Appendix to ‘The Difficulty with Conjuction’: Proofs and Simulations

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This appendix briefly describes the formalism of Bayesian networks, applies it to the difficulty with conjunction and then establishes several results about *joint* posteriors, Bayes factors and likelihood ratios. We provide analytic proofs if possible. We rely on computer simulations whenever analytic proofs become unmanageable.[1] The simulations were conducted as follows. Given a direct acyclic graph (DAG) of interest (to be defined later), 100,000 random Bayesian networks were generated whose probability measures had values sampled from the $\mathsf{Uniform}(0,1)$ distribution. For each of these networks, we calculated the relevant posterior probabilities, Bayes factors, and likelihood ratios.

added short summary here

We prove that for if the stronger independence assumptions hold, the joint Bayes factor is just the result of multiplying the individual Bayes factors. It follows that aggregation is satisfied in such cases, if individual Bayes factors are greater than one. Once the hypotheses are not independent, a weaker result can be obtained, which entails that the aggregation is satisfied for the Bayes factor, if a certain additional constraint is satisfied. In general, aggregation fails 25% of the time if the individual Bayes factors are not constrained to be greater than one, but holds once this constraint is added. Simulations reveal that there is a large class of cases in which individual Bayes factors are above one, aggregation is satisfied, but distribution fails. As for likelihood ratios, without any constraint on individual likelihood ratios, aggregation fails in 12.5% cases (twice less often than aggregation for the Bayes factor). Assuming likelihood ratios above one, around 70% of joint likelihood ratios falls between the individual ratios, and is no lower than the smaller of these. Thus, aggregation is satisfied if individual likelihood ratios equal at least one, but it no longer holds that the joint support is greater than any of the individual support levels. Still, distribution fails in a large class of cases. Finally, we identify cases in which aggregation can fail even if the individual BFs or LRs are at least one: this can happen if there is a direct dependence between the pieces of evidence.

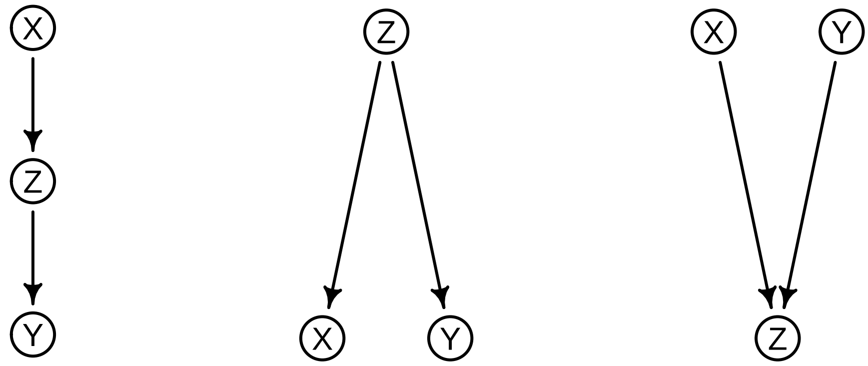
Bayesian networks and probabilistic independence

A Bayesian network consists of a graphical part—a directed acyclic graph (DAG)—and a probability measure defined over the nodes (variables) in the graph. A Bayesian network satisfies the Markov condition. That is, any node is conditionally independent of its nondescentants (including ancestors), given its parents. If a probability measure $\mathsf{P}()$ that is defined over the nodes (variables) in a graph G respects the Markov condition, $\mathsf{P}()$ is said to be compatible with G. Graph G and measure $\mathsf{P}()$ can then be combined to form a Bayesian network.

The graphical counterpart of probabilistic independence is **d-separation**, $\perp\!\!\!\perp_d$. Two nodes, X and Y, are d-separated given a set of nodes Z—$X \perp\!\!\!\perp_d Y|\mathsf{Z}$ — iff for every undirected path from X to Y there is a node Z' on the path such that either (see Figure 1):

- $Z' \in \mathsf{Z}$ and there is a **serial** connection, $\rightarrow Z' \rightarrow$, on the path (**pipe**),
- $Z' \in \mathsf{Z}$ and there is a **diverging** connection, $\leftarrow Z' \rightarrow$, on the path (**fork**),
- There is a **converging** connection $\rightarrow Z' \leftarrow$ on the path (in which case Z' is a **collider**), and neither Z' nor its descendants are in Z.

[1] The R code for the simulations, calculations and visualizations is available on the book website LINK TO DOCUMENTATION TO BE ADDED LATER.



(a) Serial: Pipe (b) Diverging: Fork (c) Converging:Collider

Figure 1: Three basic types of connections.

Serial, converging and diverging connections represent common scenarios.

Two sets of nodes, X and Y, are d-separated given Z if every node in X is d-separated from every node in Y given Z. Interestingly, it can be proven that if two sets of nodes are d-separated given a third one, they are independent given the third one, for any probabilistic measures compatible with a given DAG. However, lack of d-separation does not necessarily entail dependence for any probabilistic measure compatible with a given DAG. It only allows for it: if nodes are d-separated, there is at least one probabilistic measure fitting the DAG according to which they are dependent. So, at least, no false independencies can be inferred from the DAG, and all the dependencies are built into it.[2]

Independencies in the conjunction problem

An assumption often made in the formulation of the difficulty with conjunction is that hypotheses A and B are probabilistically independent. The theory of Bayesian networks helps to formulate this assumption precisely. The independence of the hypotheses is represented formally by DAG1 in Figure 2. The

[2]For an intuitive feel on how these types of connections should relate to independence, consider the following examples. Take the nodes:

Node	Proposition
G	The suspect is guilty.
B	The blood stain comes from the suspect.
M	The crime scene stain and the suspect's blood share the same DNA profile.

This scenarios is naturally represented by the serial connection $G \rightarrow B \rightarrow M$. If we don't know whether B holds, G has an indirect impact on the probability of M. Yet, once we find out that B is true, we expect M to be true, and whether G holds has no further impact on the probability of M.

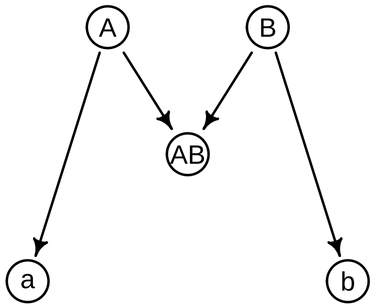
For converging connections, let G and B be as above, and let:

Node	Proposition
O	The crime scene stain comes from the offender.

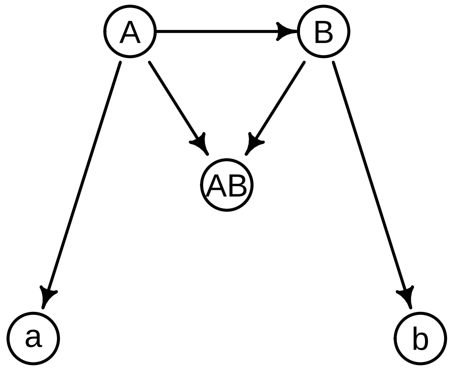
Both G and O influence B. If suspect guilty, it is more likely that the blood stain comes from him, and if the blood crime stain comes from the offender it is more likely to come from the suspect (for instance, more so than if it comes from the victim). Moreover, G and O seem independent. Whether the suspect is guilty does not have any bearing on whether the stain comes from the offender. Thus, a converging connection $G \rightarrow B \leftarrow O$ seems appropriate. However, if you do find out that B is true—that the stain comes from the suspect—then whether the crime stain comes from the offender becomes relevant for whether the suspect is guilty.

Take an example of a diverging connection. Say you have two coins, one fair, one biased. Conditional on which coin you have chosen, the results of subsequent tosses are independent. But if you don't know which coin you have chosen, the result of previous tosses give you some information about which coin it is, and this has an impact on your estimate of the probability of heads in the next toss. Whether a coin is fair, F, or not has an impact on the result of the first toss, $H1$, and on the result of the second toss, $H2$. So $H1 \leftarrow F \rightarrow H2$ seems to be appropriate. Now, on one hand, so long as you do not know whether F, the truth of $H1$ increases the probability of $H2$. On the other hand, once you know that F is true, $H1$ and $H2$ become independent, and so conditioning on the parent in a fork makes its children independent (provided there is no other open path between them in the graph).

graph contains nodes for the hypotheses A and B, the supporting evidence a and b, and the conjunctive hypothesis $A \wedge B$. The conjunction node AB is a collider, which guarantees the independence of A and B.[3] To make our discussion more general, we will also consider graph DAG2. This graph relaxes the assumption of independence by drawing an additional arrow between hypotheses A and B.



(a) DAG1



(b) DAG2

Figure 2: Two DAGs for the conjunction problem.

AB	A	B	Pr
1	1	1	1
0	1	1	0
1	0	1	0
0	0	1	1
1	1	0	0
0	1	0	1
1	0	0	0
0	0	0	1

Table 1: Conditional probability table for the conjunction node.

[3] In the Bayesian networks in this appendix, the conditional probability table for the conjunction node AB mirrors the truth table for the conjunction in propositional logic, as in Table 1.

Unsurprisingly, the relations of d-separation entailed by the two networks differ. Examples can be found in Table 2. In fact, DAG1 entails 31 d-separations, while DAG2 entails 22 of them. One warning about the notation. Nodes represent variables, and so each d-separation entails a probabilistic statement about all combination of the node states (values of variables) involved. For instance, assuming each node is binary with two possible states, 1 and 0, $B \perp\!\!\!\perp_d a$ entails that for any $B_i, a_i \in \{0,1\}$ we have $\mathsf{P}(B = B_i) = \mathsf{P}(B = B_i | a = a_i)$.

Bayesian network 1	Bayesian network 2
$A \perp\!\!\!\perp_d B$	$A \perp\!\!\!\perp_d b \vert B$
$A \perp\!\!\!\perp_d b$	$AB \perp\!\!\!\perp_d a \vert A$
$AB \perp\!\!\!\perp_d a \vert A$	$AB \perp\!\!\!\perp_d b \vert B$
$AB \perp\!\!\!\perp_d b \vert B$	$B \perp\!\!\!\perp_d a \vert A$
$B \perp\!\!\!\perp_d a$	$a \perp\!\!\!\perp_d b \vert B$
$a \perp\!\!\!\perp_d b$	$a \perp\!\!\!\perp_d b \vert A$

Table 2: Some of d-separations entailed by DAG1 and DAG2.

Turning from nodes to states (or events, propositions), Figure 3 lists the independencies between propositions.[4] It also shows which independencies are entailed by either of the two DAGs.

$A \perp\!\!\!\perp B$	DAG1	(1)	$a \perp\!\!\!\perp B \vert A$	DAG1 , DAG2	(10)
$b \perp\!\!\!\perp a$	DAG1	(2)	$a \perp\!\!\!\perp B \vert \neg A$	DAG1 , DAG2	(11)
$A \perp\!\!\!\perp b \vert a$	DAG1	(3)	$a \perp\!\!\!\perp \neg B \vert A$	DAG1 , DAG2	(12)
$B \perp\!\!\!\perp a \wedge A \vert b$	DAG1	(4)	$a \perp\!\!\!\perp \neg B \vert \neg A$	DAG1 , DAG2	(13)
$a \perp\!\!\!\perp b \vert A \wedge B$	DAG1 , DAG2	(5)	$b \perp\!\!\!\perp A \wedge a \vert B$	DAG1 , DAG2	(14)
$a \perp\!\!\!\perp b \vert A$	DAG1 , DAG2	(6)	$b \perp\!\!\!\perp \neg A \wedge a \vert B$	DAG1 , DAG2	(15)
$a \perp\!\!\!\perp b \vert \neg A$	DAG1 , DAG2	(7)	$b \perp\!\!\!\perp A \wedge a \vert \neg B$	DAG1 , DAG2	(16)
$a \perp\!\!\!\perp b \vert B$	DAG1 , DAG2	(8)	$b \perp\!\!\!\perp \neg A \wedge a \vert \neg B$	DAG1 , DAG2	(17)
$a \perp\!\!\!\perp b \vert \neg B$	DAG1 , DAG2	(9)	$b \perp\!\!\!\perp a \vert B$	DAG1 , DAG2	(18)

Figure 3: Independencies among propositions according to DAG1 and DAG2.

An ambiguity in our notation is worth mentioning. Table 2 lists independencies between *nodes*. But Figure 3 is about *states* rather than nodes. An expression such as $b \perp\!\!\!\perp A \wedge a | \neg B$ should be understood as a claim about states (events, propositions), which means the same as $\mathsf{P}(b = 1 | B = 0) = \mathsf{P}(b = 1 | A = 1, a = 1, B = 0)$. The distinction between nodes and their states (or variables and their values) matters because independence conditional on $B = 0$ doesn't entail independence given $B = 1$. For instance, one's final grade might depend on hard work if the teacher is fair, but this might fail if the teacher is not fair. We hope this ambiguity in notation will cause no confusion. Whether we talk about nodes or states (or events, propositions) should be clear from the context.

Posterior probabilities

We first examine how posterior probabilities behave in the conjunction problem. The joint posterior $\mathsf{P}(A \wedge B | a \wedge b)$ is often lower than the individual posterior $\mathsf{P}(A|a)$ and $\mathsf{P}(B|b)$, whether the hypotheses A and B are independent or not. We establish this fact via a computer simulation. We simulated 10,000 random Bayesian networks based on DAG1 (independent hypotheses) and DAG2 (dependent hypotheses). If any such network has an equal probability of occurring, the joint posterior is lower than

[4] Some caveats. In (3) the conditioning on a is not essential, because it's not on the path between the nodes: the key reason why the independence remains upon this conditioning is that there is an unconditioned collider on the path. Still, we need this independence in the proof later on. In (5) what we are conditioning on is A and B jointly. Technically, independence conditional on the conjunction node AB does not fall out of the d-separations present in the network—it follows given that AB and A, B are connected deterministically: fixing AB to true fixes both A and B to true.

both individual posteriors 68% of the time for DAG1, and around 60% for DAG2. Figure 4 displays the distributions of the distances of the joint posterior from the lowest of the individual posteriors.

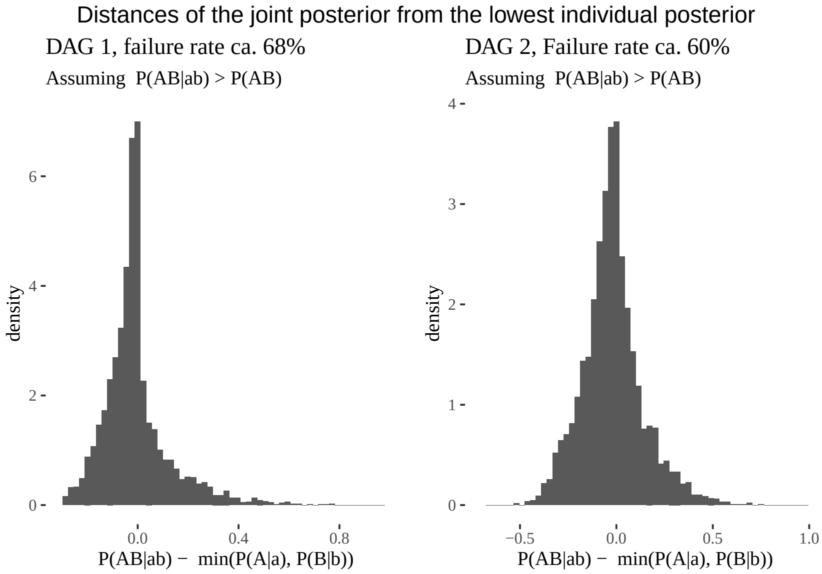


Figure 4: Even assuming the joint support is positive, the joint posterior often is lower than individual posteriors.

Bayes factor: proofs

Next, we turn to the Bayes factor. For ease of reference, we use the following abbreviations:

$$BF_A = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)},$$
$$BF_B = \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)},$$
$$BF_{AB} = \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)}$$

The objective here is to study how the combined support BF_{AB} compares to the individual supports BF_A and BF_B. We prove the following general theorem:

Theorem 1. *Given a measure* $\mathsf{P}()$ *compatible with* DAG1*, if both* BF_A *and* BF_B *are grater than one, then* $BF_{AB} \geq \max(BF_A, BF_B)$*. The same holds for any measure compatible with* DAG2*.*

In other words, the combined support BF_{AB} is never below the individual supports BF_A and BF_B, whether claims A and B are independent (DAG1) or not (DAG2).

Proof. For DAG1, the theorem holds by Fact 1 (and corollary). For DAG2, the theorem holds by Fact 2 (and corollary), Lemma 1, and Fact 3. □

Fact 1. *If the independence assumptions (1), (2), (10) and (14) hold (all of which are entailed by DAG1), then* $BF_{AB} = BF_A \times BF_B$*.*

Proof.

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(A \wedge B \wedge a \wedge b)}{\mathsf{P}(A \wedge B)} \Big/ \mathsf{P}(a \wedge b) \quad \text{(conditional probability)}$$

$$= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A \wedge B) \times \mathsf{P}(b|A \wedge B \wedge a)}{\mathsf{P}(A \wedge B)} \Big/ \mathsf{P}(a \wedge b) \quad \text{(chain rule)}$$

Next, we apply the relevant independence assumptions:

$$
\begin{aligned}
&= \frac{\mathsf{P}(A) \times \overbrace{\mathsf{P}(B|A)}^{\mathsf{P}(B)\text{ by }(1)} \times \overbrace{\mathsf{P}(a|A \wedge B)}^{\mathsf{P}(a|A)\text{ by }(10)} \times \overbrace{\mathsf{P}(b|A \wedge B \wedge a)}^{\mathsf{P}(b|B)\text{ by }(14)}}{\underbrace{\mathsf{P}(A \wedge B)}_{\mathsf{P}(A)\times\mathsf{P}(B)\text{ by }(1)}} \Bigg/ \underbrace{\mathsf{P}(a \wedge b)}_{\mathsf{P}(a)\times\mathsf{P}(b)\text{ by }(2)} \\
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B) \times \mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\mathsf{P}(A) \times \mathsf{P}(B)} \Big/ \mathsf{P}(a) \times \mathsf{P}(b) \\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} \\
&= BF_A \times BF_B
\end{aligned}
$$

□

This fact has the following straightforward consequences. They hold because, if $a = b \times c$ and $b, c > 1$, then $a > \max(b, c)$, and if $b, c < 1$, then $a < \min(b, c)$.

Corollary 1.1. *If the independence assumptions (1), (2), (10) and (14) hold, and BF_A and BF_B are both greater than 1, then BF_{AB} is greater than one. In fact, BF_{AB} is greater than $\max(BF_A, BF_B)$.*

Corollary 1.2. *If the independence assumptions (1), (2), (10) and (14) hold, and BF_A and BF_B are both strictly less than 1, then BF_{AB} is less than $\min(BF_A, BF_B)$.*

This establishes Theorem 1 for DAG 1. After dropping the independence assumptions specific to DAG1 and shifting to DAG 2, the combined BF_{AB} can no longer be obtained by multiplying the individual ones, although multiplication still provides a decent approximation (see Figure 5).

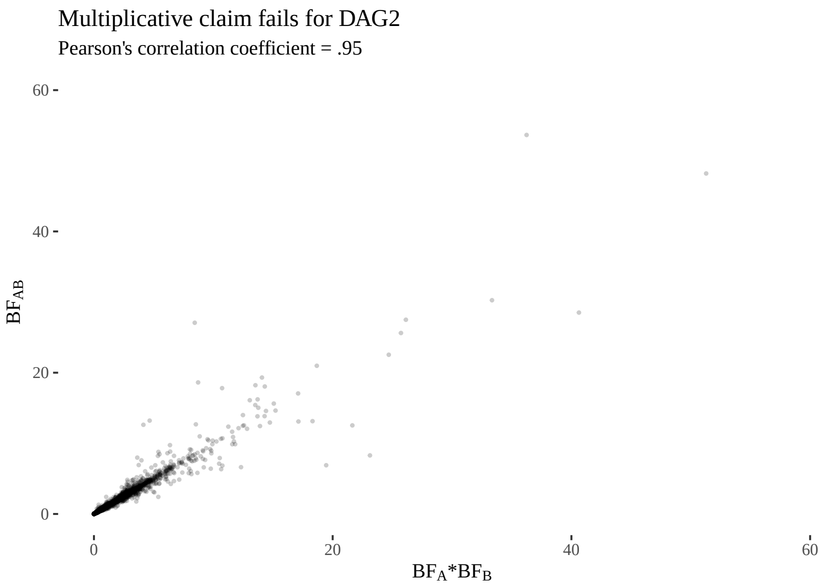


Figure 5: In DAG2, the result of multiplying individual BFs does not equal the joint BF, but often is a good approximation thereof. Axes restricted to 0, 60 (one extreme outlier lying close to the diagonal dropped).

However, if the probabilistic measure fits DAG 2, Theorem 1 is still satisfied. First, we abbreviate:

$$BF_B' = \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}$$

$$BF_A^{'} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|b)}$$

A claim weaker than Fact 1 can be proven by relying only on the independencies entailed by DAG2.

Fact 2. *If* (10) *and* (14) *hold (and they do in BNs based on DAG 2), then* $BF_{AB} = BF_A \times BF_B^{'} = BF_B \times BF_A^{'}$.

Proof. We start with the definition of conditional probability and the chain rule, as in the proof of Fact 1, but now we use fewer independencies (all of them entailed by DAG2).

$$\begin{aligned}
\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} &= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \overbrace{\mathsf{P}(a|A \wedge B)}^{\mathsf{P}(a|A) \text{ by } (10)} \times \overbrace{\mathsf{P}(b|A \wedge B \wedge a)}^{\mathsf{P}(b|B) \text{ by } (14)}}{\underbrace{\mathsf{P}(A \wedge B)}_{\mathsf{P}(A)\times\mathsf{P}(B|A) \text{ by the chain rule}}} \Bigg/ \underbrace{\mathsf{P}(a \wedge b)}_{\mathsf{P}(a)\times\mathsf{P}(b|a) \text{ by the chain rule}} \\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} \\
&= BF_A \times BF_B^{'}
\end{aligned}$$

If, instead of obtaining $\mathsf{P}(a)\mathsf{P}(b|a)$ in the denominator, we deploy the chain rule differently, resulting in $\mathsf{P}(b)\mathsf{P}(a|b)$, we end up with:

$$\begin{aligned}
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|b)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} \\
&= BF_A^{'} \times BF_B
\end{aligned}$$

□

Corollary 2.1. *Suppose* (10) *and* (14) *hold (they are entailed by DAG 2), and* $BF_{BA}, BF_B > 1$. *Then if both* $\mathsf{P}(a|b) \leq \mathsf{P}(a|A)$ *and* $\mathsf{P}(b|a) \leq \mathsf{P}(b|B)$*, we have* $BF_{AB} \geq BF_A, BF_B$.

Proof. Assume the first conjunct holds. Then $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|b)} \geq 1$ and so:

$$BF_{AB} = BF_A^{'} \times BF_B \geq BF_B$$

The argument for the other comparison is analogous. □

The proof of Theorem 1 for DAG 2 is not complete yet. This corollary relies on the additional assumptions $\mathsf{P}(a|b) \leq \mathsf{P}(a|A)$ and $\mathsf{P}(b|a) \leq \mathsf{P}(b|B)$. They seem plausible. If, say, a is used as evidence for A, we often expect A and a to be fairly strongly connected, that is, we expect $\mathsf{P}(a|A)$ to be rather high, while the connection between different pieces of evidence for different hypotheses, intuitively, is not expected to be as strong. We provide a proof of these assumptions below.

We start with the following lemma.

Lemma 1. *For any probabilistic measure* P*, if* $BF_A > 1$*, then* $LR_A > 1$.

Proof. We start with our assumption.

$$\begin{aligned}
1 &\leq \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} && (BF_A \geq 1) \\
\mathsf{P}(A) &\leq \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}\mathsf{P}(A) && \text{(algebraic manipulation)} \\
\mathsf{P}(A) &\leq \mathsf{P}(A|a) && \text{(Bayes' theorem)} \\
-\mathsf{P}(A) &\geq -\mathsf{P}(A|a) && \text{(algebraic manipulation)} \\
1-\mathsf{P}(A) &\geq 1-\mathsf{P}(A|a) && \text{(algebraic manipulation)}
\end{aligned}$$

$$
\begin{aligned}
1-\mathsf{P}(A) &\geq \mathsf{P}(\neg A|a) && \text{(algebraic manipulation)}\\
\mathsf{P}(a)\,(1-\mathsf{P}(A)) &\geq \mathsf{P}(a)\mathsf{P}(\neg A|a) && \text{(algebraic manipulation)}\\
\mathsf{P}(a) &\geq \frac{\mathsf{P}(a)\mathsf{P}(\neg A|a)}{\mathsf{P}(\neg A)} && \text{(algebraic manipulation, negation)}\\
\mathsf{P}(a) &\geq \mathsf{P}(a|\neg A) && \text{(conditional probability)}
\end{aligned}
$$

From this and our assumption that $\mathsf{P}(a|A) \geq \mathsf{P}(a)$ it follows that $\mathsf{P}(a|A) \geq \mathsf{P}(a|\neg A)$, that is, that $LR_A \geq 1$. □

Now the main claim.

Fact 3. *For any probabilistic measure* P *appropriate for* DAG 2*, if* $BF_A > 1$*, then* $\mathsf{P}(a|A) \geq \mathsf{P}(a|b)$ *and* $\mathsf{P}(b|B) \geq \mathsf{P}(b|a)$.

Proof. Let's focus on the first conjunct. First, we have:

$$
\begin{aligned}
\mathsf{P}(a|b) &= \mathsf{P}(a\wedge A|b) + \mathsf{P}(a\wedge\neg A|b) && \text{(total probability)}\\
&= \underbrace{\mathsf{P}(a|b\wedge A)}_{\mathsf{P}(a|A)\ \text{by (6)}}\mathsf{P}(A|b) + \underbrace{\mathsf{P}(a|b\wedge\neg A)}_{\mathsf{P}(a|\neg A)\ \text{by (7)}}\mathsf{P}(\neg A|b) && \text{(chain rule)}
\end{aligned}
$$

Now let's introduce some abbreviations:

$$
= \underbrace{\mathsf{P}(a|A)}_{k}\underbrace{\mathsf{P}(A|b)}_{x} + \underbrace{\mathsf{P}(a|\neg A)}_{t}\underbrace{\mathsf{P}(\neg A|b)}_{(1-x)}
$$

Note that the assumption that $BF_A \geq 1$ entails, by Lemma 1, that $k \geq t$, and so $k-t \geq 0$. Also, since x is a probability, we know $0 \leq x \leq 1$. This allows us to reason algebraically as follows:

$$
\begin{aligned}
k &\geq k\\
k &\geq t + (k-t)\\
k &\geq t + (k-t)x\\
k &\geq kx + t - tx\\
\mathsf{P}(a|A) = k &\geq kx + t(1-x) = \mathsf{P}(a|b)
\end{aligned}
$$

For the second conjunct, notice that we have a similar reasoning, albeit it relies on a different pair of independencies (which nevertheless holds in DAG1 and DAG 2).

$$
\begin{aligned}
\mathsf{P}(b|a) &= \mathsf{P}(b\wedge B|a) + \mathsf{P}(b\wedge\neg B|a) && \text{(total probability)}\\
&= \underbrace{\mathsf{P}(b|a\wedge B)}_{\mathsf{P}(b|B)\ \text{by (8)}}\mathsf{P}(B|a) + \underbrace{\mathsf{P}(b|a\wedge\neg B)}_{\mathsf{P}(b|\neg B)\ \text{by (9)}}\mathsf{P}(\neg B|a) && \text{(chain rule)}
\end{aligned}
$$

The rest of the reasoning for this case is algebraically the same as the one used above. □

Bayes factor: simulations

Computer simulations provide additional insights. The joint BF_{AB} may be lower than the individual BF_A and BF_B. Simulated cases in which $BF_{AB} < BF_A, BF_B$ are about 25% of the total (which is twice higher than for the likelihood ratio; more on this later). Such cases are displayed in Figure 6.

This result is nothing surprising. It happens when BF_A and BF_B are lower than one. When they are greater than one, the joint BF_{AB} exceeds the individual ones. The distribution of Bayes factors based

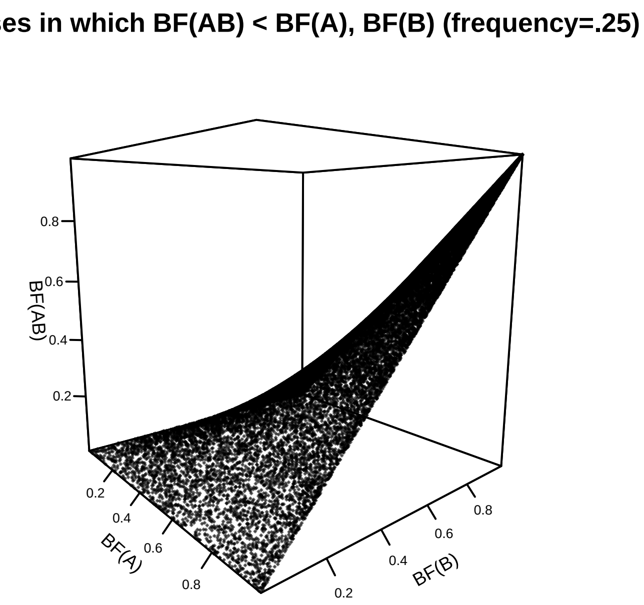


Figure 6: 25k cases (out of simulated 100k) in which the joint BF is below each of the individual BFs.

on DAG1 and DAG2 are displayed in Figure 7 and Figure 8. The distribution is unchanged in the two cases.

These simulations and the earlier theorem demonstrate that, whenever individual BF_A and BF_B are above a fixed threshold, so is the combined BF_{AB}. This fact justifies the principle of aggregation, as explained in the main text of the chapter. But the converse does not hold. Even when BF_{AB} is above a threshold, BF_A or BF_B may be below the threshold. Such cases for DAG1 and DAG2 are displayed in Figure 9. Hence, the converse of aggregation, the principle of distribution, fails.

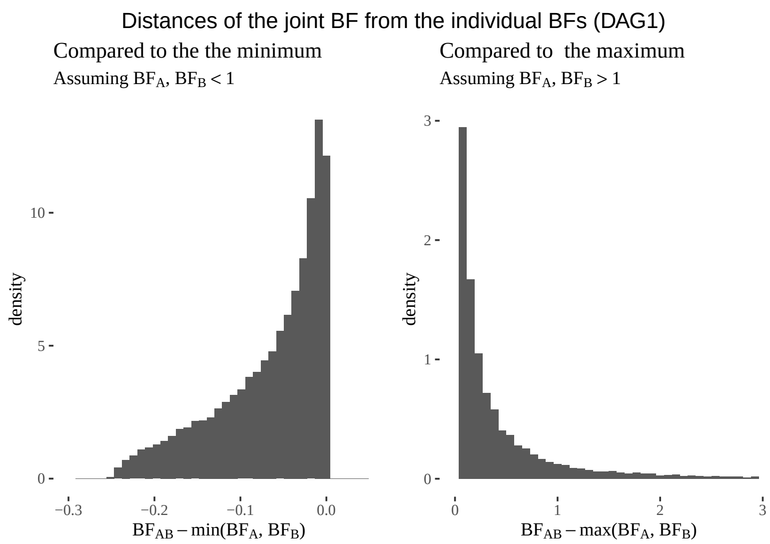


Figure 7: Distances of the joint Bayes factor from maxima and minima of individual Bayes factors, depending on whether the individual support levels are both positive or both negative. Simulation based on 100k Bayesian networks build over DAG1.

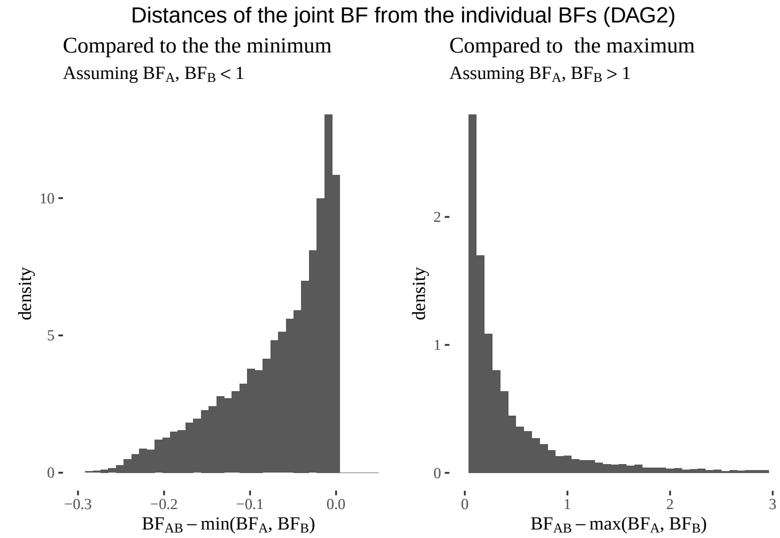


Figure 8: Distances of the joint Bayes factor from maxima and minima of individual Bayes factors, depending on whether the individual support levels are both positive or both negative. Simulation based on 100k Bayesian networks build over DAG2.

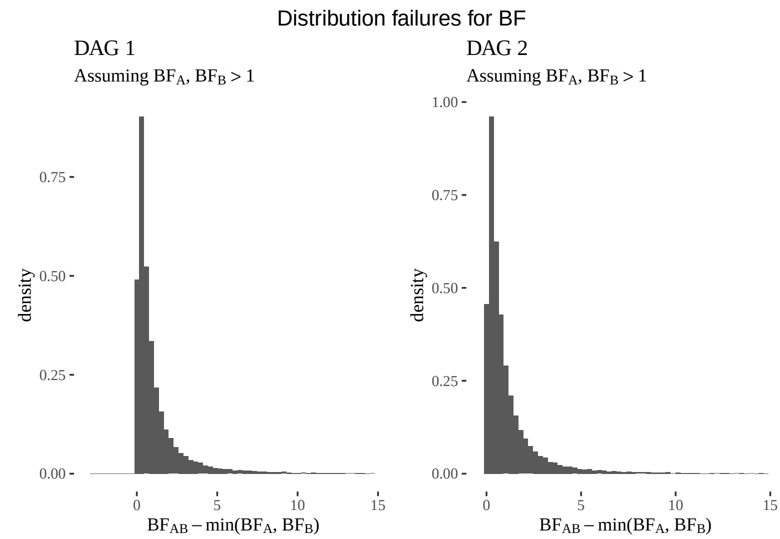


Figure 9: Distribution failure for the Bayes factor, DAG 1. The x axis restricted to $(-3, 15)$ for visibility.

Likelihood ratio: proofs

We now turn to the likelihood ratio. For ease of reference, we use the following abbreviations:

$$
\begin{aligned}
LR_{AB} &= \frac{\mathsf{P}(a \wedge b | a \wedge B)}{\mathsf{P}(a \wedge b | \neg(A \wedge B))}\\
LR_{A} &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)}\\
LR_{B} &= \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|\neg B)}.
\end{aligned}
$$

Fact 4. *If independence conditions* (10)*,* (11)*,* (12)*,* (13)*,* (14)*,* (15)*,* (16)*, and* (17) *hold, then:*

$$
LR_{AB} = \frac{\mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) + \mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A) + \mathsf{P}(A)\mathsf{P}(\neg B|A) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}}
$$

Note that these independence assumptions are entailed not only in DAG1, but also in DAG2.

Proof. Let's first compute the numerator of LR_{AB}:

$$
\begin{aligned}
\mathsf{P}(a \wedge b | A \wedge B) &= \frac{\mathsf{P}(A \wedge B \wedge a \wedge b)}{\mathsf{P}(A \wedge B)} && \text{(conditional probability)}\\
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A \wedge B) \times \mathsf{P}(b|A \wedge B \wedge a)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} && \text{(chain rule)}
\end{aligned}
$$

We deploy the relevant independencies as follows:

$$
\begin{aligned}
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \overbrace{\mathsf{P}(a|A \wedge B)}^{\mathsf{P}(a|A) \text{ by (10)}} \times \overbrace{\mathsf{P}(b|A \wedge B \wedge a)}^{\mathsf{P}(b|B) \text{ by (14)}}}{\mathsf{P}(A) \times \mathsf{P}(B|A)}\\
&= \mathsf{P}(a|A) \times \mathsf{P}(b|B) && \text{(algebraic manipulation)}
\end{aligned}
$$

The denominator of LR_{AB} is more complicated, mostly because of the conditioning on $\neg(A \wedge B)$.

$$
\begin{aligned}
\mathsf{P}(a \wedge b | \neg(A \wedge B)) &= \frac{\mathsf{P}(a \wedge b \wedge \neg(A \wedge B))}{\mathsf{P}(\neg(A \wedge B))} && \text{(conditional probability)}\\
&= \frac{\mathsf{P}(a \wedge b \wedge \neg A \wedge B) + \mathsf{P}(a \wedge b \wedge A \wedge \neg B) + \mathsf{P}(a \wedge b \wedge \neg A \wedge \neg B)}{\mathsf{P}(\neg A \wedge B) + \mathsf{P}(A \wedge \neg B) + \mathsf{P}(\neg A \wedge \neg B)} && \text{(logic \& additivity)}
\end{aligned}
$$

Now consider the first summand from the numerator:

$$
\begin{aligned}
\mathsf{P}(a \wedge b \wedge \neg A \wedge B) &= \mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A \wedge B)\mathsf{P}(b|a \wedge \neg A \wedge B) && \text{(chain rule)}\\
&= \mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) && \text{(independencies (11) and (15))}
\end{aligned}
$$

The simplification of the other two summanda is analogous (albeit with slightly different independence assumptions—(12) and (16) for the second one and (13) and (17) for the third. Once we plug these into the denominator formula we get:

$$
\begin{aligned}
\mathsf{P}(a \wedge b | \neg(A \wedge B)) &= \frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) + \mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A) + \mathsf{P}(A)\mathsf{P}(\neg B|A) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}\\
&= \frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) + \mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A) + \mathsf{P}(A)\mathsf{P}(\neg B|A) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}
\end{aligned}
$$

$\square$

Likelihod ratio: simulations

Here the analytic approach is cumbersome. Instead, we mostly rely on computer simulations. First of all, the joint LR_{AB} can be lower than any of the individual LR_A and LR_B. Based on DAG1 and DAG2, the frequency of such cases in which $LR_{AB} < LR_A, LR_B$ is about 12.5%, half the frequency for the Bayes factor. The distribution of these cases is displayed in Figure 10.

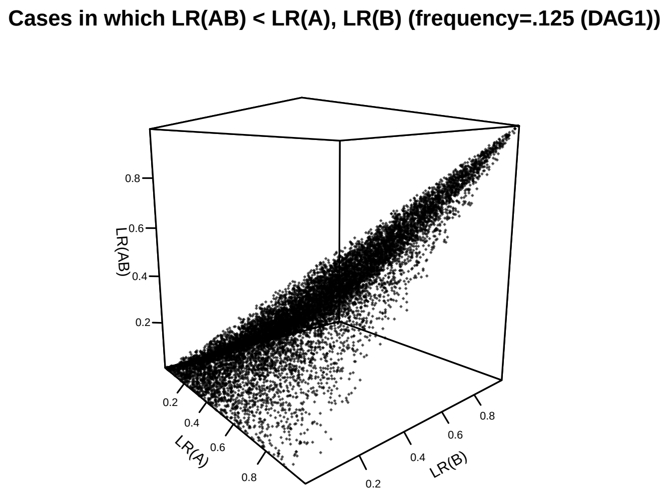


Figure 10: 12.5k cases (out of simulated 100k) in which the joint LR is below each of the individual LRs. The picture for DAG2 is very similar.

Consider now cases in which both individual likelihood ratios are above one. Interestingly, one of the individual likelihood ratios, LR_A or LR_B, may be greater than the joint LR_{AB} (see example in Figure 11). This does not happen with the Bayes factor. But even though the joint likelihood ratio can be lower than the maximum, it is never lower than the minimum of the individual likelihood ratios (Figures 12 and 13). Conversely, if both individual likelihood ratios are below one, the joint likelihood ratio can be higher than their minimum, but is never higher than their maximum (Figures 14 and 15).

A	Pr
1	0.892
0	0.108

B	A: 1	A: 0
1	0.551	0.457
0	0.449	0.543

a	A: 1	A: 0
1	0.957	0.453
0	0.043	0.547

b	B: 1	B: 0
1	0.678	0.573
0	0.322	0.427

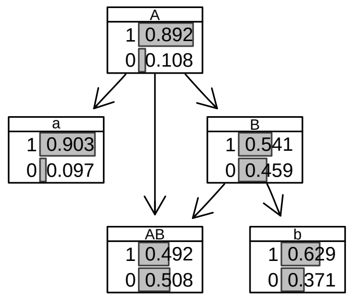


Figure 11: $LR_A \approx 2.11, LR_B \approx 1.183, LR_{AB} \approx 1.319$.
$BF_A \approx 1.06, BF_B \approx 1.076, BF_{AB} \approx 1.14$.

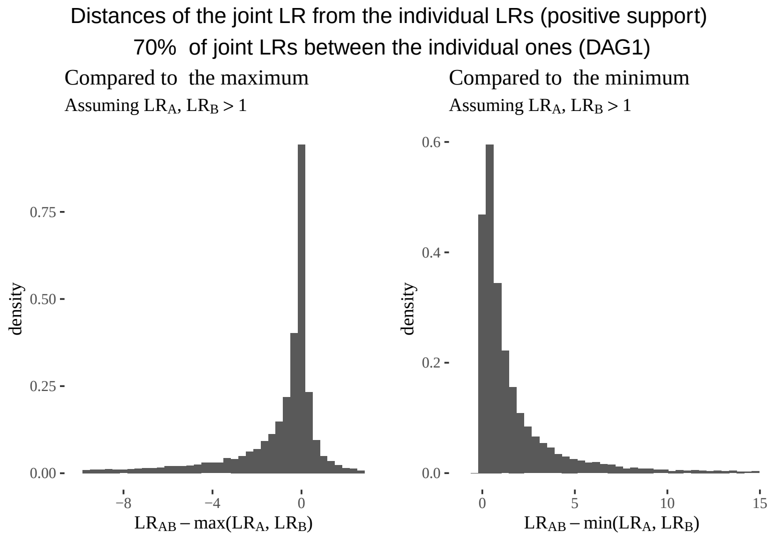


Figure 12: Distances of joint likelihood ratios for the minima and the maxima of the individual likelihood ratios if the individual likelihood ratios are above 1, DAG used in DAG1.

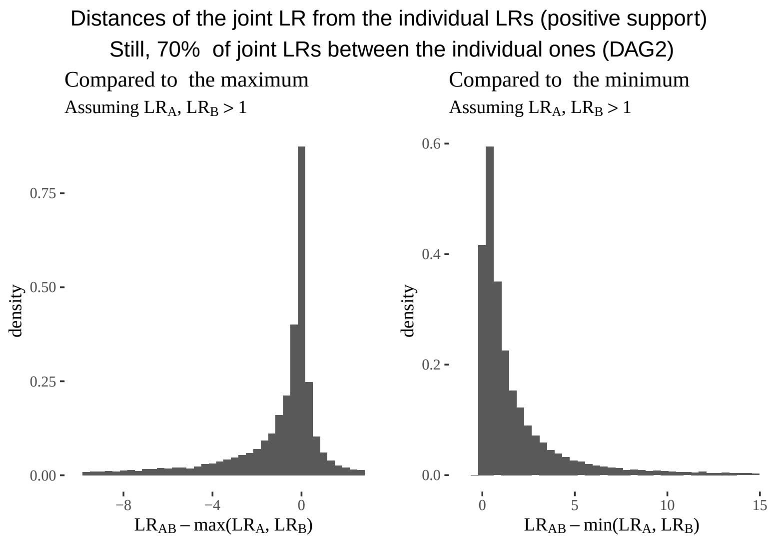


Figure 13: Distances of joint likelihood ratios for the minima and the maxima of the individual likelihood ratios if the individual likelihood ratios are above 1, DAG used in DAG2.

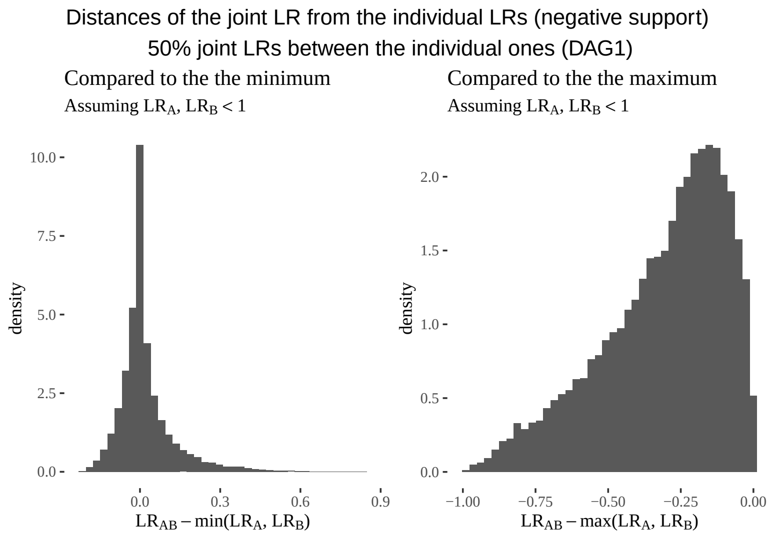


Figure 14: Distances of joint likelihood ratios for the minima and the maxima of the individual likelihood ratios if the individual likelihood ratios are below 1, DAG used in DAG1.

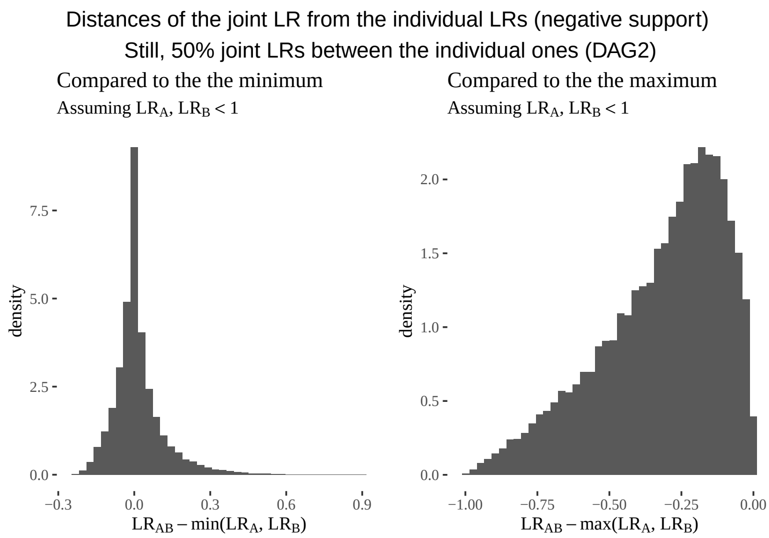


Figure 15: Distances of joint likelihood ratios for the minima and the maxima of the individual likelihood ratios if the individual likelihood ratios are below 1, DAG used in DAG2.

These simulations demonstrate that, whenever individual LR_A and LR_B are above a fixed threshold, so is the combined LR_{AB}. This fact justifies the principle of aggregation, as explained in the main text of the chapter. The converse does not hold. Even when LR_{AB} is above a threshold, LR_A or LR_B may be below the threshold. Hence, as with the Bayes factor, the principle of distribution fails in some cases.

Keeping evidence fixed

Recall the distinction between the two variants of the distribution principle:

$$S[a \wedge b, A \wedge B] \Rightarrow S[a, A] \wedge S[b, B] \tag{DIS1}$$

$$S[a \wedge b, A \wedge B] \Rightarrow S[a \wedge b, A] \wedge S[a \wedge b, B] \tag{DIS2}$$

S is a placeholder for the standard of proof. The difference is whether or not the body of evidence is held constant. The failures of distributions considered so far were failures of (DIS1). Even when the combined Bayes factor or likelihood ratio meet a fixed threshold, the individual measures may not. One wonders whether (DIS2) could be easier to justify. To this end, consider:

$$LR_A^{ab} = \frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)}$$

$$LR_B^{ab} = \frac{\mathsf{P}(a \wedge b|B)}{\mathsf{P}(a \wedge b)}$$

$$LR_{AB}^{ab} = \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)}$$

Note that the evidence $a \wedge b$ is now held constant across cases. A computer simulation shows that, even when LR_{AB}^{ab} meets a fixed threshold, the individual LR_A^{ab} and LR_B^{ab} may not. For DAG1, the joint likelihood ratio LR_{AB}^{ab} is strictly greater than both of the individual likelihood ratio LR_A^{ab} and LR_B^{ab} in 30% of the cases (22% for DAG2). A similar conclusion holds for the Bayes factor. So keeping the evidence fixed does not make it any easier to justify distribution.[5]

Dependent evidence

DAG 1 and DAG 2 ensure that the items of evidence are conditionally independent on their respective hypothesis (specifically, that a is independent both conditional on A and conditional on $\neg A$, and the same for b and B.).[6] The results established so far, then, rests on this assumption. What happens if this independence is dropped? To investigate this question, we run a simulation based on DAG 3, illustrated in Figure 16. As it turns out, the joint likelihood ratio can be lower than both individual likelihood ratios and the joint Bayes factor lower than both individual Bayes factors in 14% cases. Crucially, this occurs when the individual Bayes factor and likelihood ratios are greater than one. An example is given in Figure 17. This result is, of course, nothing surprising if we keep in mind that, in DAG 3, the items of evidence no longer count as independent lines of evidence.

[5] We run a number of computer simulations. For the sake of completeness, here are the results. If no assumption about the direction of support is made, around 12.7% of the time (twice less often than if the usual individual likelihood ratio are used), the individual LR_A^{ab} and LR_B^{ab} are both greater than the joint LR_{AB}^{ab}. This holds for DAG1. The frequency goes slightly up to around 13% if we switch to DAG2. Assuming the individual likelihood ratios are above one, only about 70% of joint likelihood ratios are between the individual ones in DAG1 (75% for DAG2) . However, no joint likelihood ratio is below the minimum of the individual ratios for DAG1, as expected. Interestingly, the join likelihood ratio may go below the minimum of the individual ones in very rare cases, 2% of the times, for DAG2. . So, strictly speaking, aggregation can fail if dependencies are present so long as the evidence is held constant.

[6] In fact, DAG 1 ensure that they are also unconditionally independent.

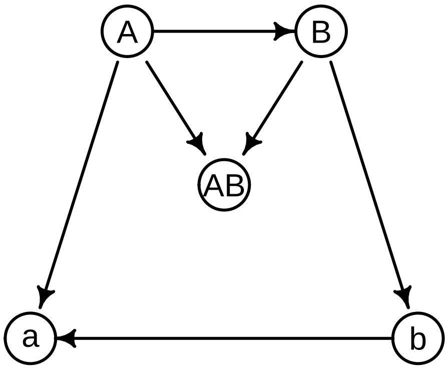


Figure 16: DAG 3 with direct dependence between the pieces of evidence.

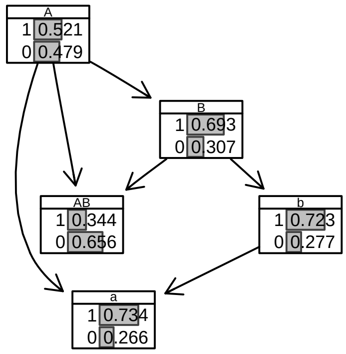
A	Pr
1	0.521
0	0.479

B	A: 1	A: 0
1	0.66	0.729
0	0.34	0.271

a	A	b	Pr
1	1	1	0.3989398
0	1	1	0.6010602
1	0	1	0.9673984
0	0	1	0.0326016
1	1	0	0.9693564
0	1	0	0.0306436
1	0	0	0.7267025
0	0	0	0.2732975

b	B: 1	B: 0
1	0.995	0.108
0	0.005	0.892

(a) Conditional probabilities for the counterexample (the one for AB does not change).



(b) Marginal probabilities.

Figure 17: A counterexample based on DAG 3, with independence between the items of evidence dropped. $LR_A \approx 1.063$, $LR_B \approx 1.159742$, $LR_{AB} \approx 0.651$. $BF_A \approx 1.022, BF_B \approx 1.079, BF_{AB} \approx 0.699$.

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