Likelihood ratio and decision thresholds

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SAMLE CHAPTER PLAN UPDATE

I am now realizing that perhpas the structure of the chapter could be further broken down into three chapters:

1. A chapter that shows how probabilistc thresholds are good as analytical tools, despite implementation or practical difficulties with them. This chapter would include discussions of expected utility, minimizing errors, signal detection theory, etc. A lot of this stuff is already in the extended version of the SEP entry. So main claim of this chayer is: yes, probabilistic threshold are not good practically, but they can still be good as analytical tools. Title: "Probability Thresholds as Analytical Models of Trial Decision Making"
2. Two chapters that looks at the two theoretical difficulties (naked stats and conjunction paradox, ad also problem of priors). One chapter on naked statistical evidence and our informal solutions to it, based either on LR or on specific narratives (this should be followed by another chapter with the formal details).
3. Another chapter on conjunction paradox and our informal solution to it, maybe in terms of LR, BF or narratives (followed by another chapter in which the formal details are spelled out).
4. A chapter that formally addresses the two theoretical difficulties, perhaps using Bayesian Networks. This need not be included in the sample chapters we sent out. Title: "Adressing the Proof Paradoxes with Bayesian Networks".

SAMPLE CHAPTER PLAN

In rethinking the sample chapter, we should perhaps stick to a simpler structure, trying to offer a more focused and compelling argument. Right now I think we have too many possible accounts under consideration, and the structure is not very tight or cohesive. It feels more like a literature review, especially the first few sections.

So here is how I proposed we do it:

1. Begin by stating the simplest probabilistic account based on a threshold for the posterior probability of guilt/liability. The threshold can be variable or not. Add brief description of decision-theoretic ways to fix the threshold. (Perhaps here we can also talk about intervals of posterior probabilities or imprecise probabilities.)
2. Formulate two common theoretical difficulties against ths posterior probability threshold view: (a) naked statistical evidence and (b) conjuction. (We should state these difficilties before we get into alternative probabilistic accounts, or else the reader might wonder why so many different variants are offerred of probabilistic accounts).

   R: Yes. That's what I thought.

   We might also want to add a third difficulty: (c) the problem of priors (if priors cannot be agreed upon then the posterior probability threshold is not functionally operative). Dahlman I think has quite a bit of stuff on the problem of priors.
3. As a first response to the difficulties, articulate the likelihood ratio account. This is the account I favor in my mind paper. Kaplow seems to do something similar. So does Sullivan. So it's a popular view, worth discusing in its own right. You say that Cheng account is one particular variant of this account, so we can talk about Cheng here, as well.
4. Examine how the likelihood ratio account fares against the two/three difficulties above. One could make an argument (not necessarily a correct one) that the likelihood ratio account can address all

the two/three difficulties. So we should say why one might think so, even thought the argument will ultimately fail. I think this will help grab the reader's attention. This is what I have in mind:
4a: the LR approach solves the naked stat problem because LR=1 (Cheng, Sullivan) or L1=unknown (Di Bello).
4b: the LR approach solves the conjuction problem because – well this is Dawid's point that we will have to make sense of the best we can
4c: the LR approach solves the priors problem b/c LR do not have priors.

5. Next, poke holes in the likelihood ratio account:
   against 4a: you do not believeLR=1 or LR=unknown , so we should talk about this
   against 4b: this is your cool argument against Dawid
   against 4c: do you believe the arguemt in 4c? we should talk about this
   In general, we will have to talk to see where we stand. As of now, I tentatively believe that the likelihood ratio account can solve (a) and (c), and you seem to disagree with that. Even if I am right, the account is still not good enough becaue it cannot solve (b).
6. Articulate (or just sketch?) a better probabilistic account overall. Use Bayesian networks, narratives, etc. I am not sure if this should be another paper. That will depend on how much we'll have to say here.

Contents

1 SAMPLE CHAPETR TITLE - "Probability Thresholds as Analytical Models of Trial Decision Making"

2 Introduction

After the evidence has been presented, examined and cross-examined at trial, trained judges or lay jurors must reach a decision. In many countries, the decision criterion is defined by law and consists of a standard of proof, also called the burden of persuasion. So long as the evidence against the defendant meets the requisite proof standard, the defendant should be found liable.

In criminal proceedings, the governing standard is 'proof beyond a reasonable doubt.' If the decision makers are persuaded beyond a reasonable doubt that the defendant is guilty, they should convict, or else they should acquit. In civil cases, the standard is typically 'preponderance of the evidence.' The latter is less demanding than the former, so the same body of evidence may meet the preponderance standard, but not meet the beyond a reasonable doubt standard. A vivid example of this difference is the 1995 trial of O.J. Simpson, who was charged with the murder of his wife. He was acquitted of the criminal charges, but when the family of the victim brought a lawsuit against him, they prevailed. O.J. Simpson did not kill his wife according to the beyond a reasonable doubt standard, but he did according to the preponderance standard. An intermediate standard, called 'clear and convincing evidence,' is sometimes used for civil proceedings in which the decision is particularly weighty, for example, a decision whether someone should be committed to a hospital facility.

Not sure if it is clear what you mean by this.

How to define standards of proof—and whether they should be even defined in the first place—remains contentious (Diamond, 1990; Horowitz & Kirkpatrick, 1996; Laudan, 2006; Newman, 1993; Walen, 2015). Judicial opinions offer different, sometimes conflicting, paraphrases of what these standards mean. The meaning of 'proof beyond a reasonable doubt' is the most controversial. It has been equated with 'moral certainty' or 'abiding conviction' (Commonwealth v. Webster, 59 Mass. 295, 320, 1850) or with 'proof of such a convincing character that a reasonable person would not hesitate to rely and act upon it in the most important of his own affairs' (US Federal Jury Practice and Instructions, 12.10, at 354, 4th ed. 1987). But courts have also cautioned that there is no need to define the term because 'jurors know what is reasonable and are quite familiar with the meaning of doubt' and attempts to define it only 'muddy the water' (U.S. v. Glass, 846 F.2d 386, 1988).

To further complicate things, differences between countries and legal traditions exist. The tripartite distinction of proof standards—beyond a reasonable doubt; preponderance; clear and convincing evidence—is common in Anglo-american jurisprudence. It is not universal, however. Different countries may use different standards. France, for example, uses the standard of 'intimate conviction' for both civil and criminal proceedings. Judges deciding cases 'must search their conscience in good faith and silently and thoughtfully ask themselves what impression the evidence given against the accused and the defence's arguments have made upon them' (French Code of Criminal Procedure, art. 353). German law is similar. Germany's Code of Civil Procedure, Sec. 286, states that 'it is for the court to decide, based on its personal conviction, whether a factual claim is indeed true or not.'

R: check the formulation in Poland

While there are inevitable differences between legal traditions, the question of how strong the evidence should be to warrant a finding of civil or criminal liability has universal appeal. Any system of adjudication whose decisions are informed by evidence will confront this question in one way or another. Not all legal systems will explicitly formulate standards of proof for trial decisions. Some legal systems may specify rules about how evidence should be weighed without formulating decision criteria such as standards of proof. But even without explicit proof standards, the triers of facts, judges or jurors, will have to decide whether the evidence is sufficient to judge the defendant legally liable.

Need to revise this when the chapter is done.

We will not survey the extensive legal literature and case law about proof standards. We will instead

examine whether or not probability theory can bring conceptual clarity to an otherwise heterogeneous legal doctrine. This chapter outlines different probabilistic approaches, formulates the most common challenges against them, and offers a number of responses from the perspective of legal probabilism. The legal and philosophical literature has focused on the theoretical and analytical challanges. We will do the same here. We will focus on two key theoretical challanges that have galvanized the philosophical literature: the problem of naked statistical evidence and the conjunction paradox. One reason to choose these two in particular is that it would be desirable to be able to handle basic conceptual difficulties before turning to more complex issues or attempting to implement probabilistic standards of proof in trial proceedings.

Here you sound like you're gonna list a bunch of reasons but you give only one. Consider adding reasons or reformulating this bit.

3 Probability thresholds

Imagine you are a trier of fact, say a judge or a juror, who is expected to make a decision about the guilt of a defendant who faces criminal charges. The prosecution presents evidence to support its accusation, and the defense offers counterevidence. As a trier of fact, you are confronted with the question whether the totality of the evidence presented at trial warrants a conviction. More specifically, the question is whether the evidence as a whole establishes the defendant's guilt beyond a reasonable doubt.

3.1 The basic idea

Legal probabilists have proposed to interpret proof beyond a reasonable doubt as the requirement that the defendant's probability of guilt, given the evidence presented at trial, meet a threshold (see Bernoulli, 1713; Dekay, 1996; Kaplan, 1968; Kaye, 1979a; Laplace, 1814; Laudan, 2006). On this interpretation, so long as the defendant's guilt is established with a sufficiently high probability, say 95%, guilt is proven beyond a reasonable doubt and the defendant should be convicted. If the probability of guilt does not reach the requisite threshold, the defendant should be acquitted. This intepretation can be spelled out more formally by means of conditional probabilities. That is, a body of evidence E establishes guilt G beyond a reasonable doubt if and only if $\mathsf{P}(G|E)$ is above the threshold.

This interpretation is, in many respects, plausible. From a legal standpoint, the requirement that guilt be established with high probability, still short of 100%, accords with the principle that proof beyond a reasonable doubt is the most stringent standard but does not require—as the Supreme Court of Canada put it—'proof to an absolute certainty' and thus 'it is not proof beyond any doubt' (R v Lifchus, 1997, 3 SCR 320, 335). The plausibility of a probabilistic intepretation is further attested by the fact that such an intepretation is tacitly assumed in empirical studies about people's understanding of proof beyond a reasonable doubt (Dhami, Lundrigan, & Mueller-Johnson, 2015). This research examines how high decision-makers set the bar for convictions, say at 80% or 90% probability, but does not question the assumption that standards of proof function as probabilistic thresholds of some kind.

Reliance on probability is even more explicit in the standard 'preponderance of the evidence'—also called 'balance of probabilities'—which governs decisions in civil disputes. This standard can be interpreted as the requirement that the plaintiff—the party making the complaint against the defendant in a civil case—establish their version of the facts with greater than 50% probability. The 50% threshold, as opposed to a more stringent threshold of 95% for criminal cases, reflects the fact that preponderance is less demanding than proof beyond a reasonable doubt. The intermediate standard 'clear and convincing evidence' is more stringent than the preponderance standard but not as stringent as the beyond a reasonable doubt standard. Since it lies in between the other two, it can be interpreted as the requirement that the plaintiff establish their versions of the facts with, say, 75-80% probability.

3.2 Mixed reactions from legal practitioners

When appellate courts have examined the question whether standards of proof can be quantified using probabilities, they have often answered in the negative. One of the clearest opposition to quantification was formulated by Germany's Supreme Court, the Federal Court of Justice, in the case of Anna Anderson who claimed to be a descendant of the Tsar family. In 1967, the Regional Court of Hamburg ruled that Anderson failed to present sufficient evidence to establish that she was Grand Duchess Anastasia Nikolayevna, the youngest daughter of Tsar Nicholas II, who allegedly escaped the murder of the Tsar family by the Bolsheviks in 1918. (Incidentally, DNA testing later demonstrated that Anna Anderson

had no relationship with the Tsar family.) Anderson appealed to Germany's Federal Court, complaining that the Regional Court had set too demanding a proof standard. Siding with the lower court, the Federal Court made clear that '[t]he law does not presuppose a belief free of all doubts', thus recognizing the inevitable fallibility of trial decisions. The Court warned, however, that it would be 'wrong' to think that a trial decision could rest on 'a probability bordering on certainty' (Federal Court of Justice, February 17, 1970; III ZR 139/67). This decision is all the more interesting as it applies to a civil case. The German court did not think trial decisions could rest on a probability, not even in a civil case.

For criminal cases, Buchak (2014) has persuasively argued that an attribution of criminal culpability is an ascription of blame which requires a full belief in someone's guilt. One is left wondering, however. If a high probability of guilt short of 100% isn't enough but absolute certainty cannot be required either, how else could the standard of proof be met? The question becomes more pressing in civil cases if we replace 'guilt' with 'civil liability'. Anticipating this worry, Germany's Federal Court in the Anderson case endorsed a conception of proof standards that acknowledges the inveitable fallibility of trial decisions while at the same time maintaining the need for certainty. The Federal Court wrote that a judge's decision must satisfy 'a degree of certainty which is useful for practical life and which makes the doubts silent without completely excluding them' (Federal Court of Justice, February 17, 1970; III ZR 139/67).

The words of Germany's Federal Court echo dilemmas that bedeviled early theorists of probability and evidence law. When Jacob Bernoulli—one of the pionerres of probability theory—discusses the requirement for a criminal conviction in his *Ars Conjectandi* (1713), he writes that 'it might be determined whether 99/100 of probability suffices or whether 999/1000 is required' (part IV). This is one of the earliest suggestions that the criminal standard of proof be equated with a threshold probability of guilt. A few decades later, the Italian legal penologist Cesare Beccaria in his celebrated treatise *On Crimes and Punishments* (1764) remarks that the certainty needed to convict is 'nothing but a probability, though a probability of such a sort to be called certainty' (chapter 14). This suggestive yet—admittedly—quite elusive remark indicates that the standard of decision in criminal trials should be a blend of probability and certainty. But what this blend of probability and certainty should amount to is unclear. At best, it brings us back to paraphrases of proof beyond a reasonable doubt such as 'moral certainty' or 'abiding conviction'.

Not all legal practitioners, however, resist a probabilistic interpretation of standards of proof. Some actually find such interpretation plausible, even obvious. For example, here is Justice Harlan of the United States Supreme Court:

> . . . in a judicial proceeding in which there is a dispute about the facts of some earlier event, the factfinder cannot acquire unassailably accurate knowledge of what happened. Instead, all the factfinder can acquire is a belief of what probably happened. The intensity of this belief – the degree to which a factfinder is convinced that a given act actually occurred – can, of course, vary. In this regard, a standard of proof represents an attempt to instruct the factfinder concerning the degree of confidence our society thinks he should have in the correctness of factual conclusions for a particular type of adjudication.[1]

After this methodological premise, Justice Harlan explicitly endorses a probabilistic interpretation of standards of proof, using the expression 'degree of confidence' instead of 'probability':

> Although the phrases 'preponderance of the evidence' and 'proof beyond a reasonable doubt' are quantitatively imprecise, they do communicate to the finder of fact different notions concerning the degree of confidence he is expected to have in the correctness of his factual conclusions.

You only talk about Harlan; it would be nice to have more examples of people embracing probabilistic explications.

3.3 Practical worries

The remarks by Justice Harlan notwithstanding, legal practioners seem in general quite opposed to quantifying standards of proof probabiistically. This resistance has many causes. One key factor is certainly the conviction that a probabilistic intepretation of proof standards is unrealistic insofar as its implementation would face unsurmountable challenges. How are probabilities—say the probability of someone's guilt—going to be quantified probabilistically? How will the triers of facts apply probabilistic thresholds? Should the application of the thresholds be automatic—that is, if the evidence is above the

[1] In re Winship, 397 U.S. 358, 370 (1970). This is landmark decision by the United States Supreme Court establishing that the beyond a reasonable doubt standard must be applied to both adults and juvenile defendants.

requisite threshold, find against the defedant (say, convict in a criminal trial) and otherwise find for the defedant (say, acquit)? The challenge, in general, is to articulate how probabilistic thresholds can be operationalized as part of trial decisions. This is by no means obvious. After all, judges and jurors do not weigh evidence in an explicitly probabilistic manner. Nor do they use probability thresholds to guide their decisions.

To alleviate the force of these worries, the probabilistic interpretion of proof standards can be broken down into two separate claims, what we might call the 'quantification claim' and the 'threshold claim'. In a criminal trial, these claims would look as follows:

QUANTIFICATION CLAIM	a probabilistic quantification of the defendant's guilt can be given through an appropriate weighing of all the evidence available (that is, of all the evidence against, and of all the evidence in defense of, the accused).
THRESHOLD CLAIM	an appropriately high threshold guilt probability, say 95%, should be the decision criterion for criminal convictions.

Those worried about implementation might reason thusly. If guilt cannot be quantified probabilistically—for example, in terms of the conditional probability of G given the total evidence E—no probabilistic threshold could ever be used as a decision criterion. Since the quantification claim is unfeasible and the threshold claim rests on the quantification claim, the threshold claim should be rejected.

One way to answer this objection is to bite the bullet. Legal probabilists can admit that probabilistic thresholds constitute a revisionist theory. If they are to be implemented in trial proceedings, they will require changes. Jurors and judges will have to become familiar with probabilistic ideas. They will have to evaluate the strength of the evidence numerically, even for evidence that is not, on its face, quantitative in nature. But this response will simply highten the resistance toward a probabilistic intepretation of proof standards. After all, the likelihood of success of such a program of radical reform of trial proceedings is uncertain. Fortunately, there is a less radical way to respond.

3.4 Idealization

Legal probabilists can admit they are not—at least, not yet—engaged with implementation or trial reform. In fact, the quantification claim can be interpreted in at least two different ways. One interpretation is that a quantification of guilt—understood as an actual reasoning process—can be effectively carried out by the fact-finders. The quantification claim can also be understood as an idealization or a regulative ideal. For instance, the authors of a book on probabilistic inference in forensic science write (Taroni, Biedermann, Bozza, Garbolino, & Aitken, 2014, (p. xv)):

> the . . . [probabilistic] formalism should primarily be considered as an aid to structure and guide one's inferences under uncertainty, rather than a way to reach precise numerical assessments. . . (Taroni et al., 2014, (p. xv))

Even from a probabilist standpoint, the quantification of guilt can well be an idealization which has, primarily, a heuristic role.

Just as the quantification claim can be interpreted in two different ways, the same can be said of the threshold claim. For one thing, we can interpret it as describing an effective decision procedure, as though the fact-finders were required to mechanically convict whenever the defendant's probability of guilt happened to meet the desired probabilistic threshold. But there is a second, and less mechanistic, interpretation of the threshold claim. On the second interpretation, the threshold claim would only describe a way to understand, or theorize about, the standard of proof or the rule of decision. The second interpretation of the threshold claim—which fits well with the 'idealization interpretation' of the quantification claim—is less likely to encounter resistance.

Lawrence Tribe, in his famous 1971 article 'Trial by Mathematics', expresses disdain for a trial process that were mechanically governed by numbers and probabilities. He claims that under this scenario judges and jurors would forget their humanizing function. He writes:

> Guided and perhaps *intimidated by the seeming inexorability of numbers*, induced by the persuasive force of formulas and the precision of decimal points to perceive themselves as performing a largely mechanical and automatic role, *few jurors ... could be relied upon to recall, let alone to perform, [their] humanizing function*. (Tribe, 1971)

But this worry does not apply if we interpret the threshold claim in a non-mechanistic way. This is the interpretation we shall adopt in this chapter. To avoid setting the bar for legal probabilism too high, we

will not be concerned with practical issues that arise if we wanted to deploy a probabilistic threshold directly. We will grant that, at least for now, successful implementation of such thresholds is not viable. For the time being, probabilistic thresholds are best understood as offerring an theoretical, analytical model of trial decisions. The fact that this theoretical model cannot be easily operationalized does not mean that the model is pointless. There are multiple ways in which such a model, even if unfit for direct deployment in trial proceedings, can offer insights into trial decision-making.

3.5 Minimizing expected costs

Here is an illustration of the analytic power of the probabilistic interpretation of proof standards. Standards of proof are usually ranked from the least demanding, such as preponderance of the evidence, to the most demanding, such as proof beyond a reasonable doubt. But why think this way? Can we give a principled justification for the use of multiple standards and this ranking? A common argument is that more is at stake in a criminal trial than in a civil trial. A mistaken conviction will injustly deprive the defedant of basic liberties or even life. Instaed, a mistaken decision in a civil trial would not encroach upon someone's basic liberties since decisions in civil trials are mostly about imposing monetary compensation.

This argument can be made precise by pairing probability thresholds with expected utility theory, a well-establish paradigm of rational decision-making used in psychology and economic theory. At its simplest, decision theory based on the maximization of expected utility states that between a number of alternative courses of action, the one with the highest expected utility (or with the lowest expected cost) should be preferred. This theory is general and can be applied to a variety of situations, including civil or criminal trials.

To see how this works, note that trial decisions can be factually erroneous in two ways. A trial decision can be a false positive—i.e. a decision to hold the defendant liable (to convict, in a criminal case) even though the defedant committed no wrong (or committed no crime). A trial decision can also be a false negative—i.e. a decision not to hold the defendant liable (or to acquit, in a criminal case) even though the defendant did commit the wrong (or committed the crime). Let $\mathsf{cost}(CI)$ and $\mathsf{cost}(AG)$ be the costs associated with the two decisional errors that can be made in a criminal trial, convicting an innocent (CI) and acquitting a guilty defendant (AG). Let $\mathsf{P}(G|E)$ and $\mathsf{P}(I|E)$ be the guilt probability and the innocence probability estimated on the basis of the evidence presented at trial. Given a simple decision-theoretic model (Kaplan, 1968), a conviction should be preferred to an acquittal whenever the expected cost resulting from a mistaken conviction—namely, $\mathsf{P}(I|E) \cdot \mathsf{cost}(CI)$—is lower than the expected cost resulting from a mistaken acquittal—namely, $\mathsf{P}(G|E) \cdot \mathsf{cost}(AG)$. That is,

$$\text{convict provided} \frac{\mathsf{cost}(CI)}{\mathsf{cost}(AG)} < \frac{\mathsf{P}(G|E)}{\mathsf{P}(I|E)}.$$ [2]

For the inequality to hold, the ratio of posterior probabilities $\frac{\mathsf{P}(G|E)}{\mathsf{P}(I|E)}$ should exceed the cost ratio $\frac{\mathsf{cost}(CI)}{\mathsf{cost}(AG)}$. So long as the costs can be quantified, the probability threshold can be determined. For example, consider a cost ratio of nine according to which a mistaken conviction is nine times as costly as a mistaken acquittal. The corresponding probability threshold will be 90%. On this reading, in order to meet the standard of proof beyond a reasonable doubt, the prosecution should provide evidence that establishes the defendant's guilt with at least 90% probability, or in formulas, $\mathsf{P}(G|E) > 90\%$. The higher the cost ratio, the higher the requisite threshold. The lower the cost ratio, the lower the requiste threshold. For example, if the cost ratio is 99, the threshold would be as high as 99%, but if the cost ratio is 2, the threshold would only be 75%.

The same line of argument applies to civil cases. Let a false attribution of liability FL be a decision to find the defendant liable when the defendant committed no civil wrong (analogous to the conviction of an innocent in a criminal case). Let a false attribution of non-liability FNL be a decision not to find the defendant liable when the defendant did commit the civil wrong (analogous to the acquittal a factually guilty defendant in a criminal case). Let $\mathsf{P}(L|E)$ and $\mathsf{P}(NL|E)$ be the liability probability and the non-liability probability given the evidence presented at trial. So long as the objective is to minimize the costs of erroneous decisions, the rule of decision would be as follows:

[2] This follows from $\mathsf{P}(I|E) \cdot \mathsf{cost}(CI) < \mathsf{P}(G|E) \cdot \mathsf{cost}(AG)$.

$$\text{find the defendant civilly liable provided} \frac{\text{cost}(FL)}{\text{cost}(PN)} < \frac{\text{P}(L|E)}{\text{P}(NL|E)}.$$[3]

If the cost ratio $\frac{\text{cost}(FP)}{\text{cost}(PN)}$ is set to 1, the threshold for liability judgments should equal 50%, a common intepretation of the preponderance standard in civil cases. This means that $\text{P}(L|E)$ should be at least 50% for a defedant to be found civilly liable.

The difference between proof standards in civil and criminal cases lies in the different cost ratios. The cost ratio in civil cases, $\frac{\text{cost}(FP)}{\text{cost}(PN)}$, is typically lower than the cost ratio in criminal cases, $\frac{\text{cost}(CI)}{\text{cost}(AG)}$, because a false positive in a criminal trial (a mistaken conviction) is considered a more harmful error than a false positive in a civil trial (a mistaken attribution of civil liability). This difference in the cost ratio can have a consequentialist or a retributivist justification (Walen, 2015). From a consequentialist perspective, the loss of personal freedom or even life can be considered a greater loss than being forced to pay an undue monetary compensation. From a retributivist perspective, the moral wrong that results from the mistaken conviction of an innocent person can be regarded as more egregious than the moral wrong that results from the mistaken attribution of civil liability. This difference in consequences or moral wrongs can be captured by positing a higher cost ratio in criminal than civil cases, $\frac{cost(FP)}{cost(PN)}$.

Justice Harlan of the United Supreme Court draws a clear differencce in the cost ratio between criminal and civil litigation:

> In a civil suit between two private parties for money damages, for example, we view it as no more serious in general for there to be an erroneous verdict in the defendant's favor than for there to be an erroneous verdict in the plaintiff's favor . . . In a criminal case, on the other hand, we do not view the social disutility of convicting an innocent man as equivalent to the disutility of acquitting someone who is guilty. In Re Winship (1970), 397 U. S. 358, 371.

To underscore the differences in the cost ratios, Harlan cites an earlier decision of the United States Supreme Court that emphasizes how a defendant's liberty has a transcending value:

> [t]here is always in litigation a margin of error . . . , representing error in factfinding, which both parties must take into account . . . [w]here one party has at stake an interest of transcending value – as a criminal defendant his liberty – . . . this margin of error is *reduced* as to him by the process of placing on the other party [i.e. the prosecutor] the standard of . . . persuading the factfinder at the conclusion of the trial of his guilt beyond a reasonable doubt. Speiser v. Randall (1958), 357 U.S. 513, 525-26.

This analysis only considers the costs of mistaken decisions, but leaves out the benefits associated with correct decisions. More comprehensive analyses would consider both, but the basic insight would remain the same (Dekay, 1996; Laudan, 2016). Trial decision-making is viewed as one instrument among others for maximizing overall social welfare (Posner, 1973).

Claims about cost ratios and their magnitude, and claims about differences between criminal and civil cases, can of course be contested. Some have argued, for example, that the standard of proof in criminal cases should be lower than commonly assumed, and they have done so by offerring a different assessment of the cost ratio (CITE LAUDAN – PERHASP LAUDAN'S DERIVATION OF A LOWERE THRESHOLD COULD BE DISCUSSED HERE IN DETAIL AS ANOTHER EXAMPLE OF THE ANALYTICAL POWER OF THE PROBABILISTIC FRAMEWORK).

Details aside, probabilistic thresholds, when paired with expected utility theory, provide an analytical framework to justify as well as meaningfully debate the different degrees of stringency necessary for decision criteria—i.e. legal proof standards—in civil or criminal trials. This analytical framework allows for even more finer distinctions, not explictily codified in the law. The law typically makes coarse distinctions between standards of proof, such as `proof beyond a reasonable doubt' for criminal cases`,` preponderance of the evidence' for civil cases and 'clear and convincing evidence' for a narrow subset of civil cases in which the accusation against the defendant is particularly serious. But for rather different crimes, associated with rather different punishments, say murder and grand theft, the same standard of proof is applied for both. It is not obvioys why this should be so, except that a finer distinction may cause more confusion than there need to be. If the probability required for a conviction or a finding of civil liability against the defendant is a function of weighing the costs and benefits that would result from true and false positives (as well as true and false negatives), the stringency of the threshold should depend on costs and benefits, and thus different cases may require

[3] This follows from $\text{P}(NL|E) \cdot \text{cost}(FP) < \text{P}(L|E) \cdot \text{cost}(FNL)$

different thresholds. Cases in which the charge is more serious than others—say, murder compared to grand theft—may require higher thresholds so long as the cost of a mistaken decision against the defendant is more significant. In countries that allow for the death penalty or life imprisonment for certain crimes but not others, the cost of a mistaken conviction would be more serious for crimes with harsher punishments, other things being equal. Thus, the threhold should be placed appropriately higher. We could even think that the threhsold should vary across individual cases even for defendants charged with the exact same crime, provided the costs are different for different individuals. Needless to say, whether or not standards of proof should vary in this way is debated (Kaplow, 2012; Picinali, 2013). Ultimately, the question is what considerations should be admissible in the calculus of costs and benefits.

3.6 SUGGESTION

MARCELLO: IF WE END UP DIVIDING THIS CHAPTER INTO TWO OR THREE SEPARATE CHAPTERS, WE COULD CONTINUE THE DISCUSISON OF THE ANALYTICAL POWER OF THE PROBABILITSTIC APPROACH MORE IN DETAIL HERE, DRAWING ON SOME OF THE MATERIALS ALREADY IN THE LONGER VERSION OF THE SEP ENTRY.

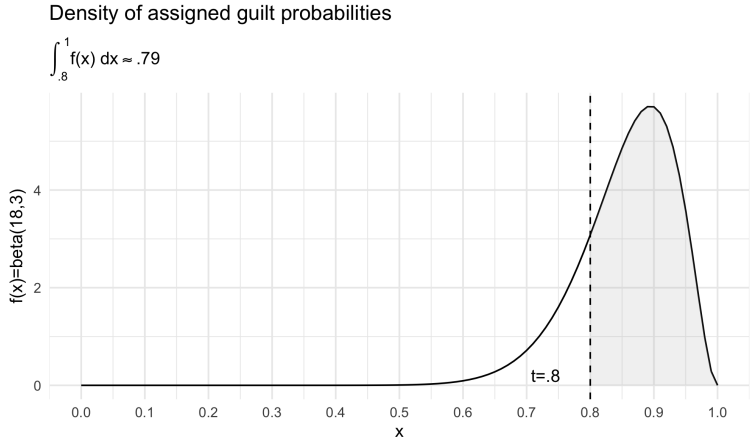
HERE IS A TENTATIVE IDEA OF WHAT TO DISCUSS:

(1) SIMPLE EXPECTED UTILITY MODEL - **DONE, SEE ABOVE**
(2) LAUDAN MODEL, THIS IS A MORE COMPLICATED EXPECTED UTILITY MODEL, PARTLY BORROWED FROM LAPLACE - **YET TO BE DONE**
(3) SIGNAL DETECTION THEORY MODEL - **YET TO BE DONE, ONLY PARTLY DONE**
(4) HAMER MODEL AND KAYE MODEL FOR ERROR MINIMIZATION (DISCUSSED IN THE SEP ENTRY, INTEGRALS, DERIVATIVES, ETC.) - **DONE SEE BELOW**
(5) GOOD AND BAD THINGS ABOUT THESE MODELS, BUT OVERALL THEY SHOW THAT THE PROBABILISTIC FRAMEWORK IS A RICH ANALYTICAL TOOL **YET TO BE DONE**

3.7 Minimizing overall errors

Instead of maximizing expected utility (or minimizing expected costs), standards of proof can be analyzed as decision criteria that have long term effects on the epistemic performance of the trial system. Think about the criminal justice as a whole, making decisions about the guilt and innocence of thousands of defendants facing trial. The system will make a number of decisional errors, committing type I and type II errors. Viewing standards of proof as probability thresholds helps to understand how decisional errors are managed and allocated at this systemic level.

Consider an idealized model of the criminal trial system. Each defendant is assigned a probability x of criminal liability (or guilt) based on the evidence presented at trial. As is customary, this probability ranges between 0 and 1, or 0% and 100%. Since over a period of time many defendants face charges, the guilt probability will have its own distribution. Extreme guilty probabilities set at 0% or 100%, presumably, are assigned rarely in trials if ever, while values between 40% and 80% are more common. A rigorous way to express this distribution is by means of a probability density function, call it $f(x)$. The figure below uses a right skewed distribution, for example, beta(18,3).



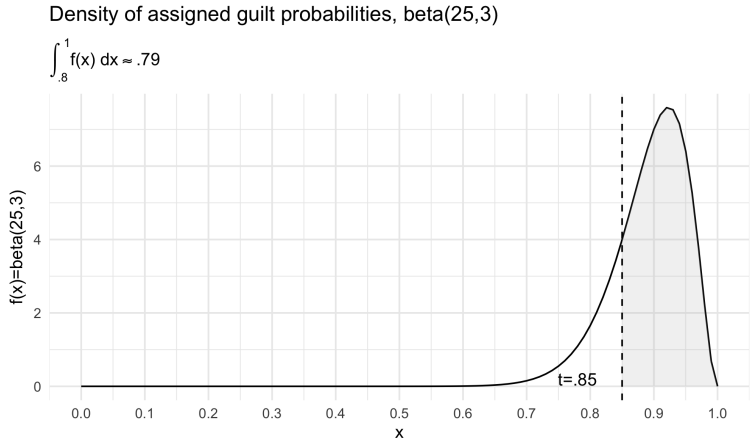
What does the distribution represent? Does it represent the probability of guilt assigned to defendants at

the beginning or the end of the trial? There should be a difference between the two—hopefully—or else trial proceedings would be useless. Suppose that the distribution represents the guilt probabilities as they are assigned to defendants at the end of the trial, once all the evidence, counterevidence, arguments and counterarguments have been proferred and weighed appropriately.

The choice of the distribution is for illustrative purposes only. There are no empirical data suggesting this is the right distribution to use. But its choice is not arbitrary either. The right skew of the distribution reflects the assumption that defendants in criminal cases are prosecuted only if the incriminating evidence against them is strong. It should be no surprise that most defendants are assigned a high probability of guilt. This is plausible in principle. For people should not be prosecuted if the evidence against them is weak. The distribution of the probability of liability in civil cases over a period of time might look quite different, perhaps centered around 50% or 60%.

In the figure above, the threshold for conviction is set at > 80%, and the area under the curve to the right of the threshold is about .79. According to this model, 79% of defendants on trial are convicted and 21% acquitted. These figures are close to the rates of conviction and acquittal in many countries (REFERENCES?). Since $f(x)$ is a probability density, the total area under the curve adds up to 1, encompassing all defendants, both convicted and acquitted defendants.

If the threshold becomes more stringent—for example, it moves up to 85%—the rate of conviction would decrease. This holds provided the underlying distribution does not change. But, if the threshold is set higher, those who are prosecuted will tend to face comparatively stronger evidence and thus the distribution will become more skewed toward the right—say beta(25,3). As a consequence, the rate of conviction could still be about 79% even with a more stringent threshold of 85%.



The two graphs above depict the rate of conviction among those who are facing trial, not the rate of conviction in the general population overall. As just shown, the rate of conviction could remain the same even if the probability threhsold is made more stringent. But, the rate of conviction in the general population is likely to diminish so long as higher thresholds, by acting as deterrents against prosecution, make it less likely that people would be prosecuted .

This formal model does not yet make any distinction between factually guilt and factually innocet defendants. But, presumably, some defendants committed the acts they are accused of and others did not. This is not a clear-cut distinction, however. Some defendants may have committed the acts they are accused of to some extent, but not to the full extent they are accused of, while others may be completely innocent of any crime whatsoever. Leaving this subtlety aside, the formal model can be refined to distinguish between factually innocent and guilty defendants.

The simplest refinement would create two separate distributions, one distribution for the factually innocent defendants and the other for the factually guilty defendants. The problem with this is that we have little idea about what these distributions should look like in the first place. Hopefully, the innocent distribution will be more left skewed and the guilty distribution more right skewed. Guilty defendants should be assigned, on average, higher guilt probabilities than innocent defendants. The two distributions could still overlap to some extent as some guilt defendants could be assigned as low guilt probabilities as some innocent defendants and conversely some innocent defendants could be assigned as high guilt probabilities as some guilty defendants. This is unfortunate, but also an inevitable consequence of the fallibility of the trial system.

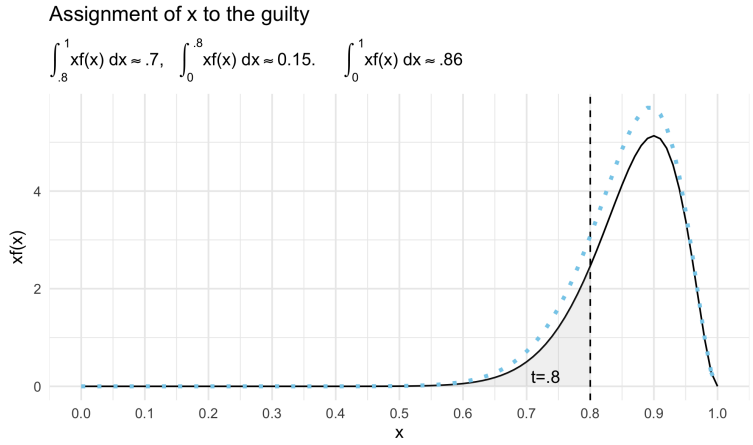
A more principled way to add two separate distributions to the model, one for guilty and another for

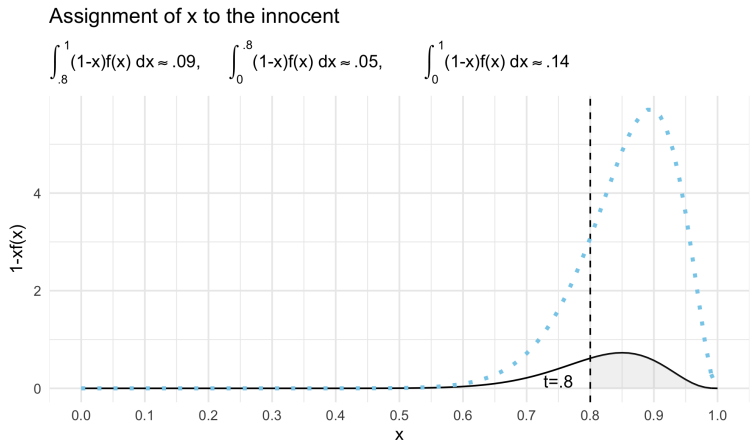
innocent defendants, would be to derive them from the overall distribution of defendants. This can be done by following the simple principle that, among those defendants who are assigned a probability of, say, 80%, there should be a corresponding proportion of 80% guilty people and 20% innocent people. These are of coruse expected values, not actual values. Say you are throwing a fair six-faced die. In the long run, you would expect that in 1/6 of the throws the die would land, say on "4".

The expected proportion of guilty and innocent defendants on trial, out of all defendants, can be inferred from the density distribution $f(x)$ under certain assumptions. Suppose each defendant is assigned a guilt probability based on the best and most complete evidence. From the perspective of judges and jurors (or anyone who has access to the evidence and evaluates it the same way), x% of defendants who are assigned x% guilt probability are expected to be guilty and $(1-x)$% innocent. For example, 85% of defendants who are assigned a 85% guilt probability are expected to be guilty and 15% innocent; 90% of defendants who are assigned a 90% guilt probability are expected to be guilty and 10% innocent; and so on.

So the expected guilty distribution as a function of x will be $xf(x)$, while the expected innocent distribution will be $(1-x)f(x)$. In other words, the function $xf(x)$ describes the (expected) assignment of guilt probabilities for guilty defendants, and similarly, $(1-x)f(x)$ the (expected) assignment of guilt probabilities for innocent defendants. Neither of these functions is a probability density, since $\int_0^1 xf(x)\,\mathrm{d}x = 0.86$ and $\int_0^1 (1-x)f(x)\,\mathrm{d}x = 0.14$. These numbers express the (expected) proportion of guilty and innocent defendants out of all defendants on trial, respectively 86% and 14%.

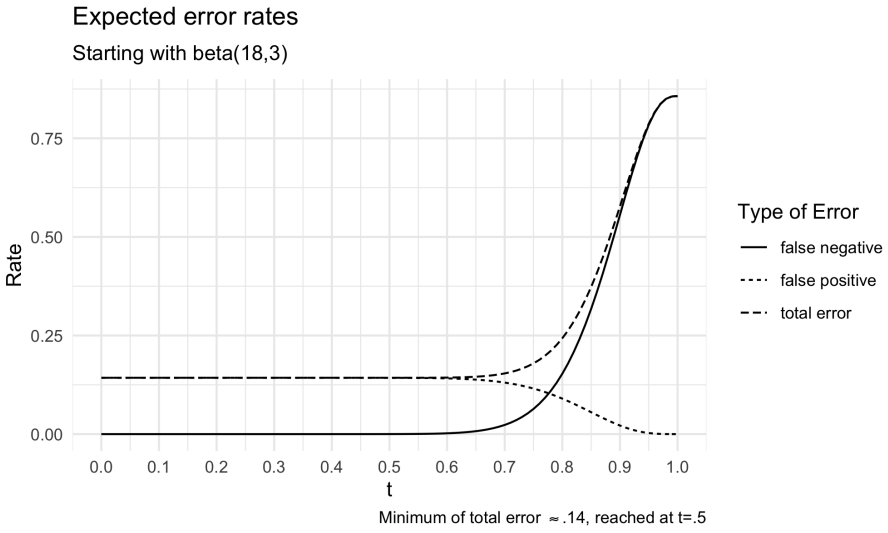
The rates of incorrect decisions—false convictions and false acquittals or more generally false positives and false negatives—can be inferred from this model as a function of the threshold t (Hamer, 2004, 2014). The integral $\int_0^t xf(x)\,\mathrm{d}x$ equals the expected rate of false acquittals, or in other words, the expected proportion of guilty defendants who fall below threshold t (out of all defendants), and the integral $\int_t^1 (1-x)f(x)\,\mathrm{d}x$ equals the expected rate of false convictions, or in other words, the expected proportion of innocent defendants who fall above threshold t (out of all defendants). The rates of correct decisions—true convictions and true acquittals or more generally true positives and true negatives—can be inferred in a similar manner. The integral $\int_t^1 xf(x)\,\mathrm{d}x$ equals the expected rate of true convictions and $\int_0^t (1-x)f(x)\,\mathrm{d}x$ the expected rate of true acquittals. In the figure below, the regions shaded in gray correspond to false negatives (false acquittals) and false positives (false convictions). The remaining white regions within the solid black curve correspond to true positives (true convictions) and true negatives (true acquittals). Note that the dotted blue curve is the original overall distribution for all defendants.





The size of the grey regions in the figures above—which correspond to false positives and false negatives—is affected by the location of threshold t. As t moves upwards, the rate of false positives decreases but the rate of false negatives increases. Conversely, as t moves downwards, the rate of false positives increases but the rate of false negatives decreases. This trade-off is inescapable so long as the underlying distribution is fixed. We have already remarked on the possibility that the distribution would change in shape as a result of changes in the probability threhsold. We will retunr to this point later in the chapter.

Below are both error rates—false positives and false negatives—and their sum plotted against a choice of t, while holding fixed the density function binom(18,3). The graph shows that any threshold that is no greater than 50% would minimize the total error rate %(comprising false positives and false negatives). A more stringent threshold, say > 90%, would instead significantly reduce the rate of false positives but also significantly increase the rate of false negatives, es expected.



In general, the threshold that minimizes the expected rate of incorrect decisions overall, no matter the underlying distribution, lies at 50%. The claim that setting threshold at $t = .5$ minimizes the expected error rate holds given the distribution $f(x)$ =beta(18,3) as well as any other distribution (**???**). To show this, let $E(t)$ %as a function of threshold t be the sum of rates of false positive and false negative decisions:

$$E(t) = \int_0^t xf(x)\,\mathrm{d}x + \int_t^1 (1-x)f(x)\,\mathrm{d}x.$$

The overall rate of error is minimized when $E(t)$ is the lowest. To determine the value of t for which $E(t)$ is the lowest, set the derivative of $E(t)$ %and $R(t)$ to zero, that is, $\frac{d}{dt}E(t) = 0$. By calculus, $t = 1/2$.[4] This claims holds when the two decisional errors are assigned the same weight, or in other words, the

[4]Note that $\frac{d}{dt}E(t)$ is the the sum of the derivatives of $\int_0^t xf(x)\,\mathrm{d}x$ and

costs of false positives and false negatives are symmetric. The > 50% threshold therefore should be most suitable for civil trials. In criminal trials, however, false convictions are typically considered significantly more costly than false acquittals, say a cost ratio of 9:1 (but see (Epps, 2015)). The sum of the two error rates can be weighted by their respective costs:

$$E(t) = \int_0^t x f(x)\,\mathrm{d}x + 9\int_t^1 (1-x) f(x)\,\mathrm{d}x.$$

Given a cost ratio of 9:1, the optimal threshold that minimizes the (weighted) overall rate of error is no longer 1/2, but rather, $t = 9/10 = 90\%$.[5]

Whenever the decision threshold is more stringent than > 50%, the overall (unweighted) error minimization may be sacrificed to pursue other goals, for example, protecting more innocents against mistaken convictions, even at the cost of making a larger number of mistaken trial decisions overall.

The standard 'proof beyond a reasonable doubt' is often paired with the Blackstone ratio, the principle that it is better that ten guilty defendants go free rather than even just one innocent be convicted. The exact ratio is a matter of controversy (Volokh, 1997). It is tempting to think that, say, a 99% threshold guarantees a 1:99 ratio between false convictions and false acquittals. But this would be hasty for at least two reasons. First, probabilistic thresholds affect the expected rate of mistaken decisions. The actual rate may deviate from its expected value (**???**-). Second, if the threshold is 99%, *at most* 1% of decision against defendants are expected to be mistaken (false convictions) and *at most* 99% of the decisions in favor of the defendant are expected to be mistaken (false acquittals). The exact ratio will depend on the probabilities assigned to defendants and how they are distributed (**?**). The (expected) rate of false positives and false negatives—and thus their ratio—depend on where the threshold is located but also on the distribution of the liability probability as given by the density function $f(x)$.

3.8 Interval thresholds (Finkelstein)

The prior probability cannot be easily determined (Friedman, 2000). Even if it can be determined, arriving at a posterior probability might be impractical because of lack of adequate quantitative information. Perhaps, decision thresholds should not rely on a unique posterior probability but on an interval of admissible probabilities given the evidence (Finkelstein & Fairley, 1970). Perhaps, the assessment of the posterior probability of guilt can be viewed as an idealized process, a regulative ideal which can improve the precision of legal reasoning. (CITE BIEDERMAN TARONI).

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4 Theoretical challenges - NEW CHAPTER WOULD STATRT HERE

Let's take stock. We briefly examined difficulties in implementation for probabilistic standards of proof and set those aside. We then offered a few illustrations how probabilistic standards can be used as analytical tools to theorize about decision-making at trial. But even if probabilistic thresholds are used solely as analytical tools, legal probabilists are not yet out of the woods. Even if the practical problems can be addressed or set aside, theoretical difficulties remain. We will focus on three in particular:

$\int_t^1 (1-x) f(x)\,\mathrm{d}x$, that is,

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_0^t x f(x)\,\mathrm{d}x + \frac{d}{dt}\int_t^1 (1-x) f(x)\,\mathrm{d}x.$$

By the fundamental theorem of calculus,

$$\frac{d}{dt}\int_0^t x f(x)\,\mathrm{d}x = t f(t) \text{ and } \frac{d}{dt}\int_t^1 (1-x) f(x)\,\mathrm{d}x = -(1-t) f(t).$$

By plugging in the values,

$$\frac{d}{dt}E(t) = t f(t) - (1-t) f(t).$$

Since $\frac{d}{dt}E(t) = 0$, then $t f(t) = (1-t) f(t)$ and thus $t = 1-t$, so $t = 1/2$ or a > 50% threshold.

[5] The proof is the same as before. Since $t f(t) = 9(1-t) f(t)$, it follows that $t = 9/10$.

the problem of priors; naked statistical evidence; and the difficulty about conjunction, also called the conjunction paradox. The latter two are difficulties that any theory of the standard of proof – not just a probabilistic theory – should be able to address. The first difficulty is peculiar to the probabilistic interpretation of standards of proof. We will examine each difficulty in turn and then examine a promising line of response within legal probabilism based on likelihood ratios instead of posterior probabilities.

4.1 The problem of priors

4.2 Naked statistical evidence

Suppose one hundred, identically dressed prisoners are out in a yard during recreation. Suddenly, ninety-nine of them assault and kill the guard on duty. We know that this is what happened from a video recording, but we do not know the identity of the ninety-nine killers. After the fact, a prisoner is picked at random and tried. Since he is one of the prisoners who were in the yard, the probability of his guilt would be 99%. But despite the high probability, many have the intuition that this is not enough to establish guilt beyond a reasonable doubt. Hypothetical scenarios of this sort suggest that a high probability of guilt, while perhaps necessary, is not sufficient to establish guilt beyond a reasonable doubt.

Perhpas, the resistance in the prisoner scenario lies in the fact that the prisoner was picked at random, and that any prisoner would be 99% likely to be one of the killers. Since the statistics cannot single out the one innocent prisoner, they are bad evidence. But consider this case. Suppose two people enter a department store. There are no other customers in the store. After they exit the store, a member of the staff finds that an item of merchandise is missing. Since no staff member could be culpable—they are strictly surveilled—the culprit must be one of the customers. One of the customers, John, has scored high in a compulsivity test and has been arrested for stealing in department stores several times in the past. The other customer, Rick, has never been arrested for stealing in a department store and shows no sign of high compulsivity. Statistics show that people with a high degree of compulsivity and who have stolen merchandise in department stores before are more likely than others to steal merchandise if they are unsupervised. So John is most likely the culprit. Suppose studies show that people like John, when unsupervised, will steal 99 times out of 100 times. Instead, people like Rick, when unsupervised, will only steal 1 time out of 100 times. So John is 99 times more likely than Rick to have stolen the merchandise. Can these statistics be enough to convict John? Again, it seems not. There is no evidence against him specifically, say, no merchandise was found on him that could link him to the crime. Many would feel uneasy about convicting John despite the fact that, between the two suspects, he is the one who is most likely the culprit.

A similar hypothetical can be constructed for civil cases. Suppose a bus company, Blue-Bus, operates 90% of the buses in town on a certain day, while Red-Bus only 10%. That day a bus injures a pedestrian. Although the buses of the two companies can be easily recognized because they are respectively painted blue and red, the pedestrian who was injured cannot remember the color of the bus involved in the accident. No other witness was around. Still, given the statistics about the market shares of the two companies, it is 90% probable that a Blue-Bus bus was involved in the accident. This is a high probability, well above the 50% threshold. Yet the 90% probability that a Blue-Bus bus was involved in the accident would seem—at least intuitively—insufficient for a judgment of liability against Blue-Bus. This intuition challenges the idea that the preponderance standard in civil cases only requires that the plaintiffi establish the facts with a probability greater than 50%.

Confronted with these hyptheticals, legal probabilists could push back. Hypotheticals rely on intuitive judgments, for example, that the high probability of the prisoners's guilt in the scenario above does not amount to proof beyond a reasonable doubt. But suppose we changed the numbers and imagined there were one thousand prisoners of whom nine hundred and ninety-nine killed the guard. The guilt probability of a prisoner picked at random would be 99.9%. Even in this situation, many would insist that guilt has not been proven beyond a reasonable doubt despite the extremely high probability of guilt. But others might say that when the guilt probability reaches such extreme values, values as high as 99.9% or higher, people's intuitive resistance to convicting should subside (Roth, 2010). A more general problem is that intuitions in such hypothetical scenarios are removed from real cases and thus are potentially unreliable as a guide to theorize about standards of proof (Allen & Leiter, 2001; Hedden & Colyvan, 2019; Lempert, 1986).

Another reason to be suspicious of these hypotheticals is that they seem to amplify biases in human reasoning. Say an eyewitness was present during the accident and testified that a Blue-Bus bus was involved. Intuitively, the testimony would be considered enough to rule against Blue-Bus, at least provided the witness survived cross-examination. We exhibit, in other words, an intuitive preference for judgments of liability based on testimonial evidence compared to judgments based on statistical evidence. This preference has been experimentally verified (Arkes, Shoots-Reinhard, & Mayes, 2012; Niedermeier, Kerr, & Messeé, 1999; Wells, 1992) and exists outside the law (Ebert, Smith, & Durbach, 2018; Friedman & Turri, 2015; Sykes & Johnson, 1999). But testimonial evidence is no less prone to error than statistical evidence. In fact, it may well be more prone to error. The unreliability of eyewitness testimony is well-known, especially when the environmental conditions are not optimal (Loftus, 1996). So are we justified in exhibiting an intuitive preference for eyewitness testimony as opposed to statistical evidence, or is this preference a cognitive bias to avoid?

These reservations notwithstanding, the puzzles about naked statistical evidence cannot be easily dismissed. Puzzles about statistical evidence in legal proof have been around for a while (Cohen, 1977; Kaye, 1979b; Nesson, 1979; Thomson, 1986). Philosophers and legal scholars have shown a renewed interest in both criminal and civil cases (Blome-Tillmann, 2017; Bolinger, 2018; Cheng, 2012; Di Bello, 2019a; Enoch, Spectre, & Fisher, 2012; Ho, 2008; Moss, 2018; Nunn, 2015; Pardo, 2018; Pritchard, 2005; Pundik, 2017; Redmayne, 2008; Roth, 2010; Smith, 2018; Stein, 2005; Wasserman, 1991). Given the growing interest in the topic, legal probabilism cannot be a defensible theoretical position without offering a story about naked statistical evidence.

4.3 Conjuction paradox

Another theoretical difficulty that any theory of the standard of proof should address is the difficulty about conjunction. First formulated by (Cohen, 1977), the difficulty about conjunction has enjoyed a great deal of scholarly attention every since (Allen, 1986; Allen & Stein, 2013; Allen & Pardo, 2019; Haack, 2014; Schwartz & Sober, 2017; Stein, 2005). This difficulty arises when an accusation of wrongdoing, in a civil or criminal proceeding, is broken down into its constituent elements. Suppose that in order to prevail in a criminal trial, the prosecution should establish by the required standard, first, that the defendant caused harm to the victim (call it claim A), and second, that the harmful act was performed with premeditation (call it claim B). Cohen (1977) argues that common law systems subscribe to a conjunction principle, that is, if A and B are established according to the governing standard of proof, so is their conjunction (and vice versa). If the conjunction principle holds, the following must be equivalent, where S is a placeholder for the standard of proof:

Separate	A is established according to S and B is established according to S
Overall	The conjunction $A \wedge B$ is established according to S

In other words, the conjunction principles requires that

$$S[A \wedge B] \text{ iff } S[A] \wedge S[B],$$

where $S[X]$ means that claim or hypothesis X is established according to standard S. The conjunction principle seems consistent with—perhaps even required by—the case law. For example, the United States Supreme Court writes that in criminal cases

> the accused [is protected] against conviction except upon proof beyond a reasonable doubt of *every fact* necessary to constitute the crime with which he is charged. In re Winship (1970), 397 U.S. 358, 364.

A plausible way to interpret this quotation is to posit this identity: to establish someone's guilt beyond a reasonable doubt *just is* to establish each element of the crime beyond a reasonable doubt. That is,

$$BARD[A \wedge B] \text{ iff } BARD[A] \wedge BARD[B],$$

where the conjunction $A \wedge B$ comprises all the material facts that, according to the applicable law, constitute the crime with with the accused is charged.

The trouble is that the conjunction principle is inconsistent with a probabilistic interpretation of the standard of proof. For suppose the prosecution presents evidence that establishes claims A and B, separately, to the required probability, say at least 95%. Has the prosecution met the burden of proof? Each claim was established to the requisite threshold, and thus it was established to the

requisite standard (assuming a probabilistic interpretation of the standard). And if each claim was established to the requisite standard, then guilt as a whole was established to the requisite standard (assuming the conjunction principle). But even though each claim was established to the requisite probability, the probability of their conjunction – assuming the two claims are independent – is only $95\% \times 95\% = 90.25\%$, below the required 95% threshold. So guilt as a whole was *not* established to the requisite standard (assuming a probabilistic interpretation). Hence, the prosecution has met and not met the burden of proof. This is a contradiction.

The difficulty about conjunction – the fact that a probabilistic interpretation of the standard of proof is inconsistent with the conjunction principle – does not subside when the number of constituent claims increases. If anything, the difficulty becomes more apparent. Say the prosecution has established three separate claims to 95% probability, the probability of their conjunction – again if the claims are independent – would be about 85%, even further below the 95% threshold. Nor does the difficulty about conjunction subsides if the claims are no longer regarded as independent. MARCELLO: CAN WE SAY WHY? HOW DO WE EXPLAIN THIS?

Legal probabilists could reject the conjunction principle outright. On its face, the conjunction principle appears to deny the fact that risks accumulate. If one is justifiably sure about the truth of each claim considered seperataly, one should not be equally sure of their conjunction. You have checked each page of a book and found no error. So, for each page, you are nearly sure there is no error. Having checked each page and found no error, can you be sure that the book as a whole contains no error? Not really. As the number of pages grows, it becomes virtually certain that there is at least one error in the book you have overlooked, although for each page you are nearly sure there is no error. The same applies to other contexts, say product quality control. You may be sure, for each product you checked, that it is free from defects. But you cannot, on this basis alone, be sure that all products you checked are free from defects. Since the risks of error accumulate, you must have missed at least one defective product.

There are reasons to push back against this line of arguemnt (more on this soon). But suppose the legal probabilist does away with conjunction principle. Now what? How should they define standards of proof? The legal probabilist has two immediate options. Neither of them is without problems. One option is that the party making the accusation should establish each claim, separately, to the requisite probability without establishing their conjunction to the requisite probability. Call this the *atomistic account*. On this view, the prosecution could establish guilt beyond a reasonable doubt without having estalished guilt with a sufficiently high probability. This accountwould allow convictions in cases in which the probability of the defendant's guilt, call it$G$, is low, just because G is a conjunction of several independent claims that separately satisfy the standard of proof. This runs counter to legal probabilism since it would allow convictions when the defendant is most likely innocent. So the atomistic account is a non-starter.

The other option is to require that the prosecution in a criminal case (or the plantiff in a civil case) establish the claim as a whole – say the cojunction of A and B – to the requisite probability. Call this the *holistic account*. This account is more promising than the atomistic account, but not without problems either. The proof of $A \wedge B$ would impose a higher requirement on the separate probabilities of the conjuncts. If the conjunction $A \wedge B$ is to be proven with at least 95% probability, the individual conjuncts should be established with probability higher than the 95% threshold. So the more conjuncts, the higher their required probability. Moreover, the standard that applies to one of the conjuncts would depend on what has been achieved for the other conjuncts. If $\mathsf{P}(A) = 0.8$, then $\mathsf{P}(B)$ must be at least 0.635 so that $\mathsf{P}(A \wedge B)$ is above a 50% threshold. If, however, $\mathsf{P}(A) = 0.6$, then $\mathsf{P}(B|E)$ must be at least 0.85 to reach the same threshold. Thus, the holistic account would require that the elements of an accusation be proven to different probabilities depeding on how well other claims have been established. This outcome seems counterintuitive.

Denying the conjunction principle, then, is not the solution. The legal probabilist should still explain how individual claims relate to larger claims in the process of legal proof. Say a piece of evidence a (defeasibly) supports hypothesis A and another piece of evidence b (defeasibly) supports B. Evidence a could be evidence that the defendant caused harm to the victim (A), and b evidence that the defendant did so intentionally (B), for example, a recorded phone conversation in which the defendant spoke of a plan to attack the victim. Does the combination of a and b (defeasibly) support the conjunction $A \wedge B$? The answer to this question can well be affirmative—and without denying that risks accumulate. No doubt the possible presence of a defeater D_a would undermine the support of a in favor of A, and another defeater D_b would undermine the support of b in favor of B. But, absent any defeater D_a or D_b,

the combination of items of evidence a and b should jointly support (again, defeasibly or to whatever suitable standard is required) the combined hypothesis $A \wedge B$.

This discussion makes it clear that the conjuction principle is really about the aggregation of different pieces of evidence and hypotheses. Understood this way, the conjuction principle for pieces of evidence a and b, hypotheses A and B, and standard of proof S can be formulated, as follows:

$$\text{S}[a, A] \text{ and } \text{S}[b, B] \text{ iff } \text{S}[a \wedge b, A \wedge B],$$

where $\text{S}[E, H]$ means that evidence E supports hypothesis H by standard S. This statement is a refinement of the earlier one that was formulated only in terms of claims A and B while disregarding the roles of the supporting evidence a and b.

This new formulation of the conjunction principle is compatible with the fact that risks accumulate. Hypothesis A and B could be attacked by a defeater D_a or D_b. The conjunction $A \wedge B$ could be attacked by a larger set of defeaters, including D_a or D_b. So $A \wedge B$ is more susceptible to attack than any individual claim A or B. In this sense, risks do accumulate. Still, even though $A \wedge B$ is more susceptible to attack, it requires a larger body of evidence $a \wedge b$ than each of the individual claim A and B. So, the larger body of evidence in support of $A \wedge B$ balances off its greater susceptibility to error.

Defeasibly logic provides a formal framework to make sense of the conjuction principle.[6] But the challenge for the legal probabilist, at this point, is to spell out a probabilistic version of the conjunction principle for the aggregation of evidence and hypotheses. To this end, the legal probabilist can pursue different strategies. One strategy is a non-starter. Suppose that $\text{P}(A|a) > t$ and $\text{P}(B|b) > t$, for a threshold t. In other words, given the supporting evidence a and b, both A and B are sufficiently probable (for a fixed threshold). It does not follow that $A \wedge B$ is sufficienlty probable given the combined evidence $a \wedge b$, or in syumbols, $\text{P}(A \wedge B|a \wedge b) > t$. By the probability calculus,

$$\begin{aligned} \text{P}(H_1 \wedge |E_1 \wedge E_2) &= \text{P}(H_1|a \wedge b) \times \text{P}(B|a \wedge b \wedge A) \\ &= \text{P}(A|a) \times \text{P}(B|b) \end{aligned}$$

The second equality holds assuming the independence of the hypotheses A and B, and the independence of A from b, and of B from a. In other words, the two hypotheses are thought to be independent of one another, and their suporting evidence are thought to be independent of the other hypothesis. These assumptions are codified in the Baayesian network in Figure 1. So it follows that

$$\text{P}(A \wedge B|a \wedge b) < \text{P}(A|a), \text{P}(B|b)$$

This is a restamenent of the difficulty about conjuction. If each piece of evidence a and b supports claims A and B to 95% probability,
the combined evidence $a \wedge b$ will not support the conjuction $A \wedge B$ to the same 95% threshold. The conjunction principle fails here.

Not all hope is lost for the legal probabilist, however. The support of piece of evidence E in favor of a hypothesis H should not be understood as a function of the conditional probability $\text{P}(H|E)$. We discussed this point extensively in earlier chapters.[7]. The degree of evidential support of E in favor of H consists in the extent to which taking into account E raises the probability that should be assigned to H. A probabilistic measure of the support of E in favor of H is the Bayesian factor $\text{P}(E|H)/\text{P}(E)$. If the Bayesian factor is greater than 1, then E positively supports H. The greater the Bayesian factor (for values above 1), the stronger the support of E in favor of H. The Bayesian factor $\text{P}(E|H)/\text{P}(E)$, unlike the conditional probability $\text{P}(H|E)$, offers a way to overcome the difficulty about conjuction.

Say a supports A and b supports B, to degrees d_a and d_b greater than 1, that is, $\text{P}(a|A)/\text{P}(a) = d_a$ and $\text{P}(b|B)/\text{P}(b) = d_b$, where $d_a, d_b > 1$. Does $a \wedge b$ provide at least as much support in favor of $A \wedge B$ as the individual support provided by the individual pieces of evidence a and b in favor of claims A and B? The support of $a \wedge b$ in favor of $A \wedge B$ should be measured by the combined Bayesian factor $\text{P}(a \wedge b|A \wedge B)/\text{P}(a \wedge b)$? Under suitable independence assumptions, the answer to this question is affirmative. By the probability calculus,

[6] REFERECES?
[7] REFER TO EARLIER CHAPTERS

$$
\begin{aligned}
\frac{\mathsf{P}(a \wedge b | A \wedge B)}{\mathsf{P}(a \wedge b)} &= \frac{\mathsf{P}(A \wedge B | a \wedge b)}{\mathsf{P}(A \wedge B)} \\
&= \frac{\mathsf{P}(A | a \wedge b) \times \mathsf{P}(B | a \wedge b \wedge A)}{\mathsf{P}(A \wedge B)} \\
&= \frac{\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \mathsf{P}(A) \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} \times \mathsf{P}(B)}{\mathsf{P}(A) \times \mathsf{P}(B)} \\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} \\
d_{ab} &= d_a \times d_b
\end{aligned}
$$

The only controversial step is the third equality. This step assumes that (i) a is probabilistically independent of B; (ii) b is probabilistically independent of A; and (iii) that A is probabilistically independent of B. These assumptions may be contested, but they were the same assumptions used to generate the difficulty about conjunction. They assumptions are also independently plausibe, and as noted before, they are codified in the Baayesian network in Figure 1.

Since the combined support d_{ab} equals $d_a \times d_b$, it will always be higher than the individual support provided by the individual pieces of evidence toward the individual hypotheses so long as d_a and d_b are greater than one. Under suitable independence assumptions, then, the difficulty about conjunction disappears if evidential support is not formalized in terms of the conditional probability $\mathsf{P}(H|E)$ but in terms of the Bayesian factor $\frac{\mathsf{P}(E|H)}{\mathsf{P}(E)}$.

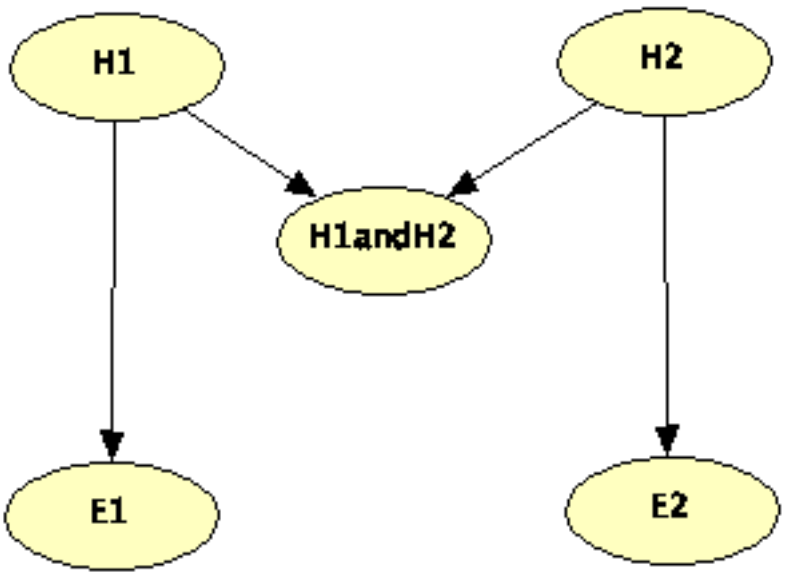

Figure 1: Bayesian network for two pieces of evidence and a combined hypothesis.

5 Specific Narratives [IDEAS OF A SOLUTION TO THEORETICAL DIFFICULTIES]

MARCELLO: I AM NOW HAVING SECOND THOUGHTS (AGAIN!) ABOUT THE STRUCTURE OF THIS CHAPTER. I FEEL THAT IN THIS CHAPTER WE SHOULD ONLY DESCRIBE THE BASIC IDEA FOR A PROBABILISTIC SOLUTION TO NAKED STATS AND CONJUCTION. I THINK I HAVE A SENSE OF WHAT THIS SOLUTION SHOULD LOOK LIKE. THERE WILL THEN BE ANOTHER CHAPTER IN WHICH THIS ROUGH, INFORMAL IEDAS IS SPELLED OUT FORMALLY USING BAYESIAN NETWORKS, BUT THAT CAN WAIT LATER WHEN WE WRITE THE BOOK.

SO, THEN, THERE WOUDL BE ANOTHER CHAPTER IN WHICH WE DISCUSS AND REJECT THE LIKELIHOOD RATIO APPROACH.

SO, BASICALLY, NOW WE HAVE TEH FOLLOWING SAMPLE CHAPTERS:

1. DISCUSSIONS OF THRESHOLDS AS ANALYTICAL TOOLS, UTILITY, ERROR, ETC. (USE LOT OF STUFF FROM ORGINAL SEP ENTRY)
2. THEORETICAL DIFFICULTIES WITH THRESHOLD AND THE PROBABILISTIC SOLUTIONS (I.E. NARRATIVES)
3. REJECTION OF LIKELIHOOD RATIOS AS A GOOD SOLUTION TO THE THEORETICAL DIFFICULTIES

So far we have assumed the most natural probabilistic interpretation of proof standards, one that posits a threshold on the posterior probabilities of a generic hypothesis such as guilt or civil liability. In criminal cases, the requirement is formulated as follows: guilt is proven beyond a reasonable doubt provided $\Pr(G|E)$ is above a suitable threshold, say 95%. The threshold IS lower in civil trials. Civil liability is proven by preponderance provided $\Pr(L|E)$ is above a suitable threshold, say 50%. The claim that the defendant is guilty or civilly liable can be replaced by a more fine-grained hypothesis, call it H_p, the hypothesis put foward by the prosecutor (or the plaintiff in a civil case), for example, the hypothesis that the defendant killed the victim with a firearm while bulglarizing the victim's apartment. H_p can be any hypothesis which, if true, would entail the defendanat is civilly or criminally liable (according to the governing law). Hypothesis H_p is a more precise description of what happened that establishes, if true, the defendant's guilt or civil liability. In defining proof standards, instead of saying – somewhat generically – that $\mathsf{P}(G|E)$ or $\mathsf{P}(L|E)$ should be above a suitable threshold, a probabilistic interpretation could read: civil or criminal liabbility is proven beyond a reasonable doubt provided $\Pr(H_d|E)$ is above a suitable threshold.

This variation may appear inconsequential. But we argue – perhpas surprisingly – it can address the naked statistical evidence problem and the difficulty about conjunction. Consider the prisoner hypothetical. It is true that the naked statistics make him 99% likely to be guilty, that is, $\mathsf{P}(G|E_s)$. It is 99% likely that he is one the prisoners who attacked and killed the guard. Notice that this a generic claim. It is odd for the prosecution to simply assert that the prisoner was one of those who killed the guard, without saying what he did, how he partook in the killing, what role he played in the attack, etc. If the prosecution offerred a more specific incriminating hypothesis, call it H_p, the probability $\mathsf{P}(H_p|E_s)$ of this hypothesis based on the naked statistical evidence E_s would be well below 99%, even though $\mathsf{P}(G|E_s) = 99\%$. The fact the prisoner on trial is most likely guilty is an artifact of the choice of a generic hypothesis G. When this hypthesis is made more specific – as it should be – this probability drops significantly. A more detailed defense of this argument is provided in the rest of this chapter.

Consider now the difficulty about conjunction, focusing again on criminal cases for the sake of concreteness. This difficulty assumes that prosecutors should establish each element of a crime in isolation. If they manage to prove each element to the desired standard, they have meet their burden. This is an artificial view of legal proof. Consider a Washington statute about negligent driving:

> (1)(a) A person is guilty of negligent driving in the first degree if he or she operates a motor vehicle in a manner that is both negligent and endangers or is likely to endanger any person or property, and exhibits the effects of having consumed liquor or marijuana or any drug or exhibits the effects of having inhaled or ingested any chemical, whether or not a legal substance, for its intoxicating or hallucinatory effects. RCW 46.61.5249

In other words, a prosecutor who wishes to establish beyond a reasonable doubt that the defendant is guilty of negligent driving should establish:

> (a) the the defendant operated a vehicle (b) that, in operating a vehicle, the defendant did

so in negligente manner (c) that, in operating a vechicle, the defendant did so in a manner likely to endanger a person pr property (d) that the defendant – presumably, immediately after the incident – exihibited the signs of intoxication by liquor or drugs

These four claims form a common narratives about what happenned. NEED TO COMPLETe THIS. BASIC IDEA IS THAT, FIRST, YOU ESTABLISH THE NARTRATIVES, AND, SECOND, THE NARRATIVE IF TRUE PROVES EACH ELEMENT. SO TO PROVE EACH ELEMENT SIMPLY MEANS TO PROVE A NARRARTIVE FROM WHICH ALL ELEMENTS FOLLOW DEDUCTIVELY. THERE IS NO PINT IN THIKING ABOUT WHETHER EACH ELEMENT HAS BEEN PROVEN.

6 The comparative stratgey

Instead of thinking in terms of absolute probability threshold, standard of proof can be understood comparatively (Cheng, 2012). Say the prosecutor or the plaintiff puts foward a hypthesis H_p about what happened. The defense offers an alternative hypothesis about what happened, call H_d. This may be more common in civil than criminal trials. On this approach, rather than directly evaluating the probability of H_Π given the evidence and comparing it to a threshold, we compare the support that the evidence provides for these hypotheses, and decide for the one for which the evidence provides better support. More specifically,given a body of evidence E and two competing hypotheses H_p and H_d, the probability $\mathsf{P}(H_p|E)$ should be suitably higher than $\mathsf{P}(H_d|E)$, or in other words, the ratio $\frac{\Pr(H_p|E)}{\Pr(H_d|E)}$ should be above a suitable threshold. Presumably, the ratio threshold shoud be higher for criminal than civil cases. In fact, in civil cases it seems enough to require that the ratio $\frac{\Pr(H_p|E)}{\Pr(H_d|E)}$ be avove 1, or in other words, $\mathsf{P}(H_p|E)$ should be higher than $\mathsf{P}(H_d|E)$. Note that H_p and H_d need not be one the negation of the other If they are one the negation of the other, then $\frac{\mathsf{P}(H_p|E)}{\mathsf{P}(H_d|E)} > 1$ implies that $\mathsf{P}(H_p|E) > 50\%$, the standard probabilistic intepretation of the preponderemca standard.

Cheng motivates this approach by the following considerations. Suppose that if the decision is correct, no costs result, but incorrect decisions have their price. Let us say that if the defendant is right and we find against them, the cost is c_1, and if the plaintiff is right and we find against them, the cost is c_2:

		Decision	
		D_Δ	D_Π
Truth	H_Δ	0	c_1
	H_Π	c_2	0

Intuitively, it seems that we want a decision rule which minimizes the expected cost. Say that given our total evidence E the relevant conditional probabilities are:

$$p_\Delta = \mathsf{P}(H_\Delta|E)$$
$$p_\Pi = \mathsf{P}(H_\Pi|E)$$

The expected costs for deciding that H_Δ and H_Π, respectively, are:

$$E(D_\Delta) = p_\Delta 0 + p_\Pi c_2 = c_2 p_\Pi$$
$$E(D_\Pi) = p_\Delta c_1 + p_\Pi 0 = c_1 p_\Delta$$

For this reason, on these assumptions, we would like to choose H_Π just in case $E(D_\Pi) < E(D_\Delta)$. This condition is equivalent to:

$$\begin{aligned} c_1 p_\Delta &< c_2 p_\Pi \\ c_1 &< \frac{c_2 p_\Pi}{p_\Delta} \\ \frac{c_1}{c_2} &< \frac{p_\Pi}{p_\Delta} \end{aligned} \tag{1}$$

Cheng (2012) (1261) insists:

> At the same time, in a civil trial, the legal system expresses no preference between finding erroneously for the plaintiff (false positives) and finding erroneously for the defendant (false negatives). The costs c_1 and c_2 are thus equal…

If we grant this assumption, $c_1 = c_2$, (1) reduces to:

$$1 < \frac{p_\Pi}{p_\Delta}$$
$$p_\Pi > p_\Delta \tag{2}$$

That is, in standard civil litigation we are to find for the plaintiff just in case H_Π is more probable given the evidence than H_Δ, which seems plausible.

This instruction is somewhat more general than the usual suggestion of the preponderance standard in civil litigation, according to which the court should find for the plaintiff just in case $\mathsf{P}(H_\Pi|E) > 0.5$. This threshold, however, results from (2) if it so happens that H_Δ is $\neg H_\Pi$, that is, if the defendant's claim is simply the negation of the plaintiff's thesis. By no means, Cheng argues, this is always the case.

How is RLP supposed to handle DAC? Consider an imaginary case, used by Cheng to discuss this issue. In it, the plaintiff claims that the defendant was speeding (S) and that the crash caused her neck injury (C). Thus, H_Π is $S \wedge C$. Suppose that given total evidence E, the conjuncts, taken separately, meet the decision standard of RLP:

$$\frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \qquad\qquad \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1$$

The question, clearly, is whether $\frac{\mathsf{P}(S \wedge C|E)}{H_\Delta|E} > 1$. But to answer it, we have to decide what H_Δ is. This is the point where Cheng's remark that H_Δ isn't normally simply $\neg H_\Pi$. Instead, he insists, there are three alternative defense scenarios: $H_{\Delta_1} = S \wedge \neg C$, $H_{\Delta_2} = \neg S \wedge C$, and $H_{\Delta_3} = \neg S \wedge \neg C$. How does H_Π compare to each of them? Cheng (assuming independence) argues:

$$\begin{aligned}
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(S \wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(S|E)\mathsf{P}(\neg C|E)} = \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1 \\
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(C|E)} = \frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \\
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)} > 1
\end{aligned} \tag{3}$$

It seems that whatever the defense story is, it is less plausible than the plaintiff's claim. So, at least in this case, whenever elements of a plaintiff's claim satisfy the decision standard proposed by RLP, then so does their conjunction.

Similarly, RLP is claimed to handle the gatecrasher paradox. It is useful to think about the problem in terms of odds and likelikoods, where the *prior odds* (before evidence E) of H_Π as compared to H_Δ, are $\frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)}$, the posterior odds of H_Δ given E are $\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)}$, and the corresponding likelihood ratio is $\frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)}$.

Now, with this notation the *odds form of Bayes' Theorem* tells us that the posterior odds equal the likelihood ratio multiplied by prior odds:

$$\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} = \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} \times \frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)}$$

[@cheng2012reconceptualizing: 1267] insists that in civil trials the prior probabilities should be equal. Granted this assumption, prior odds are 1, and we have:

$$\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} = \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} \tag{4}$$

This means that our original task of establishing that the left-hand side is greater than 1 now reduces to establishing that so is the right-hand side, which means that RLP tells us to convict just in case:

$$\mathsf{P}(E|H_\Pi) > \mathsf{P}(E|H_\Delta) \tag{5}$$

Thus, (5) tells us to convict just in case $LR(E) > 1$.

Now, in the case of the gatecrasher paradox, our evidence is statistical. In our variant E=`991 out of 1000 spectators gatecrashed''. Now pick a random spectator, call him Tom, and let $H_\Pi$`=Tom gatecrashed." (Cheng, 2012: 1270) insists:

> But whether the audience member is a lawful patron or a gatecrasher does not change the probability of observing the evidence presented.

So, on his view, in such a case, $\mathsf{P}(E|H_\Pi) = \mathsf{P}(E|H_\Delta)$, the posterior odds are, by (4), equal to 1, and conviction is unjustified.

There are various issues with how RLP has been deployed to resolve the difficulties that CLP and TLP run into.

First of all, to move from (1) to (2), Cheng assumes that the costs of wrongful decision is the same, be it conviction or acquittal. This is by no means obvious. If a poor elderly lady sues a large company for serious health damage that it supposedly caused, leaving her penniless if the company is liable is definitely not on a par with mistakenly making the company lose a small percent of their funds. Even in cases where such costs are equal, careful consideration and separate argument is needed. If, for instance, $c_1 = 5c_2$, we are to convict just in case $5 < \frac{p_\Pi}{p_\Delta}$. This limits the applicability of Cheng's reasoning about DAC, because his reasoning, if correct (and I will argue that it is not correct later on), yields only the result that the relevant posterior odds are greater than 1, not that they are greater than 5. The difficulty, however, will not have much impact on Cheng's solution of the gatecrasher paradox, as long as $c_1 \leq c_2$. This is because his reasoning, if correct (and I will argue that it is not correct later on), establishes that the relevant posterior odds are below 1, and so below any higher threshold as well.

Secondly, Cheng's resolution of DAC uses another suspicious assumption. For (3) to be acceptable we need to assume that the following pairs of events are independent conditionally on E: $\langle S, C\rangle$, $\langle S, \neg C\rangle$, $\langle \neg S, C\rangle$, $\langle \neg S, \neg C\rangle$. Otherwise, Cheng would not be able to replace conditional probabilities of corresponding conjunctions with the result of multiplication of conditional probabilities of the conjuncts. But it is far from obvious that speeding and neck injury are independent. If, for instance, the evidence makes it certain that if the car was not speeding, the neck injury was not caused by the accident, $\mathsf{P}(\neg S \wedge C|E) = 0$, despite the fact that $\mathsf{P}(\neg S|E)\mathsf{P}(C|E)$ does not have to be 0!

Without independence, the best that we can get, say for the first line of (3), is:

$$\mathsf{P}(S \wedge C|E) = \mathsf{P}(C|E)\mathsf{P}(S|C \wedge E)$$
$$\mathsf{P}(S \wedge \neg C|E) = \mathsf{P}(\neg C|E)\mathsf{P}(S|\neg C \wedge E)$$

and even if we know that $\mathsf{P}(C|E) > \mathsf{P}(\neg C|E)$, this tells us nothing about the comparison of $\mathsf{P}(S \wedge C|E)$ and $\mathsf{P}(S \wedge \neg C|E)$, because the remaining factors can make up for the former inequality.

Perhaps even more importantly, much of the heavy lifting here is done by the strategic splitting of the defense line into multiple scenarios. The result is rather paradoxical. For suppose $\mathsf{P}(H_\Pi|E) = 0.37$ and the probability of each of the defense lines given E is 0.21. This means that H_Π wins with each of the scenarios, so, according to RLP, we should find for the plaintiff. On the other hand, how eager are we to convict once we notice that given the evidence, the accusation is rather false, because $\mathsf{P}(\neg H_\Pi|E) = 0.63$?

The problem generalizes. If, as here, we individualize scenarios by boolean combinations of elements of a case, the more elements there are, into more scenarios $\neg H_\Pi$ needs to be divided. This normally would lead to the probability of each of them being even lower (because now $\mathsf{P}(\neg H_\Pi)$ needs to be "split" between more different scenarios). So, if we take this approach seriously, the more elements a case has, the more at disadvantage the defense is. This is clearly undesirable.

7 The likelihood strategy

Focusing on posterior probabilities is not the only approach that legal probabilists can pursue. By Bayes' theorem, the following holds, using G and I as competing hypotheses:

$$\frac{\Pr(G|E)}{\Pr(I|E)} = \frac{\Pr(E|G)}{\Pr(E|I)} \times \frac{\Pr(G)}{\Pr(I)},$$

or using H_p and H_d as competing hypotheses,

$$\frac{\Pr(H_p|E)}{\Pr(H_d|E)} = \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)},$$

or in words

$$\textit{posterior odds} = \textit{likelihood ratio} \times \textit{prior odds}.$$

A difficult problem is to assign numbers to the prior probabiliteis such as $\Pr(G)$ or $\Pr(H_p)$, or priors odds such as $\frac{\Pr(G)}{\Pr(I)}$ or $\frac{\Pr(H_p)}{\Pr(H_d)}$.

DISCUSS DIFFICULTIES ABOUT ASSIGNING PRIORS! WHERE? CAN WE USE IMPRECISE PROBABILKITIES T TALK ABOUT PRIORS – I.E. LOW PRIORS = TOTAL IGNORANCE = VERY IMPRECISE (LARGE INTERVAL) PRIORS? THE PROBLME WITH THIS WOULD BE THAT THERE IS NO UPDSATING POSSIBLE. ALL UPDATING WOULD STILL GET BACK TO THE STARTING POINT. DO YOU HAVE AN ANSWER TO THAT? WOULD BE INTERETSING TO DISCUSS THIS!

Given these difficulties, both practical and theoretical, one option is to dispense with priors altogether. This is not implausible. Legal disputes in both criminal and civil trials should be decided on the basis of the evidence presented by the litigants. But it is the likelihood ratio – not the prior ratio – that offers the best measure of the overall strength of the evidece presented. So it is all too natural to focus on likekihood ratios and leave the priors out of the picture. If this is the right, the question is, how would a probabilistic interpretation of standards of proof based on the likelihood rato look like? At its simplest, this stratgey will look as follows. Recall our discussion of expected utility theory:

$$\text{convict provided} \frac{cost(CI)}{cost(AG)} < \frac{\Pr(H_p|E)}{\Pr(H_d|E)},$$

which is equivalent to

$$\text{convict provided} \frac{cost(CI)}{cost(AG)} < \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)}.$$

By rearraing the terms,

$$\text{convict provided} \frac{\Pr(E|H_p)}{\Pr(E|H_d)} > \frac{\Pr(H_d)}{\Pr(H_p)} \times \frac{cost(CI)}{cost(AG)}.$$

Then, on this intepretation, the likelihood ratio should be above a suitable threshold that is a function of the cost ratio and the prior ratio. The outstanding question is how this threshold is to be determined.

7.1 Kaplow

Quite independently, a similar approach to juridical decisions has been proposed by Kaplow (2014) – we'll call it **decision-theoretic legal probabilism (DTLP)**. It turns out that Cheng's suggestion is a particular case of this more general approach. Let $LR(E) = \mathsf{P}(E|H_\Pi)/\mathsf{P}(E|H_\Delta)$. In whole generality, DTLP invites us to convict just in case $LR(E) > LR^\star$, where $LR^\star$ is some critical value of the likelihood ratio.

Say we want to formulate the usual preponderance rule: convict iff $\mathsf{P}(H_\Pi|E) > 0.5$, that is, iff $\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} > 1$. By Bayes' Theorem we have:

$$\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} = \frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)} \times \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > 1 \Leftrightarrow$$
$$\Leftrightarrow \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)}$$

So, as expected, $LR^\star$ is not unique and depends on priors. Analogous reformulations are available for thresholds other than 0.5.

Kaplow's point is not that we can reformulate threshold decision rules in terms of priors-sensitive likelihood ratio thresholds. Rather, he insists, when we make a decision, we should factor in its consequences. Let G represent potential gain from correct conviction, and L stand for the potential loss resulting from mistaken conviction. Taking them into account, Kaplow suggests, we should convict if and only if:

$$\mathsf{P}(H_\Pi|E) \times G > \mathsf{P}(H_\Delta|E) \times L \tag{6}$$

Now, (6) is equivalent to:

$$\begin{aligned}
\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} &> \frac{L}{G} \\
\frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)} \times \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} &> \frac{L}{G} \\
\frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} &> \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)} \times \frac{L}{G} \\
LR(E) &> \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)} \times \frac{L}{G} \qquad (7)
\end{aligned}$$

This is the general format of Kaplow's decision standard.

7.2 Dawid

Here is a slightly different perspective, due to Dawid (1987), that also suggests that juridical decisions should be likelihood-based. The focus is on witnesses for the sake of simplicity. Imagine the plaintiff produces two independent witnesses: W_A attesting to A, and W_B attesting to B. Say the witnesses are regarded as 70% reliable and A and B are probabilistically independent, so we infer $\mathsf{P}(A) = \mathsf{P}(B) = 0.7$ and $\mathsf{P}(A \wedge B) = 0.7^2 = 0.49$.

But, Dawid argues, this is misleading, because to reach this result we misrepresented the reliability of the witnesses: 70% reliability of a witness, he continues, does not mean that if the witness testifies that A, we should believe that $\mathsf{P}(A) = 0.7$. To see his point, consider two potential testimonies:

A_1	The sun rose today.
A_2	The sun moved backwards through the sky today.

Intuitively, after hearing them, we would still take $\mathsf{P}(A_1)$ to be close to 1 and $\mathsf{P}(A_2)$ to be close to 0, because we already have fairly strong convictions about the issues at hand. In general, how we should revise our beliefs in light of a testimony depends not only on the reliability of the witness, but also on our prior convictions.[8] And this is as it should be: as indicated by Bayes' Theorem, one and the same testimony with different priors might lead to different posterior probabilities.

So far so good. But how should we represent evidence (or testimony) strength then? Well, one pretty standard way to go is to focus on how much it contributes to the change in our beliefs in a way independent of any particular choice of prior beliefs. Let a be the event that the witness testified that A. It is useful to think about the problem in terms of *odds, conditional odds (O)* and *likelihood ratios (LR)*:

$$\begin{aligned}
O(A) &= \frac{\mathsf{P}(A)}{\mathsf{P}(\neg A)} \\
O(A|a) &= \frac{\mathsf{P}(A|a)}{\mathsf{P}(\neg A|a)} \\
LR(a|A) &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)}.
\end{aligned}$$

Suppose our prior beliefs and background knowledge, before hearing a testimony, are captured by the prior probability measure $\mathsf{P}_{prior}(\cdot)$, and the only thing that we learn is a. We're interested in what our *posterior* probability measure, $\mathsf{P}_{posterior}(\cdot)$, and posterior odds should then be. If we're to proceed with Bayesian updating, we should have:

$$\frac{\mathsf{P}_{posterior}(A)}{\mathsf{P}_{posterior}(\neg A)} = \frac{\mathsf{P}_{prior}(A|a)}{\mathsf{P}_{prior}(\neg A|a)} = \frac{\mathsf{P}_{prior}(a|A)}{\mathsf{P}_{prior}(a|\neg A)} \times \frac{\mathsf{P}_{prior}(A)}{\mathsf{P}_{prior}(\neg A)}$$

that is,

$$O_{posterior}(A) = O_{prior}(A|a) = \underbrace{LR_{prior}(a|A)}_{\text{conditional likelihood ratio}} \times O_{prior}(A) \qquad (8)$$

[8] An issue that Dawid does not bring up is the interplay between our priors and our assessment of the reliability of the witnesses. Clearly, our posterior assessment of the credibility of the witness who testified A_2 will be lower than that of the other witness.

The conditional likelihood ratio seems to be a much more direct measure of the value of a, independent of our priors regarding A itself. In general, the posterior probability of an event will equal to the witness's reliability in the sense introduced above only if the prior is $1/2$.[9]

7.3 Likelihood and DAC

But how does our preference for the likelihood ratio as a measure of evidence strength relate to DAC? Let's go through Dawid's reasoning.

A sensible way to probabilistically interpret the 70% reliability of a witness who testifies that A is to take it to consist in the fact that the probability of a positive testimony if A is the case, just as the probability of a negative testimony (that is, testimony that A is false) if A isn't the case, is 0.7:[10]

$$\mathsf{P}_{prior}(a|A) = \mathsf{P}_{prior}(\neg a|\neg A) = 0.7.$$

$\mathsf{P}_{prior}(a|\neg A) = 1 - \mathsf{P}_{prior}(\neg a|\neg A) = 0.3$, and so the same information is encoded in the appropriate likelihood ratio:

$$LR_{prior}(a|A) = \frac{\mathsf{P}_{prior}(a|A)}{\mathsf{P}_{prior}(a|\neg A)} = \frac{0.7}{0.3}$$

Let's say that *a provides (positive) support* for A in case

$$O_{posterior}(A) = O_{prior}(A|a) > O_{prior}(A)$$

that is, a testimony a supports A just in case the posterior odds of A given a are greater than the prior odds of A (this happens just in case $\mathsf{P}_{posterior}(A) > \mathsf{P}_{prior}(A)$). By (8), this will be the case if and only if $LR_{prior}(a|A) > 1$.

One question that Dawid addresses is this: assuming reliability of witnesses 0.7, and assuming that a and b, taken separately, provide positive support for their respective claims, does it follow that $a \wedge b$ provides positive support for $A \wedge B$?

Assuming the independence of the witnesses, this will hold in non-degenerate cases that do not involve extreme probabilities, on the assumption of independence of a and b conditional on all combinations: $A \wedge B$, $A \wedge \neg B$, $\neg A \wedge B$ and $\neg A \wedge \neg B$.[11,~12]

Let us see why the above claim holds. The calculations are my reconstruction and are not due to Dawid. The reader might be annoyed with me working out the mundane details of Dawid's claims, but

[9] Dawid gives no general argument, but it is not too hard to give one. Let $rel(a) = \mathsf{P}(a|A) = \mathsf{P}(\neg a|\neg A)$. We have in the background $\mathsf{P}(a|\neg A) = 1 - \mathsf{P}(\neg a|\neg A) = 1 - rel(a)$. We want to find the condition under which $\mathsf{P}(A|a) = \mathsf{P}(a|A)$. Set $\mathsf{P}(A) = p$ and start with Bayes' Theorem and the law of total probability, and go from there:

$$\begin{aligned}
\mathsf{P}(A|a) &= \mathsf{P}(a|A) \\
\frac{\mathsf{P}(a|A)p}{\mathsf{P}(a|A)p + \mathsf{P}(a|\neg A)(1-p)} &= \mathsf{P}(a|A) \\
\mathsf{P}(a|A)p &= \mathsf{P}(a|A)[\mathsf{P}(a|A)p + \mathsf{P}(a|\neg A)(1-p)] \\
p &= \mathsf{P}(a|A)p + \mathsf{P}(a|\neg A) - \mathsf{P}(a|\neg A)p \\
p &= rel(a)p + 1 - rel(a) - (1 - rel(a))p \\
p &= rel(a)p + 1 - rel(a) - p + rel(a)p \\
2p &= 2rel(a)p + 1 - rel(a) \\
2p - 2rel(a)p &= 1 - rel(a) \\
2p(1 - rel(a)) &= 1 - rel(a) \\
2p &= 1
\end{aligned}$$

First we multiplied both sides by the denominator. Then we divided both sides by $\mathsf{P}(a|A)$ and multiplied on the right side. Then we used our background notation and information. Next, we manipulated the right-hand side algebraically and moved $-p$ to the left-hand side. Move $2rel(a)p$ to the left and manipulate the result algebraically to get to the last line.

[10] In general setting, these are called the *sensitivity* and *specificity* of a test (respectively), and they don't have to be equal. For instance, a degenerate test for an illness which always responds positively, diagnoses everyone as ill, and so has sensitivity 1, but specificity 0.

[11] Dawid only talks about the independence of witnesses without reference to conditional independence. Conditional independence does not follow from independence, and it is the former that is needed here (also, four non-equivalent different versions of it).

[12] In terms of notation and derivation in the optional content that will follow, the claim holds if and only if $28 > 28p_{11} - 12p_{00}$. This inequality is not true for all admissible values of p_{11} and p_{00}. If $p_{11} = 1$ and $p_{00} = 0$, the sides are equal. However, this is a rather degenerate example. Normally, we are interested in cases where $p_{11} < 1$. And indeed, on this assumption, the inequality holds.

it turns out that in the case of Dawid's strategy, the devil is in the details. The independence of witnesses gives us:

$$\begin{aligned}
\mathsf{P}(a \wedge b | A \wedge B) &= 0.7^2 = 0.49 \\
\mathsf{P}(a \wedge b | A \wedge \neg B) &= 0.7 \times 0.3 = 0.21 \\
\mathsf{P}(a \wedge b | \neg A \wedge B) &= 0.3 \times 0.7 = 0.21 \\
\mathsf{P}(a \wedge b | \neg A \wedge \neg B) &= 0.3 \times 0.3 = 0.09
\end{aligned}$$

Without assuming A and B to be independent, let the probabilities of $A \wedge B$, $\neg A \wedge B$, $A \wedge \neg B$, $\neg A \wedge \neg B$ be $p_{11}, p_{01}, p_{10}, p_{00}$. First, let's see what $\mathsf{P}(a \wedge b)$ boils down to.

By the law of total probability we have:

$$\begin{aligned}
\mathsf{P}(a \wedge b) = {} & \mathsf{P}(a \wedge b | A \wedge B)\mathsf{P}(A \wedge B) + \\
& + \mathsf{P}(a \wedge b | A \wedge \neg B)\mathsf{P}(A \wedge \neg B) \\
& + \mathsf{P}(a \wedge b | \neg A \wedge B)\mathsf{P}(\neg A \wedge B) + \\
& + \mathsf{P}(a \wedge b | \neg A \wedge \neg B)\mathsf{P}(\neg A \wedge \neg B)
\end{aligned} \tag{9}$$

which, when we substitute our values and constants, results in:

$$= 0.49p_{11} + 0.21(p_{10} + p_{01}) + 0.09p_{00}$$

Now, note that because p_{ii}s add up to one, we have $p_{10} + p_{01} = 1 - p_{00} - p_{11}$. Let us continue.

$$\begin{aligned}
&= 0.49p_{11} + 0.21(1 - p_{00} - p_{11}) + 0.09p_{00} \\
&= 0.21 + 0.28p_{11} - 0.12p_{00}
\end{aligned}$$

Next, we ask what the posterior of $A \wedge B$ given $a \wedge b$ is (in the last line, we also multiply the numerator and the denominator by 100).

$$\begin{aligned}
\mathsf{P}(A \wedge B | a \wedge b) &= \frac{\mathsf{P}(a \wedge b | A \wedge B)\mathsf{P}(A \wedge B)}{\mathsf{P}(a \wedge b)} \\
&= \frac{49p_{11}}{21 + 28p_{11} - 12p_{00}}
\end{aligned}$$

In this particular case, then, our question whether $\mathsf{P}(A \wedge B | a \wedge b) > \mathsf{P}(A \wedge B)$ boils down to asking whether

$$\frac{49p_{11}}{21 + 28p_{11} - 12p_{00}} > p_{11}$$

that is, whether $28 > 28p_{11} - 12p_{00}$ (just divide both sides by p_{11}, multiply by the denominator, and manipulate algebraically).

Dawid continues working with particular choices of values and provides neither a general statement of the fact that the above considerations instantiate nor a proof of it. In the middle of the paper he says:

> Even under prior dependence, the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability... When the problem is analysed carefully, the 'paradox' evaporates [pp. 95-7]

where he still means the case with the particular values that he has given, but he seems to suggest that the claim generalizes to a large array of cases.

The paper does not contain a precise statement making the conditions required explicit and, *a fortriori*, does not contain a proof of it. Given the example above and Dawid's informal reading, let us develop a more precise statement of the claim and a proof thereof.

Fact 1. *Suppose that* $rel(a), rel(b) > 0.5$ *and witnesses are independent conditional on all Boolean combinations of A and B (in a sense to be specified), and that none of the Boolean combinations of A and B has an extreme probability (of 0 or 1). It follows that* $\mathsf{P}(A \wedge B | a \wedge b) > \mathsf{P}(A \wedge B)$. *(Independence of A and B is not required.)*

Roughly, the theorem says that if independent and reliable witnesses provide positive support of their separate claims, their joint testimony provides positive support of the conjunction of their claims.

Let us see why the claim holds. First, we introduce an abbreviation for witness reliability:

$$\mathbf{a} = rel(a) = \mathsf{P}(a|A) = \mathsf{P}(\neg a|\neg A) > 0.5$$
$$\mathbf{b} = rel(b) = \mathsf{P}(b|B) = \mathsf{P}(\neg b|\neg A) > 0.5$$

Our independence assumption means:

$$\mathsf{P}(a \wedge b|A \wedge B) = \mathbf{ab}$$
$$\mathsf{P}(a \wedge b|A \wedge \neg B) = \mathbf{a}(\mathbf{1} - \mathbf{b})$$
$$\mathsf{P}(a \wedge b|\neg A \wedge B) = (\mathbf{1} - \mathbf{a})\mathbf{b}$$
$$\mathsf{P}(a \wedge b|\neg A \wedge \neg B) = (\mathbf{1} - \mathbf{a})(\mathbf{1} - \mathbf{b})$$

Abbreviate the probabilities the way we already did:

$$\mathsf{P}(A \wedge B) = p_{11} \qquad \mathsf{P}(A \wedge \neg B) = p_{10}$$
$$\mathsf{P}(\neg A \wedge B) = p_{01} \qquad \mathsf{P}(\neg A \wedge \neg B) = p_{00}$$

Our assumptions entail $0 \neq p_{ij} \neq 1$ for $i, j \in \{0, 1\}$ and:

$$p_{11} + p_{10} + p_{01} + p_{00} = 1 \tag{10}$$

So, we can use this with (9) to get:

$$\begin{aligned} \mathsf{P}(a \wedge b) &= \mathbf{ab}p_{11} + \mathbf{a}(\mathbf{1} - \mathbf{b})p_{10} + (\mathbf{1} - \mathbf{a})\mathbf{b}p_{01} + (\mathbf{1} - \mathbf{a})(\mathbf{1} - \mathbf{b})p_{00} \\ &= p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab}) \end{aligned} \tag{11}$$

Let's now work out what the posterior of $A \wedge B$ will be, starting with an application of the Bayes' Theorem:

$$\begin{aligned} \mathsf{P}(A \wedge B|a \wedge b) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B)}{\mathsf{P}(a \wedge b)} \\ &= \frac{\mathbf{ab}p_{11}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})} \end{aligned} \tag{12}$$

To answer our question we therefore have to compare the content of (12) to p_{11} and our claim holds just in case:

$$\frac{\mathbf{ab}p_{11}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})} > p_{11}$$

$$\frac{\mathbf{ab}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})} > 1$$

$$p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab}) < \mathbf{ab} \tag{13}$$

Proving (13) is therefore our goal for now. This is achieved by the following reasoning:[13]

1.	$\mathbf{b} > 0.5,\ \mathbf{a} > 0.5$	assumption
2.	$2\mathbf{b} > 1,\ 2\mathbf{a} > 1$	from 1.
3.	$2\mathbf{ab} > \mathbf{a},\ 2\mathbf{ab} > \mathbf{b}$	multiplying by $\mathbf{a}$ and $\mathbf{b}$ respectively
4.	$p_{10}2\mathbf{ab} > p_{10}\mathbf{a},\ \ p_{01}2\mathbf{ab} > p_{01}\mathbf{b}$	multiplying by p_{10} and p_{01} respectively
5.	$p_{10}2\mathbf{ab} + p_{01}2\mathbf{ab} > p_{10}\mathbf{a} + p_{01}\mathbf{b}$	adding by sides, 3., 4.
6.	$1 - \mathbf{b} - \mathbf{a} < 0$	from 1.
7.	$p_{00}(1 - \mathbf{b} - \mathbf{a}) < 0$	From 6., because $p_{00} > 0$
8.	$p_{10}2\mathbf{ab} + p_{01}2\mathbf{ab} > p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1 - \mathbf{b} - \mathbf{a})$	from 5. and 7.
9.	$p_{10}\mathbf{ab} + p_{10}\mathbf{ab} + p_{01}\mathbf{ab} + p_{01}\mathbf{ab} + p_{00}\mathbf{ab} - p_{00}\mathbf{ab} > p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1 - \mathbf{b} - \mathbf{a})$	8., rewriting left-hand side
10.	$p_{10}\mathbf{ab} + p_{01}\mathbf{ab} + p_{00}\mathbf{ab} > -p_{10}\mathbf{ab} - p_{01}\mathbf{ab} + p_{00}\mathbf{ab} + p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1 - \mathbf{b} - \mathbf{a})$	9., moving from left to right
11.	$\mathbf{ab}(p_{10} + p_{01} + p_{00}) > p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})$	10., algebraic manipulation
12.	$\mathbf{ab}(1 - p_{11}) > p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})$	11. and equation (10)
13.	$\mathbf{ab} - \mathbf{ab}p_{11} > p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})$	12., algebraic manipulation
14.	$\mathbf{ab} > \mathbf{ab}p_{11} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})$	13., moving from left to right

%\end{adjustbox}

The last line is what we have been after.

Optional Content Ends

Now that we have as a theorem an explication of what Dawid informally suggested, let's see whether it helps the probabilist handling of DAC.

[13] Thanks to Pawel Pawlowski for working on this proof with me.

7.4 Kaplow

On RLP, at least in certain cases, the decision rule leads us to (5), which tells us to decide the case based on whether the likelihood ratio is greater than 1.

[14] While Kaplow did not discuss DAC or the gatecrasher paradox, it is only fair to evaluate Kaplow's proposal from the perspective of these difficulties.

Add here stuff from Marcello's Mind paper about the prisoner hypothetical. Then, discuss Rafal's critique of the likelihood ratio threshold and see where we end up.

8 Challenges (again)

8.1 Likelihood ratio and the problem of the priors

8.2 Dawid's likelihood strategy doesn't help

Recall that DAC was a problem posed for the decision standard proposed by TLP, and the real question is how the information resulting from Fact 1 can help to avoid that problem. Dawid does not mention any decision standard, and so addresses quite a different question, and so it is not clear that '`the`paradox' evaporates", as Dawid suggests.

What Dawid correctly suggests (and we establish in general as Fact 1) is that the support of the conjunction by two witnesses will be positive as soon as their separate support for the conjuncts is positive. That is, that the posterior of the conjunction will be higher that its prior. But the critic of probabilism never denied that the conjunction of testimonies might raise the probability of the conjunction if the testimonies taken separately support the conjuncts taken separately. Such a critic can still insist that Fact 1 does nothing to alleviate her concern. After all, at least *prima facie* it still might be the case that:

- the posterior probabilities of the conjuncts are above a given threshold,
- the posterior probability of the conjunction is higher than the prior probability of the conjunction,
- the posterior probability of the conjunction is still below the threshold.

That is, Fact 1 does not entail that once the conjuncts satisfy a decision standard, so does the conjunction.

At some point, Dawid makes a general claim that is somewhat stronger than the one already cited:

> When the problem is analysed carefully, the 'paradox' evaporates: suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents.
> [p. 97]

This is quite a different claim from the content of Fact 1, because previously the joint probability was claimed only to increase as compared to the prior, and here it is claimed to increase above the level of the separate increases provided by separate testimonies. Regarding this issue Dawid elaborates (we still use the p_{ij}-notation that we've already introduced):

> "More generally, let $\mathsf{P}(a|A)/\mathsf{P}(a|\neg A) = \lambda$, $\mathsf{P}(b|B)/\mathsf{P}(b|\neg B) = \mu$, with $\lambda, \mu > 0.7$, as might arise, for example, when there are several available testimonies. If the witnesses are independent, then
>
> $$\mathsf{P}(A \wedge B|a \wedge b) = \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00})$$
>
> which increases with each of λ and μ, and is never less than the larger of $\lambda p_{11}/(1 - p_{11} + \lambda p_{11}), \mu p_{11}/(1 - p_{11}1 + \mu p_{11})$, the posterior probabilities appropriate to the individual testimonies." [p. 95]

This claim, however, is false.

Optional Content Starts

Let us see why. The quoted passage is a bit dense. It contains four claims for which no arguments are given in the paper. The first three are listed below as (14), the fourth is that if the conditions in (14) hold, $\mathsf{P}(A \wedge B|a \wedge b) > max(\mathsf{P}(A|a), \mathsf{P}(B|b))$. Notice that $\lambda = LR(a|A)$ and $\mu = LR(b|B)$. Suppose the

[14] Again, the name of the view is by no means standard, it is just a term I coined to refer to various types of legal probabilism in a fairly uniform manner.

first three claims hold, that is:

$$\mathsf{P}(A \wedge B|a \wedge b) = \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00}) \tag{14}$$
$$\mathsf{P}(A|a) = \frac{\lambda p_{11}}{1 - p_{11} + \lambda p_{11}}$$
$$\mathsf{P}(B|b) = \frac{\mu p_{11}}{1 - p_{11} + \mu p_{11}}$$

Is it really the case that $\mathsf{P}(A \wedge B|a \wedge b) > \mathsf{P}(A|a), \mathsf{P}(B|b)$? It does not seem so. Let $\mathbf{a} = \mathbf{b} = 0.6$, $pr = \langle p_{11}, p_{10}, p_{01}, p_{00}\rangle = \langle 0.1, 0.7, 0.1, 0.1\rangle$. Then, $\lambda = \mu = 1.5 > 0.7$ so the assumption is satisfied. Then we have $\mathsf{P}(A) = p_{11} + p_{10} = 0.8$, $\mathsf{P}(B) = p_{11} + p_{01} = 0.2$. We can also easily compute $\mathsf{P}(a) = \mathbf{a}\mathsf{P}(A) + (1 - \mathbf{a})\mathsf{P}(\neg A) = 0.56$ and $\mathsf{P}(b) = \mathbf{b}\mathsf{P}(B) + (1 - \mathbf{b})\mathsf{P}(\neg B) = 0.44$. Yet:

$$\mathsf{P}(A|a) = \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(a)} = \frac{0.6 \times 0.8}{0.6 \times 0.8 + 0.4 \times 0.2} \approx 0.8571$$
$$\mathsf{P}(B|b) = \frac{\mathsf{P}(b|B)\mathsf{P}(B)}{\mathsf{P}(b)} = \frac{0.6 \times 0.2}{0.6 \times 0.2 + 0.4 \times 0.8} \approx 0.272$$
$$\begin{aligned}\mathsf{P}(A \wedge B|a \wedge b) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B)}{\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B) + \mathsf{P}(a \wedge b|A \wedge \neg B)\mathsf{P}(A \wedge \neg B) +} \\ &\quad + \mathsf{P}(a \wedge b|\neg A \wedge B)\mathsf{P}(\neg A \wedge B) + \mathsf{P}(a \wedge b|\neg A \wedge \neg B)\mathsf{P}(\neg A \wedge \neg B) \\ &= \frac{\mathbf{ab}p_{11}}{\mathbf{ab}p_{11} + \mathbf{a}(1-\mathbf{b})p_{10} + (1-\mathbf{a})\mathbf{b}p_{01} + (1-\mathbf{a})(1-\mathbf{b})p_{00}} \approx 0.147\end{aligned}$$

The posterior probability of $A \wedge B$ is not only lower than the larger of the individual posteriors, but also lower than any of them!

So what went wrong in Dawid's calculations in (14)? Well, the first formula is correct. However, let us take a look at what the second one says (the problem with the third one is pretty much the same):

$$\mathsf{P}(A|a) = \frac{\frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A \wedge B)}{\mathsf{P}(\neg(A \wedge B)) + \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A \wedge B)}$$

Quite surprisingly, in Dawid's formula for $\mathsf{P}(A|a)$, the probability of $A \wedge B$ plays a role. To see that it should not take any B that excludes A and the formula will lead to the conclusion that *always* $\mathsf{P}(A|a)$ is undefined. The problem with Dawid's formula is that instead of $p_{11} = \mathsf{P}(A \wedge B)$ he should have used $\mathsf{P}(A) = p_{11} + p_{10}$, in which case the formula would rather say this:

$$\begin{aligned}\mathsf{P}(A|a) &= \frac{\frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A)}{\mathsf{P}(\neg A) + \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A)} \\ &= \frac{\frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)}}{\frac{\mathsf{P}(\neg a|A)\mathsf{P}(\neg A)}{\mathsf{P}(\neg a|A)} + \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)}} \\ &= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)\mathsf{P}(\neg A) + \mathsf{P}(a|A)\mathsf{P}(A)}\end{aligned}$$

Now, on the assumption that witness' sensitivity is equal to their specificity, we have $\mathsf{P}(a|\neg A) = \mathsf{P}(\neg a|A)$ and can substitute this in the denominator:

$$= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(a|\neg A)\mathsf{P}(\neg A) + \mathsf{P}(a|A)\mathsf{P}(A)}$$

and this would be a formulation of Bayes' theorem. And indeed with $\mathsf{P}(A) = p_{11} + p_{10}$ the formula works (albeit its adequacy rests on the identity of $\mathsf{P}(a|\neg A)$ and $\mathsf{P}(\neg a|A)$), and yields the result that we already obtained:

$$\begin{aligned}\mathsf{P}(A|a) &= \frac{\lambda(p_{11} + p_{10})}{1 - (p_{11} + p_{10}) + \lambda(p_{11} + p_{10})} \\ &= \frac{1.5 \times 0.8}{1 - 0.8 + 1.5 \times 0.8} \approx 0.8571\end{aligned}$$

The situation cannot be much improved by taking $\mathbf{a}$ and $\mathbf{b}$ to be high. For instance, if they're both 0.9 and $pr = \langle 0.1, 0.7, 0.1, 0.1\rangle$, the posterior of A is ≈ 0.972, the posterior of B is ≈ 0.692, and yet the joint posterior of $A \wedge B$ is 0.525.

The situation cannot also be improved by saying that at least if the threshold is 0.5, then as soon as $\mathbf{a}$ and $\mathbf{b}$ are above 0.7 (and, *a fortriori*, so are λ and μ), the individual posteriors being above 0.5 entails the joint posterior being above 0.5 as well. For instance, for $\mathbf{a} = 0.7$ and $\mathbf{b} = 0.9$ with $pr = \langle 0.1, 0.3, 0.5, 0.1\rangle$, the individual posteriors of A and B are ≈ 0.608 and ≈ 0.931 respectively, while the joint posterior of $A \wedge B$ is ≈ 0.283.

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The situation cannot be improved by saying that what was meant was rather that the joint likelihood is going to be at least as high as the maximum of the individual likelihoods, because quite the opposite is the case: the joint likelihood is going to be lower than any of the individual ones.

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Let us make sure this is the case. We have:

$$\begin{aligned} LR(a|A) &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)} \\ &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \\ &= \frac{\mathbf{a}}{\mathbf{1}-\mathbf{a}}. \end{aligned}$$

where the substitution in the denominator is legitimate only because witness' sensitivity is identical to their specificity.

With the joint likelihood, the reasoning is just a bit more tricky. We will need to know what $\mathsf{P}(a \wedge b|\neg(A \wedge B))$ is. There are three disjoint possible conditions in which the condition holds: $A \wedge \neg B, \neg A \wedge B$, and $\neg A \wedge \neg B$. The probabilities of $a \wedge b$ in these three scenarios are respectively $\mathbf{a}(\mathbf{1}-\mathbf{b}), (\mathbf{1}-\mathbf{a})\mathbf{b}, (\mathbf{1}-\mathbf{a})(\mathbf{1}-\mathbf{b})$ (again, the assumption of independence is important), and so on the assumption $\neg(A \wedge B)$ the probability of $a \wedge b$ is:

$$\begin{aligned} \mathsf{P}(a \wedge b|\neg(A \wedge B)) &= \mathbf{a}(1-\mathbf{b}) + (1-\mathbf{a})\mathbf{b} + (1-\mathbf{a})(1-\mathbf{b}) \\ &= \mathbf{a}(1-\mathbf{b}) + (1-\mathbf{a})(\mathbf{b}+1-\mathbf{b}) \\ &= \mathbf{a}(1-\mathbf{b}) + (1-\mathbf{a}) \\ &= \mathbf{a} - \mathbf{ab} + 1 - \mathbf{a} = 1 - \mathbf{ab} \end{aligned}$$

So, on the assumption of witness independence, we have:

$$\begin{aligned} LR(a \wedge b|A \wedge B) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))} \\ &= \frac{\mathbf{ab}}{\mathbf{1}-\mathbf{ab}} \end{aligned}$$

With $0 < \mathbf{a}, \mathbf{b} < 1$ we have $\mathbf{ab} < \mathbf{a}$, $1 - \mathbf{ab} > 1 - \mathbf{a}$, and consequently:

$$\frac{\mathbf{ab}}{\mathbf{1}-\mathbf{ab}} < \frac{\mathbf{a}}{\mathbf{1}-\mathbf{a}}$$

which means that the joint likelihood is going to be lower than any of the individual ones.

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Fact 1 is so far the most optimistic reading of the claim that if witnesses are independent and fairly reliable, their testimonies are going to provide positive support for the conjunction,\footnote{And this is the reading that Dawid in passing suggests: "the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability." (Dawid, 1987: 95) and any stronger reading of Dawid's suggestions fails. But Fact 1 is not too exciting when it comes to answering the original DAC. The original question focused on the adjudication model according to which the deciding agents are to evaluate the posterior probability of the whole case conditional on all evidence, and to convict if it is above a certain threshold. The problem, generally, is that it might be the case that the pieces of evidence for particular elements of the claim can have high likelihood and posterior

probabilities of particular elements can be above the threshold while the posterior joint probability will still fail to meet the threshold. The fact that the joint posterior will be higher than the joint prior does not help much. For instance, if $\mathbf{a} = \mathbf{b} = 0.7$, $pr = \langle 0.1, 0.5, 0.3, 0.1\rangle$, the posterior of A is ≈ 0.777, the posterior of B is ≈ 0.608 and the joint posterior is ≈ 0.216 (yes, it is higher than the joint prior = 0.1, but this does not help the conjunction to satisfy the decision standard).

To see the extent to which Dawid's strategy is helpful here, perhaps the following analogy might be useful.

Imagine it is winter, the heating does not work in my office and I am quite cold. I pick up the phone and call maintenance. A rather cheerful fellow picks up the phone. I tell him what my problem is, and he reacts:

— Oh, don't worry.
— What do you mean? It's cold in here!
— No no, everything is fine, don't worry.
— It's not fine! I'm cold here!
— Look, sir, my notion of it being warm in your office is that the building provides some improvement to what the situation would be if it wasn't there. And you agree that you're definitely warmer than you'd be if your desk was standing outside, don't you? Your, so to speak, posterior warmth is higher than your prior warmth, right?

Dawid's discussion is in the vein of the above conversation. In response to a problem with the adjudication model under consideration Dawid simply invites us to abandon thinking in terms of it and to abandon requirements crucial for the model. Instead, he puts forward a fairly weak notion of support (analogous to a fairly weak sense of the building providing improvement), according to which, assuming witnesses are fairly reliable, if separate fairly reliable witnesses provide positive support to the conjuncts, then their joint testimony provides positive support for the conjunction.

As far as our assessment of the original adjudication model and dealing with DAC, this leaves us hanging. Yes, if we abandon the model, DAC does not worry us anymore. But should we? And if we do, what should we change it to, if we do not want to be banished from the paradise of probabilistic methods?

Having said this, let me emphasize that Dawid's paper is important in the development of the debate, since it shifts focus on the likelihood ratios, which for various reasons are much better measures of evidential support provided by particular pieces of evidence than mere posterior probabilities.

Before we move to another attempt at a probabilistic formulation of the decision standard, let us introduce the other hero of our story: the gatecrasher paradox. It is against DAC and this paradox that the next model will be judged.

Optional Content Starts

In fact, Cohen replied to Dawid's paper (Cohen, 1988). His reply, however, does not have much to do with the workings of Dawid's strategy, and is rather unusual. Cohen's first point is that the calculations of posteriors require odds about unique events, whose meaning is usually given in terms of potential wagers – and the key criticism here is that in practice such wagers cannot be decided. This is not a convincing criticism, because the betting-odds interpretations of subjective probability do not require that on each occasion the bet should really be practically decidable. It rather invites one to imagine a possible situation in which the truth could be found out and asks: how much would we bet on a certain claim in such a situation? In some cases, this assumption is false, but there is nothing in principle wrong with thinking about the consequences of false assumptions.

Second, Cohen says that Dawid's argument works only for testimonial evidence, not for other types thereof. But this claim is simply false – just because Dawid used testimonial evidence as an example that he worked through it by no means follows that the approach cannot be extended. After all, as long as we can talk about sensitivity and specificity of a given piece of evidence, everything that Dawid said about testimonies can be repeated *mutatis mutandis*.

Third, Cohen complaints that Dawid in his example worked with rather high priors, which according to Cohen would be too high to correspond to the presumption of innocence. This also is not a very successful rejoinder. Cohen picked his priors in the example for the ease of calculations, and the reasoning can be run with lower priors. Moreover, instead of discussing the conjunction problem, Cohen brings in quite a different problem: how to probabilistically model the presumption of innocence, and what priors of guilt should be appropriate? This, indeed, is an important problem; but it does not have

much to do with DAC, and should be discussed separately.

8.3 Problems with Cheng's relative likelihood

In the process of solving the gatecrasher paradox, to reach (4), Cheng makes another controversial assumption: that the prior odds should be one, that is, that before any evidence specific to the case is obtained, $\mathsf{P}(H_\Pi) = \mathsf{P}(H_\Delta)$. One problem with this assumption is that it is not clear how to square this with how Cheng handles DAC. For there, he insisted we need to consider *three different* defense scenarios, which we marked as $H_{\Delta_1}, H_{\Delta_2}$ and H_{Δ_3}. Now, do we take Cheng's suggestion to be that we should have

$$\mathsf{P}(H_\Pi) = \mathsf{P}(H_{\Delta_1}) = \mathsf{P}(H_{\Delta_2}) = \mathsf{P}(H_{\Delta_3})?$$

Given that the scenarios are jointly exhaustive and pairwise exclusive this would mean that each of them should have prior probability 0.25 and, in principle that the prior probability of guilt can be made lower simply by the addition of elements under consideration. This conclusion seems suboptimal.

If, on the other hand, we read Cheng as saying that we should have $\mathsf{P}(H_\Pi) = \mathsf{P}(\neg H_\Pi)$, the side-effect is that even a slightest evidence in support of H_Π will make the posterior probability of H_Π larger than that of $\neg H_\Pi$, and so the plaintiff can win their case way too easily. Worse still, if $\mathsf{P}(\neg H_\Pi)$ is to be divided between multiple defense scenarios against which H_Π is to be compared, then as soon as this division proceeds in a non-extreme fashion, the prior of each defense scenario will be lower than the prior of H_Π, and so from the perspective of RLP, the plaintiff does not have to do anything to win (as long as the defense does not provide absolving evidence), because his case is won without any evidence already!

Finally, let us play along and assume that in the gatecrasher scenario the conviction is justified just in case (5) holds. Cheng insists that it does not, because $\mathsf{P}(E|H_\Pi) = \mathsf{P}(E|H_\Delta)$. This supposedly captures the intuition that whether Tom paid has no impact on the statistics that we have.

But this is not obvious. Here is one way to think about this. Tom either paid the entrance fee or did not. Consider these two options, assuming nothing else about the case changes. If he did pay, then he is among the 9 innocent spectators. But this means that if he had not paid, there would have been 992 gatecrashers, and so E would be false (because it says there was 991 of them). If, on the other hand, Tom in reality did not pay (and so is among the 991 gatecrashers), then had he paid, there would have been only 990 gatecrashers and E would have been false, again!

So whether conviction is justified and what the relevant ratios are depends on whether Tom really paid. Cheng's criterion (5) results in the conclusion that Tom should be penalized if and only if he did not pay. But this does not help us much when it comes to handling the paradox, because the reason why we needed to rely on E was exactly that we did not know whether Tom paid.

If you are not buying into the above argument, here is another way to state the problem. Say your priors are $\mathsf{P}(E) = e$, $\mathsf{P}(H_\Pi) = \pi$. By Bayes' Theorem we have:

$$\mathsf{P}(E|H_\Pi) = \frac{\mathsf{P}(H_\Pi|E)e}{\pi}$$
$$\mathsf{P}(E|H_\Delta) = \frac{\mathsf{P}(H_\Delta|E)e}{1-\pi}$$

Assuming our posteriors are taken from the statistical evidence, we have $\mathsf{P}(H_\Pi|E) = 0.991$ and $\mathsf{P}(H_\Delta|E) = 0.009$. So we have:

$$\begin{aligned} LR(E) &= \frac{\mathsf{P}(H_\Pi|E)e}{\pi} \times \frac{1-\pi}{\mathsf{P}(H_\Delta|E)e} \\ &= \frac{\mathsf{P}(H_\Pi|E) - \mathsf{P}(H_\Pi|E)\pi}{\mathsf{P}(H_\Delta|E)\pi} \\ &= \frac{0.991 - 0.991\pi}{0.009\pi} \end{aligned} \tag{15}$$

and $LR(E)$ will be > 1 as soon as $\pi < 0.991$. This means that contrary to what Cheng suggested, in any situation in which the prior probability of guilt is less than the posterior probability of guilt, RLP tells us to convict. This, however, does not seem desirable.

8.4 Problem's with Kaplow's stuff

Kaplow does not discuss the conceptual difficulties that we are concerned with, but this will not stop us from asking whether DTLP can handle them (and answering to the negative). Let us start with DAC.

Say we consider two claims, A and B. Is it generally the case that if they separately satisfy the decision rule, then so does $A \wedge B$? That is, do the assumptions:

$$\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} > \frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)} \times \frac{L}{G}$$
$$\frac{\mathsf{P}(E|B)}{\mathsf{P}(E|\neg B)} > \frac{\mathsf{P}(\neg B)}{\mathsf{P}(B)} \times \frac{L}{G}$$

entail

$$\frac{\mathsf{P}(E|A \wedge B)}{\mathsf{P}(E|\neg(A \wedge B))} > \frac{\mathsf{P}(\neg(A \wedge B))}{\mathsf{P}(A \wedge B)} \times \frac{L}{G}?$$

Alas, the answer is negative.

Optional Content Starts

This can be seen from the following example. Suppose a random digit from 0-9 is drawn; we do not know the result; we are told that the result is < 7 ($E =$'the result is < 7'), and we are to decide whether to accept the following claims:

A	the result is < 5.
B	the result is an even number.
$A \wedge B$	the result is an even number < 5.

Suppose that $L = G$ (this is for simplicity only — nothing hinges on this, counterexamples for when this condition fails are analogous). First, notice that A and B taken separately satisfy (7). $\mathsf{P}(A) = \mathsf{P}(\neg A) = 0.5$, $\mathsf{P}(\neg A)/\mathsf{P}(A) = 1$ $\mathsf{P}(E|A) = 1$, $\mathsf{P}(E|\neg A) = 0.4$. (7) tells us to check:

$$\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} > \frac{L}{G} \times \frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)}$$
$$\frac{1}{0.4} > 1$$

so, following DTLP, we should accept A.
For analogous reasons, we should also accept B. $\mathsf{P}(B) = \mathsf{P}(\neg B) = 0.5$, $\mathsf{P}(\neg B)/\mathsf{P}(B) = 1$ $\mathsf{P}(E|B) = 0.8$, $\mathsf{P}(E|\neg B) = 0.6$, so we need to check that indeed:

$$\frac{\mathsf{P}(E|B)}{\mathsf{P}(E|\neg B)} > \frac{L}{G} \times \frac{\mathsf{P}(\neg B)}{\mathsf{P}(B)}$$
$$\frac{0.8}{0.6} > 1$$

But now, $\mathsf{P}(A \wedge B) = 0.3$, $\mathsf{P}(\neg(A \wedge B)) = 0.7$, $\mathsf{P}(\neg(A \wedge B))/\mathsf{P}(A \wedge B) = 2\frac{1}{3}$, $\mathsf{P}(E|A \wedge B) = 1$, $\mathsf{P}(E|\neg(A \wedge B)) = 4/7$ and it is false that:

$$\frac{\mathsf{P}(E|A \wedge B)}{\mathsf{P}(E|\neg(A \wedge B))} > \frac{L}{G} \times \frac{\mathsf{P}(\neg(A \wedge B))}{\mathsf{P}(A \wedge B)}$$
$$\frac{7}{4} > \frac{7}{3}$$

The example was easy, but the conjuncts are probabilistically dependent. One might ask: are there counterexamples that involve claims which are probabilistically independent?[15]

Consider an experiment in which someone tosses a six-sided die twice. Let the result of the first toss be X and the result of the second one Y. Your evidence is that the results of both tosses are greater than one ($E =: X > 1 \wedge Y > 1$). Now, let A say that $X < 5$ and B say that $Y < 5$.

[15] Thanks to Alicja Kowalewska for pressing me on this.

The prior probability of A is $2/3$ and the prior probability of $\neg A$ is $1/3$ and so $\frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)} = 0.5$. Further, $\mathsf{P}(E|A) = 0.625$, $\mathsf{P}(E|\neg A) = 5/6$ and so $\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} = 0.75$ Clearly, $0.75 > 0.5$, so A satisfies the decision standard. Since the situation with B is symmetric, so does B.

Now, $\mathsf{P}(A \wedge B) = (2/3)^2 = 4/9$ and $\mathsf{P}(\neg(A \wedge B)) = 5/9$. So $\frac{\mathsf{P}(\neg(A \wedge B))}{\mathsf{P}(A \wedge B)} = 5/4$. Out of 16 outcomes for which $A \wedge B$ holds, E holds in 9, so $\mathsf{P}(E|A \wedge B) = 9/16$. Out of 20 remaining outcomes for which $A \wedge B$ fails, E holds in 16, so $\mathsf{P}(E|\neg(A \wedge B)) = 4/5$. Thus, $\frac{\mathsf{P}(E|A \wedge B)}{\mathsf{P}(E|\neg(A \wedge B))} = 45/64 < 5/4$, so the conjunction does not satisfy the decision standard.

Optional Content Ends

Let us turn to the gatecrasher paradox.

Suppose $L = G$ and recall our abbreviations: $\mathsf{P}(E) = e$, $\mathsf{P}(H_\Pi) = \pi$. DTLP tells us to convict just in case:

$$LR(E) > \frac{1-\pi}{\pi}$$

From (15) we already now that

$$LR(E) = \frac{0.991 - 0.991\pi}{0.009\pi}$$

so we need to see whether there are any $0 < \pi < 1$ for which

$$\frac{0.991 - 0.991\pi}{0.009\pi} > \frac{1-\pi}{\pi}$$

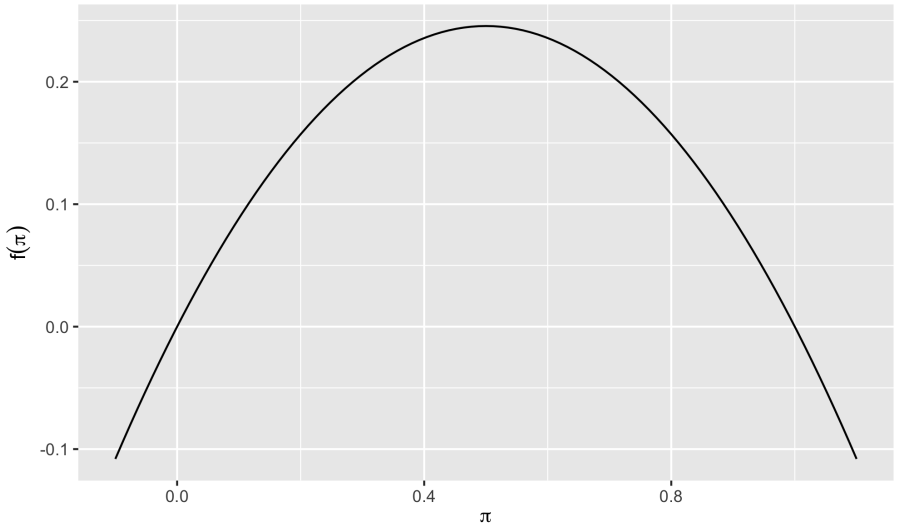
Multiply both sides first by 009π and then by π:

$$0.991\pi - 0.991\pi^2 > 0.09\pi - 0.009\pi^2$$

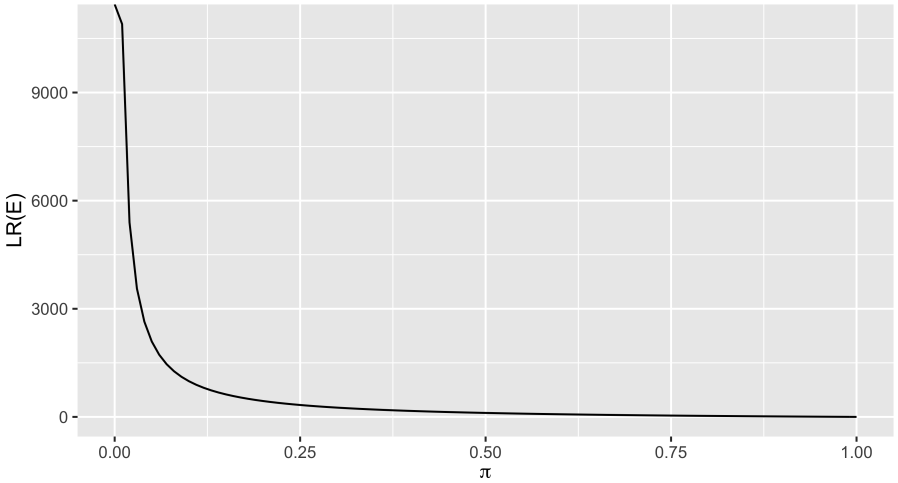
Simplify and call the resulting function f:

$$f(\pi) = -0.982\pi^2 + 0.982\pi > 0$$

The above condition is satisfied for any $0 < \pi < 1$ (f has two zeros: $\pi = 0$ and $\pi = 1$). Here is a plot of f:



Similarly, $LR(E) > 1$ for any $0 < \pi < 1$. Here is a plot of $LR(E)$ against π:



Notice that $LR(E)$ does not go below 1. This means that for $L = G$ in the gatecrasher scenario DTLP wold tell us to convict for any prior probability of guilt $\pi \neq 0, 1$.

One might ask: is the conclusion very sensitive to the choice of L and G? The answer is, not too much.

Optional Content Starts

How sensitive is our analysis to the choice of L/G? Well, $LR(E)$ does not change at all, only the threshold moves. For instance, if $L/G = 4$, instead of f we end up with

$$f'(\pi) = -0.955\pi^2 + 0.955\pi > 0$$

and the function still takes positive values on the interval $(0, 1)$. In fact, the decision won't change until L/G increases to ≈ 111. Denote L/G as ρ, and let us start with the general decision standard, plugging in our calculations for $LR(E)$:

$$\begin{aligned}
LR(E) &> \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)}\rho \\
LR(E) &> \frac{1-\pi}{\pi}\rho \\
\frac{0.991 - 0.991\pi}{0.009\pi} &> \frac{1-\pi}{\pi}\rho \\
\frac{0.991 - 0.991\pi}{0.009\pi}\frac{\pi}{1-\pi} &> \rho \\
\frac{0.991\pi - 0.991\pi^2}{0.009\pi - 0.009\pi^2} &> \rho \\
\frac{\pi(0.991 - 0.991\pi)}{\pi(0.009 - 0.009\pi)} &> \rho \\
\frac{0.991 - 0.991\pi}{0.009 - 0.009\pi} &> \rho \\
\frac{0.991(1-\pi)}{0.009(1-\pi)} &> \rho \\
\frac{0.991}{0.009} &> \rho \\
110.1111 &> \rho
\end{aligned}$$

Optional Content Ends

So, we conclude, in usual circumstances, DTLP does not handle the gatecrasher paradox.

9 Probabilistic Thresholds Revised

9.1 Likelihood ratios and naked statistical evidence

9.2 Conjunction paradox and Bayesian networks

10 Conclusions

Where are we, how did we get here, and where can we go from here? We were looking for a probabilistically explicated condition Ψ such that the trier of fact, at least ideally, should accept any relevant claim (including G) just in case $\Psi(A, E)$.

From the discussion that transpired it should be clear that we were looking for a Ψ satisfying the following desiderata:

conjunction closure If $\Psi(A, E)$ and $\Psi(B, E)$, then $\Psi(A \wedge B, E)$.

naked statistics The account should at least make it possible for convictions based on strong, but naked statistical evidence to be unjustified.

equal treatment the condition should apply to any relevant claim whatsoever (and not just a selected claim, such as G).

Throughout the paper we focused on the first two conditions (formulated in terms of the difficulty about conjunction (DAC), and the gatecrasher paradox), going over various proposals of what Ψ should be like and evaluating how they fare. The results can be summed up in the following table:

View	Convict iff	DAC	Gatecrasher
Threshold-based LP (TLP)	Probability of guilt given the evidence is above a certain threshold	fails	fails
Dawid's likelihood strategy	No condition given, focus on $\frac{\mathsf{P}(H\|E)}{\mathsf{P}(H\|\neg E)}$	- If evidence is fairly reliable, the posterior of $A \wedge B$ will be greater than the prior. - The posterior of $A \wedge B$ can still be lower than the posterior of any of A and B. - Joint likelihood, contrary do Dawid's claim, can also be lower than any of the individual likelihoods.	fails
Cheng's relative LP (RLP)	Posterior of guilt higher than the posterior of any of the defending narrations	The solution assumes equal costs of errors and independence of A and B conditional on E. It also relies on there being multiple defending scenarios individualized in terms of combinations of literals involving A and B.	Assumes that the prior odds of guilt are 1, and that the statistics is not sensitive to guilt (which is dubious). If the latter fails, tells to convict as long as the prior of guilt < 0.991.
Kaplow's decision-theoretic LP (DTLP)	The likelihood of the evidence is higher than the odds of innocence multiplied by the cost of error ratio	fails	convict if cost ratio $<$ 110.1111

Thus, each account either simply fails to satisfy the desiderata, or succeeds on rather unrealistic assumptions. Does this mean that a probabilistic approach to legal evidence evaluation should be abandoned? No. This only means that if we are to develop a general probabilistic model of legal decision standards, we have to do better. One promising direction is to go back to Cohen's pressure against **Requirement 1** and push against it. A brief paper suggesting this direction is (Di Bello, 2019b), where the idea is that the probabilistic standard (be it a threshold or a comparative wrt. defending narrations) should be applied to the whole claim put forward by the plaintiff, and not to its elements. In such a context, DAC does not arise, but **equal treatment** is violated. Perhaps, there are independent reasons to abandon it, but the issue deserves further discussion. Another strategy might be to go in the direction of employing probabilistic methods to explicate the narration theory of legal decision standards (Urbaniak, 2018), but a discussion of how this approach relates to DAC and the gatecrasher paradox lies beyond the scope of this paper.

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