\lambda}{\bar{d}_w\,g_w}\,\mathbf{n}_w\times(\mathbf{n}_w\times\mathbf{A})|_{z_{w+}} + \frac{\mathbf{n}_w\times(\nabla\times\mathbf{A})|_{z_{w-}}}{\cosh\lambda}, \tag{44}$$

$$\lambda = \sqrt{\mu_0\,R_0^2\,\bar{\sigma}_w\,\bar{d}_w^2\,\gamma}, \tag{45}$$

where γ is the growth-rate of the magnetic perturbation. Here, $\mathbf{n}_w = -\mathbf{e}_\mu$ is an outward unit normal vector to the wall. Now,

$$\nabla\times\mathbf{A} = \frac{1}{h^2\sinh\mu}\left(\frac{\partial\hat{A}_\phi}{\partial\eta} - \frac{\partial\hat{A}_\eta}{\partial\phi}\right)\mathbf{e}_\mu + \frac{1}{h^2\sinh\mu}\left(\frac{\partial\hat{A}_\mu}{\partial\phi} - \frac{\partial\hat{A}_\phi}{\partial\mu}\right)\mathbf{e}_\eta + \frac{1}{h^2}\left(\frac{\partial\hat{A}_\eta}{\partial\mu} - \frac{\partial\hat{A}_\mu}{\partial\eta}\right)\mathbf{e}_\phi, \tag{46}$$

where

$$\hat{A}_\mu = h\, A_\mu, \tag{47}$$

$$\hat{A}_\eta = h\, A_\eta, \tag{48}$$

$$\hat{A}_\phi = h\, \sinh\mu\, A_\phi. \tag{49}$$

Furthermore,

$$\mathbf{n}_w \times \mathbf{A} = -\mathbf{e}_\mu \times \mathbf{A} = A_\phi\, \mathbf{e}_\eta - A_\eta\, \mathbf{e}_\phi, \tag{50}$$

$$\mathbf{n}_w \times (\mathbf{n}_w \times \mathbf{A}) = -\mathbf{e}_\mu \times (\mathbf{n}_w \times \mathbf{A}) = -A_\eta\, \mathbf{e}_\eta - A_\phi\, \mathbf{e}_\phi, \tag{51}$$

$$\mathbf{n}_w \times (\nabla \times \mathbf{A}) = -\mathbf{e}_\mu \times (\nabla \times \mathbf{A}) = \frac{1}{h^2}\left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta}\right) \mathbf{e}_\eta - \frac{1}{h^2\, \sinh\mu}\left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu}\right) \mathbf{e}_\phi. \tag{52}$$

Thus, the wall matching conditions become

$$\hat{A}_\eta\Big|_{z_{w-}} = \frac{1}{\cosh\lambda}\, \hat{A}_\eta\Big|_{z_{w+}}, \tag{53}$$

$$\hat{A}_\phi\Big|_{z_{w+}} = \frac{1}{\cosh\lambda}\, \hat{A}_\phi\Big|_{z_{w-}}, \tag{54}$$

$$\left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta}\right)_{z_{w+}} = \frac{\lambda\, \tanh\lambda}{\bar{d}_w\, \sinh\mu_w}\, \hat{A}_\eta\Big|_{z_{w+}} + \frac{1}{\cosh\lambda}\left(\frac{\partial \hat{A}_\eta}{\partial \mu} - \frac{\partial \hat{A}_\mu}{\partial \eta}\right)_{z_{w-}}, \tag{55}$$

$$\left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu}\right)_{z_{w+}} = -\frac{\lambda\, \tanh\lambda}{\bar{d}_w\, \sinh\mu_w}\, \hat{A}_\phi\Big|_{\mu_{z+}} + \frac{1}{\cosh\lambda}\left(\frac{\partial \hat{A}_\mu}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \mu}\right)_{z_{w-}}. \tag{56}$$

Let

$$C(z,\eta,\phi) = \frac{\partial \hat{A}_\eta}{\partial \phi} - \frac{\partial \hat{A}_\phi}{\partial \eta}. \tag{57}$$

The wall matching conditions reduce to

$$C(z_{w-},\eta,\phi) = \frac{1}{\cosh\lambda}\, C(z_{w+},\eta,\phi), \tag{58}$$

$$\frac{\partial C(z_{w+},\eta,\phi)}{\partial z} = \frac{\lambda\, \tanh\lambda}{\bar{d}_w\, \sinh^2\mu_w}\, C(z_{w+},\eta,\phi) + \frac{1}{\cosh\lambda}\, \frac{\partial C(z_{w-},\eta,\phi)}{\partial z}. \tag{59}$$

However, if

$$\mathbf{b} = \mathrm{i}\, \nabla V = \nabla \times \mathbf{A} \tag{60}$$

then

$$C = -\mathrm{i}\, h\, \sinh\mu\, \frac{\partial V}{\partial \mu} = -\mathrm{i}\, h\, (z^2-1)\, \frac{\partial V}{\partial z}. \tag{61}$$

Thus,

$$C = -\mathrm{i}\,\frac{z^2-1}{z-\cos\eta}\sum_m\left[\frac{U_m}{2\,(z-\cos\eta)^{1/2}}+(z-\cos\eta)^{1/2}\,\frac{dU_m}{dz}\right]\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}, \tag{62}$$

$$\frac{\partial C}{\partial z} = -\mathrm{i}\sum_m\left[\frac{(3/4)\,\sin^2\eta}{(z-\cos\eta)^{5/2}}-\frac{(1/2)\,\cos\eta}{(z-\cos\eta)^{3/2}}+\frac{m^2+n^2/(z^2-1)}{(z-\cos\eta)^{1/2}}\right]U_m\,\mathrm{e}^{-\mathrm{i}\,(m\,\eta+n\,\phi)}. \tag{63}$$

It follows that

$$\sum_m\left[\frac{U_m}{2}+(z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w-}}\mathrm{e}^{-\mathrm{i}\,m\,\eta} = \frac{1}{\cosh\lambda}\sum_m\left[\frac{U_m}{2}+(z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w+}}\mathrm{e}^{-\mathrm{i}\,m\,\eta}, \tag{64}$$

$$\begin{aligned}\sum_m\left[\frac{3}{4}\,\sin^2\eta-\frac{1}{2}\,(z-\cos\eta)\,\cos\eta+(z-\cos\eta)^2\left(m^2+\frac{n^2}{z^2-1}\right)\right]U_m\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}\Big|_{z_{w+}} = \\ F(\lambda)\sum_m(z-\cos\eta)\left[\frac{U_m}{2}+(z-\cos\eta)\,\frac{dU_m}{dz}\right]_{z_{w+}}\mathrm{e}^{-\mathrm{i}\,m\,\eta} \\ +\frac{1}{\cosh\lambda}\sum_m\left[\frac{3}{4}\,\sin^2\eta-\frac{1}{2}\,(z-\cos\eta)\,\cos\eta+(z-\cos\eta)^2\left(m^2+\frac{n^2}{z^2-1}\right)\right]U_m\,\mathrm{e}^{-\mathrm{i}\,m\,\eta}\Big|_{z_{w-}},\end{aligned} \tag{65}$$

where

$$F(\lambda) = \frac{\lambda\,\tanh\lambda}{\bar{d}_w}. \tag{66}$$

Thus, we can write

$$\sum_{m'} I_{mm'}\,U_{m'}(z_{w-}) = \frac{1}{\cosh\lambda}\sum_{m'} I_{mm'}\,U_{m'}(z_{w+}), \tag{67}$$

$$\sum_{m'} J_{mm'}\,U_{m'}(z_{w+}) = F(\lambda)\sum_{m',m''} K_{mm''}\,I_{m''m'}\,U_{m'}(z_{w+})+\frac{1}{\cosh\lambda}\sum_{m'} J_{mm'}\,U_{m'}(z_{w-}), \tag{68}$$

where

$$I_{mm'} = \left(\frac{1}{2}+z\,\frac{d}{dz}\right)\delta_{mm'}-\frac{1}{2}\,\frac{d}{dz}\,(\delta_{m\,m'+1}+\delta_{m\,m'-1}), \tag{69}$$

$$\begin{aligned}J_{mm'} &= \left[\frac{5}{8}+\left(\frac{1}{2}+z^2\right)\left(m^2+\frac{n^2}{z^2-1}\right)\right]\delta_{mm'}-z\left[\frac{1}{4}+\left(m^2+\frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+1}+\delta_{m\,m'-1}) \\ &\quad+\left[-\frac{1}{16}+\frac{1}{4}\left(m^2+\frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+2}+\delta_{m\,m'-2}),\end{aligned} \tag{70}$$

$$K_{mm'} = z\,\delta_{mm'}-\frac{1}{2}\,(\delta_{m\,m'+1}+\delta_{m\,m'-1}). \tag{71}$$

C. Vacuum Solution

Now,

$$U_m(z) = \bar{p}_m\,\hat{p}_{m-}\,P^n_{|m|-1/2}(z) \tag{72}$$

in the region $z < z_w$, whereas

$$U_m(z) = \bar{p}_m\,\hat{p}_{m+}\,P^n_{|m|-1/2}(z) + \bar{q}_m\,\hat{q}_{m+}\,Q^n_{|m|-1/2}(z) \tag{73}$$

in the region $z > z_w$. Let $\underline{\underline{I}}_p$ be the matrix of the

$$\left\{\left[\left(\frac{1}{2} + z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\,(\delta_{m\,m'+1} + \delta_{m\,m'-1})\right] P^n_{|m'|-1/2}(z)\right\}_{z_w}\bar{p}_{m'} \tag{74}$$

values. Let $\underline{\underline{I}}_q$ be the matrix of the

$$\left\{\left[\left(\frac{1}{2} + z\,\frac{d}{dz}\right)\delta_{mm'} - \frac{1}{2}\,\frac{d}{dz}\,(\delta_{m\,m'+1} + \delta_{m\,m'-1})\right] Q^n_{|m'|-1/2}(z)\right\}_{z_w}\bar{q}_{m'} \tag{75}$$

values. Let $\underline{\underline{J}}_p$ be the matrix of the

$$\begin{aligned}\Bigg\{\left[\frac{5}{8} + \left(\frac{1}{2} + z^2\right)\left(m^2 + \frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4} + \left(m^2 + \frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+1} + \delta_{m\,m'-1})\\ + \left[-\frac{1}{16} + \frac{1}{4}\left(m^2 + \frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+2} + \delta_{m\,m'-2})\Bigg\}\,P^n_{|m'|-1/2}(z_w)\,\bar{p}_{m'}\end{aligned} \tag{76}$$

values. Let $\underline{\underline{J}}_q$ be the matrix of the

$$\begin{aligned}\Bigg\{\left[\frac{5}{8} + \left(\frac{1}{2} + z^2\right)\left(m^2 + \frac{n^2}{z^2-1}\right)\right]\delta_{mm'} - z\left[\frac{1}{4} + \left(m^2 + \frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+1} + \delta_{m\,m'-1})\\ + \left[-\frac{1}{16} + \frac{1}{4}\left(m^2 + \frac{n^2}{z^2-1}\right)\right](\delta_{m\,m'+2} + \delta_{m\,m'-2})\Bigg\}\,Q^n_{|m'|-1/2}(z_w)\,\bar{q}_{m'}\end{aligned} \tag{77}$$

values. Finally, let $\underline{\underline{K}}$ be the matrix of the

$$z_w\,\delta_{mm'} - \frac{1}{2}\,(\delta_{m\,m'+1} + \delta_{m\,m'-1}) \tag{78}$$

values. Likewise, let $\underline{\hat{p}}_+$ be the vector of the $\hat{p}_{m+}$ values, et cetera. Thus, we obtain

$$\underline{\underline{I}}_p\,\underline{\hat{p}}_- = \frac{1}{\cosh\lambda}\left(\underline{\underline{I}}_p\,\underline{\hat{p}}_+ + \underline{\underline{I}}_q\,\underline{\hat{q}}_+\right), \tag{79}$$

$$\underline{\underline{J}}_p\,\underline{\hat{p}}_+ + \underline{\underline{J}}_q\,\underline{\hat{q}}_+ = F(\lambda)\,\underline{\underline{K}}\left(\underline{\underline{I}}_p\,\underline{\hat{p}}_+ + \underline{\underline{I}}_q\,\underline{\hat{q}}_+\right) + \frac{1}{\cosh\lambda}\,\underline{\underline{J}}_p\,\underline{\hat{p}}_-, \tag{80}$$

which can be rearranged to give

$$\left[\tanh^2\lambda\,\underline{\underline{J}}_p - F(\lambda)\,\hat{\underline{\underline{I}}}_p\right]\hat{\underline{p}}_+ + \left[\underline{\underline{J}}_{pq} + \tanh^2\lambda\,\underline{\underline{J}}_{qp} - F(\lambda)\,\hat{\underline{\underline{I}}}_q\right]\hat{\underline{q}}_+, \tag{81}$$

where

$$\hat{\underline{\underline{I}}}_p = \underline{\underline{K}}\,\underline{\underline{I}}_p, \tag{82}$$

$$\hat{\underline{\underline{I}}}_q = \underline{\underline{K}}\,\underline{\underline{I}}_q, \tag{83}$$

$$\underline{\underline{J}}_{pq} = \underline{\underline{J}}_q - \underline{\underline{J}}_p\,\underline{\underline{I}}_p^{-1}\,\underline{\underline{I}}_q, \tag{84}$$

$$\underline{\underline{J}}_{qp} = \underline{\underline{J}}_p\,\underline{\underline{I}}_p^{-1}\,\underline{\underline{I}}_q. \tag{85}$$

Now, $z_w \sim 1/\bar{b}_w$, where $\bar{b}_w$ is the mean wall minor radius. In the large aspect-ratio limit $b_w \ll 1$, we have $\underline{\underline{I}}_p \sim \mathcal{O}(1)$, $\underline{\underline{I}}_q \sim \mathcal{O}(1)$, $\underline{\underline{J}}_p \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{\underline{J}}_q \sim \mathcal{O}(1/\bar{b}_w^2)$, $\underline{\underline{K}}_p \sim \mathcal{O}(1/\bar{b}_w)$, and $\underline{\underline{K}}_q \sim \mathcal{O}(1/\bar{b}_w)$. It follows that $\underline{\underline{J}}_{pq} \sim \mathcal{O}(1/\bar{b}_w^2)$ and $\underline{\underline{J}}_{qp} \sim \mathcal{O}(1/\bar{b}_w^2)$. Thus, the ratio of the first to the second term multiplying $\underline{p}_+$ in Eq. (81) is

$$\tanh\lambda\,\frac{\bar{d}_w}{\lambda\,\bar{b}_w}. \tag{86}$$

However, the wall analysis is premised on the assumption that

$$\frac{\bar{d}_w}{\lambda\,\bar{b}_w} \ll 1. \tag{87}$$

Hence, the first term is negligible with respect to the second, irrespective of the value of λ. The ratios of the three terms multiplying $\underline{q}_+$ in Eq. (81) are

$$\frac{\bar{d}_w}{\lambda\,\bar{b}_w},\;\; \tanh^2\lambda\,\frac{\bar{d}_w}{\lambda\,\bar{b}_w},\;\; \tanh\lambda. \tag{88}$$

Thus, in the thin-shell limit $\lambda \ll 1$, the second term is negligible with respect to the first. In the thick-shell limit, $\lambda \gg 1$, the third term is dominant. Thus, we can neglect the second term. Hence, we deduce that

$$\hat{\underline{q}}_+ = \underline{\underline{\mathcal{F}}}\,\hat{\underline{p}}_+, \tag{89}$$

where

$$\underline{\underline{\mathcal{F}}} = \left[\underline{\underline{J}}_{pq} - F(\lambda)\,\hat{\underline{\underline{I}}}_q\right]^{-1} F(\lambda)\,\hat{\underline{\underline{I}}}_p. \tag{90}$$

III. PLASMA/VACUUM INTERFACE

A. Matching at Plasma/Vacuum Interface

The plasma/vacuum interface lies at $r=\epsilon$. In the vacuum region between the interface and the wall,

$$V(z,\eta) = \sum_m (z-\cos\eta)^{1/2}\left[\bar{p}_m\,\hat{p}_{m+}\,P^n_{|m|-1/2}(z) + \bar{q}_m\,\hat{q}_{m+}\,Q^n_{|m|-1/2}(z)\right]\mathrm{e}^{-\mathrm{i}\,m\,\eta}. \tag{91}$$

Thus, if we write

$$V_(r,\theta) = \sum_m V_m(r)\,\mathrm{e}^{\mathrm{i}\,m\,\theta}, \tag{92}$$

$$\psi(r,\theta) = \sum_m \psi_m(r)\,\mathrm{e}^{\mathrm{i}\,m\,\theta}, \tag{93}$$

then

$$\underline{V} = \underline{\underline{\mathcal{P}}}\,\underline{\hat{p}}_+ + \underline{\underline{\mathcal{Q}}}\,\underline{\hat{q}}_+, \tag{94}$$

$$\underline{\psi} = \underline{\underline{\mathcal{R}}}\,\underline{\hat{p}}_+ + \underline{\underline{\mathcal{S}}}\,\underline{\hat{q}}_+, \tag{95}$$

where $\underline{V}$ is the vector of the $V_m(\epsilon)$ values, $\underline{\psi}$ is the vector of the ψ_m values, $\underline{\underline{\mathcal{P}}}$ is the matrix of the

$$\mathcal{P}_{mm'} = \bar{p}_{m'}\oint_{r=\epsilon}(z-\cos\eta)^{1/2}\,P^n_{|m'|-1/2}(z)\,\exp[-\mathrm{i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{96}$$

values, $\underline{\underline{\mathcal{Q}}}$ is the matrix of the

$$\mathcal{Q}_{mm'} = \bar{q}_{m'}\oint_{r=\epsilon}(z-\cos\eta)^{1/2}\,Q^n_{|m'|-1/2}(z)\,\exp[-\mathrm{i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{97}$$

values, $\underline{\underline{\mathcal{R}}}$ is the matrix of the

$$\begin{aligned}\mathcal{R}_{mm'} = \bar{p}_{m'}\oint_{r=\epsilon}&\left\{\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,P^n_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{dP^n_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z\right.\\ &\left.+\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta - \mathrm{i}\,m'\,(z-\cos\eta)^{1/2}\right]P^n_{|m'|-1/2}(z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\right\}\\ &\times\exp[-\mathrm{i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi}\end{aligned} \tag{98}$$

values, and $\underline{\underline{\mathcal{S}}}$ is the matrix of the

$$\mathcal{S}_{mm'} = \bar{q}_{m'}\oint_{r=\epsilon}\left\{\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,Q^n_{|m'|-1/2}(z) + (z-\cos\eta)^{1/2}\,\frac{dQ^n_{|m'|-1/2}}{dz}\right]\mathcal{J}\,\nabla r\cdot\nabla z\right.$$

$$+\left[\frac{1}{2}\,(z-\cos\eta)^{-1/2}\,\sin\eta - {\rm i}\,m'\,(z-\cos\eta)^{1/2}\right] Q^{\,n}_{|m'|-1/2}(z)\,\mathcal{J}\,\nabla r\cdot\nabla\eta\Bigg\}$$

$$\times\exp[-{\rm i}\,(m\,\theta+m'\,\eta)]\,\frac{d\theta}{2\pi} \tag{99}$$

Equations (89), (92), and (93) imply that

$$\underline{V} = \underline{\underline{H}}\,\underline{\psi}, \tag{100}$$

where

$$\underline{\underline{H}}\,(\underline{\underline{\mathcal{R}}}+\underline{\underline{\mathcal{S}}}\,\underline{\underline{\mathcal{F}}}) = \underline{\underline{\mathcal{P}}}+\underline{\underline{\mathcal{Q}}}\,\underline{\underline{\mathcal{F}}}. \tag{101}$$