Homework 3 (Solutions)

Due date: someday, sometime!
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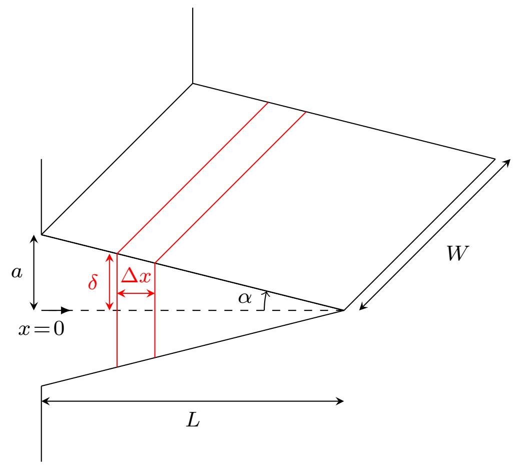
Exercise 1. [50 pts] Consider a straight rectangular heat fin illustrated below. Assuming steady-state conditions, write a thermal energy balance for the elemental volume of width Δx (from x to $x + \Delta x$) demarcated by the red solid lines. Then, let $\Delta x \to 0$ to find the governing ordinary differential equation.



You need to account for the unidirectional heat conduction within the fin, with flux $q_x = -k dT/dx$, and convection to the ambient, with flux $h(T - T_\infty)$. The sides of the fin (the two triangles with base $2a$ and height L) are insulated, and thus, does not contribute to the convection to ambient. Here, the symbols stand for thermal conductivity, k (assume constant); temperature of the fin, T; ambient temperature, T_∞; horizontal distance from the base, x; and the convective heat transfer coefficient, h.

Next, change variable $y = 2\sqrt{\gamma(L - x)}$ where $\gamma = hL/(ak\cos(\alpha))$, and $\theta = T - T_\infty$, and write the governing equation in terms of y and θ. You will reach to the modified Bessel equation of order zero, which should look like

$$y^2 \frac{d^2\theta}{dy^2} + y\frac{d\theta}{dy} - y^2\theta = 0.$$

General solution to the above Bessel equation is

$$\theta = C_1 I_0(y) + C_2 K_0(y),$$

where $I_0(y)$ and $K_0(y)$ are the modified Bessel functions of the first and second kinds. For the boundary conditions, let $T = T_0$ at the base ($x = 0$) and enforce a finite temperature at $x = L$. Find the coefficients C_1 and C_2, and express the final answer for θ in terms of x and Bessel functions. **Hint:** $K_0(y)$ blows up ($\to \infty$) at $y = 0$.

The balance equation can be expressed as

$$(q_x A)_x - (q_x A)_{x+\Delta x} - hS(T - T_\infty) = 0,$$

where $A(x) = 2\delta W$ (with $\delta = a(1 - x/L)$) and $S = 2W\Delta x/\cos(\alpha)$ are the conduction and convection area, respectively. Substitute for the area and q_x, divide by Δx, and let $\Delta x \to 0$:

$$
\begin{aligned}
&\left(-k\frac{dT}{dx}(2\delta W)\right)_x - \left(-k\frac{dT}{dx}(2\delta W)\right)_{x+\Delta x} - \frac{2hW\Delta x}{\cos(\alpha)}(T - T_\infty) = 0,\\
\Rightarrow &\lim_{\Delta x\to 0}\frac{\left(-k\frac{dT}{dx}(2\delta W)\right)_x - \left(-k\frac{dT}{dx}(2\delta W)\right)_{x+\Delta x}}{\Delta x} - \frac{2hW}{\cos(\alpha)}(T - T_\infty) = 0,\\
\Rightarrow &\lim_{\Delta x\to 0}\frac{\left(-\delta\frac{dT}{dx}\right)_x - \left(-\delta\frac{dT}{dx}\right)_{x+\Delta x}}{\Delta x} - \frac{h}{k\cos(\alpha)}(T - T_\infty) = 0,\\
\Rightarrow &-\frac{d}{dx}\left(-\delta\frac{dT}{dx}\right) - \frac{h}{k\cos(\alpha)}(T - T_\infty) = 0,\\
\Rightarrow &\frac{d}{dx}\left(a\left(1 - \frac{x}{L}\right)\frac{dT}{dx}\right) - \frac{h}{k\cos(\alpha)}(T - T_\infty) = 0.
\end{aligned}
$$

One can further simplify the final equation, the governing ODE, as

$$(L - x)\frac{d^2T}{dx^2} - \frac{dT}{dx} - \gamma(T - T_\infty) = 0.$$

with $\gamma = \dfrac{hL}{ak\cos(\alpha)}$.

Next, we let $y = 2\sqrt{\gamma(L - x)}$ and $\theta = T - T_\infty$:

$$
\begin{aligned}
\frac{dT}{dx} &= \frac{dT}{dy}\frac{dy}{dx} = -\sqrt{\gamma}(L - x)^{-\frac{1}{2}}\frac{dT}{dy},\\
\frac{d^2T}{dx^2} &= \frac{d}{dx}\left(-\sqrt{\gamma}(L - x)^{-\frac{1}{2}}\frac{dT}{dy}\right) = -\tfrac{1}{2}\sqrt{\gamma}(L - x)^{-\frac{3}{2}}\frac{dT}{dy} + \frac{\gamma}{L - x}\frac{d^2T}{dy^2}.
\end{aligned}
$$

Note that θ and T differ by only a constant, and their derivatives are the same. Substituting the new derivative formula above into the governing equation yields

$$
\begin{aligned}
&(L - x)\left[-\tfrac{1}{2}\sqrt{\gamma}(L - x)^{-\frac{3}{2}}\frac{d\theta}{dy} + \frac{\gamma}{L - x}\frac{d^2\theta}{dy^2}\right] + \sqrt{\gamma}(L - x)^{-\frac{1}{2}}\frac{d\theta}{dy} - \gamma\theta = 0,\\
\Rightarrow &\gamma\frac{d^2\theta}{dy^2} + \frac{\sqrt{\gamma}}{2\sqrt{L - x}}\frac{d\theta}{dy} - \gamma\theta = 0 \Rightarrow \gamma\frac{d^2\theta}{dy^2} + \frac{\gamma}{y}\frac{d\theta}{dy} - \gamma\theta = 0.
\end{aligned}
$$

Finally, multiplying all terms by y^2/γ gives

$$y^2\frac{d^2\theta}{dy^2} + y\frac{d\theta}{dy} - y^2\theta = 0,$$

which is the modified Bessel equation of order zero with general solution

$$\theta = C_1 I_0(y) + C_2 K_0(y) = C_1 I_0(2\sqrt{\gamma(L - x)}) + C_2 K_0(2\sqrt{\gamma(L - x)}).$$

Temperature should be finite at the tip of the fin, $x = L$, i.e., $C_1 I_0(0) + C_2 K_0(0)$ should be finite, which indicates that $C_2 = 0$ ($K_0(y)$ blows up and goes to ∞ at $y = 0$). The other boundary condition is $\theta = \theta_0 = T_0 - T_\infty$ at $x = 0$: $\theta_0 = C_1 I_0(2\sqrt{\gamma L}) \Rightarrow C_1 = \theta_0 / I_0(2\sqrt{\gamma L})$. Therefore, the solution becomes

$$\theta = \theta_0 \frac{I_0(2\sqrt{\gamma(L-x)})}{I_0(2\sqrt{\gamma L})}.$$

Exercise 2. [50 pts] Consider a heated sphere of radius R, with constant surface temperature T_R, suspended in a large, motionless (no velocity), body of fluid at $T = T_\infty$. Assuming steady state conditions, find the temperature distribution within the fluid as a function of r, distance from the center of the sphere. Start by writing a heat energy balance for a spherical shell between r and $r + \Delta r$ for an arbitrary $r > R$. Let $\Delta r \to 0$ and find the governing differential equation. Use the boundary conditions $T(R) = T_R$ and $T \to T_\infty$ as $r \to \infty$ to find the constants, and subsequently, the temperature distribution.

Next, equate the conduction heat flux at the surface of the sphere to the total heat transfer to the ambient:

$$\left(-k\frac{dT}{dr}\right)_{r=R} = h(T_R - T_\infty).$$

Having the temperature distribution $T(r)$, solve the above equation for h, the heat transfer coefficient. Then show that the dimensionless heat transfer coefficient, known as the Nusselt number, Nu, is

$$\mathrm{Nu} = \frac{hD}{k} = 2,$$

in which D is the sphere diameter.

We have

$$\left(q_r 4\pi r^2\right)_r - \left(q_r 4\pi r^2\right)_{r+\Delta r} = 0 \Rightarrow \frac{d}{dr}\left(4\pi k r^2 \frac{dT}{dr}\right) = 0 \Rightarrow r^2\frac{dT}{dr} = C_1 \Rightarrow T = -\frac{C_1}{r} + C_2.$$

Use $r \to \infty \Rightarrow T \to T_\infty$ to find $C_2 = T_\infty$. At $r = R$: $T(R) = -C_1/R + T_\infty = T_R \Rightarrow C_1 = R(T_\infty - T_R)$. Therefore, the temperature distribution is

$$T = (T_R - T_\infty)\frac{R}{r} + T_\infty.$$

The flux balance on the surface is

$$\left(-k\frac{dT}{dr}\right)_{r=R} = -k\left(-(T_R - T_\infty)\frac{R}{r^2}\right)_{r=R} = \frac{k(T_R - T_\infty)}{R} = h(T_R - T_\infty).$$

Therefore, $h = k/R$, and

$$\mathrm{Nu} = \frac{h(2R)}{k} = 2.$$