

# The difficulty with Conjunction

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# 1 The problem

A theoretical difficulty that any theory of the standard of proof should address is the the conjunction paradox or difficulty about conjunction. First formulated by J. L. Cohen (1977), the difficulty about conjunction has enjoyed a great deal of scholarly attention every since (R. J. Allen, 1986; R. J. Allen & Stein, 2013; R. Allen & Pardo, 2019; Haack, 2014; Schwartz & Sober, 2017; Stein, 2005). This difficulty arises when an accusation of wrongdoing, in a civil or criminal proceeding, is broken down into its constituent elements. The basic problem is that the probability of a conjunction is often lower than the probability of the conjuncts. Thus, even if each conjunct meets the requisite probability threshold, the conjunction does not. This chapter examines the difficulty about conjunction and how legal probabilists can respond.

here you say "any theory" but then you switch to the probabilistic ones. Is it a problem for non-probabilistic theories or not?

## 1.1 The Conjunction Principle

Suppose that in order to prevail in a criminal trial, the prosecution should establish by the required standard, first, that the defendant caused harm to the victim (call it claim *A*), and second, that the defendant had premeditated the harmful act (call it claim *B*). J. L. Cohen (1977) argues that common law systems subscribe to a conjunction principle, that is, if *A* and *B* are established according to the governing standard of proof, so is their conjunction (and vice versa). If the conjunction principle holds, the following must be equivalent, where *S* is a placeholder for the standard of proof:

<b>Separate</b>	<i>A</i> is established according to <i>S</i> and <i>B</i> is established according to <i>S</i>
<b>Overall</b>	The conjunction $A \wedge B$ is established according to <i>S</i>

Here you first fix *A* and *B* as abbreviations, and then make a general claim using those variables; this makes it seem like the conjunction principle is about two very specific sentences, which is not what you meant.

If we generalize to more than just two constituent claims, the conjunction principle requires that:

$$S[C_1 \wedge C_2 \wedge \dots \wedge C_k] \Leftrightarrow S[C_1] \wedge S[C_2] \wedge \dots \wedge S[C_k].$$

where  $S[C_i]$  means that claim or hypothesis  $C_i$  is established according to standard *S*. In what follows it will be useful to distinguish between the two directions involved: the implication from right to left will be called **distribution**, and the opposite direction will be called **aggregation**. The problem that will be discussed in this subsection is a one with aggregation, but we will later on discuss approaches which fail to satisfy distribution.

You need to be clearer about what this equivalence symbol is supposed to mean; further, later you title a subsection saying the principle is false; but if it is normative, I don't understand what you mean by saying its false.

The conjunction principle is consistent with—perhaps even required by—the case law. For example, the United States Supreme Court writes that in criminal cases

the accused [is protected] against conviction except upon proof beyond a reasonable doubt of *every fact* necessary to constitute the crime with which he is charged. In re Winship (1970), 397 U.S. 358, 364.

A plausible way to interpret this quotation is to posit this identity: to establish someone's guilt beyond a reasonable doubt *just is* to establish each element of the crime beyond a reasonable doubt. Thus,

cite properly

$$\text{BARD}[C_1 \wedge C_2 \wedge \dots \wedge C_n] \Leftrightarrow \text{BARD}[C_1] \wedge \text{BARD}[C_2] \wedge \dots \wedge \text{BARD}[C_k],$$

where the conjunction  $C_1 \wedge C_2 \wedge \dots \wedge C_n$  comprises all the material facts that, according to the applicable law, constitute the crime with which the accused is charged.

The problem for the legal probabilist is that the conjunction principle conflicts with a threshold-based probabilistic interpretation of the standard of proof. For suppose the prosecution presents evidence that establishes claims *A* and *B*, separately, to the required probability, say about 95% each. Has the prosecution met its burden of proof? Each claim was established to the requisite probability threshold, and thus it was established to the requisite standard (assuming the threshold-based interpretation of the standard of proof). And if each claim was established to the requisite standard, then (i) guilt as a whole was established to the requisite standard (assuming the conjunction principle). But even though each claim was established to the requisite probability threshold, the probability of their conjunction—assuming the two claims are independent—is only  $95\% \times 95\% = 90.25\%$ , below the required 95% threshold. So (ii) guilt as a whole was *not* established to the requisite standard (assuming a threshold-based probabilistic interpretation of the standard). Hence, we arrive at two contradictory conclusions: (i) that the prosecution met its burden of proof and (ii) that it did not meet its burden.

you didn't introduce this notion yet

I thought we agreed to use .95 notation for probabilities and keep percentages for population frequencies only, as in SEP. This is also what I have done in the LR chapter, so perhaps we should stick to this convention throughout this chapter as well.

The difficulty about conjunction—the fact that a probabilistic interpretation of the standard of proof conflicts with the conjunction principle—does not subside when the number of constituent claims increases. If anything, the difficulty becomes more apparent. Say the prosecution has established three separate claims to 95% probability. Their conjunction—again if the claims are independent—would be about 85% probable, even further below the 95% threshold. Nor does the difficulty about conjunction subside if the claims are no longer regarded as independent. The probability of the conjunction  $A \wedge B$ , without the assumption of independence, equals  $P(A|B) \times P(B)$ . But if claims  $A$  and  $B$ , separately, have been established to 95% probability, enough for each to meet the threshold, the probability of  $A \wedge B$  could still be below the 95% threshold unless  $P(A|B) = 100\%$ . For example, that someone premeditated a harmful act against another (claim  $B$ ) makes it more likely that they did cause harm in the end (claim  $A$ ). Since  $P(A|B) > P(A)$ , the two claims are not independent. Still, premeditation does not always lead to harm, so  $P(A|B)$  will often be below 100%. Consequently, in this case, the probability of the conjunction  $A \wedge B$  would often be below the 95% threshold.

## 1.2 Probabilistic (in)dependencies and Bayesian networks used

The conjunction paradox is a difficult problem, as the vast literature on the topic attests. Before we move on, it is important to become clear about the assumptions underlying the formulation of the paradox, in particular, the assumptions of probabilistic independence.

The reason why we need to be careful here is that there are various types of independence in the vicinity, and various arguments that we will look at often use some but not all of them, and we need to be clear about which assumptions are involved. Here are the main types of independence, with issues that we have to pay attention to:

- $A$  and  $B$  can simply be independent,  $A \perp\!\!\!\perp B$ . This happens iff  $P(A|B) = P(A)P(B)$ , and is equivalent to  $P(A|B) = P(A)$  if  $P(A), P(B) \neq 0$ . If  $A \perp\!\!\!\perp B$ , then so are all their literals (that is, adding a negation in front of either of them doesn't remove the independence).
- Three events  $A, B, C$ , are independent if (1) they are all pairwise independent, and moreover (2)  $P(A)P(B)P(C) = P(A \wedge B \wedge C)$ . Interestingly, pairwise independencies alone do not entail (2). Imagine tossing a coin twice, let  $A$ ='heads in the first toss,'  $B$ ='heads in the second toss,' and  $C$ ='the two results are the same.' These are pairwise independent, but fail to satisfy (2), because once you know two of them, the third one is determined. Nor does (2) entail (1). If  $P(A) = 0$ , condition (2) will be satisfied no matter what happens with  $B$  and  $C$ . The notion and non-entailment generalizes to more events.
- $A$  and  $B$  are conditionally independent given  $C$ ,  $A \perp\!\!\!\perp B|C$  iff  $P(A \wedge B|C) = P(A|C)P(B|C)$ . Conditional independence does not entail independence. Say you have two coins, one fair, one biased. Conditional on which coin you have chosen, the results of subsequent tosses are independent. But if you don't know which coin you have chosen, the result of previous tosses give you some information about which coin it is, and this has impact on your estimate of the probability of heads in the next toss. Nor does not independence entail conditional independence. For instance, suppose there are two reasons for your fire alarm to go off: malfunction, or real fire. There is a plausible scenario in which they are independent, unconditionally. However, if you conditionalize on the alarm going off and find out there is no real fire, the probability of a malfunction goes up. Moreover, conditional independence given  $C$  does not entail conditional independence given  $\neg C$ . For instance, suppose your taking classes and in some of them bad teachers give grades that don't depend on how much you work. Then your work and grade are independent conditional on the instructor being a bad teacher, but they are not independent conditional on the instructor not being a bad teacher.

One assumption often made in the formulation of the paradox is that claims  $A$  and  $B$  are probabilistically independent. This is not always the case — we have seen that the paradox does subside even if the two claims are dependent. However, we will rely on some independence assumptions in our arguments for the sake of simplicity. This has a two-fold justification. One, if we are formulating a counterexample to a claim (as is the case with the conjunction paradox), even if it satisfies a further condition (independence), it still remains a counterexample to this claim. Of course, the question remains if a weaker claim that assumes the lack of independence, but it is a different claim to be evaluated in a different way. Second, if we are making a positive claim about, say, the behavior of likelihood ratios, this is purely for illustrative purposes to make calculations and proofs easier, and the reader needs to

False in whole generality, give a counterexample with more specific numbers. M: I changed things a bit. Maybe it's clear now. The counterexample is basically  $P(A)=P(B)=0.95$ , but  $P(AB)=P(A)*P(A|B)$  and since  $P(A|B)$  is below 1, then  $P(AB)$  is below 0.95.

some flow is needed here.

moved up the independencies section

keep in mind that either the claim is restricted to situations in which independence assumptions are satisfied, or needs additional support when these assumptions are lifted.

In what follows we will be using a set up in which evidence  $a$  is meant to support claim  $A$  and another piece of evidence,  $b$  is put forward in support of claim  $B$ . The key questions then will be about the support that the combination of  $a$  and  $b$  gives to the conjunction  $A \wedge B$  and its relation to the individual support relations.

In our considerations we will sometimes use at least some of these independence assumptions, either assuming that which of these are involved is clear from the way the derivation proceeds, or being explicit about which assumptions are relied on. WE take the situation to be symmetric, that is we don't list assumptions obtained by simultaneously switching  $A$  with  $B$  and  $a$  with  $b$ .

Make a complete and clear list of all independence assumptions used

**Alicja:** confirm all are d-sep in the first BN, check which are d-sep in the second one.

- (1)  $A \perp\!\!\!\perp B$
- (2)  $A \perp\!\!\!\perp b|a$
- (3)  $B \perp\!\!\!\perp a \wedge A|b$
- (4)  $a \perp\!\!\!\perp b|A \wedge B$
- (5)  $a \perp\!\!\!\perp b|A$
- (6)  $a \perp\!\!\!\perp B|A$
- (7)  $a \perp\!\!\!\perp B|\neg A$
- (8)  $a \perp\!\!\!\perp \neg B|A$
- (9)  $a \perp\!\!\!\perp \neg B|\neg A$
- (10)  $b \perp\!\!\!\perp A \wedge a|B$
- (11)  $b \perp\!\!\!\perp \neg A \wedge a|B$
- (12)  $b \perp\!\!\!\perp A \wedge a|\neg B$
- (13)  $b \perp\!\!\!\perp \neg A \wedge a|\neg B$
- (14)  $b \perp\!\!\!\perp a|B$

Double check if we in fact use I3.

Directed Acyclic Graphs (DAGs) are useful for representing graphically these relationships of independence. The edges, intuitively, are meant to capture direct influence between the nodes. The role that such direct influence plays is that in a Bayesian network built over a DAG any node is conditionally independent of its nondescendants (including ancestors), given its parents. If this is the case for a given probabilistic measure  $P()$  and a given DAG, we say that  $P()$  is compatible with  $G$ , and they can be put together to constitute a Bayesian network.

The graphical counterpart of probabilistic independence is the so-called *d-separation*,  $\perp\!\!\!\perp_d$ . We say that two nodes,  $X$  and  $Y$ , are d-separated given a set of nodes  $Z$  —  $X \perp\!\!\!\perp_d Y|Z$  — iff for every undirected path from  $X$  to  $Y$  there is a node  $Z'$  on the path such that either:

- $Z' \in Z$  and there is a *serial* connection,  $\rightarrow Z' \rightarrow$ , on the path,
- $Z' \in Z$  and there is a *diverging* connection,  $\leftarrow Z' \rightarrow$ , on the path,
- There is a connection  $\rightarrow Z' \leftarrow$  on the path, and neither  $Z'$  nor its descendants are in  $Z$ .

Finally, two sets of nodes,  $X$  and  $Y$ , are d-separated given  $Z$  if every node in  $X$  is d-separated from every node in  $Y$  given  $Z$ . With serial connection, for instance, if:

Node	Proposition
$G$	The suspect is guilty.
$B$	The blood stain comes from the suspect.
$M$	The crime scene stain and the suspect's blood share their DNA profile.

We naturally would like to have the connection  $G \rightarrow B \rightarrow M$ . If we don't know whether  $B$  holds,  $G$  seems to have an indirect impact on the probability of  $M$ . Yet, once we find out that  $B$  is true, we expect the profile match, and whether  $G$  holds has no further impact on the probability of  $M$ .

The case of diverging connections has already been discussed when we talked about independence. Whether a coin is fair,  $F$ , or not has impact on the result of the first toss,  $H1$ , and the result of

the second toss,  $H2$ , and as long as we don't know whether  $F$ ,  $H1$  increases the probability of  $H2$ . So  $H1 \leftarrow F \rightarrow H2$  seems to be appropriate. Once we know that  $F$ , though,  $H1$  and  $H2$  become independent.

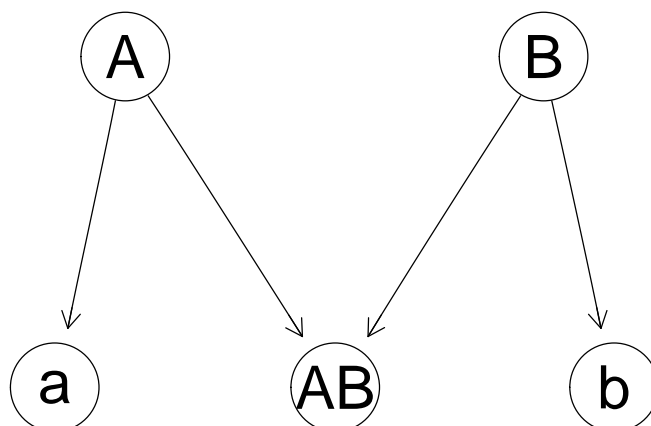
For converging connections, let  $G$  and  $B$  be as above, and let:

Node	Proposition
$O$	The crime scene stain comes from the offender.

Both  $G$  and  $O$  influence  $B$ . If he's guilty, it's more likely that the blood stain comes from him, and if the blood crime stain comes from the offender it is more likely to come from the suspect (for instance, more so than if it comes from the victim). Moreover,  $G$  and  $O$  seem independent – whether the suspect is guilty doesn't have any bearing on whether the stain comes from the offender. Thus, a converging connection  $G \rightarrow B \leftarrow O$  seem appropriate. However, if you do find out that  $B$  is true, that the stain comes from the suspect, whether the crime stain comes from the offender becomes relevant for whether the suspect is guilty.

One important reason why d-separation matters is that it can be proven that if two sets of nodes are d-separated given a third one, then they are independent given the third one, for any probabilistic measures compatible with a given DAG. Interestingly, lack of d-separation doesn't entail dependence for any probabilistic measure compatible with a given DAG. Rather, it only allows for it: if nodes are d-separated, there is at least one probabilistic measure fitting the DAG according to which they are independent. So, at least, no false independence can be inferred from the DAG, and all the dependencies are built into it.

Now, back to the independence assumptions in our take on the conjunction problem. Consider the DAG in Figure 1.



Please use this method of drawing DAGs, it's simple, uniform and requires less work.

Figure 1: DAG of the conjunction set-up, with the usual independence assumptions built in.

The structure of the network is rather natural because each piece of evidence bears on its hypothesis and is probabilistically independent conditional on one of the hypotheses. One might wonder, however, why the arrows go from  $A$  and  $B$  into the node representing the conjunction  $A \wedge B$ . This setting captures the meaning of the conjunction. The constraint that, for the conjunction to be true, both  $A$  and  $B$  have to be true, can be defined using the appropriate conditional probability table (Table 1).

This conditional probability table looks, essentially, like the truth table for conjunction. The difference is that the values 1 and 0 stand for two different things depending on where they are in the table. In the columns corresponding to the nodes they represent node states: true and false; in the  $\text{Pr}$  column they represent the conditional probability of a given state of  $AB$  given the states of  $A$  and  $B$  listed in the same row. For instance, take a look at row two. It says: if  $A$  and  $B$  are both in states 1, then the probability of  $AB$  being in state 0 is 0. In principle we could use 'true' and 'false' instead of 1 and 0 to represent states, but the numeric representation is easier to use in programming, which we do quite a bit in the background, so the reader might as well get used to this harmless ambiguity. For binary nodes, we will consistently use '1' and '0' for the states, it's just probabilities that in this case end up being extreme.

Please use the same format for tables so that it's uniform throughout the paper; I also removed the corners, it's too much fuss

	A	B	
AB			Pr
1	1	1	1
0	1	1	0
1	0	1	0
0	0	1	1
1	1	0	0
0	1	0	1
1	0	0	0
0	0	0	1

Table 1: Conditional probability table for the conjunction node.

To eliminate the assumption of independence between the two claims, while holding everything else fixed, it is enough to draw an arrow between  $A$  and  $B$  (Figure 2). The two Bayesian networks provide a compact representation of the class of models we will be concerned with in this chapter.

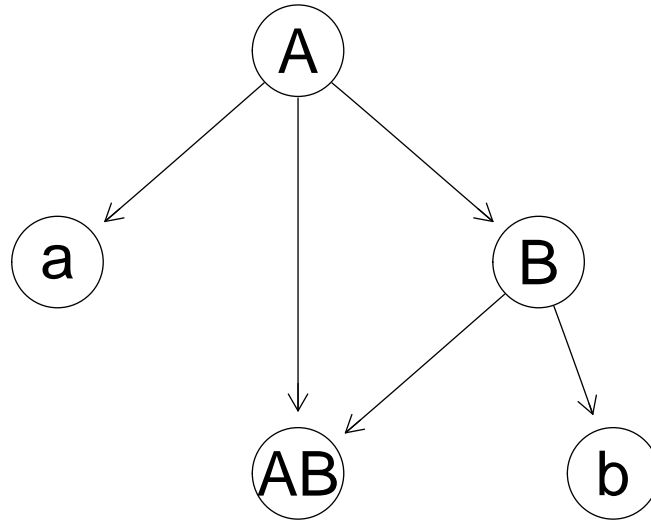


Figure 2: DAG of the conjunction set-up, without independence between  $A$  and  $B$ .

### 1.3 Conjunction principle with hypotheses and evidence

So far the discussion proceeded without mentioning explicitly the evidence presented in support of the different claims that constitute the allegation of wrongdoing. This is, however, a simplification, as no evidential relations were involved. Once we pay attention to them, a plausible candidate for the conjunction principle seems to be as follows:

$$S[a, A] \text{ and } S[b, B] \Leftrightarrow S[a \wedge b, A \wedge B],$$

where  $S[e, H]$  means that evidence  $e$  establishes hypothesis  $H$  by standard  $S$ .

Understood in terms of a threshold put on posterior probability, the conjunction principle above fails in some cases. For suppose that given their respective pieces of evidence  $a$  and  $b$ , both claims  $A$  and  $B$  are sufficiently probable (for a fixed threshold  $t$ ),  $P(A|a) > t$  and  $P(B|b) > t$ . It does not generally follow that  $A \wedge B$  is sufficiently probable given combined evidence  $a \wedge b$ . With suitable independence assumptions ((2) and (3)) we have:

$$\begin{aligned} P(A \wedge B|a \wedge b) &= P(A|a \wedge b) \times P(B|a \wedge b \wedge A) \\ &= P(A|a) \times P(B|b) \end{aligned}$$

Please use vert for conditional probability. I changed this here.

Of course, these relationships of independence do not always hold, but they do sometimes. For example, in an aggravated assault case, evidence  $a$  could be a witness testimony that the defendant physically injured the victim (claim  $A$ ), and  $b$  evidence that the defendant knew that the victim was a firefighter (claim  $B$ ), for example, another testimony that the defendant earlier called the firefighter for help. Presumably,  $P(A|a) = P(A|a \wedge b)$  because the fact that the defendant called a firefighter for help ( $b$ ) does not make it more (or less) likely that he would physically injure him ( $A$ ). Further,  $P(B|b) = P(B|a \wedge b \wedge A)$  because the fact that the defendant injured the victim ( $A$ ) and there is a testimony to that effect ( $a$ ) does not make it more (or less) likely that the victim was a firefighter ( $B$ ).

Given these assumptions, if—as is normally the case—neither  $P(A|a)$  nor  $P(B|b)$  equal 1, then

$$P(A \wedge B|a \wedge b) < P(A|a) \ \& \ P(A \wedge B|a \wedge b) < P(B|b).$$

This is another manifestation of the difficulty about conjunction. If each piece of evidence  $a$  and  $b$  establishes claims  $A$  and  $B$  with probability .95, the combined evidence  $a \wedge b$  may fail establish the conjunction  $A \wedge B$  with probability .95. The conjunction principle fails here, if the standard of proof is to be formulated in terms of a threshold on posterior probability.

Even if the independence assumptions are dropped, the difficulty about conjunction still arises in a number of circumstances. Suppose evidence  $a \wedge b$  establishes claim  $A$  and also claim  $B$ , separately, right above the probability threshold  $t$ . Since  $P(A \wedge B|a \wedge b) = P(A|a \wedge b) \times P(B|a \wedge b \wedge A)$ , it follows that  $P(A \wedge B|a \wedge b)$  might be below  $t$  if the factors on the right-hand side are below 1 and separate posteriors are very close to  $t$ .

## 2 Evidential Strength approach to the conjunction problem

We have seen that aggregation fails if the standard of proof is understood as a posterior probability threshold. Legal probabilists cannot justify aggregation if they equate the standard of proof to such threshold. The alternative is to think of proof standards as sufficiency criteria for how strong the evidence should be in order to justify a finding of criminal or civil liability. Instead of a posterior probability threshold, legal probabilists can formalize the standard of proof using a probabilistic measure of evidential strength. Will this move vindicate the conjunction principle on probabilistic grounds contrary to what Cohen thought? As we shall see, the answer to this question is complicated.

Suppose the standard of proof is no longer understood as a threshold on the posterior probability given the evidence, but rather, as a threshold on evidential strength. Two common probabilistic measures of evidential strength are the Bayes factor and the likelihood ratio. We discussed this topic in earlier chapters (REFER TO EARLIER CHAPTERS). As will become clear later, under plausible assumptions, these measures of evidential strength validate one direction of the conjunction principle: aggregation. If  $a$  is sufficiently strong evidence in favor of  $A$  and  $b$  is sufficiently strong evidence in favor of  $B$ , then  $a \wedge b$  is sufficiently strong evidence in favor of the conjunction  $A \wedge B$ . In fact, the evidential support for the conjunction will often exceed that for the individual claims, a point already made by Dawid (1987):

suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents.

Dawid thought this fact was enough for the conjunction paradox to ‘evaporate.’ This is true since so long as we are only concerned with vindicating one direction of the conjunction principle, aggregation. That is, probability theory justifies this claim: if distinct items of evidence  $a$  and  $b$  constitute sufficiently strong evidence for claims  $A$  and  $B$ , so does the conjunction  $a \wedge b$  for the composite claim  $A \wedge B$  (although, there are some caveats and extra assumptions for this to hold).

Unfortunately, we will show that on the evidential strength reading, the other direction of the conjunction principle, distribution, does not hold. If  $a \wedge b$  is sufficiently strong evidence in favor of  $A \wedge B$ , it does not follow that  $a$  is sufficiently strong evidence in favor of  $A$  or  $b$  sufficiently strong evidence in favor of  $B$ . It is not even true that, if  $a \wedge b$  is sufficiently strong evidence in favor of  $A \wedge B$ , then  $a \wedge b$  is sufficiently strong evidence in favor of  $A$  or  $B$ . This is odd. It would mean that, given a body of evidence, one can establish beyond a reasonable doubt that  $A \wedge B$  (say the defendant killed the victim *and* did so intentionally) while failing to establish by the same standard one of the conjuncts.

So we seem to be in a dilemma. If the standard of proof is understood as a threshold relative to the posterior probability, the conjunction principle fails because aggregation fails while distribution succeeds. If, on the other hand, the standard of proof is understood as a threshold relative to standard measures of



evidential strength, the conjunction principle fails because distribution fails while aggregation succeeds. From a probabilistic perspective, it seems impossible to capture both directions of the conjunction principle.

Before we discuss this more general challenge, let us explore how aggregation succeeds and distribution fails on the evidential strength approach. We will first investigate to what extent Dawid's was right in claiming that 'suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents'. Next, we will show—perhaps surprisingly—that after switching from posterior probabilities to measures of evidential support, another paradox, what we call the *distribution paradox*, arises. The argument for these two claims is tedious. The reader should arm themselves with patience or take our word for it and jump ahead to the next section.

Some bits hidden, they need to be discussed elsewhere I think.

## 2.1 Bayes factor and aggregation

We will start with the Bayes factor  $P(E|H)/P(E)$  as our measure of the support of  $E$  in favor of  $H$ . Since by Bayes' theorem

$$P(H|E) = \frac{P(E|H)}{P(E)} \times P(H),$$

the Bayes factor measures the extent to which a piece of evidence increases the probability of a hypothesis. The greater the Bayes factor (for values above one), the stronger the support of  $E$  in favor of  $H$ . Putting aside reservations about this measure of evidential support (discussed earlier in Chapter CROSSREF), the Bayes factor  $P(E|H)/P(E)$ , unlike the conditional probability  $P(H|E)$ , offers a potential way to overcome the difficulty about conjunction.

Say  $a$  and  $b$ , separately, support  $A$  and  $B$  to degree  $s_A$  and  $s_B$  respectively, that is,  $P(a|A)/P(a) = s_A$  and  $P(b|B)/P(b) = s_B$ , where both  $s_A$  and  $s_B$  are greater than one. Does the combined evidence  $a \wedge b$  provide at least as much support in favor of the joint claim  $A \wedge B$  as the individual support by  $a$  and  $b$  in favor of  $A$  and  $B$  considered separately? The combined support here is measured by the combined Bayes factor  $P(a \wedge b|A \wedge B)/P(a \wedge b)$ . The latter, under suitable independence assumptions, equals the product of the individual supports  $s_A$  and  $s_B$ . Here is the argument:

use nicefrac for inline fractions

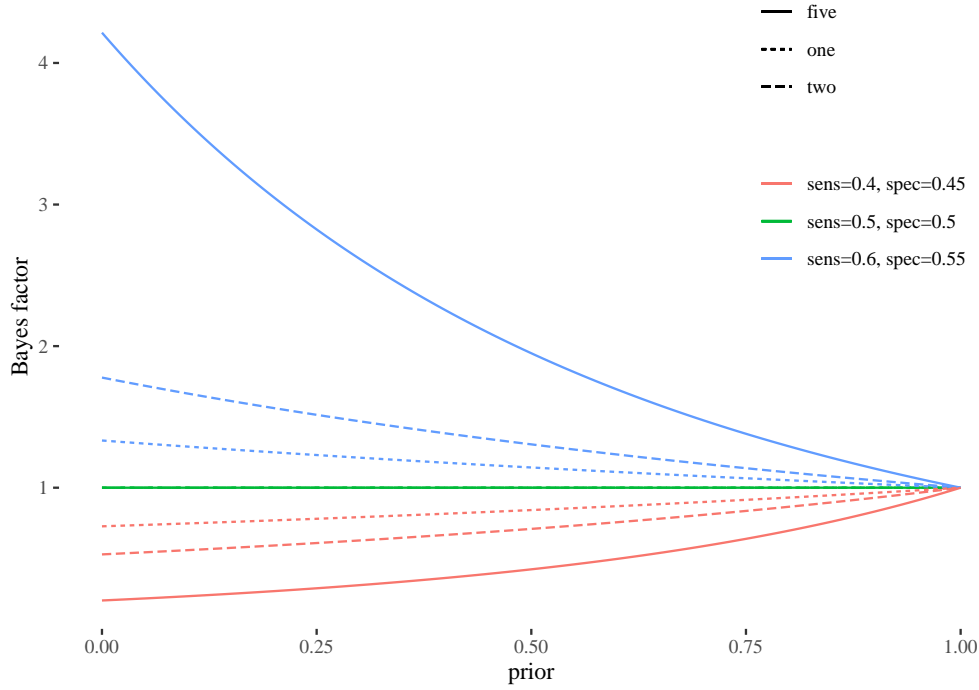
$$\begin{aligned}
 \frac{P(a \wedge b|A \wedge B)}{P(a \wedge b)} &= \frac{P(A \wedge B|a \wedge b)}{P(A \wedge B)} && \text{(Bayes's theorem)} \\
 &= \frac{\frac{P(A \wedge B \wedge a \wedge b)}{P(a \wedge b)}}{P(A \wedge B)} && \text{(definition of conditional probability)} \\
 &= \frac{\frac{P(A) \times P(B|A) \times P(a|A \wedge B) \times P(b|A \wedge B \wedge a)}{P(a) \times P(b|a)}}{P(A \wedge B)} && \text{(chain rule)} \\
 &= \frac{\frac{P(A) \times P(B) \times P(a|A) \times P(b|B)}{P(a) \times P(b)}}{P(A) \times P(B)} && \text{(independencies)} \\
 &= \frac{P(a|A)}{P(a)} \times \frac{P(b|B)}{P(b)} && \text{(algebraic manipulation)} \\
 s_{AB} &= s_A \times s_B
 \end{aligned}$$

The step marked by the asterisk rests on some of the independence assumptions codified in the Bayesian network in Figure 1, namely: (2.2), (6), and (10).

Thus, under suitable independence assumptions, the combined support  $s_{AB}$  will always be higher than the individual support so long as  $s_A$  and  $s_B$  are greater than one, that is, if the individual piece of evidence positively support their respective hypotheses. This result can be generalized beyond two pieces of evidence. Figure 3 compares the Bayes factor of one item of evidence, say  $P(a|A)/P(a)$  with the combined Bayes factor for five items of evidence, say  $P(a_1 \wedge \dots \wedge a_5|A_1 \wedge \dots \wedge A_5)/P(a_1 \wedge \dots \wedge a_5)$ , for different values of sensitivity and specificity of the evidence. The latter always exceeds the former, as soon as the individual pieces of evidence support the individual hypotheses (the order is reversed if it does not, and evidence with sensitivity=specificity=.5 results in Bayes factor 1, no matter what the prior is or how many items of evidence there are). Dawid's claim that 'the support supplied by the conjunction of



several independent testimonies exceeds that supplied by any of its constituents' holds, again granted suitable independence assumptions.



please use meaningful variable names, I revised s1 and s2 to sensitivity and specificity

Figure 3: BF for varying number of items of evidence and test qualities, with the usual independence assumptions.

What happens if  $A$  and  $B$  are not necessarily probabilistically independent as in the Bayesian network in Figure 2? The claim still holds under weaker independence assumptions in the background.

**Fact 1.** Assuming (4), (6) and (14), if  $P(B|b \wedge a) > P(B|a)$  and  $P(A|a \wedge b) > P(A|b)$ , we have:

$$S_{AB} > \frac{P(a|A)}{P(a)}, \frac{P(b|B)}{P(b)}$$

Alicja: double check, are these d-sep in the second network?

**Argument.** First, notice the following holds given (4) and (6):

$$\frac{P(a \wedge b|A \wedge B)}{P(a \wedge b)} = \frac{P(a|A)}{P(a)} \times \frac{P(b|B)}{P(b|a)} \quad (15)$$

$$s'_{AB} = s_A \times s'_B, \quad (16)$$

where factor  $s_B = P(b|B)/P(b)$  was replaced by  $s'_B = P(b|B)/P(b|a)$ .<sup>1</sup> Hence,  $s'_{AB}$  should be greater than one provided  $s_A$  and  $s_B$  are both greater than one. Note that the reasoning behind (16) is symmetric and so the assumed inequalities give the result marked by the braces:

$$\frac{P(a \wedge b|A \wedge B)}{P(a \wedge b)} = \frac{P(a|A)}{P(a)} \times \underbrace{\frac{P(b|B)}{P(b|a)}}_{>1} = \frac{P(b|B)}{P(b)} \times \underbrace{\frac{P(a|A)}{P(a|b)}}_{>1}$$

M: What does comma mean?

Alicja: check if d-sep in this fn holds in the second bn

M: Do we show that the joint support exceeds the LOWEST of the individual support? Unclear

**Argument ends.**

Introduce simulations, illustrate results for aggregation and BF without independence here

<sup>1</sup>The difference between  $s_{AB}$  and  $s'_B$  is that  $b$  need not be probabilistically independent of  $a$ , and thus there is no guarantee that  $P(b|a) = P(b)$ . In fact,  $s'_B$  is usually lower than  $s_B$  because  $P(b|a) > P(b)$  (assuming, at least,  $a$  and  $b$  are convergent pieces of evidence; SEE DISCUSSION IN EARLIER CHAPTERS ON CROSS-EXAMINATION AND CORROBORATION). At the same time,  $s'_B$  should still be greater than one if  $b$  positively supports  $B$  even conditional on  $a$ . Note that  $\frac{P(b|B)}{P(b|a)} = \frac{P(b|B \wedge a)}{P(b|a)}$ , by independence (14). Further, by Bayes' theorem,  $P(B|b \wedge a) = \frac{P(b|B \wedge a)}{P(b|a)} \times P(B|a)$ , so  $P(B|b \wedge a)$  is obtained by multiplying  $P(B|a)$  by  $s'_B$ . In other words, the claim that  $\frac{P(b|B)}{P(b|a)} > 1$  is equivalent to the claim that  $P(B|b \wedge a) > P(B|a)$ . The latter is plausible in this context, since  $b$  should still raise the probability of  $B$  even in conjunction with  $a$ , or else  $b$  would be useless evidence.

## 2.2 Likelihood ratio and aggregation

Let's now see what happens if we replace the Bayes factor with the likelihood ratio, another probabilistic measure of evidential support, extensively discussed in Chapter XXX. As we shall understand it here, the likelihood ratio compares the probability of the evidence on the assumption that a hypothesis of interest is true and the probability of the evidence on the assumption that the negation of the hypothesis is true, that is,  $P(E|H)/P(E|\neg H)$ . The greater the likelihood ratio (for values above one), the stronger the evidential support in favor of the hypothesis (as contrasted to the its negation). We can think of the the likelihood ratio as the following:

$$\frac{\text{sensitivity}}{1 - \text{specificity}}$$

Unlike the Bayes factor, the likelihood ratio is not sensitive to the priors<sup>2</sup>. In this sense, it is a more suitable measure if we want to factor out the effects that priors may have on the assessment of evidential strength. What is the combined likelihood ratio and how does it behave?

**Fact 2.** If independence assumptions (5), (7), (8), (9), (10), (11), (12) and (13) hold, the combined likelihood ratio is:

$$\frac{P(a \wedge b | A \wedge B)}{P(a \wedge b | \neg(A \wedge B))} = \frac{P(a|A) \times P(b|B)}{\frac{P(\neg A)P(B|\neg A)P(a|\neg A)P(b|B) + P(A)P(\neg B|A)P(a|A)P(b|\neg B) + P(\neg A)P(\neg B|\neg A)P(a|\neg A)P(b|\neg B)}{P(\neg A)P(B|\neg A) + P(A)P(\neg B|A) + P(\neg A)P(\neg B|\neg A)}} \quad (17)$$

If, further the hypotheses are independent in the sense of this reduces to:

$$\frac{P(a \wedge b | A \wedge B)}{P(a \wedge b | \neg(A \wedge B))} = \frac{P(a|A) \times P(b|B)}{\frac{P(\neg A)P(B)P(a|\neg A)P(b|B) + P(A)P(\neg B)P(a|A)P(b|\neg B) + P(\neg A)P(\neg B)P(a|\neg A)P(b|\neg B)}{P(\neg A)P(B) + P(A)P(\neg B) + P(\neg A)P(\neg B)}} \quad (18)$$

**Argument.** The numerator can be computed as follows:

$$\begin{aligned} P(a \wedge b | A \wedge B) &= \frac{P(A \wedge B \wedge a \wedge b)}{P(A \wedge B)} && \text{(conditional probability)} \\ &= \frac{P(A) \times P(B|A) \times P(a|A \wedge B) \times P(b|A \wedge B \wedge a)}{P(A) \times P(B|A)} && \text{(chain rule)} \\ &= \frac{P(A) \times P(B|A) \times P(a|A) \times P(b|B)}{P(A) \times P(B|A)} && \text{(independencies (5) and (10))} \\ &= P(a|A) \times P(b|B) && \text{(algebraic manipulation)} \end{aligned}$$

The numerator clearly does not depend on the prior probability of  $A \wedge B$ . We call it *combined sensitivity*, and given the independence assumptions it simply results from multiplying the sensitivity of the individual items of evidence,  $a$  and  $b$ , relative to their respective hypotheses,  $A$  and  $B$ .

The denominator is more involved:

$$\begin{aligned} P(a \wedge b | \neg(A \wedge B)) &= \frac{P(a \wedge b \wedge \neg(A \wedge B))}{P(\neg(A \wedge B))} && \text{(conditional probability)} \\ &= \frac{P(a \wedge b \wedge \neg A \wedge B) + P(a \wedge b \wedge A \wedge \neg B) + P(a \wedge b \wedge \neg A \wedge \neg B)}{P(\neg A \wedge B) + P(A \wedge \neg B) + P(\neg A \wedge \neg B)} && \text{(logic \& additivity)} \end{aligned}$$

Now consider the first summand from the numerator:

$$\begin{aligned} P(a \wedge b \wedge \neg A \wedge B) &= P(\neg A)P(B|\neg A)P(a|\neg A \wedge B)P(b|a \wedge \neg A \wedge B) && \text{(chain rule)} \\ &= P(\neg A)P(B|\neg A)P(a|\neg A)P(b|B) && \text{(independencies (7) and (11))} \end{aligned}$$

The simplification of the other two summanda is analogous (albeit with slightly different independence assumptions—(8) and (12) for the second one and (9) and (13) for the third. Once we plug these into the denominator formula we get:

<sup>2</sup>At least so long as sensitivity and specificity are not

REF

Rafal: read the sensitivity paper and check fn.

This "short lived" bit was too strong, and too distracting at this point, commented out.

Alicja: check which of these d-sep hold in the first and in the second BN

$$P(a \wedge b | \neg(A \wedge B)) = \frac{P(\neg A)P(B|\neg A)P(a|\neg A)P(b|B) + P(A)P(\neg B|A)P(a|A)P(b|\neg B) + P(\neg A)P(\neg B|\neg A)P(a|\neg A)P(b|\neg B)}{P(\neg A)P(B|\neg A) + P(A)P(\neg B|A) + P(\neg A)P(\neg B|\neg A)}$$

(18) is easily obtained from (17) by .

#### Argument ends.

Because of the many variables at play, it is not easy to compare the combined evidential support and the individual support supplied by  $a$  and  $b$  towards  $A$  and  $B$ , as measured by the individual and combined likelihood ratio. Let's work with (18). For illustrative purposes, we simplify: Let all the sensitivities and specificities involved be the same and equal  $x$ . In this simplified set-up with independence assumptions in the background, the combined likelihood ratio therefore reduces to the following, where  $P(A) = k$  and  $P(B) = t$ :

$$\frac{P(a \wedge b | A \wedge B)}{P(a \wedge b | \neg(A \wedge B))} = \frac{x^2}{\frac{(1-k)t(1-x)x + k(1-t)x(1-x) + (1-k)(1-t)(1-x)(1-x)}{(1-k)t + (1-t)k + (1-k)(1-t)}}$$

In this simplified setup, the likelihood ratios can be easily plotted. As Figure 4 shows, the combined likelihood ratio varies depending on the prior probabilities  $P(A)$  and  $P(B)$ , as expected, but always exceeds the individual likelihood ratios whenever they are greater than one (that is, the two pieces of evidence provides positive support for their respective hypothesis).

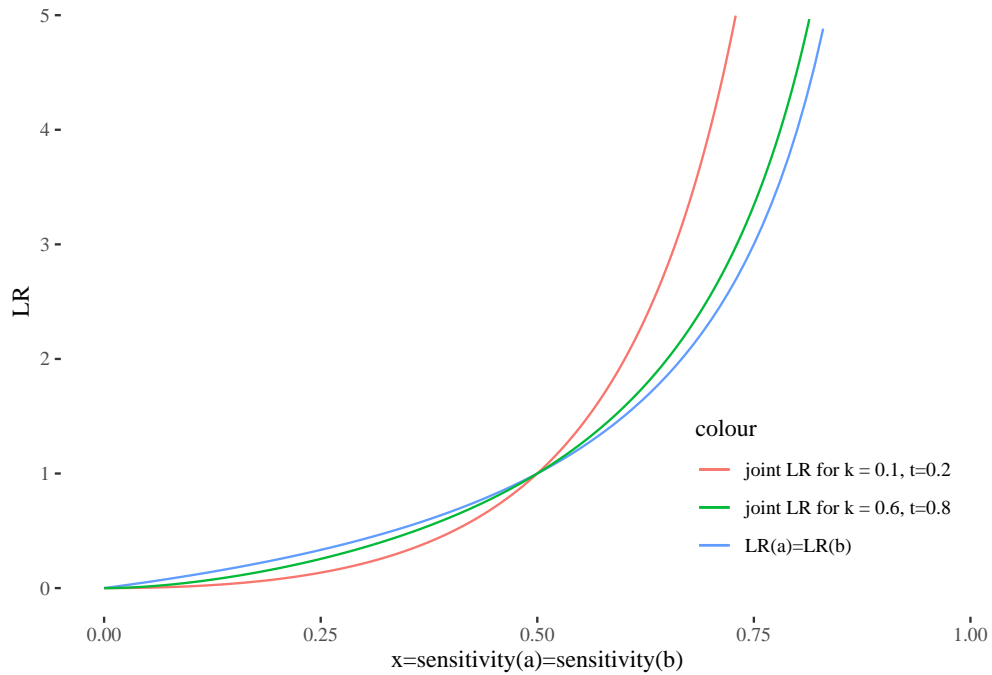


Figure 4: Combined likelihood ratios exceeds individual Likelihood ratios as soon as sensitivity is above .5. Changes in the prior probabilities  $t$  and  $k$  do not invalidate this result.

As with the Bayes factor, the combined likelihood ratio exceeds the individual likelihood ratios. However, this happens only as long as the two pieces of evidence have the same sensitivity and specificity. If the items of evidence have different levels of sensitivity and specificity, the combined likelihood ratio never goes below the lower of the two individual likelihood ratios, but can be lower than the higher one.

Notice that even if the composite likelihood ratio behaves somewhat differently than the Bayes factor in being greater than the lower of the individual likelihood ratios, rather than being greater than both of them, Dawid's claim that 'the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents', restricted to situations in which the individual support levels are in fact positive, still holds.

Need to add simulation results to make this argument fully general and drop all the simplifying assumptions (e.g. independence or equiprobability)

This also entails that any likelihood ratio threshold decision standard still satisfies aggregation: if both individual LR's are above a threshold, then so will be the composite LR, as long as one and the same threshold is to be used for the conjunction and for the conjuncts. However, whether a single threshold should be used here is somewhat unclear, and this is an issue to which we will now turn.

### 3 Variable evidential strength thresholds

Once we formulate the decision standard in terms of evidential strength instead of posterior probabilities, the standard of proof would no longer be formalized as a posterior probability threshold, but instead as a threshold for the Bayes factor or for the likelihood ratio. The threshold would no longer be a probability between 0 and 1, but rather a number somewhere above 1. The greater this number, the more stringent the standard of proof, for any value above one. In criminal trials, for example, the rule of decision would be: guilt is proven beyond a reasonable doubt if and only if the evidential support in favor of  $H$ —as measured by the Bayes factor  $P(E|H)/P(E)$  (or by the likelihood ratio  $P(E|H)/P(E|\neg H)$ )—meets a suitably high threshold  $t_{BF}$  ( $t_{LR}$ ). The obvious question at this point is, how do we identify the appropriate threshold?

One advantage of the posterior probability threshold was that in its choice one could avoid, at least to some extent, the charge of arbitrariness by plugging in decision-theoretic considerations and deriving a threshold from a cost-benefit analysis.

Can we do something analogous for BF or LR? Well, in a sense, yes. If we are able to derive  $t_{BF}$  or  $t_{LR}$  thresholds from a posterior probability threshold, they—it seems—will be as justified by decision theoretic-considerations that the threshold we started with. How to go about it?

First, Bayes factor. Since posterior = Bayes factor  $\times$  prior, the Bayes factor threshold can be determined as follows:

$$t_{BF} = \frac{t}{\text{prior}}$$

The higher the prior probability, the lower  $t_{BF}$ . Whether this is a desirable property for a decision threshold can be questioned, but a similar point holds about the posterior threshold  $t$ : the higher the prior probability, the easier to meet the posterior threshold.

Now, likelihood ratio. By the odds version of Bayes' theorem,

$$\text{posterior odds} = \text{likelihood odds} \times \text{prior odds},$$

and thus

$$\frac{\text{posterior odds}}{\text{prior odds}} = \text{likelihood ratio}.$$

If the posterior ratio is fixed at, say  $t/1-t$ ,  $t_{LR}$  can be obtained as follows:

$$t_{LR} = \frac{t/1-t}{\text{prior odds}}.$$

Just as the Bayes factor, the likelihood ratio threshold will therefore vary with the prior. The higher the prior, the lower the likelihood ratio threshold.

Now, we would like a standard of proof to be in principle applicable to all factual claims under consideration. In particular, if we consider a conjunction  $A \wedge B$ , we should have a standard that makes sense not only when applied to  $A \wedge B$ , but also when applied to  $A \vee B$ . But now notice that in the current set-up evidential strength thresholds vary with priors, and clearly the priors for  $A$  and  $B$  will differ from the priors on  $A \wedge B$ . This suggests that as long as we want  $t_{BF}$  and  $t_{LR}$  to be decision-theoretically justified by being derived from a decision-theoretically justified posterior probability threshold, the thresholds for individual claims ( $t_{BF}^A$  and  $t_{LR}^A$ ) will differ from the thresholds for the composite claim,  $t_{BF}^{A \wedge B}$  and  $t_{LR}^{A \wedge B}$ .

The conjunction principles formulated in terms of BF and LR would boil down to:

$$\begin{aligned} \frac{P(a|A)}{P(a)} > t_{BF}^A \text{ and } \frac{P(b|B)}{P(b)} > t_{BF}^B \text{ iff } \frac{P(a \wedge b|A \wedge B)}{P(a \wedge b)} > t_{BF}^{A \wedge B} \\ \frac{P(a|A)}{P(a|\neg A)} > t_{LR}^A \text{ and } \frac{P(b|B)}{P(b|\neg B)} > t_{LR}^B \text{ iff } \frac{P(a \wedge b|A \wedge B)}{P(a \wedge b|\neg(A \wedge B))} > t_{LR}^{A \wedge B} \end{aligned}$$

will at some point take a look at "Bayesian Choice"

I'm now thinking: wouldn't this make  $t$  movable as well though? Need to discuss.

Use textsf, or even better mathsf for text inside math

Now, consider the individual claims  $A$  and  $B$  and their conjunction  $A \wedge B$ , assuming the independence relations required for Facts and hold. Consider a posterior threshold of .95 as might be appropriate in a criminal case.

Alicja: REF

Alicja: REF

please use nicefrac for inline fractions

If  $A$  and  $B$  both have a prior probability of, say .1, the threshold  $t_{BF}^A = t_{BF}^B = .95/.1 = 9.5$  for  $A$  or  $B$  individually. The composite claim  $A \wedge B$  will be associated with the threshold  $t_{BF}^{A \wedge B} = .95/(.1 \times .1) = 95$ , a much higher value. But if each individual claim meets its Bayes factor threshold of 9.5 and the two claims are independent, the joint Bayes factor would result from the multiplication of the individual Bayes factors, that is,  $9.5 \times 9.5 = 90.25$ . This is not quite enough to meet  $t_{BF}^{A \wedge B} = 95$ .

The difference grows as (i) the prior probability of the individual claims decreases, (ii) as the posterior threshold  $t$  decreases, and (iii) as the number of constituents claims increases. For two constituents, the combined Bayes factor remains only 5% below the value needed to meet  $t_{BF}^{A \wedge B}$ . The difference at here is between  $t_{BF}^{A \wedge B} = .95/p^2$  and  $t_{BF}^A * t_{BF}^B = (.5/p)^2$ . Note that  $\frac{.95/p^2 - (.5/p)^2}{.95/p^2} = .05$ , for any value of the prior  $p$ . Given five constituent claims,  $\frac{.95/p^5 - (.95/p)^5}{.95/p^5} = .18$ . Now, say  $t = .5$ . Even with just two claims,  $t_{BF}^{A \wedge B} = .5/(.1 \times .1) = 50$ , but  $t_{BF}^A * t_{BF}^B = (.5/.1) * (.5/.1) = 25$ , only half of the required value.

An analogous point will hold for the likelihood ratio. Say  $A$  and  $B$  have prior probabilities of .2 and .3 respectively. On this approach, the likelihood ratio threshold for  $A$  and  $B$  will be  $t_{LR}^A \approx 76$  and  $t_{LR}^B \approx 44$ . The likelihood ratio threshold for the composite claim  $A \wedge B$  will be  $t_{LR}^{A \wedge B} \approx 297$ . Now suppose the individual likelihood ratios meet their threshold and respective sensitivities and specificities are identical. For  $t_{LR}^A$  to be met, evidence  $a$  should have sensitivity of at least 0.988. For  $t_{LR}^B$  to be met, evidence  $b$  should have sensitivity 0.978. Now with these separate sensitivities, The combined likelihood ratio equals about 145, far short that what the threshold  $t_{LR}^{A \wedge B}$  requires, namely a likelihood ratio of 297.

Things don't get better with a lower posterior threshold. Say  $t = .5$ , as might be appropriate in a civil case. The likelihood ratio thresholds for  $A$  and  $B$  will be  $t_{LR}^A \approx 4$  and  $t_{LR}^B \approx 2.3$ . The likelihood ratio threshold for the composite claim  $A \wedge B$  will be  $t_{LR}^{A \wedge B} \approx 15.6$ . To ensure that  $t_{LR}^A$  is met, evidence  $a$  should have a sensitivity of at least 0.8, and to ensure that  $t_{LR}^B$  is met, evidence  $b$  should have a sensitivity of at least 0.7. To ensure that  $t_{LR}^{A \wedge B}$  is met, evidence  $b$  should have a sensitivity of at least 0.7. With these parameters, the combined likelihood ratio equals about 5, far short that what the threshold  $t_{LR}^{A \wedge B}$  requires, namely a likelihood ratio of 15.

The general lesson here is this. If we want the choice of evidential strength threshold to be informed by decision-theoretic considerations (as might be advisable), and we do so by deriving them from decision-theoretically justified posterior thresholds, the evidential strength thresholds become variable: they depend on priors. If in fact we then do keep track of which threshold is supposed to be used for which claim, aggregation still fails: there will be cases in which the conjuncts taken separately satisfy the decision standard, while the conjunction does not.

Perhaps, this is not the path that the proponent of the evidential strength approach would take anyway. After all, if your evidential strength thresholds mirror posterior threshold, it should not be deeply surprising if they run into similar problems. So what would happen if instead, we insisted that the evidential strength thresholds should be kept fixed?

Question: how does Kaplow think about it? Does he only derive threshold for the ultimate claim? I don't remember now.

## 4 Fixed evidential strength threshold

The alternative here is to fix the evidential strength threshold regardless of the prior probability of the claim of interest. This raises the difficult question of how to fix the threshold irrespective of the priors. Standard decision theory can no longer be used. As it turns out, even if the question can be satisfactorily answered, the fixed threshold approach gives rise to a complication that proves fatal.

If the standard of proof is formalized using a fixed threshold for the Bayes factor or for the likelihood ratio, the conjunction principle would boil down to one of these:

$$\frac{P(a|A)}{P(a)} > t_{BF} \text{ and } \frac{P(b|B)}{P(b)} > t_{BF} \text{ iff } \frac{P(a \wedge b|A \wedge B)}{P(a \wedge b)} > t_{BF}$$

$$\frac{P(a|A)}{P(a|\neg A)} > t_{LR} \text{ and } \frac{P(b|B)}{P(b|B)} > t_{LR} \text{ iff } \frac{P(a \wedge b|A \wedge B)}{P(a \wedge b|\neg(A \wedge B))} > t_{LR}$$

The left-to-right direction—aggregation—will hold for any thresholds  $t_{BF}$  or  $t_{LR}$  greater than one, as Facts and entail.

This is an improvement. After all, aggregation could not be justified using posterior probabilities  $P(A|a)$  and  $P(B|b)$  nor could it be justified generally using a variable Bayes factor threshold. So it is an advantage of the fixed evidential strength threshold approach that it can justify this direction of the conjunction principle.

However, the right-to-left direction—distribution—has now become problematic. Recall that we can distinguish at least two versions of the distribution principle if we take the evidential support into account:

If  $S[a \wedge b, A \wedge B]$ , then  $S[a, A]$  and  $S[b, B]$ . (DIS1)

If  $S[a \wedge b, A \wedge B]$ , then  $S[a \wedge b, A]$  and  $S[a \wedge b, B]$ . (DIS2)

We will think of both of them. Let us start with (DIS1) first, still working with the independence assumptions in the background for the sake of simplicity. Suppose the combined Bayes factor,  $\frac{P(a \wedge b|A \wedge B)}{P(a \wedge b)}$ , barely meets the threshold. The individual support, say  $\frac{P(a|A)}{P(a)}$ , could be still below the threshold unless  $\frac{P(b|B)}{P(b)} = 1$  (which should not happen if  $b$  positively supports  $B$ ).

The problem for likelihood ratio is analogous. For suppose evidence  $a \wedge b$  supports  $A \wedge B$  to the required threshold  $t$ . The threshold in this case should be some order of magnitude greater than one. If the combined likelihood ratio meets the threshold  $t_{LR}$ , one of the individual likelihood ratios may well be below  $t_{LR}$ . So—if the standard of proof is interpreted using evidential support measured by the likelihood ratio—even though the conjunction  $A \wedge B$  was proven according to the desired standard, one of individual claims might not. (DIS1) in some cases fails.

Perhaps, the problem was with (DS1). Could this principle be rejected? Maybe it is not as essential as we thought at first. Since the evidence is not held constant, the support supplied by  $a \wedge b$  could be stronger than that supplied by  $a$  and  $b$  individually. So even when the conjunction  $A \wedge B$  is established to the requisite standard given evidence  $a \wedge b$ , it might still be that  $A$  does not meet the requisite standard (given  $a$ ) nor does  $B$  (given  $b$ ). Or at least one might try to argue so.

(DS2) is less controversial, as it holds the evidence constant. This principle is harder to deny: one would not want to claim that, holding fixed evidence  $a \wedge b$ , establishing the conjunction might not be enough for establishing one of the conjuncts. It seems that any formalization of the standard of proof should obey (DS2). Yet, (DS2) also fails both for Bayes factor and likelihood ratio. After all, if  $A$  and  $B$  are probabilistically independent, (DIS1) and (DIS2) are in fact equivalent, so the counterexamples to (DIS1) work also against (DS2).

So, curiously, on the fixed evidential strength threshold approach, no matter whether one uses Bayes factor or likelihood ratio, there would be cases in which, even though the conjunction  $A \wedge B$  is established to the desired standard of proof, one of the individual claims fails to meet the standard. This is odd, to say the least.

## 5 The comparative strategy

Instead of thinking in terms of absolute thresholds—whether relative to posterior probabilities, the Bayes factor or the likelihood ratio—the standard of proof can be understood comparatively. This suggestion has been advanced by Cheng (2012) following the theory of relative plausibility by **REFERENCE TO ALLEN AND PARDO HERE**. Say the prosecutor or the plaintiff puts forward a hypothesis  $H_p$  about what happened. The defense offers an alternative hypothesis, call it  $H_d$ . On this approach, rather than directly evaluating the support of  $H_p$  given the evidence and comparing it to a threshold, we compare

Alicja: REF

Alicja: REF, not sure how many we end up with, double check

R: I don't think this argument worked, we can discuss this. Commented out

M: What exactly did not work in the argument that you commented out? It is just a restatement of the earlier point.

M: This section seems to be missing a lot of the reasoning that was commented out. Is this still comprehensible? Not sure I can follow. Seems too brief.

Which one?



the support that the evidence provides for two competing hypotheses  $H_p$  and  $H_d$ , and decide for the one for which the evidence provides better support.

It is controversial whether this is what happens in all trial proceedings, especially in criminal trials, if one thinks of  $H_d$  as a substantial account of what has happened. The defense may elect to challenge the hypothesis put forward by the other party without proposing one of its own. For example, in the O.J. Simpson trial the defense did not advance its own story about what happened, but simply argued that the evidence provided by the prosecution, while significant on its face to establish OJ's guilt, was riddled with problems and deficiencies. This defense strategy was enough to secure an acquittal. So, in order to create a reasonable doubt about guilt, the defense does not always provide a full-fledged alternative hypothesis. The supporters of the comparative approach, however, will respond that this could happen in a small number of cases, even though in general—especially for tactical reasons—the defense will provide an alternative hypothesis. And even such cases can be construed as involving a defense hypothesis, one that is simply equivalent to the negation of  $H_p$ .

Setting aside this controversy for the time being, let's first work out the comparative strategy using posterior probabilities. More specifically, given a body of evidence  $E$  and two competing hypotheses  $H_p$  and  $H_d$ , the probability  $P(H_p|E)$  should be suitably higher than  $P(H_d|E)$ , or in other words, the ratio  $P(H_p|E)/P(H_d|E)$  should be above a suitable threshold. Presumably, the ratio threshold should be higher for criminal than civil cases. In fact, in civil cases it seems enough to require that the ratio  $P(H_p|E)/P(H_d|E)$  be above 1, or in other words, that  $P(H_p|E)$  should be higher than  $P(H_d|E)$ .<sup>3</sup>

One advantage of this approach—as Cheng shows—is that expected utility theory can set the appropriate comparative threshold  $t$  as a function of the costs and benefits of trial decisions. For simplicity, suppose that if the decision is correct, no costs result, but incorrect decisions have their price. The costs of a false positive is  $c_{FP}$  and that of a false negative is  $c_{FN}$ , both greater than zero. Intuitively, the decision rule should minimize the expected costs. That is, a finding against the defendant would be acceptable whenever its expected costs— $P(H_d|E) \times c_{FP}$ —are smaller than the expected costs of an acquittal— $P(H_p|E) \times c_{FN}$ —or in other words:

$$\frac{P(H_p|E)}{P(H_d|E)} > \frac{c_{FP}}{c_{FN}}.$$

In civil cases, it is customary to assume the costs ratio of false positives to false negatives equals one. So the rule of decision would be: find against the defendant whenever  $\frac{P(H_p|E)}{P(H_d|E)} > 1$  or in other words  $P(H_p|E)$  is greater than  $P(H_d|E)$ . In criminal trials, the costs ratio is usually considered higher, since convicting an innocent (false positive) should be more harmful or morally objectionable than acquitting a guilty defendant (false negative). Thus, the rule of decision in criminal proceedings would be: convict whenever  $P(H_p|E)$  is appropriately greater than  $P(H_d|E)$ .

Does the comparative strategy just outlined solve the difficulty about conjunction? We will work through a stylized case used by Cheng himself. Suppose, in a civil case, the plaintiff claims that the defendant was speeding ( $S$ ) and that the crash caused her neck injury ( $C$ ). Thus, the plaintiff's hypothesis  $H_p$  is  $S \wedge C$ . Given the total evidence  $E$ , the conjuncts, taken separately, meet the decision threshold:

$$\frac{P(S|E)}{P(\neg S|E)} > 1 \qquad \frac{P(C|E)}{P(\neg C|E)} > 1$$

The question is whether  $P(S \wedge C|E)/P(H_d|E) > 1$ . To answer it, we have to decide what the defense hypothesis  $H_d$  should be. Cheng reasons that there are three alternative defense scenarios:  $H_{d_1} = S \wedge \neg C$ ,  $H_{d_2} = \neg S \wedge C$ , and  $H_{d_3} = \neg S \wedge \neg C$ . How does the hypothesis  $H_p$  compare to each of them? Assuming independence between  $C$  and  $S$ , we have

I'm not convinced this footnote is needed and what its point is.

REFERENCE TO EARLIER CHAPTER FOR MORE COMPLEX COST STRUCTURE

<sup>3</sup>Note that  $H_p$  and  $H_d$  need not be one the negation of the other. Whenever two hypotheses are exclusive and exhaustive,  $P(H_p|E)/P(H_d|E) > 1$  implies that  $P(H_p|E) > .5$ , the standard probabilistic interpretation of the preponderance standard.



$$\begin{aligned}
\frac{P(S \wedge C|E)}{P(S \wedge \neg C|E)} &= \frac{P(S|E)P(C|E)}{P(S|E)P(\neg C|E)} = \frac{P(C|E)}{P(\neg C|E)} > 1 \\
\frac{P(S \wedge C|E)}{P(\neg S \wedge C|E)} &= \frac{P(S|E)P(C|E)}{P(\neg S|E)P(C|E)} = \frac{P(S|E)}{P(\neg S|E)} > 1 \\
\frac{P(S \wedge C|E)}{P(\neg S \wedge \neg C|E)} &= \frac{P(S|E)P(C|E)}{P(\neg S|E)P(\neg C|E)} > 1
\end{aligned} \tag{19}$$

So, whatever the defense hypothesis, the plaintiff's hypothesis is more probable. At least in this case, whenever the elements of a plaintiff's claim satisfy the decision threshold, so does their conjunction. The left-to-right direction of the conjunction principle—what we called aggregation—has been vindicated, at least for simple cases involving independence.

What about the opposite direction, distribution? If the threshold to be met just 1—as might be appropriate in civil cases—distribution would be satisfied. Suppose  $P(S \wedge C|E)/P(H_d|E) > 1$ , or in other words, the combined hypothesis  $S \wedge C$  has been established by preponderance of the evidence. The question is whether the individual hypotheses have been established by the same standard, specifically, whether  $\frac{P(C|E)}{P(\neg C|E)} > 1$  and  $\frac{P(S|E)}{P(\neg S|E)} > 1$ . If  $P(S \wedge C|E)/P(H_d|E) > 1$ , the combined hypothesis is assumed to be more probable than any of the competing hypotheses, in particular,  $P(S \wedge C|E)/P(\neg S \wedge C|E) > 1$ ,  $P(S \wedge C|E)/P(S \wedge \neg C|E) > 1$  and  $P(S \wedge C|E)/P(\neg S \wedge \neg C|E) > 1$ . We have

$$\begin{aligned}
1 &< \frac{P(S \wedge C|E)}{P(S \wedge \neg C|E)} = \frac{P(S|E)P(C|E)}{P(S|E)P(\neg C|E)} = \frac{P(C|E)}{P(\neg C|E)} \\
1 &< \frac{P(S \wedge C|E)}{P(\neg S \wedge C|E)} = \frac{P(S|E)P(C|E)}{P(\neg S|E)P(C|E)} = \frac{P(S|E)}{P(\neg S|E)} \\
1 &< \frac{P(S \wedge C|E)}{P(\neg S \wedge \neg C|E)} = \frac{P(S|E)P(C|E)}{P(\neg S|E)P(\neg C|E)}
\end{aligned} \tag{20}$$

In the first two cases, clearly, if the composite hypothesis meets the threshold, so do the individual claims. But now consider the third case.  $P(S|E)P(C|E)/P(\neg S|E)P(\neg C|E)$  might be strictly greater than  $P(C|E)/P(\neg C|E)$  or  $P(S|E)/P(\neg S|E)$ . So it possible that  $P(S|E)P(C|E)/P(\neg S|E)P(\neg C|E)$  is greater than one, while either  $P(C|E)/P(\neg C|E)$  or  $P(S|E)/P(\neg S|E)$  are not, say when they are 3 and 0.5, respectively. Once again, distribution fails. And the same problem would arise with a more stringent threshold as might be appropriate in criminal cases.

There is a more general problem with the comparative approach worth flagging here. Much of the heavy lifting here is done by the strategic splitting of the defense line into multiple scenarios. Now suppose, for illustrative purposes,  $P(H_p|E) = 0.37$  and the probability of each of the defense lines given  $E$  is 0.21. This means that  $H_p$  wins with each of the scenarios, so on this approach we should find against the defendant. But should we? Given the evidence, the accusation is very likely to be false, because  $P(\neg H_p|E) = 0.63$ . The problem generalizes. If, as here, we individualize scenarios by Boolean combinations of elements of a case, the more elements there are, into more scenarios  $\neg H_p$  needs to be divided. This normally would lead to the probability of each of them being even lower (because now  $P(\neg H_p)$  needs to be “split” between more scenarios). So, if we take this approach seriously, the more elements a case has, the more at a disadvantage the defense is. This seems undesirable.

MENTION THAT SINCE CHENG THOUGHT THE COMPARATIVE STRATEGY WOULD CAPTURE INFERENCE TO THE BEST EXPLANATION IN PROBABILISTIC TERMS, THE FAILURE OF THE COMPARATIVE STRATEGY TO CAPTURE THE CONJUNCTION PRINCIPLE IS ALSO A FAILURE OF INFERENCE TO THE BEST EXPLANATION IN CAPTURING THE CONJUNCTION PRINCIPLE SO LONG AS WE ASSUME THAT PLAUSIBILITY DOES NOT CONTRADICT PROBABILITY (WHICH ALLEN AND PARDO DO GRANT)

## 6 Rejecting the conjunction principle

We have seen that various strategies that a legal probabilist can try to use to handle the difficulty about conjunction are all problematic. Perhaps, a different perspective should be taken here? After all, the

problem would not arise without the conjunction principle. So could legal probabilists simply reject this principle?

add more flow once section is complete

## 6.1 Risk accumulation

In current discussions in epistemology about knowledge or justification, a principle similar to the conjunction principle has been contested because it appears to deny the fact that risks of error accumulate (Kowalewska, 2021). If one is reasonably sure about the truth of each claim considered separately, one should not be equally reasonably sure of their conjunction. You have checked each page of a book and found no error. So, for each page, you are reasonably sure there is no error. Having checked each page and found no error, can you be equally reasonably sure that the book as a whole contains no error? Not really. As the number of pages grow, it becomes virtually certain that there is at least one error in the book you have overlooked, although for each page you are reasonably sure there is no error (Makinson, 1965). A reasonable doubt about the existence of an error, in one page or another, creeps up as one considers more and more pages. The same observation applies to other contexts, say product quality control. You may be reasonably sure, for each product you checked, that it is free from defects. But you cannot, on this basis alone, be equally reasonably sure that all products you checked are free from defects. Since the risks of error accumulate, you must have missed at least one defective product.

Risk accumulation challenges aggregation: even if the probability of several claims, considered individually, is above a threshold  $t$ , their conjunction need not be above  $t$ . It does not, however, challenge distribution. If, all risks considered, you have good reasons to accept a conjunction, no further risk is involved in accepting any of the conjuncts separately. This is also mirrored by what happens with probabilities. If the probability of the conjunction of several claims is above  $t$ , so is the probability of each individual claim.

The standard of proof in criminal or civil cases can be understood as a criterion concerning the degree of risk that judicial decisions should not exceed. If this understanding of the standard of proof is correct, the phenomenon of risk accumulation would invalidate the conjunction principle, specifically, it would invalidate aggregation. It would longer be correct to assume that, if each element is proven according to the applicable standard, the case as a whole is proven according to the same standard. And, in turn, if the conjunction principle no longer holds, the conjunction paradox would disappear. Success.

R: What's our take on risk accumulation?

## 6.2 Atomistic and holistic approaches

Matters are not so straightforward, however. Suppose legal probabilists do away with the conjunction principle. Now what? How should they define standards of proof? Two immediate options come to mind, but neither is without problems.

Let's stipulate that, in order to establish the defendant's guilt beyond a reasonable doubt (or civil liability by preponderance of the evidence or clear and convincing evidence), the party making the accusation should establish each claim, separately, to the requisite probability, say at least .95 (or .5 in a civil case), without needing to establish the conjunction to the requisite probability. Call this the *atomistic account*. On this view, the prosecution could be in a position to establish guilt beyond a reasonable doubt without establishing the conjunction of different claims with a sufficiently high probability. This account would allow convictions in cases in which the probability of the defendant's guilt is relatively low, just because guilt is a conjunction of several independent claims that separately satisfy the standard of proof. For example, if each constituent claim is established with .95 probability, a composite claim consisting of five subclaims—assuming, as usual, probabilistic independence between the subclaims—would only be established with probability equal to .77, a far cry from proof beyond a reasonable doubt. This is counterintuitive, as it would allow convictions when the defendant is not very likely to have committed the crime. A similar argument can be run for the civil standard of proof 'preponderance of the evidence.' Under the atomistic account, the composite claim representing the case as a whole would often be established with a probability below the required threshold. The atomistic approach is a non-starter.

Another option is to require that the prosecution in a criminal case (or the plaintiff in a civil case) establish the accusation as a whole—say the conjunction of  $A$  and  $B$ —to the requisite probability. Call this the *holistic account*. This account is not without problems either.

The standard that applies to one of the conjuncts would depend on what has been achieved for the

other conjuncts. For instance, assuming independence, if  $P(A)$  is .96, then  $P(B)$  must be at least .99 so that  $P(A \wedge B)$  is above a .95 threshold. But if  $P(A)$  is .9999, then  $P(B)$  must only be slightly greater than .95 to reach the same threshold. Thus, the holistic account might require that the elements of an accusation be proven to different probabilities—and thus different standards—depending on how well other claims have been established. This result runs counter to the tacit assumption that each element should be established to the same standard of proof.

Fortunately, this challenge can be addressed. It is true that different elements will be established with different probabilities, depending on the probabilities of the other elements. But this follows from the fact that the prosecution or the plaintiff may choose different strategies to argue their case. They may decide that, since they have strong evidence for one element and weaker evidence for the other, one element should be established with a higher probability than the other. What matters is that the case as a whole meets the required threshold, and this objective can be achieved via different means. What will never happen is that, while the case as a whole meets the threshold, one of the constituent elements does not. As seen earlier, the probability of the conjunction never exceeds the probability of one of the conjunct, or in other words, distribution is never violated.

A more difficult challenge is the observation that the proof of  $A \wedge B$  would impose a higher requirement on the separate probabilities of the conjuncts. If the conjunction  $A \wedge B$  is to be proven with at least .95 probability, the individual conjuncts should be established with probability higher than .95. So the more constituent claims, the higher the posterior probability for each claim needed for the conjunction to meet the requisite probability threshold.

This difficulty is best appreciated by running some numbers. Assume, for the sake of illustration, the independence and equiprobability of the constituent claims. If a composite claim consists of  $k$  individual claims, these individual claims will have to be established with probability of at least  $t^{1/k}$ , where  $t$  is the threshold to be applied to the composite claim.<sup>4</sup> For example, if there are ten constituent claims, they will have to be proven with  $.5^{1/10} = .93$  even if the probability threshold is only .5. If the threshold is more stringent, as is appropriate in criminal cases, say .95, each individual claim will have to be proven with near certainty. This would make the task extremely demanding on the prosecution, if not downright impossible. If there are ten constituent claims, they will have to be proven with  $.95^{1/10} = .995$ . So the plaintiff or the prosecution would face the demanding task of establishing each element of the accusation beyond what the standard of proof would seem to require.

## [1] 0.933033

## [1] 0.9948838

## [1] 0.5987369

It is true that the individual elements (the individual conjuncts) should be established with a higher probability than the case as a whole (the conjunction). This would seem to impose an unreasonably stringent burden of proof on the prosecution or the plaintiff. But the burden might not be as unreasonable as it appears at first. As Dawid (1987) pointed out, in one of the earliest attempts to solve the conjunction paradox from a probabilistic perspective, the prior probabilities of the conjuncts will also be higher than the prior probability of their conjunction:

... it is not asking too much of the plaintiff to establish the case as a whole with a posterior probability exceeding one half, even though this means that the several component issues must be established with much larger posterior probabilities; for the *prior* probabilities of the components will also be correspondingly larger, compared with that of their conjunction (p. 97).

Dawid's proposal is compelling. Still, why exactly is it 'not asking too much' to establish the individual conjuncts by a higher threshold than the case as a whole? The prior probabilities of the conjuncts are surely higher than the prior probability of the conjunction. But what is the notion of 'not asking too much' at work here? Dawid might be recommending—as the rest of his paper suggests—that the standard of proof no longer be understood in terms of just posterior probabilities. Measures of how strongly each claim is being supported by the evidence, such as the Bayes' factor of the likelihood

<sup>4</sup>Let  $p$  the probability of each constituent claim. To meet threshold  $t$ , the probability of the composite claim,  $p^k$ , should satisfy the constraint  $p^k > t$ , or in other words,  $p > t^{1/k}$ .

ratio, account for the difference between prior and posterior probabilities. So, presumably, Dawid is recommending these measures as better suited to formalize the standard of proof.

Now, as the reader will have realized, we have pursued Dawid's strategy already. This strategy can justify, on purely probabilistic grounds, one direction of the conjunction principle: aggregation. The providential support—measured by the Bayes' factor or the likelihood ratio—for the conjunction often exceeds the individual support for the individual claims. This is a success, especially because the failure of aggregation motivated Cohen's formulation of the conjunction paradox. Unfortunately, we have also seen that this strategy invalidates a previously unchallenged direction of the conjunction principle, distribution. This outcome is very counterintuitive. Clearly, we cannot appeal to risk accumulation to challenge distribution since no additional risk arises after making an inference from a conjunction to one of its conjunct.

#### MAYBE ADD STUFF ABOUT FACTORING OPUT PRIORS

We reached an impasse. Under the atomistic approach, the standard is too lax because it allows for findings of liability when the defendant quite likely committed no wrong. Under the holistic approach, the standard is too demanding on the prosecution (or the plaintiff) because it requires the individual claims to be established with extremely high probabilities. Dawid's approach is compelling, but switching from posterior probability to measures of evidential support invalidate distribution, a seemingly unobjectionable principle. Bummer.

## 6.3 The holistic approach revised

It is time to take a fresh start on the holistic approach. When suitably formulated, we believe the holistic approach can satisfactorily address the difficulty with conjunction. Our proposal is inspired by the story model of adjudication (HASTIE, VAN KOPPENS, MACKROR) and the relative plausibility theory (ALLEN AND PARDO). It posits that prosecutors and plaintiffs should aim to establish a unified narrative of what happened or explanation of the evidence, not establish each individual element of wrongdoing separately. As we shall see, any attempt to proceed in a piecemeal manner implicitly requires, sooner or later, to weave the different elements together into a unified whole. Our argument consists of two parts. First, the guilt or civil liability of a defendant on trial cannot be equated with a generic claim of guilt or civil liability as defined in the law. Call this the specificity argument. Second, it is erroneous to think of someone's guilt or civil liability as the mere conjunction of separate claims. The separate claims must be unified, not just added up in a conjunction. Call this the unity argument.

### 6.3.1 The specificity argument

Let's start with the specificity argument. The probabilistic interpretation of proof standards usually posits a threshold that applies to the posterior probability of a *generic* hypothesis, such as the defendant is guilty of a crime, call it  $G$ , or civilly liable, call it  $L$ . In criminal cases, the requirement is formulated as follows: the evidence  $E$  presented at trial establishes guilt beyond a reasonable doubt provided  $P(G|E)$  is above a suitable threshold, say .95. The threshold is lower in civil trials. Civil liability is proven by preponderance provided  $P(L|E)$  is above a suitable threshold, say .5. In case of the clear and convincing standard, the threshold is assumed to be more stringent than the preponderance standard, but not as stringent as proof beyond a reasonable doubt.

While seemingly innocuous, this formulation conflates two things. The wrongdoing as defined in the applicable law is one thing. The way in which the wrongdoing is established in court is quite another thing. The wrongdoing is defined in a generic manner and is applicable across a large class of situations, whereas the way it is established in court is specific to a unique situation and tailored to the individual defendant. A prosecutor in a criminal case does not just establish that the defendant assaulted the victim in some way or another, but rather, that the defendant behaved in such and such a manner in this time and place, and that the specific behavior in question fulfills the legal definition of assault.

If this is correct, the probabilistic interpretation of proof standard should be revised. The generic claim that the defendant is guilty or civilly liable should be replaced by a more fine-grained hypothesis, call it  $H_p$ , the hypothesis put forward by the prosecutor (or the plaintiff in a civil case), for example, that the defendant, given reasonably well-specified circumstances, approached the victim, pushed and kicked the victim to the ground, and then run away. Hypothesis  $H_p$  is a more precise description of what happened that entails the defendant committed the wrong. In defining proof standards, instead of saying—generically—that  $P(G|E)$  or  $P(L|E)$  should be above a suitable threshold, a probabilistic

interpretation should read: civil or criminal liability is proven by the applicable standard provided  $\Pr(H_p|E)$  is above a suitable threshold, where  $H_p$  is a reasonably specific description of what happened according to the prosecutor or the plaintiff.

This revision of the probabilistic interpretation of standards of proof may appear inconsequential, but it is not. It is the revision we invoked to address the puzzles of naked statistical evidence in another chapter **REFERENCE TO EARLIER HAPER**. Recall the gist of the argument. Consider the prisoner hypothetical, a standard example of naked statistical evidence. The naked statistics  $E_s$  make the prisoner on trial .99 likely to be guilty, that is,  $P(G|E_s) = .99$ . It is .99 likely that the prisoner on trial is one of those who attacked and killed the guard. This is a generic claim. It merely asserts that the prisoner was – with very high probability – one of those who killed the guard, without specifying what he did, how he partook in the killing, what role he played in the attack, etc. If the prosecution offered a more specific incriminating hypothesis, call it  $H_p$ , the probability  $P(H_p|E_s)$  of this hypothesis based on the naked statistical evidence  $E_s$  would be well below .99, even though  $P(G|E_s) = .99$ . That the prisoner on trial is most likely guilty is an artifact of the choice of a generic hypothesis  $G$ . When this hypothesis is made more specific—as should be—this probability drops significantly. And the puzzle of naked statistical evidence disappears.

### 6.3.2 The unity argument

The specificity argument addresses the problem of naked statistical evidence, but also provides the necessary background for addressing the difficulty about conjunction. In the traditional formulation, not only are  $G$  and  $L$  understood as generic claims. They are also understood as *mere conjunctions* of simpler claims that correspond to the elements of wrongdoing in the applicable law. Since the probability of a conjunction is often lower than the probability of its conjuncts, the individual claims can be established with a suitably high probability that meets the required threshold even though the conjunction as a whole fails to meet the same threshold. The mismatch between the probability of the conjunction and the probability of its conjuncts gives rise to the difficulty with conjunction.

What triggers the difficulty is that the prosecutor or plaintiff are assumed to establish each element in isolation. If they manage to prove each element to the desired standard, they presumably have met their burden, or at least this is what the conjunction principle suggests. But here lies another conflation. It is one thing to establish that the defendant committed  $A$  and  $B$ , where the wrongdoing in question, say assault, is defined in the law as comprising two elements,  $A$  and  $B$ . It is another thing to establish that the defendant—*this* defendant—committed assault. Someone's guilt (and the same applies to civil liability) cannot be the mere conjunction of generic claims corresponding to the elements of wrongdoing as defined in the law. What is it, then? Someone's guilt is a state of affairs that is described by a well-specified series of events that possess a coherent, structured unity. These events, taken as a whole, can be subsumed under the legal definition that consists of several discrete elements.

The conjunction paradox, as is commonly formulated, is oblivious to the fact that the different elements of a crime should be part of the same episode of wrongdoing. The paradox assumes that criminal or civil wrongdoing have a simple structure and consist in a collection of elements to be proven by the required standard. The law is more complicated. The elements of wrongdoing, in a criminal or civil setting, often possess a complex structure and are not merely separate items each to be proven by the required standard. Consider, for example, the allegation of negligent misrepresentation to be established by clear and convincing evidence. These are the elements a plaintiff should establish:<sup>5</sup>

- (E1) defendant supplied false information to guide plaintiff in their business transactions;
- (E2) defendant knew the false information was supplied to guide plaintiff;
- (E3) defendant was negligent in obtaining or communicating the false information;
- (E4) plaintiff relied on the false information;
- (E5) plaintiff's reliance on the false information was reasonable; and
- (E6) the false information proximately caused damages to the plaintiff.

For the plaintiff to prevail in this type of case, they should prove each element by the required standard. What does that actually require? The plaintiff should first establish, at a minimum, that the defendant supplied false information for the guidance of the plaintiff in the course of a business transaction. Say the defendant, a owner of a vacation resort, told plaintiff, a travel agent, that the resort

<sup>5</sup>As an illustration, we follow the jury instructions of the State of Washington. EXACT REFERENCE

included amenities that were not actually there, and the plaintiff decided to book several clients at the defendant's resort instead of other resorts. The plaintiff could offer copies of emails communication or screenshots from the resort's website. This would take care of the first element. The second element qualifies the first, in that the defendant *knew* they were supplying false information to guide the business transaction. Clearly, establishing the second element presupposes having already established the first since the second element is a qualification of the first. The truth of the second element entails the truth of the first. If the defendant knowingly supplied false information (second element), they did supply false information (first element). The third element requires to show that the defendant was negligent in obtaining or communicating the false information. This is another qualification that applies to the first element and cannot be proven without having already proven the first element. The fourth and fifth element should be understood together. The plaintiff relied on the false information (fourth element) and such reliance was reasonable (fifth element). In turn, establishing the fourth and fifth element presupposes that the plaintiff did supply false information to begin with (first element). Finally, the sixth element concerns causation of harm. This is again a qualification of the first element and cannot be established without having already established the first element.

As we can see, in the case of negligent misrepresentation, the legal definition already imposes a structure on how the different elements relate to one another. They are not merely separate items that should be proven one by one.

In addition to the legal definition itself, an important theoretical, conceptual point should be made that holds regardless of how wrongdoings are actually defined in the law. The different elements should be part of the same episode or unit of wrongdoing. Even though the different elements may concern actions performed by different people, at different times, in different places, they must be part of the same wrongdoing. The false information that the defendant supplied to the plaintiff must be the information that caused damages to the plaintiff; it must be the information that the plaintiff reasonably relied on during the business transaction in question; it must be the information knowingly supplied by the defendant for the purpose of influencing the business transaction; and so on. Even though each element could be proven separately, establishing the wrongdoing requires weaving them together.

Some might object that, in principle, establishing the individual elements of wrongdoing separately might be enough, at least in some cases, for establish the wrongdoing as a whole. If there are such cases, they are likely to be more the exception than the norm. To see how fundamental it is to establish that the different elements belong to the same unit of wrongdoing, consider a much simpler example than negligent misrepresentation. This example does not impose any structure on how the elements of wrongdoing should relate to one another. Because of its simplicity, this examples is sometimes used to illustrate the conjunction paradox (SEE CHENG). Suppose only two elements must be proven. Element 1: the defendant's conduct caused a bodily injury to victim. Element 2: the defendant's conduct consisted in reckless driving. Call this criminal offense "vehicular assault."<sup>6</sup> The two elements are not independent, but they each add novel information. It could be that the defendant's driving caused an injury to victim, but the driving was not reckless, or the driving was reckless, but no injury ensued. Neither element is presupposed by the other. The law does not impose any structure between the elements, unlike the example of negligent misrepresentation.

How could one go about establishing the claim of reckless driving that caused injury to the victim? One option, following the revised holistic approach, is to offer a detailed reconstruction of what happened. The reconstruction could go something like this. The defendant was driving above the speed limit, veering left and right. The defendant's reached a school crosswalk when children were getting out of school. The defendant hit a child on the crosswalk who was then pushed against a light pole on the sidewalk incurring a head injury. This story is supported by plenty of evidence: other children, people standing around, police officers, paramedics. There is plenty of supporting evidence as the incident occurred in the middle of the day. Taken at face value, this story does establish both elements: reckless driving and cause of injury. Parts of the story are relevant for element 1 (reckless driving) and others are relevant for element 1 (cause of injury). The two cannot be neatly separated, however. But what is crucial is that the different parts of the story are part of the same episode, the same unit of wrongdoing.

Could the prosecutor prove vehicular assault in a piecemeal manner? This would be implausible from a conceptual point of view, let alone being at odds with trial practice. But suppose, for the sake of argument, the prosecutor attempted to establish vehicular assault in a piecemeal manner, by establishing, first, that the defendant drove recklessly, and second—*separately from the first element*—that the

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<sup>6</sup>CITE RELATED LAWS



defendant's reckless driving caused injury. Once both items are established by the required standard, there would still be something left to establish.

As noted before, it is not enough to establish that the defendant drove recklessly at some point in time somewhere. Nor is it enough to establish that the defendant's driving caused injury. The prosecutor should offer a specific story detailing what happened, a story relevant for the first element and a story relevant for the second element. Say this expectation of more specificity is met. Suppose the prosecutor did not simply establish generic element 1 and generic element 2 of the charge, but rather, a reasonably detailed story which, if true, would establish element 1 and a story which, if true, would establish element 2. Wouldn't that be enough? Each element—more precisely, each story associated with each element—has been established by the required standard. Still, there would be something missing here.

The prosecutor should establish that the two elements—reckless driving and injury, or the two stories associated with the two elements—are part of the same unity of wrongdoing. It must be *this* reckless driving that caused *this* injury. So, under the piecemeal approach, the prosecutor would be tasked with establishing three claims: (1) the defendant, in some well-specified circumstances, was driving recklessly; (2) the defendant, in some well-specified circumstances, caused injury to the victim; and (3) the well-specified circumstances in (1) and (2) are part of the same episode. But once (3) is established, the prosecutor would have effectively established the charge by the required standard in accordance with the holistic approach. The prosecutor did not only establish each separate element (two separate stories) but also wove the two elements (the two stories) together. Once the piecemeal approach is pursued to its logical conclusion, it coincides with the holistic approach.<sup>7</sup>

Let's recapitulate the argument in schematic form. If the prosecutor or the plaintiff is expected to establish claim *A* and *B* by the required standard, what the law actually requires—even in terms of the piecemeal approach—is (1) to establish *A*; (2) to establish *B*; and (3) to establish *A* and *B* are part of the same unit of wrongdoing by the required standard. Item (3) is often implicit, which leaves the impression that the law only requires to establish (1) and (2) separately. Interestingly, (3) entails (1) and (2). In fact, (3) amounts to establishing a unified story, narrative or explanation about what happened. Such a narrative should be subsumed under the different elements of wrongdoing as defined in the law. The piecemeal approach and the revised holistic approach, therefore, converge.

Not all wrongful acts, in civil or criminal cases, require the prosecutor or the plaintiff to establish a unified *spatio-temporal* narrative. It might not be necessary to show that all elements of an offense occurred at the same point in time or in close succession one after the other. Some wrongful acts may consist of a pattern of acts that stretches for several days, months or even years. There may be temporal and spatial gaps that cannot not be filled. We consider several of these examples in our discussion of naked statistical evidence **SEE PREVIOUS CHAPTER**. Be that as it may, an accusation of wrongdoing in a criminal or civil case should still have a degree of cohesive unity. The acts and occurrences that constitute the wrongdoing should belong to the same wrongful act. It is this unity which the plaintiff and the prosecution must establish when they make their case. One way to establish this unity is by providing a unifying narrative, but this need not be the only way. Perhaps the expressions 'theory' or 'explanation' are more apt than 'narrative' or 'story.'

### 6.3.3 Probability, specificity and completeness

There is a distinction between a narrative (or theory, story, explanation, account) and a mere conjunctions of elements of wrongdoing  $E_1 \wedge E_1 \wedge \dots \wedge E_k$ . The narrative describes one way among many of instantiating the conjunction. This distinction is important. The claims that constitute a narrative or unified explanation need not map neatly onto the elements of the wrongdoing. The narrative will comprise claims about the evidence itself and how the evidence supports other claims in the narrative, say that witnesses were standing around when the defendant's car hit the child. The narrative or explanation will not only comprise a description of what happened but also of how we know this is what happened.

The distinction between narrative and the mere conjunction of elements matters for how we should understand the standard of proof. Other things being equal, the conjunction is more probable on the evidence than the narrative, and each conjunct even more probable. But this does not mean that the

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<sup>7</sup>We should be clear that it is not enough for the prosecutor or plaintiff to provide well-specified narrative in support of their allegations, even when they are well-supported by the evidence. When the two narrative are combined into one narrative, its probability could well be below the threshold. If we only require that each element-specific narrative be proven, a defendant could be found criminally or civilly liable even though it is unlikely that they committed the alleged wrongful act. This counter-intuitive result is similar to the one that arose with the atomistic approach.



conjunction is established by a higher standard of proof than the narrative. As we argued in the chapter on naked statistical evidence **REFERENCE TO EARLIER CHAPTERS**, a highly probable narrative that nevertheless lacks the desired degree of specificity will fail to meet the standard of proof, even though a more specific narrative that is otherwise less probable might well meet the standard. On this account, the standard of proof consists of two criteria: 1. the posterior probability of the proposed narrative (or theory, story, explanation) given the evidence presented at trial; and 2. the degree of specificity, coherence and unity of the narrative or explanation.

Are we giving up on legal probabilism? We are giving up on *traditional* legal probabilism. Even though ideas such as specificity, coherence and unity cannot be captured by the posterior probability alone, they can be formalized as properties of Bayesian networks. **NEED BRIEF DISCUSSION OF HOW THIS WORKS IN GENEREAL AND REFERENCES** Further, this analysis of the standard of proof – which combines two criteria, posterior probability and specificity – can be evaluated using concepts from probability theory that are not posterior probabilities. To illustrate, compare a trial system that convicts defendants on the basis of accusatory claims that generic but highly probable, as opposed to a trial system that convicts defendants on the basis of accusatory claims that are more specific but less probable. A natural question to ask at this point is, which trial system will make fewer mistakes – fewer false convictions and false acquittals – in the long run? The answer is not obvious. But the question can be made precise in the language of probability. The question concerns the diagnostic properties of the two trial systems, specifically their rate of false positives and false negatives. We examine this question in a later chapter. **REFERENCE TO LATER CHAPTERS** To anticipate, we argue that more specific claims are liable to more extensive *direct* adversarial scrutiny than generic claims. The more specific someone's claim, the more liable to be attacked. At the same time, if a specific claim resists adversarial scrutiny, it becomes more firmly established than a less specific claim that survived scrutiny by evading the questions. So specificity plays an accuracy-conducive role even though more specific claims are, other things being equal, less probable than more generic claims.<sup>8</sup> That is why specificity should be an important ingredient in any theory of the standard of proof.

Another ingredient worth adding to posterior probability and specificity is the completeness of the evidence presented at trial. Could the probability of someone's guilt be extremely high just because the evidence presented is one-sided and missing crucial pieces of information? It certainly can. If the probability of guilt is high because of the evidence is partial, guilt was not proven beyond a reasonable doubt. It is a matter of dispute whether knowledge about the partiality of the evidence should affect the posterior probability. After all, if we know that evidence about a hypothesis is missing, shouldn't we revise the assessment of the posterior probability of the hypothesis? This may be true, but the problem is that the content of the missing evidence is unknown. The missing evidence might increase or decrease this probability. We cannot know that without knowing what the evidence turns out to be. If we knew how the missing evidence would affect our judgment about the defendant's guilt, the evidence would no longer be — strictly speaking — missing.

THINGS TO ADD: 0. POINT ABOUT UNITY 1. ROLE OF COMPLETENESS OF EVIDENCE, SEE OREGON DNA CASE, MEANT TO SHOW THAT HOLISTIC AOPPROACH IS NOT AD HOC, WEIGHT, RESILIENCE IMPORTANT FOR ASESSEING STRENGHT OF EVIDENCE AND STANDARD OF PROOF, GUILT COULD BE HIGHLY PROBABLE GIVEN AVAILABLE EVIDEMCE, BUT THIS NEED NOT BE ENOUGH IF EVIDENCE THAT SHOULD BE THERE IS MISSING 2. NOT DENYING THAT NARRATIVES ARE CONJUNCTION, BUT THE CONJUNCTS DO NOT MAP NEATLY ONTO THE ELEMENTS OF A CRIME.

### 6.3.4 The conjunction principle revised

What does this all tell us about the conjunction paradox? Say we take seriously the idea that the standard of proof is a legal device to optimize the satisfaction of the following three criteria:

1. The defendant's civil or criminal liability must be sufficiently high
2. The narrative, story, theory, explanation that details the defendant's civil or criminal liability should be sufficiently (reasonably) specific
3. The supporting evidence should be sufficiently (reasonably) complete

The conjunction paradox could hardly arise given this conception of the standards of proof. It could hardly arise because the individual elements of the crime are established via the mediation

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<sup>8</sup>CITE POPPER HERE

of a reasonably specific and sufficiently probable narrative whose supporting evidence is reasonably complete.

What about the conjunction principle? Taking seriously the idea that prosecutors and plaintiffs should aim to establish a well-specified, unified account of the wrongdoing trivializes this principle and thus dissolves the difficulty about conjunction. Suppose the prosecutor has established a narrative  $N$  to a very high probability, say above the required threshold for proof beyond a reasonable doubt. Denote the elements of wrongdoing by  $E_1, E_2, \dots$ . Then,

$$P(E_1 \wedge E_2 \wedge \dots \wedge E_k | N) = P(C_i | N) = 1 \text{ for any } i = \{1, 2, \dots, k\}.$$

Both directions of the conjunction principle, aggregation and distribution, are now trivially satisfied. Once we condition on the narrative  $N$ , each individual claim has a probability of one and thus their conjunction also has a probability of one. The narrative, however, has a probability short of one, up to whatever value is required to meet the governing standard of proof. This is the way we suggest that we make sense of the conjunction principle.

## 6.4 Notes on the argument

- Need to insist on risk accumulation. Many reasons why probability of conjunction is less than probability of conjunct. This cannot be denied. Examples. Illustrations.
- What is the role of defeaters? Say  $A$  has defeater  $D_a$  (undercutter) and not- $A$  (rebutter) and  $B$  has defeater  $D_b$  (undercutter) and not- $B$  (rebutter). If we can rule out  $D_a$  and  $D_b$  and not- $A$  and not- $B$ , could we say that we have ruled out all defeaters for  $AB$ ? Not necessarily. The rebutter not- $AB$  need not be a rebutter for  $A$  or  $B$ , and there could be an undercutter  $D_{ab}$  that does not undercut  $A$  or  $B$ . Need to give examples.
- Key move. We do not establish claims  $A$  and  $B$  in a vacuum. They need to be part of a narrative, explanation, story. The story gives the broader context, but also explains how the evidence supports the claim and how challenges are addressed. Call this  $N_a$  and  $N_b$ . The task then is to put together  $N_a$  and  $N_b$ .
- Could whole narrative  $N$  be more coherent (and thus more probable) than  $N_a$  and  $N_b$ ? Boost of probability due to coherence.
- Elements of an accusation are structured in different ways. Often we see an “accumulation” structure, where each element is intended to provide an even more specific description of the crime. So the structure is something like this. Element 1:  $A$ . Element 2:  $A + B$ . Element 3:  $(A + B) + C$ . Etc. In such cases the conjunction paradox does not apply.
- Suppose we isolate parts of the story that are relevant for Element 1, call it  $S_1$ , and parts of the story that are relevant for Element 2, call it  $S_2$ . Since both sub-stories are indexed through a time and space, it is clear that they form the same unity. Now  $S_1$  and  $S_2$  will be more likely than  $S$ . There is no doubt about that. This is a probabilistic fact. And fact about the world. But  $S_1$  and  $S_2$  will also be much less detailed than  $S$ . There would be several questions that  $S_1$  and  $S_2$  do not address.  $S_1$  will not address whether the driver hit anyone or got on the sidewalk, while  $S_2$  presumably would not address what the car was driving before hitting the child. Now specificity is part of the standard of proof. The standard of proof requires adequate probability and adequate specificity. These two directions cannot be directly compared. So, in a sense  $S_1$  and  $S_2$  are more firmly established because they are more likely, but  $S$  is more firmly established because it is more specific. The standard of proof includes probability and specificity.
- Does that mean that specificity is not a probabilistic notion? Cite Popper. How does the notion of specificity relate to coherence?
- Another dimension of the standard of proof is however resistance to challenges, or else one could just come up with a very specific story that one has completely made up. A more specific story is more liable to be proven wrong. This is basically why it is less probable, other things being equal. There are any more ways it could be false compared to a less specific story. But now compare a very specific story – that has survived all reasonable challenges directed at the story – to a less specific story that – that has survived all reasonable challenges directed at the story. Which one should we be more impressed by? It seems that the more specific story that has survived all the challenges should be at least as well (if not more) established than the less specific story that has survived all the challenges? Here is an analogy. Suppose you have gone out in a mountaineering

expedition, while your friend has just gone out to stroll. You both come back home safe (you survived the challenges), but which one is more indicative of strong survival skill? Clearly, the fact that you survived a mountaineer expedition, not just a stroll. The initial probability of survival was lower in the former than in the latter.

- But the problem recurs. What does it mean to survive challenges? Can this be explicated in probabilistic terms?
- Suppose we are understanding the standard of proof as consisting of: probability, specificity, resistance to challenge, completeness of evidence. The objection here might be that this criteria could violate accuracy maximization. Perhaps we need a simulation to show that following these four criteria is actually better for accuracy than just probability. Probability does not capture all sources of uncertainty at play and thus it is an inadequate criterion for decision.

## 6.5 What is our argument? [M:s sketch, not intended to be part of chapter]

OPTION 1. Given risk accumulation, we reject the conjunction principle (specifically, we reject aggregation). But if we reject aggregation, we are still left with two problems: (1) individual claims will have to be proven with extremely high probability, which seems to make the task too demanding for prosecutors and (2) claims will have to be proven with different standard of proof not equally, not standard would not apply uniformly across claims. Problem (1) can be addressed by noting that individual claims will have higher prior probabilities, so this would not make the task too demanding on prosecutors. We need here to develop Dawid's argument in the direction of factoring out the priors. Problem (2). Not sure.

CHALLENGE We need to come up with a formal way to factor out priors. See some attempts below. But since measures of evidential support one way or another depend on priors, it's not clear how this could be done properly. So basically, we would need to come up with a formal measure of "how hard it is to prove something" and this formal measure should capture the "burden of proof" and should return the intuitive results we expect.

OPTION 2. We switch to narratives. Prosecutor establishes a narrative to the required probability,  $P(N|E)$ . The narrative entails all the separate claims that need to be proven. Conditional on the narrative, the conjunction principle is trivially satisfied.

PROS. The switch to narratives captures the complexity of proof, different structured claims in relationships of dependence or independence.

PROBLEM. Not clear what this move actually accomplishes. Seems it changes the topic.

OPTION 3. We attack the conjunction principle upfront and show there are two different principles, each captured by probability theory – i.e. posterior probability captures distribution, while strength of evidence captures aggregation. The conjunction principle runs these two directions together as though they must be part of the same things. Perhaps not. Perhaps the standard of proof can be understood as a mixture of these two approaches.

OPTION 4. The standard of proof cannot just be based on a single criterion, say a threshold on posterior probability or a threshold on evidential strength. While these are important, most likely necessary conditions to be met, they are not sufficient. What else? The trial is an adversarial process, in which the evidence and the story or explanations proposed by one of the two parties or both are scrutinized, tested, challenged. Rebutting and undercutting defeaters must be ruled out. The most complete body of evidence, for and against the defendant, must be considered. No evidence should be left out. It is true that the posterior probability of a conjunction will tend to be lower than posterior probability of individual conjuncts. But a conjunction – the case as a whole – will typically also have a much greater degree of coherence than individual conjuncts; it will have survived more challenges as it is liable to more challenges, and the completeness of the evidence is more easily ascertained relative to the overall case. So given this multidimensional theory of the burden of proof, it is not obvious that, other things being equal, the conjunction receives a weaker support than the conjuncts or that it should receive a stronger support than the conjunct. This will depend on a variety of factors.

See Spottswood paper, Unraveling the Conjunction Paradox

OPTION 5. p-values, what about the conjunction of p-values?

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CHAPTER ENDS HERE

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## 7 Extra unstructured materials

### 7.1 Prior Probabilities

It is worth examining the holistic account more closely, focusing in particular on the role of prior probabilities, an aspect that has gone unnoticed so far. The main problem with the holistic approach is that it would require, especially in criminal cases, individual claims to be established with a very high probability, often making the task unsurmountable for the prosecution. Or so it would seem. But a composite claim such as  $A \wedge B$  will have, other things being equal, a lower prior probability than any individual claim  $A$  or  $B$ . Say a composite claim consists of  $k$  individual claims. If its prior probability is  $\pi$ , each constituent claim, assuming they are independent and equiprobable, will have a prior probability of  $\pi^{1/k}$ . The prior probability of the individual claims will approach one as the number of constituent claims increases.

Cohen worried that, as the number of constituent claims increases, the prosecution or the plaintiff would see their case against the defendant become progressively weaker and it would become impossible for them to establish liability. But this worry is an exaggeration. The paradox, as is commonly formulated, starts by assuming that the constituent claims are established by the required probability threshold and then shows that the probability of the conjunction may fall below the threshold. However, following the holistic approach, the order of presentation can be reversed. Start by assuming that the composite claim is established by the required probability threshold. No doubt the individual claims will have to be established with a higher probability, a violation of the conjunction principle. Yet, this violation is not as counterintuitive as it might first appear for two reasons. First, since risks aggregate, it is natural that the probability of a conjunction would be lower than the probability of the conjuncts. Second, the prior probabilities of the conjuncts will be higher than the prior probability of the conjunction. Thus, establishing the conjuncts with a higher probability will not be exceedingly demanding.

Along this lines, Dawid (1987), in one of the earliest attempts to solve the conjunction paradox from a probabilistic perspective, wrote:

... it is not asking too much of the plaintiff to establish the case as a whole with a posterior probability exceeding one half, even though this means that the several component issues must be established with much larger posterior probabilities; for the *prior* probabilities of the components will also be correspondingly larger, compared with that of their conjunction (p. 97).

The price of this strategy is the denial of the conjunction principle, specifically aggregation, the very motivation behind the conjunction paradox. Cohen could insist that this solution amounts to denying the paradox. To address the paradox, legal probabilists should offer a justification of the conjunction principle in probabilistic terms, something Cohen maintains cannot be done. Or can it be done?

## 8 The Conjunction Principle Is False

Neither the Bayes factor nor the likelihood ratio managed to fully justify both directions of the conjunction principle. One direction, aggregation, was justified. So the original concern that was driving Cohen's formulation of the conjunction paradox was addressed. But the other direction, distribution, failed. The failure of distribution creates a paradox of its own, what we called distribution paradox. It is odd that one could have sufficiently strong evidence in support of  $A \wedge B$ , while not having sufficiently strong evidence for  $A$  or  $B$ . This occurs even when  $A$  and  $B$  are probabilistically independent. If they were dependent of one another—say  $A$  and  $B$  were mutually reinforcing—it is possible the evidence would strongly support the conjunction, but not one of the conjuncts in isolation (because the additional support from the other claim,  $A$  or  $B$ , would be missing). But the failure of distribution manifests itself even when  $A$  and  $B$  are independent. What should we make of this? This problem exists for both the Bayes factor and the likelihood ratio.

### 8.1 Factoring Out Prior Probabilities

Let us return to the role of prior probabilities and their effect on measures of evidential strength. Dawid observed that the prior probabilities of the conjuncts are correspondingly higher than the prior probability of the conjunction. The conjunction principle, instead, ignores the role of prior probabilities and treat

the conjuncts and the conjunction only in relation to the evidence, irrespective of the prior probabilities. So, in order to capture the conjunction principle, legal probabilists should rely on probabilistic measures that are not heavily depend on prior probabilities. But, as we have seen, neither the Bayes factor nor the likelihood ratio are such measures.

We have seen that the joint Bayes factor  $P(a \wedge b|A \wedge B)/P(a \wedge b)$ , under suitable independence assumptions, is greater than the individual Bayes factors  $P(a|A)/P(a)$  and  $P(b|B)/P(b)$ . This inequality holds even if the evidence is held constant. The joint Bayes factor  $P(a \wedge b|A \wedge B)/P(a \wedge b)$ , under suitable independence assumptions, is still greater than the individual Bayes factors (for composite evidence)  $P(a \wedge b|A)/P(a \wedge b)$  and  $P(a \wedge b|B)/P(a \wedge b)$ . But the larger Bayes factor associated with the composite claim, holding the evidence fixed, need not be a sign of stronger evidence, but merely an artifact of the lower prior probability of the composite claim. The same can be said for the combined likelihood ratio. We have seen that, holding fixed the sensitivity and specificity of  $a$  and  $b$ , the combined likelihood ratio can be changed by varying the priors of  $A$  and  $B$ . The lower the priors, the stronger the likelihood ratio. Perhaps, the same body of evidence may strongly support the composite claim  $A \wedge B$ , while failing to strongly support  $A$  or  $B$  simply because  $A \wedge B$  has a lower prior probability and this lower prior probability, everything else being equal, inflates the likelihood ratio or the Bayes factor *qua* measures of evidential strength.

Intuitively, the strength of the evidence should not depend on the prior probability of the hypothesis, but solely on the quality of the evidence itself. We will later see that this intuition is not completely correct, but it has a great deal of plausibility, so it is worth taking it seriously. The prior probability of the hypothesis seems extrinsic to the quality of the evidence since the latter should solely depend on the sensitivity and specificity of the evidence relative to the hypothesis of interest. Strength of evidence determines how much the evidence changes, upwards or downwards, the probability of a hypothesis. However, as the prior probability increases, the smaller the impact that the evidence will have on the probability of the hypothesis. If the prior is close to one, the evidence would have marginal if not null impact. But this does not mean that the evidence weakens as the prior probability of the hypothesis goes up. For consider the same hypothesis which in one context has a very high prior probability and in another has a moderate prior probability (say a disease is common in a population but rare in another). The outcome of the same diagnostic test (say a positive test result) performed on two people, each drawn from two populations, should not count as stronger evidence in one case than in the other. After all, it is the same test that was performed and thus the quality of the evidence should be the same. For just one item of evidence, Bayes factor does not capture this intuition, but the likelihood ratio does, which can be considered an argument in favor of the latter and against the former measure of evidential support. However, we have seen that, for more than one item of evidence, the Bayes factor as well as the likelihood ratio are prior dependent.

To circumvent the phenomenon of prior dependency, evidential strength can be thought as a relationship between prior and posterior probabilities. The graph in Figure 5 below represents to what extent the evidence changes the probability of a select hypothesis for any value of the prior probability of the hypothesis. The graph compares the ‘base line’ (representing no change in probability) and the ‘posterior line’ (representing the posterior probability of the hypothesis as a function of the prior for a given assignment of sensitivity and specificity of the evidence). Roughly, the larger the area between the base line and the posterior line, the stronger the evidence. Crucially, this area does not depend on the prior probability of the hypothesis, but solely on the sensitivity and specificity of the evidence. As expected, any improvement in sensitivity or specificity will increase the area between the base line and the posterior line. To be sure, what matters is the ratio of sensitivity to 1-specificity, not their absolute value. So evidence with sensitivity and specificity of 0.9 and 0.9 would be equally strong as evidence with sensitivity and specificity at 0.09 and 0.09 because  $0.9/(1 - 0.9) = 0.09/(1 - 0.09)$ .

The same approach can model the joint evidential strength of two items of evidence,  $a \wedge b$ , relative to the combined hypothesis,  $A \wedge B$ . For simplicity, assume  $a$  and  $b$  are independent lines of evidence supporting their respective hypothesis  $A$  and  $B$ . Further, assume  $A$  and  $B$  are probabilistically independent of the other, as in the Bayesian network in Figure ?? (top). The graph in Figure 7 (top) shows how the prior probabilities are impacted by evidence in support of a single hypothesis—say  $a \wedge b$  supports  $A$ <sup>9</sup>—versus evidence in support of a joint hypothesis—say  $a \wedge b$  supports  $A \wedge B$ . The base line is lower in the latter than in the former case because the prior probability of  $A \wedge B$  is lower than the prior

<sup>9</sup>Given the assumptions of independence we are working with, the strength of  $a$  in support of  $A$  is the same the strength of  $a \wedge b$  in support of  $A$  since  $\frac{P(a \wedge b|A)}{P(a \wedge b)} = \frac{P(a|A)P(b|A)}{P(a)P(b)} = \frac{P(a|A)}{P(a)}$ .

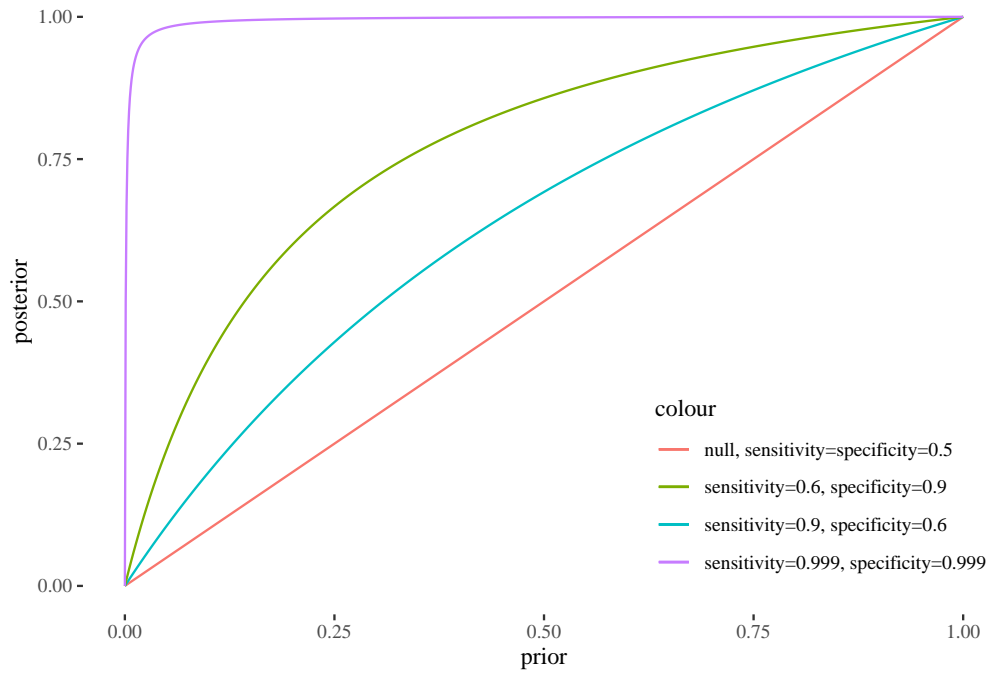


Figure 5: The further away the posterior line from the base line, the stronger the evidence irrespective of the prior probability of the hypothesis.

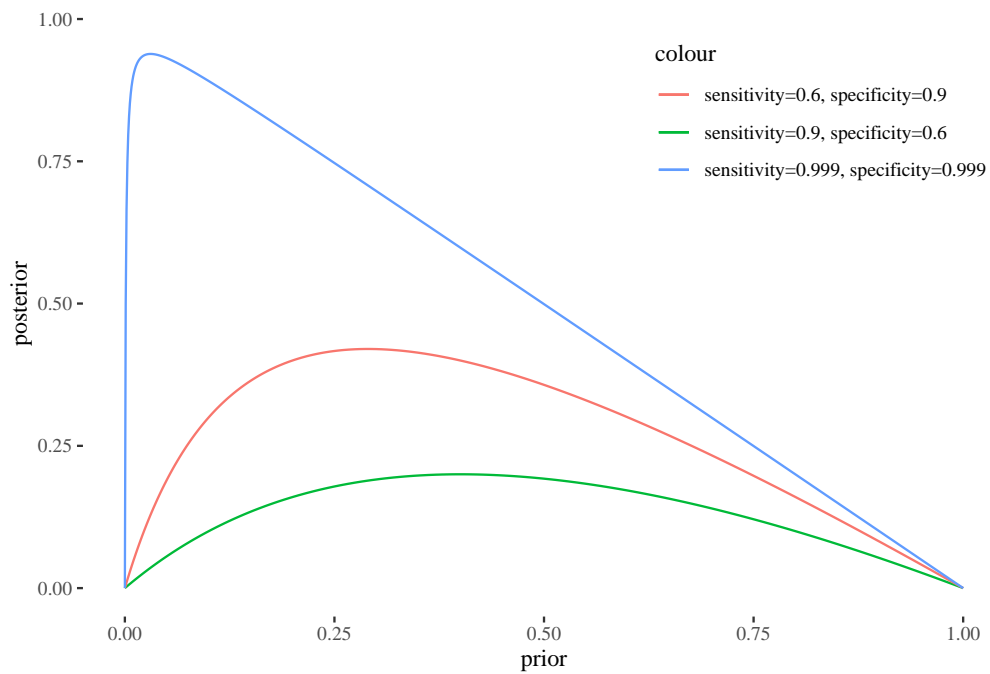


Figure 6: Difference between priors and posteriors as a measure of the strength of evidence.

probability of  $A$ . The prior of  $A$  equals  $x$  and the prior of  $A \wedge B$  equals  $x^2$  (assuming  $A$  and  $B$  have the same prior probability, and as noted before, are probabilistically independent of one another).

```
integrate(postn, 0, 1, s1 = 0.99, s2 = 0.99, n = 1)$val
```

```
## [1] 0.9628366
```

```
integrate(postn, 0, 1, s1 = 0.99, s2 = 0.99, n = 2)$val
```

```
## [1] 0.9815782
```

```
integrate(postn, 0, 1, s1 = 0.99, s2 = 0.99, n = 5)$val
```

```
## [1] 0.9876204
```

```
integrate(postn, 0, 1, s1 = 0.99, s2 = 0.99, n = 10)$val
```

```
## [1] 0.9889299
```

```
integrate(postn, 0, 1, s1 = 0.99, s2 = 0.99, n = 20)$val
```

```
## [1] 0.989491
```

```
integrate(postn, 0, 1, s1 = 0.5, s2 = 0.5, n = 1)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 = 0.5, s2 = 0.5, n = 2)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 = 0.5, s2 = 0.5, n = 5)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 = 0.5, s2 = 0.5, n = 10)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 = 0.5, s2 = 0.5, n = 20)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 = 0.99, s2 = 0.99, n = 1)$val - integrate(postn, 0, 1, s1
```

```
## [1] 0.4628366
```

```
integrate(postn, 0, 1, s1 = 0.99, s2 = 0.99, n = 2)$val - integrate(postn, 0, 1, s1
```

```
## [1] 0.4815782
```



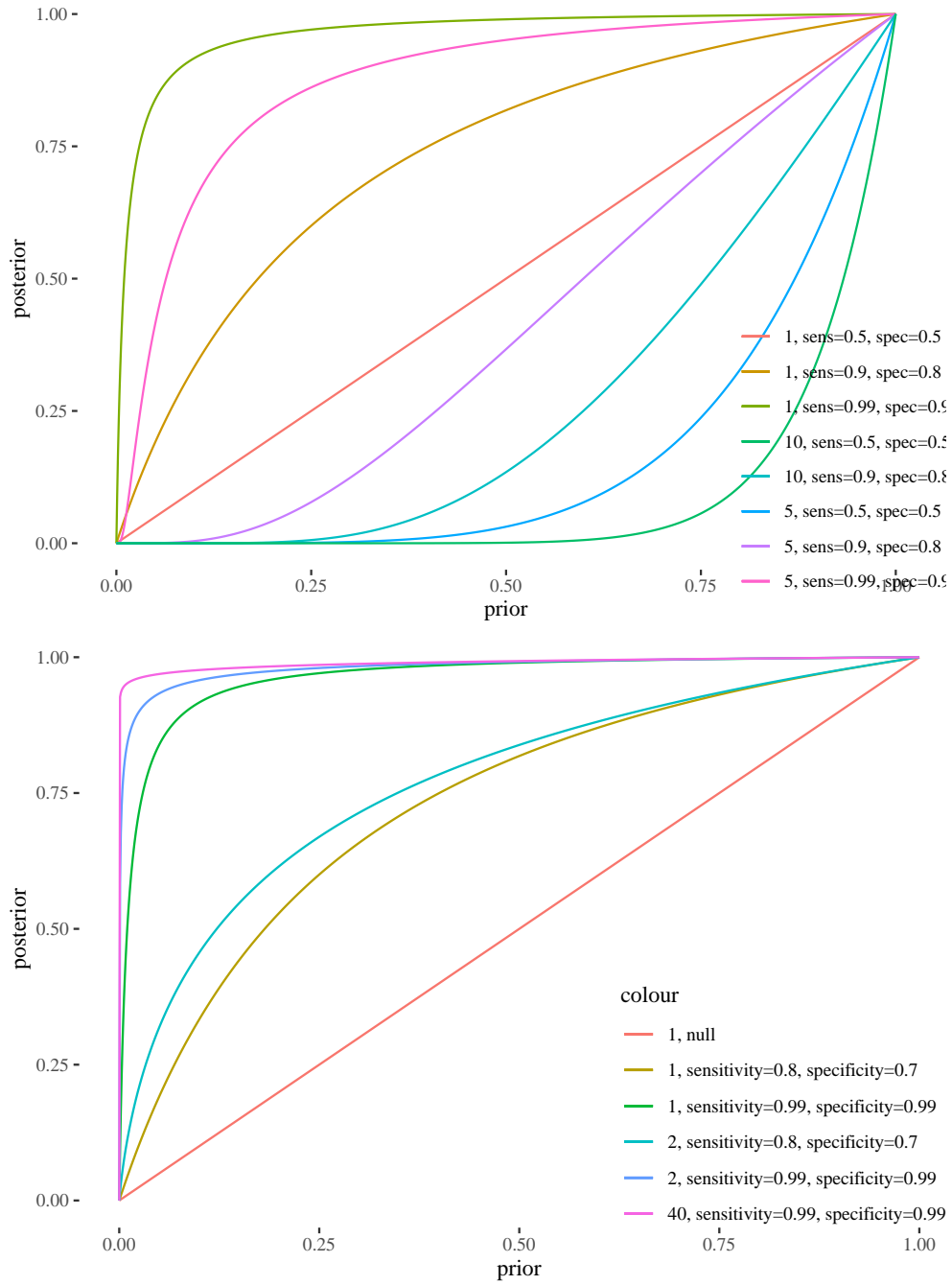


Figure 7: The comparison is between individual support (marked by 1, for one individual hypothesis) and joint support (marked by 2, for a two-claim composite claim). Top graph: The base line for joint support ( $y = x * x$ ) is below the base line for individual support ( $y = x$ ). Bottom graph: the two base lines are equalized and the posterior lines adjusted accordingly. The posterior lines for individual and joint support get closer especially for high posterior probability values.

```

integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 5)$val - integrate(postn, 0, 1, s1
## [1] 0.4876204

integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 10)$val - integrate(postn, 0, 1, s1
## [1] 0.4889299

integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 20)$val - integrate(postn, 0, 1, s1
## [1] 0.489491

integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 25)$val - integrate(postn, 0, 1, s1
## [1] 0.4895968

integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 40)$val - integrate(postn, 0, 1, s1
## [1] 0.4897516

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 1)$val
## [1] 0.6137056

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 2)$val
## [1] 0.6355324

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 5)$val
## [1] 0.65284

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 10)$val
## [1] 0.6595075

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 20)$val
## [1] 0.663025

integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 1)$val
## [1] 0.5

integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 2)$val
## [1] 0.5

integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 5)$val
## [1] 0.5

```

```

integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 10)$val

## [1] 0.5

integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 20)$val

## [1] 0.5

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 1)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 1)$val

## [1] 0.1137056

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 2)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 2)$val

## [1] 0.1355324

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 5)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 5)$val

## [1] 0.15284

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 10)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 10)$val

## [1] 0.1595075

integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 20)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 20)$val

## [1] 0.163025

integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 1)$val

## [1] 0.7331961

integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 2)$val

## [1] 0.7717311

integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 5)$val

## [1] 0.7990992

integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 10)$val

## [1] 0.8086466

integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 20)$val

## [1] 0.8134273

integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 1)$val

## [1] 0.5

```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 2)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 5)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 10)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 20)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 1)$val - integrate(postn, 0, 1, s1 =0
```

```
## [1] 0.2331961
```

```
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 2)$val - integrate(postn, 0, 1, s1 =0
```

```
## [1] 0.2717311
```

```
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 5)$val - integrate(postn, 0, 1, s1 =0
```

```
## [1] 0.2990992
```

```
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 10)$val - integrate(postn, 0, 1, s1 =
```

```
## [1] 0.3086466
```

```
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 20)$val - integrate(postn, 0, 1, s1 =
```

```
## [1] 0.3134273
```

```
integrate(post, 0, 1, s1 =0.99 , s2 =0.99)$val
```

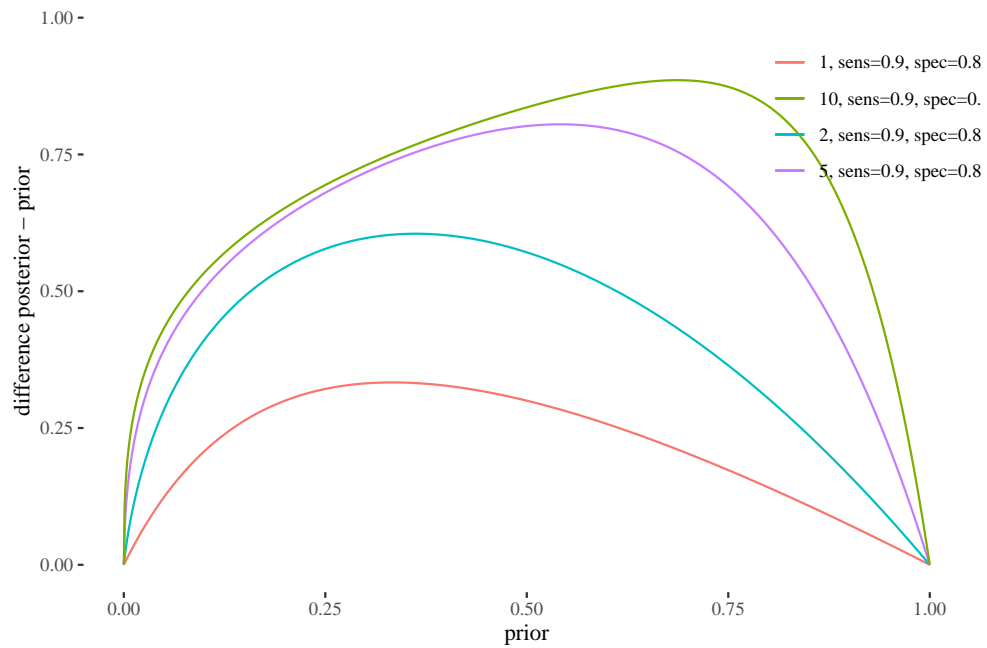
```
## [1] 0.9628366
```

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99 , n = 2)$val
```

```
## [1] 0.9815782
```

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99 , n = 5)$val
```

```
## [1] 0.9876204
```



```
prior_a <- 0.6
prior_b <- 0.4
sen_a <- 0.85 # Pr(a given A)
spe_a <- 0.75 # Pr(not-a given not-A)
sen_b <- 0.7
spe_b <- 0.75

prior_ab <- prior_a*prior_b

bf_a <- sen_a / ((sen_a*prior_a) + ((1-spe_a) * (1-prior_a)))
bf_b <- sen_b / ((sen_b*prior_b) + ((1-spe_b) * (1-prior_b)))
bf_ab <- bf_a*bf_b

post_a <- bf_a*prior_a
post_b <- bf_b*prior_b
post_ab <- bf_ab*prior_ab

prior_a
```

```
## [1] 0.6
```

```
prior_b
```

```
## [1] 0.4
```

```
prior_ab
```

```
## [1] 0.24
```

```
post_a
```

```
## [1] 0.8360656
```

```
post_b
```

```
## [1] 0.6511628
```

```
post_ab
```

```
## [1] 0.5444148
```

```
prior_a <- 0.24
```

```
prior_b <- 0.24
```

```
bf_a <- sen_a / ((sen_a*prior_a) + ((1-spe_a)*(1-prior_a)))
```

```
bf_b <- sen_b / ((sen_b*prior_b) + ((1-spe_b)*(1-prior_b)))
```

```
bf_ab <- bf_a*bf_b
```

```
post_a <- bf_a*prior_a
```

```
post_b <- bf_b*prior_b
```

```
post_a
```

```
## [1] 0.5177665
```

```
post_b
```

```
## [1] 0.4692737
```

What happens if we make the same comparison between individual and composite claims by equalizing their prior probability? If the claims are independent and equiprobable, let  $x$  be the prior probability of an individual claim (when it is considered in isolation) and let  $x^{1/k}$  the prior probability of the same individual claim when it is part of a composite claim that consists of  $k$  claims. In this way—and again, assuming independence and equiprobability of the hypotheses—the prior probability of the composite claim equals the prior probability of the individual claim since  $(x^{1/k})^k = x$ , as desired. These different claims are then plotted having the same priors. Here we are explicitly factoring out the role of prior probabilities. Figure 7 (bottom) shows the result of this process of equalization.

We observe two things. First, the difference in posterior probability, though still present, is less significant, especially for values above the 75% threshold and even more clearly above the 95% threshold. Second, whatever remaining difference in posterior probability is now reversed, that is, a composite claim supported by several items of evidence has a higher posterior probability compared to an individual claim supported by one item of evidence. This second observation agrees with the analysis based on the Bayes factor and the likelihood ratio in the earlier section. That analysis showed that the support for a composite claim by a joint body of evidence often exceeds the support for an individual claim.

These two observations establish that, by factoring out prior probabilities and under certain independence assumptions, whenever the individual claims meet the applicable posterior threshold, so does the composite claim. This verifies aggregation. Conversely, whenever the composite claim, say  $A \wedge B$ , meets the applicable posterior threshold, so do the individual claims insofar as the threshold is about 75% or higher. This verifies—to some approximation and in a limited class of cases—the other direction of the conjunction principle, what we called distribution.

Has the distribution paradox been eliminated then? The approach we have just described—equalizing the prior probabilities across individual and composite claims—does not entirely eliminate the paradox. There are still cases in which a composite hypothesis, say  $A \wedge B$ , receives stronger support than an individual hypothesis, given the same body of evidence. Sensitivity to priors seems to play a role. But it cannot be the only factor at play, or else the equalization of the prior probabilities would have eliminated the paradox entirely. So what else is going on?

## 8.2 Weaker Claims Weaken Sensitivity

Let's examine more closely the Bayes factor and the likelihood ratio as measures of evidential strength. Likelihood ratios are comparative in nature. Suppose we compare claim  $A$  and claim  $A \wedge B$  relative to

the same body of evidence  $a \wedge b$ . Thinks of  $A$  as ‘the defendant physically injured the victim’ while  $B$  as ‘the defendant knew the victim was a firefighter.’ Think of  $a$  and  $b$  as testimonies each supporting one of the two hypotheses. We are still working with the Bayesian networks in Figure ??.

Consider the combined body of evidence  $a \wedge b$ . Which claim between  $A$  and  $A \wedge B$  will receive more support by evidence  $a \wedge b$ ? Intuitively, one might think that  $A$  should receive more or at least equal support compared to  $A \wedge B$ . After all,  $A \wedge B$  is a stronger claim than  $A$  and thus more difficult to establish than  $A$ , other things being equal. In terms of posterior probabilities, this is true. The posterior probability of  $A \wedge B$  should not be higher than the posterior probability of  $A$  alone given evidence  $a \wedge b$ .

Let’s now think in terms of evidential support. Formally, the question is whether  $\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)}$  is less than one or greater than one. Given the customary independencies between evidence and hypotheses,

$$\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)} = \frac{P(a|A)P(b|A)}{P(a|A)P(b|B)} = \frac{P(b|A)}{P(b|B)} < 1.$$

The reason is that  $P(b|A) < P(b|B)$  since the sensitivity of  $b$  relative to  $B$  should be higher than the sensitivity of  $b$  relative to  $A$ .<sup>10</sup> This is obvious if  $A$  and  $b$  are independent claims, as in the Bayesian networks in Figure ?? (bottom). In this case,  $P(b|A) = P(b)$ .

So  $\frac{P(b)}{P(b|B)} < 1$  since  $b$ , by assumption, positively supports  $B$ , or in other words,  $\frac{P(b|B)}{P(b)} > 1$ . Thus,  $a \wedge b$  supports  $A \wedge B$  more than it supports  $A$  alone. This is not what one would expect intuitively.

The same conclusion holds using the Bayes factor. If  $\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)} < 1$ , then

$$\frac{P(a \wedge b|A)}{P(a \wedge b)} < \frac{P(a \wedge b|A \wedge B)}{P(a \wedge b)}$$

Thus, evidence  $a \wedge b$  better supports, even on an absolute scale,  $A \wedge B$  compared to  $A$ . Note that, even if the Baye factor depends on the priors, the difference here is not due to the difference between the priors of  $A$  and the priors of  $A \wedge B$  since the denominator is simply  $P(a \wedge b)$ .<sup>11</sup>

What we have said so far agrees with the claim defended in the previous section. That is, even when  $a \wedge b$  strongly supports  $A \wedge B$ , the same evidence need *not* strongly support  $A$ . Formally,

$$\frac{P(a \wedge b|A)}{P(a \wedge b|\neg A)} < \frac{P(a \wedge b|A \wedge B)}{P(a \wedge b|\neg(A \wedge B))}$$

In other words, even if  $a \wedge b$  favors  $A \wedge B$  over its negation to a very high degree, it need not favor equally strongly  $A$  over its negation. This is a comparative claim about two comparative claims, and as such, it may not be easy to parse. Evidential support, when it is formalized by the likelihood ratio, is always relative to a contrast class. In comparing the support that the same body of evidence provides to  $A$  as contrasted to  $A \wedge B$ , it might be better to include these two hypotheses in the contrast class. So the expression  $\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)}$  is more straightforward.

No matter the formulation, the same conclusion holds. Evidence  $a \wedge b$  supports the composite claim  $A \wedge B$  more than it supports the weaker claim  $A$ , even assuming that  $A$  and  $B$  are independent of one another and thus not mutually reinforcing. This conclusion seems paradoxical. How should we make sense of it? At first, we thought the paradox could be due to prior dependency since the combined likelihood ratio  $\frac{P(a \wedge b|A \wedge B)}{P(a \wedge b|\neg(A \wedge B))}$  varies depending on the priors of  $A$  and  $B$ . But this argument seems to no longer holds since in case of  $\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)}$  any prior dependency seems to have been eliminated. After all, if  $\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)} < 1$ , the sensitivity of  $a \wedge b$  must be worse relative to  $A$  than the composite claim  $A \wedge B$ .

Sensitivity is a crucial property of the quality of the evidence. Everything else being equal, the lower the sensitivity of the evidence, the lower its evidential strength. The importance of sensitivity as a factor for assessing the strength of the evidence is hard to dispute. So why is the sensitivity of  $a \wedge b$  worse relative to  $A$  than  $A \wedge B$ ? Suppose  $A$  is the case. If  $A$  is the case, in order for  $a \wedge b$  to arise, both  $a$  and  $b$  should pick up on  $A$ . If  $b$  fails to pick up on  $A$ , then  $A \wedge b$  would not arise even if  $a$  pick up on  $A$ .<sup>12</sup>

<sup>10</sup>GIVE PROOF OF THIS

<sup>11</sup>Note that  $P(a \wedge b|A)P(A) + P(a \wedge b|\neg A)P(\neg A)$  is the same as  $P(a \wedge b|(A \wedge B))P(A \wedge B) + P(a \wedge b|\neg(A \wedge B))P(\neg(A \wedge B))$ . **NEED A PROOF FOR THIS BUT IT SHOULD HOLD, RIGHT?**

<sup>12</sup>The occurrence of  $a \wedge b$  is less likely to occur than just  $a$  alone picking up on  $A$  because  $b$  may fail—and fail more often than  $a$  would—in picking up on  $A$ .



Suppose instead  $A \wedge B$  holds. In this case,  $a \wedge b$  would arise even if  $b$  fails to pick up on  $A$  so long as  $a$  picks up on  $A$  and  $b$  picks up on  $B$ . Now of course  $b$  could also fail to pick on  $B$  just like  $a$  could fail to pick up on  $B$ . But we are assuming that  $b$  is better than  $a$  at tracking  $B$ . So  $b$  will fail less often than  $a$  at picking up on  $B$ . Thus, the sensitivity of  $a \wedge b$  relative to  $A \wedge B$  is better than the sensitivity of  $a$  relative to  $A$  alone. This is a subtle point that probability theory helps to bring out clearly.

How big are these variations? **PLOT GRAPH TO GET A SENSE OF VARIATIONS OF SENSITIVITY**

One explanation of the paradox, then, is the difference in sensitivity. The sensitivity of  $a \wedge b$  relative to  $A \wedge B$  is better than the sensitivity of the same evidence relative to  $A$ . Consequently, other things being equal, evidence  $a \wedge b$  supports  $A \wedge B$  better than  $A$ . However counterintuitive this might seem, we should accept this fact and admit that our intuitions were wrong. So the fallacy seems to be to assume that the sensitivity of  $a \wedge b$  relative to  $A$  cannot be lower than the sensitivity of  $a \wedge b$  relative to  $A \wedge B$ . The thought would be something like this: if  $a \wedge b$  tracks  $A \wedge B$  to some degree, it surely must be able to track  $A$  alone, at least as well. But we have just shown that we cannot assume that  $P(a \wedge b|A) \geq P(a \wedge b|A \wedge B)$  and in fact the opposite is the case,  $P(a \wedge b|A) < P(a \wedge b|A \wedge B)$ .

Another way to convince ourselves this is the case is to run a simulation. Suppose we are deciding about the truth of  $A$  and the truth of  $A \wedge B$ , and we have a fixed body of evidence, say,  $a \wedge b$  that speaks in favor of both claims.

We should circumscribe the point we just made since it does not always hold. Suppose, as in Figure 8, that  $H$  is a claim unrelated to  $A$  and evidence  $a$ . Evidence  $a$  supports  $A$ . Would the composite claim  $A \wedge H$  be better supported by  $a$  than  $A$  alone? It would not. Mere tagging an unrelated hypothesis does not strengthen the evidence. Note that  $P(a|A) = P(a|A \wedge H)$  because  $H$  is independent from everything else. Thus,

$$\frac{P(a|A)}{P(a|A \wedge H)} = \frac{P(a|A)}{P(a|A)} = 1.$$

Tagging an unrelated claim  $H$  does not strengthen the evidence, but leaves it unchanged. Similarly, suppose  $B$  constitutes one element of a crime and  $A$  constitutes the other element. The two claims are independent, each supported by items of evidence  $a$  and  $b$  respectively. This is our standard set up. If  $a$  supports  $A$ , would  $a$  support  $A \wedge B$  more strongly than it supports  $A$  alone? Here we no longer have  $a \wedge b$ , but instead,  $a$  alone. The question is whether  $\frac{P(a|A)}{P(a|A \wedge B)} > 1$  or  $\frac{P(a|A)}{P(a|A \wedge B)} < 1$ . Given the usual independencies between evidence and hypotheses,

$$\frac{P(a|A)}{P(a|A \wedge B)} = \frac{P(a|A)}{P(a|A)} = 1$$

Evidence  $a$  supports  $A$  and  $A \wedge B$  to the same extent. One might complain that this is counterintuitive. How can it be that  $a$  supports  $A$  to the same degree that it supports the more demanding claim  $A \wedge H$  or  $A \wedge B$ ? For suppose we have evidence  $a$  in favor of  $A$  and then wonder whether we could use that evidence in support of another claim  $H$  or  $B$ . By tagging  $H$  or  $B$  to  $A$ , we can at least say that we have evidence  $a$  for  $A \wedge H$  or  $A \wedge B$  that is at least as strong as evidence  $a$  in support of  $A$ . But note that  $\frac{P(a|A)}{P(a|negA)} > 1$  even though  $\frac{P(a|H)}{P(a|negH)=1}$  and  $\frac{P(a|B)}{P(a|negB)=1}$  (assuming  $A$  and  $B$  are independent). So  $a$  does not support  $H$  or  $B$  to the same degree that it supports  $A$ . However,  $a$  supports  $A$  to the same degree that it supports  $A \wedge H$  or  $A \wedge B$ .

What should we make of this? The commonality between  $H$  and  $B$  is that they are irrelevant for  $a$ . Evidence  $a$  does not increase nor decrease their probability so the evidence is irrelevant for them. So the upshot here is that, tagging an irrelevant hypothesis does not change evidential support. The other upshot is that tagging a relevant hypothesis—a hypothesis that does bear on the evidence in one way or another, such as  $B$  relative to  $a \wedge b$ —does increase evidential support.

The intuition that the same evidence tracks the conjunction  $A \wedge B$  better than one of the conjuncts  $A$  and  $B$  might rest on another model of what is going on. Say  $ab$  is an item of evidence that arises when  $A$  or  $B$  occurs with 60% probability. That is,  $P(ab|A) = P(ab|B) = 60\%$ . What would be the sensitivity of  $ab$  relative to the conjunction  $A \wedge B$ ,  $P(ab|A \wedge B)$ ? We can represent this set up in Figure 9. Intuitively, one might reason as follows. There are two paths leading to  $ab$ , one path starts with  $A$  and another path

M: Run simulation to show that same diagnostic test for composite claim would perform better than when applied to individual claim (worse LR). How do to do this? HELP!

M: How to explain this better? Can we make this plausible? HELP!

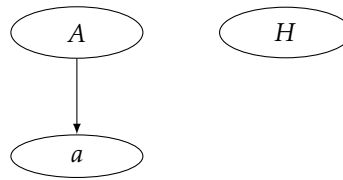


Figure 8: Bayesian network with wholly unrelated claim  $H$ .

starts with  $B$ . When both these paths are active, since we are assuming  $A \wedge B$ , then the probability of  $ab$  must be higher than if just one of the two paths is active. Hence,  $P(ab|A \wedge B) \geq 60\%$ .

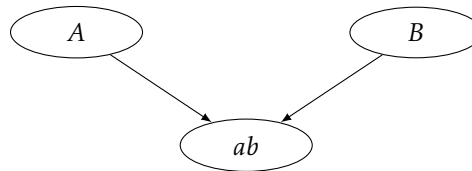


Figure 9: Bayesian network with  $ab$  resulting from  $A$  and  $B$ .

M: Is this true probabilistically? If not, what assumptions are required? HELP!

### 8.3 But Sensitivity Depends On Prior Probabilities

We have shown, given a suitable number of assumptions, that the sensitivity of  $a \wedge b$  can be greater relative to  $A \wedge B$  than  $A$  alone. This explains why the evidential support of  $a \wedge b$  is greater in favor of  $A \wedge B$  than alone  $A$ . Therefore—one might conclude—even factoring out differences in prior probabilities could still lead to differences in evidential strength simply due to difference in sensitivity. But note that this argument assumes that sensitivity—or specificity—have nothing to do with prior probabilities.

Things are somewhat more complicated. After all,  $\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)} = \frac{P(b|A)}{P(b|B)}$ . The denominator, which tracks the sensitivity of  $a \wedge b$  relative to  $A \wedge B$  does not depend on the priors of  $A \wedge B$ , but solely on the sensitivity of  $a$  and  $b$  relative to  $A$  and  $B$ . However, the numerator, which tracks the sensitivity of  $a \wedge b$  relative to  $a$ , does depend on the priors of  $A$  (or whatever other hypothesis one chooses to compute  $P(b|A)$  or  $P(b)$ ). This means that the denominator can vary depending on the prior probability of a chosen hypothesis. Thus, the lower the prior of  $A$ , the lower the probability of  $b$ . One could still insist that here we are not comparing the prior of a hypothesis, say  $B$ , and the prior of another hypothesis, say  $A \wedge B$ . Whatever the difference in sensitivity, it cannot be due to the difference in prior probabilities of the hypotheses.

On other hand hand,  $\frac{P(a \wedge b|A)}{P(a \wedge b|A \wedge B)}$  is comparing two quantities, one dependent on the priors and the other not dependent on the priors.

More generally, the intuition that characteristics of the evidence such as sensitivity and specificity should be independent of the probability of the hypothesis of interest turns out to be incorrect empirically. One study in the medical literature has shown, surprisingly, that the sensitivity of a diagnostic test is independent of the prior of the hypothesis being tested—say whether the patient has a medical condition. However, specificity is dependent on the prior of the hypothesis:

Overall, specificity tended to be lower with higher disease prevalence; there was no such systematic effect for sensitivity (page E537). Source: Variation of a test's sensitivity and specificity with disease prevalence Mariska M.G. Leeflang, Anne W.S. Rutjes, Johannes B. Reitsma, Lotty Hooft and Patrick M.M. Bossuyt CMAJ August 06, 2013 185 (11) E537-E544; DOI: <https://doi.org/10.1503/cmaj.121286>

The authors of the study, however, caution that

Because sensitivity is estimated in people with the disease of interest and specificity in people without the disease of interest, changing the relative number of people with and without the disease of interest should not introduce systematic differences. Therefore, the

effects that we found may be generated by other mechanisms that affect both prevalence and accuracy.

So, according to the authors, changes in prevalence need not directly affect specificity since variations in prevalence and variation in specificity may have a common cause. Our earlier calculations about combined specificity and sensitivity agree with experimental results, namely, only specificity depends on the priors. Our calculations, in fact, show that different priors for the individual claim do affect specificity. The variation of specificity in the result of splitting the negation of the composite hypothesis  $\neg(A \wedge B)$  into three further scenarios,  $\neg A \wedge B$ ,  $B \wedge \neg A$  and  $\neg A \wedge \neg B$ . This prior sensitivity, of course, only applies to composite hypotheses, but to some extent, any hypothesis can be analyzed as a composite hypothesis. The claim that the defendant was running down 5th avenue can be broken down in the conjunction that the defendant was running and that the defendant was at 5th avenue. Any claim, under some level of description, is a composite hypothesis. So, perhaps, the quality or strength of the evidence should depend on the priors whether the hypothesis is composite or not. Is this another example of base rate neglect?

Let's grant that the quality of the evidence should depend, contrary to our initial intuition, on the prior probability of the hypotheses. If that is so, it would not be natural to see that evidence – the same evidence – strongly favors  $A \wedge B$  without strongly favoring  $A$  or  $B$ . Perhaps we can make sense of this if we keep in mind the comparison between hypothesis we are making here.

NOT SURE HOW TO CONTINUE HERE THOUGH!

TO DO:

1. NOTE THAT EVEN BY EQUALIZING PRIORS, THE DISTRIBUTION PARADOX DOES NOT GO AWAY. SO WHAT ELSE IS GOING ON HERE? NEED TO MAKE COMPARISON BETWEEN HYPOTHESES. NEED TO FIGURE THIS OUT!
2. TRY TO MAKE SENSE OF THIS, IT IS INTUITIVELY ACCEPTABLE THAT SUPPORT FOR COMBINED CLAIM, EVEN HOLDING FIXED THE SAME EVIDENCE, SHOULD BE STRONGER THEN SUPPORT FOR INDIVIDUAL CLAIM? THAT IS CLEARLY ODD AND GOES AGAINST COMMON ASSUMPTIONS.
3. **HYPOTHESIS: WHEN THE HYPOTHESIS IS NOT MEDIATED (AS IN  $a$  TOWARD  $A$  OR  $b$  TOWARD  $B$ , AS OPPOSED TO  $a$  TOWARD  $A$ -AND- $B$ ), THEN SENSITIVITY OR SPECIFICITY IS NOT DEPENDENT ON PRIORS**

M: Might be good to have a simulation here that makes vivid why combined specificity is in fact dependent on the priors. Maybe it is, after all, a fallacy to think that the quality/strength of the evidence should be independent of the priors. HELP!

## 8.4 Which Measure of (Combined) Evidential Support?

THINGS TO ADD:

1. THE MIN SEEMS TO BE THE MEASURE FOR COMPOSITE CLAIMS THAT CAPTURES AGGREGATION AND DISTRIBUTION BEST. SO THE QUESTION IS WHAT PROBABILISTIC MEASURE CAPTURES MIN?
2. IS DEPENDENCY ON PRIOR ANOTHER EXAMPLE OF BASE RATE NEGLECT? WE NEGLECT BASE RATE IN CALCULATING POSTERIOR BUT ALSO IN CALCULATING STRENGTH OF EVIDENCE? WE JUST NEED TO LIVE WITH THE FACT THAT WE HAVE A POOR UNDERSTANDING OF EVIDENCE BUT PERHAPS GOOD ENOUGH TO GET BY IN THE WORLD. CONNECT TO POINT 1 AND MIN FUNCTION.

## 8.5 The likelihood strategy

Focusing on posterior probabilities is not the only approach that legal probabilists can pursue. By Bayes' theorem, the following holds, using  $G$  and  $I$  as competing hypotheses:

$$\frac{\Pr(G|E)}{\Pr(I|E)} = \frac{\Pr(E|G)}{\Pr(E|I)} \times \frac{\Pr(G)}{\Pr(I)},$$

or using  $H_p$  and  $H_d$  as competing hypotheses,

$$\frac{\Pr(H_p|E)}{\Pr(H_d|E)} = \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)},$$

or in words

$$\text{posterior odds} = \text{likelihood ratio} \times \text{prior odds}.$$

A difficult problem is to assign numbers to the prior probabilities such as  $\Pr(G)$  or  $\Pr(H_p)$ , or prior odds such as  $\frac{\Pr(G)}{\Pr(I)}$  or  $\frac{\Pr(H_p)}{\Pr(H_d)}$ .

DISCUSS DIFFICULTIES ABOUT ASSIGNING PRIORS! WHERE? CAN WE USE IMPRECISE PROBABILITIES TALK ABOUT PRIORS – I.E. LOW PRIORS = TOTAL IGNORANCE = VERY IMPRECISE (LARGE INTERVAL) PRIORS? THE PROBLEM WITH THIS WOULD BE THAT THERE IS NO UPDATING POSSIBLE. ALL UPDATING WOULD STILL GET BACK TO THE STARTING POINT. DO YOU HAVE AN ANSWER TO THAT? WOULD BE INTERESTING TO DISCUSS THIS!

Given these difficulties, both practical and theoretical, one option is to dispense with priors altogether. This is not implausible. Legal disputes in both criminal and civil trials should be decided on the basis of the evidence presented by the litigants. But it is the likelihood ratio – not the prior ratio – that offers the best measure of the overall strength of the evidence presented. So it is all too natural to focus on likelihood ratios and leave the priors out of the picture. If this is the right, the question is, how would a probabilistic interpretation of standards of proof based on the likelihood ratio look like? At its simplest, this strategy will look as follows. Recall our discussion of expected utility theory:

$$\text{convict provided } \frac{\text{cost}(CI)}{\text{cost}(AG)} < \frac{\Pr(H_p|E)}{\Pr(H_d|E)},$$

which is equivalent to

$$\text{convict provided } \frac{\text{cost}(CI)}{\text{cost}(AG)} < \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)}.$$

By rearranging the terms,

$$\text{convict provided } \frac{\Pr(E|H_p)}{\Pr(E|H_d)} > \frac{\Pr(H_d)}{\Pr(H_p)} \times \frac{\text{cost}(CI)}{\text{cost}(AG)}.$$

Then, on this interpretation, the likelihood ratio should be above a suitable threshold that is a function of the cost ratio and the prior ratio. The outstanding question is how this threshold is to be determined.

### 8.5.1 Kaplow

Quite independently, a similar approach to juridical decisions has been proposed by Kaplow (2014) – we'll call it **decision-theoretic legal probabilism (DTLP)**. It turns out that Cheng's suggestion is a particular case of this more general approach. Let  $LR(E) = P(E|H_{\Pi})/P(E|H_{\Delta})$ . In whole generality, DTLP invites us to convict just in case  $LR(E) > LR^*$ , where  $LR^*$  is some critical value of the likelihood ratio.

Say we want to formulate the usual preponderance rule: convict iff  $P(H_{\Pi}|E) > 0.5$ , that is, iff  $\frac{P(H_{\Pi}|E)}{P(H_{\Delta}|E)} > 1$ . By Bayes' Theorem we have:

$$\begin{aligned} \frac{P(H_{\Pi}|E)}{P(H_{\Delta}|E)} &= \frac{P(H_{\Pi})}{P(H_{\Delta})} \times \frac{P(E|H_{\Pi})}{P(E|H_{\Delta})} > 1 \Leftrightarrow \\ &\Leftrightarrow \frac{P(E|H_{\Pi})}{P(E|H_{\Delta})} > \frac{P(H_{\Delta})}{P(H_{\Pi})} \end{aligned}$$

So, as expected,  $LR^*$  is not unique and depends on priors. Analogous reformulations are available for thresholds other than 0.5.

Kaplow's point is not that we can reformulate threshold decision rules in terms of priors-sensitive likelihood ratio thresholds. Rather, he insists, when we make a decision, we should factor in its consequences. Let  $G$  represent potential gain from correct conviction, and  $L$  stand for the potential loss resulting from mistaken conviction. Taking them into account, Kaplow suggests, we should convict if and only if:

$$P(H_{\Pi}|E) \times G > P(H_{\Delta}|E) \times L \quad (21)$$

Now, (21) is equivalent to:

$$\begin{aligned}
\frac{P(H_{\Pi}|E)}{P(H_{\Delta}|E)} &> \frac{L}{G} \\
\frac{P(H_{\Pi})}{P(H_{\Delta})} \times \frac{P(E|H_{\Pi})}{P(E|H_{\Delta})} &> \frac{L}{G} \\
\frac{P(E|H_{\Pi})}{P(E|H_{\Delta})} &> \frac{P(H_{\Delta})}{P(H_{\Pi})} \times \frac{L}{G} \\
LR(E) &> \frac{P(H_{\Delta})}{P(H_{\Pi})} \times \frac{L}{G}
\end{aligned} \tag{22}$$

This is the general format of Kaplow's decision standard.

### 8.5.2 Dawid

Here is a slightly different perspective, due to Dawid (1987), that also suggests that juridical decisions should be likelihood-based. The focus is on witnesses for the sake of simplicity. Imagine the plaintiff produces two independent witnesses:  $W_A$  attesting to  $A$ , and  $W_B$  attesting to  $B$ . Say the witnesses are regarded as 70% reliable and  $A$  and  $B$  are probabilistically independent, so we infer  $P(A) = P(B) = 0.7$  and  $P(A \wedge B) = 0.7^2 = 0.49$ .

But, Dawid argues, this is misleading, because to reach this result we misrepresented the reliability of the witnesses: 70% reliability of a witness, he continues, does not mean that if the witness testifies that  $A$ , we should believe that  $P(A) = 0.7$ . To see his point, consider two potential testimonies:

$A_1$	The sun rose today.
$A_2$	The sun moved backwards through the sky today.

Intuitively, after hearing them, we would still take  $P(A_1)$  to be close to 1 and  $P(A_2)$  to be close to 0, because we already have fairly strong convictions about the issues at hand. In general, how we should revise our beliefs in light of a testimony depends not only on the reliability of the witness, but also on our prior convictions.<sup>13</sup> And this is as it should be: as indicated by Bayes' Theorem, one and the same testimony with different priors might lead to different posterior probabilities.

So far so good. But how should we represent evidence (or testimony) strength then? Well, one pretty standard way to go is to focus on how much it contributes to the change in our beliefs in a way independent of any particular choice of prior beliefs. Let  $a$  be the event that the witness testified that  $A$ . It is useful to think about the problem in terms of *odds*, *conditional odds* ( $O$ ) and *likelihood ratios* ( $LR$ ):

$$\begin{aligned}
O(A) &= \frac{P(A)}{P(\neg A)} \\
O(A|a) &= \frac{P(A|a)}{P(\neg A|a)} \\
LR(a|A) &= \frac{P(a|A)}{P(a|\neg A)}.
\end{aligned}$$

Suppose our prior beliefs and background knowledge, before hearing a testimony, are captured by the prior probability measure  $P_{prior}(\cdot)$ , and the only thing that we learn is  $a$ . We're interested in what our *posterior* probability measure,  $P_{posterior}(\cdot)$ , and posterior odds should then be. If we're to proceed with Bayesian updating, we should have:

$$\frac{P_{posterior}(A)}{P_{posterior}(\neg A)} = \frac{P_{prior}(A|a)}{P_{prior}(\neg A|a)} = \frac{P_{prior}(a|A)}{P_{prior}(a|\neg A)} \times \frac{P_{prior}(A)}{P_{prior}(\neg A)}$$

that is,

$$O_{posterior}(A) = O_{prior}(A|a) = \underbrace{LR_{prior}(a|A)}_{\text{conditional likelihood ratio}} \times O_{prior}(A) \tag{23}$$

<sup>13</sup> An issue that Dawid does not bring up is the interplay between our priors and our assessment of the reliability of the witnesses. Clearly, our posterior assessment of the credibility of the witness who testified  $A_2$  will be lower than that of the other witness.

The conditional likelihood ratio seems to be a much more direct measure of the value of  $a$ , independent of our priors regarding  $A$  itself. In general, the posterior probability of an event will equal to the witness's reliability in the sense introduced above only if the prior is  $1/2$ .<sup>14</sup>

## 8.6 Likelihood and DAC

But how does our preference for the likelihood ratio as a measure of evidence strength relate to DAC? Let's go through Dawid's reasoning.

A sensible way to probabilistically interpret the 70% reliability of a witness who testifies that  $A$  is to take it to consist in the fact that the probability of a positive testimony if  $A$  is the case, just as the probability of a negative testimony (that is, testimony that  $A$  is false) if  $A$  isn't the case, is 0.7.<sup>15</sup>

$$P_{prior}(a|A) = P_{prior}(\neg a|\neg A) = 0.7.$$

$P_{prior}(a|\neg A) = 1 - P_{prior}(\neg a|\neg A) = 0.3$ , and so the same information is encoded in the appropriate likelihood ratio:

$$LR_{prior}(a|A) = \frac{P_{prior}(a|A)}{P_{prior}(a|\neg A)} = \frac{0.7}{0.3}$$

Let's say that  $a$  provides (positive) support for  $A$  in case

$$O_{posterior}(A) = O_{prior}(A|a) > O_{prior}(A)$$

that is, a testimony  $a$  supports  $A$  just in case the posterior odds of  $A$  given  $a$  are greater than the prior odds of  $A$  (this happens just in case  $P_{posterior}(A) > P_{prior}(A)$ ). By (23), this will be the case if and only if  $LR_{prior}(a|A) > 1$ .

One question that Dawid addresses is this: assuming reliability of witnesses 0.7, and assuming that  $a$  and  $b$ , taken separately, provide positive support for their respective claims, does it follow that  $a \wedge b$  provides positive support for  $A \wedge B$ ?

Assuming the independence of the witnesses, this will hold in non-degenerate cases that do not involve extreme probabilities, on the assumption of independence of  $a$  and  $b$  conditional on all combinations:  $A \wedge B$ ,  $A \wedge \neg B$ ,  $\neg A \wedge B$  and  $\neg A \wedge \neg B$ .<sup>16, ~17</sup>

Let us see why the above claim holds. The calculations are my reconstruction and are not due to Dawid. The reader might be annoyed with me working out the mundane details of Dawid's claims, but

<sup>14</sup>Dawid gives no general argument, but it is not too hard to give one. Let  $rel(a) = P(a|A) = P(\neg a|\neg A)$ . We have in the background  $P(a|\neg A) = 1 - P(\neg a|\neg A) = 1 - rel(a)$ . We want to find the condition under which  $P(A|a) = P(a|A)$ . Set  $P(A) = p$  and start with Bayes' Theorem and the law of total probability, and go from there:

$$\begin{aligned} P(A|a) &= P(a|A) \\ \frac{P(a|A)p}{P(a|A)p + P(a|\neg A)(1-p)} &= P(a|A) \\ P(a|A)p &= P(a|A)[P(a|A)p + P(a|\neg A)(1-p)] \\ p &= P(a|A)p + P(a|\neg A) - P(a|\neg A)p \\ p &= rel(a)p + 1 - rel(a) - (1 - rel(a))p \\ p &= rel(a)p + 1 - rel(a) - p + rel(a)p \\ 2p &= 2rel(a)p + 1 - rel(a) \\ 2p - 2rel(a)p &= 1 - rel(a) \\ 2p(1 - rel(a)) &= 1 - rel(a) \\ 2p &= 1 \end{aligned}$$

First we multiplied both sides by the denominator. Then we divided both sides by  $P(a|A)$  and multiplied on the right side. Then we used our background notation and information. Next, we manipulated the right-hand side algebraically and moved  $-p$  to the left-hand side. Move  $2rel(a)p$  to the left and manipulate the result algebraically to get to the last line.

<sup>15</sup>In general setting, these are called the *sensitivity* and *specificity* of a test (respectively), and they don't have to be equal. For instance, a degenerate test for an illness which always responds positively, diagnoses everyone as ill, and so has sensitivity 1, but specificity 0.

<sup>16</sup>Dawid only talks about the independence of witnesses without reference to conditional independence. Conditional independence does not follow from independence, and it is the former that is needed here (also, four non-equivalent different versions of it).

<sup>17</sup>In terms of notation and derivation in the optional content that will follow, the claim holds if and only if  $28 > 28p_{11} - 12p_{00}$ . This inequality is not true for all admissible values of  $p_{11}$  and  $p_{00}$ . If  $p_{11} = 1$  and  $p_{00} = 0$ , the sides are equal. However, this is a rather degenerate example. Normally, we are interested in cases where  $p_{11} < 1$ . And indeed, on this assumption, the inequality holds.



it turns out that in the case of Dawid's strategy, the devil is in the details. The independence of witnesses gives us:

$$\begin{aligned} P(a \wedge b|A \wedge B) &= 0.7^2 = 0.49 \\ P(a \wedge b|A \wedge \neg B) &= 0.7 \times 0.3 = 0.21 \\ P(a \wedge b|\neg A \wedge B) &= 0.3 \times 0.7 = 0.21 \\ P(a \wedge b|\neg A \wedge \neg B) &= 0.3 \times 0.3 = 0.09 \end{aligned}$$

Without assuming  $A$  and  $B$  to be independent, let the probabilities of  $A \wedge B$ ,  $\neg A \wedge B$ ,  $A \wedge \neg B$ ,  $\neg A \wedge \neg B$  be  $p_{11}, p_{01}, p_{10}, p_{00}$ . First, let's see what  $P(a \wedge b)$  boils down to.

By the law of total probability we have:

$$\begin{aligned} P(a \wedge b) &= P(a \wedge b|A \wedge B)P(A \wedge B) + \\ &\quad + P(a \wedge b|A \wedge \neg B)P(A \wedge \neg B) \\ &\quad + P(a \wedge b|\neg A \wedge B)P(\neg A \wedge B) + \\ &\quad + P(a \wedge b|\neg A \wedge \neg B)P(\neg A \wedge \neg B) \end{aligned} \tag{24}$$

which, when we substitute our values and constants, results in:

$$= 0.49p_{11} + 0.21(p_{10} + p_{01}) + 0.09p_{00}$$

Now, note that because  $p_{ii}$ s add up to one, we have  $p_{10} + p_{01} = 1 - p_{00} - p_{11}$ . Let us continue.

$$\begin{aligned} &= 0.49p_{11} + 0.21(1 - p_{00} - p_{11}) + 0.09p_{00} \\ &= 0.21 + 0.28p_{11} - 0.12p_{00} \end{aligned}$$

Next, we ask what the posterior of  $A \wedge B$  given  $a \wedge b$  is (in the last line, we also multiply the numerator and the denominator by 100).

$$\begin{aligned} P(A \wedge B|a \wedge b) &= \frac{P(a \wedge b|A \wedge B)P(A \wedge B)}{P(a \wedge b)} \\ &= \frac{49p_{11}}{21 + 28p_{11} - 12p_{00}} \end{aligned}$$

In this particular case, then, our question whether  $P(A \wedge B|a \wedge b) > P(A \wedge B)$  boils down to asking whether

$$\frac{49p_{11}}{21 + 28p_{11} - 12p_{00}} > p_{11}$$

that is, whether  $28 > 28p_{11} - 12p_{00}$  (just divide both sides by  $p_{11}$ , multiply by the denominator, and manipulate algebraically).

Dawid continues working with particular choices of values and provides neither a general statement of the fact that the above considerations instantiate nor a proof of it. In the middle of the paper he says:

Even under prior dependence, the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability. . . . When the problem is analysed carefully, the 'paradox' evaporates [pp. 95-7]

where he still means the case with the particular values that he has given, but he seems to suggest that the claim generalizes to a large array of cases.

The paper does not contain a precise statement making the conditions required explicit and, *a fortiori*, does not contain a proof of it. Given the example above and Dawid's informal reading, let us develop a more precise statement of the claim and a proof thereof.

**Fact 3.** *Suppose that  $\text{rel}(a), \text{rel}(b) > 0.5$  and witnesses are independent conditional on all Boolean combinations of  $A$  and  $B$  (in a sense to be specified), and that none of the Boolean combinations of  $A$  and  $B$  has an extreme probability (of 0 or 1). It follows that  $P(A \wedge B|a \wedge b) > P(A \wedge B)$ . (Independence of  $A$  and  $B$  is not required.)*

Roughly, the theorem says that if independent and reliable witnesses provide positive support of their separate claims, their joint testimony provides positive support of the conjunction of their claims.

Let us see why the claim holds. First, we introduce an abbreviation for witness reliability:

$$\mathbf{a} = \text{rel}(a) = P(a|A) = P(\neg a|\neg A) > 0.5$$

$$\mathbf{b} = \text{rel}(b) = P(b|B) = P(\neg b|\neg B) > 0.5$$

Our independence assumption means:

$$P(a \wedge b|A \wedge B) = \mathbf{a}\mathbf{b}$$

$$P(a \wedge b|A \wedge \neg B) = \mathbf{a}(1 - \mathbf{b})$$

$$P(a \wedge b|\neg A \wedge B) = (1 - \mathbf{a})\mathbf{b}$$

$$P(a \wedge b|\neg A \wedge \neg B) = (1 - \mathbf{a})(1 - \mathbf{b})$$

Abbreviate the probabilities the way we already did:

$$\begin{aligned} P(A \wedge B) &= p_{11} & P(A \wedge \neg B) &= p_{10} \\ P(\neg A \wedge B) &= p_{01} & P(\neg A \wedge \neg B) &= p_{00} \end{aligned}$$

Our assumptions entail  $0 \neq p_{ij} \neq 1$  for  $i, j \in \{0, 1\}$  and:

$$p_{11} + p_{10} + p_{01} + p_{00} = 1 \quad (25)$$

So, we can use this with (24) to get:

$$\begin{aligned} P(a \wedge b) &= \mathbf{a}\mathbf{b}p_{11} + \mathbf{a}(1 - \mathbf{b})p_{10} + (1 - \mathbf{a})\mathbf{b}p_{01} + (1 - \mathbf{a})(1 - \mathbf{b})p_{00} \\ &= p_{11}\mathbf{a}\mathbf{b} + p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b}) \end{aligned} \quad (26)$$

Let's now work out what the posterior of  $A \wedge B$  will be, starting with an application of the Bayes' Theorem:

$$\begin{aligned} P(A \wedge B|a \wedge b) &= \frac{P(a \wedge b|A \wedge B)P(A \wedge B)}{P(a \wedge b)} \\ &= \frac{\mathbf{a}\mathbf{b}p_{11}}{p_{11}\mathbf{a}\mathbf{b} + p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b})} \end{aligned} \quad (27)$$

To answer our question we therefore have to compare the content of (27) to  $p_{11}$  and our claim holds just in case:

$$\begin{aligned} \frac{\mathbf{a}\mathbf{b}p_{11}}{p_{11}\mathbf{a}\mathbf{b} + p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b})} &> p_{11} \\ \frac{\mathbf{a}\mathbf{b}}{p_{11}\mathbf{a}\mathbf{b} + p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b})} &> 1 \\ p_{11}\mathbf{a}\mathbf{b} + p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b}) &< \mathbf{a}\mathbf{b} \end{aligned} \quad (28)$$

Proving (28) is therefore our goal for now. This is achieved by the following reasoning:<sup>18</sup>

- |     |  |   |
|-----|--|---|
| 1.  | $\mathbf{b} > 0.5, \mathbf{a} > 0.5$   | assumption  |
| 2.  | $2\mathbf{b} > 1, 2\mathbf{a} > 1$   | from 1.   |
| 3.  | $2\mathbf{a}\mathbf{b} > \mathbf{a}, 2\mathbf{a}\mathbf{b} > \mathbf{b}$   | multiplying by $\mathbf{a}$ and $\mathbf{b}$ respectively |
| 4.  | $p_{10}2\mathbf{a}\mathbf{b} > p_{10}\mathbf{a}, p_{01}2\mathbf{a}\mathbf{b} > p_{01}\mathbf{b}$   | multiplying by $p_{10}$ and $p_{01}$ respectively         |
| 5.  | $p_{10}2\mathbf{a}\mathbf{b} + p_{01}2\mathbf{a}\mathbf{b} > p_{10}\mathbf{a} + p_{01}\mathbf{b}$  | adding by sides, 3., 4.                                   |
| 6.  | $1 - \mathbf{b} - \mathbf{a} < 0$  | from 1.   |
| 7.  | $p_{00}(1 - \mathbf{b} - \mathbf{a}) < 0$  | From 6., because $p_{00} > 0$                             |
| 8.  | $p_{10}2\mathbf{a}\mathbf{b} + p_{01}2\mathbf{a}\mathbf{b} > p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1 - \mathbf{b} - \mathbf{a})$  | from 5. and 7.  |
| 9.  | $p_{10}\mathbf{a}\mathbf{b} + p_{10}\mathbf{a}\mathbf{b} + p_{01}\mathbf{a}\mathbf{b} + p_{01}\mathbf{a}\mathbf{b} + p_{00}\mathbf{a}\mathbf{b} - p_{00}\mathbf{a}\mathbf{b} > p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1 - \mathbf{b} - \mathbf{a})$  | 8., rewriting left-hand side                              |
| 10. | $p_{10}\mathbf{a}\mathbf{b} + p_{01}\mathbf{a}\mathbf{b} + p_{00}\mathbf{a}\mathbf{b} > -p_{10}\mathbf{a}\mathbf{b} - p_{01}\mathbf{a}\mathbf{b} + p_{00}\mathbf{a}\mathbf{b} + p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1 - \mathbf{b} - \mathbf{a})$ | 9., moving from left to right                             |
| 11. | $\mathbf{a}\mathbf{b}(p_{10} + p_{01} + p_{00}) > p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b})$  | 10., algebraic manipulation                               |
| 12. | $\mathbf{a}\mathbf{b}(1 - p_{11}) > p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b})$  | 11. and equation (25)                                     |
| 13. | $\mathbf{a}\mathbf{b} - \mathbf{a}\mathbf{b}p_{11} > p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b})$   | 12., algebraic manipulation                               |
| 14. | $\mathbf{a}\mathbf{b} > \mathbf{a}\mathbf{b}p_{11} + p_{10}(\mathbf{a} - \mathbf{a}\mathbf{b}) + p_{01}(\mathbf{b} - \mathbf{a}\mathbf{b}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{a}\mathbf{b})$   | 13., moving from left to right                            |

<sup>18</sup>Thanks to Pawel Pawlowski for working on this proof with me.

%\end{adjustbox}

The last line is what we have been after.

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OPTIONAL CONTENT ENDS

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Now that we have as a theorem an explication of what Dawid informally suggested, let's see whether it helps the probabilist handling of DAC.

### 8.6.1 Kaplow

On RLP, at least in certain cases, the decision rule leads us to (??), which tells us to decide the case based on whether the likelihood ratio is greater than 1.

<sup>19</sup> While Kaplow did not discuss DAC or the gatecrasher paradox, it is only fair to evaluate Kaplow's proposal from the perspective of these difficulties.

Add here stuff from Marcello's Mind paper about the prisoner hypothetical. Then, discuss Rafal's critique of the likelihood ratio threshold and see where we end up.

## 8.7 Dawid's likelihood strategy doesn't help

Recall that DAC was a problem posed for the decision standard proposed by TLP, and the real question is how the information resulting from Fact 3 can help to avoid that problem. Dawid does not mention any decision standard, and so addresses quite a different question, and so it is not clear that 'the paradox' evaporates', as Dawid suggests.

What Dawid correctly suggests (and we establish in general as Fact 3) is that the support of the conjunction by two witnesses will be positive as soon as their separate support for the conjuncts is positive. That is, that the posterior of the conjunction will be higher than its prior. But the critic of probabilism never denied that the conjunction of testimonies might raise the probability of the conjunction if the testimonies taken separately support the conjuncts taken separately. Such a critic can still insist that Fact 3 does nothing to alleviate her concern. After all, at least *prima facie* it still might be the case that:

- the posterior probabilities of the conjuncts are above a given threshold,
- the posterior probability of the conjunction is higher than the prior probability of the conjunction,
- the posterior probability of the conjunction is still below the threshold.

That is, Fact 3 does not entail that once the conjuncts satisfy a decision standard, so does the conjunction.

At some point, Dawid makes a general claim that is somewhat stronger than the one already cited:

When the problem is analysed carefully, the 'paradox' evaporates: suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents.

[p. 97]

This is quite a different claim from the content of Fact 3, because previously the joint probability was claimed only to increase as compared to the prior, and here it is claimed to increase above the level of the separate increases provided by separate testimonies. Regarding this issue Dawid elaborates (we still use the  $p_{ij}$ -notation that we've already introduced):

“More generally, let  $P(a|A)/P(a|\neg A) = \lambda$ ,  $P(b|B)/P(b|\neg B) = \mu$ , with  $\lambda, \mu > 0.7$ , as might arise, for example, when there are several available testimonies. If the witnesses are independent, then

$$P(A \wedge B|a \wedge b) = \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00})$$

which increases with each of  $\lambda$  and  $\mu$ , and is never less than the larger of  $\lambda p_{11}/(1 - p_{11} + \lambda p_{11})$ ,  $\mu p_{11}/(1 - p_{11} + \mu p_{11})$ , the posterior probabilities appropriate to the individual testimonies.” [p. 95]

This claim, however, is false.

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<sup>19</sup> Again, the name of the view is by no means standard, it is just a term I coined to refer to various types of legal probabilism in a fairly uniform manner.

Let us see why. The quoted passage is a bit dense. It contains four claims for which no arguments are given in the paper. The first three are listed below as (29), the fourth is that if the conditions in (29) hold,  $P(A \wedge B|a \wedge b) > \max(P(A|a), P(B|b))$ . Notice that  $\lambda = LR(a|A)$  and  $\mu = LR(b|B)$ . Suppose the first three claims hold, that is:

$$\begin{aligned} P(A \wedge B|a \wedge b) &= \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00}) \\ P(A|a) &= \frac{\lambda p_{11}}{1 - p_{11} + \lambda p_{11}} \\ P(B|b) &= \frac{\mu p_{11}}{1 - p_{11} + \mu p_{11}} \end{aligned} \quad (29)$$

Is it really the case that  $P(A \wedge B|a \wedge b) > P(A|a), P(B|b)$ ? It does not seem so. Let  $\mathbf{a} = \mathbf{b} = 0.6$ ,  $pr = \langle p_{11}, p_{10}, p_{01}, p_{00} \rangle = \langle 0.1, 0.7, 0.1, 0.1 \rangle$ . Then,  $\lambda = \mu = 1.5 > 0.7$  so the assumption is satisfied. Then we have  $P(A) = p_{11} + p_{10} = 0.8$ ,  $P(B) = p_{11} + p_{01} = 0.2$ . We can also easily compute  $P(a) = \mathbf{a}P(A) + (1 - \mathbf{a})P(\neg A) = 0.56$  and  $P(b) = \mathbf{b}P(B) + (1 - \mathbf{b})P(\neg B) = 0.44$ . Yet:

$$\begin{aligned} P(A|a) &= \frac{P(a|A)P(A)}{P(a)} = \frac{0.6 \times 0.8}{0.6 \times 0.8 + 0.4 \times 0.2} \approx 0.8571 \\ P(B|b) &= \frac{P(b|B)P(B)}{P(b)} = \frac{0.6 \times 0.2}{0.6 \times 0.2 + 0.4 \times 0.8} \approx 0.272 \\ P(A \wedge B|a \wedge b) &= \frac{P(a \wedge b|A \wedge B)P(A \wedge B)}{P(a \wedge b|A \wedge B)P(A \wedge B) + P(a \wedge b|A \wedge \neg B)P(A \wedge \neg B) + P(a \wedge b|\neg A \wedge B)P(\neg A \wedge B) + P(a \wedge b|\neg A \wedge \neg B)P(\neg A \wedge \neg B)} \\ &= \frac{\mathbf{a}\mathbf{b}p_{11}}{\mathbf{a}\mathbf{b}p_{11} + \mathbf{a}(1 - \mathbf{b})p_{10} + (1 - \mathbf{a})\mathbf{b}p_{01} + (1 - \mathbf{a})(1 - \mathbf{b})p_{00}} \approx 0.147 \end{aligned}$$

The posterior probability of  $A \wedge B$  is not only lower than the larger of the individual posteriors, but also lower than any of them!

So what went wrong in Dawid's calculations in (29)? Well, the first formula is correct. However, let us take a look at what the second one says (the problem with the third one is pretty much the same):

$$P(A|a) = \frac{\frac{P(a|A)}{P(\neg a|A)} \times P(A \wedge B)}{P(\neg(A \wedge B)) + \frac{P(a|A)}{P(\neg a|A)} \times P(A \wedge B)}$$

Quite surprisingly, in Dawid's formula for  $P(A|a)$ , the probability of  $A \wedge B$  plays a role. To see that it should not take any  $B$  that excludes  $A$  and the formula will lead to the conclusion that *always*  $P(A|a)$  is undefined. The problem with Dawid's formula is that instead of  $p_{11} = P(A \wedge B)$  he should have used  $P(A) = p_{11} + p_{10}$ , in which case the formula would rather say this:

$$\begin{aligned} P(A|a) &= \frac{\frac{P(a|A)}{P(\neg a|A)} \times P(A)}{P(\neg A) + \frac{P(a|A)}{P(\neg a|A)} \times P(A)} \\ &= \frac{\frac{P(a|A)P(A)}{P(\neg a|A)}}{\frac{P(\neg a|A)P(\neg A)}{P(\neg a|A)} + \frac{P(a|A)P(A)}{P(\neg a|A)}} \\ &= \frac{P(a|A)P(A)}{P(\neg a|A)P(\neg A) + P(a|A)P(A)} \end{aligned}$$

Now, on the assumption that witness' sensitivity is equal to their specificity, we have  $P(a|\neg A) = P(\neg a|A)$  and can substitute this in the denominator:

$$= \frac{P(a|A)P(A)}{P(a|\neg A)P(\neg A) + P(a|A)P(A)}$$

and this would be a formulation of Bayes' theorem. And indeed with  $P(A) = p_{11} + p_{10}$  the formula works (albeit its adequacy rests on the identity of  $P(a|\neg A)$  and  $P(\neg a|A)$ ), and yields the result that we already obtained:

$$\begin{aligned} P(A|a) &= \frac{\lambda(p_{11} + p_{10})}{1 - (p_{11} + p_{10}) + \lambda(p_{11} + p_{10})} \\ &= \frac{1.5 \times 0.8}{1 - 0.8 + 1.5 \times 0.8} \approx 0.8571 \end{aligned}$$

The situation cannot be much improved by taking **a** and **b** to be high. For instance, if they're both 0.9 and  $pr = \langle 0.1, 0.7, 0.1, 0.1 \rangle$ , the posterior of  $A$  is  $\approx 0.972$ , the posterior of  $B$  is  $\approx 0.692$ , and yet the joint posterior of  $A \wedge B$  is 0.525.

The situation cannot also be improved by saying that at least if the threshold is 0.5, then as soon as **a** and **b** are above 0.7 (and, *a fortiori*, so are  $\lambda$  and  $\mu$ ), the individual posteriors being above 0.5 entails the joint posterior being above 0.5 as well. For instance, for **a** = 0.7 and **b** = 0.9 with  $pr = \langle 0.1, 0.3, 0.5, 0.1 \rangle$ , the individual posteriors of  $A$  and  $B$  are  $\approx 0.608$  and  $\approx 0.931$  respectively, while the joint posterior of  $A \wedge B$  is  $\approx 0.283$ .

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OPTIONAL CONTENT ENDS

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The situation cannot be improved by saying that what was meant was rather that the joint likelihood is going to be at least as high as the maximum of the individual likelihoods, because quite the opposite is the case: the joint likelihood is going to be lower than any of the individual ones.

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OPTIONAL CONTENT STARTS

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Let us make sure this is the case. We have:

$$\begin{aligned} LR(a|A) &= \frac{P(a|A)}{P(a|\neg A)} \\ &= \frac{P(a|A)}{P(\neg a|A)} \\ &= \frac{\mathbf{a}}{1 - \mathbf{a}}. \end{aligned}$$

where the substitution in the denominator is legitimate only because witness' sensitivity is identical to their specificity.

With the joint likelihood, the reasoning is just a bit more tricky. We will need to know what  $P(a \wedge b | \neg(A \wedge B))$  is. There are three disjoint possible conditions in which the condition holds:  $A \wedge \neg B$ ,  $\neg A \wedge B$ , and  $\neg A \wedge \neg B$ . The probabilities of  $a \wedge b$  in these three scenarios are respectively  $\mathbf{a}(1 - \mathbf{b})$ ,  $(1 - \mathbf{a})\mathbf{b}$ ,  $(1 - \mathbf{a})(1 - \mathbf{b})$  (again, the assumption of independence is important), and so on the assumption  $\neg(A \wedge B)$  the probability of  $a \wedge b$  is:

$$\begin{aligned} P(a \wedge b | \neg(A \wedge B)) &= \mathbf{a}(1 - \mathbf{b}) + (1 - \mathbf{a})\mathbf{b} + (1 - \mathbf{a})(1 - \mathbf{b}) \\ &= \mathbf{a}(1 - \mathbf{b}) + (1 - \mathbf{a})(\mathbf{b} + 1 - \mathbf{b}) \\ &= \mathbf{a}(1 - \mathbf{b}) + (1 - \mathbf{a}) \\ &= \mathbf{a} - \mathbf{ab} + 1 - \mathbf{a} = 1 - \mathbf{ab} \end{aligned}$$

So, on the assumption of witness independence, we have:

$$\begin{aligned} LR(a \wedge b | A \wedge B) &= \frac{P(a \wedge b | A \wedge B)}{P(a \wedge b | \neg(A \wedge B))} \\ &= \frac{\mathbf{ab}}{1 - \mathbf{ab}} \end{aligned}$$

With  $0 < \mathbf{a}, \mathbf{b} < 1$  we have  $\mathbf{ab} < \mathbf{a}$ ,  $1 - \mathbf{ab} > 1 - \mathbf{a}$ , and consequently:

$$\frac{\mathbf{ab}}{1 - \mathbf{ab}} < \frac{\mathbf{a}}{1 - \mathbf{a}}$$

which means that the joint likelihood is going to be lower than any of the individual ones.

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OPTIONAL CONTENT ENDS

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Fact 3 is so far the most optimistic reading of the claim that if witnesses are independent and fairly reliable, their testimonies are going to provide positive support for the conjunction.<sup>Footnote</sup> And this is the reading that Dawid in passing suggests: “the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability.” (Dawid, 1987: 95) and any stronger reading of Dawid’s suggestions fails. But Fact 3 is not too exciting when it comes to answering the original DAC. The original question focused on the adjudication model according to which the deciding agents are to evaluate the posterior probability of the whole case conditional on all evidence, and to convict if it is above a certain threshold. The problem, generally, is that it might be the case that the pieces of evidence for particular elements of the claim can have high likelihood and posterior probabilities of particular elements can be above the threshold while the posterior joint probability will still fail to meet the threshold. The fact that the joint posterior will be higher than the joint prior does not help much. For instance, if  $\mathbf{a} = \mathbf{b} = 0.7$ ,  $pr = \langle 0.1, 0.5, 0.3, 0.1 \rangle$ , the posterior of  $A$  is  $\approx 0.777$ , the posterior of  $B$  is  $\approx 0.608$  and the joint posterior is  $\approx 0.216$  (yes, it is higher than the joint prior = 0.1, but this does not help the conjunction to satisfy the decision standard).

To see the extent to which Dawid’s strategy is helpful here, perhaps the following analogy might be useful.

Imagine it is winter, the heating does not work in my office and I am quite cold. I pick up the phone and call maintenance. A rather cheerful fellow picks up the phone. I tell him what my problem is, and he reacts:

- Oh, don’t worry.
- What do you mean? It’s cold in here!
- No no, everything is fine, don’t worry.
- It’s not fine! I’m cold here!
- Look, sir, my notion of it being warm in your office is that the building provides some improvement to what the situation would be if it wasn’t there. And you agree that you’re definitely warmer than you’d be if your desk was standing outside, don’t you? Your, so to speak, posterior warmth is higher than your prior warmth, right?

Dawid’s discussion is in the vein of the above conversation. In response to a problem with the adjudication model under consideration Dawid simply invites us to abandon thinking in terms of it and to abandon requirements crucial for the model. Instead, he puts forward a fairly weak notion of support (analogous to a fairly weak sense of the building providing improvement), according to which, assuming witnesses are fairly reliable, if separate fairly reliable witnesses provide positive support to the conjuncts, then their joint testimony provides positive support for the conjunction.

As far as our assessment of the original adjudication model and dealing with DAC, this leaves us hanging. Yes, if we abandon the model, DAC does not worry us anymore. But should we? And if we do, what should we change it to, if we do not want to be banished from the paradise of probabilistic methods?

Having said this, let me emphasize that Dawid’s paper is important in the development of the debate, since it shifts focus on the likelihood ratios, which for various reasons are much better measures of evidential support provided by particular pieces of evidence than mere posterior probabilities.

Before we move to another attempt at a probabilistic formulation of the decision standard, let us introduce the other hero of our story: the gatecrasher paradox. It is against DAC and this paradox that the next model will be judged.

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OPTIONAL CONTENT STARTS

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In fact, Cohen replied to Dawid’s paper (L. J. Cohen, 1988). His reply, however, does not have much to do with the workings of Dawid’s strategy, and is rather unusual. Cohen’s first point is that the calculations of posteriors require odds about unique events, whose meaning is usually given in terms of potential wagers – and the key criticism here is that in practice such wagers cannot be decided. This is not a convincing criticism, because the betting-odds interpretations of subjective probability do not require that on each occasion the bet should really be practically decidable. It rather invites one to imagine a possible situation in which the truth could be found out and asks: how much would we bet on a certain claim in such a situation? In some cases, this assumption is false, but there is nothing in principle wrong with thinking about the consequences of false assumptions.

Second, Cohen says that Dawid’s argument works only for testimonial evidence, not for other types thereof. But this claim is simply false – just because Dawid used testimonial evidence as an example

that he worked through it by no means follows that the approach cannot be extended. After all, as long as we can talk about sensitivity and specificity of a given piece of evidence, everything that Dawid said about testimonies can be repeated *mutatis mutandis*.

Third, Cohen complains that Dawid in his example worked with rather high priors, which according to Cohen would be too high to correspond to the presumption of innocence. This also is not a very successful rejoinder. Cohen picked his priors in the example for the ease of calculations, and the reasoning can be run with lower priors. Moreover, instead of discussing the conjunction problem, Cohen brings in quite a different problem: how to probabilistically model the presumption of innocence, and what priors of guilt should be appropriate? This, indeed, is an important problem; but it does not have much to do with DAC, and should be discussed separately.

## 8.8 Problem's with Kaplow's stuff

Kaplow does not discuss the conceptual difficulties that we are concerned with, but this will not stop us from asking whether DTLP can handle them (and answering to the negative). Let us start with DAC.

Say we consider two claims,  $A$  and  $B$ . Is it generally the case that if they separately satisfy the decision rule, then so does  $A \wedge B$ ? That is, do the assumptions:

$$\begin{aligned}\frac{P(E|A)}{P(E|\neg A)} &> \frac{P(\neg A)}{P(A)} \times \frac{L}{G} \\ \frac{P(E|B)}{P(E|\neg B)} &> \frac{P(\neg B)}{P(B)} \times \frac{L}{G}\end{aligned}$$

entail

$$\frac{P(E|A \wedge B)}{P(E|\neg(A \wedge B))} > \frac{P(\neg(A \wedge B))}{P(A \wedge B)} \times \frac{L}{G}?$$

Alas, the answer is negative.

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OPTIONAL CONTENT STARTS

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This can be seen from the following example. Suppose a random digit from 0-9 is drawn; we do not know the result; we are told that the result is  $< 7$  ( $E = \text{'the result is } < 7\text{'}$ ), and we are to decide whether to accept the following claims:

$A$	the result is $< 5$ .
$B$	the result is an even number.
$A \wedge B$	the result is an even number $< 5$ .

Suppose that  $L = G$  (this is for simplicity only — nothing hinges on this, counterexamples for when this condition fails are analogous). First, notice that  $A$  and  $B$  taken separately satisfy (22).  $P(A) = P(\neg A) = 0.5$ ,  $P(\neg A)/P(A) = 1$   $P(E|A) = 1$ ,  $P(E|\neg A) = 0.4$ . (22) tells us to check:

$$\begin{aligned}\frac{P(E|A)}{P(E|\neg A)} &> \frac{L}{G} \times \frac{P(\neg A)}{P(A)} \\ \frac{1}{0.4} &> 1\end{aligned}$$

so, following DTLP, we should accept  $A$ .

For analogous reasons, we should also accept  $B$ .  $P(B) = P(\neg B) = 0.5$ ,  $P(\neg B)/P(B) = 1$   $P(E|B) = 0.8$ ,  $P(E|\neg B) = 0.6$ , so we need to check that indeed:

$$\begin{aligned}\frac{P(E|B)}{P(E|\neg B)} &> \frac{L}{G} \times \frac{P(\neg B)}{P(B)} \\ \frac{0.8}{0.6} &> 1\end{aligned}$$



But now,  $P(A \wedge B) = 0.3$ ,  $P(\neg(A \wedge B)) = 0.7$ ,  $P(\neg(A \wedge B))/P(A \wedge B) = 2\frac{1}{3}$ ,  $P(E|A \wedge B) = 1$ ,  $P(E|\neg(A \wedge B)) = 4/7$  and it is false that:

$$\frac{P(E|A \wedge B)}{P(E|\neg(A \wedge B))} > \frac{L}{G} \times \frac{P(\neg(A \wedge B))}{P(A \wedge B)}$$

$$\frac{7}{4} > \frac{7}{3}$$

The example was easy, but the conjuncts are probabilistically dependent. One might ask: are there counterexamples that involve claims which are probabilistically independent?<sup>20</sup>

Consider an experiment in which someone tosses a six-sided die twice. Let the result of the first toss be  $X$  and the result of the second one  $Y$ . Your evidence is that the results of both tosses are greater than one ( $E =: X > 1 \wedge Y > 1$ ). Now, let  $A$  say that  $X < 5$  and  $B$  say that  $Y < 5$ .

The prior probability of  $A$  is  $2/3$  and the prior probability of  $\neg A$  is  $1/3$  and so  $\frac{P(\neg A)}{P(A)} = 0.5$ . Further,  $P(E|A) = 0.625$ ,  $P(E|\neg A) = 5/6$  and so  $\frac{P(E|A)}{P(E|\neg A)} = 0.75$ . Clearly,  $0.75 > 0.5$ , so  $A$  satisfies the decision standard. Since the situation with  $B$  is symmetric, so does  $B$ .

Now,  $P(A \wedge B) = (2/3)^2 = 4/9$  and  $P(\neg(A \wedge B)) = 5/9$ . So  $\frac{P(\neg(A \wedge B))}{P(A \wedge B)} = 5/4$ . Out of 16 outcomes for which  $A \wedge B$  holds,  $E$  holds in 9, so  $P(E|A \wedge B) = 9/16$ . Out of 20 remaining outcomes for which  $A \wedge B$  fails,  $E$  holds in 16, so  $P(E|\neg(A \wedge B)) = 4/5$ . Thus,  $\frac{P(E|A \wedge B)}{P(E|\neg(A \wedge B))} = 45/64 < 5/4$ , so the conjunction does not satisfy the decision standard.

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OPTIONAL CONTENT ENDS

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Let us turn to the gatecrasher paradox.

Suppose  $L = G$  and recall our abbreviations:  $P(E) = e$ ,  $P(H_{\Pi}) = \pi$ . DTLT tells us to convict just in case:

$$LR(E) > \frac{1 - \pi}{\pi}$$

From (??) we already now that

$$LR(E) = \frac{0.991 - 0.991\pi}{0.009\pi}$$

so we need to see whether there are any  $0 < \pi < 1$  for which

$$\frac{0.991 - 0.991\pi}{0.009\pi} > \frac{1 - \pi}{\pi}$$

Multiply both sides first by  $0.009\pi$  and then by  $\pi$ :

$$0.991\pi - 0.991\pi^2 > 0.09\pi - 0.009\pi^2$$

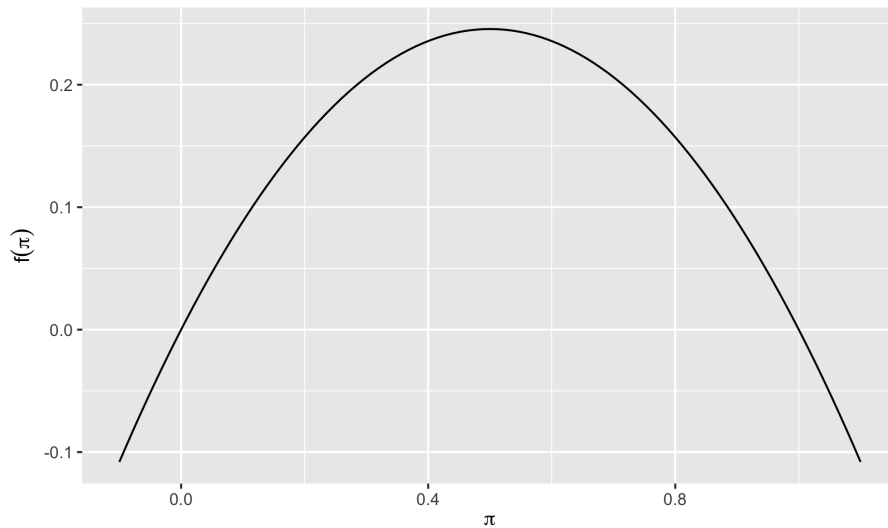
Simplify and call the resulting function  $f$ :

$$f(\pi) = -0.982\pi^2 + 0.982\pi > 0$$

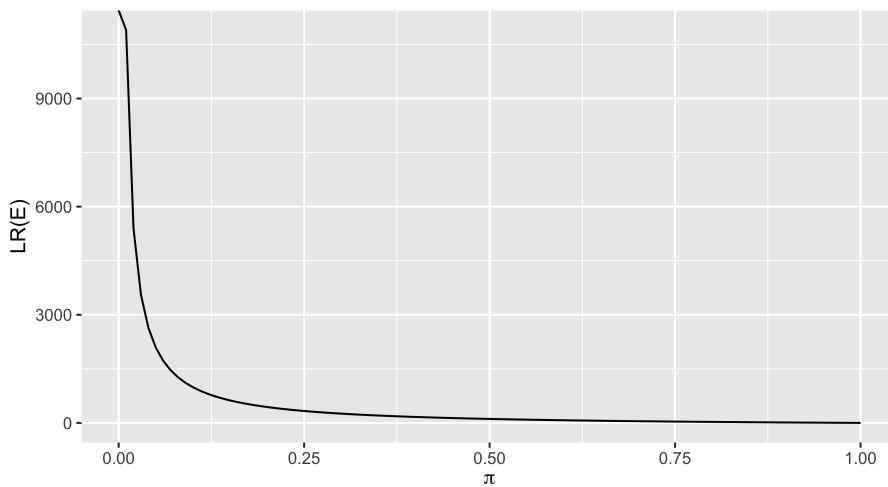
The above condition is satisfied for any  $0 < \pi < 1$  ( $f$  has two zeros:  $\pi = 0$  and  $\pi = 1$ ). Here is a plot of  $f$ :

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<sup>20</sup>Thanks to Alicja Kowalewska for pressing me on this.



Similarly,  $LR(E) > 1$  for any  $0 < \pi < 1$ . Here is a plot of  $LR(E)$  against  $\pi$ :



Notice that  $LR(E)$  does not go below 1. This means that for  $L = G$  in the gatecrasher scenario DTLP would tell us to convict for any prior probability of guilt  $\pi \neq 0, 1$ .

One might ask: is the conclusion very sensitive to the choice of  $L$  and  $G$ ? The answer is, not too much.

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OPTIONAL CONTENT STARTS

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How sensitive is our analysis to the choice of  $L/G$ ? Well,  $LR(E)$  does not change at all, only the threshold moves. For instance, if  $L/G = 4$ , instead of  $f$  we end up with

$$f'(\pi) = -0.955\pi^2 + 0.955\pi > 0$$

and the function still takes positive values on the interval  $(0, 1)$ . In fact, the decision won't change until  $L/G$  increases to  $\approx 111$ . Denote  $L/G$  as  $\rho$ , and let us start with the general decision standard, plugging

in our calculations for  $LR(E)$ :

$$\begin{aligned}
LR(E) &> \frac{P(H_{\Delta})}{P(H_{\Pi})} \rho \\
LR(E) &> \frac{1-\pi}{\pi} \rho \\
\frac{0.991-0.991\pi}{0.009\pi} &> \frac{1-\pi}{\pi} \rho \\
\frac{0.991-0.991\pi}{0.009\pi} \frac{\pi}{1-\pi} &> \rho \\
\frac{0.991\pi-0.991\pi^2}{0.009\pi-0.009\pi^2} &> \rho \\
\frac{\pi(0.991-0.991\pi)}{\pi(0.009-0.009\pi)} &> \rho \\
\frac{0.991-0.991\pi}{0.009-0.009\pi} &> \rho \\
\frac{0.991(1-\pi)}{0.009(1-\pi)} &> \rho \\
\frac{0.991}{0.009} &> \rho \\
110.1111 &> \rho
\end{aligned}$$

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OPTIONAL CONTENT ENDS

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So, we conclude, in usual circumstances, DTLP does not handle the gatecrasher paradox.

## 8.9 Conclusions

Where are we, how did we get here, and where can we go from here? We were looking for a probabilistically explicated condition  $\Psi$  such that the trier of fact, at least ideally, should accept any relevant claim (including  $G$ ) just in case  $\Psi(A, E)$ .

From the discussion that transpired it should be clear that we were looking for a  $\Psi$  satisfying the following desiderata:

**conjunction closure** If  $\Psi(A, E)$  and  $\Psi(B, E)$ , then  $\Psi(A \wedge B, E)$ .

**naked statistics** The account should at least make it possible for convictions based on strong, but naked statistical evidence to be unjustified.

**equal treatment** the condition should apply to any relevant claim whatsoever (and not just a selected claim, such as  $G$ ).

Throughout the paper we focused on the first two conditions (formulated in terms of the difficulty about conjunction (DAC), and the gatecrasher paradox), going over various proposals of what  $\Psi$  should be like and evaluating how they fare. The results can be summed up in the following table:

View	Convict iff	DAC	Gatecrasher
Threshold-based LP (TLP)	Probability of guilt given the evidence is above a certain threshold	fails	fails
Dawid's likelihood strategy	No condition given, focus on $\frac{P(H E)}{P(H \neg E)}$	<ul style="list-style-type: none"> <li>- If evidence is fairly reliable, the posterior of <math>A \wedge B</math> will be greater than the prior.</li> <li>- The posterior of <math>A \wedge B</math> can still be lower than the posterior of any of <math>A</math> and <math>B</math>.</li> <li>- Joint likelihood, contrary to Dawid's claim, can also be lower than any of the individual likelihoods.</li> </ul>	fails
Cheng's relative LP (RLP)	Posterior of guilt higher than the posterior of any of the defending narrations	The solution assumes equal costs of errors and independence of $A$ and $B$ conditional on $E$ . It also relies on there being multiple defending scenarios individualized in terms of combinations of literals involving $A$ and $B$ .	Assumes that the prior odds of guilt are 1, and that the statistics is not sensitive to guilt (which is dubious). If the latter fails, tells to convict as long as the prior of guilt $< 0.991$ .
Kaplow's decision-theoretic LP (DTLP)	The likelihood of the evidence is higher than the odds of innocence multiplied by the cost of error ratio	fails	convict if cost ratio $< 110.1111$

Thus, each account either simply fails to satisfy the desiderata, or succeeds on rather unrealistic assumptions. Does this mean that a probabilistic approach to legal evidence evaluation should be abandoned? No. This only means that if we are to develop a general probabilistic model of legal decision standards, we have to do better. One promising direction is to go back to Cohen's pressure against **Requirement 1** and push against it. A brief paper suggesting this direction is (Di Bello, 2019), where the idea is that the probabilistic standard (be it a threshold or a comparative wrt. defending narrations) should be applied to the whole claim put forward by the plaintiff, and not to its elements. In such a context, DAC does not arise, but **equal treatment** is violated. Perhaps, there are independent reasons to abandon it, but the issue deserves further discussion. Another strategy might be to go in the direction of employing probabilistic methods to explicate the narration theory of legal decision standards (Urbaniak, 2018), but a discussion of how this approach relates to DAC and the gatecrasher paradox lies beyond the scope of this paper.

## References

- Allen, R. J. (1986). A reconceptualization of civil trials. *Boston University Law Review*, 66, 401–437.
- Allen, R. J., & Stein, A. (2013). Evidence, probability and the burden of proof. *Arizona Law Journal*, 55, 557–602.
- Allen, R., & Pardo, M. (2019). Relative plausibility and its critics. *The International Journal of Evidence & Proof*, 23(1-2), 5–59. <https://doi.org/10.1177/1365712718813781>
- Cheng, E. (2012). Reconceptualizing the burden of proof. *Yale LJ*, 122, 1254.
- Cohen, J. L. (1977). *The probable and the provable*. Oxford University Press. <https://doi.org/10.2307/2219193>
- Cohen, L. J. (1988). The difficulty about conjunction in forensic proof. *The Statistician*, 37(4/5), 415. <https://doi.org/10.2307/2348767>
- Dawid, A. P. (1987). The difficulty about conjunction. *The Statistician*, 91–97.
- Di Bello, M. (2019). Probability and plausibility in juridical proof. *International Journal of Evidence and Proof*.
- Haack, S. (2014). Legal probabilism: An epistemological dissent. In *Haack2014-HAAEMS* (pp. 47–77).
- Kaplow, L. (2014). Likelihood ratio tests and legal decision rules. *American Law and Economics Review*, 16(1), 1–39.
- Kowalewska, A. (2021). Reasoning without the conjunction closure. *Episteme*, 1–14. <https://doi.org/10.1017/epi.2020.53>

- Makinson, D. C. (1965). The paradox of the preface. *Analysis*, 25(6), 205–207.
- Schwartz, D. S., & Sober, E. R. (2017). The Conjunction Problem and the Logic of Jury Findings. *William & Mary Law Review*, 59(2), 619–692.
- Stein, A. (2005). *Foundations of evidence law*. Oxford University Press.
- Urbaniak, R. (2018). Narration in judiciary fact-finding: A probabilistic explication. *Artificial Intelligence and Law*, 1–32. <https://doi.org/10.1007/s10506-018-9219-z>