Higher-order Probabilism

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Abstract. Rational agents are often uncertain about the truth of many propositions. To represent this uncertainty, it is natural to rely on probability theory. Two options are typically on the table, precise and imprecise probabilism, but both fall short in some respect. Precise probabilism is not expressive enough, while imprecise probabilism suffers from belief inertia and the impossibility of proper scoring rules. We put forward a novel version of probabilism, higher-order probabilism, and we argue that it outperforms existing alternatives.
Keywords: Probabilism; Imprecise probabilities; Evidence; Probability; Belief inertia; Bayesian networks; Proper scores.

1 Introduction

As rational agents, we are uncertain about the truth of many propositions since our evidence is often fallible. To represent this uncertainty, it is natural to rely on probabilities. Two options are typically on the table: precise and imprecise probabilism. Precise probabilism models an agent's state of uncertainty (or credal state) with a single probability measure: each proposition in the algebra is assigned a probability value between 0 and 1 (a sharp credence). The problem is that a single probability measure is not expressive enough to distinguish between intuitively different states of uncertainty (§2). To avoid this problem, a *set* of probability measures can be used to represent the uncertainty of a rational agent. This approach is known as imprecise probabilism. It outperforms precise probabilism in some respects, but also runs into its own problems, such as belief inertia and the impossibility of defining proper scoring rules (§3).

To make progress, this paper argues that a rational agent's uncertainty should be represented by a set of precise probability measures defined over an algebra of propositions, and in addition, a probability distribution over parameter values interpreted as probability measures. We call this proposal *high-order probabilism*. The theory we propose is not mathematically novel, nor are philosophers unfamiliar with higher-order probabilities **CITE SOME: Skyrms, Gaifmann, Domotor, Uchii, Pearl, Peijnenburg and Atkinson**. Here we address specifically how higher-order probabilism can overcome the problems that plague precise and imprecise probabilism (§**??** and §**??**). We also argue that higher-order probabilism fares better than existing versions of probabilism when the probability of multiple propositions, dependent or independent, is to be assessed (§**??** and **??**). Many of the examples in this paper are about coin tosses, but in the final two sections, we discuss legal examples and illustrate the broader applicability of higher-order probabilism.

2 Precise probabilism

Precise probabilism holds that a rational agent's uncertainty about a proposition is to be represented as a single, precise probability measure. Bayesian updating regulates how the prior probability measure should change in light of new evidence that the agent learns. This is an elegant and simple theory, but representing our uncertainty about a proposition with a single, precise probability measure fails to capture an important dimension of how our fallible beliefs reflect the evidence we have (or have not) obtained. A couple of stylized examples featuring coin tosses should make the point clear.

> **No evidence v. fair coin** You are about to toss a coin but have no evidence about its bias. You are completely ignorant. Compare this to the situation in which you know, based on overwhelming evidence, that the coin is fair.

On precise probabilism, both scenarios are represented by assigning a probability of .5 to the proposition *the coin will land heads on the next toss* (or *heasd* for short). If you are completely ignorant, the principle of insufficient evidence suggests that you assign .5 to this proposition. Similarly, if you are sure the coin is fair, assigning again .5 seems the best way to quantify your uncertainty about the outcome. The agent's evidence in the two scenarios is quite different, but the precise probability of *heasd* cannot capture this difference.

> **Learning from ignorance** You toss a coin with unknown bias. You toss it 10 times and observe 5 heads. You toss it further and observe 50 heads in 100 tosses.

Since the coin initially had an unknown bias, you should presumably assign a probability of .5 to the proposition *heads*, if you stick with precise probabilism. After the 10 tosses, you again assess the probability to be .5. You must have learned something, but whatever that is, it is not reflected in the precise probability of *heads*. When you toss the coin 100 times and observe 50 heads, you also learn something new, but your precise probability does not change.[1]

These examples show that the precise probability of *heads* is not appropriately responsive to evidence. But instead of focusing on this probability, the precise probabilist might respond that we should extend the algebra and include propositions about the bias of the coin. As evidence accumulates that the coin is fair, the (precise) probability of the proposition *the coin has a .5 bias* should go up, even though the (precise) probability of *heads* should remain .5. We think this response is on the right track, but how does it generalize beyond cases of coin tosses?

[1] Another problem for precise probabilism is known as *sweetening* (Hare, 2010). Imagine a rational agent who does not know the bias of the coin. For precise probabilism, this state of uncertainty is represented by a .5 probability assignment to heads. Next, the agent learns that the bias towards heads, whatever the bias is, has been slightly increased, say by .001. Intuitively, the new information should leave the agent equally undecided about betting on heads or tails. After sweetening, the agent still does not know much about the actual bias of the coin. But, according to precise probabilism, sweetening should make the agent bet on heads: if the probability of heads was initially .5, it should now be slightly above .5.

Suppose that, given a certain stock of evidence, A is more likely than B. Further, suppose that the acquisition of new evidence does not change the probabilities. Admittedly, something has changed in the agent's state of uncertainty: the quantity of evidence on which the agent can assess whether A is more probable than B has become larger. And yet, this change is not reflected in the precise probabilities assigned to A and B.[2] The precise probabilist might recommend we extend the algebra and include propositions of the form *the probability of event X is such and such*. Since such propositions contain a probability, assigning a probability to them effectively amounts to using higher-order probabilities.[3] Before going higher-order, however, we should explore another view in the literature.

3 Imprecise probabilism

Imprecise probabilism holds that a rational agent's credal stance towards a hypothesis is to be represented by a set of probability measures, typically called a representor $\mathbb{P}$, rather than a single measure P. The representor should include all and only those probability measures which are compatible with the evidence (more on this point later).[4] Modeling an agent's credal state by sets of probability measures can easily accommodate the scenarios in the previous section without adding propositions about coin biases. For instance, if an agent is sure the coin is fair, their credal state would be represented by the singleton set $\{\mathrm{P}\}$, where P is a probability measure that assigns .5 to *heads*. If, on the other hand, the agent knows nothing about the coin's bias, their credal state would be represented by the set of all probabilistic measures, since none of them is excluded by the available evidence.

So far so good. But now consider this scenario:

> **Even v. uneven bias:** You have two coins and you are sure the probability of *heads* is .4, if you toss one coin, and .6, if you toss the other. But you cannot tell which is which. You pick one coin at random and toss it. Contrast this with an uneven case. You have four coins and you are sure three of them have bias .4 and one of them bias .6. You pick a coin at random and toss it. You should be three times more confident that the probability of getting heads is .4. rather than .6.

The first scenario can be easily modeled by imprecise probabilism. The representor would contain two probability measures; one assigns .4. and the other assigns .6 to the proposition *the coin will land heads*. Yet imprecise probabilism cannot model the second scenario. Since the probability measures in the set are all compatible with the agent's evidence, no probability

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[2] The distinction here is sometimes formulated in terms of the *balance* of the evidence (that is, whether the evidence available tips in favor a hypothesis or another) as opposed to its *weight* (that is, the overall quantity of evidence regardless of its balance); see Keynes (1921) and Joyce (2005).

[3] Interestingly, **PERAL ADD REFERENCE** argues that the semantics of higher-order probability statements can still be expressed in the language of first-order probability.

[4] For the development of imprecise probabilism, see Keynes (1921); Levi (1974); Gärdenfors & Sahlin (1982); Kaplan (1968); Joyce (2005); Fraassen (2006); Sturgeon (2008); Walley (1991). Bradley (2019) is a good source of further references. Imprecise probabilism is closely related to what we might call interval probabilism (Kyburg, 1961; Kyburg Jr & Teng, 2001). In interval probabilism, precise probabilities are replaced by intervals of probabilities. Imprecise probabilism is more general since the representor set need not be an interval.

measure can be assigned a greater (higher-order) probability than any other.[5]

These scenarios show that imprecise probabilism is not always expressive enough. Worst still, imprecise probabilism suffers from three shortcomings that do not affect precise probabilism: first, the idea of compatibility between the available evidence and the probability measures in the representator set is not clearly defined; second, updating imprecise probabilities can run into belief inertia; and third, no proper scoring rules exist for imprecise probabilities. We consider each in turn.

The first shortcoming has not received extensive discussion in the literature. For imprecise probabilism, an agent's state of uncertainty is represented by those probability measures that are *compatible* with the agent's evidence, but the notion of compatibility lacks a clear definition in the literature. If we think of it as the fact that the agent's evidence does not outright rule out the probability measure in question, the problem is that observations, evidence and data will often allow almost any probability measure. Admittedly, there will be clear-cut cases: if you see the outcome of a coin toss is heads, you reject the measure with $\mathrm{P}(H) = 0$, and similarly for tails. Another class of cases might arise while randomly drawing objects from a finite set where the objective chances are known.[6] But clear-cut cases aside, what else?

A second, related problem for imprecise probabilism is belief inertia (Levi, 1980). Precise probabilism offers an elegant model of learning from evidence: Bayesian updating. Imprecise probabilism, at least *prima facie*, offers an equally elegant model of learning from evidence, richer and more nuanced. . When faced with new evidence E between time t_0 and t_1, the representor set should be updated point-wise, running the standard Bayesian updating on each probability measure in the representor:

$$\mathbb{P}_{t_1} = \{\mathrm{P}_{t_1} | \exists\, \mathrm{P}_{t_0} \in \mathbb{P}_{t_0}\ \forall H\ \left[\mathrm{P}_{t_1}(H) = \mathrm{P}_{t_0}(H|E)\right]\}.$$

Belief inertia arises in situations in which no amount of evidence can lead the agent to change their initial belief state, according to a given modeling strategy. Consider a situation in which you start tossing a coin knowing nothing about its bias, so the range of possibilities is $[0, 1]$. After a few tosses, if you observed at least one tail and one head, you can exclude the measures assigning 0 or 1 to *heads*. But else what can you learn? For any sequence of outcomes that you can obtain and any probability value in $(0, 1)$, there will exist a probability measure (conditional on the outcomes) that assigns that probability to *heads*. Consequently, the edges of your resulting interval will remain the same. Some downplay the problem of belief inertia and blame the uniform prior [0, 1].[7] Just as contingent propositions should not be assigned

[5]Other scenarios can be constructed in which imprecise probabilism fails to capture distinctive intuitions about evidence and uncertainty; see, for example, (Rinard, 2013).

[6]Probability measures can be inconsistent with evidential constraints that agents believe to be true (Bradley, 2012), for example, *structural constraints* such as "X and Y are independent". But, unless they come from an oracle, there will usually be some degree of uncertainty about the acceptability of these constraints.

[7]Another strategy is to say that, in a state of ignorance, a special updating rule should be deployed. Elkin (2017) suggests the rule of *credal set replacement* that recommends that upon receiving evidence the agent should drop measures rendered implausible, and add all non-extreme plausible probability measures. This, however, is tricky. One needs a separate account of what makes a distribution plausible from a principled account of why one should use a separate special update rule when starting with complete ignorance.

(precise) probabilities of 0 or 1 since these extreme values are unrevisable, by the same token the uniform prior [0, 1] should not be used for imprecise probabilities whenever it is impervious to revision (Moss, 2020). More generally, it is unsurprising you will not learn anything when you start from a place of ignorance (Joyce, 2010). But, as we will see in the next section, uniform priors are not necessarily unrevisable and can be a starting point for learning. The problem lies with imprecise probabilities, not with uniform priors as such.

The third problem for imprecise probabilism is to find suitable proper scoring rules. In the precise case, scoring rules measure the distance between a rational agent's probability measure and the actual value. The Brier score is the most common scoring rule for precise probabilities.[8] A requirement of scoring rules is *propriety*: any rational agent will expect their probability measure to be more accurate than any other. A bit more formally, the expected inaccuracy of p from the perspective of p should always be smaller than the expected inaccuracy of p from the perspective of another distribution q. After all, if an agent thought a different probability measure was more accurate, they should switch to it. Well-known results demonstrate the strict propriety of the Brier score for precise probabilities.[9] Can similar results be established for imprecise probabilities? The answer is likely to be negative. In the precise case, let $I(p, w)$ be an inaccuracy score of a probability distribution p relative to the true state w. Its expected inaccuracy, from the perspective of p equals

$$\sum_{w \in W} p(w) I(p, w).$$

Now, let $I([p_-, p_+], w)$ be the inaccuracy score for the interval $[p_-, p_+]$. What is its expected inaccuracy from the perspective of, say, the interval $[p_-, p_+]$ itself? There is no straightforward answer to this question because $I([p_-, p_+], w)$ cannot be multiplied by $[p_-, p_+]$, in the way in which $I(p, w)$ can be multiplied by $p(w)$. The expected inaccuracy of $[p_-, p_+]$ can be evaluated from the perspective of the precise probabilities at its edges, either p_- or p_+.[10] The problem is that, in this case, finding a proper inaccuracy score that is also continuous is mathematically impossible (Seidenfeld, Schervish, & Kadane, 2012).

Proper scoring rules are often used to formulate accuracy-based arguments for precise probabilism. These arguments show (roughly) that, if your precise measure follows the axioms of probability theory, no other non-probabilistic measure is going to be more accurate than yours whatever the facts are. So, without proper scoring rules for imprecise probabilities, the prospects for an accuracy-based argument for imprecise probabilism look dim (Campbell-Moore, 2020; Mayo-Wilson & Wheeler, 2016).[11]

[8]The Brier score is defined as the squared distance between the true state and the probability forecast, or formally, $(p(x) - V(x, w))^2$, where $p(x)$ is the probability forecast and $V(x, w)$ determines if a proposition obtains at w ($V(x, w) = 1$) or not ($V(x, w) = 0$). If, for example, the proposition 'rain' obtains at w, the forecast 'rain with .9 probability' would be more accurate at w than the forecast 'rain with .8 probability'. If, on the other hand, the proposition 'not rain' obtained at w, the latter forecast would be more accurate.

[9]Besides propriety, other common requirements (which the Brier score also satisfies) are: the score $I(p, w)$ should be a function of the probability distribution p and the true state w (extensionality); and the score should be a continuous function around p (continuity).

[10]So the expected inaccuracy would equal $\sum_{w \in W} p_-(w) I([p_-, p_+], w)$ or $\sum_{w \in W} p_+(w) I([p_-, p_+], w)$.

[11]Moreover, as shown by Schoenfield (2017), if an accuracy measure satisfies certain plausible formal constraints,

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it will never strictly recommend an imprecise stance, as for any imprecise stance there will be a precise one with at least the same accuracy.

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