MATH-UA.0251 Intro Math Modeling

Aref Hashemi
NYU Courant

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S1 Dimensional Analysis, Buckingham π theorem

Characteristics of a physical system

- qualitative: length, time, stress, velocity, etc
- quantitative: 2 m, 5 s, 100 Pa, etc.

2 ← a number; m ← a standard (unit)

given in terms of "primary" quantities: L, T, M, Θ, C, ...

(L: length, T: time, M: mass, Θ: temperature, C: charge) ← dimensions

(*) primary quantities can then be used to provide qualitative description of any "secondary" quantity.

EX. Area $\doteq L^2$, Velocity $\doteq LT^{-1}$, ...

the dimensions of an area is length x length

Dimensional Homogeneity
⇝ dimensions of LHS $\doteq$ dimensions of RHS
- X + Y → X and Y must have the same dimensions. (e.g., L + M is meaningless!)

EX. Gravitational force: $F = G\frac{m_1 m_2}{r^2}$

m_1: mass of object 1 $\doteq M$
m_2: mass of object 2 $\doteq M$
r : distance between the two objects $\doteq L$
F : force $\doteq MLT^{-2}$

$$\rightsquigarrow MLT^{-2} \doteq \underbrace{[G]}_{\text{dimensions of } G} \frac{M\times M}{L^2} \longrightarrow [G] = L^3M^{-1}T^{-2}$$

quantitative? $\approx 6.67\times10^{-11}\ \frac{m^3}{kg\,s^2}$

Similitude

- Analytical solution to a problem → an explicit formula
- Many laws in physics
- simple models

EX. $F = ma$ ⟹ How does "a" depend on "F" and "m"? Easy!

What if we don't have such formula?
Experiments? Numerical solution?

EX. Flow of a fluid through a long, horizontal, circular pipe $\rightarrow$ our concern: pressure drop per unit length

Question: what are the factors that will have an effect on the pressure drop?

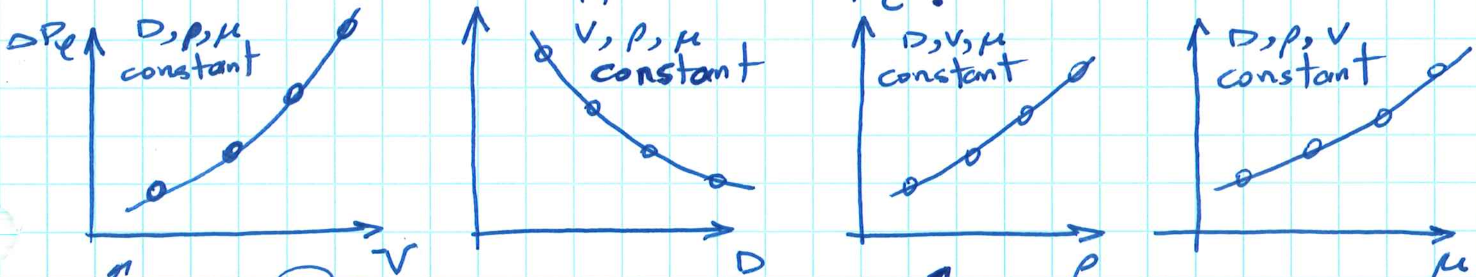
$$\Delta P_\ell = f(D, \rho, \mu, V)$$

viscosity, velocity, diameter, density

$\rightarrow$ ΔP_ℓ is some function of D, ρ, μ, V.

Find the nature of this function by experiments:

change one of the parameters and keep all others constant $\rightarrow$ what happens to ΔP_ℓ?



* valid for the specific type of fluid used

* how possibly can one change ρ and keep μ constant?

* Even if you find all of the fitting curves, how do you combine these data to obtain the desired general functional relationship which woul be valid for any similar pipe system? e.g., water vs oil?

* too many parameters

concept of similitude

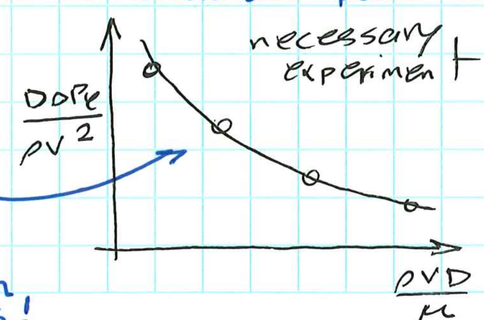
Fortunately, there is a much simpler approach to this problem $\rightarrow$ 2 dimensionless groups instead of 5 dimensional parameters

EX. If two systems have the same $\frac{\rho V D}{\mu}$, they will have the same $\frac{D \Delta P_\ell}{\rho V^2}$ as well.

$$\frac{D \Delta P_\ell}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

universal (valid for any similar system)

$\rightarrow$ not dependent on the system of units!



→ dimensional analysis → Buckingham π theorem

* If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k−r independent dimensionless groups, where r is the minum number of reference dimensions required.

π terms

$$u_1 = f(u_2, u_3, \ldots, u_k) \rightsquigarrow \Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

Determination of Π terms

- List all the variables that are involved in the problem.
- Express each of the variables in terms of basic dimensions.
- Determine the required number of Π terms: k−r
- select "r" repeating variables (independent) that cover all of the required dimensions together. → (not the dependent one)
- Form a Π term by multiplying one of the nonrepeating variables by the product of repeating variables each raised to an exponent.
- Find the exponents so that each Π term is dimensionless.

EX. $\Delta P_e = f(D, \rho, \mu, V)$

$\Delta P_e \doteq FL^{-3} \doteq ML^{-2}T^{-2}$; $D \doteq L$; $\rho \doteq ML^{-3}$; $\mu \doteq ML^{-1}T^{-1}$; $V \doteq LT^{-1}$

$r = 3 \Rightarrow$ number of Π terms $= 5 - 3 = 2$

repeating variables: take D, ρ, & V

$$\rightsquigarrow \Pi_1 = \Delta P_e D^a V^b \rho^c \doteq ML^{-2}T^{-2}L^a L^b T^{-b} M^c L^{-3c} \doteq M^0 L^0 T^0$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1 \Rightarrow \Pi_1 = \frac{D \Delta P_e}{\rho V^2}$$

$$\Pi_2 = \mu D^a V^b \rho^c \doteq ML^{-1}T^{-1}L^a L^b T^{-b} M^c L^{-3c}$$

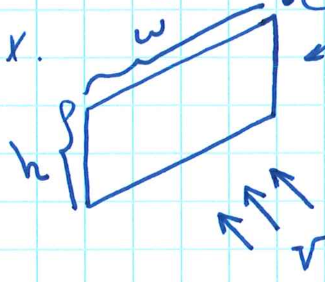
$$\Rightarrow a = -1,\ b = -1,\ c = -1 \Rightarrow \Pi_2 = \frac{\mu}{DV\rho}$$

$$\rightsquigarrow \underbrace{\frac{D \Delta P_e}{\rho V^2}}_{\Pi_1} = \phi\left(\frac{\mu}{DV\rho}\right) = \bar{\phi}\left(\underbrace{\frac{\rho V D}{\mu}}_{\Pi_2}\right)$$

S2

- Buckingham π theorem (Applications)
- Coordinate systems & shell Blance

EX.



a thin plate located normal to a moving stream of fluid

$$\rightsquigarrow F_D = f(w, h, \rho, \mu, V)$$

F_D: drag force

$F_D \doteq MLT^{-2}$, $w \doteq L$, $h \doteq L$, $\rho \doteq ML^{-3}$, $\mu \doteq ML^{-1}T^{-1}$, $V = LT^{-1}$

$\left.\begin{array}{l}\#\text{ of variables} = 6 \\ \#\text{ of basic dimensions} = 3\end{array}\right\} \Rightarrow \#\text{ of } \pi \text{ terms} = 6-3 = 3$

Repeating variables: w, V, ρ

$$\rightsquigarrow \Pi_1 = F_D w^a V^b \rho^c \doteq M^0 L^0 T^0 \Rightarrow a=-2,\ b=-2,\ c=-1$$

$$\to \Pi_1 = \frac{F_D}{w^2 V^2 \rho}$$

$$\Pi_2 = h w^a V^b \rho^c \doteq M^0 L^0 T^0 \Rightarrow a=-1,\ b=0,\ c=0$$

$$\to \Pi_2 = \frac{h}{w}$$ (could we guess this?)

$$\Pi_3 = \mu w^a V^b \rho^c \doteq M^0 L^0 T^0 \Rightarrow a=-1,\ b=-1,\ c=-1$$

$$\to \Pi_3 = \frac{\mu}{wV\rho}$$ (what about this one?)

$$\Rightarrow \frac{F_D}{w^2V^2\rho} = \phi\left(\frac{h}{w}, \frac{\mu}{wV\rho}\right) = \bar{\phi}\left(\frac{w}{h}, \frac{wV\rho}{\mu}\right) = \hat{\phi}\left(\frac{w}{h}, \frac{hV\rho}{\mu}\right) = \ldots$$

one can have new Π terms from the old ones.

Theory of Models

prototype: $\underbrace{\Pi_1}_{\text{containing the variable that is to be predicted}} = \phi(\Pi_2, \Pi_3, \ldots, \Pi_n)$

model: $\Pi_{1m} = \underbrace{\phi}_{\text{the same for the two systems}}(\Pi_{2m}, \Pi_{3m}, \ldots, \Pi_{nm})$

$\to \Pi_{2m}=\Pi_2,\ \Pi_{3m}=\Pi_3, \dots, \Pi_{nm}=\Pi_n \implies \Pi_1=\Pi_{1m}$ cool!

EX. Reconsider the previous example:

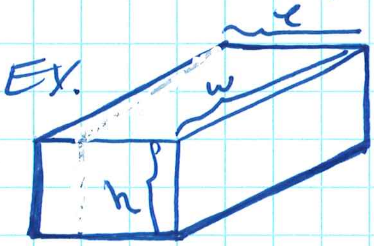
$$\frac{F_D}{w^2V^2\rho}=\Phi\left(\frac{w}{h},\frac{\rho V w}{\mu}\right) \quad \text{vs} \quad \frac{F_{Dm}}{w_m^2V_m^2\rho_m}=\Phi\left(\frac{w_m}{h_m},\frac{\rho_m V_m w_m}{\mu_m}\right)$$

$$\implies \frac{w_m}{h_m}=\frac{w}{h} \text{ and } \frac{\rho_m V_m w_m}{\mu_m}=\frac{\rho V w}{\mu} \implies \frac{F_D}{w^2V^2\rho}=\frac{F_{Dm}}{w_m^2V_m^2\rho_m}$$

$$w_m=\frac{h_m}{h}w \to V_m=\frac{\mu_m}{\mu}\frac{\rho}{\rho_m}\frac{w}{w_m}V \implies F_D=\left(\frac{w}{w_m}\right)^2\left(\frac{\rho}{\rho_m}\right)\left(\frac{V}{V_m}\right)^2F_{Dm}$$

(select an h_m and a model fluid that are easy to work with)

EX.



$$\leadsto \frac{F_D}{w^2V^2\rho}=\phi\left(\frac{w}{h},\frac{w}{\ell},\frac{\rho V w}{\mu}\right)$$

$\propto C_D$ (drag coefficient)

what if $w=h=\ell \implies \frac{F_D}{w^2V^2\rho}=\phi\left(\frac{\rho V w}{\mu}\right)$

Flow around a sphere: $\frac{F_D}{D^2V^2\rho}=\phi\left(\frac{\rho V D}{\mu}\right)$

Convention for C_D: $F_D=\frac{C_D}{2}\rho A V^2 \implies C_D=\frac{F_D}{\frac{1}{2}\rho A V^2}$

A: projected area

$$\leadsto \frac{\frac{1}{2}C_D\cancel{\rho}\frac{\pi \cancel{D^2}}{4}\cancel{V^2}}{\cancel{D^2}\cancel{V^2}\cancel{\rho}}=\frac{\pi}{8}C_D=\phi\left(\frac{\rho V D}{\mu}\right)$$

$$\implies C_D=\psi\left(\frac{\rho V D}{\mu}\right)$$

Experiments:

* For low Re number $\leadsto C_D\downarrow$ as $Re\uparrow$
* For high Re number $\leadsto C_D=\text{constant}\neq f(Re)$

correct for other geometric shapes as well

$\to 10^3 - 2\times10^5$ for a smooth sphere

Stokes flow: $Re \lesssim 1 \to$ density does not matter (given information)

$\implies F_D=f(D,\mu,V) \leadsto r=3 \implies$ # of π terms $=4-3=1$

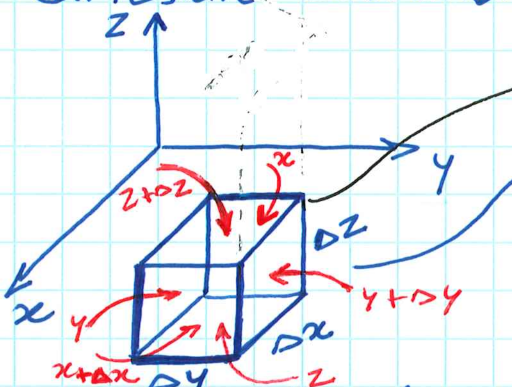
$\leadsto \Pi_1=\frac{F_D}{\mu V D}=C \implies F_D=C\mu V D$

$C = 3\pi$ (from theory & experiment)

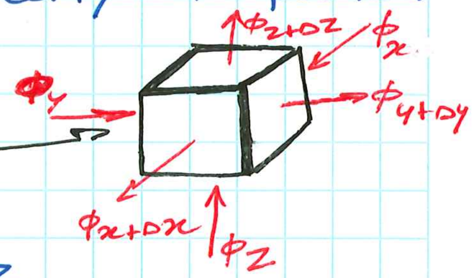
Q: Given $F_D = 3\pi\mu v D$, find the function ϕ so that $C_D = \phi(Re)$ for low Re numbers.

Coordinate Systems → a system that uses one–three numbers to uniquely specify the position of points.

- Cartesian Coordinate system



elemental volumes



$V = \Delta x \Delta y \Delta z$

$A_x = \Delta y \Delta z$, $A_y = \Delta x \Delta z$, $A_z = \Delta x \Delta y$

$\phi_{x,y,z}$: flux of a quantity (mass, energy, momentum, ...)
→ quantity per unit area per time

$\phi_x = \rho u_x$ ← velocity in the x direction

ϕ_x: flux of mass; ρ: density

flux x Area ⇒ transfer rate of the quantity

Shell Balance → $\begin{Bmatrix}\text{rate of}\\ \text{change}\end{Bmatrix} = \begin{Bmatrix}\text{in}\end{Bmatrix} - \begin{Bmatrix}\text{out}\end{Bmatrix} + \begin{Bmatrix}\text{generation}\end{Bmatrix}$

$$\rightarrow \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = (\rho u_x \Delta y \Delta z)\Big|_x + (\rho u_y \Delta x \Delta z)\Big|_y + (\rho u_z \Delta x \Delta y)\Big|_z$$

in ↗

out → $-(\rho u_x \Delta y \Delta z)\Big|_{x+\Delta x} - \quad '' \quad \Big|_{y+\Delta y} - \quad '' \quad \Big|_{z+\Delta z}$

generation → $+0$

→ divide by $\Delta x \Delta y \Delta z$ →

$$\frac{\partial \rho}{\partial t} = \frac{(\rho u_x)|_x - (\rho u_x)|_{x+\Delta x}}{\Delta x} + \frac{(\rho u_y)|_y - (\rho u_y)|_{y+\Delta y}}{\Delta y} + \frac{(\rho u_z)|_z - (\rho u_z)|_{z+\Delta z}}{\Delta z}$$

let $\Delta x, \Delta y, \Delta z \to 0$ ⇝ $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = 0$

Continuity eqn in Cartesian coordinates

$\phi_{x,y,z}$ ≡ thermal energy flux → Heat eqn

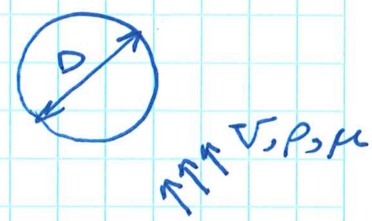
$\phi_{x,y,z}$ ≡ momentum flux → Navier–Stokes eqn

S2'

Clarifications re S2:

1) For the drag force on a smooth sphere, we showed

$$\frac{F_D}{\rho V^2 D^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$



Equivalently, one can write

$$\frac{F_D}{\frac{1}{2}\rho V^2 \frac{\pi D^2}{4}} = \psi\left(\frac{\rho V D}{\mu}\right) \quad (*)$$

where the difference between functions ϕ and ψ is the sole constant $\frac{\pi}{8}$.

Now, the LHS of eqn (*) is termed (by convention) as the drag coefficient C_D:

$$C_D = \psi\left(\frac{\rho V D}{\mu}\right), \text{ where } C_D = \frac{F_D}{\frac{1}{2}\rho V^2 \frac{\pi D^2}{4}}$$

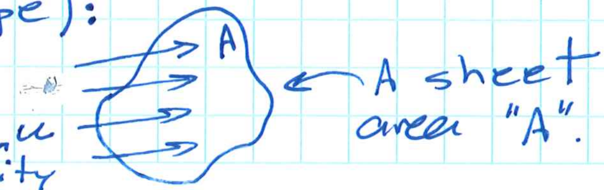
we can generalize this relationship to other objects as well:

$$C_D = \psi\left(\frac{\rho V D}{\mu}\right) = \frac{F_D}{\frac{1}{2}\rho V^2 A}$$

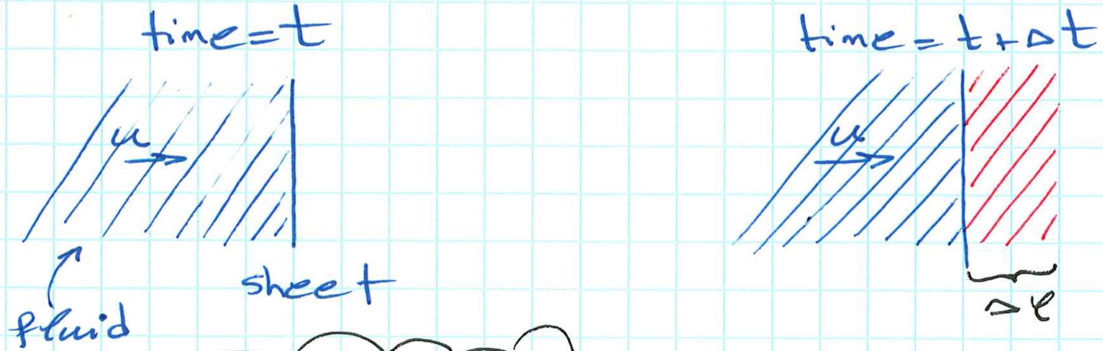
A: projected area $= \frac{\pi D^2}{4}$ for sphere

2) the flowrate: $\dfrac{\text{Volume of fluid displaced}}{\text{Time}}$

There are better but more complicated methods to explain this. I reiterate what we discussed in class. Consider a fluid passing through an area A (with arbitrary shape):



Looking at the sheet from a side:



$\Delta V = A\Delta \ell$
displaced volume of the fluid (volume of fluid passed through area A)

$$\frac{\text{Volume passed}}{\text{Time}} = \frac{A\Delta \ell}{\Delta t}$$

$$\text{flowrate} = \frac{\Delta V}{\Delta t}$$

$$\Rightarrow \frac{\Delta V}{\Delta t} = \frac{A\Delta \ell}{\Delta t}$$

take limit $\Delta t \to 0$

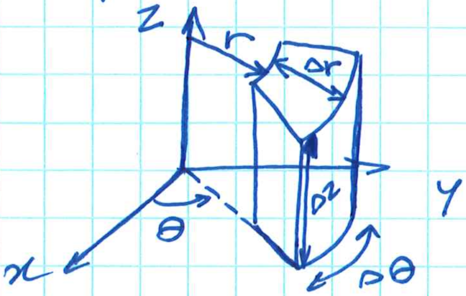
$$\Rightarrow \underbrace{\lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t}}_{\text{flowrate}} = A \underbrace{\lim_{\Delta t \to 0} \frac{\Delta \ell}{\Delta t}}_{\text{velocity}} \Rightarrow \boxed{Q = Au}$$

$\rightarrow$ mass flowrate $= \rho Q = \rho A u$

mass flux $= \dfrac{\text{mass flowrate}}{\text{unit area}} = \rho u$

S3 Coordinate Systems & Shell Balance

Cylindrical Coordinates



for small $\Delta r \rightsquigarrow V = r\Delta\theta\,\Delta r\,\Delta z$

$$A_r = r\Delta\theta\,\Delta z$$

$$A_\theta = \Delta r\,\Delta z$$

$$A_z = r\Delta\theta\,\Delta r$$

$$\rightsquigarrow \frac{\partial}{\partial t}(r\Delta\theta\Delta r\Delta z\,\rho) = (\rho u_r\, r\Delta\theta\Delta z)\big|_r - (\rho u_r\, r\Delta\theta\Delta z)\big|_{r+\Delta r} + (\rho u_\theta\,\Delta r\Delta z)\big|_\theta - (\rho u_\theta\,\Delta r\Delta z)\big|_{\theta+\Delta\theta} + (\rho u_z\, r\Delta\theta\Delta r)\big|_z - (\rho u_z\, r\Delta\theta\Delta r)\big|_{z+\Delta z}$$

divide by $\Delta r\Delta\theta\Delta z$
let $\Delta r, \Delta\theta, \Delta z \to 0$

$$\Longrightarrow \frac{\partial}{\partial t}(r\rho) = \frac{(\rho u_r r)|_r - (\rho u_r r)_{r+\Delta r}}{\Delta r} + \frac{(\rho u_\theta)|_\theta - (\rho u_\theta)|_{\theta+\Delta\theta}}{\Delta\theta} + r\,\frac{(\rho u_z)|_z - (\rho u_z)|_{z+\Delta z}}{\Delta z}$$

$$\rightsquigarrow r\frac{\partial\rho}{\partial t} + \frac{\partial(r\rho u_r)}{\partial r} + \frac{\partial(\rho u_\theta)}{\partial\theta} + r\frac{\partial(\rho u_z)}{\partial z} = 0$$

$$\rightsquigarrow \frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial\theta} + \frac{\partial(\rho u_z)}{\partial z} = 0$$

Continuity eq in cylindrical coordinates

Could we find this from the eqn of the cartesian coordinates?

Cartesian: $\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_x}{\partial x} + \frac{\partial\rho u_y}{\partial y} + \frac{\partial\rho u_z}{\partial z} = 0$

$\rightsquigarrow$ • express u_x, u_y, u_z in terms of the scalar velocities in the cylindrical coordinates.

see S3' for detail.

$u_x = u_r\cos\theta - u_\theta \sin\theta \qquad u_y = u_r \sin\theta + u_\theta\cos\theta \qquad u_z = u_z$

- express x, y, z in terms of r, θ, z and vice versa.

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \longrightarrow r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$z = z$

$$\leadsto \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta ; \quad \frac{\partial r}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r} ; \quad \frac{\partial\theta}{\partial y} = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u_x)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}(\rho u_x)\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}(\rho u_x)\underbrace{\frac{\partial z}{\partial x}}_{0}$$
$$+ \frac{\partial}{\partial r}(\rho u_y)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}(\rho u_y)\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z}(\rho u_y)\underbrace{\frac{\partial z}{\partial y}}_{0}$$
$$+ \frac{\partial}{\partial r}(\rho u_z)\underbrace{\frac{\partial r}{\partial z}}_{0} + \frac{\partial}{\partial \theta}(\rho u_z)\underbrace{\frac{\partial \theta}{\partial z}}_{0} + \frac{\partial}{\partial z}(\rho u_z)\underbrace{\frac{\partial z}{\partial z}}_{1}$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u_r\cos\theta - \rho u_\theta\sin\theta)(\cos\theta) + \frac{\partial}{\partial\theta}(\rho u_r\cos\theta - \rho u_\theta \sin\theta)\left(\frac{-\sin\theta}{r}\right)$$
$$+ \frac{\partial}{\partial r}(\rho u_r\sin\theta + \rho u_\theta\cos\theta)(\sin\theta) + \frac{\partial}{\partial\theta}(\rho u_r \sin\theta + \rho u_\theta\cos\theta)\left(\frac{\cos\theta}{r}\right)$$
$$+ \frac{\partial}{\partial z}(\rho u_z)$$

$$= \frac{\partial\rho}{\partial t} + \cos\theta\left[\cos\theta\frac{\partial\rho u_r}{\partial r} - \cancel{\sin\theta\frac{\partial \rho u_\theta}{\partial r}}\right] - \frac{\sin\theta}{r}\left[\cancel{\cos\theta\frac{\partial\rho u_r}{\partial\theta}} - \sin\theta\,\rho u_r - \sin\theta\frac{\partial\rho u_\theta}{\partial\theta} - \cancel{\cos\theta\,\rho u_\theta}\right]$$
$$+ \sin\theta\left[\sin\theta\frac{\partial\rho u_r}{\partial r} + \cancel{\cos\theta\frac{\partial\rho u_\theta}{\partial r}}\right] + \frac{\cos\theta}{r}\left[\cancel{\sin\theta\frac{\partial\rho u_r}{\partial\theta}} + \cos\theta\,\rho u_r + \cos\theta\frac{\partial\rho u_\theta}{\partial\theta} - \cancel{\sin\theta\,\rho u_\theta}\right]$$
$$+ \frac{\partial}{\partial z}(\rho u_z) = 0$$

$$\leadsto \frac{\partial\rho}{\partial t} + \frac{\partial\rho u_r}{\partial r}(\cos^2\theta + \sin^2\theta) + \frac{\partial \rho u_\theta}{\partial\theta}\left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right) + \cancel{\frac{\partial\rho u_r}{\partial\theta}(0)} + \cancel{\frac{\partial\rho u_\theta}{\partial r}(0)}$$
$$+ \rho u_r\left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right) + \cancel{\rho u_\theta(0)} + \frac{\partial \rho u_z}{\partial z} = 0$$

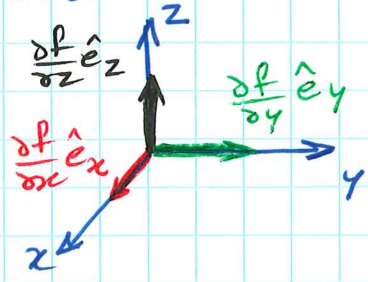
$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial \rho u_\theta}{\partial \theta} + \frac{\partial \rho u_z}{\partial z} = 0 \qquad \checkmark$$

Gradient of a scalar ∇f

Cartesian coordinates $\Rightarrow \nabla f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z$

a vector

$\hat{e}_x, \hat{e}_y, \hat{e}_z$: unit vectors in x, y, z directions



Cylindrical system? $\hat{e}_x = \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta$

$\hat{e}_y = \hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta$

$\hat{e}_z = \hat{e}_z$

$$\rightsquigarrow \nabla f = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}(\hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta) + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}(\hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta)$$
$$+ \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}(\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta) + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}(\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta)$$
$$+ \frac{\partial f}{\partial z}\hat{e}_z$$

$$= \frac{\partial f}{\partial r}(\cos\theta)(\hat{e}_r\cos\theta \cancel{- \hat{e}_\theta\sin\theta}) + \frac{\partial f}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)(\cancel{\hat{e}_r\cos\theta} - \hat{e}_\theta\sin\theta)$$
$$+ \frac{\partial f}{\partial r}(\sin\theta)(\hat{e}_r\sin\theta \cancel{+ \hat{e}_\theta\cos\theta}) + \frac{\partial f}{\partial \theta}\left(\frac{\cos\theta}{r}\right)(\cancel{\hat{e}_r\sin\theta} + \hat{e}_\theta\cos\theta) + \frac{\partial f}{\partial z}\hat{e}_z$$

$$= \frac{\partial f}{\partial r}\left[\cos^2\theta\,\hat{e}_r \cancel{- \sin\theta\cos\theta\,\hat{e}_\theta} + \sin^2\theta\,\hat{e}_r \cancel{+ \sin\theta\cos\theta\,\hat{e}_\theta}\right]$$
$$+ \frac{\partial f}{\partial \theta}\left[\cancel{\frac{-\sin\theta\cos\theta}{r}\hat{e}_r} + \frac{\sin^2\theta}{r}\hat{e}_\theta + \cancel{\frac{\sin\theta\cos\theta}{r}\hat{e}_r} + \frac{\cos^2\theta}{r}\hat{e}_\theta\right] + \frac{\partial f}{\partial z}\hat{e}_z$$

$$\Rightarrow \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z$$

S3'

How u_x, u_y, u_z depend on u_r, u_θ, u_z

u_x: scalar component of velocity in the x direction

dot product

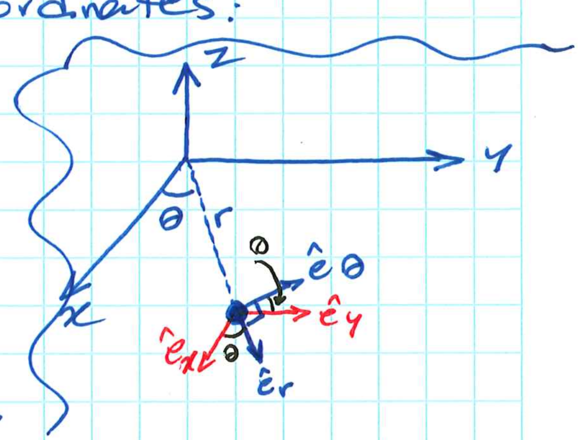
unit vector in the x direction

$\Rightarrow u_x = \underline{u} \cdot \hat{e}_x$

vector of velocity

we can express $\underline{u}$ in terms of scalar velocity components in the cylindrical coordinates:

$$\underline{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z$$



Insert into the u_x eqn above:

$$u_x = (u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z) \cdot \hat{e}_x$$

$$= u_r \hat{e}_r \cdot \hat{e}_x + u_\theta \hat{e}_\theta \cdot \hat{e}_x + u_z \hat{e}_z \cdot \hat{e}_x$$

dot product of two unit vectors is simply the cosine of the angle between them.

$$\Rightarrow u_x = u_r(\cos\theta) + u_\theta\left(\cos\left(\frac{\pi}{2}+\theta\right)\right) + \cancelto{0}{u_z\left(\cos\frac{\pi}{2}\right)}$$

$$\Rightarrow \boxed{u_x = u_r \cos\theta - u_\theta \sin\theta}$$

Similarly: $u_y = (u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z) \cdot \hat{e}_y$

$$= u_r \hat{e}_r \cdot \hat{e}_y + u_\theta \hat{e}_\theta \cdot \hat{e}_y + \cancelto{0}{u_z \hat{e}_z \cdot \hat{e}_y}$$

$$= u_r \cos\left(\frac{\pi}{2}-\theta\right) + u_\theta \cos\theta + 0$$

$$\Rightarrow \boxed{u_y = u_r \sin\theta + u_\theta \cos\theta}$$

S4 Heat eqn ≡ species continuity eqn

Recall the continuity eqn in the Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = 0$$

More generally:

$$\frac{\partial \psi}{\partial t} + \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z} = S$$

generation source (quantity per unit volume per unit time)

ψ: quantity of interest per unit volume
e.g., Mass/Volume = Density (ρ)

$\phi_{x,y,z}$: flux of the quantity in x,y,z directions

Ex. For total mass (quantity):

$S=0$, $\psi=\rho$, $\phi_x=\rho u_x$, $\phi_y=\rho u_y$, $\phi_z=\rho u_z$ → Continuity eqn

Cylindrical coordinates? Easy!

$$\frac{\partial \psi}{\partial t} + \frac{1}{r}\frac{\partial \phi_r}{\partial r} + \frac{1}{r}\frac{\partial \phi_\theta}{\partial \theta} + \frac{\partial \phi_z}{\partial z} = S$$

In vector form → $\frac{\partial \psi}{\partial t} + \underline{\nabla} \cdot \underline{\phi} = S$

Heat eqn in Cartesian coordinates:

$$\psi = \frac{\text{Energy}}{\text{Volume}} = \frac{m C_p T}{V} = \rho C_p T$$

$$\phi_x = -K\frac{\partial T}{\partial x} + \rho u_x C_p T$$

y,z y,z y,z

Fourier's law (conduction)   flux of energy carried by fluid

vector form

$$(\underline{\phi} = -K\nabla T + \rho \underline{u} C_p T)$$

$$\rightarrow \frac{\partial \rho c_p T}{\partial t} + \frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x} + \rho u_x c_p T\right) + \frac{\partial}{\partial y}\left(-k\frac{\partial T}{\partial y} + \rho u_y c_p T\right)$$
$$+ \frac{\partial}{\partial z}\left(-k\frac{\partial T}{\partial z} + \rho u_z c_p T\right) = S$$

$$\rightsquigarrow \frac{\partial \rho c_p T}{\partial t} + \frac{\partial \rho c_p u_x T}{\partial x} + \frac{\partial \rho c_p u_y T}{\partial y} + \frac{\partial \rho c_p u_z T}{\partial z}$$
$$= \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + S$$

For constant k, ρ, c_p:

$$\rho c_p\left(\frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

Heat eqn

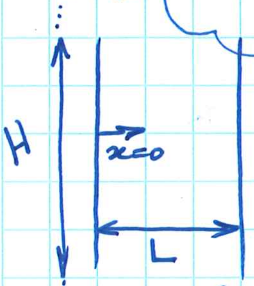
S5 1D, steady-state transport problems

Heat eqn: $\rho c_p \left(\frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T \right) = k \nabla^2 T + S$

Temp distribution in a plane wall

$\underline{u} = 0$, steady-state $\rightarrow \frac{\partial T}{\partial t} = 0$, no source $\Rightarrow S = 0$

$\Rightarrow k \nabla^2 T = 0 \rightarrow k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = 0$



very large in other directions

It can be assumed that $\frac{\partial^2 T}{\partial x^2}$ is $\gg \frac{\partial^2 T}{\partial y^2}$ and $\frac{\partial^2 T}{\partial z^2}$.

why? Let $\tilde{x} = \frac{x}{L}$, $\tilde{y} = \frac{y}{H}$, $\tilde{z} = \frac{z}{W}$

size in x, y, z directions

$\rightarrow k \left(\frac{\partial^2 T}{L^2 \partial \tilde{x}^2} + \frac{\partial^2 T}{H^2 \partial \tilde{y}^2} + \frac{\partial^2 T}{W^2 \partial \tilde{z}^2} \right) = 0$

$\tilde{x} = O(1)$, $\tilde{y} = O(1)$, $\tilde{z} = O(1) \rightsquigarrow \frac{\partial^2 T}{\partial \tilde{x}^2} = O(T)$

$\frac{\partial^2 T}{\partial \tilde{y}^2} = O(T)$, $\frac{\partial^2 T}{\partial \tilde{z}^2} = O(T) \rightsquigarrow \frac{1}{L^2} \gg \frac{1}{H^2}$ & $\frac{1}{L^2} \gg \frac{1}{W^2}$

$\Rightarrow$ only the first term matters

$\Rightarrow \cancel{k} \frac{d^2 T}{dx^2} = 0 \rightsquigarrow \frac{dT}{dx} = C_1 \rightsquigarrow \boxed{T = C_1 x + C_2}$

Boundary Conditions

value of the variable is given

Dirichlet $\rightarrow$ at $x = 0 \Rightarrow T = T_0$ (I)

at $x = L \Rightarrow T = T_L$ (II)

$\rightarrow$ (I): $T_0 = C_2$; (II): $T_L = C_1 L + T_0 \rightarrow C_1 = \frac{T_L - T_0}{L}$

$\rightsquigarrow T = \frac{T_L - T_0}{L} x + T_0$

Neumann

at $x=0 \Rightarrow T=T_0$ ①
at $x=L \Rightarrow \left.\frac{dT}{dx}\right|_{x=L}=0$ ② gradient is given

①: $C_2=T_0$; ②: $\frac{dT}{dx}=C_1=0$

$\rightarrow \boxed{T=T_0}$

Robin

both value and gradient are involved!

at $x=0 \Rightarrow T=T_0$ ①
at $x=L \Rightarrow -k\left.\frac{dT}{dx}\right|_{x=L} = h\left(\left.T\right|_{x=L}-T_\infty\right)$ ②

$h(T_w-T_\infty)$ is known as the Newton's cooling law $\longrightarrow$ heat flux from a surface to a medium is proportional to the temp. difference

convection heat-transfer coefficient

①: $C_2=T_0$; ②: $-kC_1 = h(C_1L+T_0-T_\infty)$

$$\Rightarrow C_1(hL+k) = h(T_\infty-T_0) \rightarrow C_1=\frac{h(T_\infty-T_0)}{hL+k}$$

$$\Rightarrow \boxed{T=\frac{h(T_\infty-T_0)}{hL+k}x+T_0}$$

Homogeneous Neumann at $x=0$ & L

at $x=0 \Rightarrow \frac{dT}{dx}=0$
at $x=L \Rightarrow$ " $=0$
$\Big\} \longrightarrow C_1=0$; $C_2=$?!!!

Note: heat flux through a solid can be expressed as $-k\nabla T \longrightarrow$ 1D system $\rightsquigarrow q=-k\frac{dT}{dx}$

$\frac{dT}{dx}=0 \equiv -k\frac{dT}{dx} \Rightarrow$ insulation $\rightarrow$ total energy is conserved

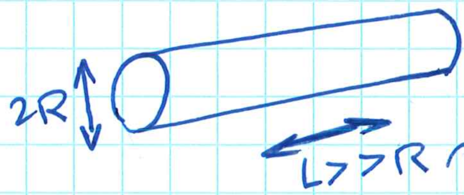
$$C_2=\frac{1}{L}\int_0^L T_{initial}\,dx$$

(depends on initial conditions)

average temp. at $t=0$

A cylinder with heat source

$T = T_w$ at $r = R$



only the r-direction term matters

symmetric/uniform in θ direction

$$\rightarrow \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + S = 0 \rightarrow d\left(r\frac{dT}{dr}\right) = -\frac{S}{k} r\, dr$$

$$\Rightarrow r\frac{dT}{dr} = -\frac{S}{2k} r^2 + C_1 \rightarrow T = -\frac{S}{4k} r^2 + C_1 \ln r + C_2$$

T should be finite at $r=0$

$\rightarrow C_1 = 0$

$$r = R \Rightarrow T_w = -\frac{SR^2}{4k} + C_2 \Rightarrow C_2 = T_w + \frac{SR^2}{4k}$$

$$\Rightarrow T = -\frac{S}{4k}\left(r^2 - R^2\right) + T_w$$

Consistency check $\rightarrow$ Heat generated = Heat out

$$\cancel{S}\cancel{\pi}R^{\cancel{2}}\cancel{L} = -k\left.\frac{dT}{dr}\right|_{r=R} 2\pi R L$$

$$= -\cancel{k}\left(-\frac{2\cancel{S}\cancel{R}}{4\cancel{k}}\right)\cancel{2\pi}\cancel{R}\cancel{L} \quad \checkmark$$

(*) Heat and Mass transfer equations are mathematically "identical"!

Conduction - Convection Systems (Fin)

T_∞

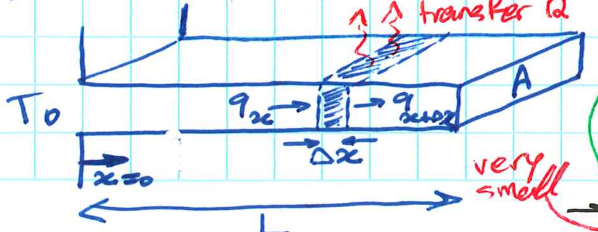
A, T_0 $\quad Q = hA(T_0 - T_\infty)$

h is hard to change.

$T_0 - T_\infty$ is usually constant.

hot water? $\quad$ room temp

An idea $\rightarrow$ increase A



temp. difference in y,z directions

Assumption: temp. changes only in the x direction.

$\hookrightarrow -k\nabla T \sim h(T - T_\infty)$

very small

$\rightarrow \Delta T \sim \frac{h\ell}{k}(T - T_\infty)$ $\quad$ length scale in y,z directions energy balance $\rightarrow q_x = \left(-k\frac{dT}{dx}A\right)_x$; $q_{x+\Delta x} = \left(-k\frac{dT}{dx}A\right)_{x+\Delta x}$

$Q = hP\Delta x\,(T - T_\infty)$

$$\Rightarrow \left(-k\frac{dT}{dx}A\right)_x - \left(-k\frac{dT}{dx}A\right)_{x+\Delta x} = hP\Delta x\,(T-T_\infty)$$

constant k and A $\Rightarrow$
$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T-T_\infty) = 0 \qquad m^2 = \frac{hP}{kA}$$

Let $\theta = T - T_\infty \rightarrow \dfrac{d^2\theta}{dx^2} - m^2\theta = 0$

Guess $\theta = e^{ax} \Rightarrow a = \pm m \Rightarrow \theta = C_1 e^{mx} + C_2 e^{-mx}$

BCs: at $x = 0 \rightarrow T = T_0 \rightarrow \theta = T_0 - T_\infty = \theta_0 \Rightarrow C_1 + C_2 = \theta_0$

Assume $L \rightarrow \infty \Rightarrow \theta = 0$ as $x \rightarrow \infty \Rightarrow C_1 = 0$

the surface temp. reaches to the ambient temp for large L

$$\rightarrow \theta = \theta_0 e^{-mx} \rightsquigarrow \frac{T-T_\infty}{T_0-T_\infty} = e^{-mx}$$

Total heat into the ambient $Q = -kA\left.\dfrac{dT}{dx}\right|_{x=0} = \displaystyle\int_0^L hP\theta\,dx$

$$= -kA\left.\frac{d\theta}{dx}\right|_{x=0} = \sqrt{kAhP}\,\theta_0$$

S6 An evaporating water droplet

concentration : $\frac{\text{amount of material}}{\text{unit volume}}$

(e.g., concentration of water vapor per unit volume of air

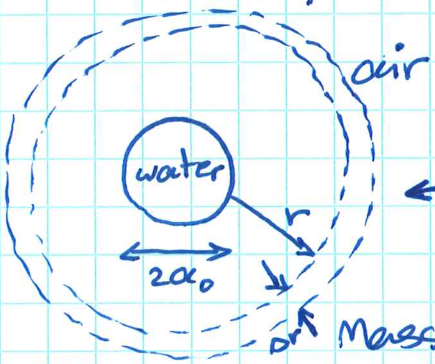
diffusivity coefficient

flux of mass $\equiv J_A = \underbrace{-D\nabla C_A}_{\text{Fick's law}} + uC_A$

mass/mole of a species (A) in a medium (e.g., water vapor in air)

diffusion

convection



an elemental volume

Question: How long does it take for the radius to drop to some $a_c < a_0$ due to evaporation?

Mass balance: $(J_{A_r} 4\pi r^2)_r - (J_{A_r} 4\pi r^2)_{r+\Delta r} = 0$

*pseudo steady state

* clearly a (radius of the droplet) is changing due to evaporation. But we assume that happens very slowly so that the concentration profile around the droplet is almost steady-state at all times. ⟶ diffusion time scale ≪ evaporation time scale

$$\frac{d}{dr}\left(J_{A_r} 4\pi r^2\right) = 0 \rightsquigarrow -D\frac{dC_A}{dr}(4\pi r^2) = k_1$$

$$J_{A_r} = -D\frac{dC_A}{dr} \qquad \Rightarrow C_A = \frac{k_1}{4\pi D r} + k_2$$

BCs: at $r = a \Rightarrow C_A = C_A^*$ ← saturation concentration (function of temp)

as $r \to \infty \Rightarrow C_A \to C_A^\infty$ ← concentration of water vapor in the ambient

$$\Rightarrow \frac{C_A - C_A^\infty}{C_A^* - C_A^\infty} = \frac{a}{r}$$

Now we write a balance for the droplet:

$$\underbrace{\frac{d}{dt}\left(\frac{4}{3}\pi a^3 \rho/_{MW}\right)}_{\text{time derivative of the total moles of water in the droplet}} = -\left(J_{A_r} 4\pi r^2\right)_{r=a} = \left[\frac{D a (C_A^* - C_A^\infty)}{\cancel{r^2}} 4\pi \cancel{r^2}\right]_{r=a}$$

$$\Rightarrow \cancel{4\pi} a^{\cancel{2}} \frac{\rho}{MW}\frac{da}{dt} = -D \cancel{a} (C_A^* - C_A^\infty)\cancel{(4\pi)}$$

$$\rightarrow a\frac{da}{dt} = \frac{-D\,MW\,(C_A^* - C_A^\infty)}{\rho}$$

Integrate $\leadsto a^2 = a_0^2\left(1 - \underbrace{\frac{2D\,MW\,(C_A^* - C_A^\infty)}{\rho a_0^2}}\, t\right)$

$$\tau_e = \frac{\rho a_0^2}{2D\,MW\,(C_A^* - C_A^\infty)}$$ has dimensions $\frac{1}{\text{time}}$

time scale of evaporation

$$\tau_D = \frac{a_0^2}{D}$$

time scale of diffusion

$$\frac{\rho a_0^{\cancel{2}}}{2\cancel{D}\,MW\,(C_A^* - C_A^\infty)} >> \frac{\cancel{a_0^2}}{\cancel{D}}$$

$$\rightarrow \frac{\rho}{2MW(C_A^* - C_A^\infty)} >> 1 \checkmark$$

reasonable

How much does it take for the droplet radius to decrease to $a = a_c$

$$\Rightarrow t_c = \left(1 - \left(\frac{a_c}{a_0}\right)^2\right)\frac{\rho a_0^2}{2D\,MW\,(C_A^* - C_A^\infty)}$$

Let $\rho = 1000\ \frac{kg}{m^3}$, $D = 2.5\times10^{-5}\ \frac{m^2}{s}$, $MW = 18\times10^{-3}\ \frac{kg}{mol}$

$C_A^* = 1.2\ \frac{mol}{m^3}$, $C_A^\infty = \frac{1}{2}C_A^*$ $\leadsto$ $\left.\begin{array}{l} a_0 = 5\ \mu m \\ a_c = 2.5\ \mu m \end{array}\right\} \rightarrow \underline{\underline{t_c = 34\ ms}}$

$\frac{P_A^*}{RT} \leftarrow 3\times10^3\ Pa$ $\qquad$ % 50 Relative Humidity

respiratory droplet size for breathing and talking

How long does it take for the respiratory drop to fall on the ground?

Stokes? $V = \frac{4a_0^2(\rho - \rho_{air})g}{18\mu_{air}} \approx 2.7\ \frac{mm}{s} \rightarrow Re = \frac{\rho_{air}V(2a_0)}{\mu_{air}} \approx 0.0014 << 1 \checkmark$

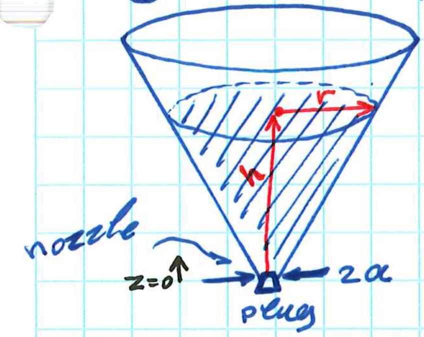
falling time $\sim \frac{H}{V} \approx 740\ s$

$H \sim 2m$

it is actually more than this because $a < a_0$ and $\rho < \rho_{water}$

S7

* A draining cone-shape reservoir
* Bernoulli eqn
* Finite difference method & Euler method



We remove the plug at $t=0$; how long does it take for the height of the liquid (h) to drop to $h'<h$?

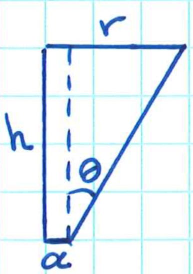
Bernoulli eqn (steady, inviscid, incompressible flow)

$$\underset{\text{pressure}}{P} + \tfrac{1}{2}\rho \underset{\text{velocity}}{u^2} + \underset{\text{head}}{\rho g z} = \text{constant}$$

point 1: on the surface of liquid at $z=h$, $P=P_{atm}$, $u=u_1$

" 2: at the outlet (nozzle) at $z=0$, $P=P_{atm}$, $u=u_2$

$$\cancel{P_{atm}} + \tfrac{1}{2}\rho u_1^2 + \rho g h = \cancel{P_{atm}} + \tfrac{1}{2}\rho u_2^2 + 0 \rightsquigarrow gh = \tfrac{1}{2}u_2^2\left(1-\left(\tfrac{u_1}{u_2}\right)^2\right) \quad (*)$$



$$\rightarrow \tan\theta = \frac{r-a}{h} \longrightarrow r = a + h\tan\theta$$

Conservation of mass: $A_1 u_1 = A_2 u_2$

$$\Rightarrow \cancel{\pi}(a+h\tan\theta)^2 u_1 = \cancel{\pi} a^2 u_2$$

$$\Rightarrow \frac{u_1}{u_2} = \frac{a^2}{(a+h\tan\theta)^2} = \left(\frac{1}{1+\frac{h\tan\theta}{a}}\right)^2 \ll 1 \quad \text{for } a \ll h$$

$$(*) \Rightarrow gh \approx \tfrac{1}{2}u_2^2 \Rightarrow u_2 = \sqrt{2gh}$$

Volume of fluid (a truncated cone)

$$V = \frac{1}{3}\pi \frac{h}{r-a}(r^3-a^3) = \frac{\pi}{3\tan\theta}(r^3-a^3)$$

Balance for the volume of fluid:

$$\frac{dV}{dt} = -A_2 u_2 \rightsquigarrow \frac{d}{dt}\left(\frac{\pi}{3\tan\theta}(r^3-a^3)\right) = -\pi a^2\sqrt{2gh} = -\pi a^2\sqrt{2g\frac{r-a}{\tan\theta}}$$

$$\Rightarrow \frac{\cancel{\pi}}{\cancel{3}\tan\theta}\cancel{3}r^2\frac{dr}{dt} = -\cancel{\pi}a^2\sqrt{2g\frac{r-a}{\tan\theta}} \longrightarrow r^2\frac{dr}{dt} = -\underbrace{a^2\sqrt{2g\tan\theta}}_{\text{constant: } \gamma}\sqrt{r-a} \quad (*)$$

we need to solve this ODE to find r vs t

Approach 1: $r^2 \frac{dr}{dt} = -\gamma\sqrt{r-\alpha}$ → integrate to find r as a function of t

→ focus of next session

Approach 2: Numerical solution to ⊛

$$\frac{dr}{dt} = -\frac{\gamma\sqrt{r-\alpha}}{r^2} \qquad \text{Recall} \qquad \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}$$

what if Δt is something small but not $\to 0$?

$$\Rightarrow \frac{dr}{dt} \approx \frac{r(t+\Delta t)-r(t)}{\Delta t} = -\frac{\gamma\sqrt{r-\alpha}}{r^2}$$

what about this one?

$\underbrace{r \approx r(t)}_{\text{Euler's method}}$ or $r(t+\Delta t)$

$$\Rightarrow \frac{r(t+\Delta t)-r(t)}{\Delta t} = -\frac{\gamma\sqrt{r(t)-\alpha}}{r^2(t)}$$

→ if we know $r(t)$, we can find $r(t+\Delta t)$

* More generally: $\frac{r_{i+1}-r_i}{\Delta t} = -\gamma\frac{\sqrt{r_i-\alpha}}{r_i^2}$

$r_{i+1} = r(t_{i+1})$
$r_i = r(t_i)$
$t_{i+1} = t_i + \Delta t$

$t_0 \quad t_1 \quad t_2 \quad \dots \quad t_i \quad \dots$
$0 \quad \Delta t \quad 2\Delta t \quad \dots \quad i\Delta t \dots$

→ $r_{i+1} = r_i - \Delta t\, \gamma \frac{\sqrt{r_i-\alpha}}{r_i^2}$

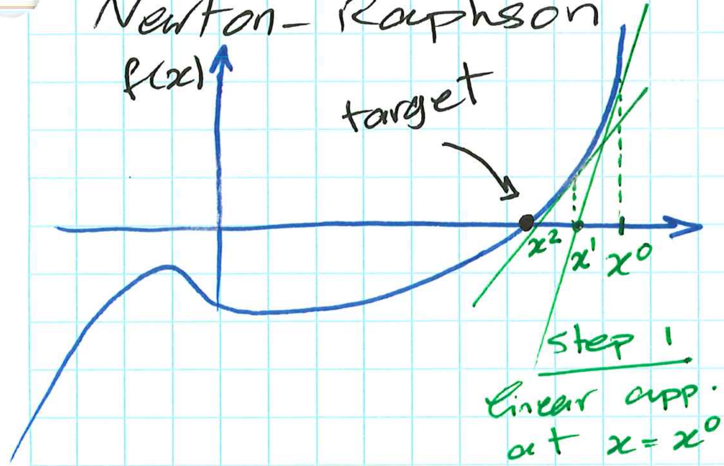
r_0 is given by the initial conditions:
at $t=0$, $h=h_0 \Rightarrow r_0 = \alpha + h_0 \tan\theta$

* see the code drainingReservoir.py

S8

Newton–Raphson
Projectile motion with drag
A radiating body

Newton–Raphson



step 1: $f'(x^0) = \dfrac{\overset{f(x^0)}{L(x^0)} - \overset{0}{L(x^1)}}{x^0 - x^1}$

$\Rightarrow x^1 = x^0 - \dfrac{f(x^0)}{f'(x^0)}$

$\rightsquigarrow x^{k+1} = x^k - \dfrac{f(x^k)}{f'(x^k)}$ in each iteration

x^0 (initial guess) $\rightarrow x^1, x^2, \dots$

$|f(x^k)| < \epsilon$ (tolerance) small value

Recall from last session:

$r^2 \dfrac{dr}{dt} = -\gamma \sqrt{r-a} \rightarrow \displaystyle\int \frac{r^2}{\sqrt{r-a}} dr = -\gamma t + C$ ← $r - a = u^2$

from the IC $r = r_0$ at $t = 0$

$\Rightarrow \frac{2}{15}\sqrt{r-a}\,(8a^2 + 4ar + 3r^2) + \gamma t - C = 0$

$\rightarrow$ at each time $t = t_i$ $(t_0, t_1, t_2, \dots)$

$f(r) = \frac{2}{15}\sqrt{r-a}\,(8a^2 + 4ar + 3r^2) + \gamma t_i - C = 0$

$f'(r) = \frac{2}{15}\left[\frac{2}{\sqrt{r-a}}(8a^2 + 4ar + 3r^2) + \sqrt{r-a}\,(4a + 6r)\right]$

$r^{k+1} = r^k - \dfrac{f(r^k)}{f'(r^k)}$, $r^0 = r_{i-1}$ ← value from the previous time

when $|f(r^k)| < \epsilon \rightsquigarrow r_i = r^k$ see code drainingReservoirNewoton.py

Projectile

$\downarrow g$

initial velocity $= v_0$ $\uparrow v(t)$

$\uparrow z = 0$ ......

$m\ddot{z} = -mg - \epsilon \dot{z}^3$

$\rightsquigarrow m\dot{v} = -mg - \epsilon v^3$

If $\epsilon = 0 \Rightarrow m\dot{v} = -mg \rightarrow \dot{v} = -g \rightarrow v = -gt + v_0$

$\rightsquigarrow z = -\frac{1}{2}gt^2 + v_0 t$

Note: max height happens when $v = 0 \Rightarrow t = \frac{v_0}{g}$

when $\epsilon \neq 0 \rightarrow$ numerical (or perturbation)

focus of another session

$$m \frac{dv}{dt} = -mg - \epsilon v^3$$

Euler's method: $\dfrac{v_{i+1} - v_i}{\Delta t} = -g - \dfrac{\epsilon}{m} v_i^3$

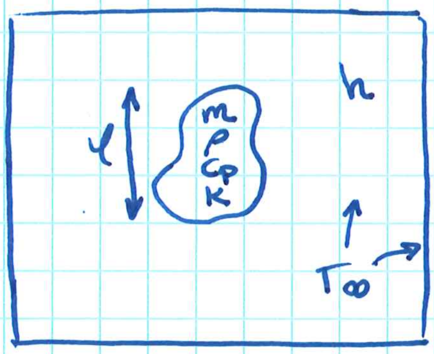
$$\Rightarrow v_{i+1} = v_i - \Delta t \left(g + \underbrace{\frac{\epsilon}{m}}_{k} v_i^3 \right)$$

$$\frac{dz}{dt} = v \Rightarrow \frac{z_{i+1} - z_i}{\Delta t} = v_i \Rightarrow z_{i+1} = z_i + \Delta t\, v_i$$

see code projectile.py

A radiating body

Consider a body of material (e.g., a sphere, cylinder, cubes etc.) in a large room.



If the conduction time-scale within the body is small, i.e., heat transfers fast in the body.

we can assume the T is uniform inside the body

$$\tau_{cond} \sim \frac{\ell^2}{\alpha}$$ ← thermal diffusivity $\frac{K}{\rho C_p}$

Energy balance

$\sigma \simeq 5.7 \times 10^{-8} \frac{W}{m^2 \cdot K^4}$

$$\frac{d}{dt}(m C_p T) = -hA(T - T_\infty) - \sigma \epsilon (T^4 - T_\infty^4) A$$

emissivity =1 for black body

radiation

Stefan-Boltzmann equation

$$\Rightarrow \frac{dT}{dt} = -\frac{hA}{mC_p}(T - T_\infty) - \frac{\sigma \epsilon A}{mC_p}(T^4 - T_\infty^4)$$

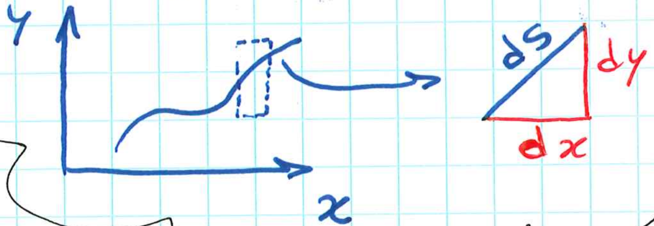
Euler? $T_{i+1} = T_i - \Delta t \left[\dfrac{hA}{mC_p}(T - T_\infty) + \dfrac{\sigma \epsilon A}{mC_p}(T^4 - T_\infty^4) \right]$

see code radiatingBody.py

$$\tau_{conv} \sim \frac{mC_p}{hA}$$

S9 Flows on the line

A digression on calculus (re HW #2)



$$\rightsquigarrow ds = \sqrt{dx^2 + dy^2}$$
$$= |dx| \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \to \ldots$$

$\dot{x} = f(x)$

$\cancel{\dot{x} = f(x,t)}$? not for now

$x(t) \to$ a real valued function of t
$f(x) \to$ a " (smooth) " " " x

usually nonlinear

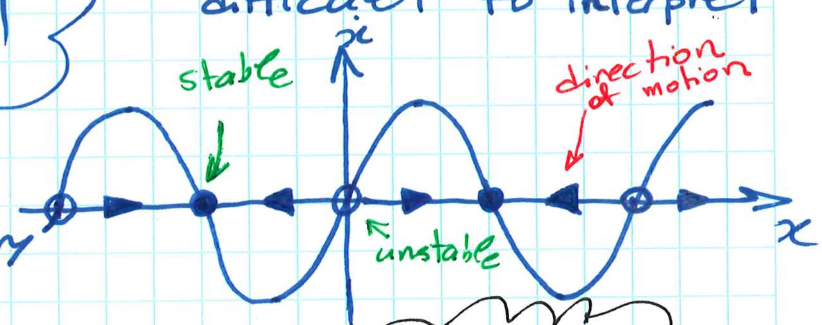
Interpreting a differential eqn as a vector field

$$\dot{x} = \sin x \rightsquigarrow dt = \frac{dx}{\sin x} \rightsquigarrow t = \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx \to \ldots$$
$$t = \ln\left|\tan\frac{x}{2}\right| + C$$

Let $x = x_0$ at $t = 0 \Rightarrow C = -\ln\left|\tan\frac{x_0}{2}\right|$

$$\Rightarrow t = \ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x_0}{2}}\right|$$ ← exact but very difficult to interpret

Alternative:



$t \to$ time
$x \to$ position of an imaginary particle moving along the real line
$\dot{x} \to$ velocity of that particle

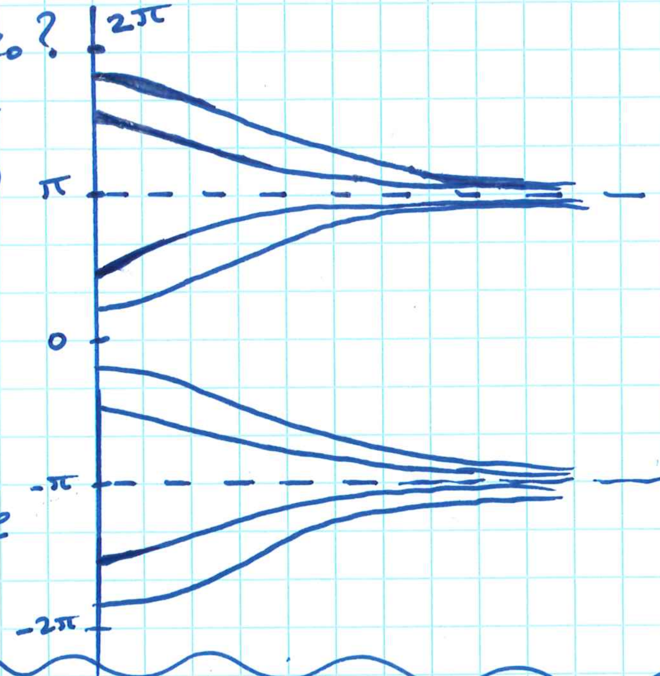
a vector field on the line → dictates the velocity vector at each x

$\dot{x} > 0 \to$ flow to the right
$\dot{x} < 0 \to$ " " " left
$\dot{x} = 0 \to$ no flow (fixed points)

Stable FP: attractors, sinks Unstable FP: repellers, sources

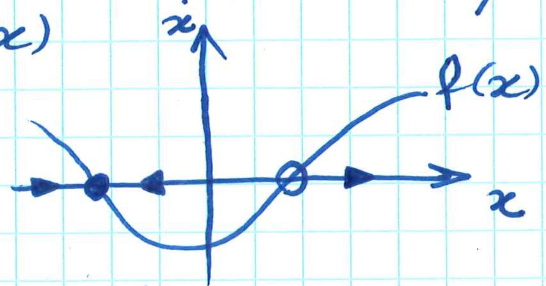
Q: Suppose $x_0 = \frac{\pi}{4}$ → describe the qualitative features of the solution $x(t)$ for all $t > 0$. What happens as $t \to \infty$?

Q: What about an arbitrary x_0?



Fixed Points & Stability

$\dot{x} = f(x)$

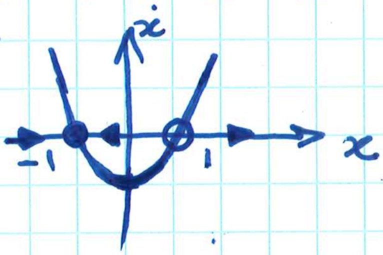


phase point: our imaginary particle
trajectory: $x(t)$ (based on x_0)
phase portrait

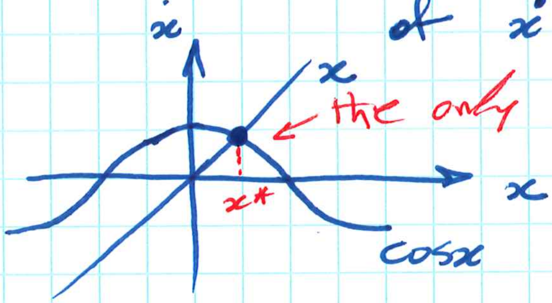
EX. Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

$f(x^*) = 0 \leadsto x^* = \pm 1$

fixed point



EX. Determine the stability of the fixed points of $\dot{x} = x - \cos x$



the only fixed point: $x^* = \cos x^* \Rightarrow \dot{x} = 0$ ✓

$$\left.\begin{array}{l} x > x^* \Rightarrow \dot{x} > 0 \\ x < x^* \Rightarrow \dot{x} < 0 \end{array}\right\} \Rightarrow \text{unstable}$$

but what is the value of x^*?

S10

Population Growth
Linear Stability Analysis
potentials

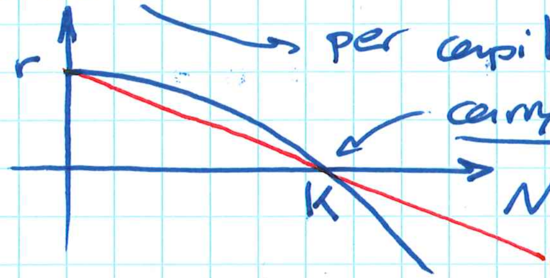
population of organisms at time t

$\dot{N} = rN$ → $r > 0$ is the growth rate

$\leadsto \dfrac{dN}{dt} = rN \Rightarrow \boxed{N = N_0 e^{rt}}$ unreasonable!
* the growth rate cannot be constant

growth rate → per capita growth rate, $\dfrac{\dot{N}}{N}$

carrying capacity



a simple model (linear)

$\leadsto \dfrac{\dot{N}}{N} = r\left(1 - \dfrac{N}{K}\right) \Rightarrow \boxed{\dot{N} = rN\left(1 - \dfrac{N}{K}\right)}$

Analytical solution: $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right) \to \dfrac{dN}{N\left(1 - \frac{N}{K}\right)} = r\,dt$

partial fractions:

$$\frac{1}{N\left(1-\frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}} = \frac{A - \frac{A}{K}N + BN}{N\left(1-\frac{N}{K}\right)} \Rightarrow \begin{array}{l} A = 1 \\ B = \frac{A}{K} = \frac{1}{K} \end{array}$$

$$\Rightarrow \int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int \frac{1}{N}\,dN + \int \frac{\frac{1}{K}}{1-\frac{N}{K}}\,dN = \boxed{\ln\left|\frac{N}{1-\frac{N}{K}}\right| = rt + C}$$

Alternative → Let $x = \dfrac{1}{N} \Rightarrow \ldots$

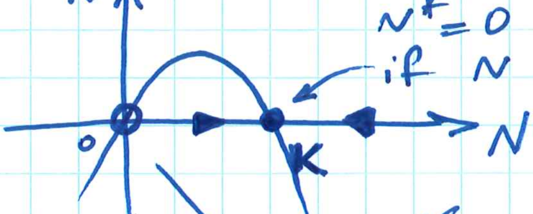
Graphical:

unstable stable

$N^* = 0$ and $N^* = K$ are the fixed points

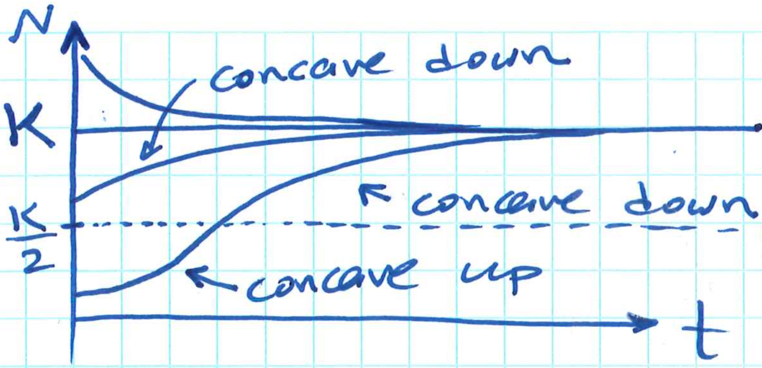
if N is disturbed from $K \Rightarrow N \to K$ as $t \to \infty$

monotonically



a small population grows exponentially fast

$N_0 = 0$? ⇒ there's nobody around to reproduce!



$$\dot{N} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{f(N)}$$

$$f'(N) = r\left(1-\frac{N}{K}\right) - \frac{rN}{K}$$

$$f'(0) = r > 0, \quad f'(K) = -r < 0$$

Let $n = \frac{N}{K} \Rightarrow \frac{dN}{dt} = \cancel{K}\frac{dn}{dt} = r\cancel{K}n(1-n)$

$\Rightarrow \frac{dn}{dt} = rn(1-n)$ see populationGrowth.py

$\frac{1}{r}$ is the time scale

Linear stability Analysis

Let x^* be a fixed point of the system $\dot{x} = f(x)$.
Let $\eta = x - x^*$ be a small perturbation away from x^*.

$\dot{\eta} = \dot{x} - 0 = \dot{x} = f(x) = f(x^* + \eta)$

x^* is constant

From Taylor series expansion: $f(x^*+\eta) = \underbrace{f(x^*)}_{0} + f'(x^*)\eta + \underbrace{O(\eta^2)}_{\text{negligible}}$

Linearization

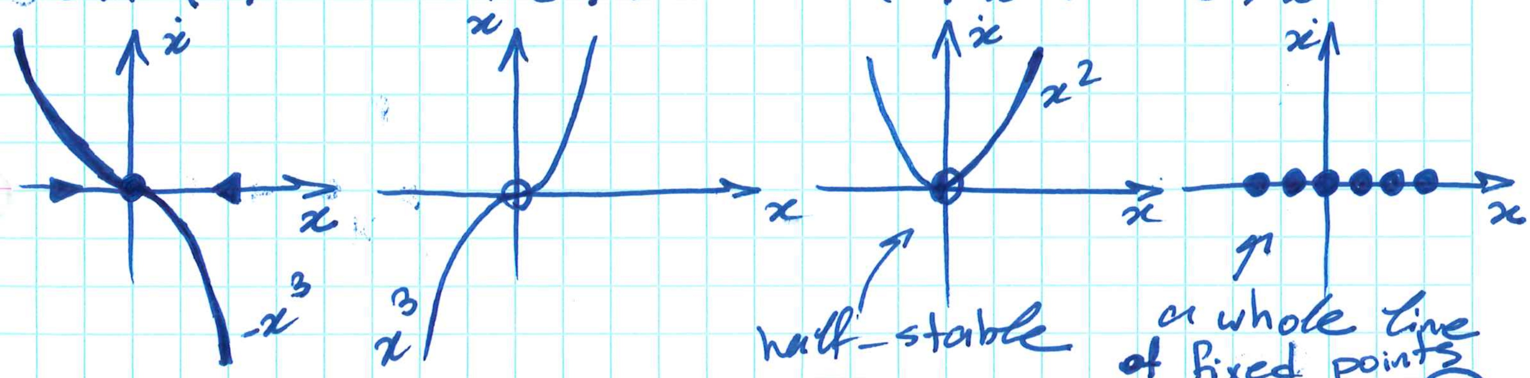
$\Rightarrow \dot{\eta} \approx f'(x^*)\eta \longrightarrow \eta(t) = \eta_0 \exp(f'(x^*)t)$

$f'(x^*) < 0 \longrightarrow$ decays (stable) ; $f'(x^*) > 0 \longrightarrow$ grows (unstable)

If $f'(x^*) = 0$, then $O(\eta^2)$ is not negligible and a nonlinear analysis is needed. (or via a case-by-case analysis)

Q: what does the magnitude of $f'(x^*)$ tell us?

EX. (a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = 0$

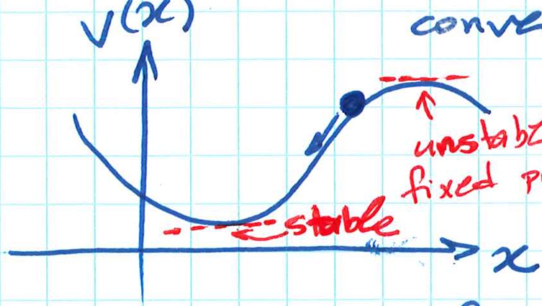


Potentials

$$\dot{x} = f(x) = -\frac{dV}{dx}$$

potential energy

convention: particle always move downhill



* no inertia → impossibility of oscillation (the only force on the particle is from the potential energy)

topic of a future session

see x as a function of t

$$\rightarrow \frac{dV(x(t))}{dt} = \frac{dV}{dx}\frac{dx}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

$\Rightarrow$ V(t) decreases along trajectories!

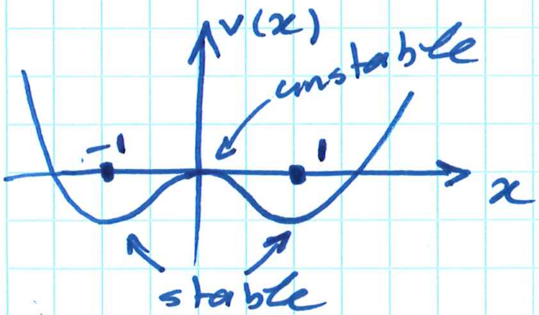
EX. Graph the potential for the system $\dot{x} = -x$

$$-\frac{dV}{dx} = -x \Rightarrow V = \frac{1}{2}x^2 + \cancel{C}$$

set 0 for convenience

(potential is only defined up to an additive constant)

EX. $\dot{x} = x - x^3 \Rightarrow -\frac{dV}{dx} = x - x^3 \Rightarrow V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \cancel{C}$ 0



- double-well potential
- the system is bistable

S11 Language Death
Laser Threshold

Language Death

Let X and Y be two languages competing for speakers in a given society.

Assume everybody is monolingual and the population well-mixed

$$\dot{x} = -x P_{XY} + (1-x) P_{YX}$$

$0 \leq x \leq 1$: fraction of the population speaking X

$1-x$: " " " " " Y

P_{YX} : rate at which individuals switch from Y to X

↳ attractiveness of the language X

$P_{YX} = s x^{\alpha}$ ← α an adjustable parameter > 1

$0 \leq s \leq 1$: perceived status, relects the social or economic opportunities afforded to the speakers of X

By symmetry : $P_{XY} = (1-s)(1-x)^{\alpha}$

$$\rightsquigarrow \dot{x} = -x(1-s)(1-x)^{\alpha} + (1-x) s x^{\alpha} = f(x)$$

Fixed Points: $f(x) = 0$

$$\Rightarrow x(1-x) \underbrace{\left[-(1-s)(1-x)^{\alpha-1} + s x^{\alpha-1}\right]}_{} = 0$$

$$\Rightarrow x^* = 0 \,;\, x^* = 1$$

$$\hookrightarrow = 0 \Rightarrow \left(\frac{1-x^*}{x^*}\right)^{\alpha-1} = \frac{s}{1-s}$$

$$\Rightarrow \frac{1}{x^*} - 1 = \left(\frac{s}{1-s}\right)^{\frac{1}{\alpha-1}} \qquad \Rightarrow x^* = \frac{1}{1+\left(\frac{s}{1-s}\right)^{\frac{1}{\alpha-1}}} \leftarrow 0 < \alpha < 1$$

$$f'(x) = (1-x)\left[-(1-s)(1-x)^{\alpha-1} + s x^{\alpha-1}\right]$$
$$- x \left[-(1-s)(1-x)^{\alpha-1} + s x^{\alpha-1}\right]$$
$$+ x(1-x)\left[(\alpha-1)(1-s)(1-x)^{\alpha-2} + (\alpha-1) s x^{\alpha-2}\right]$$

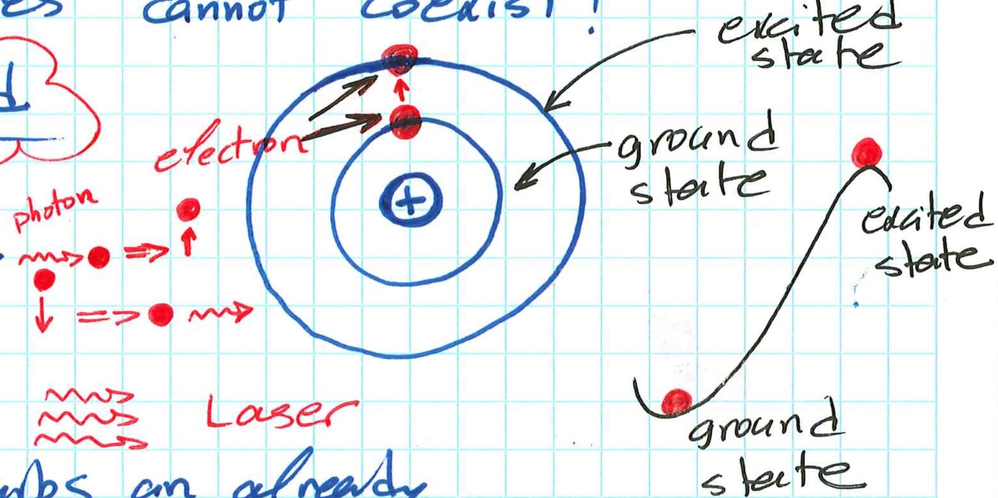
$x^* = 0 \Rightarrow f'(x^*) = s-1 < 0$ stable

$x^* = 1 \Rightarrow f'(x^*) = -s < 0$ stable

$x^* = \alpha \Rightarrow f'(x^*) > 0$ (check) unstable

$\rightsquigarrow$ two languages cannot coexist!

Laser threshold



Stimulated Absorption

Spontaneous Emission

stimulated emission

a photon disturbs an already excited electron.

$\dot{n} = GnN - kn$

G: gain coefficient

kn: loss of photon $\to \tau = \frac{1}{k}$ lifetime

n: number of photons

N: number of excited atoms

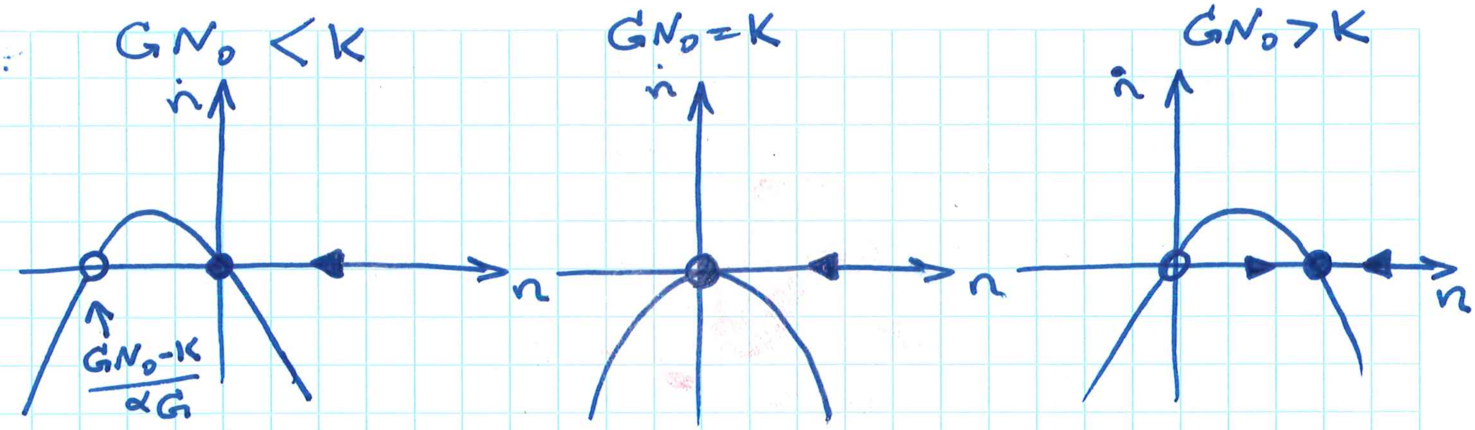
Note: N decreases by the emission of photons

Let $N(t) = N_0 - \alpha n$

N_0: number of excited atoms in the absence of laser action (fixed)

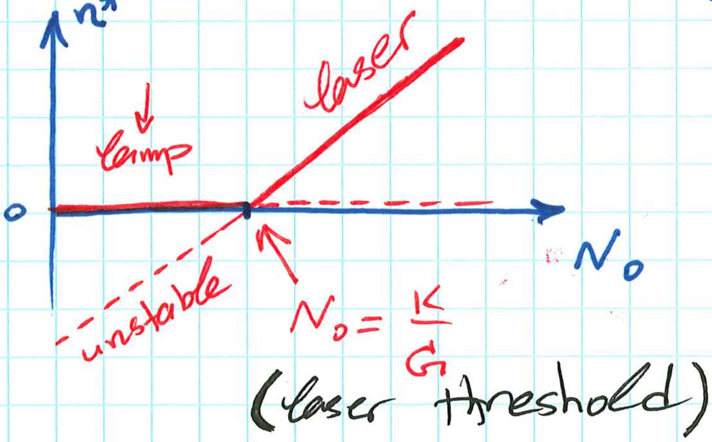
$\rightsquigarrow \dot{n} = Gn(N_0 - \alpha n) - kn$

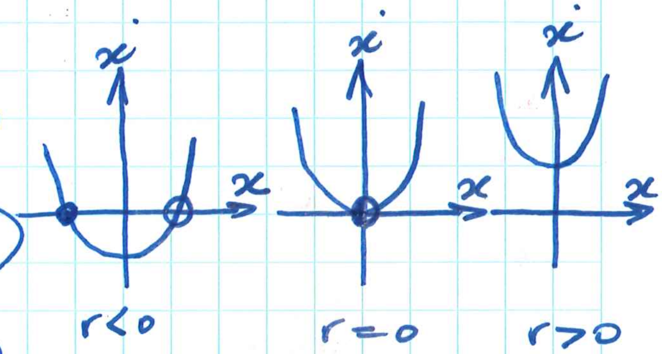
$$\rightarrow \dot{n} = (GN_0 - k)n - \alpha G n^2$$



Transcritical Bifurcation

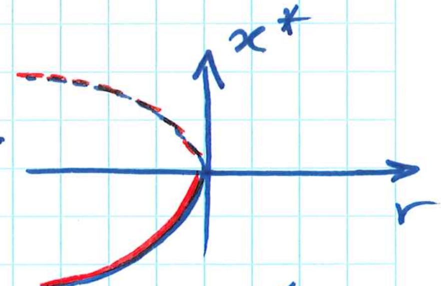
Bifurcation Diagram:



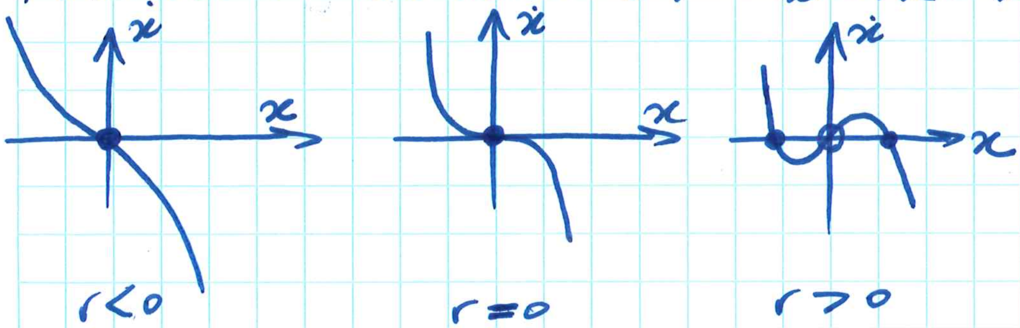


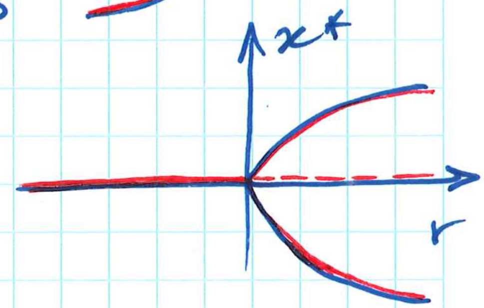
Other Bifurcations

Saddle-node bifurcation: $\dot{x} = r + x^2$



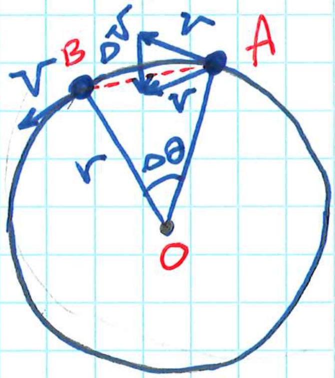
Pitchfork bifurcation: $\dot{x} = rx - x^3$





S12 Overdamped Bead on a Rotating Hoop

Centrifugal force:



the triangles OAB and the little triangle are congruent:

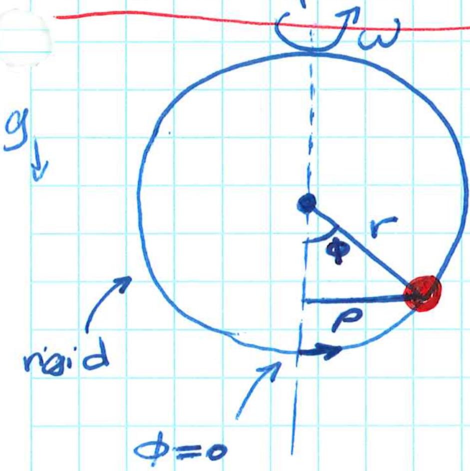
$$\frac{\overline{AB}}{r} = \frac{\Delta v}{v} \qquad \overline{AB} = 2r\sin\frac{\Delta\theta}{2}$$

$$\Rightarrow \text{for } \Delta\theta \to 0 \to \overline{AB} = r\Delta\theta = v\Delta t$$

$$\leadsto \frac{v\Delta t}{r} = \frac{\Delta v}{v} \Rightarrow \frac{\Delta v}{\Delta t} = \frac{v^2}{r} \leadsto a = \frac{v^2}{r} = r\omega^2$$

centrifugal acceleration

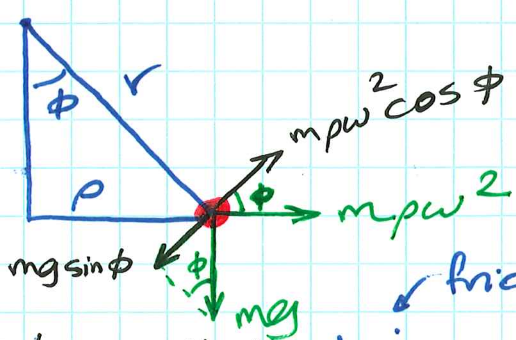
Overdamped bead on a rotating hoop



this hoop rotates around the z (vertical axis)

$\phi \in (-\pi, \pi]$

Force balance in the tangential direction:



note that the rotation occurs around the z axis.

$\underset{\sim}{a}$

$$\leadsto mr\ddot{\phi} = m\rho\omega^2\cos\phi - mg\sin\phi - b\dot{\phi}$$

$$\Rightarrow mr\ddot{\phi} = mr\omega^2\sin\phi\cos\phi - mg\sin\phi - b\dot{\phi}$$

this force balance is oversimplified!

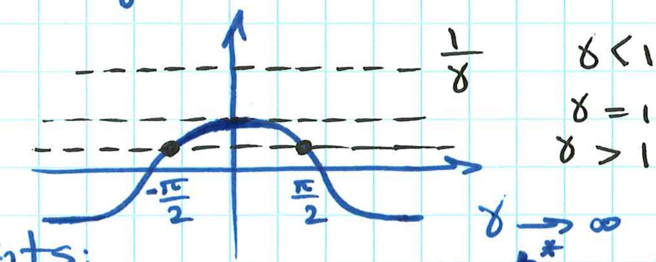
Let $mr\ddot{\phi} \approx 0$ for now! We will get back to this point later.

$$\Rightarrow b\dot{\phi} = mg\sin\phi\left[\underbrace{\frac{r\omega^2}{g}}_{\gamma}\cos\phi - 1\right] = f(\phi)$$

$f(\phi^*) = 0 \Rightarrow \phi^* = 0$ and π

$\phi^* = \pm\cos^{-1}\frac{1}{\gamma}$ for $\gamma \geq 1$

$[\ldots] \Rightarrow \cos\phi = \frac{1}{\gamma}$



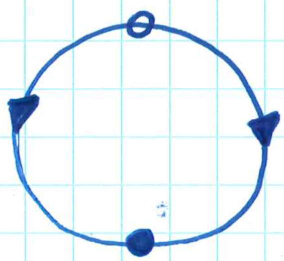
$\gamma \to \infty \Rightarrow \phi^* \to \pm\frac{\pi}{2}$

Stability of the fixed points:

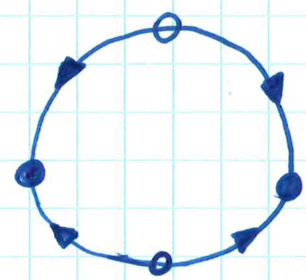
$$\frac{df}{d\phi} = mg\cos\phi\left[\gamma\cos\phi - 1\right] + mg\sin\phi\left[-\gamma\sin\phi\right]$$

$$\phi^* = \pi \Rightarrow \frac{df(\pi)}{d\phi} = -mg\cdot(-\gamma - 1) = mg(\gamma + 1) > 0$$

$$\phi^* = 0 \Rightarrow \frac{df(0)}{d\phi} = mg(\gamma - 1) \longrightarrow \begin{array}{l} > 0, \text{ unstable if } \gamma > 1 \\ < 0, \text{ stable if } \gamma < 1 \end{array}$$

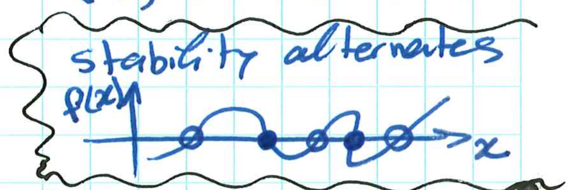


$\gamma < 1$



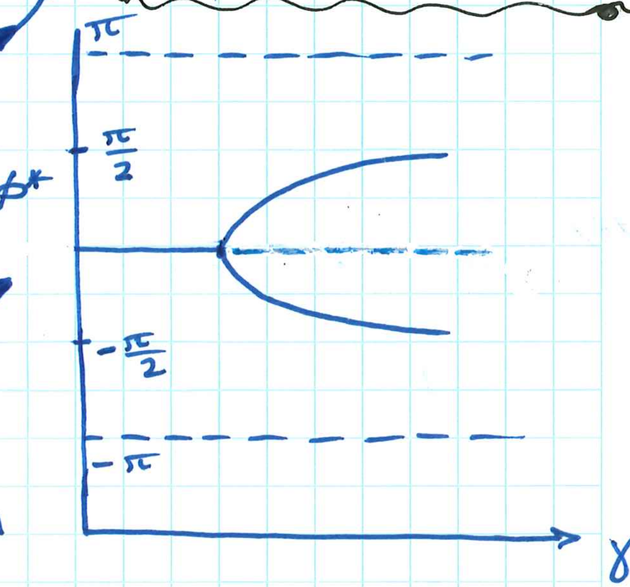
$\gamma > 1$

stability alternates



Bifurcation Diagram:

pitchfork bifurcation



Question: when is it valid to neglect the inertial term $mr\ddot{\phi}$? $m = 0$? Nope!

It helps to write the equation in dimensionless form $\to$ scales, meaning of small, orders

Define a dimensionless time $\tau = \frac{t}{T}$ ← time scale (obvious sometimes)

$O(1)$

$$mr\frac{d^2\phi}{dt^2} = mr\omega^2 \sin\phi\cos\phi - mg\sin\phi - b\frac{d\phi}{dt}$$

$$t = T\tau \Rightarrow \frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = mr\omega^2\sin\phi\cos\phi - mg\sin\phi - \frac{b}{T}\frac{d\phi}{d\tau}$$

divide by mg:

$$\frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = \underbrace{\frac{r\omega^2}{g}}_{\gamma}\sin\phi\cos\phi - \sin\phi - \frac{b}{mgT}\frac{d\phi}{d\tau}$$

we want the left hand side to be small compared to the right hand side terms.

we need $\underbrace{\frac{b}{mgT} \sim O(1)}$ and $\frac{r}{gT^2} \ll 1$

Let $T = \frac{b}{mg}$ (It is also useful to get rid of the parameters)

$$\Rightarrow \frac{r}{gT^2} = \underbrace{\frac{rm^2g}{b^2}}_{\epsilon} \ll 1$$

(a dimensionless group)

$$\Rightarrow \boxed{\epsilon\frac{d^2\phi}{d\tau^2} = \gamma\sin\phi\cos\phi - \sin\phi - \frac{d\phi}{d\tau}}$$

for $\epsilon \ll 1 \rightarrow$ we can neglect the inertia term.

S13 Insect Outbreak (Ludwig et al. 1978)

→ outbreak of an insect called the spruce budworm ↗ a serious pest in eastern Canada

they can defoliate and kill most of the fir trees in the forest in about four years!

Time scales:
- budworm population (fast) (population can increase five fold in a year) → $\tau \sim$ months
- trees population (slow) (trees can completely replace their foliage in $\sim$7–10 years; their lifespan in the absence of budworm $\sim$100–150 years)

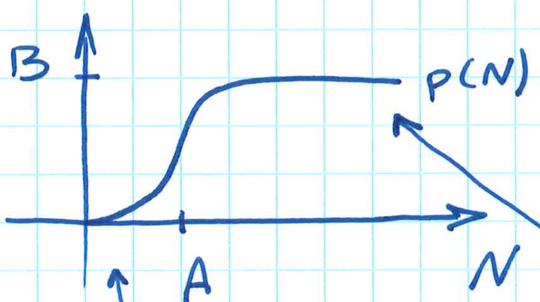
forest variables can be assumed constant

Proposed Model

predators kill budworms! → birds

$$\dot{N} = \underbrace{RN\left(1 - \frac{N}{K}\right)}_{\text{logistic eqn}} - p(N)$$

carrying capacity depends on the amount of foliage left on the trees



Qualitative shape of p(N)

saturation: the birds are eating as fast as they can!

no predation when budworms are scarce (the birds look for food elsewhere)

$$p(N) = \frac{BN^2}{A^2 + N^2} \qquad A, B > 0$$

$$\Rightarrow \dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}$$

Outbreak → as parameters drift, the budworm population, N, suddenly jumps from a low to a high level.

?↗ ?↗

Dimensionless Form

Let $x = \frac{N}{A} \rightarrow N = xA$; $\tau = \frac{t}{T} \rightarrow t = \tau T$

T to be determined

$$\leadsto \frac{A}{T}\frac{dx}{d\tau} = RAx\left(1 - \frac{Ax}{K}\right) - \frac{BA^2x}{A^2 + A^2x^2}$$

$$\rightarrow \underbrace{\frac{A}{BT}}\frac{dx}{d\tau} = \frac{RA}{B}x\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2}$$

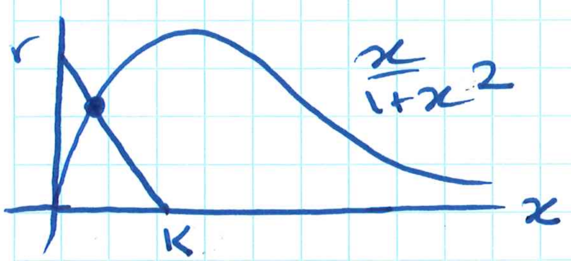
$$\hookrightarrow T = \frac{A}{B} \Rightarrow \boxed{\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}}$$

$r = \frac{RA}{B}$, $k = \frac{K}{A}$

Fixed Points

$x^* = 0$ → it is always unstable

$$\frac{dx}{d\tau} = 0 \Rightarrow r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$$ ← independent of the parameters!

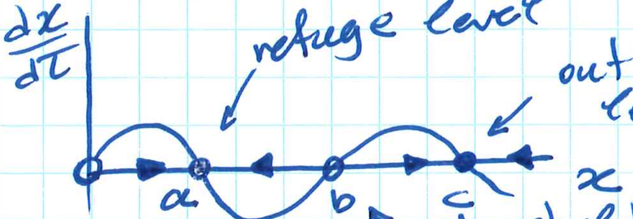


sufficiently small K (stable ?)

Bifurcations:

- b & c coalesce as r ↓
- a & b " " r ↑

stability alternates



threshold: $x_0 > b$ → outbreak!

large r ("a" disappears) leads to outbreak too

Bifurcation Curves $\to$ curves in (K, r) space where the system undergoes bifurcations.

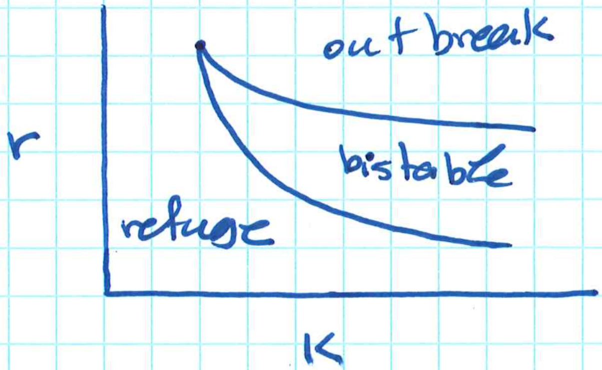
$\to$ we can't find an explicit formula

Bifurcation occurs when the two curves are tangent to each other (intersect tangentially)

$$\to r\left(1-\frac{x}{K}\right)=\frac{x}{1+x^2} \quad \text{and} \quad \frac{d}{dx}\left[r\left(1-\frac{x}{K}\right)\right]=\frac{d}{dx}\left[\frac{x}{1+x^2}\right]$$

$$\boxed{\Rightarrow r=\frac{2x^3}{(1+x^2)^2} \quad \text{and} \quad K=\frac{2x^3}{x^2-1} \quad x>1}$$

For each $x>1$, we plot $(K(x), r(x))$ in the (K, r) plane:



see insectOutbreak.py

Can you find the r, K, and x of the cusp point?

S14

- 2D Flows
- Simple harmonic oscillator

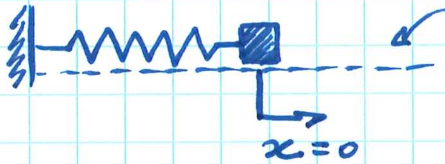
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= f(x,y) = ax + by \\ \dot{y} &= g(x,y) = cx + dy \end{aligned} \rightsquigarrow \dot{X} = AX$$

linear: if X_1 and X_2 are solutions, then any linear combination $c_1 X_1 + c_2 X_2$ is a solution as well.

Solutions can be visualized as trajectories moving on the (x,y) plane (phase plane in this context)

Simple harmonic oscillator



no friction

spring force: $-kx \Rightarrow m\ddot{x} + kx = 0$

Analytical solution? easy! more on that later...

* We want to develop methods for deducing the behavior of systems like this without actually solving them.

x and $v = \dot{x}$ determine the state of the system:

let $\omega^2 = \frac{k}{m}$

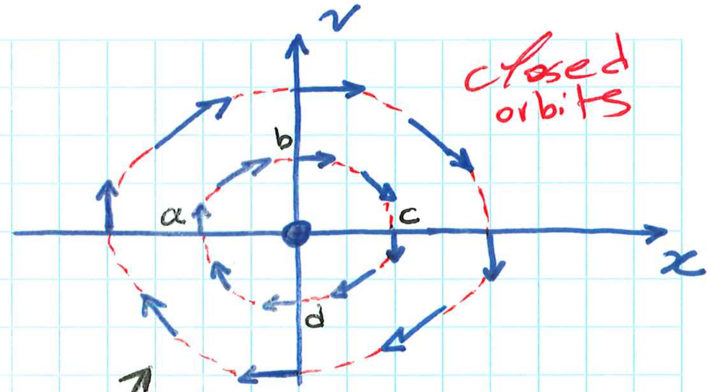
$$m\ddot{x} = -kx \rightarrow \ddot{x} = -\frac{k}{m}x \rightsquigarrow \boxed{\begin{aligned} \dot{v} &= -\omega^2 x \\ \dot{x} &= v \end{aligned}}$$

the system assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point $(x,v) \rightarrow$ represents a vector field on the phase plane.

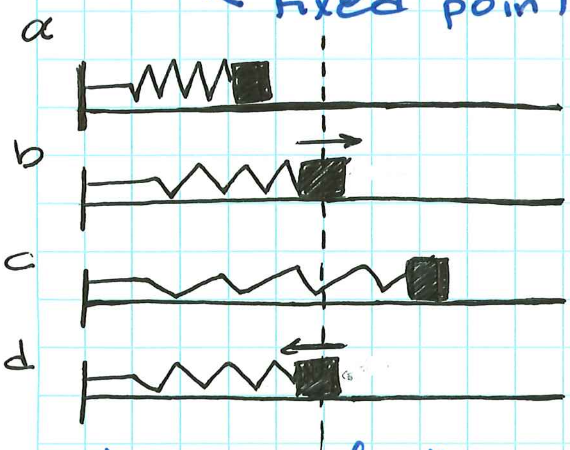
$(\dot{x}, \dot{v})$: local velocity of the imaginary particle in the phase plane

$v=0 \rightarrow (\dot{x}, \dot{v}) = (0, -\omega^2 x)$
$x=0 \rightarrow (\dot{x}, \dot{v}) = (v, 0)$

the flow of the imaginary particle swirls about the origin.
↖ fixed point



phase portrait of the system: overall picture of the trajectories



* closed orbits correspond to the periodic motions of the mass.

Shape of the closed orbits?

Let's solve it analytically: $\ddot{x} + \omega^2 x = 0$

guess $x = e^{\alpha t} \Rightarrow \alpha^2 e^{\alpha t} + \omega^2 e^{\alpha t} = 0 \rightarrow \alpha = \pm i\omega$

$\Rightarrow x = Ae^{i\omega t} + Be^{-i\omega t} = A' \cos\omega t + B' \sin\omega t = C\cos(\omega t - \phi)$

$\Rightarrow x = C\cos(\omega t - \phi) \Rightarrow v = \dot{x} = -C\omega \sin(\omega t - \phi)$

$$\rightarrow \left\{ \begin{aligned} \omega x &= C\omega\cos(\omega t - \phi) \\ v &= -C\omega \sin(\omega t - \phi) \end{aligned} \right\} \Rightarrow \omega^2 x^2 + v^2 = C^2\omega^2 \left[\cos^2(\omega t - \phi) + \sin^2(\omega t - \phi)\right]$$

$$\Rightarrow \omega^2 x^2 + v^2 = K^2$$

ellipses

Example: $\dot{x} = ax$, $\dot{y} = -y$ → Graph the phase portrait as a varies from $-\infty$ to ∞, showing the qualitatively different cases.

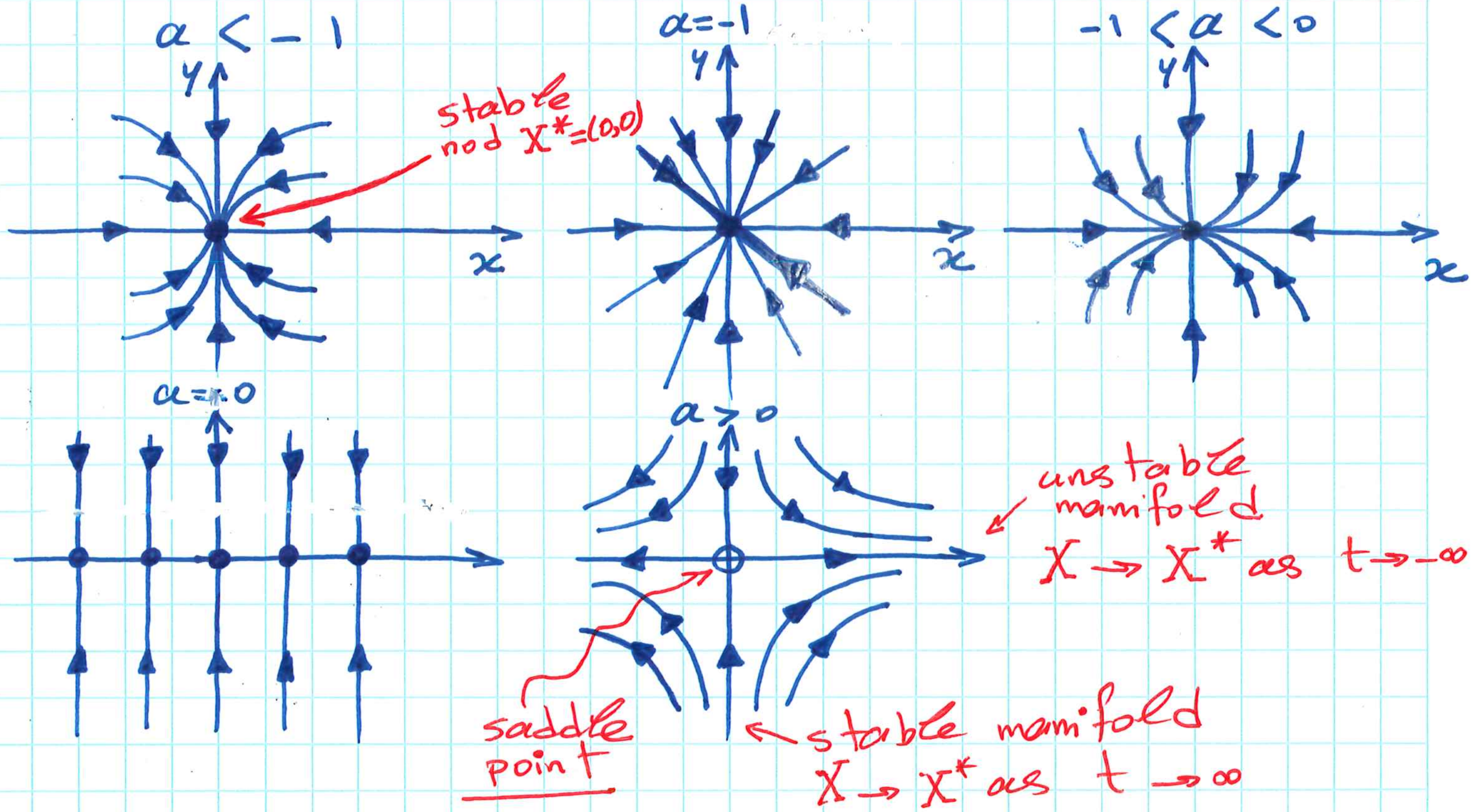
$\rightarrow x = x_0 e^{at}$
$y = y_0 e^{-t}$

$$\rightarrow \frac{dy}{dx} = -\frac{y_0}{a x_0} e^{(-a-1)t}$$

* $a<0 \rightarrow$ as $t \rightarrow \infty$
$\Rightarrow x, y \rightarrow 0$
↗ stable node

* $a=0 \rightarrow x$ does not change

* $a>0 \rightarrow x$ increases exponentially



$a < -1$
y
x
stable nod $X^* = (0,0)$
$a = -1$
y
x
$-1 < a < 0$
y
x
$a = 0$
$a > 0$
unstable manifold
$X \to X^*$ as $t \to -\infty$
saddle point
stable manifold
$X \to X^*$ as $t \to \infty$

S15
- Classification of 2D Linear systems
- Love affairs

(*) $\dot{X} = AX$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

straight-line trajectories are determined by the eigenvectors of the matrix A.

eigensolutions $X(t) = e^{\lambda t} V$

eigenvalue (λ), eigenvector (V, constant)

Insert into the eqn (*):

$\lambda \cancel{e^{\lambda t}} V = A \cancel{e^{\lambda t}} V \implies \boxed{AV = \lambda V} \to (A - \lambda I)V = 0$ (*)

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\leadsto \det(A - \lambda I) = 0$

$\leadsto \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = 0 \to \lambda^2 - (a+d)\lambda + ad - bc = 0$

$\to$ roots are λ_1 & λ_2

Our focus $\to$ typical situation $\lambda_1 \neq \lambda_2$

Find V_1 and V_2 from λ_1 and λ_2:

(*) $\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 0 \to \alpha_1 = ?\ \ \beta_1 = ? \implies V_1 = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$

$\begin{bmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = 0 \to \alpha_2 = ?\ \ \beta_2 = ? \implies V_2 = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$

$\leadsto X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$

To sketch the phase portrait

1) Find λ_1 and λ_2
2) Find the corresponding eigenvectors V_1 and V_2
3) Draw straight lines V_1 and V_2 on the phase plane
4) Direction of flow is determined by the sign of λ
5) Draw (qualitatively) the other trajectories

Love Affairs (Strogatz 1988)

$R(t)$: Romeo's love/hate for Juliet at time t
$J(t)$: Juliet's " " Romeo " " "

→ For two identically cautious lovers:

$\dot{R} = aR + bJ$ $\quad a<0$: measure of cautiousness
$\dot{J} = bR + aJ$ $\quad b>0$: " " responsiveness

Sketch the phase portrait and interpret the results.

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \to \det\left(\begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix}\right) = 0 \Rightarrow (a-\lambda)^2 - b^2 = 0$$

$\Rightarrow \lambda_1 = a+b, \quad \lambda_2 = a-b$

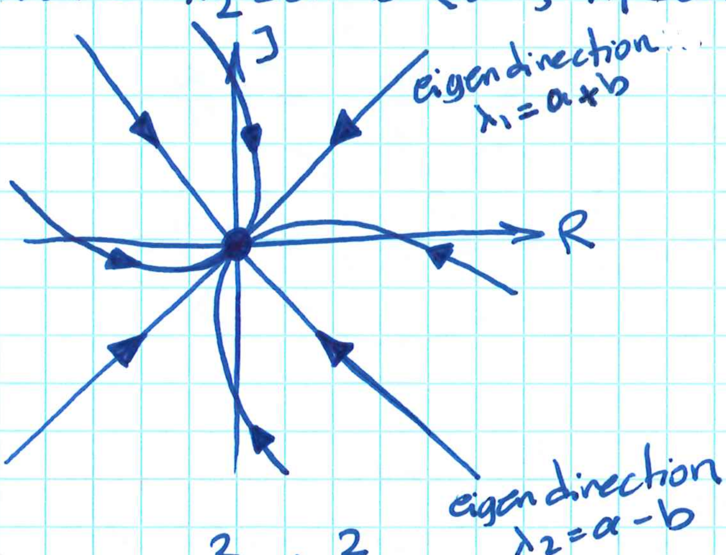
$$\begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{cases} (a-\lambda)x + by = 0 \\ bx + (a-\lambda)y = 0 \end{cases}$$

$$\lambda = \lambda_1 = a+b \Rightarrow \begin{cases} -bx + by = 0 \\ bx - by = 0 \end{cases} \to x=1, y=1 \Rightarrow V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

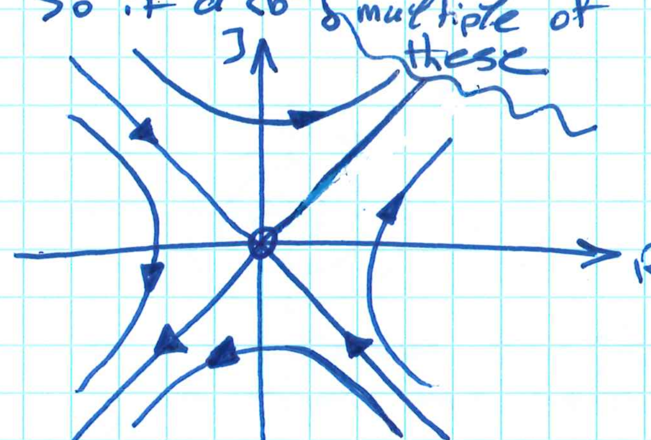
$$\lambda = \lambda_2 = a-b \Rightarrow \begin{cases} bx + by = 0 \\ bx + by = 0 \end{cases} \to x=1, y=-1 \Rightarrow V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or any multiple of these

Note: $\lambda_2 = a-b < 0$; $\lambda_1 = a+b$ <0 if $a^2 > b^2$
>0 if $a^2 < b^2$



$a^2 > b^2$



$a^2 < b^2$

$|a-b| > |a+b|$

$a^2 > b^2$: the relationship fizzles out to mutual indifference
$a^2 < b^2$: love fest or war!

→ Juliet is a fickle lover
Romeo tends to echo Juliet

↳ he warms up when she loves her and grows cold when she hates him.

the more Romeo loves her, the more Juliet wants to run away!

$$\left.\begin{aligned}\dot{R} &= aJ \\ \dot{J} &= -bR\end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix} \rightarrow \lambda = \pm i\sqrt{ab}$$

$$\lambda_1 = i\sqrt{ab} \Rightarrow v_1 = \begin{bmatrix} -i\sqrt{\frac{a}{b}} \\ 1 \end{bmatrix}$$

$$\lambda_2 = -i\sqrt{ab} \Rightarrow v_2 = \begin{bmatrix} i\sqrt{\frac{a}{b}} \\ 1 \end{bmatrix}$$

$$\rightsquigarrow \mathbf{X} = c_1 e^{i\sqrt{ab}t} \begin{bmatrix} -i\sqrt{\frac{a}{b}} \\ 1 \end{bmatrix} + c_2 e^{-i\sqrt{ab}t} \begin{bmatrix} i\sqrt{\frac{a}{b}} \\ 1 \end{bmatrix}$$

$$\Rightarrow R = -i\sqrt{\frac{a}{b}}\, c_1 e^{i\sqrt{ab}t} + i\sqrt{\frac{a}{b}}\, c_2 e^{-i\sqrt{ab}t}$$

$$i\sqrt{\frac{a}{b}}\, J = i\sqrt{\frac{a}{b}}\, c_1 e^{i\sqrt{ab}t} + i\sqrt{\frac{a}{b}}\, c_2 e^{-i\sqrt{ab}t}$$

$$\Rightarrow R^2 = -\frac{a}{b} c_1^2 e^{2i\sqrt{ab}t} - \frac{a}{b} c_2^2 e^{-2i\sqrt{ab}t} + 2\frac{a}{b} c_1 c_2$$

$$-\frac{a}{b} J^2 = -\frac{a}{b} c_1^2 e^{2i\sqrt{ab}t} - \frac{a}{b} c_2^2 e^{-2i\sqrt{ab}t} - 2\frac{a}{b} c_1 c_2$$

$$\Rightarrow \boxed{R^2 + \frac{a}{b} J^2 = 4\frac{a}{b} c_1 c_2} \longrightarrow$$

a neverlasting cycle of love and hate!

Question: what if $\lambda_{1,2} = \alpha \pm i\omega$?

S16

- Perturbation solution
- Projectile
- weakly damped linear oscillator

nonlinear

Projectile: $m\dot{v} = -mg - \delta v^3$

Let $\tilde{v} = \dfrac{v}{v_0}$, $\tilde{t} = \dfrac{t}{\tau}$ $\qquad v(0) = v_0$

$$\Rightarrow \frac{m v_0}{\tau}\frac{d\tilde{v}}{d\tilde{t}} = -mg - \delta v_0^3 \tilde{v}^3 \Rightarrow \frac{v_0}{g\tau}\frac{d\tilde{v}}{d\tilde{t}} = -1 - \frac{\delta v_0^3}{mg}\tilde{v}^3$$

Let $\tau = \dfrac{v_0}{g}$ $\Rightarrow$ $\dfrac{d\tilde{v}}{d\tilde{t}} = -1 - \epsilon\tilde{v}^3$ with $\tilde{t} = \dfrac{tg}{v_0}$, $\tilde{v} = \dfrac{v}{v_0}$

For very small ϵ values ($\epsilon \ll 1$), we seek solutions in the form of a power series in ϵ:

$$v(t) = \underbrace{\epsilon^0 v^{(0)}(t) + \epsilon^1 v^{(1)}(t)}_{\text{first-order perturbation}} + \epsilon^2 v^{(2)}(t) + \dots$$

(dropped ~ sign for simplicity)

substitute into the eqn and conditions and equate the coefficients of each power of ϵ to zero separately. (Equate the terms with common power of ϵ)

$$\frac{d}{dt}\left(\epsilon^0 v^{(0)} + \epsilon^1 v^{(1)}\right) = -1 - \epsilon\left(\epsilon^0 v^{(0)} + \epsilon^1 v^{(1)}\right)^3$$

$$\rightsquigarrow \frac{dv^{(0)}}{dt} + \epsilon\frac{dv^{(1)}}{dt} = -1 - \epsilon\left[{v^{(0)}}^3 + 3{v^{(0)}}^2 \epsilon v^{(1)} + 3v^{(0)}\epsilon^2 {v^{(1)}}^2 + \epsilon^3 {v^{(1)}}^3\right]$$

$$= -1 - \epsilon {v^{(0)}}^3 - 3\epsilon^2 {v^{(0)}}^2 v^{(1)} - 3\epsilon^3 v^{(0)} {v^{(1)}}^2 - \epsilon^4 {v^{(1)}}^3$$

$$\Rightarrow \frac{dv^{(0)}}{dt} + \epsilon\frac{dv^{(1)}}{dt} = -1 - \epsilon {v^{(0)}}^3 + O(\epsilon^2)$$

$$\Rightarrow \left[\frac{dv^{(0)}}{dt} + 1\right] + \epsilon\left[\frac{dv^{(1)}}{dt} + {v^{(0)}}^3\right] + O(\epsilon^2) = 0$$

initial conditions: $\epsilon^0 v^{(0)} + \epsilon^1 v^{(1)} = 1 \Rightarrow \left[v^{(0)} - 1\right] + \epsilon v^{(1)} = 0$

zeroth-order $\frac{dv^{(0)}}{dt} = -1$; $v^{(0)}(0) = 1$

$\longrightarrow v^{(0)} = -t + 1$

first-order $\frac{dv^{(1)}}{dt} = -{v^{(0)}}^3 = -(1-t)^3$; $v^{(1)}(0) = 0$

$\longrightarrow v^{(1)} = \frac{1}{4}(t-1)^4 + C$ $\quad C = -\frac{1}{4}$ $\longrightarrow v^{(1)} = \frac{1}{4}\left[(t-1)^4 - 1\right]$

$\Longrightarrow v(t) = 1 - t + \frac{\epsilon}{4}\left[(t-1)^4 - 1\right] + O(\epsilon^2)$

see "perturbation_projectile.py"

Higher orders? Why not? It is like a domino!

Weakly damped linear oscillator

$\ddot{x} + 2\varepsilon\dot{x} + x = 0$; $x(0) = 0$; $\dot{x}(0) = 1$

Analytical solution: $r^2 + 2\varepsilon r + 1 = 0 \Longrightarrow r = -\varepsilon \pm i\sqrt{1-\varepsilon^2}$

$\Longrightarrow x = e^{-\varepsilon t}\left[A\cos\sqrt{1-\varepsilon^2}\,t + B\sin\sqrt{1-\varepsilon^2}\,t\right]$

$x(0) = 0 \Longrightarrow A = 0$; $\dot{x}(0) = 1 \Longrightarrow -\varepsilon e^{-\varepsilon t} B \sin\sqrt{1-\varepsilon^2}\,t + \sqrt{1-\varepsilon^2}\,e^{-\varepsilon t} B\cos\sqrt{1-\varepsilon^2}\,t = 1$ at 0

$\longrightarrow \sqrt{1-\varepsilon^2}\,B = 1 \Longrightarrow B = \frac{1}{\sqrt{1-\varepsilon^2}}$

$$\Longrightarrow x = \frac{e^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}} \sin\sqrt{1-\varepsilon^2}\,t$$

Let $x = x^{(0)} + \varepsilon x^{(1)} + O(\varepsilon^2)$

$\longrightarrow \ddot{x}^{(0)} + \varepsilon\ddot{x}^{(1)} + 2\varepsilon(\dot{x}^{(0)} + \varepsilon\dot{x}^{(1)}) + x^{(0)} + \varepsilon x^{(1)} + O(\varepsilon^2) = 0$

$\longrightarrow \left[\ddot{x}^{(0)} + x^{(0)}\right] + \varepsilon\left[\ddot{x}^{(1)} + 2\dot{x}^{(0)} + x^{(1)}\right] + O(\varepsilon^2) = 0$

Initial conditions: $x^{(0)}(0) + \varepsilon x^{(1)}(0) = 0$

$\dot{x}^{(0)}(0) + \varepsilon\dot{x}^{(1)}(0) = 1$

zeroth-order $\ddot{x}^{(0)} + x^{(0)} = 0$; $x^{(0)}(0) = 0$; $\dot{x}^{(0)}(0) = 1$

$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow x^{(0)} = \underbrace{\cancel{A}}_{0}\cos t + \underbrace{B}_{1}\sin t \Rightarrow x^{(0)} = \sin t$

first-order $\ddot{x}^{(1)} + 2\dot{x}^{(0)} + x^{(1)} = 0$; $x^{(1)}(0) = 0$; $\dot{x}^{(1)}(0) = 0$

$\rightarrow \ddot{x}^{(1)} + x^{(1)} = -2\cos t \rightarrow x^{(1)} = -t\sin t$

grows with time without bound!

$\Rightarrow x = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$

see "perturbation_damped.py"

S17 Two-timing

ODE: $\ddot{x} + 2\varepsilon\dot{x} + x = 0 \longrightarrow$ analytical solution : $x = \frac{e^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}} \sin(\sqrt{1-\varepsilon^2}\, t)$

* the true solution exhibits two time scales:

- fast $O(1)$ for the oscillation
- slow $O(\frac{1}{\varepsilon})$ over which the amplitude decays

kicks in at large times

Our perturbation solution misrepresents the slow time scale behavior

$x = \sin t - \varepsilon t \sin t$

Two-timing

$\tau = t$ fast time

$T = \varepsilon t$ slow time

Note: functions of T (slow time) can be regarded as constant on the fast time scale, e.g., your height is constant on the time scale of a day

$$x(t) = x^{(0)}(\tau, T) + \varepsilon x^{(1)}(\tau, T) + O(\varepsilon^2)$$

chain rule $\frac{dx}{dt} = \frac{\partial x}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t} = \underline{\frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial T}}$

$$\Rightarrow \frac{dx}{dt} = \frac{\partial x^{(0)}}{\partial \tau} + \varepsilon \frac{\partial x^{(1)}}{\partial \tau} + \varepsilon \frac{\partial x^{(0)}}{\partial T} + \varepsilon^2 \frac{\partial x^{(1)}}{\partial T}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\partial x^{(0)}}{\partial \tau} + \varepsilon\left(\frac{\partial x^{(1)}}{\partial \tau} + \frac{\partial x^{(0)}}{\partial T}\right) + O(\varepsilon^2) \qquad (*)$$

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x^{(0)}}{\partial \tau^2} + \varepsilon\left(\frac{\partial^2 x^{(1)}}{\partial \tau^2} + \frac{\partial^2 x^{(0)}}{\partial \tau \partial T}\right) + \varepsilon\left[\frac{\partial^2 x^{(0)}}{\partial \tau \partial T} + \varepsilon\left(\frac{\partial^2 x^{(1)}}{\partial \tau \partial T} + \frac{\partial^2 x^{(0)}}{\partial T^2}\right)\right] + O(\varepsilon^2)$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{\partial^2 x^{(0)}}{\partial \tau^2} + \varepsilon\left(\frac{\partial^2 x^{(1)}}{\partial \tau^2} + 2\frac{\partial^2 x^{(0)}}{\partial \tau \partial T}\right) + O(\varepsilon^2) \qquad (**)$$

* & ** → ODE:

$$\frac{\partial^2 x^{(0)}}{\partial \tau^2} + \varepsilon\left(\frac{\partial^2 x^{(1)}}{\partial \tau^2} + 2\frac{\partial^2 x^{(0)}}{\partial \tau \partial T}\right) + 2\varepsilon\left(\frac{\partial x^{(0)}}{\partial \tau} + \varepsilon\left(\frac{\partial x^{(1)}}{\partial \tau} + \frac{\partial x^{(0)}}{\partial T}\right)\right)$$
$$+ x^{(0)} + \varepsilon x^{(1)} = 0$$

functions of T — I

$O(\varepsilon^0)$: $\dfrac{\partial^2 x^{(0)}}{\partial \tau^2} + x^{(0)} = 0 \Rightarrow x^{(0)} = A\sin\tau + B\cos\tau$

$O(\varepsilon^1)$: $\dfrac{\partial^2 x^{(1)}}{\partial \tau^2} + 2\left(\dfrac{\partial^2 x^{(0)}}{\partial \tau \partial T} + \dfrac{\partial x^{(0)}}{\partial \tau}\right) + x^{(1)} = 0$

$$\Rightarrow \frac{\partial^2 x^{(1)}}{\partial \tau^2} + x^{(1)} = -2\frac{\partial}{\partial \tau}\left[\frac{\partial x^{(0)}}{\partial T} + x^{(0)}\right]$$
$$= -2\frac{\partial}{\partial \tau}\left[A'\sin\tau + B'\cos\tau + A\sin\tau + B\cos\tau\right]$$
$$\Rightarrow \frac{\partial^2 x^{(1)}}{\partial \tau^2} + x^{(1)} = -2(A'+A)\cos\tau + 2(B'+B)\sin\tau$$

As it is, the equation leads to the same problem we had because the RHS is a "resonant forcing" → solutions of the form $\tau\sin\tau$ and $\tau\cos\tau$

Unless $\left.\begin{array}{l} A'+A=0 \\ B'+B=0 \end{array}\right\}$ $\begin{array}{l} \to A = A(0)e^{-T} \\ \to B = B(0)e^{-T} \end{array}$ $\qquad x^{(1)} = a\sin\tau + b\cos\tau$ — II

Initial conditions: $x(0)=0$, $\dot{x}(0)=1$

$$x(t) = x^{(0)}(\tau, T) + \varepsilon x^{(1)}(\tau, T)$$

$$\Rightarrow x(0) = 0 = x^{(0)}(0,0) + \varepsilon x^{(1)}(0,0) \Rightarrow \begin{array}{l} x^{(0)}(0,0) = 0 \quad ① \\ x^{(1)}(0,0) = 0 \quad ② \end{array}$$

$$\dot{x}(t) = \frac{\partial x^{(0)}(\tau,T)}{\partial \tau} + \varepsilon\left(\frac{\partial x^{(1)}(\tau,T)}{\partial \tau} + \frac{\partial x^{(0)}(\tau,T)}{\partial T}\right)$$

$$\Rightarrow \dot{x}(0) = 1 = \frac{\partial x^{(0)}(0,0)}{\partial \tau} + \varepsilon\left(\frac{\partial x^{(1)}(0,0)}{\partial \tau} + \frac{\partial x^{(0)}(0,0)}{\partial T}\right)$$

$$\Rightarrow \frac{\partial x^{(0)}(0,0)}{\partial \tau} = 1 \quad ③, \qquad \frac{\partial x^{(1)}(0,0)}{\partial \tau} + \frac{\partial x^{(0)}(0,0)}{\partial T} = 0 \quad ④$$

I ①: $0 = A(0)\sin 0 + B(0)\cos(0) \Rightarrow B(0) = 0 \Rightarrow \boxed{B=0}$

③: $1 = A(0)\cos 0 \Rightarrow A(0) = 1 \Rightarrow \boxed{A = e^{-T}}$

II ②: $0 = a\sin 0 + b\cos 0 \Rightarrow \boxed{b=0}$

④: $\underbrace{a\cos 0}_{\frac{\partial x^{(1)}(0,0)}{\partial \tau}} - \underbrace{e^{-0}\sin 0}_{\frac{\partial x^{(0)}(0,0)}{\partial T}} \Rightarrow \boxed{a=0}$

$$\Rightarrow x = e^{-\varepsilon t}\sin t \quad \text{vs} \quad \frac{e^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}}\sin\left(\sqrt{1-\varepsilon^2}\,t\right)$$

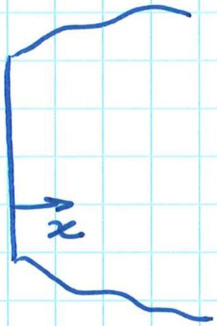
see "perturbation_damped.py"

with twoTiming = True

S18

- semi-infinite regions
- Combination of variables
- Stefan problem

Heating of a semi-infinite slab



$x \in [0, \infty)$ $\quad$ $T(x,t)$ for $x \to \infty$ $= T_0$

$T(x,0) = T_0$ $\qquad$ Find $T(x,t)$

$T(0,t) = T_W$

Assume that the domain goes to infinity in all directions except the one at $x=0$ $\to$ like a thick, tall, wide, wall! $\to$ one-dimensional heat eqn:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \to T = T(x,t,\alpha)$$

Let $\theta = \frac{T-T_0}{T_W - T_0} \to \frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \to \theta = \theta(\eta)$ dimensionless combination of the quantities

$[x] \doteq$ length $\qquad [t] \doteq$ time $\qquad [\alpha] = \frac{\text{length}^2}{\text{time}}$

$\Rightarrow \quad \eta = \frac{cx}{\sqrt{\alpha t}}$ $\quad$ some number (will be chosen in such way to simplify the equations)

Chain rule: $\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial \theta}{\partial \eta}\left(-\frac{1}{2}\frac{cx}{\sqrt{\alpha}} t^{-3/2}\right)$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \theta}{\partial \eta}\frac{c}{\sqrt{\alpha t}}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial \theta}{\partial \eta}\frac{c}{\sqrt{\alpha t}}\right)\frac{\partial \eta}{\partial x} = \frac{\partial^2 \theta}{\partial \eta^2}\frac{c^2}{\alpha t}$$

$$\Rightarrow -\frac{1}{2}\frac{cx}{\sqrt{\alpha}} t^{-3/2}\frac{d\theta}{d\eta} = \alpha \frac{c^2}{\alpha t}\frac{d^2\theta}{d\eta^2}$$

$$\frac{d^2\theta}{d\eta^2} + \underbrace{\frac{1}{2c}\frac{x}{\sqrt{\alpha t}}}\frac{d\theta}{d\eta} = 0$$

$\hookrightarrow$ take $c^2 = \frac{1}{4} \Rightarrow c = \frac{1}{\sqrt{4}}$

$\rightsquigarrow \frac{d^2\theta}{d\eta^2} + 2\eta\frac{d\theta}{d\eta} = 0, \quad \eta = \frac{x}{\sqrt{4\alpha t}}$, at $\eta = \infty \Rightarrow \theta = 0$
at $\eta = 0 \Rightarrow \theta = 1$

$\theta = \frac{T - T_0}{T_W - T_0}$

To solve, let $\psi = \frac{d\theta}{d\eta} \Rightarrow \frac{d\psi}{d\eta} + 2\eta\psi = 0$

$\Rightarrow \frac{d\psi}{\psi} = -2\eta\, d\eta \Rightarrow \psi = c_1 e^{-\eta^2}$

$\rightsquigarrow \frac{d\theta}{d\eta} = c_1 e^{-\eta^2} \Rightarrow d\theta = c_1 e^{-\eta^2} d\eta$

$$\mathrm{erf}(\eta) = \frac{\int_0^{\eta} e^{-t^2} dt}{\frac{\sqrt{\pi}}{2}}$$

$$\Rightarrow \theta = c_1 \int_0^{\eta} e^{-\bar{\eta}^2} d\bar{\eta} + c_2 = c_3\, \mathrm{erf}(\eta) + c_2$$

$\eta = 0 \Rightarrow \theta = 1 \Rightarrow \boxed{c_2 = 1}$

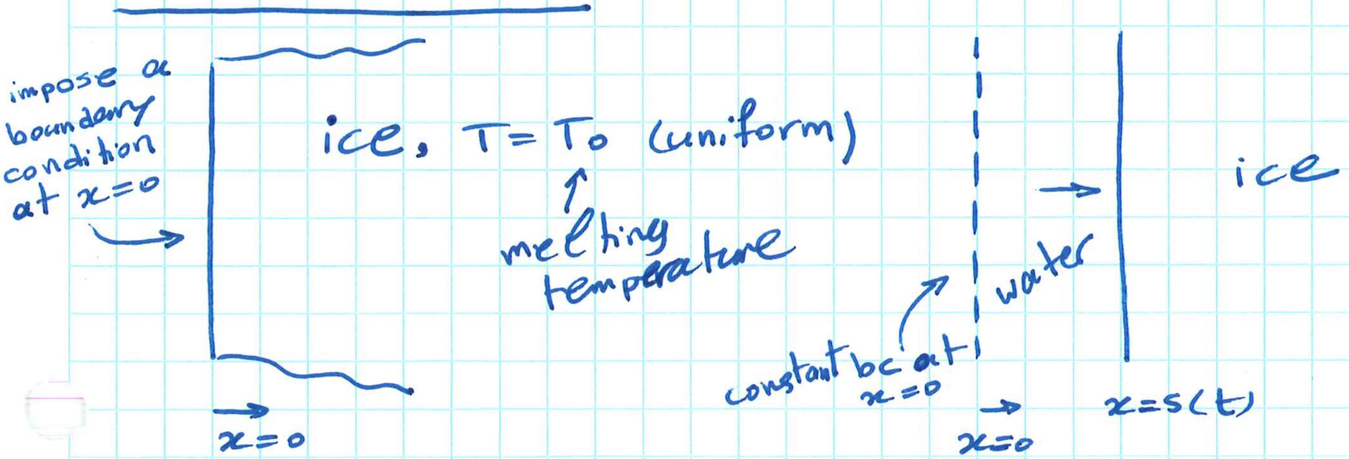
$\eta = \infty \Rightarrow \theta = 0 \Rightarrow 0 = c_3 + 1 \Rightarrow \boxed{c_3 = -1}$

$\mathrm{erf}(\infty) = 1$ (Can you prove this?)

$$\Rightarrow \theta = 1 - \mathrm{erf}(\eta) = 1 - \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

see "semiInfinite.py" for a demonstration

Stefan Problem



Question: find the temp distribution in water and s as a function of time

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (i) \qquad x \in [0, s(t)]$$

$T = T_W$ at $x = 0$ & $T = T_0$ at $x = s(t)$

the melting process:

the heat conducted to the interface balances with the heat required to melt the ice ($\underbrace{\text{latent heat}}$)

no temperature change

this is why winds make you feel cooler: evaporation from the skin removes heat

$$\Rightarrow -K \frac{dT(s(t))}{dx} \cancel{A} = q \rho_i \cancel{A} \frac{ds}{dt}$$

q: $\frac{\text{energy}}{\text{unit mass}}$; ρ_i: density of ice

$$\leadsto -K \frac{dT(s(t))}{dt} = q \rho_i \frac{ds}{dt} \quad (*)$$

We solve (i) using combination of variables:

$$\theta = A \,\mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) + B \qquad \begin{array}{lll} \theta = 1 & \text{at} & x = 0 \quad (1) \\ \theta = 0 & \text{at} & x = s(t) \quad (2) \end{array}$$

$\theta = \frac{T - T_0}{T_W - T_0}$

(1): $1 = A\,\mathrm{erf}(0) + B \Rightarrow \boxed{B = 1}$

$dT = (T_W - T_0)\, d\theta$

(2): $0 = A\,\mathrm{erf}\Big(\underbrace{\frac{s(t)}{\sqrt{4\alpha t}}}_{\lambda}\Big) + 1 \Rightarrow \boxed{A = -\frac{1}{\mathrm{erf}(\lambda)}}$

Note: since A is a constant, λ must also be a constant $\leadsto s(t) \propto \sqrt{t}$ (we will see if it makes sense)

$\lambda = ?$

$$(*) \quad -K(T_W - T_0) \frac{d}{dt}\left(1 - \frac{1}{\mathrm{erf}(\lambda)} \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right)\Bigg|_{x = s(t)} = q \rho_i \frac{d}{dt}\left(\lambda \sqrt{4\alpha t}\right)$$

$$\leadsto -K(T_W - T_0)\left(-\frac{1}{\mathrm{erf}(\lambda)} \frac{\cancel{2}}{\sqrt{\cancel{4}\alpha t \pi}} e^{-\frac{x^2}{4\alpha t}}\right)\Bigg|_{x = s(t)} = q \rho_i \lambda \sqrt{\cancel{4}\alpha}\, \frac{1}{\cancel{2}} t^{-1/2}$$

$$\frac{K(T_W - T_0)}{\mathrm{erf}(\lambda)\sqrt{\alpha \cancel{t} \pi}} e^{-\lambda^2} = \frac{q \rho_i \lambda \sqrt{\alpha}}{\sqrt{t}} \Longrightarrow \lambda\, \mathrm{erf}(\lambda)\, e^{\lambda^2} = \frac{K(T_W - T_0)}{\alpha \rho_i q \sqrt{\pi}}$$

Summary → Neumann solution

$$\begin{cases} \theta(x,t) = 1 - \dfrac{1}{\operatorname{erf}(\lambda)} \operatorname{erf}\left(\dfrac{x}{\sqrt{4\alpha t}}\right) \\ s(t) = 2\lambda\sqrt{\alpha t} \\ \lambda \operatorname{erf}(\lambda) e^{\lambda^2} = \dfrac{K(T_w - T_0)}{\alpha \rho_i q \sqrt{\pi}} = \dfrac{c_p (T_w - T_0)}{q\sqrt{\pi}} = \beta \end{cases}$$

Stefan number

$\alpha = \dfrac{K}{\rho c_p}$ and $\rho_i \approx \rho$

we implicitly assumed that there is no volume change

$q \approx 336000 \ \dfrac{J}{kg}$

$c_p \approx 4184 \ \dfrac{J}{kg \cdot K}$

→ for water, $\beta \approx 0.007 (T_w - T_0)$

S19 : • Stefan problem (continued)
• film condensation

Stefan problem

we had $\underbrace{\lambda\, \mathrm{erf}(\lambda) e^{\lambda^2}}$ $= \beta \rightarrow s(t) = 2\lambda\sqrt{\alpha t}$

$\rightarrow$ strictly increasing

$$\theta(x,t) = 1 - \frac{1}{\mathrm{erf}(\lambda)}\, \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

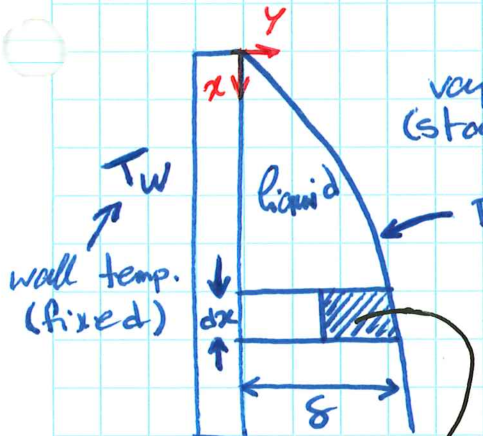
find λ using Newton's method $\rightarrow s\checkmark$ $\theta\checkmark$

Note that $\underline{\theta(s(t), t) = 0}$

see "stefan.py"

film condensation

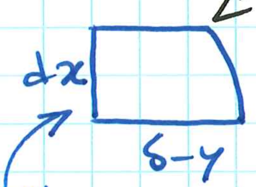
(first proposed by Nusselt) 1916



T_g : equilibrium temp (e.g., 100°C for water at 1 atm)

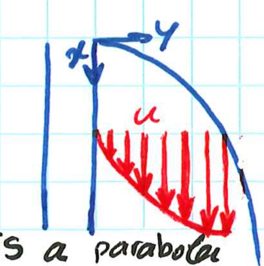
$T_W < T_g$

wall removes heat from vapor $\rightarrow$ vapor turns to liquid



Weight of the element: $W = dx(\delta - y)\rho g$
Buoyancy force : $B = dx(\delta - y)\rho_v g$

there is a gradient of velocity in the x direction $\rightarrow$ it increases from the wall to the interface



$\rightarrow$ the element will experience a shear force pulling it up:
$S = \mu \frac{du}{dy} dx$

u(y) is a parabola

At the interface, we assume negligible shear stress by the vapor phase.

$$\rightarrow d\cancel{x}(\delta - y)\rho g = d\cancel{x}(\delta - y)\rho_v g + \mu \frac{du}{dy} d\cancel{x}$$

$$\Rightarrow \boxed{u = \frac{g}{\mu}(\rho - \rho_v)\left(\delta y - \frac{1}{2}y^2\right)} + \cancel{C} \rightarrow 0$$

velocity in the x direction

becaus $u(0) = 0$ (no-slip boundery condition)

mass flow of condensate at each x:

$$\dot{m} = \int_0^\delta \rho u(y)\, dy = \frac{\rho(\rho - \rho_v)g}{\mu}\left[\delta \frac{\delta^2}{2} - \frac{1}{2}\frac{\delta^3}{3}\right]$$

$$\Rightarrow \boxed{\dot{m} = \frac{\rho(\rho - \rho_v)g\delta^3}{3\mu}}$$

the amount of condensate added between x and $x + dx$ is the differential of $\dot{m}$:

$$d\dot{m} = \frac{\rho(\rho - \rho_v)g\delta^2}{\mu} d\delta$$

Energy needs to be removed from vapor to condense it. this energy will coduct to the wall:

$$(d\dot{m}) \times h = (q_{-y}\, dx)_{y=\delta} = k \left.\frac{dT}{dy}\right|_{y=\delta} dx$$

$\approx 2260 \frac{kJ}{kg}$ for water

latent heat of condensation

heat flux in the "$-y$" direction

Assume a linear temperature distribution between wall and interface $\rightarrow \frac{dT}{dy} = \frac{T_g - T_w}{\delta}$

$$\Rightarrow \frac{h\rho(\rho - \rho_v)g\delta^2}{\mu} d\delta = k\frac{T_g - T_w}{\delta} dx$$

$$\Rightarrow \delta^3 d\delta = \frac{\mu k (T_g - T_w)}{h \rho (\rho - \rho_v) g} dx$$

$$\Rightarrow \frac{1}{4}\delta^4 = \frac{\mu k (T_g - T_w)}{h \rho (\rho - \rho_v) g} x + C$$

at $x = 0 \Rightarrow \delta = 0 \Rightarrow C = 0$

$$\Rightarrow \delta = \left[\frac{4 \mu k (T_g - T_w)}{h \rho (\rho - \rho_v) g} x \right]^{1/4}$$

S20 Angular Momentum & Fixed Axis Rotation

$\vec{x}$ is a vector
x is $|\vec{x}|$

Translational Motion:
→ force, linear momentum, center of mass
linear momentum → $m\vec{v}$

Rotational Motion:
→ torque, angular momentum, moment of inertia

Angular Momentum of a Particle

← not an extended body

→ angular momentum $\vec{L}$ of a particle that has momentum $\vec{p} = m\vec{v}$ and is at position $\vec{r}$ with respect to a given origin:

$$\vec{L} = \vec{r} \times \vec{p}$$

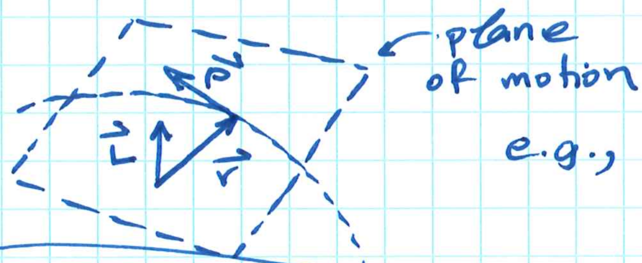
$$\hookrightarrow |\vec{L}| = L = rp\sin\alpha$$

α: angle between $\vec{r}$ and $\vec{p}$

$$[\vec{L}] = \frac{ML^2}{T}$$

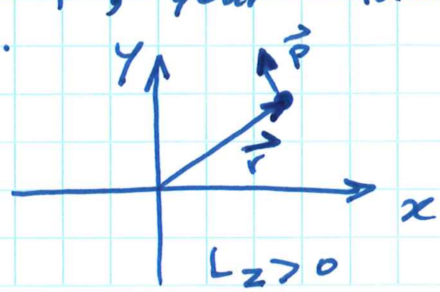
Remarks:

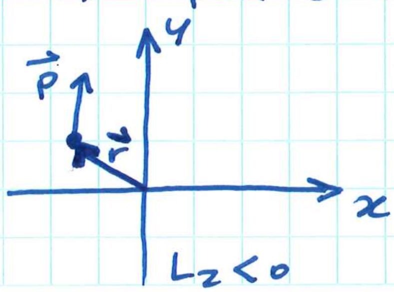
- $\vec{p}$ is independent of the coordinate system but $\vec{L}$ is not!
- $\vec{L}$ is perpendicular to the plane of motion



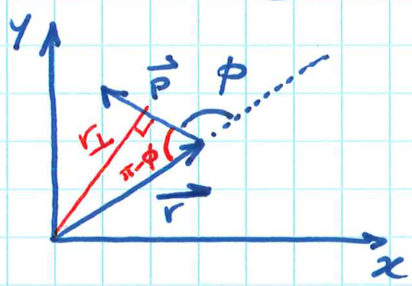
e.g., if $\vec{r}$ and $\vec{p}$ lie in the xy plane, $\vec{L}$ lies along the z direction.

Right hand rule determines if it is in the positive or negative z directions: point your fingers (right hand) along $\vec{r}$ and orient your hand so that you can bend your fingers toward $\vec{p}$; your thumb then points in the direction of $\vec{L}$.





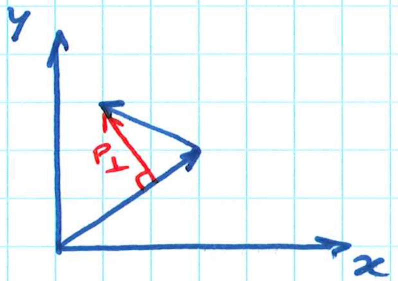
Geometrical Understanding



- Decompose $\vec{r}$ into $r_\perp$ that is perpendicular to the trajectory and $r_\parallel$ that is parallel.

$$r_\perp = r\sin(\pi-\phi) = r\sin\phi$$

$$\Rightarrow L_z = rp\sin\phi = \underline{\underline{r_\perp p}}$$



- Decompose $\vec{p}$ into $p_\perp$ that is perpendicular to $\vec{r}$ and $p_\parallel$ that is parallel.

$$p_\perp = p\sin(\pi-\phi) = p\sin\phi$$

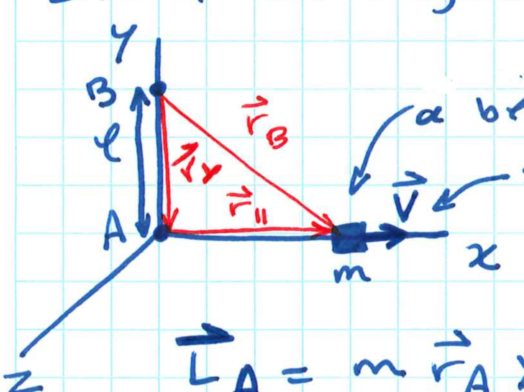
$$\Rightarrow L_z = rp\sin\phi = \underline{\underline{r\,p_\perp}}$$

algebraically:

$$\vec{r} = (x, y, 0)\ , \quad \vec{p} = (mv_x, mv_y, 0)$$

$$\Rightarrow \vec{L} = \vec{r}\times\vec{p} = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ v_x & v_y & 0 \end{vmatrix} = m\,(xv_y - yv_x)\,\hat{k}$$

Example: Angular Momentum of a sliding Block



a block of mass m and negligible dimensions sliding freely in the x direction ($\vec{V} = v\,\hat{i}$)

$$\vec{L}_A = m\,\vec{r}_A \times \vec{V} = 0$$

$$\vec{L}_B = m\,\vec{r}_B\times\vec{V} = m\,(\vec{r}_\parallel + \vec{r}_\perp)\times\vec{V} = m\ell v\,\hat{k}$$

Or, $$\vec{L}_B = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & -\ell & 0 \\ v & 0 & 0 \end{vmatrix} = m\ell v\,\hat{k}$$

Fixed Axis Rotation

direction of the axis of rotation is always along the same line: e.g., a car wheel attached to an axle undergoes fixed axis rotation as long as the car drives straight.

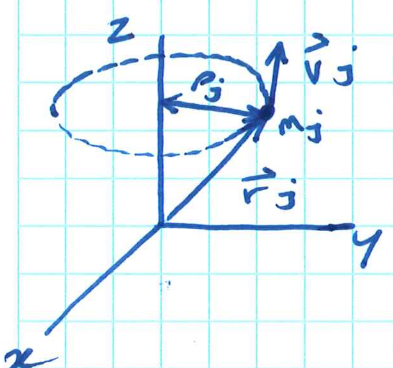
around z for example (z is the axis of rotation)

- When a rigid body rotates around an axis, every particle in the body remains at a fixed distance from the axis.

→ a coordinate system with its origin on the axis, $|\vec{r}| = \text{constant}$ for every particle

⇒ $\vec{r}$ changes while $|\vec{r}|$ remains constant: velocity is perpendicular to $\vec{r}$.

Consider a body rotating around the z axis:



ρ_j: perpendicular distance to the axis of rotation from particle m_j $= \sqrt{x_j^2 + y_j^2}$

$|\vec{v}_j| = \rho_j \, \omega_j$ ← rate of rotation (angular speed)

Angular momentum of the jth particle:

$$\vec{L}_j = \vec{r}_j \times m_j \vec{v}_j$$ → not exactly in the z direction!

Our focus: the component of angular momentum along the axis of rotation (z here)

$$\Rightarrow L_{j,z} = \rho_j \, m_j \, v_j = m_j \rho_j^2 \, \omega_j$$

For the whole body: $L_z = \sum_j L_{j,z} = \sum_j m_j \rho_j^2 \, \omega$

sum over all particles of the body

ω is constant (rigid body)

S21 Moment of Inertia Parallel Axis theorem

analogy: $\vec{p} = m\vec{v}$
$\vec{L} = I\vec{\omega}$

$$L_z = \sum_j L_{j,z} = \sum_j m_j \rho_j^2 \omega \equiv I\,\omega$$

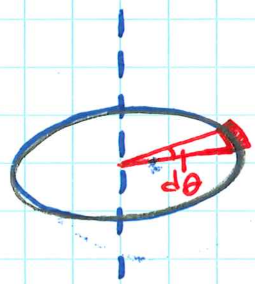
$$I = \sum_j m_j \rho_j^2$$

I: geometrical quantity called "moment of inertia"

For continuously distributed matter: $I = \int_M \rho^2 dm$

Moments of inertia of some simple objects

(a) Uniform thin ring of mass M and radius R, around the axis of symmetry of the ring

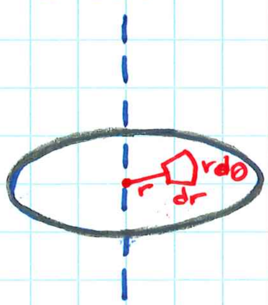


Let $\lambda = \frac{M}{2\pi R}$ mass per unit length of the ring

$\Rightarrow dm = R\,d\theta\,\lambda$

$$\rightarrow I = \int_0^{2\pi} R^2 \cdot (R\lambda\,d\theta) = 2\pi R^3 \lambda = MR^2$$

(b) Uniform thin disk of mass M and radius R, around the axis of symmetry of the disk



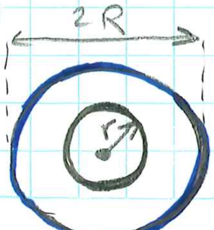
$\lambda = \frac{M}{\pi R^2}$ mass per unit area of the disk

$dm = r\,d\theta\,dr\,\lambda$

$$\rightarrow I = \int_0^{2\pi}\int_0^R r^2 \cdot (r\lambda\,dr\,d\theta) = 2\pi\lambda \frac{R^4}{4}$$

Alternatively: divide the disk into rings with radius $r \in [0, R]$

top view



$$\rightarrow I = \int dI = \int_0^R 2\pi r^3 \lambda\,dr = 2\pi\lambda\frac{R^4}{4} = \frac{1}{2}MR^2$$

moment of inertia for a ring of mass $2\pi r\,dr\,\lambda$ and radius r

(c) Uniform thin stick of mass M and length L, around a perpendicular axis through its midpoint

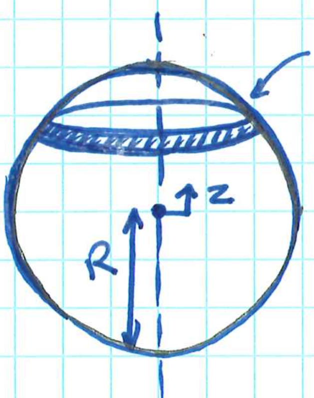
$$\lambda = \frac{M}{L}$$

$$\rightarrow I = \int_{-L/2}^{L/2} x^2 \lambda \, dx = \frac{1}{12} \lambda L^3 = \frac{1}{12} M L^2$$

(d) Uniform thin stick around a perpendicular axis at its end.

$$I = \int_0^L x^2 \lambda \, dx = \frac{1}{3} \lambda L^3 = \frac{1}{3} M L^2$$

(e) Uniform sphere of mass M and radius R, around an axis through its center



disks of radius $r(z)$ and thickness dz

$$I = 2\int dI = 2\int_0^R \frac{1}{2} \pi r^4 \lambda \, dz = \pi \lambda \int_0^R r^4 \, dz$$

moment of intertia for a disk of mass $\pi r^2 dz$ and radius r

$$\lambda = \frac{M}{\frac{4}{3}\pi R^3}$$

$$r = \sqrt{R^2 - z^2} \rightsquigarrow I = \pi \lambda \int_0^R (R^2 - z^2)^2 \, dz = \frac{2}{5} M R^2$$

Parallel Axis Theorem

$\rightarrow$ tells us I, the moment of inertia around any axis, provided that we know I_0, the moment of inertia around an axis through the center of mass parallel to the first.

$$\rightarrow I = I_0 + M\ell^2$$

ℓ: distance between the two axes

$$\vec{X} = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j$$

(center of mass for N masses) of mass m_j and location $\vec{r}_j$

M: total mass

In continuous form $\rightarrow$ $\underbrace{\vec{X} = \frac{1}{M}\int \vec{r}\, dm}$

Example: moment of inertia of a $\underbrace{M}$ disk around axis at the rim

the center of mass for a disk is at its center:

can you prove it?

$$I_0 = \frac{1}{2} M R^2$$

radius of the disk

$$\Rightarrow I = I_0 + M\ell^2$$

$$= \frac{1}{2} M R^2 + M R^2 = \frac{3}{2} M R^2$$

S22

Torque
Torque & Angular Momentum
Yo-yo motion

Torque

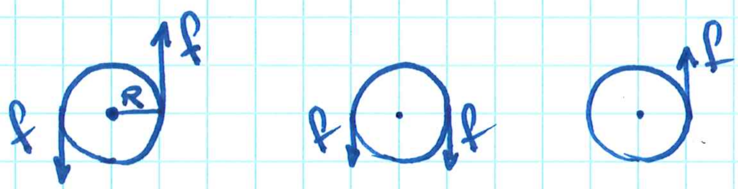
$\vec{\tau} = \vec{r} \times \vec{F}$ torque due to force $\vec{F}$ that acts on a particle at position $\vec{r}$

$$|\vec{\tau}| = |\vec{r}_\perp||\vec{F}| = |\vec{r}||\vec{F}_\perp|$$

Also,
$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$
$\rightsquigarrow$ torque depends on the origin we choose but force does not.

$\vec{\tau}$ and $\vec{F}$ are always mutually perpendicular.

- Force and torque are inherently different quantities.



$F = 0$, $\tau = 2Rf$ | $F = 2f$, $\tau = 0$ | $F = f$, $\tau = Rf$

three different cases of τ, F combinations (τ is evaluated around the center of the disk)

Torque due to gravity

For a uniform gravitational field: $\vec{\tau} = \vec{R} \times \vec{W}$

$\vec{R}$: vector to the center of mass; $\vec{W}$: weight

Proof $\rightsquigarrow$ $\vec{\tau}_j = \vec{r}_j \times m_j \vec{g} = m_j \vec{r}_j \times \vec{g}$

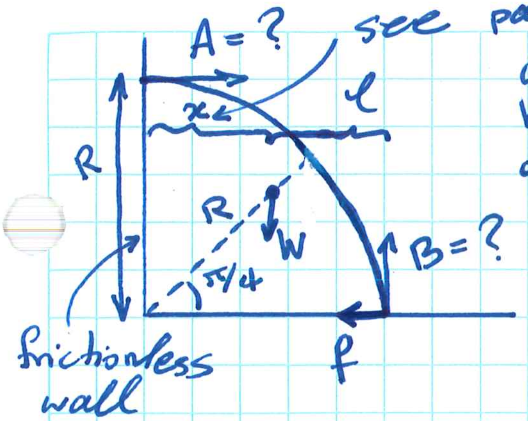
$$\Rightarrow \vec{\tau} = \sum_j \vec{\tau}_j = \underbrace{\left(\sum_j m_j \vec{r}_j\right)}_{M\vec{R}} \times \vec{g} \;\rightarrow\; \vec{\tau} = \vec{R} \times M\vec{g} \;\checkmark$$

Torque and Force in Equilibrium

equilibrium $\rightarrow$ total force $\equiv 0$
total torque $\equiv 0$

around any origin (better to choose a point where several forces act, since then torques due to these forces all vanish.)

see page 67 for detail



a uniform rod of length $\frac{\pi R}{2}$ bent in the shape of a quadrant of radius R

force balance in the x and y directions:

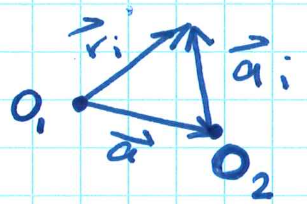
$$x: \quad A = f$$
$$y: \quad B = W$$

torque about the point where the quadrant rests:

$$\tau = \ell W - RA = 0 \implies A = \frac{\ell W}{R} \qquad \ell = R - x = \left(1 - \frac{2}{\pi}\right) R$$

$$\rightsquigarrow A = \frac{W}{\cancel{R}}\left(1 - \frac{2}{\pi}\right)\cancel{R} \approx 0.36\, W$$

* If the total torque is zero about a point, it will be zero around any other point given that $\sum F = 0$ (total force is zero)



$$\sum_i \vec{r}_i \times \vec{F}_i = 0 \qquad \text{(about } O_1\text{)}$$

force index

$$\rightsquigarrow \sum_i \vec{r}_i \times \vec{F}_i = \sum_i (\vec{a} + \vec{a}_i) \times \vec{F}_i$$
$$= \sum_i \vec{a} \times \vec{F}_i + \sum \vec{a}_i \times \vec{F}_i$$
$$= \vec{a} \times \underbrace{\cancel{\left(\sum \vec{F}_i\right)}}_{0} + \sum \vec{a}_i \times \vec{F}_i$$
$$\implies \sum_i \vec{r}_i \times \vec{F}_i = \sum \vec{a}_i \times \vec{F}_i = 0$$

Torque & Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p} \implies \frac{d\vec{L}}{dt} = \overbrace{\frac{d\vec{r}}{dt}}^{\vec{v}} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \underbrace{\vec{v} \times \vec{p}}_{0} + \vec{r} \times \frac{d\vec{p}}{dt}$$

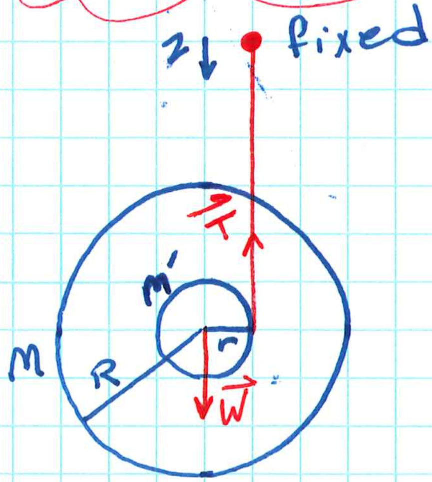
$$\vec{F} = \frac{d\vec{p}}{dt}$$

(by Newton's second law)

$$\implies \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$
$$\frac{d\vec{p}}{dt} = \vec{F}$$

Yo-yo motion



total moment of inertia:

$$I = 2\left(\frac{1}{2}MR^2\right) + \frac{1}{2}m'r^2 \qquad (m = M + m')$$

$\underbrace{\phantom{2(\tfrac{1}{2}MR^2)}}_{\text{two large cylinders}}$ $\underbrace{\phantom{\tfrac{1}{2}m'r^2}}_{\text{one small cylinder}}$

- Linear momentum:

velocity of the center of mass

$$\frac{d\vec{P}}{dt} = \vec{W} + \vec{T} \Rightarrow m\frac{d\vec{v}}{dt} = \vec{W} + \vec{T}$$

$$\Rightarrow m\frac{dv}{dt} = W - T \qquad (i)$$

- Angular momentum: (about the center of mass)

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = \vec{r} \times \vec{T} \rightsquigarrow I\frac{d\omega}{dt} = rT \qquad (ii)$$

Note that $v = \dfrac{dz}{dt} = r\omega \rightsquigarrow \dot{\omega} = \dfrac{\dot{v}}{r}$ (*)

$$(*) \to (ii) \Rightarrow \frac{I}{r}\dot{v} = rT \overset{(i)}{=\!=} r(W - m\dot{v})$$

$$\rightsquigarrow \frac{I}{r}\dot{v} = rmg - rm\dot{v} \Rightarrow \dot{v}\left(\frac{I}{r} + rm\right) = rmg$$

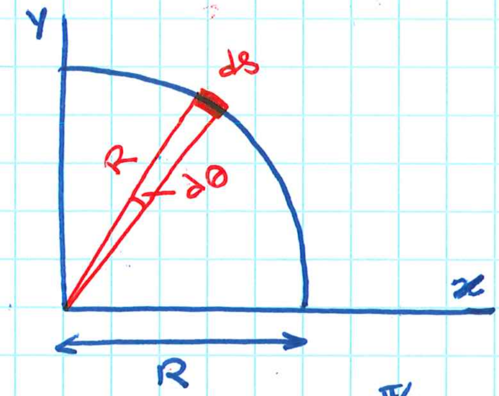
$$\Rightarrow \dot{v} = \frac{g}{1 + \frac{I}{r^2 m}}$$

acceleration is lower than gravitational

If we had only one cylinder with radius r and mass m $\to I = \frac{1}{2}mr^2$

$$\Rightarrow \dot{v} = \frac{g}{1 + \frac{1}{2}} = \frac{2}{3}g$$

Center of mass of a bent rod (example from session 22)



$$\vec{X} = \frac{1}{m}\int \vec{r}\,dm \qquad \lambda = \frac{m}{\frac{\pi R}{2}}$$

$$dm = \lambda\, ds = \lambda R\, d\theta$$

$$\vec{r} = R\hat{r} = R(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$\rightsquigarrow \vec{X} = \frac{1}{m}\int_0^{\pi/2} R(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\,\lambda R\,d\theta = \frac{\lambda R^2}{m}\int_0^{\pi/2}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\,d\theta$$

$$\Rightarrow \vec{X} = \frac{\lambda R^2}{m}(\hat{i} + \hat{j}) = \frac{2R}{\pi}(\hat{i} + \hat{j})$$

$$\rightsquigarrow \ell = R - x = R - \frac{2}{\pi}R \cong 0.36\,R$$

S23

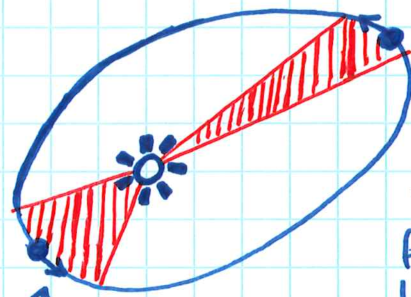
- Conservation of angular momentum
- law of equal areas (Kepler's 2nd law)
- effective area of a far-off planet

$\frac{d\vec{L}}{dt} = \vec{\tau} \rightsquigarrow \vec{\tau} = 0 \Rightarrow \vec{L}$ is constant and the angular momentum is conserved.

Law of equal areas (Kepler's second law)

Explanation:
earth is moving under a central force (gravity, but can be extended to any central force):

$\vec{F}(\vec{r}) = f(r)\,\hat{r}$ ← unit vector in the radial direction



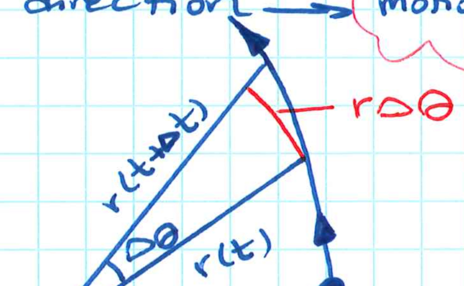
the area swept by the earth for a given time is constant.

- shorter radius
- higher speed

$\rightsquigarrow \underset{\text{around sun}}{\vec{\tau}} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = 0$

→ the angular momentum is conserved

- $\vec{L}$ is therfore constant in both magnitude and direction → $\boxed{\text{motion is confined to a plane!}}$



For small $\Delta\theta$, the area swept by earth can be approximated as

$$\Delta A \approx \frac{1}{2}(r(t+\Delta t)) \cdot (r\Delta\theta)$$

$$= \frac{1}{2}(r+\Delta r)\cdot(r\Delta\theta)$$

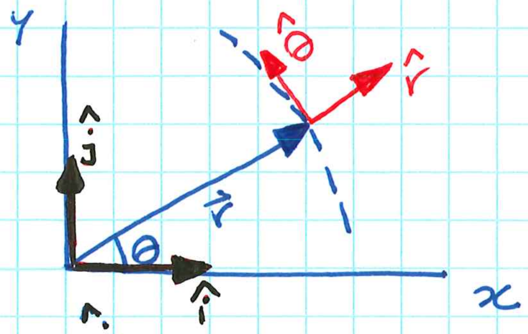
$$= \frac{1}{2}r^2\Delta\theta + \frac{1}{2}r\Delta r\Delta\theta$$

the rate at which area is swept is

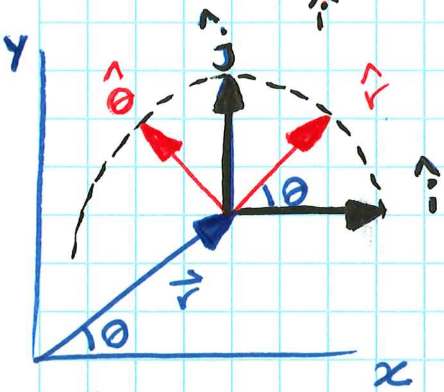
$$\frac{dA}{dt} = \lim_{\Delta t \to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\left[\frac{1}{2}r^2\frac{\Delta\theta}{\Delta t} + \cancel{\frac{1}{2}r\frac{\Delta r\Delta\theta}{\Delta t}}^{\to 0}\right]$$

$$\rightsquigarrow \boxed{\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}} \quad (*)$$

A digression to polar coordinates



Fundamental difference: the directions of $\hat{r}$ or $\hat{\theta}$ vary with position, whereas $\hat{i}$ and $\hat{j}$ have fixed directions



$$\rightsquigarrow \hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$
$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

$$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$
$$= r\,\underbrace{(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})}_{\hat{r}} = r\,\hat{r}$$

Velocity in polar coordinates:

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \longrightarrow \frac{d\hat{r}}{dt} = \frac{d}{dt}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$
$$= -\dot{\theta}\sin\theta\,\hat{i} + \dot{\theta}\cos\theta\,\hat{j}$$
$$= \dot{\theta}\,\underbrace{(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})}_{\hat{\theta}}$$
$$\Rightarrow \frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$$

$$\rightsquigarrow \boxed{\frac{d\vec{r}}{dt} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}}$$

$$\vec{L} = \vec{r} \times m\vec{v} = r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$
$$= \underbrace{\cancel{m r \dot{r}\,\hat{r}\times\hat{r}}}_{0} + m r^2\dot{\theta}\,\underbrace{\hat{r}\times\hat{\theta}}_{\hat{k}} \Rightarrow \vec{L} = m r^2\dot{\theta}\,\hat{k}$$

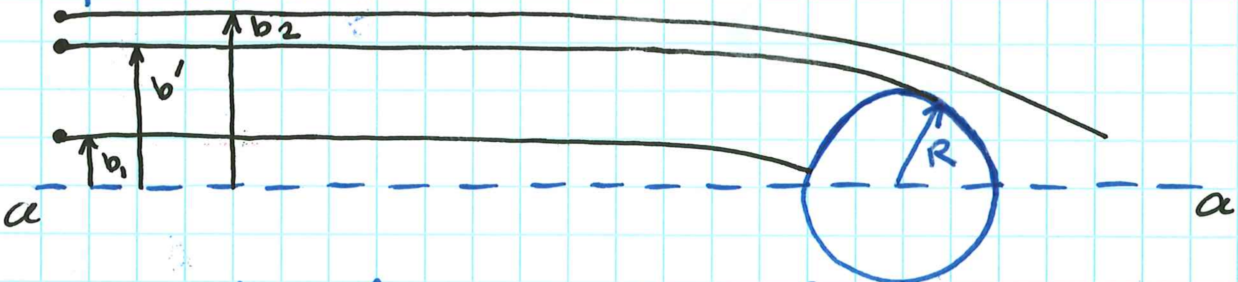
$$\rightsquigarrow L_z = m r^2 \dot{\theta}$$

$$(*) \rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L_z}{2m}$$ constant for any central force

$$\Rightarrow \frac{dA}{dt} = \text{constant}\ \checkmark$$

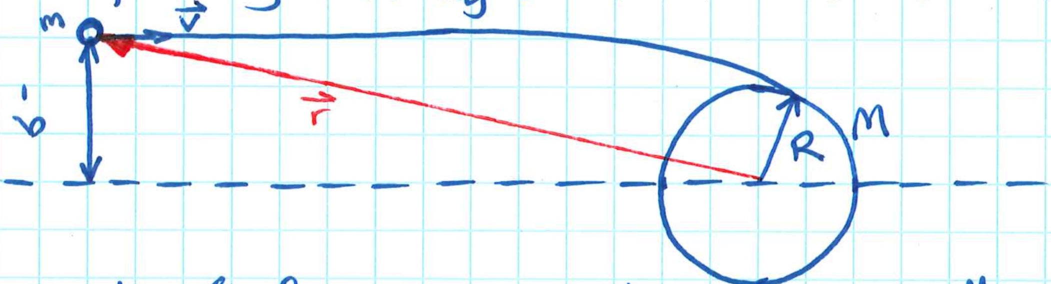
Effective area of a far planet

How accurately must you aim the trajectory of an unpowered space-craft to hit a far-off planet?



geometrical area: $A_g = \pi R^2$
effective area: $A_e = \pi b'^2$
$\Big\}$ $A_e > A_g$ due to gravity

Distance between the launch point and the planet is $\gg R$ $\rightarrow$ different trajectories are parallel to "aa" passing through the center of the planet.



central force (gravity) $= -G\frac{mM}{r^2}\hat{r}$

$\vec{\tau} = \vec{r} \times \left(-G\frac{mM}{r^2}\hat{r}\right) = 0 \rightarrow$ conservation of angular momentum around the center of the planet

Conservation of energy: $E = \frac{1}{2}mv^2 - \underbrace{G\frac{mM}{r}}$ $\vec{F} = -\nabla U$ potential energy

Initially: $E_i = \frac{1}{2}mv_0^2 - 0$ $\leftarrow r \rightarrow \infty$

$L_i = -b'mv_0$

At the point of closest approach, $\vec{v}$ and $\vec{r}$ are perpendicular upon collision:

$$L_c = -mRv(R); \quad E_c = \frac{1}{2}mv^2(R) - G\frac{mM}{R}$$

$L_i = L_c \Rightarrow v(R) = v_0\frac{b'}{R}$

$$E_i = E_c \Rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} m v^2(R) - G\frac{mM}{R}$$

$$= \frac{1}{2} m v_0^2 \frac{b'^2}{R^2} - G\frac{mM}{R}$$

$- U(R)$

$$\Rightarrow \frac{1}{2} m v_0^2 \left(\frac{b'^2}{R^2} - 1\right) = G\frac{mM}{R} \Rightarrow b'^2 = R^2\left(1 + \frac{G\frac{mM}{R}}{\frac{1}{2} m v_0^2}\right)$$

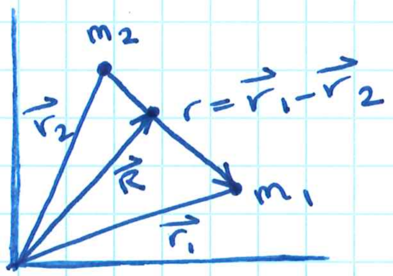
E_i

$$A_e = \pi b'^2 = A_g\left(1 - \frac{U(R)}{E_i}\right)$$

S24

- Central force motion
- Universal features of central force motion
- Energy eqn and energy diagrams

Central force motion as a one-body problem



An isolated system of two particles interacting under a central force $f(r)\hat{r}$

equations of motion: $m_1 \ddot{\vec{r}}_1 = f(r)\hat{r}$ ①

$m_2 \ddot{\vec{r}}_2 = -f(r)\hat{r}$ ②

$f(r) < 0 \rightarrow$ attractive
$f(r) > 0 \rightarrow$ repulsive

Write ① & ② in terms of $\vec{r} = \vec{r}_1 - \vec{r}_2$ and

$\vec{R} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ (center of mass)

$\vec{r}$: divide ① by m_1 and ② by m_2 to get

$$\ddot{\vec{r}}_1 = \frac{f(r)}{m_1}\hat{r}, \quad \ddot{\vec{r}}_2 = \frac{f(r)}{m_2}\hat{r} \implies \underbrace{\frac{m_1 m_2}{m_1 + m_2}}_{\mu \text{ (reduced mass)}} \ddot{\vec{r}} = f(r)\hat{r}$$

$$\implies \mu \ddot{\vec{r}} = f(r)\hat{r} \quad (*)$$

$\vec{R}$: add ① & ② and divide by $m_1 + m_2$: $\ddot{\vec{R}} = 0$

$\implies \vec{R} = \vec{R}_0 + \vec{V} t$ — origin at the center of mass? $\vec{R}_0 = 0$

center of mass is stationary? $\vec{V} = 0$

(*): eqn of motion for a single particle. (not generalizable to systems with more than two particles)

$\vec{r}$ & $\vec{R}$ are known $\rightarrow \vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$ & $\vec{r}_2 = \vec{R} - \left(\frac{m_1}{m_1+m_2}\right)\vec{r}$

Conservation of angular momentum (see session 23)

$\leadsto$ the angular momentum is conserved
$\Rightarrow \vec{L} =$ constant and the motion is confined to a plane
$\Rightarrow$ Law of equal areas

Conservation of energy

the kinetic energy of μ is (written in polar coordinates)

$$K = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})^2 = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$$

there is also a potential energy associated with the central force $f(r)$:

$$f(r) = -\frac{dU(r)}{dr} \rightarrow U(r) = -\int_{\infty}^{r} f(r)\,dr$$

$$\Rightarrow E = K + U = \frac{1}{2}\mu v^2 + U(r) \qquad (U \to 0 \text{ as } r \to \infty$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r)$$

$$= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{1}{2}\frac{L^2}{\mu r^2} + U(r)}_{U_{eff}(r)} \qquad L = \mu r^2\dot{\theta}$$

centrifugal potential

true potential

$$\Rightarrow \boxed{E = K + U_{eff} = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)}$$

energy eqn for a particle moving in one dimension.

all reference to θ is gone!

Energy eqn and energy diagrams

$$E = \underbrace{\frac{1}{2}\mu v^2 + U(r)} = \underbrace{\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)}$$

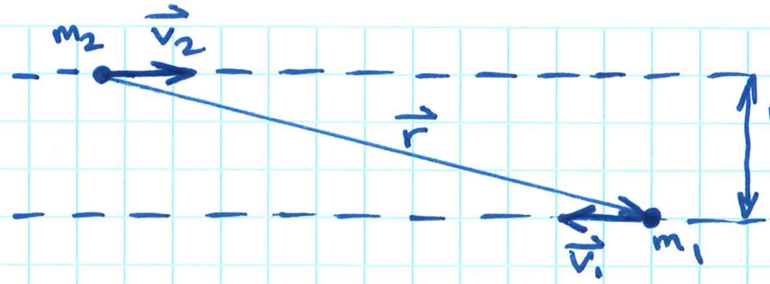
can be used in energy diagrams versus r

depends on a single coordinate only: r

handy for evaluating E; all we need to know is the relative speed and position at some instant.

Example

non-interacting particles ($U(r)=0$)



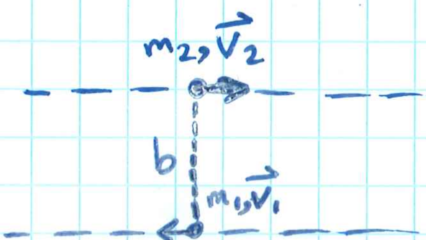
Goal is to plot U_{eff} versus r and interpret the system behavior.

$$U_{eff} = \frac{1}{2}\frac{L^2}{\mu r^2} + \cancel{U(r)}^{\,0} \qquad E = \frac{1}{2}mv^2 + \cancel{U(r)}^{\,0} = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

$$\rightsquigarrow \frac{1}{2}mv^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{L^2}{\mu r^2}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{v}_1 - \vec{v}_2 = \vec{v}_0 \quad \text{(constant)}$$

$$\Rightarrow \boxed{\frac{1}{2}mv_0^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{L^2}{\mu r^2}}$$

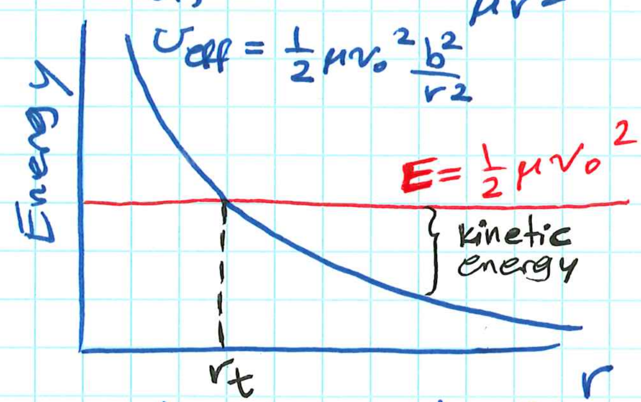


when m_1 and m_2 pass each other, $r=b$ and $\dot{r}=0$

$$\Rightarrow \frac{1}{2}mv_0^2 = 0 + \frac{1}{2}\frac{L^2}{\mu b^2} \Rightarrow L = \mu b v_0$$

Alternatively $\rightarrow$ $\vec{L} = \vec{r} \times \mu\dot{\vec{r}} = \mu b v_0$ (try it!)

$$\rightsquigarrow U_{eff}(r) = \frac{1}{2}\frac{L^2}{\mu r^2} = \frac{1}{2}\mu v_0^2 \frac{b^2}{r^2}$$



the kinetic energy associated with the radial motion is

$$\frac{1}{2}\mu\dot{r}^2 = -U_{eff} + E$$

can never be negative

$\Rightarrow$ motion is restricted to regions where $E - U_{eff} \geq 0$.

Initially r is very large and $U_{eff} \approx 0$. As the particles approach, the kinetic energy decreases, vanishing at the "turning point" where the radial velocity is zero:

$$E = U_{eff}(r_t)$$

$$\Rightarrow \frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu v_0^2 \frac{b^2}{r_t^2} \Rightarrow r_t = b$$

distance of the closest approach of the particles

S25

- Energy diagram of planetary motion
- Perturbed circular orbit

Gravitational force (always attractive): $f(r) = -G\frac{m_1 m_2}{r^2}$

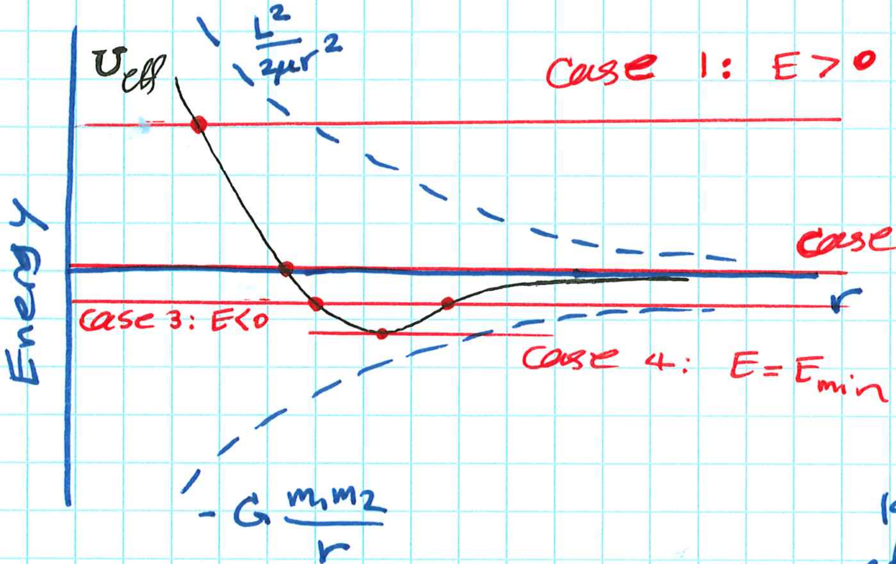
Associating potential $\rightarrow$ $f(r) = -\frac{dU(r)}{dr}$

$$\Rightarrow U(r) - U(\infty) = -\int_{\infty}^{r} f(r')\,dr' = -G\frac{m_1 m_2}{r}$$

$$\rightsquigarrow U_{eff} = \frac{L^2}{2\mu r^2} - G\frac{m_1 m_2}{r}$$

$L=0$? collision along a straight line (no barrier)

$L \neq 0 \rightarrow$
- repulsive centrifugal potential dominates at small r
- attractive gravitational " " " larger r



motion is restricted to regions where $E - U_{eff} \geq 0$

$K = E - U_{eff}$

kinetic energy of radial motion ≥ 0

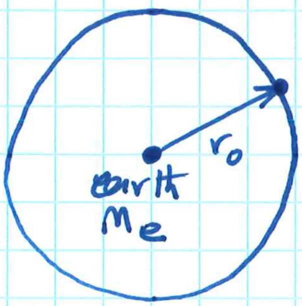
Case 1: ($E>0$) (hyperbola) r is unbounded for large values but cannot be less than a certain minimum $\rightarrow$ the particles are kept apart by a "centrifugal barrier."

Case 2: ($E=0$) (parabola) quantitatively similar to case 1 but at the border!

Case 3: ($E<0$) (ellipse) the motion is bounded for both small and large r values. the two particles form a bound system.

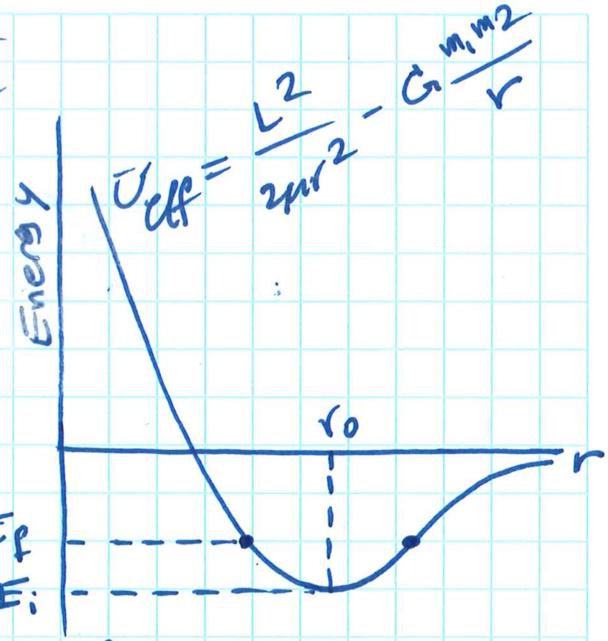
Case 4: $E=E_{min}$ (circle) r is restricted to one value. particles stay a constant distance from one another.

Perturbed Circular Orbit



satellite of mass m

orbit is a <u>circle</u> of radius r_0



At $t=0$, one of its engines is fired briefly toward the center of the earth, changing the energy of the satellite but not its angular momentum $\rightarrow$ find the new orbit.

$M_e >> m \leadsto$ the earth is relatively fixed (the center of mass is $\approx$ at the center of earth!)

If E_f is not much larger than E_i, the energy diagram shows that r never differs from r_0 much.

Approximate $U_{eff}(r)$ in the neighborhood of r_0 by a parabola:

(drop "eff" for simplicity) $$U(r) \approx U(r_0) + \underbrace{(r-r_0)\left.\frac{dU}{dr}\right|_{r_0}}_{\substack{0 \\ \text{(minimum)}}} + \frac{1}{2}(r-r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r_0}$$

$$\Rightarrow U(r) \approx U(r_0) + \frac{1}{2}(r-r_0)^2\left.\frac{d^2U}{dr^2}\right|_{r_0}$$

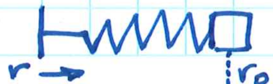
$$\frac{dU}{dr} = -\frac{L^2}{\mu r^3} + \frac{Gm_1m_2}{r^2}$$

$$\left.\frac{dU}{dr}\right|_{r_0} = 0 \Rightarrow L = \sqrt{\mu G m_1 m_2 r_0}$$

$$\left.\frac{d^2U}{dr^2}\right|_{r_0} = \underbrace{\frac{3L^2}{\mu r_0^4} - \frac{2Gm_1m_2}{r_0^3}}_{= \frac{Gm_1m_2}{r_0^3}} = K$$

$$\leadsto \underbrace{U(r) \approx U(r_0) + \frac{K}{2}(r-r_0)^2}$$

potential energy of a harmonic oscillator



$\Rightarrow F(r) = -K(r-r_0)$

$\Rightarrow$ the system reacts as

$$\mu \frac{d^2 r}{dt^2} = -K(r - r_0) \xrightarrow{u = r - r_0} \mu \frac{d^2 u}{dt} + Ku = 0$$

$\beta = \frac{L}{\mu r_0^2}$

$\beta = \sqrt{\frac{K}{\mu}}$

$\rightsquigarrow u = A \sin \beta t + B \cos \beta t \Rightarrow r = r_0 + A \sin \beta t + B \cos \beta t$

at $t = 0$, $r = r_0 \Rightarrow B = 0 \Rightarrow r = r_0 + A \sin \beta t$

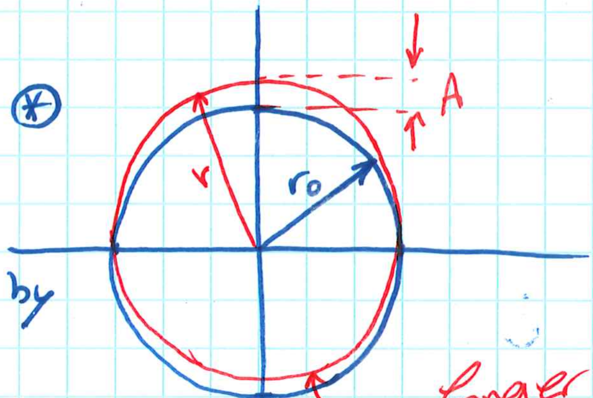
$A \ll r_0$ for $E_f \approx E_i$

To find the new orbit we need r as a function of θ. For a circular orbit:

$$L \approx \mu r_0^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{L}{\mu r_0^2} \Rightarrow \theta = \frac{L}{\mu r_0^2} t = \beta t$$

Note: frequency of rotation is equal to the frequency of radial oscillation.

$$\Rightarrow r = r_0 + A \sin \theta \quad (*)$$



the exact orbit is given by

$$r = \frac{r_0}{1 - \frac{A}{r_0} \sin \theta} \leftarrow \text{an } \underline{\text{ellipse}}$$

no longer centered on the earth.

$\approx (*)$ for small $\frac{A}{r_0}$

S26

- Planetary motion
- Elliptic orbits
- Kepler's first & third laws

Goal: finding the orbit for a planet of mass m moving about a star of mass M under the gravitational interaction.

$$\to U(r) = -G\frac{mM}{r} \equiv -\frac{C}{r}$$

the results would also apply to a satellite of mass m orbiting a planet of mass M, or even a binary star system, Coulomb scattering, ...?

Recall: $E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{\mu}(E - U_{eff})}$$

$$L = \mu r^2 \frac{d\theta}{dt}$$

$$\Rightarrow \dot{\theta} = \frac{L}{\mu r^2}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{L}{r^2\sqrt{2\mu(E-U_{eff})}}$$

$$\leadsto \theta - \theta_0 = L\int_{r_0}^{r}\frac{dr}{r^2\sqrt{2\mu(E-U_{eff})}}$$

$\theta \in [-\pi, 0]$
$\Rightarrow \dot{r} \geq 0$

$$U_{eff} = \frac{L^2}{2\mu r^2} - \frac{C}{r} \leadsto \theta - \theta_0 = L\int_{r_0}^{r}\frac{dr}{r^2\sqrt{2\mu\left(E - \frac{L^2}{2\mu r^2} + \frac{C}{r}\right)}}$$

$$\Rightarrow \theta - \theta_0 = L\int_{r_0}^{r}\frac{dr}{r\sqrt{2\mu E r^2 - L^2 + 2\mu C r}} \quad (*)$$

Evaluating the integral (*) → see Kleppner Ch 10

$$\Rightarrow r = \frac{L^2/(\mu C)}{1 - \sqrt{1 + (2EL^2/\mu C^2)}\,\sin(\theta - \theta_0)}$$

Let,

$$r_0 \equiv L^2/\mu C$$

$$\theta_0 = -\frac{\pi}{2}$$

$$\epsilon \equiv \sqrt{1 + \frac{2EL^2}{\mu C^2}} \leftarrow \text{eccentricity}$$

$$\Rightarrow r = \frac{r_0}{1 - \epsilon\cos\theta}$$

r at θ = θ₀ (also, radius of the circular orbit corresponding to L, μ, C)

In Cartesian coordinates: $r=\sqrt{x^2+y^2}$, $x=r\cos\theta$

$$\rightsquigarrow r_0 = r-\epsilon x = \sqrt{x^2+y^2}-\epsilon x$$

$$\Rightarrow r_0^2 = x^2+y^2+\epsilon^2x^2-2\epsilon r x$$

$$= (1-\epsilon^2)x^2+y^2+\underbrace{2\epsilon^2x^2-2\epsilon r x}_{=2\epsilon x(\underbrace{\epsilon x - r}_{-r_0})}$$

$$\Rightarrow (1-\epsilon^2)x^2+y^2-2\epsilon r_0 x = r_0^2$$

The shape of the orbit depends on $\epsilon=\sqrt{1+\frac{2EL^2}{\mu C^2}}$.

$\epsilon>1$ $(E>0) \rightarrow y^2-Ax^2-Bx=\text{constant}$ (hyperbola)

$\epsilon=1$ $(E=0) \rightarrow x=\frac{y^2}{2r_0}-\frac{r_0}{2}$ (parabola)

$0<\epsilon<1$ $\left(-\mu C^2/2L^2 < E<0\right) \rightarrow$ system is bounded

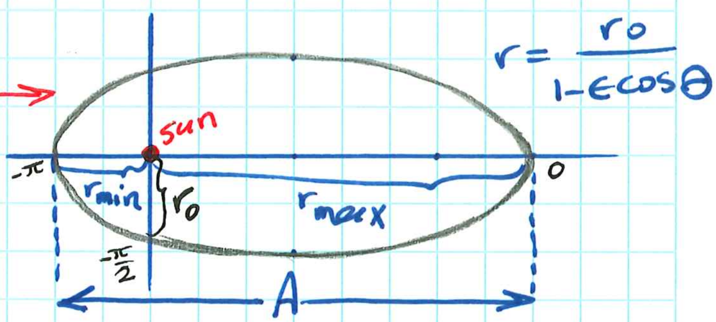
$y^2+Ax^2-Bx=\text{constant}$ (ellipse)

$\epsilon=0$ $\left(E=-\mu C^2/2L^2\right) \rightarrow x^2+y^2=r_0^2$ (circle)

minimum energy

Elliptic orbits

Kepler's first law

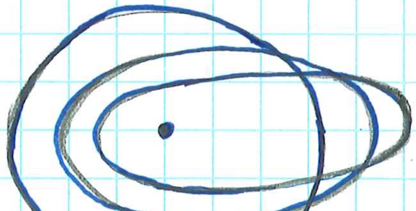


It can be shown that a focus of the ellipse is at the origin (the center of mass) → Solar system? The sun!

$$r_{min}=\frac{r_0}{1+\epsilon},\quad r_{max}=\frac{r_0}{1-\epsilon}$$

the length of the major axis:

$$A=r_{min}+r_{max}=\frac{2r_0}{1-\epsilon^2}=\frac{\cancel{2}\frac{\cancel{L^2}}{\cancel{\mu C}}}{-\frac{\cancel{2}E\cancel{L^2}}{\cancel{\mu} C^{\cancel{2}}}}=-\frac{C}{E} \quad \text{independent of } L$$



All orbits in the sketch correspond to the same value of E but different L.

ϵ is a measure of eccentricity

$\Rightarrow$ $\epsilon = 0 \Rightarrow$ circle

$\epsilon \to 1 \Rightarrow$ elongated ellipse

$$E = \frac{1}{2}\mu v^2 + U = \frac{1}{2}\mu v^2 - \frac{C}{r}$$

$\uparrow$ $-\frac{C}{A}$

$$\Rightarrow \boxed{v^2 = \frac{2C}{\mu}\left(\frac{1}{r} - \frac{1}{A}\right)}$$

orbital speed at any radial position (cool!)

The period of an elliptic orbit

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{eff})} \rightsquigarrow \frac{dt}{dr} = \frac{\mu r}{\sqrt{2\mu E r^2 - L^2 + 2\mu C r}}$$

$$\rightsquigarrow t_b - t_a = \underbrace{\frac{\sqrt{2\mu E r^2 + 2\mu C r - L^2}}{2E}\Bigg|_{r_a}^{r_b}}_{\to 0 \ (r_a = r_b)}$$

$$- \left(\frac{\mu C}{2E}\right)\frac{1}{\sqrt{-2\mu E}} \underbrace{\sin^{-1}\left(\frac{-2\mu E r - \mu C}{\sqrt{\mu^2 C^2 + 2\mu E L^2}}\right)\Bigg|_{r_a}^{r_b}}_{\to 2\pi \text{ (full cycle)}}$$

$T = t_b - t_a$ (period)

$$\Rightarrow T = \frac{\pi\mu C}{-E}\frac{1}{\sqrt{-2\mu E}} \longrightarrow T^2 = \frac{\pi^2 \mu C^2}{-2E^3}$$

$$\Rightarrow \boxed{T^2 = \frac{\pi^2 \mu}{2C} A^3} \quad \text{using } A = -\frac{C}{E}$$

Kepler's second law

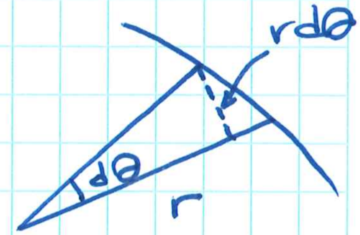
$T^2 = kA^3$ (k the same for all

Table 10.1 of Kleppner

	ϵ	A, km	T, s	A^3/T^2
Mercury	0.206	1.16×10^8	7.62×10^6	2.69×10^{10}
Earth	0.017	2.99×10^8	3.16×10^7	2.68×10^{10}
Mars	0.093	4.56×10^8	5.93×10^7	2.7×10^{10}
Jupiter	0.048	1.557×10^9	3.74×10^8	2.69×10^{10}
Neptune	0.007	9.05×10^9	5.25×10^9	2.69×10^{10}

wow!

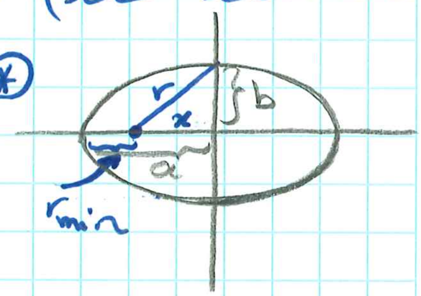
A simpler way to calculate the period:



$$L=\mu r^2\frac{d\theta}{dt} \Rightarrow \frac{L}{2\mu}dt=\underbrace{\frac{1}{2}r^2d\theta}_{\text{area element in polar coordinates}}$$ (see session 23)

$$\Rightarrow \frac{L}{2\mu}dt=dA$$

$$\Rightarrow \frac{L}{2\mu}T=\text{area of ellipse}=\pi ab \quad (*)$$



$$a=\frac{A}{2}=\frac{r_0}{1-\epsilon^2}=-\frac{C}{2E} \quad \text{(i)}$$

$$x=a-r_{min}=\frac{r_0}{1-\epsilon^2}-\frac{r_0}{1+\epsilon}=\frac{r_0\epsilon}{1-\epsilon^2}$$

$$r=\frac{r_0}{1-\epsilon\cos\theta}=\frac{r_0}{1-\epsilon\frac{x}{r}} \Rightarrow r=r_0+\epsilon x=\frac{r_0}{1-\epsilon^2}$$

$$\Rightarrow b=\sqrt{r^2-x^2}=\frac{r_0}{\sqrt{1-\epsilon^2}}=\frac{L}{\sqrt{-2\mu E}} \quad \text{(ii)}$$

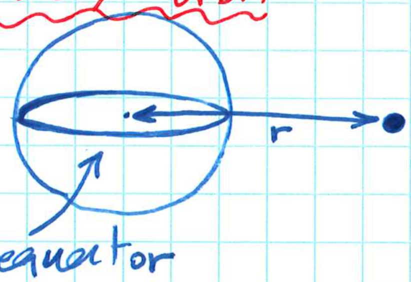
$$(*): T=\frac{2\mu}{L}\pi ab=\frac{\cancel{2}\mu}{\cancel{L}}\pi\frac{-C}{\cancel{2}E}\frac{\cancel{L}}{\sqrt{-2\mu E}}=\frac{-\mu\pi C}{E\sqrt{-2\mu E}}$$

$$\leadsto T^2=\frac{\mu\pi^2}{2C}A^3 \quad \checkmark$$

S27

- Geostationary orbit
- Satellite orbit transfer

Geostationary orbit



a satellite with orbit in the equatorial plane (geostationary)

for communication purposes, the orbital speed is set to match the angular velocity of the rotating earth, Ω_e. (satellite is stationary above a point on earth)

$A = 2r$, $\mu \approx m$, $v = r\Omega_e$

$$\Rightarrow v^2 = \frac{2C}{m}\left(\frac{1}{r} - \frac{1}{2r}\right)$$

$$\rightsquigarrow r^3 = \frac{C}{m\Omega_e^2} = \frac{gR_e^2}{\Omega_e^2}$$

On earth: $F_{Gravitational} = -G\frac{mM}{r^2}$

$$= -G\frac{mM}{R_e^2}$$

$$= -mg \Rightarrow g = \frac{GM}{R_e^2}$$

$$\Rightarrow C = GmM = \underline{mgR_e^2}$$

$$\Omega_e = \frac{2\pi}{24 \times 60 \times 60} = \frac{2\pi}{86400} \ \frac{rad}{s}$$

$g = 9.8 \ m/s^2$, $R_e = 6400$ km

$\Rightarrow r \approx 42250$ km

altitude $\rightsquigarrow h = 42250 - 6400 = \underline{35\,850 \text{ km}}$

orbital speed: $v = r\Omega_e = 3070$ m/s $\approx$ **6870 mi/hr**

Satellite orbit transfer

the most energy efficient approach to put a satellite into circular orbit:

step 1) launch it into an elliptical transfer orbit whose apogee is at the desired final radius

step 2) when the satellite is at apogee, it is accelerated tangentially into the circular orbit

Step 1

earth: $M \gg m$, $R_e = 6400$ km

Suppose that the launching angle, angular momentum, etc. were chosen so that the satellite heights (altitude) at perigee and apogee are 1100 km and 35850 km, respectively.

↖ ~~radius~~ altitude of desired geostationary orbit

kinetic energy because of the earth rotation

E_{launch}? ⇝ $E_{ground} = K_0 + U(R_e)$

$$= \frac{1}{2} m (R_e \Omega_e)^2 - \frac{C}{R_e} \quad \frac{C}{R_e} = -mgR_e$$

$$\Rightarrow \boxed{E_{ground} = -1.25 \times 10^{11}\ \text{J}}$$

$A = 1100 + 35850 + 2R_e = 5 \times 10^7$ m

$$\Rightarrow E_{orb} = -\frac{C}{A} = -\frac{mgR_e^2}{A} = \boxed{-1.61 \times 10^{10}\ \text{J}}$$

⇒ the energy needed to launch the satellite:

$$E_{launch} = E_{orb} - E_{ground} = \boxed{1.09 \times 10^{11}\ \text{J}}$$

ϵ? ⇝ $r = \dfrac{r_0}{1 - \epsilon \cos\theta} \Rightarrow r_{min} = \dfrac{r_0}{1+\epsilon}$, $r_{max} = \dfrac{r_0}{1-\epsilon}$

$$\Longrightarrow r_0 = (1+\epsilon) r_{min} = (1-\epsilon) r_{max} \leadsto \epsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$

$$\longrightarrow \boxed{\epsilon = 0.7}$$

$L = ?$ ⇝ $\epsilon = \sqrt{1 + \dfrac{2EL^2}{\mu C^2}}$ ⇝ $\boxed{L = 1.43 \times 10^{14}\ \dfrac{\text{kg} \cdot \text{m}^2}{\text{s}}}$

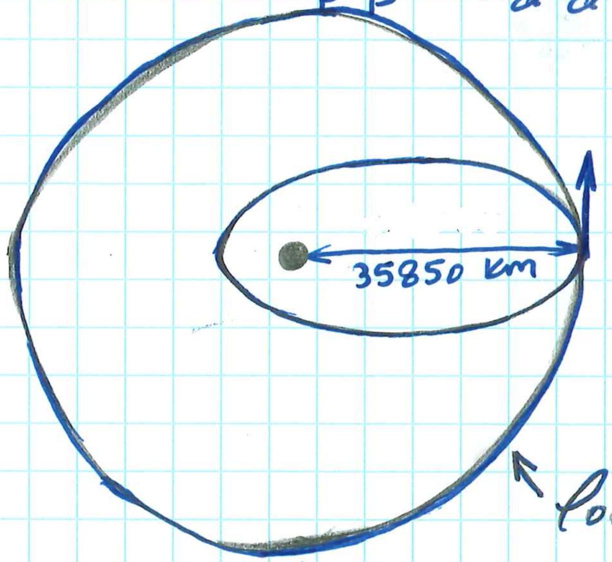
v_p and v_a ?

At the perigee and apogee: $r_p = 1100 + 6400 = 7.5 \times 10^6 m$

$r_a = 35850 + 6400 = 4.225 \times 10^7 m$

$L = m r_p v_p = m r_a v_a \Rightarrow v_p = 9530 \ m/s = 21300 \ mi/hr$

$v_a = 1690 \ m/s = 3800 \ mi/hr$



the engine gives a burst to increase the speed from $v_a = 1690$ m/s to $V = 3070$ m/s (the geostationary speed at altitude 35850 km)

looks like a potato!

desired geostationary orbit

$$\Delta E = -C \left(\frac{1}{A_f} - \frac{1}{A_i} \right) = -mgR_e^2 \left(\frac{1}{A_f} - \frac{1}{A_i} \right)$$

$A_i = 5.0 \times 10^7 m$, $A_f = 2 \times (35850 + 6400) Km = 8.45 \times 10^7 m$

$\Rightarrow \Delta E = 6.6 \times 10^9 J$

$E_i \longrightarrow E_f$

$-16.1 \times 10^9 J \qquad -9.5 \times 10^9 J$

Similar considerations apply when a spacecraft on a space mission returns to earth:

step 1) the spacecraft is slowed enough to be captured in a circular orbit.

step 2) Then it is transferred to an elliptic orbit that intersects the Earth.