

Simplified Forced Magnetic Reconnection

1 Simplified Analytic Calculation

1.1 Analysis

Let $\hat{t} = t/(S^{1/3} \tau_H)$, where t represents time, S is the Lundquist number, and τ_H the hydromagnetic timescale. Let $\Psi_b(\hat{t})$ be the driving flux, and $\Psi_0(\hat{t})$ the reconnected flux. Let E_{sb} be a (real) dimensionless coupling constant, and E_{ss} the (real) dimensionless tearing stability index.

Let

$$\bar{\Psi}_b(g) = \int_0^\infty \Psi_b(\hat{t}) e^{-g\hat{t}} d\hat{t}, \quad (1)$$

$$\bar{\Psi}_0(g) = \int_0^\infty \Psi_0(\hat{t}) e^{-g\hat{t}} d\hat{t}. \quad (2)$$

In fact, if we assume that $\Psi_b(\hat{t}) = 0$ for $\hat{t} < 0$, and

$$\Psi_b(\hat{t}) = \Psi_{b0} \quad (3)$$

for $\hat{t} \geq 0$, then

$$\bar{\Psi}_b(g) = \frac{\Psi_{b0}}{g}. \quad (4)$$

The Laplace transformed simplified layer equation is

$$\frac{d}{dp} \left(\frac{p^2}{Q + p^2} \frac{d\bar{\phi}}{dp} \right) - Q p^2 \bar{\phi} = 0, \quad (5)$$

where

$$Q = g + i Q_E. \quad (6)$$

Here, $Q_E = S^{1/3} \omega_E \tau_H$, where ω_E is the E-cross-B frequency. Equation (5) must be solved subject to the boundary conditions that $\bar{\phi}(p) \rightarrow 0$ as $p \rightarrow \infty$, and

$$\bar{\phi}(p) \rightarrow \frac{\hat{\Delta}_s(g)}{\pi p} + 1 + \mathcal{O}(p) \quad (7)$$

as $p \rightarrow 0$. Furthermore,

$$F_s(g) = - \int_0^\infty \frac{p^2}{Q + p^2} \frac{d\bar{\phi}}{dp} dp. \quad (8)$$

Finally,

$$\bar{\Psi}_0(g) = \frac{E_{sb} F_s(g) \bar{\Psi}_b(g)}{g [S^{1/3} \hat{\Delta}_s(g) - E_{ss}]}. \quad (9)$$

It can be demonstrated that

$$\frac{p^2}{Q + p^2} \frac{d\bar{\phi}}{dp} = \frac{Q^{1/4} \Gamma(a) \Gamma(5/2)}{3\pi} e^{-z/2} U(a, -1/2, z), \quad (10)$$

where $U(a, b, z)$ is a confluent hypergeometric function, and

$$z = Q^{1/2} p^2, \quad (11)$$

$$a = \frac{1}{4} (Q^{3/2} - 1). \quad (12)$$

It follows that

$$F_s(g) = - \frac{\Gamma(a) \Gamma(5/2)}{6\pi} \int_0^\infty z^{-1/2} e^{-z/2} U(a, -1/2, z) dz. \quad (13)$$

However, according to G&R 7.621.6,

$$\int_0^\infty z^{-1/2} e^{-z/2} U(a, -1/2, z) dz = \frac{\Gamma(1/2) \Gamma(2)}{\Gamma(a+2)} F(1/2, 2; a+2; 1/2), \quad (14)$$

where $F(a, b; c; z)$ is a hypergeometric function. Thus,

$$F_s(g) = - \frac{\Gamma(a) \Gamma(5/2)}{6\pi} \frac{\Gamma(1/2) \Gamma(2)}{\Gamma(a+2)} F(1/2, 2; a+2; 1/2). \quad (15)$$

Finally,

$$\hat{\Delta}_s(g) = - \frac{\pi}{8} \frac{\Gamma(a) Q^{5/4}}{\Gamma(a+3/2)}. \quad (16)$$

1.2 Non-Constant- ψ Limit

Suppose that $|Q| \gg 1$, which implies that $|a| \gg 1$. It follows that

$$\begin{aligned} F_s(g) &\simeq -\frac{\Gamma(a)\Gamma(5/2)}{6\pi} \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(a+2)} \simeq -\frac{\Gamma(5/2)\Gamma(1/2)\Gamma(2)}{6\pi a^2} \\ &= -\frac{1}{8a^2} \simeq -\frac{2}{Q^3}. \end{aligned} \quad (17)$$

Furthermore,

$$\hat{\Delta}_s \simeq -\frac{\pi Q^{5/4}}{8a^{3/2}} \simeq -\frac{\pi}{Q}. \quad (18)$$

Let us assume that $S^{1/3}/|Q| \gg 1 \sim \mathcal{O}(E_{ss})$, because if $S^{1/3}/|Q| \ll 1$ then the layer width is not much smaller than the plasma minor radius, which invalidates asymptotic matching. Thus, we find that

$$\bar{\Psi}_0(g) = \frac{2}{\pi} \frac{E_{sb}\Psi_{b0}}{S^{1/3}} \frac{1}{g(g + iQ_E)^2}. \quad (19)$$

If we assume that $|Q_E| \sim \mathcal{O}(1)$ then we get

$$\bar{\Psi}_0(g) \simeq \frac{2}{\pi} \frac{E_{sb}\Psi_{b0}}{S^{1/3}} \frac{1}{g^3}, \quad (20)$$

for

$$1 \ll g \ll S^{1/3}. \quad (21)$$

It follows that

$$\Psi_0(\hat{t}) = \frac{E_{sb}\Psi_{b0}}{\pi S^{1/3}} \hat{t}^2, \quad (22)$$

which is valid for

$$S^{-1/3} \ll \hat{t} \ll 1. \quad (23)$$

More generally, if $|Q_E| \gg 1$ then (Erdélyi 5.2.8)

$$\Psi_0(\hat{t}) = \frac{2E_{sb}\Psi_{b0}}{\pi S^{1/3}} \frac{(1 + iQ_E\hat{t})e^{-iQ_E\hat{t}} - 1}{Q_E^2}. \quad (24)$$

1.3 Constant- ψ Limit

Suppose that $|Q| \ll 1$, which implies that $a = -1/4$. If we assume that $|Q_E| \ll 1$ then this limit corresponds to $|g| \gg 1$, or

$$1 \ll \hat{t}. \quad (25)$$

It follows that

$$F_s(g) = -\frac{\Gamma(-1/4)\Gamma(5/2)}{6\pi} \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(7/4)} F(1/2, 2; 7/4; 1/2) = 1. \quad (26)$$

Moreover,

$$\hat{\Delta}_s = -\frac{\pi}{8} \frac{\Gamma(-1/4)}{\Gamma(5/4)} Q^{5/4} = \frac{2\pi\Gamma(3/4)}{\Gamma(1/4)} Q^{5/4}. \quad (27)$$

It follows that

$$\bar{\Psi}_0(g) = \frac{E_{sb} \Psi_{b0}}{g [c S^{1/3} (g + i Q_E)^{5/4} - E_{ss}]}, \quad (28)$$

where

$$c = \frac{2\pi\Gamma(3/4)}{\Gamma(1/4)}. \quad (29)$$

Now,

$$\Psi_0(\hat{t}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{E_{sb} \Psi_{b0} e^{g\hat{t}}}{g [c S^{1/3} (g + i Q_E)^{5/4} - E_{ss}]} dg, \quad (30)$$

which can be written

$$\Psi_0(\hat{t}) = \frac{E_{sb} \Psi_{b0}}{(-E_{ss})} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{(p-iQ_E)\hat{t}}}{(p-iQ_E)(\lambda p^{5/4} + 1)} dp, \quad (31)$$

where

$$\lambda = \frac{c S^{1/3}}{(-E_{ss})}. \quad (32)$$

Here, we are assuming that $E_{ss} < 0$. The contour integral in Eq. (23) can be decomposed into elements from the three poles (at $p_E = i Q_E$, and $p_{\pm} = e^{\pm i 4\pi/5} \lambda^{-4/5}$), respectively) plus a contribution from a branch cut along the $p = e^{i\pi} u$ axis (where u is real and positive). The contribution from the pole at $p = p_E$ is

$$\frac{E_{sb} \Psi_{b0}}{(-E_{ss})} \frac{1}{1 + e^{i 5\pi/8} \lambda Q_E^{5/4}}. \quad (33)$$

The contribution from the pole at $p = p_+$ is

$$-\frac{4}{5} \frac{E_{sb} \Psi_{b0}}{(-E_{ss})} \frac{p_+ e^{(p_+ - i Q_E) \hat{t}}}{p_+ - i Q_E}. \quad (34)$$

Likewise, the contribution from the pole at $p = p_-$ is

$$-\frac{4}{5} \frac{E_{sb} \Psi_{b0}}{(-E_{ss})} \frac{p_- e^{(p_- - i Q_E) \hat{t}}}{p_- - i Q_E}. \quad (35)$$

Finally, the contribution from the branch cut is

$$\frac{E_{sb} \Psi_{b0}}{(-E_{ss})} \frac{\lambda}{\sqrt{2} \pi} \int_0^\infty \frac{e^{-(u+i Q_E) \hat{t}} u^{5/4} du}{(u + i Q_E) (1 - \sqrt{2} \lambda u^{5/4} + \lambda^2 u^{5/2})}. \quad (36)$$

Thus,

$$\begin{aligned} \Psi_0(\hat{t}) = \frac{E_{sb} \Psi_{b0}}{(-E_{ss})} \left\{ \frac{1}{1 + e^{i 5\pi/8} \lambda Q_E^{5/4}} - \frac{4}{5} \left[\frac{p_+ e^{(p_+ - i Q_E) \hat{t}}}{p_+ - i Q_E} + \frac{p_- e^{(p_- - i Q_E) \hat{t}}}{p_- - i Q_E} \right] \right. \\ \left. + \frac{\lambda}{\sqrt{2} \pi} \int_0^\infty \frac{e^{-(u+i Q_E) \hat{t}} u^{5/4} du}{(u + i Q_E) (1 - \sqrt{2} \lambda u^{5/4} + \lambda^2 u^{5/2})} \right\}. \end{aligned} \quad (37)$$

1.4 Numerical Calculation

Let

$$\frac{E_{sb} \bar{\Psi}_{b0}}{(-E_{ss})} = 1. \quad (38)$$

Let

$$\frac{S^{1/3}}{(-E_{ss})} = \Sigma. \quad (39)$$

Thus,

$$\Psi_0(\hat{t}) = \frac{1}{2\pi} \int_{-\infty}^\infty \bar{\Psi}_0(g) e^{g \hat{t}} dx, \quad (40)$$

where $g = \sigma + i x$, σ is real and positive, and

$$\bar{\Psi}_0(g) = \frac{F_s(g)}{g \left[\Sigma \hat{\Delta}_s(g) + 1 \right]}. \quad (41)$$

Here,

$$F_s(g) = -\frac{1}{8} \sum_{n=0,\infty} \frac{(1/2)_n (n+1)_1}{(a)_{2+n} 2^n}, \quad (42)$$

where

$$a = \frac{1}{4} (Q^{3/2} - 1), \quad (43)$$

$$Q = g + i Q_E, \quad (44)$$

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}. \quad (45)$$

In fact,

$$(z)_0 = 1, \quad (46)$$

$$(z)_n = (z+n-1) (z)_{n-1} \quad (47)$$

for $n > 0$. Furthermore,

$$\hat{\Delta}_s(g) = -\frac{\pi}{8} \frac{Q^{5/4}}{a+1/2} \frac{\Gamma(a)}{\Gamma(a+1/2)}. \quad (48)$$

2 Solution of Simplified Layer Equations

2.1 Simplified Layer Equation

The simplified layer equation is

$$\frac{d}{dp} \left(\frac{p^2}{Q+p^2} \frac{d\bar{\phi}}{dp} \right) - Q p^2 \bar{\phi} = 0, \quad (49)$$

which can also be written

$$\frac{d^2 \bar{\phi}}{dp^2} + \frac{2Q}{p(Q+p^2)} \frac{d\bar{\phi}}{dp} - Q(Q+p^2) \bar{\phi} = 0. \quad (50)$$

The boundary conditions are $\bar{\phi}(p) \rightarrow 0$ as $p \rightarrow \infty$, and

$$\bar{\phi}(p) = \frac{\hat{\Delta}_s}{\pi p} + 1 + \mathcal{O}(p) \quad (51)$$

as $p \rightarrow 0$.

Let

$$\chi(p) = \frac{p^2}{Q + p^2} \frac{d\bar{\phi}}{dp}. \quad (52)$$

It follows that

$$\frac{d\chi}{dp} = Q p^2 \bar{\phi}. \quad (53)$$

It is also easily seen that $\chi(p)$ satisfies

$$\frac{d}{dp} \left(\frac{1}{p^2} \frac{d\chi}{dp} \right) - \frac{Q(Q + p^2)}{p^2} \chi = 0, \quad (54)$$

which can also be written

$$\frac{d^2\chi}{dp^2} - \frac{2}{p} \frac{d\chi}{dp} - Q(Q + p^2) \chi = 0. \quad (55)$$

2.2 Small- p Limit

Let

$$\bar{\phi}(p) = \frac{\hat{\Delta}_s}{\pi p} + 1 + a p + \mathcal{O}(p^2). \quad (56)$$

Substituting into the layer equation, (50), we get

$$\bar{\phi}(p) = \frac{\hat{\Delta}_s}{\pi p} + 1 + \frac{\hat{\Delta}_s}{\pi} \left(\frac{Q^2}{2} - \frac{1}{Q} \right) p + \mathcal{O}(p^2). \quad (57)$$

Substituting into the alternative layer equation, (55), and also making use of Eqs. (53) and (57), we obtain

$$\chi(p) = -\frac{\hat{\Delta}_s}{Q \pi} + \frac{\hat{\Delta}_s Q}{2\pi} p^2 + \frac{Q}{3} p^3 + \mathcal{O}(p^4). \quad (58)$$

2.3 Large- p Limit

Let us write

$$\bar{\phi}(p) = A \exp \left(-\frac{Q^{1/2} p^2}{2} \right) f(p). \quad (59)$$

At large p , the layer equation, (50), yields

$$2p \frac{df}{dp} + (1 + Q^{3/2}) f = 0. \quad (60)$$

The solution is

$$f(p) = p^{-\alpha}, \quad (61)$$

where

$$\alpha = \frac{1}{2} (1 + Q^{3/2}). \quad (62)$$

It follows that

$$\bar{\phi}(p) = A p^{-\alpha} \exp \left(-\frac{Q^{1/2} p^2}{2} \right). \quad (63)$$

Let us write

$$\chi(p) = B \exp \left(-\frac{Q^{1/2} p^2}{2} \right) g(p). \quad (64)$$

At large p , the alternative layer equation, (55), yields

$$2p \frac{dg}{dp} + (-1 + Q^{3/2}) g = 0. \quad (65)$$

The solution is

$$g(p) = p^{\beta}, \quad (66)$$

where

$$\beta = \frac{1}{2} (1 - Q^{3/2}). \quad (67)$$

It follows that

$$\chi(p) = B p^{\beta} \exp \left(-\frac{Q^{1/2} p^2}{2} \right). \quad (68)$$

In order to be in the large- p limit, we require

$$p^2 \gg |Q|, \quad (69)$$

$$|Q|^{1/2} p^2 \gg 1, \quad (70)$$

or

$$p \gg |Q|^{1/2}, |Q|^{-1/4}, \quad (71)$$

or

$$p \gg \frac{(1 + |Q|^{3/2})^{1/2}}{|Q|^{1/4}}. \quad (72)$$

2.4 Riccati Transformation

Let

$$W(p) = \frac{p}{\bar{\phi}} \frac{d\bar{\phi}}{dp}. \quad (73)$$

It follows from Eq. (57) that

$$W(p) = -1 + \frac{\pi p}{\hat{\Delta}_s} + \mathcal{O}(p^2) \quad (74)$$

at small p . Furthermore, Eq. (63) implies that

$$W(p) = -Q^{1/2} p^2 - \alpha \quad (75)$$

at large p . Substituting into the layer equation, (50), we deduce that the differential equation that governs $W(p)$ is

$$\frac{dW}{dp} = -\frac{1}{p} \left(\frac{Q - p^2}{Q + p^2} \right) W - \frac{W^2}{p} + Q(Q + p^2)p. \quad (76)$$

The equation is integrated from large p , subject to the boundary condition (75), to small p . At small p ,

$$\hat{\Delta}_s = \frac{\pi}{dW/dp}. \quad (77)$$

Let

$$V(p) = \frac{p}{\chi} \frac{d\chi}{dp}. \quad (78)$$

Substituting into the alternative layer equation, (55), we deduce that the differential equation that governs $V(p)$ is

$$\frac{dV}{dp} = \frac{3V}{p} - \frac{V^2}{p} + Q(Q + p^2)p. \quad (79)$$

According to Eq. (58),

$$V(p) = -Q^2 p^2 - \frac{Q^2 \pi}{\hat{\Delta}_s} p^3 \quad (80)$$

at small p . Moreover, according to Eq. (68),

$$V(p) = -Q^{1/2} p^2 + \beta \quad (81)$$

at large p .

2.5 Plan of Action

2.5.1 Stage 1

Launch solutions of Eqs. (76) and (79) from large p , subject to the respective boundary conditions (75) and (81), and integrate to small p . Save $V(p)$ onto a grid. Deduce the value of $\hat{\Delta}_s(g)$ from Eq. (77).

2.5.2 Stage 2

Launch the following system of equations from small p ,

$$\frac{dU}{dp} = \frac{V(p)}{p}, \quad (82)$$

$$\frac{dF}{dp} = \exp[U(p)], \quad (83)$$

subject to the boundary conditions

$$U(0) = 0, \quad (84)$$

$$F(0) = 0, \quad (85)$$

and integrate to large p . Here, $V(p)$ is interpolated from the grid. Then

$$F_s(g) = \frac{\hat{\Delta}_s}{Q\pi} F(\infty). \quad (86)$$

2.6 Stage 3

Inverse Laplace transform:

$$\Psi_0(\hat{t}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_s(\sigma + i\omega) e^{(\sigma + i\omega)\hat{t}}}{(\sigma + i\omega) [\Sigma \hat{\Delta}_s(\sigma + i\omega) + 1]} d\omega. \quad (87)$$

3 Solution of Full Layer Equation

3.1 Full Layer Equation

The full layer equation is

$$\frac{d}{dp} \left(A \frac{dY_e}{dp} \right) - \frac{B}{C} p^2 Y_e, \quad (88)$$

where $Y_e(p)$ is the Fourier-Laplace transformed electron stream-function, and

$$A = \frac{p^2}{g + i(Q_E + Q_e) + p^2}, \quad (89)$$

$$B = (g + iQ_E)[g + i(Q_E + Q_i)] + [g + i(Q_E + Q_i)](P_\varphi + P_\perp)p^2 + P_\varphi P_\perp p^4, \quad (90)$$

$$C = g + i(Q_E + Q_e) + \{P_\perp + [g + i(Q_E + Q_i)D^2]\}p^2 + \iota_e^{-1} P_\varphi D^2 p^4. \quad (91)$$

The boundary conditions are $Y_e \rightarrow 0$ as $p \rightarrow \infty$, and

$$Y_e(p) = \frac{\hat{\Delta}_s}{\pi p} + 1 + \mathcal{O}(p) \quad (92)$$

as $p \rightarrow 0$. We also need

$$F_s(g) = - \int_0^\infty A \frac{dY_e}{dp} dp. \quad (93)$$

Let

$$\chi(p) = A \frac{dY_e}{dp}. \quad (94)$$

It follows from Eqs. (88) and the previous two equations that

$$A \frac{d}{dp} \left(\frac{C}{B p^2} \frac{d\chi}{dp} \right) - \chi = 0, \quad (95)$$

as well as

$$F_s = - \int_0^\infty \chi(p) dp. \quad (96)$$

3.2 Small- p Limit

Let

$$Y_e(p) = \frac{\hat{\Delta}_s}{\pi p} + 1 + a p + b p^2 + \mathcal{O}(p^3). \quad (97)$$

Substituting into the generalized layer equation, (88), we get

$$Y_e(p) = \frac{\hat{\Delta}_s}{\pi p} + 1$$

$$\begin{aligned}
& + \frac{\hat{\Delta}_s}{\pi} \left\{ \frac{1}{2} (g + i Q_E) [g + i (Q_E + Q_i)] - \frac{1}{g + i (Q_E + Q_e)} \right\} p \\
& + \frac{1}{6} (g + i Q_E) [g + i (Q_E + Q_i)] p^2 + \mathcal{O}(p^3).
\end{aligned} \tag{98}$$

Substituting into Eq. (94), we get

$$\begin{aligned}
\chi(p) = & - \frac{\hat{\Delta}_s}{[g + i (Q_E + Q_e)] \pi} + \frac{\Delta_s}{2\pi} \frac{(g + i Q_E) [g + i (Q_E + Q_i)]}{g + i (Q_E + Q_e)} p^2 \\
& + \frac{1}{3} \frac{(g + i Q_E) [g + i (Q_E + Q_i)]}{g + i (Q_E + Q_e)} p^3 + \mathcal{O}(p^4).
\end{aligned} \tag{99}$$

3.3 Large- p Limit

In the large- p limit, if we write

$$A = 1 + \frac{\alpha}{p^2}, \tag{100}$$

$$\frac{B}{C} = \beta + \frac{\gamma}{p^2}, \tag{101}$$

and look for a solution of Eq. (94) of the form

$$Y_e(p) \propto p^x \exp \left(\frac{-\sqrt{\beta} p^2}{2} \right) \tag{102}$$

then we find that

$$x = \frac{\gamma - \sqrt{\beta} (1 - \sqrt{\beta} \alpha)}{2\sqrt{\beta}}. \tag{103}$$

It is easily seen that

$$\alpha = -[g + i (Q_E + Q_e)], \tag{104}$$

$$\beta = \frac{\iota_e P_\perp}{D^2}, \tag{105}$$

$$\begin{aligned}
\gamma = & \frac{\iota_e P_\perp}{D^2} \left(1 + [g + i (Q_E + Q_i)] \frac{P_\varphi + P_\perp}{P_\varphi P_\perp} \right. \\
& \left. - \{P_\perp + [g + i (Q_E + Q_i)] D^2\} \frac{\iota_e}{P_\varphi D^2} \right).
\end{aligned} \tag{106}$$

Finally, it is easily seen from Eq. (94) that

$$\chi(p) \propto p^{x+1} \exp\left(\frac{-\sqrt{\beta} p^2}{2}\right) \quad (107)$$

at large- p .

In order to be in the large- p limit, we require

$$p \gg |g + i(Q_E + Q_e)|^{1/2}, \quad (108)$$

$$p \gg \left| \frac{[g + i(Q_E + Q_i)](P_\varphi + P_\perp)}{P_\varphi P_\perp} \right|^{1/2}, \quad (109)$$

$$p \gg \left| \frac{(g + iQ_E)[g + i(Q_E + Q_i)]}{P_\varphi P_\perp} \right|^{1/4}, \quad (110)$$

$$p \gg \left| \frac{(P_\perp + [g + i(Q_E + Q_i) D^2])}{\iota_e^{-1} P_\varphi D^2} \right|^{1/2}, \quad (111)$$

$$p \gg \left| \frac{g + i(Q_E + Q_e)}{\iota_e^{-1} P_\varphi D^2} \right|^{1/4}, \quad (112)$$

$$p \gg \left(\frac{\iota_e^{-1} P_\varphi D^2}{P_\varphi P_\perp} \right)^{1/4}. \quad (113)$$

3.4 Low- D Limit

If

$$D^2 \ll \left| \frac{P_\perp}{\hat{g} + i(Q_E + Q_e)} \right|, \quad \frac{\iota_e P_\perp}{P_\varphi^{2/3}} \quad (114)$$

then the terms in the layer equations involving D^2 are negligible. In this case, in the large- p limit, we can write

$$A = 1 + \frac{\alpha}{p^2}, \quad (115)$$

$$\frac{B}{C} = \beta p^2 + \gamma, \quad (116)$$

where

$$\alpha = -[\hat{g} + i(Q_E + Q_e)], \quad (117)$$

$$\beta = P_\varphi, \quad (118)$$

$$\gamma = -i(Q_e - Q_i) \frac{P_\varphi}{P_\perp} + \hat{g} + i(Q_E + Q_i). \quad (119)$$

The solution of Eq. (94) becomes

$$\hat{Y}_e \propto p^{-1} \exp \left(x p - \frac{\sqrt{\beta} p^3}{3} \right) \quad (120)$$

where

$$x = \frac{\alpha \beta - \gamma}{2\sqrt{\beta}}. \quad (121)$$

Finally, it is easily seen from Eq. (94) that

$$\chi(p) \propto p \exp \left(x p - \frac{\sqrt{\beta} p^3}{3} \right), \quad (122)$$

at large- p .

In order to be in the large- p limit, we require

$$p \gg |\hat{g} + i(Q_E + Q_e)|^{1/2}, \quad (123)$$

$$p \gg \left| \frac{[\hat{g} + i(Q_E + Q_i)](P_\varphi + P_\perp)}{P_\varphi P_\perp} \right|^{1/2}, \quad (124)$$

$$p \gg \left| \frac{(\hat{g} + iQ_E)[\hat{g} + i(Q_E + Q_i)]}{P_\varphi P_\perp} \right|^{1/4}, \quad (125)$$

$$p \gg \left| \frac{\hat{g} + i(Q_E + Q_e)}{P_\perp} \right|^{1/2}, \quad (126)$$

$$p \gg P_\varphi^{-1/6}. \quad (127)$$

3.5 Ricatti Transformation

Let

$$W = \frac{p}{Y_e} \frac{dY_e}{dp}. \quad (128)$$

The generalized layer equation, (88), transforms to give

$$\frac{dW}{dp} = -\frac{A'}{p} W - \frac{W^2}{p} + \frac{B}{AC} p^3, \quad (129)$$

where

$$A' = \frac{g + i(Q_E + Q_e) - p^2}{g + i(Q_E + Q_e) + p^2}. \quad (130)$$

This equation must be solved subject to the boundary condition that

$$W(p) = x - \sqrt{\beta} p^2 \quad (131)$$

at large- p , and

$$W(p) = -1 + \frac{\pi p}{\hat{\Delta}_s} \quad (132)$$

at small- p . However, in the low- D limit,

$$W(p) = -1 + x p - \sqrt{\beta} p^3 \quad (133)$$

at large- p .

Let

$$V = \frac{p}{\chi} \frac{d\chi}{dp}. \quad (134)$$

Equation (95) transforms to give

$$\frac{dV}{dp} = 2p(B' - C')V + \frac{3V}{p} - \frac{V^2}{p} + \frac{B}{AC}p^3, \quad (135)$$

where

$$B' = \frac{[g + i(Q_E + Q_i)](P_\varphi + P_\perp) + 2P_\varphi P_\perp p^2}{(g + iQ_E)[g + i(Q_E + Q_i)] + [g + i(Q_E + Q_i)](P_\varphi + P_\perp)p^2 + P_\varphi P_\perp p^4}, \quad (136)$$

$$C' = \frac{\{P_\perp + [g + i(Q_E + Q_i)]D^2\} + 2\iota_e^{-1}P_\varphi D^2 p^2}{g + i(Q_E + Q_e) + \{P_\perp + [g + i(Q_E + Q_i)]D^2\}p^2 + \iota_e^{-1}P_\varphi D^2 p^4}. \quad (137)$$

This equation must be solved subject to the boundary condition that

$$V(p) = 1 + x - \sqrt{\beta} p^2 \quad (138)$$

at large- p . However, in the low- D limit,

$$V(p) = 1 + x p - \sqrt{\beta} p^3 \quad (139)$$

at large- p .

3.6 Revised Plan of Action

3.6.1 Stage 1

Launch solutions of Eqs. (129) and (135) from large p , subject to the respective boundary conditions (131) and (138), and integrate to small p . Save $V(p)$ onto a grid. Deduce the value of $\hat{\Delta}_s(g)$ from Eq. (132).

3.6.2 Stage 2

Launch the following system of equations from small p ,

$$\frac{dU}{dp} = \frac{V(p)}{p}, \quad (140)$$

$$\frac{dF}{dp} = \exp[U(p)], \quad (141)$$

subject to the boundary conditions

$$U(0) = 0, \quad (142)$$

$$F(0) = 0, \quad (143)$$

and integrate to large p . Here, $V(p)$ is interpolated from the grid. Then

$$F_s(g) = \frac{\hat{\Delta}_s}{[g + i(Q_E + Q_e)]\pi} F(\infty). \quad (144)$$

3.7 Stage 3

Inverse Laplace transform:

$$\Psi_0(\hat{t}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_s(\sigma + i\omega) e^{(\sigma + i\omega)\hat{t}}}{(\sigma + i\omega) [\Sigma \hat{\Delta}_s(\sigma + i\omega) + 1]} d\omega. \quad (145)$$