Homework 5 (Solutions)

Due date: someday, sometime!
Submit on NYU Brightspace.

Exercise 1. [100 pts] Consider the following population growth models:

(i) $\dfrac{\dot{N}}{N} = -a\ln(bN)$
with a, b as positive constants. Interpret the constants a and b biologically. (You might find it helpful to analyze their dimensions.)

(ii) $\dfrac{\dot{N}}{N} = r - a(N-b)^2$
with r, a, b as positive constants. Find a relation between these constants, so that the per capita growth rate $\dot{N}/N \to 0$ as $N \to 0$.

For each of the above models,

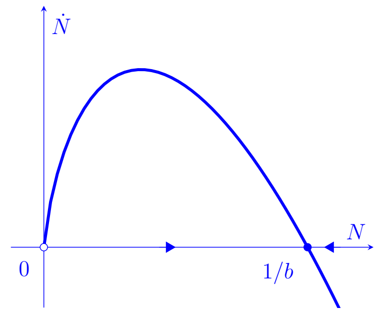
- Analytically find the fixed points N^* by letting $\dot{N} = 0$. Hint: sometimes, $\dot{N}$ is not defined at a fixed point (in particular unstable fixed points); under such circumstances, look for N^* for which $\lim_{N\to N^*} \dot{N} = 0$.
- Use linear stability analysis to identify the stability of the fixed points. If not possible, find the stability by a graphical analysis from the next steps.
- Sketch (by hand) $\dot{N}$ versus N, show the fixed points on the graph, along with the flow field. (Do **NOT** use computer to make this plot! Instead, use your understanding from calculus and make a qualitative plot.)
- Sketch (by hand) $N(t)$ versus time for various initial values N_0. (Again, do **NOT** use computer.)

notes: Solutions without details of the work and interpretation of the results will not receive full credits. Your sketches should show the important qualitative features of each growth model (e.g., concavities, approach to fixed points). They should also clearly demonstrate the differences, if any, between the two models.

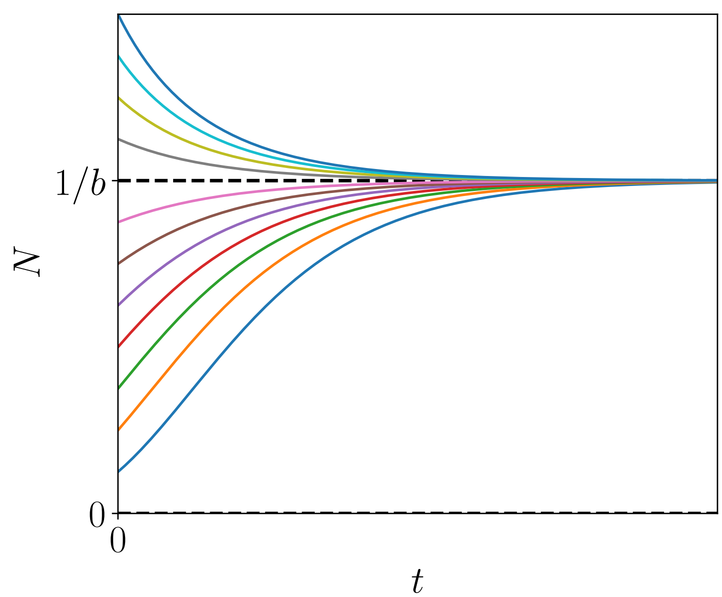
(i) The dimensions of a are {numbers/time}, and hence, a corresponds to the growth rate scale. b is dimensionless {1/numbers}, and $1/b$ is the carrying capacity. See below for more details.

- Let $\dot{N} = -aN\ln(bN) = 0$; clearly, $N^* = 1/b$ is a fixed point. Furthermore, note that
$$\lim_{N\to 0}(-aN\ln(bN)) = 0,$$
and, therefore, $N^* = 0$ is the other fixed point of the system.
- We evaluate $f'(N)$ where $f(N) = -aN\ln(bN)$:
$$f'(N) = -a\ln(bN) - a.$$
We note that $\lim_{N^*\to 0} f'(N^*) \to \infty$ ($N^* = 0$ is unstable), and $f'(1/b) = -a$ ($N^* = 1/b$ is stable).

- $\dot{N}$ versus N



- $N(t)$ versus t for different N_0



(ii) We need $\dot{N}/N = r - a(N-b)^2$ to be zero at $N = 0$:

$$0 = r - ab^2 \Rightarrow b = \sqrt{r/a}.$$

- Let $\dot{N} = N\left[r - a(N-b)^2\right] = 0$; $N^* = 0$ is clearly a fixed point. Equating the expression in the brackets to zero yields

$$r - a(N-b)^2 = 0 \Rightarrow N^* = b \pm \sqrt{\frac{r}{a}}.$$

Note that the minus sign gives $N^* = 0$ since $b = \sqrt{r/a}$. Therefore, we have two fixed points $N^* = 0$ and $N^* = 2b$.

- For $f(N) = N\left[r - a(N-b)^2\right]$,

$$f'(N) = r - a(N-b)^2 - 2aN(N-b)$$

We note that

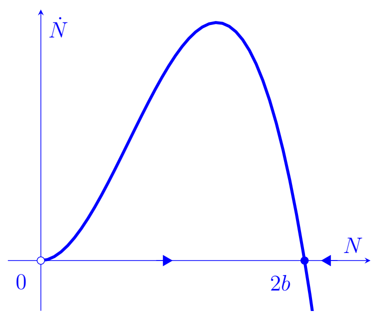
$$f'(2b) = r - a(2b-b)^2 - 2a \cdot (2b)(2b-b) = r - ab^2 - 4ab^2 = -4ab^2 < 0,$$

and so, $N^* = 2b$ is stable. However, at $N^* = 0$,

$$f'(0) = r - a(0-b)^2 - 0 = r - ab^2 = 0.$$

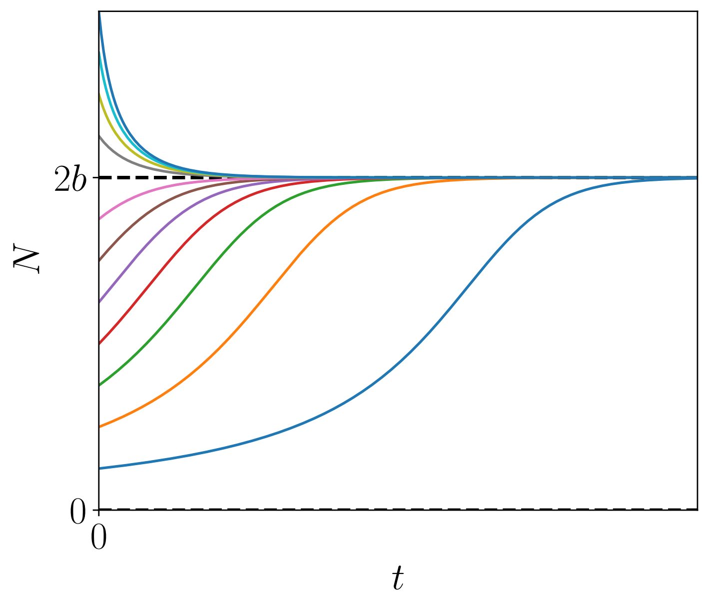
Therefore, linear stability analysis does not provide any information as to whether $N^* = 0$ is stable or unstable.

- $\dot{N}$ versus N



From this figure, it is evident that $N^* = 0$ is unstable.

- $N(t)$ versus t for different N_0



A key feature here, is that, compared to case (i), it takes much longer for a small population to start growing fast. Note that we can find the inflection point by

$$\begin{aligned}\frac{df}{dN} =& r - a(N-b)^2 - 2aN(N-b) = 0\\ \Rightarrow& \frac{r}{a} - (N-b)^2 - 2N(N-b) = b^2 - (N-b)^2 - 2N(N-b) = N(-3N+4b) = 0\\ \Rightarrow& N = 0, \quad N = \tfrac{4}{3}b.\end{aligned}$$