Homework 1 (Solutions)

Due date: someday, sometime!
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Exercise 1. [50 pts] We wish to use the Buckingham π theorem to estimate the drag force on an airplane cruising at 400 km/hr in standard air (atmospheric pressure) using a scaled $1:10$ down model (geometrical dimensions of the model are 0.1 of that of the prototype). Let the air speed in the wind tunnel be 400 km/hr as well. Assuming the same air temperature for the model and prototype, determine the required pressure in the tunnel, and the drag on the prototype corresponding to 1 N on the model.

Hints:

(i) The drag force, F, is expected to depend on the following parameters: V (wind speed), μ (viscosity), ρ (density), ℓ (length of the airplane), and w (wing span of the airplane).

(ii) Assume that the air viscosity does not depend on the air pressure.

(iii) Use Ideal Gas Law $pMW = \rho RT$ to relate the air density to pressure. (p: pressure, R: universal gas constant, MW: molecular weight of air (constant), T: absolute temperature)

We have $F = f(V, \mu, \rho, \ell, w)$. We use the Buckingham π theorem to find the dimensionless group.

$$\begin{aligned} F &\doteq MLT^{-2}, \\ V &\doteq LT^{-1}, \\ \mu &\doteq ML^{-1}T^{-1}, \\ \rho &\doteq ML^{-3}, \\ \ell &\doteq L, \\ w &\doteq L. \end{aligned}$$

Therefore, the number of reference dimensions is $r = 3$, which, by Buckingham π theorem, yields $6 - 3 = 3$ dimensionless groups. We pick ℓ, V, and ρ as the repeating variables:

$$\begin{aligned} \Pi_1 &= F\ell^a V^b \rho^c = M^0L^0T^0 \Rightarrow a = -2, b = -2, c = -1 \Rightarrow \Pi_1 = \frac{F}{\ell^2 V^2 \rho}, \\ \Pi_2 &= \mu\ell^a V^b \rho^c = M^0L^0T^0 \Rightarrow a = -1, b = -1, c = -1 \Rightarrow \Pi_2 = \frac{\mu}{\ell V \rho}, \\ \Pi_3 &= w\ell^a V^b \rho^c = M^0L^0T^0 \Rightarrow a = -1, b = 0, c = 0 \Rightarrow \Pi_3 = \frac{\ell}{w}. \end{aligned}$$

Finally, we can express the results of the dimensional analysis as

$$\Pi_1 = \phi\left(\Pi_2, \Pi_3\right) \Rightarrow \Pi_1 = \psi\left(1/\Pi_2, \Pi_3\right) \Rightarrow \frac{F}{\ell^2 V^2 \rho} = \psi\left(\frac{\ell V \rho}{\mu}, \frac{\ell}{w}\right).$$

Now, for the model to represent the prototype, we need

$$\frac{\ell_m V_m \rho_m}{\mu_m} = \frac{\ell V \rho}{\mu}, \quad \frac{\ell}{w} = \frac{\ell_m}{w_m},$$

where the subscript m denotes the model. The second condition is automatically satisfied since the model is a $1:10$ scaled down version of the prototype: $\ell_m = 0.1\ell$ and $w_m = 0.1w$. Consequently, we only need to ensure that the Reynolds numbers are the same:

$$\frac{\ell_m V_m \rho_m}{\mu_m} = \frac{\ell V \rho}{\mu} \Rightarrow \frac{\rho_m}{\rho} = \frac{\mu_m}{\mu}\frac{V}{V_m}\frac{\ell}{\ell_m} = \frac{\mu_m}{\mu} \cdot 1 \cdot 10 = 10\frac{\mu_m}{\mu}.$$

As this result indicates, the same fluid with $\rho_m = \rho$ and $\mu_m = \mu$ cannot be used if the Reynolds number similarity is to be maintained. We can pressurize the wind tunnel to increase the density ρ_m. We assume that μ_m does not change with pressure significantly. Therefore, $\mu_m = \mu$, and we need

$$\frac{\rho_m}{\rho} = 10 \Rightarrow \frac{p_m MW/RT_m}{pMW/RT} = \frac{p_m}{p} = 10.$$

So, the pressure needed to ensure that $\mathrm{Re}_m = \mathrm{Re}$ is 10 times higher than the pressure at which the actual prototype operates. Given that $p = 1$ atm, $p_m = 10$ atm.

Now that $\Pi_{1m} = \Pi_1$ and $\Pi_{2m} = \Pi_2$, we conclude that

$$\frac{F}{\ell^2 V^2 \rho} = \frac{F_m}{\ell_m^2 V_m^2 \rho_m} \Rightarrow F = \left(\frac{\ell}{\ell_m}\right)^2 \left(\frac{V}{V_m}\right)^2 \left(\frac{\rho}{\rho_m}\right) F_m = 10^2 \cdot 1 \cdot 0.1 F_m = 10F_m.$$

So for a drag of $F_m = 1$ N on the model, the corresponding drag on the prototype is $F = 10$ N.

Exercise 2. [50 pts] From dimensional analysis, we know that the drag coefficient C_D of a smooth sphere moving in a fluid is a function of Reynolds number:

$$C_D = f(\mathrm{Re}) = f\left(\frac{\rho V D}{\mu}\right),$$

where ρ and μ are the density and viscosity of the fluid (e.g., air), D is the diameter of the sphere, and V is its velocity. The drag force can then be expressed as

$$F = \tfrac{1}{2} C_D \rho A V^2$$

with A as the projected area. Note that you can derive the Stokes drag force formula (valid when $\mathrm{Re} \lesssim 1$) from the above equation by inserting $C_D = 24/\mathrm{Re}$.

Now consider two raindrops, one with diameter 2 mm and another one with diameter 4 mm. Which one falls faster in the air and by how much?

Hints:

(i) Assume that the drops fall at their 'terminal' velocities, which is the velocity at which the drag force and gravitational forces cancel each other: $F = \frac{1}{6}\pi D^3(\rho_s - \rho)g$, where $\frac{1}{6}\pi D^3$ is the volume of the droplet, ρ_s is its density (≈ 1000 kg/m^3), and $g \approx 9.8$ m/s^2 is the gravitational acceleration. Also, for the air, use $\rho = 1$ kg/m^3 and $\mu = 2\times 10^{-5}$ Pa $\cdot$ s.

(ii) Initially assume that the drops fall in the Stokes regime (i.e., Re $\lesssim 1$) and use the Stokes drag formula. After finding the terminal velocity calculate the Re number to verify your initial assumption.

(iii) In case the Stokes regime is not valid: for high Reynolds number $10^3 \lesssim \text{Re} \lesssim 10^5$, one can assume that C_D is constant and not a function of Re. Use $C_D \approx 0.4$ and redo the above process. Verify if the Reynolds number is within the range.

We first derive the Stokes drag formula:

$$F = \tfrac{1}{2}C_D\rho A V^2 = \tfrac{1}{2}\left(\tfrac{24\mu}{\rho V D}\right)\rho\left(\tfrac{1}{4}\pi D^2\right)V^2 = 3\pi\mu D V.$$

Now, we find the terminal velocity by balancing the Stokes drag and gravitational forces:

$$3\pi\mu D V = \tfrac{1}{6}\pi D^3(\rho_s - \rho)g \Rightarrow V = \frac{D^2(\rho_s-\rho)g}{18\mu}.$$

We see that the terminal velocity is proportional to the square of the diameter. Therefore,

$$\frac{V(D = 4\text{ mm})}{V(D = 2\text{ mm})} = 2^2 = 4.$$

In other words, the 4 mm-droplet falls 4 times faster than the 2 mm-droplet. We now need to verify that we could use the Stokes drag formula by calculating the Reynolds number:

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{\rho(\rho_s-\rho)D^3 g}{18\mu^2}.$$

Using $\rho_s = 1000$ kg/m$^3, \rho = 1$ kg/m$^3, \mu = 2\times 10^{-5}$ Pa $\cdot$ s, the Reynolds number become 10885 and 87082 for $D = 2$ mm and 4 mm, respectively. This is **very** high! So the Stokes law is not valid.

Now we let $C_D = 0.4$ and constant:

$$\tfrac{1}{2}C_D\rho A V^2 = \tfrac{1}{6}\pi D^3(\rho_s-\rho)g \Rightarrow V = \sqrt{\frac{4D(\rho_s-\rho)g}{3C_D\rho}}.$$

Therefore, the falling velocity is proportional to the square of the diameter:

$$\frac{V(D = 4\text{ mm})}{V(D = 2\text{ mm})} = \sqrt{2} = 1.41.$$

Using the new formula for the velocity to calculate the Reynolds number yields Re ≈ 808 and ≈ 2285 for $D = 2$ mm and 4 mm, respectively. This is approximately within the range for which C_D remains constant. Finally, we find the falling velocities as 8.08 m/s and 11.43 m/s for the 2-mm and 4-mm raindrops, respectively.

Exercise 3. [10+10+10+10 pts] Heat transfer flux from a surface at temperature T_s to an environment at temperature T_∞ is given by $h(T_s - T_\infty)$, where h is the heat transfer coefficient with dimensions energy/(time · area · temperature). For a sphere suspended in a still air, the heat transfer coefficient depends on the sphere diameter, D, and conductivity of the air, k, (with dimensions energy/(time · length · temperature)). Hence, one can write $h = f(k, D)$.

(i) List all of the parameters and their dimensions. Hint: energy $\doteq$ force · length.

(ii) Find the number of *reference* dimensions, and subsequently, the number of dimensionless groups.

(iii) Find the dimensionless groups using the method described in class.

(iv) Interpret the result.

We use M, L, T, θ to denote the primary dimensions mass, length, time, and temperature.

$$h \doteq \frac{\text{energy}}{\text{time} \cdot \text{area} \cdot \text{temperature}} \doteq \frac{\text{force} \cdot \text{length}}{\text{time} \cdot \text{area} \cdot \text{temperature}} \doteq \frac{MLT^{-2}L}{TL^2\theta} \doteq MT^{-3}\theta^{-1},$$
$$k \doteq \frac{\text{energy}}{\text{time} \cdot \text{length} \cdot \text{temperature}} \doteq \frac{\text{force} \cdot \text{length}}{\text{time} \cdot \text{length} \cdot \text{temperature}} \doteq \frac{MLT^{-2}L}{TL\theta} \doteq MLT^{-3}\theta^{-1},$$
$$D \doteq L.$$

A simple inspection shows that h and k differ from each other by a length scale. Indeed, the dimensions required to fully describe the system are $MLT^{-3}\theta^{-1}$ and L. Hence, the total number of dimensionless group(s) is $3 - 2 = 1$. We can take D and k as the repeating parameters and write the dimensionless group as

$$\Pi = hk^a D^b$$

It is easy to find that we need $a = -1, b = 1$ for Π to be dimensionless, i.e., $\Pi = M^0L^0T^0\theta^0$. Since the problem is characterized by only one dimensionless number, we conclude that

$$\Pi = \frac{hD}{k} = \text{constant}.$$

The dimensionless group $h\ell/k$ is known as the Nusselt number, where ℓ is a length scale (D for a sphere). For a sphere in still air, one can show (by solving transport equations) that $hD/k = 2$.