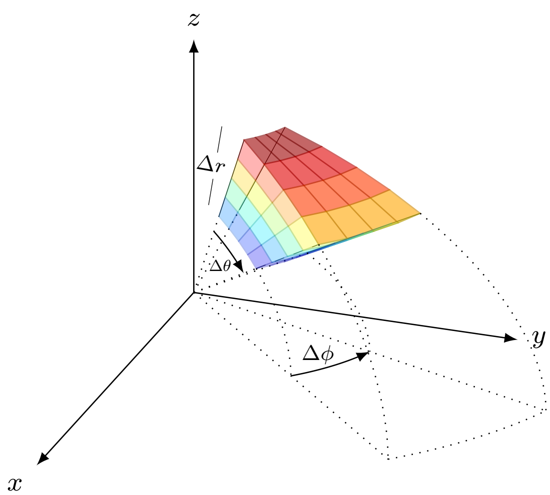
Homework 2 (Solutions)

Due date: someday, sometime!
Submit on NYU Brightspace.

Spherical Coordinates: The spherical coordinate system uses a distance r from the origin along with two angles θ and ϕ to uniquely locate a point in space. An elemental volume in spherical coordinates is illustrated below:



The radial distance r is measured from the origin; the angle $\phi \in [0, 2\pi)$ varies in the xy plane and is zero on the x axis; the angle $\theta \in [0, \pi]$ measures the deviation from the z axis. The projected radius onto the xy plane is $r\sin(\theta)$. The radii of curvature for angles θ and ϕ are r and $r\sin(\theta)$, respectively. Therefore, we have the following formula for the volume of the element, V, and area in different directions (treating the elemental volume as a rectangular box for small $\Delta r, \Delta\theta, \Delta\phi$):

$$\begin{aligned}
V &= (\Delta r) \cdot (r\Delta\theta) \cdot (r\sin(\theta)\Delta\phi) = r^2 \sin(\theta)\Delta r\Delta\theta\Delta\phi, \\
A_r &= (r\Delta\theta) \cdot (r\sin(\theta)\Delta\phi) = r^2 \sin(\theta)\Delta\theta\Delta\phi, \\
A_\theta &= (\Delta r) \cdot (r\sin(\theta)\Delta\phi) = r\sin(\theta)\Delta r\Delta\phi, \\
A_\phi &= (\Delta r) \cdot (r\Delta\theta) = r\Delta r\Delta\theta.
\end{aligned}$$

Here, $A_r,\ A_\theta,\ A_\phi$ denote the area in the $r,\ \theta,\ \phi$ directions.

Exercise 1. [50 pts] The mass flux (mass per unit area per unit time) can be expressed as

$$\rho\mathbf{u} = \rho u_r \hat{e}_r + \rho u_\theta \hat{e}_\theta + \rho u_\phi \hat{e}_\phi,$$

where ρ is density of the material and $\mathbf{u}$ is the vector of velocity with scalar components $u_r,\ u_\theta,\ u_\phi$ in $r,\ \theta,\ \phi$ directions, respectively.

Write a shell balance for the elemental volume to derive the differential form of the continuity equation in spherical coordinates. Next, simplify your answer for constant ρ.

We write the shell balance as

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho r^2 \sin(\theta)\Delta r\Delta\theta\Delta\phi) = \\
&+ (\rho u_r r^2 \sin(\theta)\Delta\theta\Delta\phi)_r - (\rho u_r r^2 \sin(\theta)\Delta\theta\Delta\phi)_{r+\Delta r} \\
&+ (\rho u_\theta r \sin(\theta)\Delta r\Delta\phi)_\theta - (\rho u_\theta r \sin(\theta)\Delta r\Delta\phi)_{\theta+\Delta\theta} \\
&+ (\rho u_\phi r\Delta r\Delta\theta)_\phi - (\rho u_\phi r\Delta r\Delta\theta)_{\phi+\Delta\phi}.
\end{aligned}$$

Divide both sides by $\Delta r\Delta\theta\Delta\phi$ and let $\Delta r, \Delta\theta, \Delta\phi \to 0$:

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho r^2 \sin(\theta)) &= \lim_{\Delta r\to 0}\frac{(\rho u_r r^2 \sin(\theta))_r - (\rho u_r r^2 \sin(\theta))_{r+\Delta r}}{\Delta r} \\
&+ \lim_{\Delta\theta\to 0}\frac{(\rho u_\theta r \sin(\theta))_\theta - (\rho u_\theta r \sin(\theta))_{\theta+\Delta\theta}}{\Delta\theta} \\
&+ \lim_{\Delta\phi\to 0}\frac{(\rho u_\phi r)_\phi - (\rho u_\phi r)_{\phi+\Delta\phi}}{\Delta\phi}. \\
\Rightarrow r^2\sin(\theta)\frac{\partial\rho}{\partial t} &= -\frac{\partial}{\partial r}(\rho u_r r^2 \sin(\theta)) - \frac{\partial}{\partial\theta}(\rho u_\theta r\sin(\theta)) - \frac{\partial}{\partial\phi}(\rho u_\phi r).
\end{aligned}$$

Finally, divide both sides by $r^2\sin(\theta)$ to get

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho u_r r^2) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\rho u_\theta \sin(\theta)) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}(\rho u_\phi) = 0.$$

If density is constant (i.e, not a function of time or location), the continuity equation simplifies to

$$\frac{1}{r^2}\frac{\partial}{\partial r}(u_r r^2) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(u_\theta \sin(\theta)) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}(u_\phi) = 0.$$

Exercise 2. [50 pts] The heat flux (thermal energy per unit area per unit time) can be expressed as

$$\rho\mathbf{u}C_pT - k\nabla T,$$

where the new parameters $C_p,\ k,\ T$ are the specific heat capacity, thermal conductivity, and temperature, respectively, and the gradient operator in spherical coordinates is

$$\nabla T = \frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial T}{\partial\phi}\hat{e}_\phi.$$

Write a shell balance (with a heat source s) for the elemental volume to derive the differential form of the heat equation in spherical coordinates. Next, simplify your answer for constant $\rho,\ C_p,\ k$.

The total heat flux in each direction, $\gamma_{r,\theta,\phi}$, can be expressed as

$$\begin{aligned}
\gamma_r &= \rho u_r C_p T - k\frac{\partial T}{\partial r},\\
\gamma_\theta &= \rho u_\theta C_p T - \frac{k}{r}\frac{\partial T}{\partial \theta},\\
\gamma_\phi &= \rho u_\phi C_p T - \frac{k}{r\sin(\theta)}\frac{\partial T}{\partial \phi}.
\end{aligned}$$

We then write the energy balance as

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho r^2\sin(\theta)\Delta r\Delta\theta\Delta\phi C_p T) = s r^2\sin(\theta)\Delta r\Delta\theta\Delta\phi\\
&+ (\gamma_r r^2\sin(\theta)\Delta\theta\Delta\phi)_r - (\gamma_r r^2\sin(\theta)\Delta\theta\Delta\phi)_{r+\Delta r}\\
&+ (\gamma_\theta r\sin(\theta)\Delta r\Delta\phi)_\theta - (\gamma_\theta r\sin(\theta)\Delta r\Delta\phi)_{\theta+\Delta\theta}\\
&+ (\gamma_\phi r\Delta r\Delta\theta)_\phi - (\gamma_\phi r\Delta r\Delta\theta)_{\phi+\Delta\phi}.
\end{aligned}$$

Following the same procedure as in Exercise 1, divide both sides by $r^2\sin(\theta)\Delta r\Delta\theta\Delta\phi$, let $\Delta r, \Delta\theta, \Delta\phi \to 0$, and use the definition of derivative to get:

$$\frac{\partial(\rho C_p T)}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\gamma_r r^2) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\gamma_\theta\sin(\theta)) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}(\gamma_\phi) = s.$$

Now substituting for $\gamma_{r,\theta,\phi}$ yields

$$\begin{aligned}
&\frac{\partial(\rho C_p T)}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\rho u_r C_p T - k\frac{\partial T}{\partial r}\right]\right)\\
&+\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\left[\rho u_\theta C_p T - \frac{k}{r}\frac{\partial T}{\partial\theta}\right]\right)\\
&+\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}\left(\rho u_\phi C_p T - \frac{k}{r\sin(\theta)}\frac{\partial T}{\partial\phi}\right) = s.
\end{aligned}$$

Upon some rearrangements we obtain the heat balance equation as

$$\begin{aligned}
&\frac{\partial(\rho C_p T)}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho u_r C_p T\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\rho u_\theta C_p T\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}\left(\rho u_\phi C_p T\right)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(k\sin(\theta)\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + s.
\end{aligned}$$

For constant $\rho,\ C_p,\ k$, the equation simplifies to

$$\begin{aligned}
&\frac{\partial T}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r T\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\theta T\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi}\left(u_\phi T\right)\\
&= \frac{k}{\rho C_p}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 T}{\partial\phi^2}\right] + \frac{s}{\rho C_p}.
\end{aligned}$$

One can further simplify the left hand side by using the product rule, and then the continuity equation for constant density:

$$
\begin{aligned}
&\frac{\partial T}{\partial t} + \frac{1}{r^2}\left[r^2 u_r \frac{\partial T}{\partial r} + T\frac{\partial r^2 u_r}{\partial r}\right] + \frac{1}{r\sin(\theta)}\left[\sin(\theta)u_\theta \frac{\partial T}{\partial \theta} + T\frac{\partial \sin(\theta) u_\theta}{\partial \theta}\right] + \frac{1}{r\sin(\theta)}\left[u_\phi \frac{\partial T}{\partial \phi} + T\frac{\partial u_\phi}{\partial \phi}\right]\\
=&\frac{\partial T}{\partial t} + T\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}(u_\theta \sin(\theta)) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}(u_\phi)\right] + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial T}{\partial \phi}\\
=&\frac{\partial T}{\partial t} + T\left[0\right] + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial T}{\partial \phi}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\partial T}{\partial t} + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial T}{\partial \phi}\\
&= \frac{k}{\rho C_p}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 T}{\partial \phi^2}\right] + \frac{s}{\rho C_p}.
\end{aligned}
$$