The difficulty with Conjunction

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Contents

1 The problem

A theoretical difficulty that any theory of the standard of proof should address is the the conjunction paradox or difficulty about conjunction. First formulated by J. Cohen (1977), the difficulty about conjunction has enjoyed a great deal of scholarly attention every since (R. J. Allen, 1986; R. J. Allen &

here you say "any theory" but then you switch to the probabilistic ones. Is it a problem for non-probabilistic theories or not?

Stein, 2013; R. Allen & Pardo, 2019; Haack, 2014; Schwartz & Sober, 2017; Stein, 2005). This difficulty arises when an accusation of wrongdoing, in a civil or criminal proceeding, is broken down into its constituent elements. The basic problem is that the probability of a conjuction is often lower than the probability of the conjuncts. Thus, even if each conjunct meets the requisite probability threshold, the conjunction does not. This chapter examines the difficulty about conjuction and how legal probabilists can respond.

1.1 The Conjunction Principle

Suppose that in order to prevail in a criminal trial, the prosecution should establish by the required standard, first, that the defendant caused harm to the victim (call it claim A), and second, that the defendant had premeditated the harmful act (call it claim B). J. Cohen (1977) argues that common law systems subscribe to a conjunction principle, that is,if A and B are established according to the governing standard of proof, so is their conjunction (and vice versa). If the conjunction principle holds, the following must be equivalent, where S is a placeholder for the standard of proof:

Here you first fix A and B as abbreviations, and then make a general claim using those variables; this makes it seem like the conjunction principle is about two very specific sentences, which is not what you meant.

Separate	A is established according to S and B is established according to S
Overall	The conjunction $A \wedge B$ is established according to S

If we generalize to more than just two constituent claims, the conjunction principle requires that:

You need to be clearer about what this equivalence symbol is supposed to mean; further, later you title a subsection saying the principle is false; but if it is normative, I don't understand what you mean by saying its false.

$$S[C_1 \wedge C_2 \wedge \cdots \wedge C_k] \Leftrightarrow S[C_1] \wedge S[C_2] \wedge \cdots \wedge S[C_k].$$

where $S[C_i]$ means that claim or hypothesis C_i is established according to standard S. In what follows it will be useful to distinguish between the two directions involved: the implication from right to left will be called **distribution**, and the opposite direction will be called **aggregation**. The problem that will be discussed in this subsection is a one with aggregation, but we will later on discuss approaches which fail to satisfy distribution.

The conjunction principle is consistent with—perhaps even required by—the case law. For example, the United States Supreme Court writes that in criminal cases

> the accused [is protected] against conviction except upon proof beyond a reasonable doubt of *every fact* necessary to constitute the crime with which he is charged. In re Winship (1970), 397 U.S. 358, 364.

cite properly

A plausible way to interpret this quotation is to posit this identity: to establish someone's guilt beyond a reasonable doubt *just is* to establish each element of the crime beyond a reasonable doubt. Thus,

$$\mathsf{BARD}[C_1 \wedge C_2 \wedge \cdots \wedge C_n] \Leftrightarrow \mathsf{BARD}[C_1] \wedge \mathsf{BARD}[C_2] \wedge \cdots \wedge \mathsf{BARD}[C_k],$$

where the conjunction $C_1 \wedge C_2 \cdots \wedge C_n$ comprises all the material facts that, according to the applicable law, constitute the crime with with the accused is charged.

The problem for the legal probabilist is that the conjunction principle conflicts with a threshold-basedprobabilistic interpretation of the standard of proof. For suppose the prosecution presents evidence that establishes claims A and B, separately, to the required probability, say about 95% each.Has the prosecution met its burden of proof? Each claim was established to the requisite probability threshold, and thus it was established to the requisite standard (assuming the threshold-based interpretation of the standard of proof). And if each claim was established to the requisite standard, then (i) guilt as a whole was established to the requisite standard (assuming the conjunction principle). But even though each claim was established to the requisite probability threshold, the probability of their conjunction—assuming the two claims are independent—is only $95\% \times 95\% = 90.25\%$, below the required 95% threshold. So (ii) guilt as a whole was *not* established to the requisite standard (assuming a threshold-based probabilistic interpretation of the standard). Hence, we arrive at two contradictory conclusions: (i) that the prosecution met its burden of proof and (ii) that it did not meet its burden.

you didn't introduce this notion yet

I thought we agreed to use .95 notation for probabilities and keep percentages for population frequencies only, as in SEP. This is also what I have done in the LR chapter, so perhaps we should stick to this convention throughout this chapter as well.

The difficulty about conjunction—the fact that a probabilistic interpretation of the standard of proof conflicts with the conjunction principle—does not subside when the number of constituent claims increases. If anything, the difficulty becomes more apparent. Say the prosecution has established three separate claims to 95% probability. Their conjunction—again if the claims are independent—would be about 85% probable, even further below the 95% threshold. Nor does the difficulty about conjunction subside if the claims are no longer regarded as independent. The probabilty of the conjunction $A \wedge B$, without the assumption of independence, equals $\mathsf{P}(A|B) \times \mathsf{P}(B)$. But if claims A and B, separately, have been established to 95% probability, enough for each to meet the threshold, the probability of $A \wedge B$ could still be below the 95% threshold unless $\mathsf{P}(A|B) = 100\%$. For example, that someone premediated a harmful act against another (claim B) makes it more likely that they did cause harm in the end (claim A). Since $\mathsf{P}(A|B) > \mathsf{P}(A)$, the two claims are not independent. Still, premeditation does not always lead to harm, so $\mathsf{P}(A|B)$ will often be below 100%. Consequently, in this case, the probability of the conjunction $A \wedge B$ would often be below the 95% threhsold.

False in whole generality, give a counterexample with more specific numbers. M: I changed things a bit. Maybe it's clear now. The counterexample is basically P(A)=P(B)=0.95, but P(AB)=Pr(A)*P(A|B) and since P(A|B) is below 1, then P(AB) is below 0.95.

some flow is needed here.

moved up the independencies section

1.2 Probabilistic (in)dependencies and Bayesian networks used

The conjunction paradox is a difficult problem, as the vast literature on the topic attests. Before we move on, it is important to become clear about the assumptions underlying the formulation of the paradox, in particular, the assumptions of probabilistic independence.

The reason why we need to be careful here is that there are various types of independence in the vincinity, and various arguments that we will look at often use some but not all of them, and we need to be clear about which assumptions are involved. Here are the main types of independence, with issues that we have to pay attention to:

- A and B can simply be independent, $A \perp\!\!\!\perp B$. This happens iff $\mathsf{P}(A|B) = \mathsf{P}(A)\mathsf{P}(B)$, and is equivalent to $\mathsf{P}(A|B) = \mathsf{P}(A)$ if $\mathsf{P}(A), \mathsf{P}(B) \neq 0$. If $A \perp\!\!\!\perp B$, then so are all their literals (that is, adding a negation in front of either of them doesn't remove the independence).
- Three events A, B, C, are independent if (1) they are all pairwise independent, and moreover (2) $\mathsf{P}(A)\mathsf{P}(B)\mathsf{P}(C) = \mathsf{P}(A \wedge B \wedge C)$. Interestingly, pairwise independencies alone do not entail (2). Imagine tossing a coin twice, let A='heads in the first toss', B='heads in the second toss', and C='the two results are the same'. These are pairwise independent, but fail to satisfy (2), because once you know two of them, the third one is determined. Nor does (2) entail (1). If $\mathsf{P}(A) = 0$, condition (2) will be satisfied no matter what happens with B and C. The notion and non-entailment generalizes to more events.
- A and B are conditionally independent given C, $A \perp\!\!\!\perp B|C$ iff $\mathsf{P}(A \wedge B|C) = \mathsf{P}(A|C)\mathsf{P}(B|C)$. Conditional independence does not entail independence. Say you have two coins, one fair, one biased. Conditional on which coin you have chosen, the results of subsequent tosses are independent. But if you don't know which coin you have chosen, the result of previous tosses give you some information about which coin it is, and this has impact on your estimate of the probability of heads in the next toss. Nor does not independence entail conditional independence. For instance, suppose there are two reasons for your fire alarm to go off: malfunction, or real fire. There is a plausible scenario in which they are independent, unconditionally. However, if you conditionalize on the alarm going off and find out there is no real fire, the probability of a malfunction goes up. Morever, conditional independence given C does not entail conditional independence given $\neg C$. For instance, suppose your taking classes and in some of them bad teachers give grades that don't depend on how much you work. Then your work and grade are independent conditional on the instructor being a bad teacher, but they are not independent conditional on the instructor not being a bad teacher.

One assumption often made in the formulation of the paradox is that claims A and B are probabilistically independent. This is not always the case — we have seen that the paradox does subside even if the two claims are dependent. However, we will rely on some independence assumptions in our arguments for the same of simplicity. This has a two-fold justification. One, if we are formulating a counterexample to a claim (as is the case with the conjunction paradox), even if it satisfises a further condition (independence), it still remains a counterexample to this claim. Of course, the question remain if a weaker claim that assumes the lack of independence, but it is a different claim to be evaluated in a different way. Second, if we are making a positive claim about, say, the behavior of likelihood ratios, this is purely for illustrative purposes to make calculations and proofs easier, and the reader needs to keep in mind that either the claim is restricted to situations in which independence assumptions are satisfied, or needs additional support when these assumptions are lifted.

In what follows we will be using a set up in which evidence a is meant to support claim A and another piece of evidence, b is pur forward in support of claim B. The key questions then will be about the support that the combination of a and b gives to the conjunction $A \wedge B$ and its relation to the individual support relations.

In our considerations we will sometimes use at least some of these independence assumptions, either assuming that which of these are involved is clear from the way the derivation proceeds, or being explicit about which assumptions are relied on. WE take the situation to be symmetric, that is we don't list assumptions obtained by simultaneously switching A with B and a with b.

Make a complete and clear list of all independence assumptions used

Alicja: confirm all are d-sep in the first BN, check which are d-sep in the second one.

$$A \perp\!\!\!\perp B \tag{1}$$
$$A \perp\!\!\!\perp b|a \tag{2}$$
$$B \perp\!\!\!\perp a \wedge A|b \tag{3}$$
$$a \perp\!\!\!\perp b|A \wedge B \tag{4}$$
$$a \perp\!\!\!\perp B|A \tag{5}$$
$$b \perp\!\!\!\perp A \wedge a|B \tag{6}$$
$$b \perp\!\!\!\perp a|B \tag{7}$$

Double check if we in fact use I3.

Directed Acyclic Graphs (DAGs) are useful for representing graphically these relationships of independence. The edges, intuitively, are meant to capture direct influence between the nodes. The role that such direct influence plays is that in a Bayesian network built over a DAG any node is conditionaly independent of its nondescentants (including ancestors), given its parents. If this is the case for a given probabilistic measure P() and a given DAG, we say that P() is compatible with G, and they can be put together to constitute a Bayesian network.

The graphical counterpart of probabilistic independence is the so-called *d-separation*, $\perp\!\!\!\perp_d$. We say that two nodes, X and Y, are d-separated given a set of nodes Z—$X \perp\!\!\!\perp_d Y|$Z — iff for every undirected path from X to Y there is a node Z' on the path such that either:

- $Z' \in$ Z and there is a *serial* connection, $\rightarrow Z' \rightarrow$, on the path,
- $Z' \in$ Z and there is a diverging connection, $\leftarrow Z' \rightarrow$, on the path,
- There is a connection $\rightarrow Z' \leftarrow$ on the path, and neither Z' nor its descendants are in Z.

Finally, two sets of nodes, X and Y, are d-separated given Z if every node in X is d-separated from every node in Y given Z. With serial connection, for instance, if:

Node	Proposition
G	The suspect is guilty.
B	The blood stain comes from the suspect.
M	The crime scene stain and the suspect's blood share their DNA profile.

We naturally would like to have the connection $G \rightarrow B \rightarrow M$. If we don't know whether B holds, G seems to have an indirect impact on the probability of M. Yet, once we find out that B is true, we expect the profile match, and whether G holds has no further impact on the probability of M.

The case of diverging connections has already been discussed when we talked about idependence. Whether a coin is fair, F, or not has impact on the result of the first toss, $H1$, and the result of the second toss, $H2$, and as long as we don't know whether F, $H1$ increases the probability of $H2$. So $H1 \leftarrow F \rightarrow H2$ seems to be appropriate. Once we know that F, though, $H1$ and $H2$ become independent.

For converging connections, let G and B be as above, and let:

Node	Proposition
O	The crime scene stain comes from the offender.

Both G and O influence B. If he's guilty, it's more likely that the blood stain comes from him, and if the blood crime stain comes from the offender it is more likely to come from the suspect (for instance, more so than if it comes from the victim). Moreover, G and O seem independent – whether the suspect is guilty doesn't have any bearing on whether the stain comes from the offender. Thus, a converging connection $G \rightarrow B \leftarrow O$ seem appropriate. However, if you do find out that B is true, that the stain

AB	A	B	Pr
1	1	1	1
0	1	1	0
1	0	1	0
0	0	1	1
1	1	0	0
0	1	0	1
1	0	0	0
0	0	0	1

Table 1: Conditional probability table for the conjunction node.

comes from the suspect, whether the crime stain comes from the offender becomes relevant for whether the suspect is guilty.

One important reason why d-separation matters is that it can be proven that if two sets of nodes are d-separated given a third one, then they are independent given the third one, for any probabilistic measures compatible with a given DAG. Interestingly, lack of d-separation doesn't entail dependence for any probabilistic measure compatible with a given DAG. Rather, it only allows for it: if nodes are d-separated, there is at least one probabilistic measure fitting the DAG according to which they are independent. So, at least, no false independence can be inferred from the DAG, and all the dependencies are built into it.

Now, back to the independence assumptions in our take on the conjunction problem. Consider the DAG in Figure 1.

Please use this method of drawing DAGs, it's simple, uniform and requires less work.

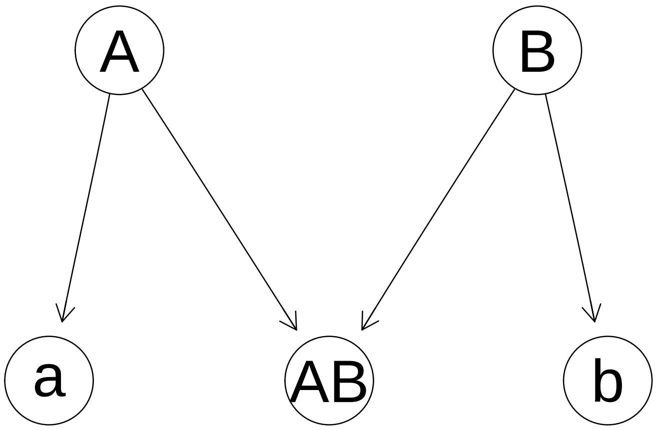

Figure 1: DAG of the conjunction set-up, with the usual independence assumptions built in.

The structure of the network is rather natural because each piece of evidence bears on its hypothesis and is probabilistically independent conditional on one of the hypotheses. One might wonder, however, why the arrows go from A and B into the node representing the conjunction $A \wedge B$. This setting captures the meaning of the conjunction. The constraint that, for the conjunction to be true, both A and B have to be true, can be defined using the appropriate conditional probability table (Table 1).

Please use the same format for tables so that it's uniform throughout the paper; I also removed the corners, it's too much fuss

This conditional probability table looks, essentially, like the truth table for conjunction. The difference is that the values 1 and 0 stand for two different things depending on where they are in the table. In the columns corresponding to the nodes they represent node states: true and false; in the Pr column they represent the conditional probability of a given state of AB given the states of A and B listed in the same row. For instance, take a look at row two. It says: if A and B are both in states 1, then the probability of AB being in state 0 is 0. In principle we could use 'true' and 'false' instead of 1 and 0 to represent states, but the numeric representation is easier to use in programming, which we do quite a bit in the background, so the reader might as well get used to this harmless ambiguity. For binary nodes, we will consistently use '1' and '0' for the states, it's just probabilities that in this case end up being extreme.

To eliminate the assumption of independence between the two claims, while holding everything else fixed, it is enough to draw an arrow between A and B (Figure 2). The two Bayesian networks provide a compact representation of the class of models we will be concerned with in this chapter.

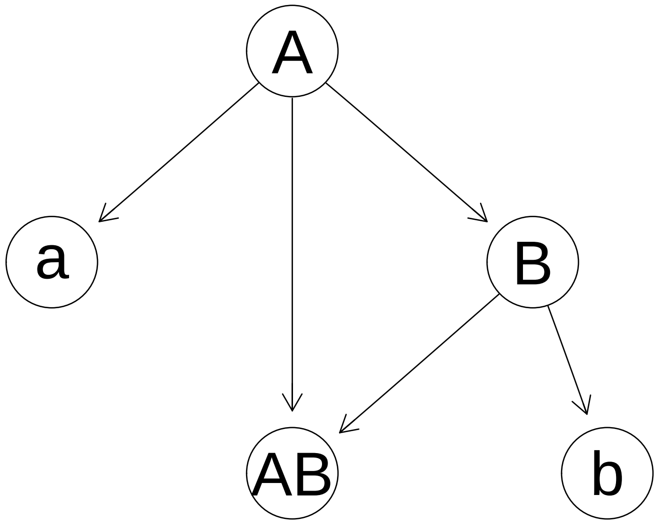

Figure 2: DAG of the conjunction set-up, without independence between A and B.

1.3 Conjunction principle with hypotheses and evidence

So far the discussion proceeded without mentioning explicitly the evidence proferred in support of the different claims that constitute the allegation of wrongdoing. This is, however, a simplification, as no evidential relations were involved. Once we pay attention to them, a plausible candidate for the conjunction principle seems to be as follows:

$$\mathrm{S}[a, A] \text{ and } \mathrm{S}[b, B] \Leftrightarrow \mathrm{S}[a \wedge b, A \wedge B],$$

where $\mathrm{S}[e, H]$ means that evidence e establishes hypothesis H by standard S.

Understood in terms of a threshold put on posterior probability, the conjunction principle above fails in some cases.For suppose that given their respective pieces of evidence a and b, both claims A and B are sufficiently probable (for a fixed threshold t), $\mathsf{P}(A|a) > t$ and $\mathsf{P}(B|b) > t$. It does not generally follow that $A \wedge B$ is sufficiently probable given combined evidence $a \wedge b$. With suitable indepedence assumptions ((2) and (3)) we have:

Please use vert for conditional probability. I changed this here.

$$\begin{aligned} \mathsf{P}(A \wedge B|a \wedge b) &= \mathsf{P}(A|a \wedge b) \times \mathsf{P}(B|a \wedge b \wedge A) \\ &= \mathsf{P}(A|a) \times \mathsf{P}(B|b) \end{aligned}$$

Of course, these relationships of independence do not always hold, but they do sometimes. For example, in an aggravated assault case, evidence a could be a witness testimony that the defendant physically injured the victim (claim A), and b evidence that the defendant knew that the victim was a firefighter (claim B), for example, another testimony that the defendant earlier called the firefighther for help. Presumably, $\mathsf{P}(A|a) = \mathsf{P}(A|a \wedge b)$ because the fact that the defendant called a firefighter for help (b) does not make it more (or less) likely that he would physically injure him (A). Further, $\mathsf{P}(B|b) = \mathsf{P}(B|a \wedge b \wedge A)$ because the fact that the defendant injured the victim (A) and there is a testimony to that effect (a) does not make it more (or less) likely that the victim was a firefighter (B).

Given these assumptions, if—as is normally the case—neither $\mathsf{P}(A|a)$ nor $\mathsf{P}(B|b)$ equal 1, then

$$\mathsf{P}(A \wedge B|a \wedge b) < \mathsf{P}(A|a) \;\;\&\;\; \mathsf{P}(A \wedge B|a \wedge b) < \mathsf{P}(B|b).$$

This is another manifestation of the difficulty about conjuction. If each piece of evidence a and b establishes claims A and B with probability .95, the combined evidence $a \wedge b$ may fail establish the conjuction $A \wedge B$ with probability .95. The conjunction principle fails here, if the standard of proof is to be formulated in terms of a threshold on posterior probability.

Even if the independence assumptions are dropped, the difficulty about conjunction still arises in a number of circumstances. Suppose evidence $a \wedge b$ establishes claim A and also claim B, separately, right above the probability threshold t. Since $\mathsf{P}(A \wedge B|a \wedge b) = \mathsf{P}(A|a \wedge b) \times \mathsf{P}(B|a \wedge b \wedge A)$, it follows that $\mathsf{P}(A \wedge B|a \wedge b)$ might be below t if the factors on the right-hand side are below 1 and separate posteriors are very close to t.

2 Evidential Strength approach to the conjunction problem

We have seen that aggregation fails if the standard of proof is understood as a posterior probability threshold. So legal probabilists cannot justify aggregation if they equate the standard of proof to such threshold. The alternative is to think of proof standards as sufficiency criteria for how strong the evidence should be in order to justify a finding of criminal or civil liability.So, instead of a posterior probability threhsold, legal probabilists could try formalize the standard of proof using a probabilistic measure of evidential strength. Will this move vindicate the cojunction principle on probabilistic grounds contrary to what Cohen thought? As we shall see, the answer to this question is complicated.

Suppose the standard of proof is no longer understood as a threshold on the posterior probability given the evidence, but rather, as a threshold on evidential strength. From a probabilistic perspective, the notion of the strength of the evidence in favor of a hypothesis should not be understood as the posterior probability of the hypothesis given the evidence. Two common probabilistic measures of evidential strength are the Bayes factor and the likelihood ratio. We discussed this topic in earlier chapters (REFER TO EARLIER CHAPTERS). As we will show in detail later, under plausible assumptions, these measures of evidential strength validate one direction of the conjunction principle:aggregation. If a is sufficiently strong evidence in favor of A and b is sufficiently strong evidence in favor of B, then $a \wedge b$ is sufficiently strong evidence in favor of the conjunction $A \wedge B$. In fact, the evidential support for the conjunction will often exceed that for the individual claims, a point already made by Dawid (1987):

> suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents.

Dawid thought this fact was enough for the conjuction paradox to 'evaporate'. To some extent, this is true since we are vindicating one direction of the conjunction principle, aggregation. That is, probability theory justifies this claim: if distinct items of evidence a and b constitute sufficiently strong evidence for claims A and B, so does the conjunction $a \wedge b$ for the composite claim $A \wedge B$ (although, there are some caveats and extra assumptions for this to hold).

Unfortunately, we will show that on the evidential strength reading, the other direction of the cojunction principle, distribution, does not hold. If $a \wedge b$ is sufficiently strong evidence in favor of $A \wedge B$, it does not follow that a is sufficiently strong evidence in favor of A or b sufficiently strong evidence in favor of B. It is not even true that, if $a \wedge b$ is sufficiently strong evidence in favor of $A \wedge B$, then $a \wedge b$ is sufficiently strong evidence in favor of A or B. This is odd. It would mean that, given a body of evidence, one can establish beyond a reasonable doubt that $A \wedge B$ (say the defendant killed the victim *and* did so intentionally) while failing to establish by the same standard one of the conjuncts.

So we seem to be in a dilemma. If the standard of proof is understood as a threshold relative to the posterior probability, the conjunction principle fails because aggregation fails while distribution succeeds. If, on the other hand, the standard of proof is understood as a threshold relative to standard measures of evidential strength, the conjuction principle fails because distribution fails while aggregation succeeds. From a probabilistic perspective, it might seems impossible to capture both directions of the conjunction principle into one unified account.

Before we discuss this more general challenge, let us explore how aggregation succeeds and distribution fails on the evidential strength approaches. We will first investigate to what extent Dawid's was right in claiming that 'suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents'. Next, we will show—perhaps surprisingly—that after switching from posterior probabilities to measures of evidential support, another paradox, what we call the *distribution paradox*, arises. The argument for these two claims is somewhat tedious. The reader should arm themselves with patience or take our word for it and jump ahead to the next section.

Some bits hidden, they need to be discussed elsewhere I think.

2.1 Bayes factor and aggregation

A common probabilistic measure of the support of E in favor of H is the Bayes factor $\mathsf{P}(E|H)/\mathsf{P}(E)$. Since by Bayes' theorem

$$\mathsf{P}(H|E) = \frac{\mathsf{P}(E|H)}{\mathsf{P}(E)} \times \mathsf{P}(H),$$

the Bayes factor measures the extent to which a piece of evidence increases the probability of a hypothesis. The greater the Bayes factor (for values above one), the stronger the support of E in favor of H. Putting aside reservations about this measure of evidential support (discussed earlier in Chapter CROSSREF), the Bayes factor $\mathsf{P}(E|H)/\mathsf{P}(E)$, unlike the conditional probability $\mathsf{P}(H|E)$, offers a potential way to overcome the difficulty about conjunction.

Say a and b, separately, support A and B to degree s_A and s_B respectively, that is, $\mathsf{P}(a|A)/\mathsf{P}(a) = s_A$ and $\mathsf{P}(b|B)/\mathsf{P}(b) = s_B$, where both s_A and s_B are greater than one. Does the combined evidence $a \wedge b$ provide at least as much support in favor of the combined claim $A \wedge B$ as the individual support by a and b in favor of A and B considered separately? The combined support here is measured by the combined Bayes factor $\mathsf{P}(a \wedge b|A \wedge B)/\mathsf{P}(a \wedge b)$. The latter, under suitable independence assumptions, equals the product of the individual supports s_A and s_B. Here is the argument:

use nicefrac for inline fractions

$$\begin{aligned}
\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} &= \frac{\mathsf{P}(A \wedge B|a \wedge b)}{\mathsf{P}(A \wedge B)} && \text{(Bayes's theorem)}\\
&= \frac{\frac{\mathsf{P}(A \wedge B \wedge a \wedge b)}{\mathsf{P}(a \wedge b)}}{\mathsf{P}(A \wedge B)} && \text{(definition of conditional probability)}\\
&= \frac{\frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A \wedge B) \times \mathsf{P}(b|A \wedge B \wedge a)}{\mathsf{P}(a) \times \mathsf{P}(b|a)}}{\mathsf{P}(A \wedge B)} && \text{(chain rule)}\\
&=^{*} \frac{\frac{\mathsf{P}(A) \times \mathsf{P}(B) \times \mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\mathsf{P}(a) \times \mathsf{P}(b)}}{\mathsf{P}(A) \times \mathsf{P}(B)} && \text{(independencies)}\\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} && \text{(algebraic manipulation)}\\
s_{AB} &= s_A \times s_B
\end{aligned}$$

The step marked by the asterisk rests on some of the independence assumptions codified in the Bayesian network in Figure 1, namely: (1), (5), and (6).

Thus, given the independence assumptions the combined support s_{AB} will always be higher than the individual support so long as s_A and s_B are greater than one, that is if the individual piece of evidence in fact support their relevant hypotheses.

This result can be generalized beyond two pieces of evidence. Figure 3 compares the Bayes factor of one item of evidence, say $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}$ with the combined Bayes factor for five items of evidence, say $\frac{\mathsf{P}(a_1 \wedge \cdots \wedge a_5|A_1 \wedge \cdots \wedge A_5)}{\mathsf{P}(a_1 \wedge \cdots \wedge a_5)}$, for different values of sensitivity and specificity of the evidence. The latter always exceeds the fomer, as soon as the individual pieces of evidence support the individual hypotheses (the order is reversed if it does not, and evidence with sensitivity=specificity= .5 results in Bayes factor 1, no matter what the prior is or how many items of evidence there are).

please use meaningful variable names, I revised s1 and s2 to sensitivity and specificity

Dawid's claim that 'the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents' holds, at least if restricted to supportive evidence with independence assumptions.

What happens if A and B are not necessarily probabilistically independent as in the Bayesian network in Figure 2? Assuming (4) and (5), the following holds:

Alicja: double check, are these d-sep in the second network?

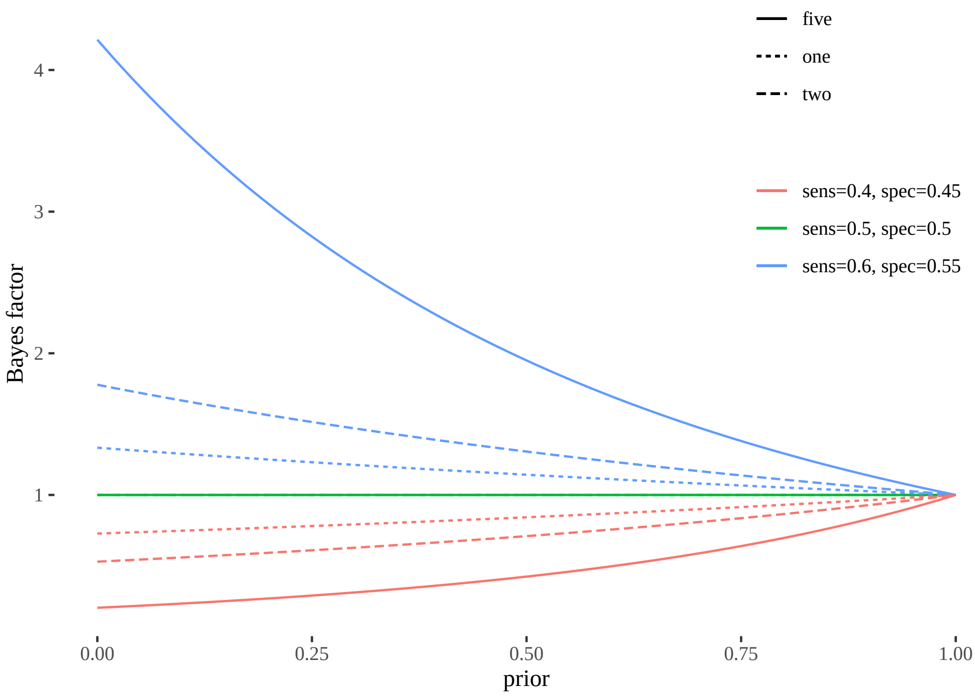

Figure 3: BF for varying number of items of evidence and test qualities, with the usual independence assumptions.

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} \tag{8}$$

$$s'_{AB} = s_A \times s'_B \tag{9}$$

Note that factor $s_B = \mathsf{P}(b|B)/\mathsf{P}(b)$ was replaced by $s'_B = \mathsf{P}(b|B)/\mathsf{P}(b|a)$.[1] Now, s'_B is usually lower than s_B because $\mathsf{P}(b|a) > \mathsf{P}(b)$ (assuming, at least, a and b are convergent pieces of evidence; SEE DISCUSSION IN EARLIER CHAPTERS).At the same time, s'_B should still be greater than one if b positively supports B even conditional on a.[2] Hence, s'_{AB} should still be greater than one provided s_A and S_B are both greater than one.

However, it might not always exceed the support supplied by s_A and s_B individually. For suppose $s_A = 2$ and $s_B = 3$, but $s'_B = 1.2$.Then, $s'_{AB} = 2 \times 1.2 = 2.4$, which is below the individual support s_B, but still above s_A. For the case in which A and B are probabilistically dependent, Dawid's claim should be amended as follows. Even though the support supplied by the conjunction of several independent testimonies need not always exceed that supplied by any of its constituents, it is always at least as great as the smallest support supplied by its constituents, if the individual Bayes factors are $>$1.

Fact 1. *Assuming* (4) *and* (5)*, if* $\mathsf{P}(B|b \wedge a) > \mathsf{P}(B|a)$ *and* $\mathsf{P}(A|a \wedge b) > \mathsf{P}(A|b)$*, we have:*

$$S_{AB} > \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}, \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)}$$

Proof. Note that the reasoning behind (9) is symmetric and so the assumed inequalities give the result

which one do you mean, corroboration?

Alicja: check if d-sep in this fn holds in the second bn

checked and rewrote the footnote

Give a clear example showing that this might happen, I now think this is mathematically impossible!

This needs to be reconsidered.

[1] Given the second Bayesian network, b need not be probabilistically independent of a, and thus there is no guarantee that $\mathsf{P}(b|a) = \mathsf{P}(a)$.

[2] Note that $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} = \frac{\mathsf{P}(b|B \wedge a)}{\mathsf{P}(b|a)}$, by independence (7). Now, observe that by Bayes' theorem by $\mathsf{P}(B|b \wedge a) = \frac{\mathsf{P}(b|B \wedge a)}{\mathsf{P}(b|a)} \times \mathsf{P}(B|a)$, so $\mathsf{P}(B|b \wedge a)$ is obtained by multiplying $\mathsf{P}(B|a)$ by, effectively, s'_B. So the assumption that $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} > 1$ is equivalent to $\mathsf{P}(B|b \wedge a) > \mathsf{P}(B|a)$, which is a fairly natural one in this context, since b should still raise the probability of B even in cojunction with a, or else b would be useless evidence.

marked by the braces:

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \underbrace{\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}}_{>1} = \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} \times \underbrace{\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|b)}}_{>1}$$

which clearly lead to the conclusion. □

2.1.1 Variable Threshold

If the combined support equals $s_A \times s_B$ or $s_A \times s_B^{'}$, does the difficulty about conjunction evaporate, as Dawid thought? One hurdle here is that the standard of proof would no longer be formalized as a posterior probability threshold, but instead as a threshold about the Bayes factor. The threshold would no longer be a probability between 0% and 100%, but rather a number somewhere above 1. The greater this number, the more stringent the standard of proof, for any value above one. In criminal trials, for example, the rule of decision would be: guilt is proven beyond a reasonable doubt if and only if the evidential support in favor of G—as measured by the Bayes factor $\frac{\mathsf{P}(E|G)}{\mathsf{P}(E)}$—meets a suitably high threshold t_{BF}. The obvious question at this point is, how do we identify the appropriate threshold?

good question, will think about it, will need to take a look at "Bayesian Choice", also need to think about a counterexample

One strategy is to derive the Bayes factor threshold, call it t_{BF}, from the posterior threshold t. Since $posterior = Bayes\,factor \times prior$, the Bayes factor threshold can be determined as follows:

$$\frac{t}{prior} = t_{BF}$$

The higher the prior probability, the lower t_{BF}. Whether this is a desirable property for a decision threshold can be questioned, but the same can be said about the posterior threshold t. The higher the prior probability, the easier to meet the posterior threshold.

Presumably, the threshold t_{BF} should be applied to individual as well as composite claims. Since the threshold varies depending on the priors, the thresholds for the individual claims A and B, denoted by t_{BF}^A and t_{BF}^B, will differ from the threshold for the composite claim $A \wedge B$, denoted by $t_{BF}^{A\wedge B}$.

At issue here is whether the conjunction principle can be formalized in a plausible manner with the Bayes factor. Unfortuantely, the answer is negative. To see why, first recall the conjunction principle:

$$\mathsf{S}[a, A] \text{ and } \mathsf{S}[b, B] \text{ iff } \mathsf{S}[a \wedge b, A \wedge B],$$

where $\mathsf{S}[E, H]$ means that evidence E establishes hypothesis H by standard S. If the standard of proof is formalized using the Bayes factor, the conjunction principle would boil down to:

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} > t_{BF}^A \text{ and } \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} > t_{BF}^B \text{ iff } \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} > t_{BF}^{A\wedge B}$$

Consider a posterior threshold $t = 0.95$, as might be appropriate in a criminal case. If A and B both have a prior probability of 10%, the threshold $t_{BF}^A = t_{BF}^B = 0.95/0.1 = 9.5$ for A or B individually. Assuming independence of A and B, the composite claim $A \wedge B$ will be associated with the threshold $t_{BF}^{A\wedge B} = 0.95/(0.1 * 0.1) = 95$, a much higher value. But if each individual claim meets its Bayes factor threshold of 9.5 and the two claims are independent, the joint Bayes factor would equal the multiplication of the individual Bayer factors, that is, $9.5 * 9.5 = 90.25$. This is not quite enough to meet $t_{BF}^{A\wedge B} = 95$, but is fairly close. The difference in absolute terms grows as the prior probability of the individual claims becomes lower, but the combined Bayes factor remains only 5% below the value needed to meet $t_{BF}^{A\wedge B}$.[3] Perhaps this is a good enough approximation. However, as the number of constituent claims grows, the difference becomes larger.[4] In addition, the difference becomes larger with a lower posterior probability threshold, say 0.5. Even with just two claims, $t_{BF}^{A\wedge B} = 0.5/(0.1 * 0.1) = 50$, but $t_{BF}^A * t_{BF}^B = (0.5/0.1) * (0.5/0.1) = 25$, only half the required value. The conjunction principle therefore fails in a large number of cases even using the Bayes factor threshold.

[3]The difference at here is between $t_{BF}^{A\wedge B} = 0.95/p^2$ and $t_{BF}^A * t_{BF}^B = (0.5/p)^2$. Note that $\frac{0.95/p^2-(0.5/p)^2}{0.95/p^2} = 5\%$, for any value of the prior p.

[4]Given five constituent claims, $\frac{0.95/p^5-(0.95/p)^5}{0.95/p^5} = 18\%$.

```
## [1] 9.5
## [1] 95
## [1] 90.25
## [1] 95
## [1] 9500
## [1] 9025
## [1] 950000
## [1] 902500
## [1] 5
## [1] 50
## [1] 25
```

2.1.2 Fixed Threshold

The alternative here is to fix the Bays factor threshold regardless of the prior probability of the claim of interest. This raises the difficult question of how to fix the Bayesian factor threshold irrespective of the priors. Standard decision theory can no longer be used. As it turns out, even if the question can be satisfactorily answered, the fixed threshold approach gives rise to a complication that proves fatal. If the standard of proof is formalized using a fixed Bayes factor threshold t_{BF}, the conjunction principle would boil down to:

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} > t_{BF} \text{ and } \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} > t_{BF} \text{ iff } \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} > t_{BF}$$

The left-to-right direction—aggregation—is likely to hold for any threshold t_{BF} greater than one. As shown earlier, the combined evidential support is greater than the individual evidential support if A and B are independent, or greater than the smallest individual support if A and B are dependent. Aggregation could not be justified using posterior probabilities $\mathsf{P}(A|a)$ and $\mathsf{P}(B|b)$ nor could it be justified generally using a variable Bayes factor threshold. So it is an advantage of the fixed Bayes factor threshold that it can justify this direction of the conjunction principle.

However, the right-to-left direction—distribution—has now become problematic. For suppose the combined evidential support, $\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)}$, barely meets the threshold. This implies that the individual support, say $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}$, could be be below the threshold unless $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} = 1$ (which should not happen if b positively supports B). So, curiously, there would be cases in which, even though the conjunction $A \wedge B$ is established to the desired standard of proof, one of the individual claims fails to meet the standard. This is odd. More specifically, the following distribution principle fails in some cases:

Not always true; just give a specific numerical counterexample. M: I added 'often'. Is this enough?

$$\text{If } \mathsf{S}[a \wedge b, A \wedge B], \text{ then } \mathsf{S}[a, A] \text{ and } \mathsf{S}[b, B]. \tag{DIS1}$$

Could this principle be rejected? Perhaps, it is not as essential as we thought at first. Since the evidence is not held constant, the support supplied by $a \wedge b$ could be stronger than that supplied by a and b individually. So even when the conjunction $A \wedge B$ is established to the requisite standard given evidence $a \wedge b$, it might still be that A does not meet the requisite standard (given a) nor does B (given b).

But consider a less controversial version, holding the evidence constant:

$$\text{If } \mathsf{S}[a \wedge b, A \wedge B], \text{ then } \mathsf{S}[a \wedge b, A] \text{ and } \mathsf{S}[a \wedge b, B]. \tag{DIS2}$$

This principle is harder to deny. That is, one would not want to claim that, holding fixed evidence $a \wedge b$, establishing the conjunction might not be enough for establishing one of the conjuncts. One cannot be willing to assent to the conjunction without being willing to assent to one of the conjuncts against a fixed body of evidence. Certainly any formalization of the standard of proof should obey (DIS2). And yet, it is this very principle that we should deny if we understand the standard of proof using the Bayes factor. [5] In case A and B are probabilistically independent, (DIS1) and (DIS2) are in fact equivalent, so

[5] To show that (DIS2) fails, it is enough to show that $\frac{\mathsf{P}(B|a \wedge b \wedge A)}{\mathsf{P}(B|A)} > 1$ because $S[a \wedge b, A \wedge B] > S[a \wedge b, A]$ iff $\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} > \frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)}$ iff $\frac{\mathsf{P}(A \wedge B|a \wedge b)}{\mathsf{P}(A \wedge B)} > \frac{\mathsf{P}(A|a \wedge b)}{\mathsf{P}(A)}$ iff $\frac{\mathsf{P}(A|a \wedge b) \times \mathsf{P}(B|a \wedge b \wedge A)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} > \frac{\mathsf{P}(A|a \wedge b)}{\mathsf{P}(A)}$. Now, $\frac{\mathsf{P}(B|a \wedge b \wedge A)}{\mathsf{P}(B|A)} > 1$ so long as $a \wedge b$ positively

rejectign one requires rejecting the other.[6] The intuitive reason for this— perhpas surprising—result is that $A \wedge B$ has a much lower prior probability than A (or B) considered separately. Thus, the same body of evidence is going to have a larger impact on a hypothesis with a lower prior probability, other things being equal. This larger impact on the prior is reflected in a larger Bayes factor. We will examine this point in greater detail in later sections.

All in all, using Bayes factor to understad the standard of proof has counterintuitive consequences, what we will call the distribution paradox. For suppose the prosecution provided evidence for claim A, but this evidence still falls short of the threshold t (a certain number above 1). Just by tagging an additional claim B and without doing any further evidentiary work, the prosecution could provide sufficiently strong evidence (which meets the threshold t) in favor of claim $A \wedge B$. So, it could well happen that, while the prosecution failed to prove beyond a reasonable doubt that the defendant injured the victim, the prosecution could nevertheless prove beyond a reasonable doubt that the defendant injured the victim and did so intentionally. This is odd.

2.2 Likelihood ratio threshold

Let's now replace the Bayes factor with the likelihood ratio, another probabilistic measure of evidential support. As we shall understand it for now, the likelihood ratio compares the probability of the evidence on the assumption that a hypothesis of interest is true and the probability of the evidence on the assumption that the negation of the hypothesis is true, that is, $\frac{\mathsf{P}(E|H)}{\mathsf{P}(E|\neg H)}$. The greater the likelihood ratio (for values above one), the stronger the evidential support in favor of the hypothesis (as contrasted to the its negation). We discussed extensively the advantages and limitations of this account in **REFERENCE TO EARLIER CHAPTER**.

We can think of the the likelihood rato as the following:

$$\frac{\textit{sensitivity}}{\textit{1- specificity}}$$

Unlike the Bayes factor, the likleihood ratio is not sensitive to the priors so long as sensitivity and specificity are not. In this sense, it is a more suitable measure if we want to factor out the effects that priors may have on the assessment of evidential strength. But this advantage is short lived. The likelihood ratio is sensitive to priors when one considers a composite claim instead of an individual claim.

2.2.1 Aggregating evidence

To see why, consider the combined ikelihood ratio $\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}$. The numerator can be computed easily:[7]

---

supports B even under the assumption of A, unless A entailed B. We naturally exclude the situation in which one claim entails the other because otherwise there would be no need to establish the two claims. Establishing one claim alone would suffice. To see why $a \wedge b$ positively supports B (or A), note that $S[a, A] = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \leq \frac{\mathsf{P}(a|A) \times \mathsf{P}(b|A \wedge a)}{\mathsf{P}(a) \times \mathsf{P}(b|a)} = \frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)} = S[a \wedge b, A]$. The key step here is $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \leq \frac{\mathsf{P}(a|A) \times \mathsf{P}(b|A \wedge a)}{\mathsf{P}(a) \times \mathsf{P}(b|a)}$. The latter holds because $\frac{\mathsf{P}(b|A \wedge a)}{\mathsf{P}(b|a)} \geq 1$. It is useful to distinguish two cases. First, if A and B are probabilistically independent, as in the Bayesian network in Figure **??** (top), then $\frac{\mathsf{P}(b|A \wedge a)}{\mathsf{P}(b|a)} = \frac{\mathsf{P}(b)}{\mathsf{P}(b)} = 1$. Second, if A and B are probabilistically dependent, as in the Bayesian network in Figure **??** (bottom), evidence b positively supports claim A (even conditional on a) so long as b positively supports B. The assumption is that claim A and B are positively correlated, and thus, any evidence that supports one of the claims is going to support the other claim, as well. **SEE EARLIER CHAPTERS FOR A MORE RIGOROUS PROOF OF THIS LAST POINT.**

6 $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)\mathsf{P}(b|A)}{\mathsf{P}(a)\mathsf{P}(b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}$. In other words, $S[a \wedge b, A] = S[a, A]$.

7

$$\begin{aligned}
\mathsf{P}(a \wedge b|A \wedge B) &= \frac{\mathsf{P}(A \wedge B \wedge a \wedge b)}{\mathsf{P}(A \wedge B)} \\
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A \wedge B) \times \mathsf{P}(b|A \wedge B \wedge a)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} \\
&=^{*} \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} \\
&= \mathsf{P}(a|A) \times \mathsf{P}(b|B)
\end{aligned}$$

The asterisk marks the step that requires the indepedence assumptions in Figure **??**.

$$P(a \wedge b|A \wedge B) = P(a|A) \times P(b|B)$$

The equality requires the independence assumptions codified in the Bayesian networks in Figure **??**. That is, the two items of evidence should be independent of one another conditional on the hypothesis they support. The numerator does not depend on the priors associated with $A \wedge B$. Call it *combined sensitivity*, simply resulting from multiplying the sensitivity of the individual items of evidence, a and b, relative to their respective hyptheses, A and B.

The denominator is more involved:[8]

$$P(a \wedge b|\neg(A \wedge B)) = \frac{P(\neg A)P(B|\neg A)P(a|\neg A)P(b|B) + P(A)P(\neg B|A)P(a|A)P(b|\neg B) + P(\neg A)P(\neg B|\neg A)P(a|\neg A)P(b|\neg B)}{P(\neg A)P(B|\neg A) + P(A)P(\neg B|A) + P(\neg A)P(\neg B|\neg A)}$$

The same independence assumptions invoked before are needed here. Unlike the numerator, the denominator—call it *combined specificity*—depends on the priors of A and B and thus on the priors of $A \wedge B$. Putting numerator and denominator together yields the formula for the combined likelihood ratio.

$$\frac{P(a \wedge b|A \wedge B)}{P(a \wedge b|\neg(A \wedge B))} = \frac{P(a|A) \times P(b|B)}{\frac{P(\neg A)P(B|\neg A)P(a|\neg A)P(b|B)+P(A)P(\neg B|A)P(a|A)P(b|\neg B)+P(\neg A)P(\neg B|\neg A)P(a|\neg A)P(b|\neg B)}{P(\neg A)P(B|\neg A)+P(A)P(\neg B|A)+P(\neg A)P(\neg B|\neg A)}}$$

As with the Bayes factor, under suitable independence assumptions, the combined likelihood ratio exceeds the individual likelihood ratio so logn as the two pieces of evidence have the same sensitivity and specificity. If they have differnt levels of sensitivity and specificity, the combined likelihood ratio never goes below the lowest of the two individual likelihood ratios.

Becasue of the many variables at play, it is not easy to compare the combined evidential support and the individual support supplied by a and b towards A and B, as measured by the individual and combined likelihood ratio. To circumvent this difficulty, we make three simplifying assumptions. First, the sensitivity of a piece of evidence, say $P(a|A)$, is the same as its specificity, $P(\neg a|\neg A)$. Let $P(a|A) = x$ and $P(b|B) = y$. So $P(a|\neg A) = 1-x$ and $P(b|\neg B) = 1-y$. Finally, the sensitivity (and thus the specificity) of the two pieces of evidence is the same, that is, $P(a|A) = x = P(b|B) = y$. Finally, as is customary, claims A and B are independent of one another. The combined likelihood ratio therefore reduces to the following, where $P(A) = k$ and $P(B) = t$:

$$\frac{P(a \wedge b|A \wedge B)}{P(a \wedge b|\neg(A \wedge B))} = \frac{xx}{\frac{(1-k)t(1-x)x+k(1-t)x(1-x)+(1-k)(1-t)(1-x)(1-x)}{(1-k)t+(1-t)k+(1-k)(1-t)}}$$

The graph of the combined likelihood ratio can now be easily plotted against the single likelihood ratios. As Figure 4 shows, the combined likelihood ratio varies depeding on the prior probabilities $P(A)$ and $P(B)$, as expected, but always execeeds the individual likelihood ratios whenever they are greater than one (that is, the two piece of evidence provides positive support for their respective hypothesis)

What happens if we relax the three simplifying assumptions? Suppose the sensitivity and specificity of the two pieces of evidence are not the same. Their likelihood ratios will then also be different. In this case, the combined likelihood ratio is not always greater than the individual ratios, but it is always greater than the smallest of the two provided the individual likelihood ratios are greater than one.

M: Need to add simulation results to make this argument fully general and drop all the simplifying assumptions (.e.g. independence or equiprobability). HELP!

The argument, once again, verified Dawid's claim that 'the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents'. One caveat is

8

$$\begin{aligned} P(a \wedge b|\neg(A \wedge B)) &= \frac{P(a \wedge b \wedge \neg(A \wedge B))}{P(\neg(A \wedge B))} \\ &= \frac{P(a \wedge b \wedge \neg A \wedge B) + P(a \wedge b \wedge A \wedge \neg B) + P(a \wedge b \wedge \neg A \wedge \neg B)}{P(\neg A \wedge B) + P(A \wedge \neg B) + P(\neg A \wedge \neg B)} \\ &=^* \frac{P(\neg A)P(B|\neg A)P(a|\neg A)P(b|B) + P(A)P(\neg B|A)P(a|A)P(b|\neg B) + P(\neg A)P(\neg B|\neg A)P(a|\neg A)P(b|\neg B)}{P(\neg A)P(B|\neg A) + P(A)P(\neg B|A) + P(\neg A)P(\neg B|\neg A)} \end{aligned}$$

The asterisk marks the step that requires the indepedence assumptions in Figure **??**.

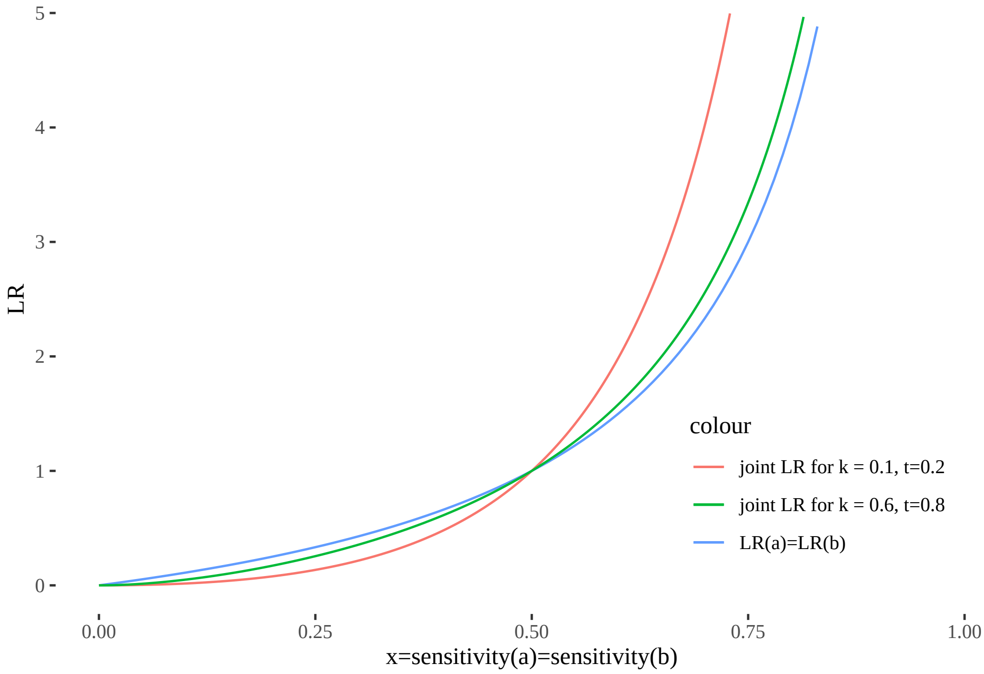

Figure 4: Combined likelihood ratios exceeds individual Likelihood ratios. Changes in the prior probabilities t and k do not invalidate this result.

that, if evidential support is measured by the likelihood ratio, the support supplied by the conjunction of different independent pieces of evidence always exceeds the smallest of the support supplied by its consitituents, but there are cases in which it does not exceed the support supplied by some of its constituents.

2.2.2 Variable Threshold

Like the Bayes factor, the likelihood ratio can be used to formalize the standard of proof by equating the standard to a threshold t above one. The greater the threshold, the more stringent the standard. In criminal trials, for example, the rule of decision would be: guilt is proven beyond a reasonable doubt if and only if the evidential support in favor of G—as measured by the likelihood ratio $\frac{\mathsf{P}(E|G)}{\mathsf{P}(E|\neg G)}$—meets a suitably high threshold t above one.

By the ratio version of Bayes' theorem,

$$\textit{posterior ratio} = \textit{likelihood ratio} \times \textit{prior ratio},$$

and thus

$$\frac{\textit{posterior ratio}}{\textit{prior ratio}} = \textit{likelihood ratio}.$$

If the posterior ratio is fixed at, say $t/1-t$, the threshold on the likelihood ratio, call it t_{LR} is defined as:

$$\frac{t/1-t}{\textit{prior ratio}} = t_{LR}.$$

The likelihood ratio threshold will vary depending on the prior. The higher the prior, the lower the likelihood ratio threshold. So

Consider the individual claim A and B and compared them with the composite claim $A \wedge B$. Suppose the two claims are independent. Consider a posterior threshold of 95% as might be appropriate in a criminal case. Assuming A and B have a prior probability of 20% and 30% respectively, the likelihood ratio threshold for A and B will be $t^{A}_{LR} \approx 76$ and $t^{B}_{LR}a \approx 44$. The likelihood ratio threshold for the composite claim $A \wedge B$ will be $t^{A\wedge B}_{LR} \approx 297$.

```
## [1] 76
## [1] 44.33333
## [1] 297.6667
```

Consider a posterior threshold of 50% as might be appropriate in a civil case. Assuming A and B have a prior probability of 20% and 30% respectively, the likelihood ratio threshold for A and B will be $t_{LR}^{A} \approx 4$ and $t_{LR}^{B}a \approx 2$. The likelihood ratio threshold for the composite claim $A \wedge B$ will be $t_{LR}^{A \wedge B} \approx 15$.

```
## [1] 4
## [1] 2.333333
## [1] 15.66667
```

As expected, other things being equal, the likelihood ratio threshold is lower for civil than criminal cases. The threshold is variable and depends on then prior, so claim that have higher priors are asociated with lower likelihood ratio threshold, such as A and B, than claims associated with lower priors such as $A \wedge B$.

Now suppoose the individual likelihood ratios meet the threshold t_{LR}^{A} and t_{LR}^{B} give a posterior threshold of 0.95. To ensure that t_{LR}^{A} is met, evidence a should have a sensitivity of at least (roughly) 0.99 (and a specificity of 1-sensitivity). To ensure that t_{LR}^{B} is met, evidence b should have a sensitivity of at least (roughly) 0.98 (and a specificity of 1-sensitivity). Holding fixed the values for sensitivity and specificity, does the combined likelihood ratio meet the threshold $t_{LR}^{A \wedge B}$? Not quite. The combined likelihood ratio equals about 145, far short that what the threshold $t_{LR}^{A \wedge B}$ requires, namely a likelihood ratio of 297

```
## [1] 76
## [1] 75.92308
## [1] 44.33333
## [1] 44.45455
## [1] 145.1446
```

Things do not look any better for a lower threshold. Now suppoose the individual likelihood ratios meet the threshold t_{LR}^{A} and t_{LR}^{B} give a posterior threshold of 0.5. To ensure that t_{LR}^{A} is met, evidence a should have a sensitivity of at least 0.8 (and a specificity of 1-sensitivity). To ensure that t_{LR}^{B} is met, evidence b should have a sensitivity of at least 0.7 (and a specificity of 1-sensitivity). To ensure that t_{LR}^{B} is met, evidence b should have a sensitivity of at least 0.7 (and a specificity of 1-sensitivity). Holding fixed the values for sensitivity and specificity, does the combined likelihood ratio meet the threshold $t_{LR}^{A \wedge B}$? Note quite. The combined likelihood ratio equals about 5, far short that what the threshold $t_{LR}^{A \wedge B}$ requires, namely a likelihood ratio of 15.

```
## [1] 4
## [1] 4
## [1] 2.333333
## [1] 2.333333
## [1] 5.222222
```

2.2.3 Fixed Threshold

By using a likelihood ratio threshold that is prior dependent, the conjunction principle—in particular, what we called aggregation—fails. The alternative is to fix a likelihood ratio threshold irrespective of the prior. Setting aside the problem of how ot identify the appropriate threshold, we will see that this approach is able to justify one direction of the conjunction principle—what we called aggregation—but still fails to justify the other direction—what we called distribution So the distribution paradox arises here again.

Consider aggregation first. Say both individual likelihood ratios $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)}$ and $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|\neg B)}$ are above the requisite threshold t for meeting the standard of proof. Will the combined likelihood ratio $\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}$ also be above the threshold? The answer is affirmative. As we argued earlier, the combined likelihood ratio is never below the lowest of the individual likelihood ratio. If both individual likelihood ratios meet the threhsold, so does the combined likelihood ratio. But this approach faces another problem, the same problem that plagues the Bayes factor. That is, likelihood ratios still fail to capture the other direction of the conjunction principle, what we called distribution.

For suppose evidence $a \wedge b$ supports $A \wedge B$ to the required threshold t. If evidential support is measured by the likelihood ratio, the threshold in this case should be some order of magnitude greater than one. If the combined likelihood ratio meets the threshold t_{LR}, one of the individual likelihood ratios may well be below t_{LR} So—if the standard of proof is interpreted using evidential support measured by the likelihood ratio—even though the conjuction $A \wedge B$ was proven according to the desired standard, one of individual claims might not. The right-to-left direction of the conjunction principle—that is, if $S[a \wedge b, A \wedge B]$, then $S[a, A] \wedge S[b, B]$, what we called earlier the distribution principle (DIS1)—fails.

Consider now the second and less objectionale extrapolation principle discussed earlier. This is the principle that holds the evidence fixed throughout, repeated below for convenience:

$$\text{If S}[a \wedge b, A \wedge B]\text{, then S}[a \wedge b, A]\text{ and S}[b \wedge b, B]. \tag{DIS2}$$

Here again, the support of $a \wedge b$ in favor of $A \wedge B$ could exceed that of $a \wedge b$ in favor of A alone (or B alone) if evidential support is measured using likelihood ratios.[9] Even this second, seemingly unobjectionable version of extrapolation fails. The same counterintuitive consequences that arose with Bayes factor manifest themselves here. The distribution paradox persists.

M: This whole section should be generalized to the case in which A and B are not independent using the simulation data.

M: Might be goot to emphasize how devastaing this finding is for legal probabilists who endorsed likelihood ratios as the solution to many problems with legal probabilism.

2.3 The comparative strategy

Instead of thinking in terms of absolute thresholds—whether relative to posterior probabilities, the Bayes factor or the likelihood ratio—the standard of proof can be understood comparatively. This suggestion has been advanced by Cheng (2012) following the theory of relative plausibility by **REFERENCE TO ALLEN AND PARDO HERE**. Say the prosecutor or the plaintiff puts foward a hypothesis H_p about what happened. The defense offers an alternative hypothesis, call it H_d. On this approach, rather than directly evaluating the support of H_p given the evidence and comparing it to a threshold, we compare the support that the evidence provides for two competing hypotheses H_p and H_d, and decide for the one for which the evidence provides better support.

It is controversial whether this is what happens in all trial proceedings, especially in criminal trials. The defense may elect to challenge the hypothesis put foward by the other party without proposing one of its own. For example, in the O.J. Simpson trial the defense did not advance its own story about what happened, but simply argued that the evidence provided by the prosecution, while significant on its face to establish OJ's guilt, was riddled with problems and deficencies. This defense strategy was enough to secure an acquittal. So, in order to create a reasonable doubt about guilt, the defense does not always provide a full-fledged alternative hypothesis. The supporters of the comparative approach, however, will respond that this could happen in a small number of cases, even though in general—especially for tactical reasons—the defense will provide an alternative hypothesis. After all, not to provide one would usually amount to an admission of criminal or civil liability.

Setting aside this controversy for the time being, let's first work out the comparative strategy using posterior probabilities. More specifically, given a body of evidence E and two competing hypotheses H_p and H_d, the probability $\mathsf{P}(H_p|E)$ should be suitably higher than $\mathsf{P}(H_d|E)$, or in other words, the ratio $\frac{\Pr H_p|E}{\Pr H_d|E}$ should be above a suitable threshold. Presumably, the ratio threshold shoud be higher for criminal than civil cases. In fact, in civil cases it seems enough to require that the ratio $\frac{\Pr H_p|E}{\Pr H_d|E}$ be avove 1, or in other words, $\mathsf{P}(H_p|E)$ should be higher than $\mathsf{P}(H_d|E)$. Note that H_p and H_d need not be one the negation of the other. Whenever two hypotheses are one the negation of the other, $\frac{\mathsf{P}(H_p|E)}{\mathsf{P}(H_d|E)} > 1$ implies that $\mathsf{P}(H_p|E) > 50\%$, the standard probabilistic intepretation of the preponderemca standard.

[9] Note that, assuming either of the Baysian networks in Figure **??**, $S[a \wedge b, A] = \frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|\neg A)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)} \times \frac{\mathsf{P}(b|A)}{\mathsf{P}(b|\neg A)}$, where

$$\frac{\mathsf{P}(b|A)}{\mathsf{P}(b|\neg A)} = \frac{\mathsf{P}(B|A) \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|\neg B)} + (1 - \mathsf{P}(B|A))}{\mathsf{P}(B|\neg A) \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|\neg B)} + (1 - \mathsf{P}(B|\neg A))}.$$

SEE PROOF IN EARLIER CHAPTERS. If A and B are assumed to be probabilistically independent, the numerator and the denominator will be the same, so $\frac{\mathsf{P}(b|A)}{\mathsf{P}(b|\neg A)} = 1$. Thus, $S[a \wedge b, A] = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)} = S[a, A]$. Since $S[a, A] < S[a \wedge b, A \wedge B]$ (see Figure 4), it follows $S[a \wedge b, A] < S[a \wedge b, A \wedge B]$. **What if A and B are dependent? Need simulation data here.**

One advantage of this approach—as Cheng shows—is that expected utility theory can set the appropriate comparative threshold t as a function of the costs and benefits of trial decisions. For simplicity, suppose that if the decision is correct, no costs result, but incorrect decisions have their price (**REFERENCE TO EARLIER CHAPTER FOR MORE COMPLEX COST STRUCTURE**). The costs of a false positive is c_{FP} and false negative is C_{FN}, both greater than zero. Intuitively, the decision rule should minimize the expected costs. That is, a finding against the defendant would be acceptable whenever its expected costs—$\mathsf{P}(H_d|E) \times c_{FP}$—are smaller than the expected costs of an acquittal—$\mathsf{P}(H_p|E) \times c_{FN}$— or in other words:

$$\frac{\mathsf{P}(H_p|E)}{\mathsf{P}(H_d|E)} > \frac{c_{FP}}{c_{FN}}.$$

In civil cases, it is customary to assume the costs ratio of false postives to false negatives equals one. So the rule of decision would be: Find against the defedant whenever $\frac{\mathsf{P}(H_p|E)}{\mathsf{P}(H_d|E)} > 1$ or in other words $\mathsf{P}(H_p|E)$ is greater than $\mathsf{P}(H_d|E)$. In criminal trials, the costs ratio is usually considered higher, since convicting an innocent (false positive) should be more harmful or morally objectionable than acquitting a guilty defendant (false negative). Thus, the rule of decision in criminal proceedings would be: Convict whenever $\mathsf{P}(H_p|E)$ is significantly greater than $\mathsf{P}(H_d|E)$.

Does the comparative strategy just outlined solve the difficulty about conjuction? We will work through a stylized case used by Cheng himself. Suppose, in a civil case, the plaintiff claims that the defendant was speeding (S) and that the crash caused her neck injury (C). Thus, the plaintiff's hypothesis H_p is $S \wedge C$. Given the total evidence E, the conjuncts, taken separately, meet the decision threshold:

$$\frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \qquad\qquad \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1$$

The question is whether $\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(H_d|E)} > 1$. To answer it, we have to decide what the defense hypothesis H_d should. Cheng reasons that there are three alternative defense scenarios: $H_{d_1} = S \wedge \neg C$, $H_{d_2} = \neg S \wedge C$, and $H_{d_3} = \neg S \wedge \neg C$. How does the hypothesis H_p compare to each of them? Assuming independence between C and S, we have

$$\begin{aligned}
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(S \wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(S|E)\mathsf{P}(\neg C|E)} = \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1 \\
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(C|E)} = \frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \\
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)} > 1
\end{aligned} \tag{10}$$

So, whatever the defense hypothesis, the plaintiff's hypothesis is more probable. At least in this case, whenever the elements of a plaintiff's claim satisfy the decision threshold, so does their conjunction. The left-to-right direction of the conjunction principle—what we called aggregation—has been vidicated. But what about the opposite direction, what we called extrapolation? Interestingly, if the threshold is just 1—as might be appropriate in civil cases—extrapolation would be satisfied. Even if $\frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)}$ might be strictly greater than $\frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)}$ or $\frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)}$, wehenver the former is greater than one the latter must be greater than one. However, suppose the threshold is more stringent than one, as might be appropriate for criminal cases. For some constituent claims A and B in a criminal case, whenever $\frac{\mathsf{P}(A|E)\mathsf{P}(B|E)}{\mathsf{P}(\neg A|E)\mathsf{P}(\neg B|E)}$ barely meet the threhold t, $\frac{\mathsf{P}(A|E)}{\mathsf{P}(\neg A|E)}$ or $\frac{\mathsf{P}(B|E)}{\mathsf{P}(\neg B|E)}$ could be below t. Since the evidence is held fixed throughout, this would be a violation of the extrapolation principle (EXT2).

Another problem with this approach is that much of the heavy lifting here is done by the strategic splitting of the defense line into multiple scenarios. Now suppose $\mathsf{P}(H_p|E) = 0.37$ and the probability of each of the defense lines given E is 0.21. This means that H_p wins with each of the scenarios, so we should find against the defendant. But should we? Given the evidence, the accusation is very likely to be false, because $\mathsf{P}(\neg H_p|E) = 0.63$? The problem generalizes. If, as here, we individualize scenarios by boolean combinations of elements of a case, the more elements there are, into more scenarios $\neg H_p$ needs to be divided. This normally would lead to the probability of each of them being even lower (because now $\mathsf{P}(\neg H_p)$ needs to be “split” between more scenarios). So, if we take this approach seriously, the more elements a case has, the more at disadvantage the defense is. This seems undesirable.

REPEAT SAME ARGUMENT USING LR AND COMPARATIVE STRATEGY

3 Rejecting the Conjunction Principle?

Let’s now turn to possible strategies that legal probabilists can pursue to address the conjunction paradox. The first thing to note is that the paradox would not arise without the conjunction principle. So could legal probabilists reject this principle and let the paradox disappear?

add more flow once section is complete

3.1 Risk Accumulation

In current discussions in epistemology about knowledge or justification, a principle similar to the conjunction principle has been contested because it appears to deny the fact that risks of error accumulate (CITE). If one is justifiably sure about the truth of each claim considered seperataly, one should not be equally sure of their conjunction. You have checked each page of a book and found no error. So, for each page, you are nearly sure there is no error. Having checked each page and found no error, can you be sure that the book as a whole contains no error? Not really. As the number of pages grow, it becomes virtually certain that there is at least one error in the book you have overlooked, although for each page you are nearly sure there is no error. (ADD CITATION ABOUT PREFACE PARADOX) The same applies to other contexts, say product quality control. You may be sure, for each product you checked, that it is free from defects. But you cannot, on this basis alone, be sure that all products you checked are free from defects. Since the risks of error accumulate, you must have missed at least one defective product.

Hey, we can quote a paper that’s out by Alicja!:) Also, I guess you want me to find the right refs? M: Yes, if you can

Risk accumulation challenges one direction of the conjunction principle, call it *aggregation*:

$$S[C_1 \wedge C_2 \wedge \cdots \wedge C_k] \Leftarrow S[C_1] \wedge S[C_2] \wedge \cdots \wedge S[C_k].$$

Even if the probability of several claims, considered individually, is above a threhsold t, their conjuction need not be above t. Risk accumulation, howvere, does not challenge the other direction of the conjunction principle, call it *distribution*:

$$S[C_1 \wedge C_2 \wedge \cdots \wedge C_k] \Rightarrow S[C_1] \wedge S[C_2] \wedge \cdots \wedge S[C_k].$$

Probability theory ensures that, if the probabiliity of the conjunction of several claims is above t, so is the probability of each individual claim.

3.2 Atomistic and Holistic Approches

Suppose the legal probabilist does away with the conjunction principle. Now what? How should they define standards of proof? Two immediate options come to mind, but neither is without problems.

Let’s stipulates that, in order to establish the defendant’s guilt beyond a reasonable doubt (or civil liability by preponderance of the evidence), the party making the accusation should establish each claim, separately, to the requisite probability, say at least 95%, without needing to establish the conjunction to the requisite probability. Call this the *atomistic account*. On this view, the prosecution could be in a position to establish guilt beyond a reasonable doubt without estalishing the conjunction of different claims with a sufficiently high probability. This account would allow convictions in cases in which the probability of the defendant’s guilt is relatively low, just because guilt is a conjunction of several independent claims that separately satisfy the standard of proof. For example, if each constituent claim is established with 95% probability, a composite claim consisting of five subclaims—assuming, as usual, probabilistic independence between individual claims—would only be established with 77% probability, a far cry from proof beyond a reasonable doubt. This is counterintuitve as it would allow convictions when the defendant is not very likely to be guilty. Under the atomistic account, the composite claim representing the case as a whole would often be established with a probability below the required threshold. The atomistic approach is a non-starter.

The other option is to require that the prosecution in a criminal case (or the plantiff in a civil case) establish the accusation as a whole—say the cojunction of A and B—to the requisite probability. Call this the *holistic account*. This account is not without problems either. The proof of $A \wedge B$ would impose a higher requirement on the separate probabilities of the conjuncts. If the conjunction $A \wedge B$ is to be proven with at least 95% probability, the individual conjuncts should be established with probability higher than 95%. So the more constituent claims, the higher the posterior probability for each claim needed to meet the requisite probability threhsold.

We can get a sense of the difficulty by running in some numbers. Assume, for the sake of illustration, the independence and equiprobability of the constituent claims. If a composite claim consists of k individual claims, these individual claims will have to be established with proabilility of at least $t^{1/k}$, where t is the threhsold applicable to the composite claim.[10] For example, if there are ten constituent claims, they will have to be proven with $0.5^{1/10} = 0.93$ even if the probability threshold is only 50%. If the threshold is more stringent, as is appropriate in criminal cases, say 95%, each individual claim will have to be proven with near certainity. This would make the task extremely demanding on the prosecution. If there are ten constituent claims, they will have to be proven with $0.95^{1/10} = 0.995$. So the plaintiff or the prosecution would face the demanding task of establishing each element of the accusation beyond what the standard of proof would seem to require.

Another problem with the holistic account is that the standard that applies to one of the conjuncts would depend on what has been achieved for the other conjuncts. For instance, assuming independence, if $\mathsf{P}(A)$ is 96%, then $\mathsf{P}(B)$ must be at least 99% so that $\mathsf{P}(A \wedge B)$ is above a 95% threshold. But if $\mathsf{P}(A)$ is 99.99%, then $\mathsf{P}(B)$ must only be greater than 95% to reach the same threshold. Thus, the holistic account would require that the elements of an accusation be proven to different probabilities—and thus different standards—depeding on how well other claims have been established. This result runs counter to the tacit assumption that each element should be established to the same standard of proof.

```
## [1] 0.933033
## [1] 0.9948838
## [1] 0.5987369
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We have a dilemma here: either (under the holistic approach) the standard is too demanding on the prosecution (or the plantiff) because it would require the individual claims to be established to extremely high probabilities, or (under the atomistic approach) the standard is too lax because it would allow findings of liability when the defendant most likely committed no wrong. Denying the conjunction principle, then, is not without difficulties of its own. Absent the conjuction principle, legal probabilists should still explain how individual claims relate to larger claims in the process of legal proof.

3.3 Prior Probabilities

It is worth examining the holistic account more closely, focusing in particular on the role of prior probabilities, an aspect that has gone unnoticed so far. The main problem with the holistic approach is that it would require, especially in criminal cases, individual claims to be established with a very high probability, often making the task unsurmountable for the prosecution. Or so it would seem. But a composite claim such as $A \wedge B$ will have, other things being equal, a lower prior probability than any individual claim A or B. Say a composite claim consists of k individual claims. If its prior probability is π, each constituent claim, assuming they are independent and equiprobable, will have a prior probability of $\pi^{1/k}$. The prior probability of the individual claims will approach one as the number of constituent claims increases.

Cohen worried that, as the number of constituent claims increases, the prosection or the plaintiff would see their case against the defendant become progressively weaker and it would become impossisble for them to establish liability. But this worry is an exaggeration. The paradox, as is commonly formulated, starts by assuming that the constituent claims are established by the required probability threshold and then shows that the probability of the conjuction may fall below the threshold. However, following the holistic approach, the order of presentation can be reversed. Start by assuming that the composite claim is established by the required probability threshold. No doubt the individual claims will have to be established with a higher probability, a violation of the conjuction principle. Yet, this violation is not as counterintuitive as it might first appear for two reasons. First, since risks aggregate, it is natural

[10] Let p the probability of each consitutuent claim. To meet threshold t, the probability of the composite claim, p^k, should satisfy the constraint $p^k > t$, or in other words, $p > t^{1/k}$.

that the probability of a conjunction would be lower than the probability of the conjuncts. Second, the prior probabilities of the conjuncts will be higher than the prior probability of the conjunction. Thus, establishing the conjuncts with a higher probability will not be exceedingly demanding.

Along this lines, Dawid (1987), in one of the earliest attempts to solve the conjunction paradox from a probabilistic perspective, wrote:

> ...it is not asking too much of the plaintiff to establish the case as a whole with a posterior probability exceeding one half, even though this means that the several component issues must be established with much larger posterior probabilities; for the *prior* probabilities of the components will also be correspondingly larger, compared with that of their conjunction (p. 97).

The price of this strategy is the denial of the conjunction principle, specifically aggregation, the very motivation behind the conjunction paradox. Cohen could insist that this solution amounts to denying the paradox. To address the paradox, legal probabilists should offer a justification of the conjunction principle in probabilistic terms, something Cohen maintains cannot be done. Or can it be done?

4 The Conjunction Principle Is False

Neither the Bayes factor nor the likelihood ratio managed to fully justify both directions of the conjunction principle. One direction, aggregation, was justified. So the original concern that was driving Cohen's formulation of the conjuction paradox was addressed. But the other direction, distribution, failed. The failure of distribution creates a paradox of its own, what we called distribution paradox. It is odd that one could have sufficiently strong evidence in support of $A \wedge B$, while not having sufficiently strong evidence for A or B. This occurs even when A and B are probabilistically independent. If they were dependent of one another—say A and B were mutually reinforcing—it is possible the evidence would strongly support the conjunction, but not one of the conjuncts in isolation (becuase the additional support from the other claim, A or B, would be missing). But the failure of distribution manifests itself even when A and B are independent. What should we make of this? This problem exists for both the Bayes factor and the likelihood ratio.

4.1 Factoring Out Prior Probabilities

Let us return to the role of prior probabilities and their effect on measures of evidential strength. Dawid observed that the prior probabilities of the conjuncts are correspondingly higher than the prior probability of the conjunction. The conjunction principle, instead, ignores the role of prior probabilities and treat the conjuncts and the conjuction only in relation to the evidence, irrespective of the prior probabilities. So, in order to capture the conjunction principle, legal probabilists should rely on probabilistic measures that are not heavily depend on prior probabilities. But, as we have seen, neither the Bayes factor nor the likelihood ratio are such measures.

We have seen that the joint Bayes factor $\mathsf{P}(a \wedge b|A \wedge B)/\mathsf{P}(a \wedge b)$, under suitable independence assumptions, is greter than the individual Bayes factors $\mathsf{P}(a|A)/\mathsf{P}(a)$ and $\mathsf{P}(b|B)/\mathsf{P}(b)$. This inequality holds even if the evidence is held constant. The joint Bayes factor $\mathsf{P}(a \wedge b|A \wedge B)/\mathsf{P}(a \wedge b)$, under suitable independence assumptions, is still greter than the individual Bayes factors (for composite evidence) $\mathsf{P}(a \wedge b|A)/\mathsf{P}(a \wedge b)$ and $\mathsf{P}(a \wedge b|B)/\mathsf{P}(a \wedge b)$. But the larger Bayes factor associated with the composite claim, holding the evidence fixed, need not be a sign of stronger evidence, but merely an artifact of the lower prior probability of the composite claim. The same can be said for the combined likelihood ratio. We have seen that, holding fixed the sensitivty and specificity of a and b, the combined likelihood ratio can be changed by varying the priors of A and B. The lower the priors, the stronger the likelihood ratio. Perhpas, the same body of evidence may strongly suport the composite claim $A \wedge B$, while failing to strongly support A or B simply because $A \wedge B$ has a lower prior probability and this lower prior probability, everything else being equal, inflates the likelihood ratio or the Bayes factor *qua* measures of evidential strength.

Intuitively, the strength of the evidence should not depend on the prior probability of the hypothesis, but solely on the quality of the evidence itself. We will later see that this intuition is not completely correct, but it has a great deal of plausibility, so it is worth taking it seriously. The prior probability of the hypothesis seems extrinsic to the quality of the evidence since the latter should solely depend on the sensitivity and specificity of the evidence relative to the hypothesis of interest. Strength of evidence determines how much the evidence changes, upwards or downwards, the probability of a hypothesis. However, as the prior probability increases, the smaller the impact that the evidence will have on the probability of the hypothesis. If the prior is close to one, the evidence would have marginal if not null impact. But this does not mean that the evidence weakens as the prior probability of the hypothesis goes up. For consider the same hypothesis which in one context has a very high prior probability and in another has a moderate prior probability (say a disease is common in a population but rare in another). The outcome of the same diagnostic test (say a positive test result) performed on two people, each drawn from two populations, should not count as stronger evidence in one case than in the other. After all, it is the same test that was performed and thus the quality of the evidence should be the same. For just one item of evidence, Bayes factor does not capture this intuition, but the likelihood ratio does, which can be considered an argument in favor of the latter and against the former measure of evidenetial support. However, we have seen that, for more than one item of evidence, the Bayes factor as well as the likelihood ratio are prior dependent.

To circumvent the phenomenon of prior dependency, evidential strength can be thought as a relationship between prior and posterior probabilities. The graph in Figure 5 below represents to what extent the evidence changes the probability of a select hypothesis for any value of the prior probability of the hypothesis. The graph compares the 'base line' (representing no change in probability) and the 'posterior line' (representing the posterior probability of the hypothesis as a function of the prior for a given assignment of sensitivity and specificity of the evidence). Roughly, the larger the area between the base line and the posterior line, the stronger the evidence. Crucially, this area does not depend on the prior probability of the hypothesis, but solely on the sensitivity and specificity of the evidence. As expected, any improvement in sensitivity or specificity will increase the area between the base line and the posterior line. To be sure, what matters is the ratio of sensitivity to 1-specificity, not their absolute value. So evidence with sensitivity and specificity of 0.9 and 0.9 would be equally strong as evidence with sensitivity and specificity at 0.09 and 0.09 becaue $0.9/(1-0.9) = 0.09/(1-0.09)$.

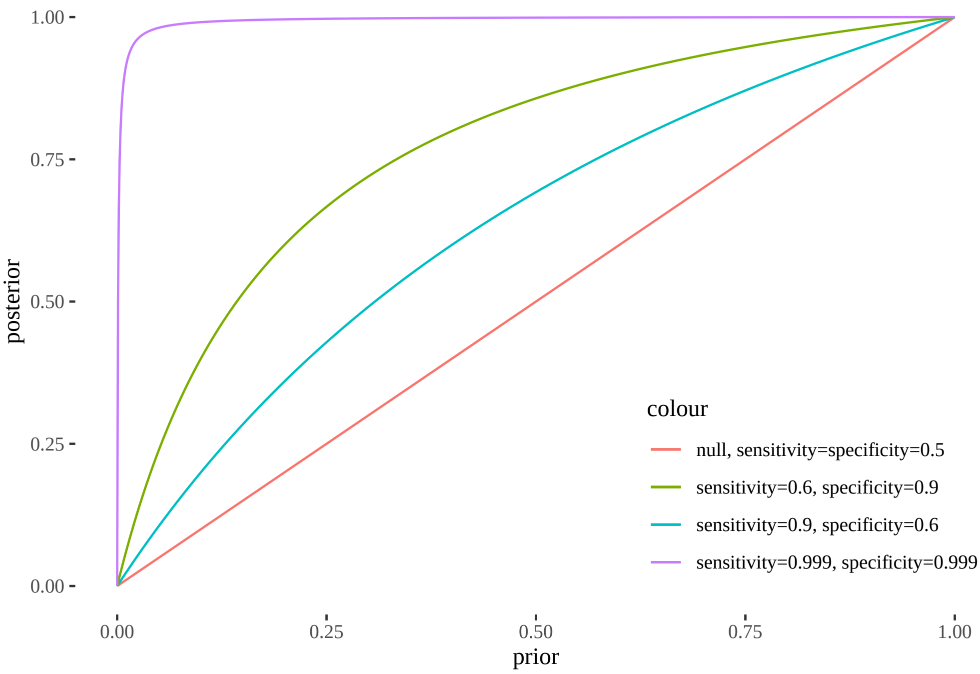

Figure 5: The further away the posterior line from the base line, the stronger the evidence irrespective of the prior probability of the hypothesis.

The same approch can model the joint evidential strength of two items of evidence, $a \wedge b$, relative to the combined hypothesis, $A \wedge B$. For simplicity, assume a and b are independent lines of evidence supporting their respective hypothesis A and B. Further, assume A and B are probabilistically independent of the other, as in the Bayesian network in Figure **??** (top). The graph in Figure 6 (top) shows how the prior probabilities are impacted by evidence in support of a single hypothesis—say $a \wedge b$ supports A[11]—versus

[11] Given the assumptions of independence we are working with, the strength of a in support of A is the same the strength of $a \wedge b$

evidence in support of a joint hypothesis—say $a \wedge b$ supports $A \wedge B$. The base line is lower in the latter than in the former case because the prior probability of $A \wedge B$ is lower than the prior probability of A. The prior of A equals x and the prior of $A \wedge B$ equals x^2 (assuming A and B have the same prior probability, and as noted before, are probabilistically independent of one another).

What happens if we make the same comparison between individual and composite claims by equalizing their prior probability? If the claims are independent and equiprobable, let x be the prior probability of an individual claim (when it is considered in isolation) and let $x^{1/k}$ the prior probability of the same individual claim when it is part of a composite claim that consists of k claims. In this way—and again, assuming independence and equiprobability of the hypotheses—the prior probability of the composite claim equals the prior probability of the individual claim since $(x^{1/k})^k = x$, as desired. These different claims are then plotted having the same priors. Here we are explicitly factoring out the role of prior probabilities. Figure 6 (bottom) shows the result of this process of equalization.

We observe two things. First, the difference in posterior probability, though still present, is less significant, especially for values above the 75% threshold and even more clearly above the 95% threshold. Second, whatever remaining difference in posterior probability is now reversed, that is, a composite claim supported by several items of evidence has a higher posterior probability compared to an individual claim supported by one item of evidence. This second observation agrees with the analysis based on the Bayes factor and the likelihood ratio in the earlier section. That analysis showed that the support for a composite claim by a joint body of evidence often exceeds the support for an individual claim.

These two observations establish that, by factorig out prior probabilities and under certain independence assumptions, whenever the individual claims meet the applicable posterior threhsold, so does the composite claim. This verifies aggregation. Conversely, whenever the composite claim, say $A \wedge B$, meets the applicable posterior threshold, so do the individual claims insofar as the threshold is about 75% or higher. This verifies—to some approximation and in a limited class of cases—the other direction of the conjunction principle, what we called distribution.

Has the distribution paradox been eliminated then? The approach we have just described—equalizing the prior probabilities across individual and composite claims—does not entirely eliminate the paradox. There are still cases in which a composite hypothesis, say $A \wedge B$, receives stronger support than an individual hypothesis, given the same body of evidence. Sensitivity to priors seems to play a role. But it cannot be the only factor at play, or else the equalization of the prior probabilities would have eliminated the paradox entirely. So what else is going on?

4.2 Weaker Claims Weaken Sensitivity

Let's examine more closely the Bayes factor and the likelihood ratio as measures of evidential strength. Likelihood ratios are comparative in nature. Suppose we compare claim A and claim $A \wedge B$ relative to the same body of evidence $a \wedge b$. Thinks of A as 'the defendant physically injured the victim' while B as 'the defendant knew the victim was a firefighter'. Think of a and b as testimonies each supporting one of the two hypotheses. We are still working with the Bayesian networks in Figure **??**.

Consider the combined body of evidence $a \wedge b$. Which claim between A and $A \wedge B$ will receive more support by evidence $a \wedge b$? Intuitively, one might think that A should receive more or at least equal support compared to $A \wedge B$. After all, $A \wedge B$ is a stronger claim than A and thus more difficult to establish than A, other things being equal. In terms of posterior probabilities, this is true. The posterior probability of $A \wedge B$ should not be higher than the posterior probability of A alone given evidence $a \wedge b$.

Let's now think in terms of evidential support. Formally, the question is whether $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ is less than one or greater than one. Given the customary independencies between evidence and hypotheses,

$$\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} = \frac{\mathsf{P}(a|A)\mathsf{P}(b|A)}{\mathsf{P}(a|A)\mathsf{P}(b|B)} = \frac{\mathsf{P}(b|A)}{\mathsf{P}(b|B)} < 1.$$

The reason is that $\mathsf{P}(b|A) < \mathsf{P}(b|B)$ since the sensitivity of b relative to B should be higher than the sensitivity of b relative to A.[12] This is obvious if A and b are independent claims, as in the Bayesian networks in Figure **??** (bottom). In this case, $\mathsf{P}(b|A) = \mathsf{P}(b)$.

in support of A since $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)\mathsf{P}(b|A)}{\mathsf{P}(a)\mathsf{P}(b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}$.

[12]GIVE PROOF OF THIS

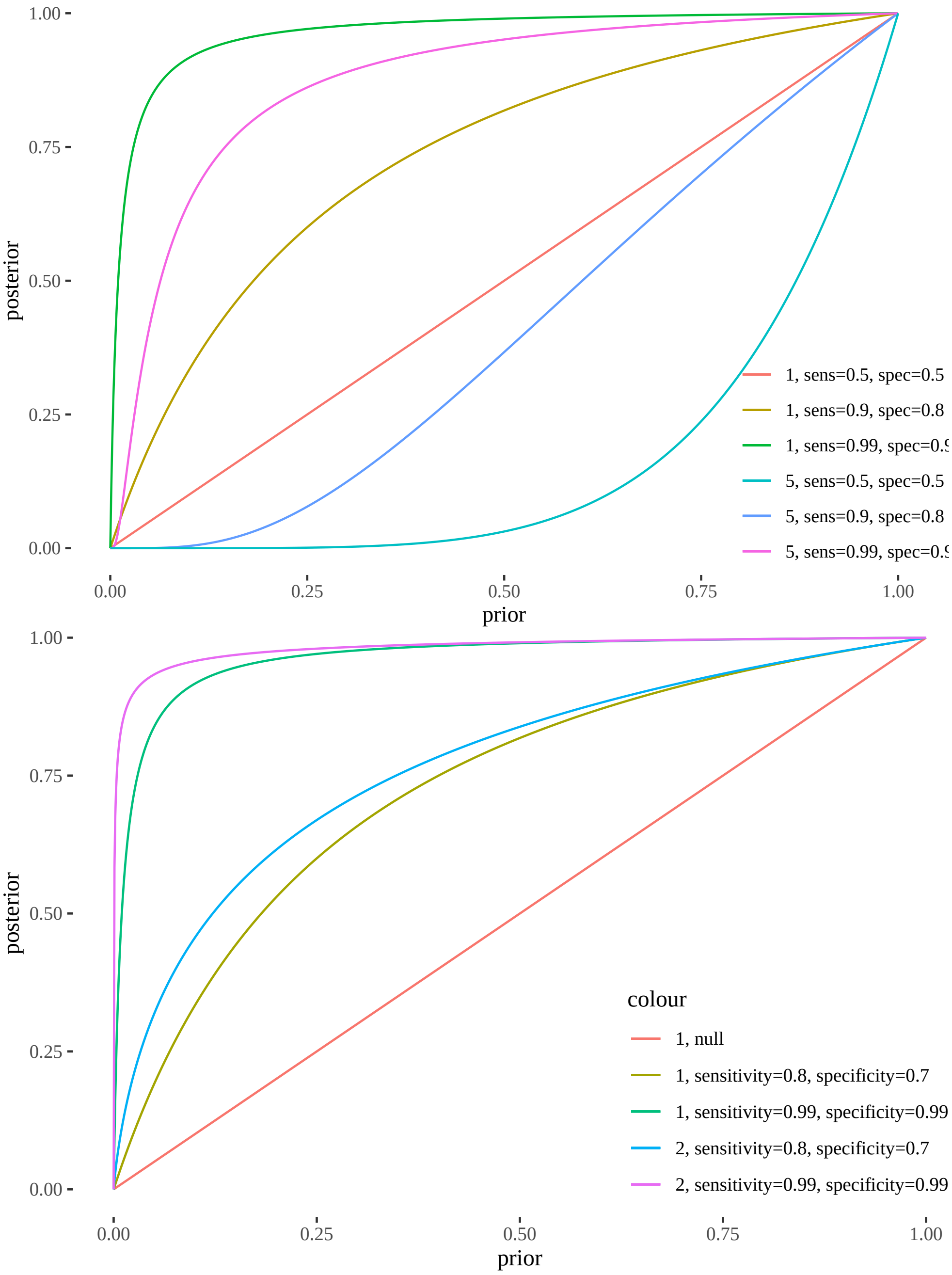

Figure 6: The comparison is betwen individual support (marked by 1, for one individual hypothesis) and joint support (marked by 2, for a two-claim composite claim). Top graph: The base line for joint support ($y = x * x$) is below the base line for individual support ($y = x$). Bottom graph: the two base lines are equalized and the posterior lines adjusted accordingly. The posterior lines for individual and joint support get closer especially for high posterior probability values.

So $\frac{\mathsf{P}(b)}{\mathsf{P}(b|B)} < 1$ since b, by assumption, positively supports B, or in other words, $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} > 1$. Thus, $a \wedge b$ supports $A \wedge B$ more than it supports A alone. This is not what one would expect intuitively.

The same conslusion holds using the Bayes factor. If $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} < 1$, then

$$\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)} < \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)}$$

Thus, evidence $a \wedge b$ better supports, even on an absolute scale, $A \wedge B$ compared to A. Note that, even if the Baye factor depends on the priors, the difference here is not due to the difference between the priors of A and the priors of $A \wedge B$ since the denominator is simply $\mathsf{P}(a \wedge b)$.[13]

What we have said so far agrees with the claim defended in the previous section. That is, even when $a \wedge b$ strongly supports $A \wedge B$, the same evidence need *not* strongly support A. Formally,

$$\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|\neg A)} < \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}$$

In other words, even if $a \wedge b$ favors $A \wedge B$ over its negation to a very high degree, it need not favor equally strongly A over its negation. This is a comparative claim about two compartive claims, and as such, it may not be easy to parse. Evidential support, when it is formalized by the likelihoo ratio, is always relative to a contrast class. In comparing the support that the same body of evidence provides to A as contrasted to $A \wedge B$, it might be better to include these two hypotheses in the contrast class. So the expression $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ is more straightforward.

No matter the formulation, the same conclusion holds. Evidence $a \wedge b$ supports the composite claim $A \wedge B$ more than it supports the weaker claim A, even assuming that A and B are independent of one another and thus not mutually reinforcing. This conclusion seems paradoxical. How should we make sense of it? At first, we thought the paradox could be due to prior dependency since the combined likelihood ratio $\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}$ varies depending on the priors of A and B. But this argument seems to no longer holds since in case of $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ any prior dependency seems to have been eliminated. After all, if $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} < 1$, the sensitivity of $a \wedge b$ must be worse relative to A than the composite claim $A \wedge B$.

Sensitivity is a crucial property of the quality of the evidence. Everything else being equal, the lower the sensitivity of the evidence, the lower its evidential strength. The importance of sensitivity as a factor for assessing the strength of the evidence is hard to dispute. So why is the sensitivity of $a \wedge b$ worse relative to A than $A \wedge B$? Suppose A is the case. If A is the case, in order for $a \wedge b$ to arise, both a and b should pick up on A. If b fails to pick up on A, then $A \wedge b$ would not arise even if a pick up on A.[14] Suppose instead $A \wedge B$ holds. In this case, $a \wedge b$ would arise even if b fails to pick up on A so long a picks up on A and b picks up on B. Now of course b could also fail to pick on B just like a could fail to pick up on B. But we are assuming that b is better than a at tracking B. So b will fail less often than a at picking up on B. Thus, the sensitivity of $a \wedge b$ relative to $A \wedge B$ is better than the sensitivity of a relative to A alone. This is a subtle point that probability theory helps to bring out clearly.

How big are these variations? **PLOT GRAPH TO GET A SENSE OF VARIATIONS OF SENSITIVITY**

One explanation of the paradox, then, is the difference in sensitivity. The sensitivity of $a \wedge b$ relative to $A \wedge B$ is better than the sensitivity of the same evidence relative to A. Consequently, other thigns being equal, evidence $a \wedge b$ supports $A \wedge B$ better than A. However counterintuitive this might seem, we should accept this fact and admit that our intuitions were wrong. So the fallacy seems to be to assume that the sensitivity of $a \wedge b$ relative to A cannot be lower than the senstivity of $a \wedge b$ relative to $A \wedge B$. The thought would be something like this: if $a \wedge b$ tracks $A \wedge B$ to some degree, it surely must be able to track A alone, at least as well. But we have just shown that we cannot assume that $\mathsf{P}(a \wedge b|A) \geq \mathsf{P}(a \wedge b|A \wedge B)$ and in fact the opposite is the case, $\mathsf{P}(a \wedge b|A) < \mathsf{P}(a \wedge b|A \wedge B)$.

Another way to convince ourselves this is the case is to run a simulation. Suppose we are deciding about the truth of A and the the truth of $A \wedge B$, and we have a fixed body of evidence, say, $a \wedge b$ that speaks in favor of both claims.

M: Run simulation to show that same diagnostic test for composite claim would perform better then when applied to individual claim (worse LR). How do to do this? HELP!

[13] Note that $\mathsf{P}(a \wedge b|A)\mathsf{P}(A) + \mathsf{P}(a \wedge b|\neg A)\mathsf{P}(\neg)A$ is the same as $\mathsf{P}(a \wedge b|(A \wedge B))\mathsf{P}(A \wedge B) + \mathsf{P}(a \wedge b|\neg(A \wedge B))\mathsf{P}(\neg(A \wedge B))$. **NEED A PROOF FOR THIS BUT IT SHOULD HOLD, RIGHT?**

[14] The occurrance of $a \wedge b$ is less likely to occur than just a alone picking up on A because b may fail—and fail more often than a would—in picking up on A.

We should circumscribe the point we just made since it does not always hold. Suppose, as in Figure 7, that H is a claim unrelated to A and evidence a. Evidence a supports A. Would the composite claim $A \wedge H$ be better supported by a than A alone? It would not. Mere tagging an unrelated hypothesis does not strengthen the evidence. Note that $\mathsf{P}(a|A) = \mathsf{P}(a|A \wedge H)$ because H is indepedent from everything else. Thus,

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge H)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A)} = 1.$$

Tagging an unrelated claim H does not strengthen the evidence, but leaves it unchanged. Similarly, suppose B constitutes one element of a crime and A constitutes the other element. The two claims are independent, each supported by items of evidence a and b respectively. This is our standard set up. If a supports A, would a support $A \wedge B$ more strongly than it supports A alone? Here we no longer have $a \wedge b$, but instead, a alone. The question is whether $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge B)} > 1$ or $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge B)} < 1$. Given the usual independencies between evidence and hypotheses,

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge B)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A)} = 1$$

Evidence a supports A and $A \wedge B$ to the same extent. One might complain that this is counterintuitive. How can it be that a supports A to the same degree that it supports the more demanding claim $A \wedge H$ or $A \wedge B$? For suppose we have evidence a in favor of A and then wonder whether we could use that evidence in support of another claim H or B. By tagging H or B to A, we can at least say that we have evidence a for $A \wedge H$ or $A \wedge B$ that is at least as strong as evidence a in support of A. But note that $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|negA)} > 1$ even though $\frac{\mathsf{P}(a|H)}{\mathsf{P}(a|negH)=1}$ and $\frac{\mathsf{P}(a|B)}{\mathsf{P}(a|negB)=1}$ (assuming A and B are independent). So a does not support H or B to the same degree that it supports A. However, a supports A to the same degree that it supports $A \wedge H$ or $A \wedge B$.

What should we make of this? The commonality between H and B is that they are irrelevant for a. Evidence a does not increase nor decrease their probability so the evidence is irrelevant for them. So the upshot here is that, tagging an irrelevant hypothesis does not change evidential support. The other upshot is that tagging a relevant hypothesis—a hypthesis that does bear on the evidence in one way or another, such as B relative to $a \wedge b$—does increase evidential support.

M: How to explain this better? Can we make this plausible? HELP!

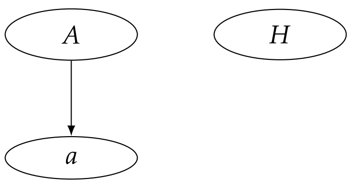

Figure 7: Bayesian network with wholly unrelated claim H.

The intuition that the same evidence tracks the conjunction $A \wedge B$ better than one of the conjuncts A and B might rest on another model of what is going on. Say ab is an item of evidence that arises when A or B occurs with 60% probability. That is, $\mathsf{P}(ab|A) = \mathsf{P}(a|B) = 60\%$. What would be the sensitivity of ab relative to the conjunction $A \wedge B$, $\mathsf{P}(ab|A \wedge B)$? We can represent this set up in Figure 8. Intuitvely, one might reason as follows. There are two paths leading to ab, one path starts with A and another path starts with B. When both these pathst are active, since we are assuming $A \wedge B$, then the probability of ab must be higher than if just one of the two paths is active. Hence, $\mathsf{P}(ab|A \wedge B) \geq 60\%$.

M: Is this true probabilistically? If not, what assumptions are required? HELP!

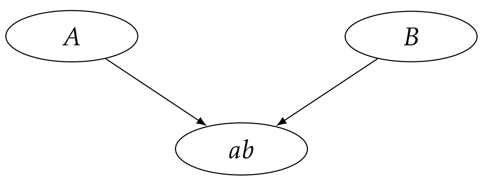

Figure 8: Bayesian network with ab resulting from A and B.

4.3 But Sensitivity Depends On Prior Probabilities

We have shown, given a suitable number of assumptions, that the sensitivity of $a \wedge b$ can be greater relative to $A \wedge B$ than A alone. This explains why the evidential support of $a \wedge b$ is greater in favor of $A \wedge B$ than alone A. Therefore—one might conclude—even factoring out differences in priors probabiliies could still lead to differences in evidential strength simply due to difference in sensitivity. But note that this argument assumes that sensitivity—or specificity—have nothing to do with prior probabilities.

Things are somewhat more complicated. After all, $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} = \frac{\mathsf{P}(b|A)}{\mathsf{P}(b|B)}$. The denominator, which tracks the sensitivity of $a \wedge b$ relative to $A \wedge B$ does not depend on the priors of $A \wedge B$, but solely on the sensitivty of a and b relative to A and B. However, the numerator, which tracks the sensitivity of $a \wedge b$ relative to a, does depend on the priors of A (or whatvere other hypothesis one choses to computer $\mathsf{P}(b|A)$ or $\mathsf{P}(b)$). This means that the denominator can vary depeding on the prior probability of a chosen hypothesis. Thus, the lower the prior of A, the lower the probability of b. One could still insist that here we are not comparing the prior of a hypothesis, say B, and the prior of another hypothesis, say $A \wedge B$. Whatever the difference in sensitivity, it cannot be due to the difference in prior probabilities of the hypotheses.

On other hand hand, $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ is comparing two quantitities, one dependent on the priors and the other not dependent on the priors.

More generally, the intuition that characteristics of the evidence such as sensitivity and specificity should be indepedent of the probability of the hypothesis of iterest turns out to be incorrect empirically. One study in the medical literature has shown, surprisingly, that the sensitivity of a diagnostic test is independent of the prior of the hypothesis beign tested—say whether the patient has a medical condition. However, specificity is dependent on the prior of the hypothesis:

> Overall, specificity tended to be lower with higher disease prevalence; there was no such systematic effect for sensitivity (page E537). Source: Variation of a test's sensitivity and specificity with disease prevalence Mariska M.G. Leeflang, Anne W.S. Rutjes, Johannes B. Reitsma, Lotty Hooft and Patrick M.M. Bossuyt CMAJ August 06, 2013 185 (11) E537-E544; DOI: https://doi.org/10.1503/cmaj.121286

The authors of the study, however, caution that

> Because sensitivity is estimated in people with the disease of interest and specificity in people without the disease of interest, changing the relative number of people with and without the disease of interest should not introduce systematic differences. Therefore, the effects that we found may be generated by other mechanisms that affect both prevalence and accuracy.

So, according to the authors, changes in prevalence need not directly affect specificity since variations in prevalence and variation in specificity may have a common cause. Our earlier calculations about combined specificity and sensitivity agree with experimental results, namely, only specificity depends on the priors. Our calculations, in fact, show that different priors for the individual claim do affect specificity. The variation of specificity in the result of of splitting the negation of the composite hypthesis $\neg(A \wedge B)$ into three further scenarios, $\neg A \wedge B$, $B \wedge \neg A$ and $\neg A \wedge negB$. This prior sensitivity, of course, only applies to composite hyotheses, but to some extent, any hypothesis can be analyzed as a composite hypothesis. The claim that the defedant was running down 5th avnue can be broken down in the conjuction that the defendant was running and that the defendant was at 5th avenue. Any claim, under some level of description, is a composite hypothesis. So, perhpas, the quality or strength of the evidence should depend on the the priors whether the hypothesis is composite or not. Is this another example of base rate neglect?

M: Might be good to have a simulation here that makes vidid why combined specificity is in fact dependent on the priors. Maybe it is, after all, a fallacy to think that the quality/strength of the evidence should be independent of the priors. HELP!

Let's grant that the quality of the evidence should depend, contrary to our initial intuition, on the prior probability of the hypotheses. If that is so, it would not be natural to see that evidence – the same evidence – strongly favors $A \wedge B$ without strongly favoring A or B. Perhaps we can make sense of this if we keep in mind the comparison between hypothesis we are making here.

NOT SURE HOW TO CONTINUE HERE THOUGH!

TO DO:

1. NOTE THAT EVEN BY EQUALIZING PRIORS, TE DISTRBUTION PARADOX DOES NOT GO AWAY. SO WHAT ELSE IS GOING ON HERE? NEED TO MAKE COMPATISON BETWEEN HYPOTHESES. NEED TO FIGURE THIS OUT!
2. TRY TO MAKE SENSE OF THIS, IT IS INTITVELY ACCEPTABLE THAT SUPPORT FOR COMBINED CLAIM, EVEN HOLDING FIXED THE SAME EVIDENCE, SHOULD BE STRONGER THEN SUPPORT FOR INDIVIDUAL CLAIM? THAT IS CLEARLY ODD AND GOES AGAINST COMMON ASSUMPTIONS.
3. **HYPOTHESIS: WHEN THE HYPTHESIS IS NOT MEDIATED (AS IN a toward A OR b TOWARD B, AS OPPOSED TO a TOWARD A-AND-B), THEN SENSITIVITY OR SPECICITIY IS NOT DEPEDENT ON PRIORS**

4.4 Which Measure of (Combined) Evidential Support?

THIGNS TO ADD:

1. THE MIN SEEMS TO BE THE MEASURE FOR COMPOSITE CLAIMS THAT CAPTURES AGGREGATION AND DISTRIBUTION BEST. SO THE QUESTION IS WHAT PROBABILISTIC MEASURE CAPTURES MIN?
2. IS DEPEDENCY ON PRIOR ANOTHER EXAMPLE OF BASE RATE NEGLECT? WE NEGLECT BAE RATE IN CALCULATING POSTERIOR BUT ALSO IN CALCULATING STRENGTH OF EVIDENCE? WE JUST NEED TO LIVE WITH THE FACT THAT WE HAVE A POOR UNDERSTANDING OF EVIDENCE BUT PERHPAS GOOD ENOUGH TO GET BY IN THE WORLD. CONNECT TO POINT 1 AND MIN FUNCTION.

5 Revising the Holistic Approach

The other route for legal probabilists is to reject aggregation and hold on to distribution. This can be accomplished by viewing the standard of proof as a posterior probability threshold. Following Dawid, rejecting aggregation does not make legal probabilism an unworkable theory. The conjunction paradox begins with the assumption that each element of the allegation has been established by the required standard. Then, one shows that it does not follow that the conjunction has been established by the required standard. Aggreation fails. This is the paradox. But the force of this paradox—as Dawid showed—can be limited by noting that each element will be established by higher probability than the conjunction because each element will have a higher prior probability, other things being equal. We think this is the right strategy. Here we offer a revised version of the holistic approahc elaborating on Dawid's insight.

So far we have assumed the most natural probabilistic interpretation of proof standards, one that posits a threshold on the posterior probabilities of a generic hypothesis such as guilt or civil liability. In criminal cases, the requirement is formulated as follows: guilt is proven beyond a reasonable doubt provided $\mathsf{P}(G|E)$ is above a suitable threshold, say 95%. The threshold is lower in civil trials. Civil liability is proven by preponderance provided $\mathsf{P}(L|E)$ is above a suitable threshold, say 50%. The general claim G or L is split into several consituent claims, depending on the defintion of the wrongful act in the governing law. But this framing misses a crucial ingredient.

The claim that the defendant is guilty or civilly liable can be replaced by a more fine-grained hypothesis, call it H_p, the hypothesis put foward by the prosecutor (or the plaintiff in a civil case), for example, the hypothesis that the defendant killed the victim with a firearm while bulglarizing the victim's apartment. H_p can be any hypothesis which, if true, would entail the defendanat is civilly or criminally liable (according to the governing law). Hypothesis H_p is a more precise description of what happened that establishes, if true, the defendant's guilt or civil liability. In defining proof standards, instead of saying – somewhat generically – that $\mathsf{P}(G|E)$ or $\mathsf{P}(L|E)$ should be above a suitable threshold, a probabilistic interpretation could read: civil or criminal liabbility is proven beyond a reasonable doubt provided $\Pr(H_d|E)$ is above a suitable threshold.

This variation may appear inconsequential. But we argue – perhpas surprisingly – it can address the naked statistical evidence problem and the difficulty about conjunction. We have examined the naked statistical evidence problem in another chapter **REFERENCE TO EARLIER HAPER**. The basic argument was this. Consider the prisoner hypothetical. It is true that the naked statistics make him 99% likely to be guilty, that is, $\mathsf{P}(G|E_s)$. It is 99% likely that he is one the prisoners who attacked and killed the guard. Notice that this a generic claim. It is odd for the prosecution to simply assert that the prisoner was one of those who killed the guard, without saying what he did, how he partook in the killing, what role he played in the attack, etc. If the prosecution offerred a more specific incriminating hypothesis, call it H_p, the probability $\mathsf{P}(H_p|E_s)$ of this hypothesis based on the naked statistical evidence E_s would be well below 99%, even though $\mathsf{P}(G|E_s) = 99\%$. The fact the prisoner on trial is most likely guilty is an artifact of the choice of a generic hypothesis G. When this hypthesis is made more specific – as it should be – this probability drops significantly.

The same approach can address the difficulty about conjunction. The tacit assumption is that prosecutors or plaintiffs should establish each element in isolation. If they manage to prove each element to the desired standard, they have meet their burden. This is artificial. What prosecutors or plaintiffs should do instead is to establish a specific hypothesis from which the constituent elements follow (almost) deductively. To illustrate, consider a Washington statute about negligent driving:

> (1)(a) A person is guilty of negligent driving in the first degree if he or she operates a motor vehicle in a manner that is both negligent and endangers or is likely to endanger any person or property, and exhibits the effects of having consumed liquor or marijuana or any drug or exhibits the effects of having inhaled or ingested any chemical, whether or not a legal substance, for its intoxicating or hallucinatory effects. RCW 46.61.5249

In other words, a prosecutor who wishes to establish beyond a reasonable doubt that the defendant is guilty of negligent driving should establish:

> (a) that the defendant operated a vehicle (b) that, in operating a vehicle, the defendant did so in negligente manner (c) that, in operating a vechicle, the defendant did so in a manner likely to endanger a person pr property (d) that the defendant – presumably, immediately after the incident – exihibited the signs of intoxication by liquor or drugs

It would be odd for the prosecutor to go about establishig each claim in isolation, especially because these four claims are specific to a time and place. Clearly, the prosecutor cannot simply establish that the defendant was operating a vehicle at some point in time and at some other point in time the defedant exhibited signs of intoxication. That would establish nothing. The elements to be established must be combined in a coherent spatio-temporal narrative. So establishing them in isolation makes little sense. Prosecutors could establish, say, that the defendant was driving on a busy Highway 1 north San Francisco at about 8:30 PM; the car was moving erratically left and right, cutting across other lanes; the defendant was stopped by a police officer who conducted a field sobriety test and used a breathylizer, both tests showing a higher-than-normal quantity of alchool. This narrative, if true, establishes each element of the offense. The prosecutor's task would be to establish the narrative as a whole. Presumably, the prosecutor could establish item (a) and (b) separately, but then, would have to show that they happened as part of the same driving episode. So they would have to be unified somehow. The same applies to the other elements of the offense.

Suppose the prosecutor has established a narrative N to a very high probability, say above the required threshold for proof beyond a reasonable doubt. Then,

$$\mathsf{P}(C_a \wedge C_b \wedge C_c \wedge C_d|N) = \mathsf{P}(C_i|N) = 1 \text{ for any } i = \{a, b, c, d\}.$$

Both direction of the conjunction principle, aggregation and distributin, are now trivially satisfied. Once we condition on the narrative N, each individual claim has a probability of one and thus their conjunction also has a probability of one. The narrative, however, has a probability short of one, up to whatever value is required to meet the governing standard of proof.

To be sure, not all wrongful acts, in civil or criminal cases, require the prosecutor or the plaintiff to establish a spatio-temporal narrative. It might not be necessary to show that all elements of an offense occurred at the same point in time or in close succession one after the other. Some wrongful acts may consist of a pattern of acts that stretches for several days, months or even years. There may be temporal and spatial gaps that cannot not be filled. We consider several of these examples in our discussion of naked statistical evidence **SEE PREVIOUS CHAPTER**. Be that as it may, an accusation of wrongdoing in a criminal or civil case should still have a degree of cohesive unity. The acts and occurences that constitute the wrongdoing should belong to the same wrongful act. It is this unity which the plaintiff and the prosecution must establish when they make their case. One way to establish this unity is by providing a unifying narrative, but this need not be the only way. Perhaps the expresion 'theory' or 'explanation' are better and more general than 'narrative'.

One upshot of this holistic approach is that it higlights a distinction that is often made, between statements of fact and statements of law. That the defendant was driving on highway 1 and the car was moving erratically left and right is a statement of fact. That such an occurrence counts as 'negligent driving' is a statement of law. The instance 'car moving erratically' is subsumed under the category 'driving erratically'. A defense lawyer could challange the inference. Perhaps the car was moving erratically because of other reasons, say, mulfunctioning of the brakes due to manifacturing defects not attributable to the driver. **TO BE COMPLETED**

6 Extra unstructured materials

6.1 The likelihood strategy

Focusing on posterior probabilities is not the only approach that legal probabilists can pursue. By Bayes' theorem, the following holds, using G and I as competing hypotheses:

$$\frac{\Pr(G|E)}{\Pr(I|E)} = \frac{\Pr(E|G)}{\Pr(E|I)} \times \frac{\Pr(G)}{\Pr(I)},$$

or using H_p and H_d as competing hypotheses,

$$\frac{\Pr(H_p|E)}{\Pr(H_d|E)} = \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)},$$

or in words

$$\textit{posterior odds} = \textit{likelihood ratio} \times \textit{prior odds}.$$

A difficult problem is to assign numbers to the prior probabiliteis such as $\Pr(G)$ or $\Pr(H_p)$, or priors odds such as $\frac{\Pr(G)}{\Pr(I)}$ or $\frac{\Pr(H_p)}{\Pr(H_d)}$.

DISCUSS DIFFICULTIES ABOUT ASSIGNING PRIORS! WHERE? CAN WE USE IMPRECISE PROBABILKITIES T TALK ABOUT PRIORS – I.E. LOW PRIORS = TOTAL IGNORANCE = VERY IMPRECISE (LARGE INTERVAL) PRIORS? THE PROBLME WITH THIS WOULD BE THAT THERE IS NO UPDSATING POSSIBLE. ALL UPDATING WOULD STILL GET BACK TO THE STARTING POINT. DO YOU HAVE AN ANSWER TO THAT? WOULD BE INTERETSING TO DISCUSS THIS!

Given these difficulties, both practical and theoretical, one option is to dispense with priors altogether. This is not implausible. Legal disputes in both criminal and civil trials should be decided on the basis of the evidence presented by the litigants. But it is the likelihood ratio – not the prior ratio – that offers the best measure of the overall strength of the evidece presented. So it is all too natural to focus on likekihood ratios and leave the priors out of the picture. If this is the right, the question is, how would a probabilistic interpretation of standards of proof based on the likelihood rato look like? At its simplest, this stratgey will look as follows. Recall our discussion of expected utility theory:

$$\text{convict provided} \frac{cost(CI)}{cost(AG)} < \frac{\Pr(H_p|E)}{\Pr(H_d|E)},$$

which is equivalent to

$$\text{convict provided} \frac{cost(CI)}{cost(AG)} < \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)}.$$

By rearraing the terms,

$$\text{convict provided} \frac{\Pr(E|H_p)}{\Pr(E|H_d)} > \frac{\Pr(H_d)}{\Pr(H_p)} \times \frac{cost(CI)}{cost(AG)}.$$

Then, on this intepretation, the likelihood ratio should be above a suitable threshold that is a function of the cost ratio and the prior ratio. The outstanding question is how this threshold is to be determined.

6.1.1 Kaplow

Quite independently, a similar approach to juridical decisions has been proposed by Kaplow (2014) – we'll call it **decision-theoretic legal probabilism (DTLP)**. It turns out that Cheng's suggestion is a particular case of this more general approach. Let $LR(E) = \mathsf{P}(E|H_\Pi)/\mathsf{P}(E|H_\Delta)$. In whole generality, DTLP invites us to convict just in case $LR(E) > LR^\star$, where $LR^\star$ is some critical value of the likelihood ratio.

Say we want to formulate the usual preponderance rule: convict iff $\mathsf{P}(H_\Pi|E) > 0.5$, that is, iff $\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} > 1$. By Bayes' Theorem we have:

$$\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} = \frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)} \times \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > 1 \Leftrightarrow$$
$$\Leftrightarrow \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)}$$

So, as expected, $LR^\star$ is not unique and depends on priors. Analogous reformulations are available for thresholds other than 0.5.

Kaplow's point is not that we can reformulate threshold decision rules in terms of priors-sensitive likelihood ratio thresholds. Rather, he insists, when we make a decision, we should factor in its consequences. Let G represent potential gain from correct conviction, and L stand for the potential loss resulting from mistaken conviction. Taking them into account, Kaplow suggests, we should convict if and only if:

$$\mathsf{P}(H_\Pi|E) \times G > \mathsf{P}(H_\Delta|E) \times L \tag{11}$$

Now, (11) is equivalent to:

$$\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} > \frac{L}{G}$$
$$\frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)} \times \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > \frac{L}{G}$$
$$\frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)} \times \frac{L}{G}$$
$$LR(E) > \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)} \times \frac{L}{G} \tag{12}$$

This is the general format of Kaplow's decision standard.

6.1.2 Dawid

Here is a slightly different perspective, due to Dawid (1987), that also suggests that juridical decisions should be likelihood-based. The focus is on witnesses for the sake of simplicity. Imagine the plaintiff produces two independent witnesses: W_A attesting to A, and W_B attesting to B. Say the witnesses are regarded as 70% reliable and A and B are probabilistically independent, so we infer $\mathsf{P}(A) = \mathsf{P}(B) = 0.7$ and $\mathsf{P}(A \wedge B) = 0.7^2 = 0.49$.

But, Dawid argues, this is misleading, because to reach this result we misrepresented the reliability of the witnesses: 70% reliability of a witness, he continues, does not mean that if the witness testifies that A, we should believe that $\mathsf{P}(A) = 0.7$. To see his point, consider two potential testimonies:

A_1	The sun rose today.
A_2	The sun moved backwards through the sky today.

Intuitively, after hearing them, we would still take $\mathsf{P}(A_1)$ to be close to 1 and $\mathsf{P}(A_2)$ to be close to 0, because we already have fairly strong convictions about the issues at hand. In general, how we should revise our beliefs in light of a testimony depends not only on the reliability of the witness, but also on our prior convictions.[15] And this is as it should be: as indicated by Bayes' Theorem, one and the same testimony with different priors might lead to different posterior probabilities.

[15] An issue that Dawid does not bring up is the interplay between our priors and our assessment of the reliability of the witnesses. Clearly, our posterior assessment of the credibility of the witness who testified A_2 will be lower than that of the other witness.

So far so good. But how should we represent evidence (or testimony) strength then? Well, one pretty standard way to go is to focus on how much it contributes to the change in our beliefs in a way independent of any particular choice of prior beliefs. Let a be the event that the witness testified that A. It is useful to think about the problem in terms of *odds, conditional odds (O)* and *likelihood ratios (LR)*:

$$\begin{aligned}
O(A) &= \frac{\mathsf{P}(A)}{\mathsf{P}(\neg A)}\\
O(A|a) &= \frac{\mathsf{P}(A|a)}{\mathsf{P}(\neg A|a)}\\
LR(a|A) &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)}.
\end{aligned}$$

Suppose our prior beliefs and background knowledge, before hearing a testimony, are captured by the prior probability measure $\mathsf{P}_{prior}(\cdot)$, and the only thing that we learn is a. We're interested in what our *posterior* probability measure, $\mathsf{P}_{posterior}(\cdot)$, and posterior odds should then be. If we're to proceed with Bayesian updating, we should have:

$$\frac{\mathsf{P}_{posterior}(A)}{\mathsf{P}_{posterior}(\neg A)} = \frac{\mathsf{P}_{prior}(A|a)}{\mathsf{P}_{prior}(\neg A|a)} = \frac{\mathsf{P}_{prior}(a|A)}{\mathsf{P}_{prior}(a|\neg A)} \times \frac{\mathsf{P}_{prior}(A)}{\mathsf{P}_{prior}(\neg A)}$$

that is,

$$O_{posterior}(A) = O_{prior}(A|a) = \underbrace{LR_{prior}(a|A)}_{\text{conditional likelihood ratio}} \times O_{prior}(A) \tag{13}$$

The conditional likelihood ratio seems to be a much more direct measure of the value of a, independent of our priors regarding A itself. In general, the posterior probability of an event will equal to the witness's reliability in the sense introduced above only if the prior is $1/2$.[16]

6.2 Likelihood and DAC

But how does our preference for the likelihood ratio as a measure of evidence strength relate to DAC? Let's go through Dawid's reasoning.

A sensible way to probabilistically interpret the 70% reliability of a witness who testifies that A is to take it to consist in the fact that the probability of a positive testimony if A is the case, just as the probability of a negative testimony (that is, testimony that A is false) if A isn't the case, is 0.7:[17]

$$\mathsf{P}_{prior}(a|A) = \mathsf{P}_{prior}(\neg a|\neg A) = 0.7.$$

[16] Dawid gives no general argument, but it is not too hard to give one. Let $rel(a) = \mathsf{P}(a|A) = \mathsf{P}(\neg a|\neg A)$. We have in the background $\mathsf{P}(a|\neg A) = 1 - \mathsf{P}(\neg a|\neg A) = 1 - rel(a)$. We want to find the condition under which $\mathsf{P}(A|a) = \mathsf{P}(a|A)$. Set $\mathsf{P}(A) = p$ and start with Bayes' Theorem and the law of total probability, and go from there:

$$\begin{aligned}
\mathsf{P}(A|a) &= \mathsf{P}(a|A)\\
\frac{\mathsf{P}(a|A)p}{\mathsf{P}(a|A)p + \mathsf{P}(a|\neg A)(1-p)} &= \mathsf{P}(a|A)\\
\mathsf{P}(a|A)p &= \mathsf{P}(a|A)[\mathsf{P}(a|A)p + \mathsf{P}(a|\neg A)(1-p)]\\
p &= \mathsf{P}(a|A)p + \mathsf{P}(a|\neg A) - \mathsf{P}(a|\neg A)p\\
p &= rel(a)p + 1 - rel(a) - (1 - rel(a))p\\
p &= rel(a)p + 1 - rel(a) - p + rel(a)p\\
2p &= 2rel(a)p + 1 - rel(a)\\
2p - 2rel(a)p &= 1 - rel(a)\\
2p(1 - rel(a)) &= 1 - rel(a)\\
2p &= 1
\end{aligned}$$

First we multiplied both sides by the denominator. Then we divided both sides by $\mathsf{P}(a|A)$ and multiplied on the right side. Then we used our background notation and information. Next, we manipulated the right-hand side algebraically and moved $-p$ to the left-hand side. Move $2rel(a)p$ to the left and manipulate the result algebraically to get to the last line.

[17] In general setting, these are called the *sensitivity* and *specificity* of a test (respectively), and they don't have to be equal. For instance, a degenerate test for an illness which always responds positively, diagnoses everyone as ill, and so has sensitivity 1, but specificity 0.

$P_{prior}(a|\neg A) = 1 - P_{prior}(\neg a|\neg A) = 0.3$, and so the same information is encoded in the appropriate likelihood ratio:

$$LR_{prior}(a|A) = \frac{P_{prior}(a|A)}{P_{prior}(a|\neg A)} = \frac{0.7}{0.3}$$

Let's say that *a provides (positive) support* for A in case

$$O_{posterior}(A) = O_{prior}(A|a) > O_{prior}(A)$$

that is, a testimony a supports A just in case the posterior odds of A given a are greater than the prior odds of A (this happens just in case $P_{posterior}(A) > P_{prior}(A)$). By (13), this will be the case if and only if $LR_{prior}(a|A) > 1$.

One question that Dawid addresses is this: assuming reliability of witnesses 0.7, and assuming that a and b, taken separately, provide positive support for their respective claims, does it follow that $a \wedge b$ provides positive support for $A \wedge B$?

Assuming the independence of the witnesses, this will hold in non-degenerate cases that do not involve extreme probabilities, on the assumption of independence of a and b conditional on all combinations: $A \wedge B$, $A \wedge \neg B$, $\neg A \wedge B$ and $\neg A \wedge \neg B$.[18,19]

Let us see why the above claim holds. The calculations are my reconstruction and are not due to Dawid. The reader might be annoyed with me working out the mundane details of Dawid's claims, but it turns out that in the case of Dawid's strategy, the devil is in the details. The independence of witnesses gives us:

$$\begin{aligned}
P(a \wedge b|A \wedge B) &= 0.7^2 = 0.49 \\
P(a \wedge b|A \wedge \neg B) &= 0.7 \times 0.3 = 0.21 \\
P(a \wedge b|\neg A \wedge B) &= 0.3 \times 0.7 = 0.21 \\
P(a \wedge b|\neg A \wedge \neg B) &= 0.3 \times 0.3 = 0.09
\end{aligned}$$

Without assuming A and B to be independent, let the probabilities of $A \wedge B$, $\neg A \wedge B$, $A \wedge \neg B$, $\neg A \wedge \neg B$ be $p_{11}, p_{01}, p_{10}, p_{00}$. First, let's see what $P(a \wedge b)$ boils down to.

By the law of total probability we have:

$$\begin{aligned}
P(a \wedge b) = {} & P(a \wedge b|A \wedge B)P(A \wedge B) + \\
& + P(a \wedge b|A \wedge \neg B)P(A \wedge \neg B) \\
& + P(a \wedge b|\neg A \wedge B)P(\neg A \wedge B) + \\
& + P(a \wedge b|\neg A \wedge \neg B)P(\neg A \wedge \neg B)
\end{aligned} \tag{14}$$

which, when we substitute our values and constants, results in:

$$= 0.49p_{11} + 0.21(p_{10} + p_{01}) + 0.09p_{00}$$

Now, note that because p_{ii}s add up to one, we have $p_{10} + p_{01} = 1 - p_{00} - p_{11}$. Let us continue.

$$\begin{aligned}
&= 0.49p_{11} + 0.21(1 - p_{00} - p_{11}) + 0.09p_{00} \\
&= 0.21 + 0.28p_{11} - 0.12p_{00}
\end{aligned}$$

Next, we ask what the posterior of $A \wedge B$ given $a \wedge b$ is (in the last line, we also multiply the numerator and the denominator by 100).

[18] Dawid only talks about the independence of witnesses without reference to conditional independence. Conditional independence does not follow from independence, and it is the former that is needed here (also, four non-equivalent different versions of it).

[19] In terms of notation and derivation in the optional content that will follow, the claim holds if and only if $28 > 28p_{11} - 12p_{00}$. This inequality is not true for all admissible values of p_{11} and p_{00}. If $p_{11} = 1$ and $p_{00} = 0$, the sides are equal. However, this is a rather degenerate example. Normally, we are interested in cases where $p_{11} < 1$. And indeed, on this assumption, the inequality holds.

$$\begin{aligned}\mathsf{P}(A\wedge B|a\wedge b) &= \frac{\mathsf{P}(a\wedge b|A\wedge B)\mathsf{P}(A\wedge B)}{\mathsf{P}(a\wedge b)}\\ &= \frac{49p_{11}}{21+28p_{11}-12p_{00}}\end{aligned}$$

In this particular case, then, our question whether $\mathsf{P}(A\wedge B|a\wedge b) > \mathsf{P}(A\wedge B)$ boils down to asking whether

$$\frac{49p_{11}}{21+28p_{11}-12p_{00}} > p_{11}$$

that is, whether $28 > 28p_{11} - 12p_{00}$ (just divide both sides by p_{11}, multiply by the denominator, and manipulate algebraically).

Dawid continues working with particular choices of values and provides neither a general statement of the fact that the above considerations instantiate nor a proof of it. In the middle of the paper he says:

> Even under prior dependence, the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability. . . When the problem is analysed carefully, the 'paradox' evaporates [pp. 95-7]

where he still means the case with the particular values that he has given, but he seems to suggest that the claim generalizes to a large array of cases.

The paper does not contain a precise statement making the conditions required explicit and, *a fortriori*, does not contain a proof of it. Given the example above and Dawid's informal reading, let us develop a more precise statement of the claim and a proof thereof.

Fact 2. *Suppose that* $rel(a), rel(b) > 0.5$ *and witnesses are independent conditional on all Boolean combinations of A and B (in a sense to be specified), and that none of the Boolean combinations of A and B has an extreme probability (of 0 or 1). It follows that* $\mathsf{P}(A\wedge B|a\wedge b) > \mathsf{P}(A\wedge B)$. *(Independence of A and B is not required.)*

Roughly, the theorem says that if independent and reliable witnesses provide positive support of their separate claims, their joint testimony provides positive support of the conjunction of their claims.

Let us see why the claim holds. First, we introduce an abbreviation for witness reliability:

$$\begin{aligned}\mathbf{a} = rel(a) = \mathsf{P}(a|A) = \mathsf{P}(\neg a|\neg A) > 0.5\\ \mathbf{b} = rel(b) = \mathsf{P}(b|B) = \mathsf{P}(\neg b|\neg A) > 0.5\end{aligned}$$

Our independence assumption means:

$$\begin{aligned}\mathsf{P}(a\wedge b|A\wedge B) &= \mathbf{ab}\\ \mathsf{P}(a\wedge b|A\wedge\neg B) &= \mathbf{a(1-b)}\\ \mathsf{P}(a\wedge b|\neg A\wedge B) &= \mathbf{(1-a)b}\\ \mathsf{P}(a\wedge b|\neg A\wedge\neg B) &= \mathbf{(1-a)(1-b)}\end{aligned}$$

Abbreviate the probabilities the way we already did:

$$\begin{array}{ll}\mathsf{P}(A\wedge B) = p_{11} & \mathsf{P}(A\wedge\neg B) = p_{10}\\ \mathsf{P}(\neg A\wedge B) = p_{01} & \mathsf{P}(\neg A\wedge\neg B) = p_{00}\end{array}$$

Our assumptions entail $0 \neq p_{ij} \neq 1$ for $i, j \in \{0, 1\}$ and:

$$p_{11} + p_{10} + p_{01} + p_{00} = 1 \tag{15}$$

So, we can use this with (14) to get:

$$\begin{aligned}\mathsf{P}(a\wedge b) &= \mathbf{ab}p_{11} + \mathbf{a(1-b)}p_{10} + \mathbf{(1-a)b}p_{01} + \mathbf{(1-a)(1-b)}p_{00}\\ &= p_{11}\mathbf{ab} + p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})\end{aligned} \tag{16}$$

Let's now work out what the posterior of $A \wedge B$ will be, starting with an application of the Bayes' Theorem:

$$\begin{aligned}\mathsf{P}(A \wedge B|a \wedge b) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B)}{\mathsf{P}(a \wedge b)} \\ &= \frac{\mathbf{ab}p_{11}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})}\end{aligned} \tag{17}$$

To answer our question we therefore have to compare the content of (17) to p_{11} and our claim holds just in case:

$$\frac{\mathbf{ab}p_{11}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})} > p_{11}$$

$$\frac{\mathbf{ab}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})} > 1$$

$$p_{11}\mathbf{ab} + p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab}) < \mathbf{ab} \tag{18}$$

Proving (18) is therefore our goal for now. This is achieved by the following reasoning:[20]

1.	$\mathbf{b} > 0.5,\ \mathbf{a} > 0.5$	assumption
2.	$2\mathbf{b} > 1,\ 2\mathbf{a} > 1$	from 1.
3.	$2\mathbf{ab} > \mathbf{a},\ 2\mathbf{ab} > \mathbf{b}$	multiplying by **a** and **b** respectively
4.	$p_{10}2\mathbf{ab} > p_{10}\mathbf{a},\quad p_{01}2\mathbf{ab} > p_{01}\mathbf{b}$	multiplying by p_{10} and p_{01} respectively
5.	$p_{10}2\mathbf{ab} + p_{01}2\mathbf{ab} > p_{10}\mathbf{a} + p_{01}\mathbf{b}$	adding by sides, 3., 4.
6.	$1-\mathbf{b}-\mathbf{a} < 0$	from 1.
7.	$p_{00}(1-\mathbf{b}-\mathbf{a}) < 0$	From 6., because $p_{00} > 0$
8.	$p_{10}2\mathbf{ab} + p_{01}2\mathbf{ab} > p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1-\mathbf{b}-\mathbf{a})$	from 5. and 7.
9.	$p_{10}\mathbf{ab} + p_{10}\mathbf{ab} + p_{01}\mathbf{ab} + p_{01}\mathbf{ab} + p_{00}\mathbf{ab} - p_{00}\mathbf{ab} > p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1-\mathbf{b}-\mathbf{a})$	8., rewriting left-hand side
10.	$p_{10}\mathbf{ab} + p_{01}\mathbf{ab} + p_{00}\mathbf{ab} > -p_{10}\mathbf{ab} - p_{01}\mathbf{ab} + p_{00}\mathbf{ab} + p_{10}\mathbf{a} + p_{01}\mathbf{b} + p_{00}(1-\mathbf{b}-\mathbf{a})$	9., moving from left to right
11.	$\mathbf{ab}(p_{10}+p_{01}+p_{00}) > p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	10., algebraic manipulation
12.	$\mathbf{ab}(1-p_{11}) > p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	11. and equation (15)
13.	$\mathbf{ab}-\mathbf{ab}p_{11} > p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	12., algebraic manipulation
14.	$\mathbf{ab} > \mathbf{ab}p_{11} + p_{10}(\mathbf{a}-\mathbf{ab}) + p_{01}(\mathbf{b}-\mathbf{ab}) + p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	13., moving from left to right

%\end{adjustbox}

The last line is what we have been after.

OPTIONAL CONTENT ENDS

Now that we have as a theorem an explication of what Dawid informally suggested, let's see whether it helps the probabilist handling of DAC.

6.2.1 Kaplow

On RLP, at least in certain cases, the decision rule leads us to (**??**), which tells us to decide the case based on whether the likelihood ratio is greater than 1.

[21] While Kaplow did not discuss DAC or the gatecrasher paradox, it is only fair to evaluate Kaplow's proposal from the perspective of these difficulties.

Add here stuff from Marcello's Mind paper about the prisoner hypothetical. Then, discuss Rafal's critique of the likelihood ratio threshold and see where we end up.

6.3 Dawid's likelihood strategy doesn't help

Recall that DAC was a problem posed for the decision standard proposed by TLP, and the real question is how the information resulting from Fact 2 can help to avoid that problem. Dawid does not mention any decision standard, and so addresses quite a different question, and so it is not clear that '`the`paradox' evaporates", as Dawid suggests.

[20] Thanks to Pawel Pawlowski for working on this proof with me.

[21] Again, the name of the view is by no means standard, it is just a term I coined to refer to various types of legal probabilism in a fairly uniform manner.

What Dawid correctly suggests (and we establish in general as Fact 2) is that the support of the conjunction by two witnesses will be positive as soon as their separate support for the conjuncts is positive. That is, that the posterior of the conjunction will be higher that its prior. But the critic of probabilism never denied that the conjunction of testimonies might raise the probability of the conjunction if the testimonies taken separately support the conjuncts taken separately. Such a critic can still insist that Fact 2 does nothing to alleviate her concern. After all, at least *prima facie* it still might be the case that:

- the posterior probabilities of the conjuncts are above a given threshold,
- the posterior probability of the conjunction is higher than the prior probability of the conjunction,
- the posterior probability of the conjunction is still below the threshold.

That is, Fact 2 does not entail that once the conjuncts satisfy a decision standard, so does the conjunction.

At some point, Dawid makes a general claim that is somewhat stronger than the one already cited:

> When the problem is analysed carefully, the 'paradox' evaporates: suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents.
> [p. 97]

This is quite a different claim from the content of Fact 2, because previously the joint probability was claimed only to increase as compared to the prior, and here it is claimed to increase above the level of the separate increases provided by separate testimonies. Regarding this issue Dawid elaborates (we still use the p_{ij}-notation that we've already introduced):

> "More generally, let $\mathsf{P}(a|A)/\mathsf{P}(a|\neg A) = \lambda$, $\mathsf{P}(b|B)/\mathsf{P}(b|\neg B) = \mu$, with $\lambda, \mu > 0.7$, as might arise, for example, when there are several available testimonies. If the witnesses are independent, then
>
> $$\mathsf{P}(A \wedge B|a \wedge b) = \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00})$$
>
> which increases with each of λ and μ, and is never less than the larger of $\lambda p_{11}/(1 - p_{11} + \lambda p_{11}), \mu p_{11}/(1 - p_{11}1 + \mu p_{11})$, the posterior probabilities appropriate to the individual testimonies." [p. 95]

This claim, however, is false.

Optional Content Starts

Let us see why. The quoted passage is a bit dense. It contains four claims for which no arguments are given in the paper. The first three are listed below as (19), the fourth is that if the conditions in (19) hold, $\mathsf{P}(A \wedge B|a \wedge b) > max(\mathsf{P}(A|a), \mathsf{P}(B|b))$. Notice that $\lambda = LR(a|A)$ and $\mu = LR(b|B)$. Suppose the first three claims hold, that is:

$$\begin{aligned} \mathsf{P}(A \wedge B|a \wedge b) &= \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00}) \\ \mathsf{P}(A|a) &= \frac{\lambda p_{11}}{1 - p_{11} + \lambda p_{11}} \\ \mathsf{P}(B|b) &= \frac{\mu p_{11}}{1 - p_{11} + \mu p_{11}} \end{aligned} \tag{19}$$

Is it really the case that $\mathsf{P}(A \wedge B|a \wedge b) > \mathsf{P}(A|a), \mathsf{P}(B|b)$? It does not seem so. Let $\mathbf{a} = \mathbf{b} = 0.6$, $pr = \langle p_{11}, p_{10}, p_{01}, p_{00}\rangle = \langle 0.1, 0.7, 0.1, 0.1\rangle$. Then, $\lambda = \mu = 1.5 > 0.7$ so the assumption is satisfied. Then we have $\mathsf{P}(A) = p_{11} + p_{10} = 0.8$, $\mathsf{P}(B) = p_{11} + p_{01} = 0.2$. We can also easily compute $\mathsf{P}(a) = \mathbf{a}\mathsf{P}(A) + (1 - \mathbf{a})\mathsf{P}(\neg A) = 0.56$ and $\mathsf{P}(b) = \mathbf{b}\mathsf{P}(B) + (1 - \mathbf{b})\mathsf{P}(\neg B) = 0.44$. Yet:

$$
\begin{aligned}
\mathsf{P}(A|a) &= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(a)} = \frac{0.6\times 0.8}{0.6\times 0.8 + 0.4\times 0.2} \approx 0.8571\\
\mathsf{P}(B|b) &= \frac{\mathsf{P}(b|B)\mathsf{P}(B)}{\mathsf{P}(b)} = \frac{0.6\times 0.2}{0.6\times 0.2 + 0.4\times 0.8} \approx 0.272\\
\mathsf{P}(A\wedge B|a\wedge b) &= \frac{\mathsf{P}(a\wedge b|A\wedge B)\mathsf{P}(A\wedge B)}{\begin{array}{l}\mathsf{P}(a\wedge b|A\wedge B)\mathsf{P}(A\wedge B) + \mathsf{P}(a\wedge b|A\wedge\neg B)\mathsf{P}(A\wedge \neg B) + \\ \quad + \mathsf{P}(a\wedge b|\neg A\wedge B)\mathsf{P}(\neg A\wedge B) + \mathsf{P}(a\wedge b|\neg A\wedge \neg B)\mathsf{P}(\neg A\wedge\neg B)\end{array}}\\
&= \frac{\mathbf{ab}p_{11}}{\mathbf{ab}p_{11} + \mathbf{a}(1-\mathbf{b})p_{10} + (1-\mathbf{a})\mathbf{b}p_{01} + (1-\mathbf{a})(1-\mathbf{b})p_{00}} \approx 0.147
\end{aligned}
$$

The posterior probability of $A\wedge B$ is not only lower than the larger of the individual posteriors, but also lower than any of them!

So what went wrong in Dawid's calculations in (19)? Well, the first formula is correct. However, let us take a look at what the second one says (the problem with the third one is pretty much the same):

$$
\mathsf{P}(A|a) = \frac{\frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)}\times \mathsf{P}(A\wedge B)}{\mathsf{P}(\neg(A\wedge B)) + \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)}\times \mathsf{P}(A\wedge B)}
$$

Quite surprisingly, in Dawid's formula for $\mathsf{P}(A|a)$, the probability of $A\wedge B$ plays a role. To see that it should not take any B that excludes A and the formula will lead to the conclusion that *always* $\mathsf{P}(A|a)$ is undefined. The problem with Dawid's formula is that instead of $p_{11} = \mathsf{P}(A\wedge B)$ he should have used $\mathsf{P}(A) = p_{11} + p_{10}$, in which case the formula would rather say this:

$$
\begin{aligned}
\mathsf{P}(A|a) &= \frac{\frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)}\times \mathsf{P}(A)}{\mathsf{P}(\neg A) + \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)}\times \mathsf{P}(A)}\\
&= \frac{\frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)}}{\frac{\mathsf{P}(\neg a|A)\mathsf{P}(\neg A)}{\mathsf{P}(\neg a|A)} + \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)}}\\
&= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)\mathsf{P}(\neg A) + \mathsf{P}(a|A)\mathsf{P}(A)}
\end{aligned}
$$

Now, on the assumption that witness' sensitivity is equal to their specificity, we have $\mathsf{P}(a|\neg A) = \mathsf{P}(\neg a|A)$ and can substitute this in the denominator:

$$
= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(a|\neg A)\mathsf{P}(\neg A) + \mathsf{P}(a|A)\mathsf{P}(A)}
$$

and this would be a formulation of Bayes' theorem. And indeed with $\mathsf{P}(A) = p_{11} + p_{10}$ the formula works (albeit its adequacy rests on the identity of $\mathsf{P}(a|\neg A)$ and $\mathsf{P}(\neg a|A)$), and yields the result that we already obtained:

$$
\begin{aligned}
\mathsf{P}(A|a) &= \frac{\lambda(p_{11}+p_{10})}{1-(p_{11}+p_{10}) + \lambda(p_{11}+p_{10})}\\
&= \frac{1.5\times 0.8}{1-0.8+1.5\times 0.8} \approx 0.8571
\end{aligned}
$$

The situation cannot be much improved by taking $\mathbf{a}$ and $\mathbf{b}$ to be high. For instance, if they're both 0.9 and $pr = \langle 0.1, 0.7, 0.1, 0.1\rangle$, the posterior of A is ≈ 0.972, the posterior of B is ≈ 0.692, and yet the joint posterior of $A\wedge B$ is 0.525.

The situation cannot also be improved by saying that at least if the threshold is 0.5, then as soon as $\mathbf{a}$ and $\mathbf{b}$ are above 0.7 (and, *a fortriori*, so are λ and μ), the individual posteriors being above

0.5 entails the joint posterior being above 0.5 as well. For instance, for $\mathbf{a} = 0.7$ and $\mathbf{b} = 0.9$ with $pr = \langle 0.1, 0.3, 0.5, 0.1\rangle$, the individual posteriors of A and B are ≈ 0.608 and ≈ 0.931 respectively, while the joint posterior of $A \wedge B$ is ≈ 0.283.

Optional Content Ends

The situation cannot be improved by saying that what was meant was rather that the joint likelihood is going to be at least as high as the maximum of the individual likelihoods, because quite the opposite is the case: the joint likelihood is going to be lower than any of the individual ones.

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Let us make sure this is the case. We have:

$$\begin{aligned}
LR(a|A) &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)}\\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)}\\
&= \frac{\mathbf{a}}{\mathbf{1-a}}.
\end{aligned}$$

where the substitution in the denominator is legitimate only because witness' sensitivity is identical to their specificity.

With the joint likelihood, the reasoning is just a bit more tricky. We will need to know what $\mathsf{P}(a \wedge b|\neg(A \wedge B))$ is. There are three disjoint possible conditions in which the condition holds: $A \wedge \neg B, \neg A \wedge B$, and $\neg A \wedge \neg B$. The probabilities of $a \wedge b$ in these three scenarios are respectively $\mathbf{a}(\mathbf{1-b}), (\mathbf{1-a})\mathbf{b}, (\mathbf{1-a})(\mathbf{1-b})$ (again, the assumption of independence is important), and so on the assumption $\neg(A \wedge B)$ the probability of $a \wedge b$ is:

$$\begin{aligned}
\mathsf{P}(a \wedge b|\neg(A \wedge B)) &= \mathbf{a}(1-\mathbf{b}) + (1-\mathbf{a})\mathbf{b} + (1-\mathbf{a})(1-\mathbf{b})\\
&= \mathbf{a}(1-\mathbf{b}) + (1-\mathbf{a})(\mathbf{b} + 1 - \mathbf{b})\\
&= \mathbf{a}(1-\mathbf{b}) + (1-\mathbf{a})\\
&= \mathbf{a} - \mathbf{ab} + 1 - \mathbf{a} = 1 - \mathbf{ab}
\end{aligned}$$

So, on the assumption of witness independence, we have:

$$\begin{aligned}
LR(a \wedge b|A \wedge B) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}\\
&= \frac{\mathbf{ab}}{\mathbf{1-ab}}
\end{aligned}$$

With $0 < \mathbf{a}, \mathbf{b} < 1$ we have $\mathbf{ab} < \mathbf{a}$, $1 - \mathbf{ab} > 1 - \mathbf{a}$, and consequently:

$$\frac{\mathbf{ab}}{\mathbf{1-ab}} < \frac{\mathbf{a}}{\mathbf{1-a}}$$

which means that the joint likelihood is going to be lower than any of the individual ones.

Optional Content Ends

Fact 2 is so far the most optimistic reading of the claim that if witnesses are independent and fairly reliable, their testimonies are going to provide positive support for the conjunction,\footnote{And this is the reading that Dawid in passing suggests: "the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability." (Dawid, 1987: 95) and any stronger reading of Dawid's suggestions fails. But Fact 2 is not too exciting when it comes to answering the original DAC. The original question focused on the adjudication model according to which the deciding agents are to evaluate the posterior probability of the whole case conditional on all evidence, and to convict if it is above a certain threshold. The problem, generally, is that it might be the case that the pieces of evidence for particular elements of the claim can have high likelihood and posterior probabilities of particular elements can be above the threshold while the posterior joint probability will still fail to meet the threshold. The fact that the joint posterior will be higher than the joint prior does not help much. For instance, if $\mathbf{a} = \mathbf{b} = 0.7$, $pr = \langle 0.1, 0.5, 0.3, 0.1\rangle$, the posterior of A is ≈ 0.777, the posterior of B is ≈ 0.608 and the joint posterior is ≈ 0.216 (yes, it is higher than the joint prior $= 0.1$, but this does not help the conjunction to satisfy the decision standard).

To see the extent to which Dawid's strategy is helpful here, perhaps the following analogy might be useful.

Imagine it is winter, the heating does not work in my office and I am quite cold. I pick up the phone and call maintenance. A rather cheerful fellow picks up the phone. I tell him what my problem is, and he reacts:

— Oh, don't worry.
— What do you mean? It's cold in here!
— No no, everything is fine, don't worry.
— It's not fine! I'm cold here!
— Look, sir, my notion of it being warm in your office is that the building provides some improvement to what the situation would be if it wasn't there. And you agree that you're definitely warmer than you'd be if your desk was standing outside, don't you? Your, so to speak, posterior warmth is higher than your prior warmth, right?

Dawid's discussion is in the vein of the above conversation. In response to a problem with the adjudication model under consideration Dawid simply invites us to abandon thinking in terms of it and to abandon requirements crucial for the model. Instead, he puts forward a fairly weak notion of support (analogous to a fairly weak sense of the building providing improvement), according to which, assuming witnesses are fairly reliable, if separate fairly reliable witnesses provide positive support to the conjuncts, then their joint testimony provides positive support for the conjunction.

As far as our assessment of the original adjudication model and dealing with DAC, this leaves us hanging. Yes, if we abandon the model, DAC does not worry us anymore. But should we? And if we do, what should we change it to, if we do not want to be banished from the paradise of probabilistic methods?

Having said this, let me emphasize that Dawid's paper is important in the development of the debate, since it shifts focus on the likelihood ratios, which for various reasons are much better measures of evidential support provided by particular pieces of evidence than mere posterior probabilities.

Before we move to another attempt at a probabilistic formulation of the decision standard, let us introduce the other hero of our story: the gatecrasher paradox. It is against DAC and this paradox that the next model will be judged.

Optional Content Starts

In fact, Cohen replied to Dawid's paper (L. J. Cohen, 1988). His reply, however, does not have much to do with the workings of Dawid's strategy, and is rather unusual. Cohen's first point is that the calculations of posteriors require odds about unique events, whose meaning is usually given in terms of potential wagers – and the key criticism here is that in practice such wagers cannot be decided. This is not a convincing criticism, because the betting-odds interpretations of subjective probability do not require that on each occasion the bet should really be practically decidable. It rather invites one to imagine a possible situation in which the truth could be found out and asks: how much would we bet on a certain claim in such a situation? In some cases, this assumption is false, but there is nothing in principle wrong with thinking about the consequences of false assumptions.

Second, Cohen says that Dawid's argument works only for testimonial evidence, not for other types thereof. But this claim is simply false – just because Dawid used testimonial evidence as an example that he worked through it by no means follows that the approach cannot be extended. After all, as long as we can talk about sensitivity and specificity of a given piece of evidence, everything that Dawid said about testimonies can be repeated *mutatis mutandis*.

Third, Cohen complaints that Dawid in his example worked with rather high priors, which according to Cohen would be too high to correspond to the presumption of innocence. This also is not a very successful rejoinder. Cohen picked his priors in the example for the ease of calculations, and the reasoning can be run with lower priors. Moreover, instead of discussing the conjunction problem, Cohen brings in quite a different problem: how to probabilistically model the presumption of innocence, and what priors of guilt should be appropriate? This, indeed, is an important problem; but it does not have much to do with DAC, and should be discussed separately.

6.4 Problem's with Kaplow's stuff

Kaplow does not discuss the conceptual difficulties that we are concerned with, but this will not stop us from asking whether DTLP can handle them (and answering to the negative). Let us start with DAC.

Say we consider two claims, A and B. Is it generally the case that if they separately satisfy the decision rule, then so does $A \wedge B$? That is, do the assumptions:

$$\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} > \frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)} \times \frac{L}{G}$$
$$\frac{\mathsf{P}(E|B)}{\mathsf{P}(E|\neg B)} > \frac{\mathsf{P}(\neg B)}{\mathsf{P}(B)} \times \frac{L}{G}$$

entail

$$\frac{\mathsf{P}(E|A \wedge B)}{\mathsf{P}(E|\neg(A \wedge B))} > \frac{\mathsf{P}(\neg(A \wedge B))}{\mathsf{P}(A \wedge B)} \times \frac{L}{G}?$$

Alas, the answer is negative.

OPTIONAL CONTENT STARTS

This can be seen from the following example. Suppose a random digit from 0-9 is drawn; we do not know the result; we are told that the result is < 7 ($E =$'the result is < 7'), and we are to decide whether to accept the following claims:

A	the result is < 5.
B	the result is an even number.
$A \wedge B$	the result is an even number < 5.

Suppose that $L = G$ (this is for simplicity only — nothing hinges on this, counterexamples for when this condition fails are analogous). First, notice that A and B taken separately satisfy (12). $\mathsf{P}(A) = \mathsf{P}(\neg A) = 0.5$, $\mathsf{P}(\neg A)/\mathsf{P}(A) = 1$ $\mathsf{P}(E|A) = 1$, $\mathsf{P}(E|\neg A) = 0.4$. (12) tells us to check:

$$\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} > \frac{L}{G} \times \frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)}$$
$$\frac{1}{0.4} > 1$$

so, following DTLP, we should accept A.
For analogous reasons, we should also accept B. $\mathsf{P}(B) = \mathsf{P}(\neg B) = 0.5$, $\mathsf{P}(\neg B)/\mathsf{P}(B) = 1$ $\mathsf{P}(E|B) = 0.8$, $\mathsf{P}(E|\neg B) = 0.6$, so we need to check that indeed:

$$\frac{\mathsf{P}(E|B)}{\mathsf{P}(E|\neg B)} > \frac{L}{G} \times \frac{\mathsf{P}(\neg B)}{\mathsf{P}(B)}$$
$$\frac{0.8}{0.6} > 1$$

But now, $\mathsf{P}(A \wedge B) = 0.3$, $\mathsf{P}(\neg(A \wedge B)) = 0.7$, $\mathsf{P}(\neg(A \wedge B))/\mathsf{P}(A \wedge B) = 2\frac{1}{3}$, $\mathsf{P}(E|A \wedge B) = 1$, $\mathsf{P}(E|\neg(A \wedge B)) = 4/7$ and it is false that:

$$\frac{\mathsf{P}(E|A \wedge B)}{\mathsf{P}(E|\neg(A \wedge B))} > \frac{L}{G} \times \frac{\mathsf{P}(\neg(A \wedge B))}{\mathsf{P}(A \wedge B)}$$
$$\frac{7}{4} > \frac{7}{3}$$

The example was easy, but the conjuncts are probabilistically dependent. One might ask: are there counterexamples that involve claims which are probabilistically independent?[22]

[22]Thanks to Alicja Kowalewska for pressing me on this.

Consider an experiment in which someone tosses a six-sided die twice. Let the result of the first toss be X and the result of the second one Y. Your evidence is that the results of both tosses are greater than one ($E =: X > 1 \wedge Y > 1$). Now, let A say that $X < 5$ and B say that $Y < 5$.

The prior probability of A is $2/3$ and the prior probability of $\neg A$ is $1/3$ and so $\frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)} = 0.5$. Further, $\mathsf{P}(E|A) = 0.625$, $\mathsf{P}(E|\neg A) = 5/6$ and so $\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} = 0.75$ Clearly, $0.75 > 0.5$, so A satisfies the decision standard. Since the situation with B is symmetric, so does B.

Now, $\mathsf{P}(A \wedge B) = (2/3)^2 = 4/9$ and $\mathsf{P}(\neg(A \wedge B)) = 5/9$. So $\frac{\mathsf{P}(\neg(A \wedge B))}{\mathsf{P}(A \wedge B)} = 5/4$. Out of 16 outcomes for which $A \wedge B$ holds, E holds in 9, so $\mathsf{P}(E|A \wedge B) = 9/16$. Out of 20 remaining outcomes for which $A \wedge B$ fails, E holds in 16, so $\mathsf{P}(E|\neg(A \wedge B)) = 4/5$. Thus, $\frac{\mathsf{P}(E|A \wedge B)}{\mathsf{P}(E|\neg(A \wedge B))} = 45/64 < 5/4$, so the conjunction does not satisfy the decision standard.

Optional Content Ends

Let us turn to the gatecrasher paradox.

Suppose $L = G$ and recall our abbreviations: $\mathsf{P}(E) = e$, $\mathsf{P}(H_\Pi) = \pi$. DTLP tells us to convict just in case:

$$LR(E) > \frac{1-\pi}{\pi}$$

From (**??**) we already now that

$$LR(E) = \frac{0.991 - 0.991\pi}{0.009\pi}$$

so we need to see whether there are any $0 < \pi < 1$ for which

$$\frac{0.991 - 0.991\pi}{0.009\pi} > \frac{1-\pi}{\pi}$$

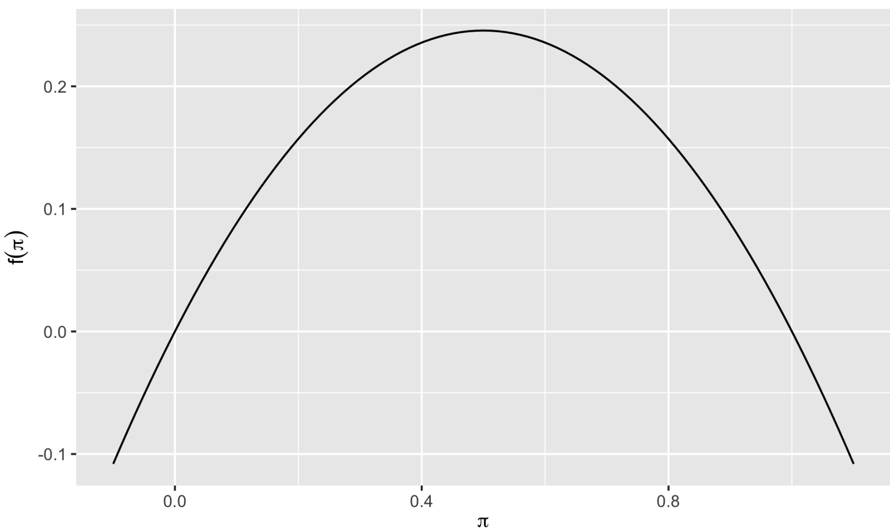
Multiply both sides first by 009π and then by π:

$$0.991\pi - 0.991\pi^2 > 0.09\pi - 0.009\pi^2$$

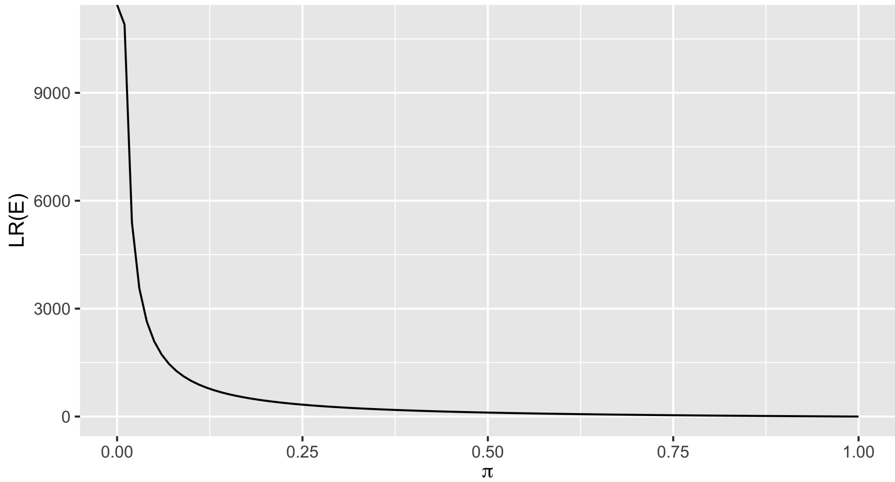
Simplify and call the resulting function f:

$$f(\pi) = -0.982\pi^2 + 0.982\pi > 0$$

The above condition is satisfied for any $0 < \pi < 1$ (f has two zeros: $\pi = 0$ and $\pi = 1$). Here is a plot of f:



Similarly, $LR(E) > 1$ for any $0 < \pi < 1$. Here is a plot of $LR(E)$ against π:



Notice that $LR(E)$ does not go below 1. This means that for $L = G$ in the gatecrasher scenario DTLP wold tell us to convict for any prior probability of guilt $\pi \neq 0, 1$.

One might ask: is the conclusion very sensitive to the choice of L and G? The answer is, not too much.

Optional Content Starts

How sensitive is our analysis to the choice of L/G? Well, $LR(E)$ does not change at all, only the threshold moves. For instance, if $L/G = 4$, instead of f we end up with

$$f'(\pi) = -0.955\pi^2 + 0.955\pi > 0$$

and the function still takes positive values on the interval $(0, 1)$. In fact, the decision won't change until L/G increases to ≈ 111. Denote L/G as ρ, and let us start with the general decision standard, plugging in our calculations for $LR(E)$:

$$\begin{aligned}
LR(E) &> \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)}\rho \\
LR(E) &> \frac{1-\pi}{\pi}\rho \\
\frac{0.991 - 0.991\pi}{0.009\pi} &> \frac{1-\pi}{\pi}\rho \\
\frac{0.991 - 0.991\pi}{0.009\pi}\frac{\pi}{1-\pi} &> \rho \\
\frac{0.991\pi - 0.991\pi^2}{0.009\pi - 0.009\pi^2} &> \rho \\
\frac{\pi(0.991 - 0.991\pi)}{\pi(0.009 - 0.009\pi)} &> \rho \\
\frac{0.991 - 0.991\pi}{0.009 - 0.009\pi} &> \rho \\
\frac{0.991(1-\pi)}{0.009(1-\pi)} &> \rho \\
\frac{0.991}{0.009} &> \rho \\
110.1111 &> \rho
\end{aligned}$$

Optional Content Ends

So, we conclude, in usual circumstances, DTLP does not handle the gatecrasher paradox.

6.5 Conclusions

Where are we, how did we get here, and where can we go from here? We were looking for a probabilistically explicated condition Ψ such that the trier of fact, at least ideally, should accept any relevant claim (including G) just in case $\Psi(A, E)$.

From the discussion that transpired it should be clear that we were looking for a Ψ satisfying the following desiderata:

conjunction closure If $\Psi(A, E)$ and $\Psi(B, E)$, then $\Psi(A \wedge B, E)$.

naked statistics The account should at least make it possible for convictions based on strong, but naked statistical evidence to be unjustified.

equal treatment the condition should apply to any relevant claim whatsoever (and not just a selected claim, such as G).

Throughout the paper we focused on the first two conditions (formulated in terms of the difficulty about conjunction (DAC), and the gatecrasher paradox), going over various proposals of what Ψ should be like and evaluating how they fare. The results can be summed up in the following table:

View	Convict iff	DAC	Gatecrasher
Threshold-based LP (TLP)	Probability of guilt given the evidence is above a certain threshold	fails	fails
Dawid's likelihood strategy	No condition given, focus on $\frac{\mathsf{P}(H\vert E)}{\mathsf{P}(H\vert\neg E)}$	- If evidence is fairly reliable, the posterior of $A \wedge B$ will be greater than the prior. - The posterior of $A \wedge B$ can still be lower than the posterior of any of A and B. - Joint likelihood, contrary do Dawid's claim, can also be lower than any of the individual likelihoods.	fails
Cheng's relative LP (RLP)	Posterior of guilt higher than the posterior of any of the defending narrations	The solution assumes equal costs of errors and independence of A and B conditional on E. It also relies on there being multiple defending scenarios individualized in terms of combinations of literals involving A and B.	Assumes that the prior odds of guilt are 1, and that the statistics is not sensitive to guilt (which is dubious). If the latter fails, tells to convict as long as the prior of guilt < 0.991.
Kaplow's decision-theoretic LP (DTLP)	The likelihood of the evidence is higher than the odds of innocence multiplied by the cost of error ratio	fails	convict if cost ratio < 110.1111

Thus, each account either simply fails to satisfy the desiderata, or succeeds on rather unrealistic assumptions. Does this mean that a probabilistic approach to legal evidence evaluation should be abandoned? No. This only means that if we are to develop a general probabilistic model of legal decision standards, we have to do better. One promising direction is to go back to Cohen's pressure against **Requirement 1** and push against it. A brief paper suggesting this direction is (Di Bello, 2019), where the idea is that the probabilistic standard (be it a threshold or a comparative wrt. defending narrations) should be applied to the whole claim put forward by the plaintiff, and not to its elements. In such a context, DAC does not arise, but **equal treatment** is violated. Perhaps, there are independent reasons to abandon it, but the issue deserves further discussion. Another strategy might be to go in the direction of employing probabilistic methods to explicate the narration theory of legal decision standards (Urbaniak, 2018), but a discussion of how this approach relates to DAC and the gatecrasher paradox lies beyond the scope of this paper.

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