Simplified Forced Magnetic Reconnection

1 Simplified Analytic Calculation

1.1 Analysis

Let $\hat{t} = t/(S^{1/3}\,\tau_H)$, where t represents time, S is the Lundquist number, and τ_H the hydromagnetic timescale. Let $\Psi_b(\hat{t})$ be the driving flux, and $\Psi_0(\hat{t})$ the reconnected flux. Let E_{sb} be a (real) dimensionless coupling constant, and E_{ss} the (real) dimesionless tearing stability index.

Let

$$\bar{\Psi}_b(g) = \int_0^\infty \Psi_b(\hat{t})\,\mathrm{e}^{-g\,\hat{t}}\,d\hat{t}, \tag{1}$$

$$\bar{\Psi}_0(g) = \int_0^\infty \Psi_0(\hat{t})\,\mathrm{e}^{-g\,\hat{t}}\,d\hat{t}. \tag{2}$$

In fact, if we assume that $\Psi_b(\hat{t}) = 0$ for $\hat{t} < 0$, and

$$\Psi_b(\hat{t}) = \Psi_{b\,0} \tag{3}$$

for $\hat{t} \geq 0$, then

$$\bar{\Psi}_b(g) = \frac{\Psi_{b\,0}}{g}. \tag{4}$$

The Laplace transformed simplified layer equation is

$$\frac{d}{dp}\left(\frac{p^2}{Q+p^2}\,\frac{d\bar{\phi}}{dp}\right) - Q\,p^2\,\bar{\phi} = 0, \tag{5}$$

where

$$Q = g + \mathrm{i}\,Q_E. \tag{6}$$

Here, $Q_E = S^{1/3}\,\omega_E\,\tau_H$, where ω_E is the E-cross-B frequency. Equation (5) must be solved subject to the boundary conditions that $\bar{\phi}(p)\rightarrow 0$ as $p\rightarrow\infty$, and

$$\bar{\phi}(p)\rightarrow\frac{\hat{\Delta}_s(g)}{\pi\,p}+1+{\cal O}(p) \tag{7}$$

as $p\rightarrow 0$. Furthermore,

$$F_s(g) = -\int_0^\infty\frac{p^2}{Q+p^2}\,\frac{d\bar{\phi}}{dp}\,dp. \tag{8}$$

Finally,

$$\bar{\Psi}_0(g) = \frac{E_{sb}\,F_s(g)\,\bar{\Psi}_b(g)}{g\,[S^{1/3}\,\hat{\Delta}_s(g)-E_{ss}]}. \tag{9}$$

It can be demonstrated that

$$\frac{p^2}{Q+p^2}\,\frac{d\bar{\phi}}{dp} = \frac{Q^{1/4}\,\Gamma(a)\,\Gamma(5/2)}{3\pi}\,{\rm e}^{-z/2}\,U(a,-1/2,z), \tag{10}$$

where $U(a,b,z)$ is a confluent hypergeometric function, and

$$z = Q^{1/2}\,p^2, \tag{11}$$

$$a = \frac{1}{4}\,(Q^{3/2}-1). \tag{12}$$

It follows that

$$F_s(g) = -\frac{\Gamma(a)\,\Gamma(5/2)}{6\pi}\int_0^\infty z^{-1/2}\,{\rm e}^{-z/2}\,U(a,-1/2,z)\,dz. \tag{13}$$

However, according to G&R 7.621.6,

$$\int_0^\infty z^{-1/2}\,{\rm e}^{-z/2}\,U(a,-1/2,z)\,dc = \frac{\Gamma(1/2)\,\Gamma(2)}{\Gamma(a+2)}\,F(1/2,2;a+2;1/2), \tag{14}$$

where $F(a,b;c;z)$ is a hypergeometric function. Thus,

$$F_s(g) = -\frac{\Gamma(a)\,\Gamma(5/2)}{6\pi}\,\frac{\Gamma(1/2)\,\Gamma(2)}{\Gamma(a+2)}\,F(1/2,2;a+2;1/2). \tag{15}$$

Finally,

$$\hat{\Delta}_s(g) = -\frac{\pi}{8}\,\frac{\Gamma(a)\,Q^{5/4}}{\Gamma(a+3/2)}. \tag{16}$$

1.2 Non-Constant-ψ Limit

Suppose that $|Q|\gg 1$, which implies that $|a|\gg 1$. It follows that

$$\begin{aligned}F_s(g) &\simeq -\frac{\Gamma(a)\,\Gamma(5/2)}{6\pi}\,\frac{\Gamma(1/2)\,\Gamma(2)}{\Gamma(a+2)}\simeq -\frac{\Gamma(5/2)\,\Gamma(1/2)\,\Gamma(2)}{6\pi\,a^2}\\ &= -\frac{1}{8\,a^2}\simeq -\frac{2}{Q^3}.\end{aligned}\tag{17}$$

Furthermore,

$$\hat{\Delta}_s\simeq -\frac{\pi\,Q^{5/4}}{8\,a^{3/2}}\simeq -\frac{\pi}{Q}.\tag{18}$$

Let us assume that $S^{1/3}/|Q|\gg 1\sim\mathcal{O}(E_{ss})$, because if $S^{1/3}/|Q|\ll 1$ then the layer width is not much smaller than the plasma minor radius, which invalidates asymptotic matching. Thus, we find that

$$\bar{\Psi}_0(g)=\frac{2}{\pi}\,\frac{E_{sb}\,\Psi_{b\,0}}{S^{1/3}}\,\frac{1}{g\,(g+\mathrm{i}\,Q_E)^2}.\tag{19}$$

If we assume that $|Q_E|\sim\mathcal{O}(1)$ then we get

$$\bar{\Psi}_0(g)\simeq\frac{2}{\pi}\,\frac{E_{sb}\,\Psi_{b\,0}}{S^{1/3}}\,\frac{1}{g^3},\tag{20}$$

for

$$1\ll g\ll S^{1/3}.\tag{21}$$

It follows that

$$\Psi_0(\hat{t})=\frac{E_{sb}\,\Psi_{b\,0}}{\pi\,S^{1/3}}\,\hat{t}^{\,2},\tag{22}$$

which is valid for

$$S^{-1/3}\ll\hat{t}\ll 1.\tag{23}$$

More generally, if $|Q_E|\gg 1$ then (Erdélyi 5.2.8)

$$\Psi_0(\hat{t})=\frac{2\,E_{sb}\,\Psi_{b\,0}}{\pi\,S^{1/3}}\,\frac{(1+\mathrm{i}\,Q_E\,\hat{t})\,\mathrm{e}^{-\mathrm{i}\,Q_E\,\hat{t}}-1}{Q_E^2}.\tag{24}$$

1.3 Constant-ψ Limit

Suppose that $|Q| \ll 1$, which implies that $a = -1/4$. If we assume that $|Q_E| \ll 1$ then this limit corresponds to $|g| \gg 1$, or

$$1 \ll \hat{t}. \tag{25}$$

It follows that

$$F_s(g) = -\frac{\Gamma(-1/4)\,\Gamma(5/2)}{6\pi}\,\frac{\Gamma(1/2)\,\Gamma(2)}{\Gamma(7/4)}\,F(1/2,2;7/4;1/2) = 1. \tag{26}$$

Moreover,

$$\hat{\Delta}_s = -\frac{\pi}{8}\,\frac{\Gamma(-1/4)}{\Gamma(5/4)}\,Q^{5/4} = \frac{2\pi\,\Gamma(3/4)}{\Gamma(1/4)}\,Q^{5/4}. \tag{27}$$

It follows that

$$\bar{\Psi}_0(g) = \frac{E_{sb}\,\Psi_{b\,0}}{g\,[c\,S^{1/3}\,(g+\mathrm{i}\,Q_E)^{5/4} - E_{ss}]}, \tag{28}$$

where

$$c = \frac{2\pi\,\Gamma(3/4)}{\Gamma(1/4)}. \tag{29}$$

Now,

$$\Psi_0(\hat{t}) = \frac{1}{2\pi\,\mathrm{i}}\int_{-\mathrm{i}\,\infty}^{\mathrm{i}\,\infty}\frac{E_{sb}\,\Psi_{b\,0}\,\mathrm{e}^{\,g\,\hat{t}}}{g\,[c\,S^{1/3}\,(g+\mathrm{i}\,Q_E)^{5/4} - E_{ss}]}\,dg, \tag{30}$$

which can be written

$$\Psi_0(\hat{t}) = \frac{E_{sb}\,\Psi_{b\,0}}{(-E_{ss})}\,\frac{1}{2\pi\,\mathrm{i}}\int_{-\mathrm{i}\,\infty}^{\mathrm{i}\,\infty}\frac{\mathrm{e}^{\,(p-\mathrm{i}\,Q_E)\,\hat{t}}}{(p-\mathrm{i}\,Q_E)\,(\lambda\,p^{5/4}+1)}\,dp, \tag{31}$$

where

$$\lambda = \frac{c\,S^{1/3}}{(-E_{ss})}. \tag{32}$$

Here, we are assuming that $E_{ss} < 0$. The contour integral in Eq. (23) can be decomposed into elements from the three poles (at $p_E = \mathrm{i}\,Q_E$, and $p_\pm = \mathrm{e}^{\pm\,\mathrm{i}\,4\pi/5}\,\lambda^{-4/5}$), respectively) plus a contribution from a branch cut along the $p = \mathrm{e}^{\mathrm{i}\,\pi}\,u$ axis (where u is real and positive). The contribution from the pole at $p = p_E$ is

$$\frac{E_{sb}\,\Psi_{b\,0}}{(-E_{ss})}\,\frac{1}{1+\mathrm{e}^{\,\mathrm{i}\,5\pi/8}\,\lambda\,Q_E^{5/4}}. \tag{33}$$

The contribution from the pole at $p=p_+$ is

$$-\frac{4}{5}\,\frac{E_{sb}\,\Psi_{b\,0}}{(-E_{ss})}\,\frac{p_+\,{\rm e}^{\,(p_+-{\rm i}\,Q_E)\,\hat{t}}}{p_+-{\rm i}\,Q_E}. \tag{34}$$

Likewise, the contribution from the pole at $p=p_-$ is

$$-\frac{4}{5}\,\frac{E_{sb}\,\Psi_{b\,0}}{(-E_{ss})}\,\frac{p_-\,{\rm e}^{\,(p_--{\rm i}\,Q_E)\,\hat{t}}}{p_--{\rm i}\,Q_E}. \tag{35}$$

Finally, the contribution from the branch cut is

$$\frac{E_{sb}\,\Psi_{b\,0}}{(-E_{ss})}\,\frac{\lambda}{\sqrt{2}\,\pi}\int_0^\infty\frac{{\rm e}^{-(u+{\rm i}\,Q_E)\,\hat{t}}\,u^{5/4}\,du}{(u+{\rm i}\,Q_E)\left(1-\sqrt{2}\,\lambda\,u^{5/4}+\lambda^2\,u^{5/2}\right)}. \tag{36}$$

Thus,

$$\Psi_0(\hat{t})=\frac{E_{sb}\,\Psi_{b\,0}}{(-E_{ss})}\left\{\frac{1}{1+{\rm e}^{\,{\rm i}\,5\pi/8}\,\lambda\,Q_E^{5/4}}-\frac{4}{5}\left[\frac{p_+\,{\rm e}^{\,(p_+-{\rm i}\,Q_E)\,\hat{t}}}{p_+-{\rm i}\,Q_E}+\frac{p_-\,{\rm e}^{\,(p_--{\rm i}\,Q_E)\,\hat{t}}}{p_--{\rm i}\,Q_E}\right]\right.$$
$$\left.+\frac{\lambda}{\sqrt{2}\,\pi}\int_0^\infty\frac{{\rm e}^{-(u+{\rm i}\,Q_E)\,\hat{t}}\,u^{5/4}\,du}{(u+{\rm i}\,Q_E)\left(1-\sqrt{2}\,\lambda\,u^{5/4}+\lambda^2\,u^{5/2}\right)}\right\}. \tag{37}$$

1.4 Numerical Calculation

Let

$$\frac{E_{sb}\,\bar{\Psi}_{b\,0}}{(-E_{ss})}=1. \tag{38}$$

Let

$$\frac{S^{1/3}}{(-E_{ss})}=\Sigma. \tag{39}$$

Thus,

$$\Psi_0(\hat{t})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\Psi}_0(g)\,{\rm e}^{\,g\,\hat{t}}\,dx, \tag{40}$$

where $g=\sigma+{\rm i}\,x$, σ is real and positive, and

$$\bar{\Psi}_0(g)=\frac{F_s(g)}{g\left[\Sigma\,\hat{\Delta}_s(g)+1\right]}. \tag{41}$$

Here,

$$F_s(g) = -\frac{1}{8}\sum_{n=0,\infty}\frac{(1/2)_n\,(n+1)_1}{(a)_{2+n}\,2^n}, \tag{42}$$

where

$$a = \frac{1}{4}\,(Q^{3/2}-1), \tag{43}$$

$$Q = g + \mathrm{i}\,Q_E, \tag{44}$$

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}. \tag{45}$$

In fact,

$$(z)_0 = 1, \tag{46}$$

$$(z)_n = (z+n-1)\,(z)_{n-1} \tag{47}$$

for $n>0$. Furthermore,

$$\hat{\Delta}_s(g) = -\frac{\pi}{8}\,\frac{Q^{5/4}}{a+1/2}\,\frac{\Gamma(a)}{\Gamma(a+1/2)}. \tag{48}$$

2 Solution of Simplified Layer Equations

2.1 Simplified Layer Equation

The simplified layer equation is

$$\frac{d}{dp}\left(\frac{p^2}{Q+p^2}\,\frac{d\bar{\phi}}{dp}\right) - Q\,p^2\,\bar{\phi} = 0, \tag{49}$$

which can also be written

$$\frac{d^2\bar{\phi}}{dp^2} + \frac{2\,Q}{p\,(Q+p^2)}\,\frac{d\bar{\phi}}{dp} - Q\,(Q+p^2)\,\bar{\phi} = 0. \tag{50}$$

The boundary conditions are $\bar{\phi}(p)\to 0$ as $p\to\infty$, and

$$\bar{\phi}(p) = \frac{\hat{\Delta}_s}{\pi\,p} + 1 + \mathcal{O}(p) \tag{51}$$

as $p \to 0$.

Let

$$\chi(p) = \frac{p^2}{Q + p^2}\,\frac{d\bar{\phi}}{dp}. \tag{52}$$

It follows that

$$\frac{d\chi}{dp} = Q\,p^2\,\bar{\phi}. \tag{53}$$

It is also easily seen that $\chi(p)$ satisfies

$$\frac{d}{dp}\left(\frac{1}{p^2}\,\frac{d\chi}{dp}\right) - \frac{Q\,(Q+p^2)}{p^2}\,\chi = 0, \tag{54}$$

which can also be written

$$\frac{d^2\chi}{dp^2} - \frac{2}{p}\,\frac{d\chi}{dp} - Q\,(Q+p^2)\,\chi = 0. \tag{55}$$

2.2 Small-p Limit

Let

$$\bar{\phi}(p) = \frac{\hat{\Delta}_s}{\pi\,p} + 1 + a\,p + {\cal O}(p^2). \tag{56}$$

Substituting into the layer equation, (50), we get

$$\bar{\phi}(p) = \frac{\hat{\Delta}_s}{\pi\,p} + 1 + \frac{\hat{\Delta}_s}{\pi}\left(\frac{Q^2}{2} - \frac{1}{Q}\right)p + {\cal O}(p^2). \tag{57}$$

Substituting into the alternative layer equation, (55), and also making use of Eqs. (53) and (57), we obtain

$$\chi(p) = -\frac{\hat{\Delta}_s}{Q\,\pi} + \frac{\hat{\Delta}_s\,Q}{2\pi}\,p^2 + \frac{Q}{3}\,p^3 + {\cal O}(p^4). \tag{58}$$

2.3 Large-p Limit

Let us write

$$\bar{\phi}(p) = A\,\exp\left(-\frac{Q^{1/2}\,p^2}{2}\right)f(p). \tag{59}$$

At large p, the layer equation, (50), yields

$$2\,p\,\frac{df}{dp}+(1+Q^{3/2})\,f=0. \tag{60}$$

The solution is

$$f(p)=p^{-\alpha}, \tag{61}$$

where

$$\alpha=\frac{1}{2}\,(1+Q^{3/2}). \tag{62}$$

It follows that

$$\bar{\phi}(p)=A\,p^{-\alpha}\,\exp\left(-\frac{Q^{1/2}\,p^2}{2}\right). \tag{63}$$

Let us write

$$\chi(p)=B\,\exp\left(-\frac{Q^{1/2}\,p^2}{2}\right)g(p). \tag{64}$$

At large p, the alternative layer equation, (55), yields

$$2\,p\,\frac{dg}{dp}+(-1+Q^{3/2})\,g=0. \tag{65}$$

The solution is

$$g(p)=p^{\beta}, \tag{66}$$

where

$$\beta=\frac{1}{2}\,(1-Q^{3/2}). \tag{67}$$

It follows that

$$\chi(p)=B\,p^{\beta}\,\exp\left(-\frac{Q^{1/2}\,p^2}{2}\right). \tag{68}$$

In order to be in the large-p limit, we require

$$p^2\gg|Q|, \tag{69}$$

$$|Q|^{1/2}\,p^2\gg 1, \tag{70}$$

or

$$p\gg|Q|^{1/2},|Q|^{-1/4}, \tag{71}$$

or

$$p\gg\frac{(1+|Q|^{3/2})^{1/2}}{|Q|^{1/4}}. \tag{72}$$

2.4 Riccati Transformation

Let

$$W(p) = \frac{p}{\bar{\phi}}\,\frac{d\bar{\phi}}{dp}. \tag{73}$$

It follows from Eq. (57) that

$$W(p) = -1 + \frac{\pi\,p}{\hat{\Delta}_s} + {\cal O}(p^2) \tag{74}$$

at small p. Furthermore, Eq. (63) implies that

$$W(p) = -Q^{1/2}\,p^2 - \alpha \tag{75}$$

at large p. Substituting into the layer equation, (50), we deduce that the differential equation that governs $W(p)$ is

$$\frac{dW}{dp} = -\frac{1}{p}\left(\frac{Q-p^2}{Q+p^2}\right)W - \frac{W^2}{p} + Q\,(Q+p^2)\,p. \tag{76}$$

The equation is integrated from large p, subject to the boundary condition (75), to small p. At small p,

$$\hat{\Delta}_s = \frac{\pi}{dW/dp}. \tag{77}$$

Let

$$V(p) = \frac{p}{\chi}\,\frac{d\chi}{dp}. \tag{78}$$

Substituting into the alternative layer equation, (55), we deduce that the differential equation that governs $V(p)$ is

$$\frac{dV}{dp} = \frac{3\,V}{p} - \frac{V^2}{p} + Q\,(Q+p^2)\,p. \tag{79}$$

According to Eq. (58),

$$V(p) = -Q^2\,p^2 - \frac{Q^2\,\pi}{\hat{\Delta}_s}\,p^3 \tag{80}$$

at small p. Moreover, according to Eq. (68),

$$V(p) = -Q^{1/2}\,p^2 + \beta \tag{81}$$

at large p.

2.5 Plan of Action

2.5.1 Stage 1

Launch solutions of Eqs. (76) and (79) from large p, subject to the respective boundary conditions (75) and (81), and integrate to small p. Save $V(p)$ onto a grid. Deduce the value of $\hat{\Delta}_s(g)$ from Eq. (77).

2.5.2 Stage 2

Launch the following system of equations from small p,

$$\frac{dU}{dp} = \frac{V(p)}{p}, \tag{82}$$

$$\frac{dF}{dp} = \exp[U(p)], \tag{83}$$

subject to the boundary conditions

$$U(0) = 0, \tag{84}$$

$$F(0) = 0, \tag{85}$$

and integrate to large p. Here, $V(p)$ is interpolated from the grid. Then

$$F_s(g) = \frac{\hat{\Delta}_s}{Q\,\pi}\,F(\infty). \tag{86}$$

2.6 Stage 3

Inverse Laplace transform:

$$\Psi_0(\hat{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_s(\sigma+\mathrm{i}\,\omega)\,\mathrm{e}^{(\sigma+\mathrm{i}\,\omega)\,\hat{t}}}{(\sigma+\mathrm{i}\,\omega)\left[\Sigma\,\hat{\Delta}_s(\sigma+\mathrm{i}\,\omega)+1\right]}\,d\omega. \tag{87}$$

3 Solution of Full Layer Equation

3.1 Full Layer Equation

The full layer equation is

$$\frac{d}{dp}\left(A\,\frac{dY_e}{dp}\right) - \frac{B}{C}\,p^2\,Y_e, \tag{88}$$

where $Y_e(p)$ is the Fourier-Laplace transformed electron stream-function, and

$$A = \frac{p^2}{g + \mathrm{i}\,(Q_E + Q_e) + p^2}, \tag{89}$$

$$B = (g + \mathrm{i}\,Q_E)[g + \mathrm{i}\,(Q_E + Q_i)] + [g + \mathrm{i}\,(Q_E + Q_i)]\,(P_\varphi + P_\perp)\,p^2 + P_\varphi\,P_\perp\,p^4, \tag{90}$$

$$C = g + \mathrm{i}\,(Q_E + Q_e) + \{P_\perp + [g + \mathrm{i}\,(Q_E + Q_i)\,D^2]\}\,p^2 + \iota_e^{-1}\,P_\varphi\,D^2\,p^4. \tag{91}$$

The boundary conditions are $Y_e \to 0$ as $p \to \infty$, and

$$Y_e(p) = \frac{\hat{\Delta}_s}{\pi\,p} + 1 + \mathcal{O}(p) \tag{92}$$

as $p \to 0$. We also need

$$F_s(g) = -\int_0^\infty A\,\frac{dY_e}{dp}\,dp. \tag{93}$$

Let

$$\chi(p) = A\,\frac{dY_e}{dp}. \tag{94}$$

It follows from Eqs. (88) and the previous two equations that

$$A\,\frac{d}{dp}\left(\frac{C}{B\,p^2}\,\frac{d\chi}{dp}\right) - \chi = 0, \tag{95}$$

as well as

$$F_s = -\int_0^\infty \chi(p)\,dp. \tag{96}$$

3.2 Small-p Limit

Let

$$Y_e(p) = \frac{\hat{\Delta}_s}{\pi\,p} + 1 + a\,p + b\,p^2 + \mathcal{O}(p^3). \tag{97}$$

Substituting into the generalized layer equation, (88), we get

$$Y_e(p) = \frac{\hat{\Delta}_s}{\pi\,p} + 1$$

$$+ \frac{\hat{\Delta}_s}{\pi} \left\{ \frac{1}{2} (g + \mathrm{i}\, Q_E) \, [g + \mathrm{i}\, (Q_E + Q_i)] - \frac{1}{g + \mathrm{i}\, (Q_E + Q_e)} \right\} p$$

$$+ \frac{1}{6} (g + \mathrm{i}\, Q_E) \, [g + \mathrm{i}\, (Q_E + Q_i)] \, p^2 + \mathcal{O}(p^3). \tag{98}$$

Substituting into Eq. (94), we get

$$\chi(p) = -\frac{\hat{\Delta}_s}{[g + \mathrm{i}\, (Q_E + Q_e)] \, \pi} + \frac{\Delta_s}{2\pi} \frac{(g + \mathrm{i}\, Q_E) \, [g + \mathrm{i}\, (Q_E + Q_i)]}{g + \mathrm{i}\, (Q_E + Q_e)} \, p^2$$

$$+ \frac{1}{3} \frac{(g + \mathrm{i}\, Q_E) \, [g + \mathrm{i}\, (Q_E + Q_i)]}{g + \mathrm{i}\, (Q_E + Q_e)} \, p^3 + \mathcal{O}(p^4). \tag{99}$$

3.3 Large-p Limit

In the large-p limit, if we write

$$A = 1 + \frac{\alpha}{p^2}, \tag{100}$$

$$\frac{B}{C} = \beta + \frac{\gamma}{p^2}, \tag{101}$$

and look for a solution of Eq. (94) of the form

$$Y_e(p) \propto p^x \, \exp\left(\frac{-\sqrt{\beta}\, p^2}{2} \right) \tag{102}$$

then we find that

$$x = \frac{\gamma - \sqrt{\beta}\, (1 - \sqrt{\beta}\, \alpha)}{2\sqrt{\beta}}. \tag{103}$$

It is easily seen that

$$\alpha = -[g + \mathrm{i}\, (Q_E + Q_e)], \tag{104}$$

$$\beta = \frac{\iota_e \, P_\perp}{D^2}, \tag{105}$$

$$\gamma = \frac{\iota_e \, P_\perp}{D^2} \left(1 + [g + \mathrm{i}\, (Q_E + Q_i)] \, \frac{P_\varphi + P_\perp}{P_\varphi \, P_\perp} \right.$$

$$\left. - \{ P_\perp + [g + \mathrm{i}\, (Q_E + Q_i) \, D^2] \} \, \frac{\iota_e}{P_\varphi \, D^2} \right). \tag{106}$$

Finally, it is easily seen from Eq. (94) that

$$\chi(p) \propto p^{x+1}\,\exp\left(\frac{-\sqrt{\beta}\,p^2}{2}\right) \tag{107}$$

at large-p.

In order to be in the large-p limit, we require

$$p \gg |g + \mathrm{i}\,(Q_E + Q_e)|^{1/2}, \tag{108}$$

$$p \gg \left|\frac{[g + \mathrm{i}\,(Q_E + Q_i)]\,(P_\varphi + P_\perp)}{P_\varphi\,P_\perp}\right|^{1/2}, \tag{109}$$

$$p \gg \left|\frac{(g + \mathrm{i}\,Q_E)\,[g + \mathrm{i}\,(Q_E + Q_i)]}{P_\varphi\,P_\perp}\right|^{1/4}, \tag{110}$$

$$p \gg \left|\frac{(P_\perp + [g + \mathrm{i}\,(Q_E + Q_i)\,D^2])}{\iota_e^{-1}\,P_\varphi\,D^2}\right|^{1/2}, \tag{111}$$

$$p \gg \left|\frac{g + \mathrm{i}\,(Q_E + Q_e)}{\iota_e^{-1}\,P_\varphi\,D^2}\right|^{1/4}, \tag{112}$$

$$p \gg \left(\frac{\iota_e^{-1}\,P_\varphi\,D^2}{P_\varphi\,P_\perp}\right)^{1/4}. \tag{113}$$

3.4 Ricatti Transformation

Let

$$W = \frac{p}{Y_e}\,\frac{dY_e}{dp}. \tag{114}$$

The generalized layer equation, (88), transforms to give

$$\frac{dW}{dp} = -\frac{A'}{p}\,W - \frac{W^{\,2}}{p} + \frac{B}{A\,C}\,p^3, \tag{115}$$

where

$$A' = \frac{g + \mathrm{i}\,(Q_E + Q_e) - p^2}{g + \mathrm{i}\,(Q_E + Q_e) + p^2}. \tag{116}$$

This equation must be solved subject to the boundary condition that

$$W(p) = x - \sqrt{\beta}\,p^2 \tag{117}$$

at large-p, and

$$W(p) = -1 + \frac{\pi\, p}{\hat{\Delta}_s} \tag{118}$$

at small-p.

Let

$$V = \frac{p}{\chi}\,\frac{d\chi}{dp}. \tag{119}$$

Equation (95) transforms to give

$$\frac{dV}{dp} = 2\,p\,(B' - C')\,V + \frac{3\,V}{p} - \frac{V^2}{p} + \frac{B}{A\,C}\,p^3, \tag{120}$$

where

$$B' = \frac{[g + {\rm i}\,(Q_E+Q_i)]\,(P_\varphi+P_\perp) + 2\,P_\varphi\,P_\perp\,p^2}{(g+{\rm i}\,Q_E)[g+{\rm i}\,(Q_E+Q_i)] + [g+{\rm i}\,(Q_E+Q_i)]\,(P_\varphi+P_\perp)\,p^2 + P_\varphi\,P_\perp\,p^4}, \tag{121}$$

$$C' = \frac{\{P_\perp + [g+{\rm i}\,(Q_E+Q_i)\,D^2\} + 2\,\iota_e^{-1}\,P_\varphi\,D^2\,p^2}{g+{\rm i}\,(Q_E+Q_e) + \{P_\perp + [g+{\rm i}\,(Q_E+Q_i)\,D^2\}\,p^2 + \iota_e^{-1}\,P_\varphi\,D^2\,p^4}. \tag{122}$$

This equation must be solved subject to the boundary condition that

$$V(p) = 1 + x - \sqrt{\beta}\,p^2 \tag{123}$$

at large-p.

3.5 Revised Plan of Action

3.5.1 Stage 1

Launch solutions of Eqs. (115) and (120) from large p, subject to the respective boundary conditions (117) and (123), and integrate to small p. Save $V(p)$ onto a grid. Deduce the value of $\hat{\Delta}_s(g)$ from Eq. (118).

3.5.2 Stage 2

Launch the following system of equations from small p,

$$\frac{dU}{dp} = \frac{V(p)}{p}, \tag{124}$$

$$\frac{dF}{dp} = \exp[U(p)], \tag{125}$$

subject to the boundary conditions

$$U(0) = 0, \tag{126}$$

$$F(0) = 0, \tag{127}$$

and integrate to large p. Here, $V(p)$ is interpolated from the grid. Then

$$F_s(g) = \frac{\hat{\Delta}_s}{[g + \mathrm{i}\,(Q_E + Q_e)]\,\pi}\,F(\infty). \tag{128}$$

3.6 Stage 3

Inverse Laplace transform:

$$\Psi_0(\hat{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F_s(\sigma + \mathrm{i}\,\omega)\,\mathrm{e}^{(\sigma+\mathrm{i}\,\omega)\,\hat{t}}}{(\sigma + \mathrm{i}\,\omega)\left[\Sigma\,\hat{\Delta}_s(\sigma + \mathrm{i}\,\omega) + 1\right]}\,d\omega. \tag{129}$$