The difficulty with Conjunction

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1 Introduction

According to a common probabilistic reading, the standard of proof functions has a probability threshold. In criminal cases, the standard is very stringent, so the probability threshold is high—say the defendant's criminal liability should be established with at least .95 probability. In civil cases, the standard is less stringent, so the probability threshold is lower—say the defendant civil liability should be established with at least .5 probability.

In Chapter (**REFER TO EARLIER CHAPTER**), we discussed how such thresholds can be justified, for example, by minimizing expected costs. In Chapter (**REFER TO EARLIER CHAPTER**), we discussed a theoretical problem for this probabilistic interpretation: the paradox of naked statistical evidence. In this chapter, we discuss a second theoretical problem, which we call **the difficulty with conjunction**, also known as the conjunction paradox.

The chapter is structured as follows. First, we describe the difficulty with conjunction. Next, we explore different strategies that legal probabilists can pursue to respond to this difficulty. These strategies are promising and worth examining. But we show that they are ultimately unsatisfactory. Finally, we articulate our proposal.

Our solution to the difficulty with conjunction complements our solution to the problem of naked statistical evidence. Both problems—we maintain—suggest that the claim of wrongdoing, criminal or civil liability, should not be understood as an abstract, general proposition, but as a specific narrative, theory or explanation of the facts tailored to the individual defendant on trial. This insight—which can be formalized using Bayesian networks, as done in other chapters **REFER TO EARLIER CHAPTER**—contributes to a more adequate understanding of the standard of proof in civil and criminal cases.

2 The problem

First formulated by Cohen (1977), the difficulty with conjunction has enjoyed a great deal of scholarly attention every since (Ronald J. Allen, 1986; Ronald J. Allen & Stein, 2013; R. Allen & Pardo, 2019; Haack, 2014; Schwartz & Sober, 2017; Stein, 2005). This difficulty arises when a claim of wrongdoing, in a civil or criminal proceeding, is broken down into its constituent elements. By the probability calculus, the probability of a conjunction is often lower than the probability of the conjuncts. So, according to the probabilistic interpretation of the standard of proof, even when each constituent element (each individual conjunct) is established by the required standard of proof, the overall claim of wrongdoing (the conjunction) will usually fail to meet the required standard. Many consider this outcome counter-intuitive and contrary to trial practice.

Alicja: M: Check format of references. We do not want initilas of authors, only their last name.

2.1 Simple formulation

We begin with a simple formulation of the problem. An example will help to fix ideas. Suppose that, in order to prevail in a criminal trial, the prosecution should establish two claims by the required standard: first, that the defendant caused harm to the victim; and second, that the defendant's action was premeditated. Cohen (1977) argues that common law systems subscribe to what he calls a *conjunction principle*, that is, if two claims—call them A and B—are established according to the governing standard of proof, so is their conjunction $A \wedge B$ (and vice versa). If the conjunction principle holds, the following must be equivalent, where S is a placeholder for the standard of proof:

Separate	A is established according to S and B is established according to S
Overall	The conjunction $A \wedge B$ is established according to S

If we generalize to more than just two constituent claims, the conjunction principle requires that:

$$S[C_1 \wedge C_2 \wedge \cdots \wedge C_k] \Leftrightarrow S[C_1] \wedge S[C_2] \wedge \cdots \wedge S[C_k].$$

where $S[C_i]$ means that claim or hypothesis C_i is established according to standard S. The principle goes in both directions: call the implication from left to right **distribution**, and the opposite direction **aggregation**. Aggregation posits that establishing the individual claims by the requisite standard is enough to establish the conjunction by the same standard. Distribution posits that establishing the conjunction by the requisite standard is enough to establish the individual claims by the same standard. Aggregation and distribution identify properties that the standard of proof should possess. The difficulty with conjunction is traditionally concerned with the failure of aggregation, but we will see later on under what circumstances distribution can fail, as well.

There is some disagreement among legal scholars about the tenability of conjunction principle (more on this toward the end of this chapter). For the time being, however, let us assume the principle correctly captures features of the standard of proof. The principle has some degree of plausibility and is consistent with the case law. For example, the United States Supreme Court writes that in criminal cases

> the accused [is protected] against conviction except upon proof beyond a reasonable doubt of *every fact* necessary to constitute the crime with which he is charged. (In re Winship, 397 U.S. 358, 364, 1970)

A plausible way to interpret this quotation is to posit this identity: to establish someone's guilt beyond a reasonable doubt *just is* to establish each element of the crime beyond a reasonable doubt. Thus,

A: shouldn't this shortcut BARD be written in parenthesis before using in formula?

$$\mathsf{BARD}[C_1 \wedge C_2 \wedge \cdots \wedge C_n] \Leftrightarrow \mathsf{BARD}[C_1] \wedge \mathsf{BARD}[C_2] \wedge \cdots \wedge \mathsf{BARD}[C_k],$$

where the conjunction $C_1 \wedge C_2 \cdots \wedge C_n$ comprises all the material facts that, according to the applicable law, constitute the crime with with the accused is charged. A similar argument could be run for the standard of proof in civil cases, preponderance of the evidence or clear and convincing evidence.

The problem for the legal probabilist is that the conjunction principle conflicts with a threshold-based probabilistic interpretation of the standard of proof. For suppose the prosecution in a criminal case presents evidence that establishes claims A and B, separately, by the required probability, say about .95 each. Has the prosecution met its burden of proof? For one thing, if each claim is established by the requisite probability threshold, each claim is established by the requisite standard (assuming the threshold-based interpretation of the standard of proof). And if each claim is established by the requisite standard, then (i) liability as a whole is established by the requisite standard (assuming the conjunction principle). And yet, even though each claim is established by the requisite probability threshold, the probability of their conjunction—assuming the two claims are independent—is only $.95 \times .95 \approx .9$, below the required .95 threshold. So (ii) liability as a whole is *not* established by the requisite standard (assuming a threshold-based probabilistic interpretation of the standard). Hence, (i) the prosecution met its burden of proof and (ii) it did not meet its burden. Contradiction.

The difficulty with conjunction can be stated more plainly as the fact that a threshold-based interpretation of the standard of proof violates the conjunction principle, specifically, aggregation. Even though aggregation posits that establishing each conjunct by the required standard of proof is enough to establish the conjunction as a whole, the probability of each conjunct, considered in isolation, can be meet the required probability threshold without the conjunction as a whole meeting the requisite probability threshold.

This difficulty is quite persistent. It does not subside as the number of constituent claims increases. If anything, the difficulty becomes more apparent. Say the prosecution has established three separate claims to .95 probability. Their conjunction—again if the claims are independent—would be about .85 probable, even further below the .95 threshold.

Nor does the difficulty with conjunction subside if the claims are no longer regarded as independent. The probability of the conjunction $A \wedge B$, without the assumption of independence, equals $\mathsf{P}(A|B) \times \mathsf{P}(B)$. But if claims A and B, separately, are established with .95 probability, enough for each to meet the threshold, the probability of $A \wedge B$ should still be below the .95 threshold unless $\mathsf{P}(A|B) = 1$. For

Alicja: R: double-check that abbreviations in formulae are in textsf (outside of math) and mathsf inside math throughout the chapter.

example, that someone premeditated a harmful act against another (call it premed) makes it more likely that they did cause harm in the end (call it harm). Since $\mathsf{P}(\mathsf{harm}|\mathsf{premed}) > \mathsf{P}(\mathsf{harm})$, the two claims are not independent. Still, premeditation does not always lead to harm, so $\mathsf{P}(\mathsf{harm}|\mathsf{premed})$ will often be below 1. If both claims are established with .95, the probability of the conjunction $\mathsf{harm} \wedge \mathsf{premed}$ should still be below the .95 threshold so long as $\mathsf{P}(\mathsf{harm}|\mathsf{premed})$ is still below 1.

2.2 Adding the evidence

The discussion so far proceeded without mentioning the evidence in support of the claims that constitute the wrongdoing. This is a simplification. To be more precise, the conjunction paradox and the conjunction principle must be formulated by including the supporting evidence. For two claims, the conjunction principle can be formulated as follows:

$$\mathrm{S}[a,A] \text{ and } \mathrm{S}[b,B] \Leftrightarrow \mathrm{S}[a \wedge b, A \wedge B],$$

where a and b denote the evidence for claims A and B respectively, and S denote the standard by which the evidence establishes the claim in question. In the case of more than two claims, the formulation of the conjunction principle should be modified accordingly.

Does a threshold-based probabilistic interpretation of the standard of proof also conflict with this revised version of the conjunction principle? We should check whether, whenever both $\mathsf{P}(A|a)$ and $\mathsf{P}(B|b)$ meet the threshold, say .95, then so does $\mathsf{P}(A \wedge B|a \wedge b)$. The answer is negative, but seeing why requires some work. We are no longer just comparing the probability of $A \wedge B$ to the probability of A and the probability of B as such. Rather, we are comparing the probability of $A \wedge B$ given the combined evidence $a \wedge b$ to the probability of A given evidence a and the probability of B given evidence b.

To fix ideas, consider an example. In an aggravated assault case, the prosecution should establish two claims: first, that the defendant injured the victim; and second, that the defendant knew he was interacting with a public official. Let $\mathsf{witness}$ denote a testimony that the defendant injured the victim, call this claim injury. Let call denote the fact that the defendant made a call to an emergency number. This is evidence that the defendant knew he was dealing with a firefighter, call this claim $\mathsf{firefighther}$. If $\mathsf{P}(\mathsf{injury}|\mathsf{witness})$ and $\mathsf{P}(\mathsf{firefighter}|\mathsf{call})$ both meet the required probability threshold, does $\mathsf{P}(\mathsf{injury} \wedge \mathsf{firefighter}|\mathsf{witness} \wedge \mathsf{call})$ also meet the threshold?

The answer is negative provided two probabilistic independence assumptions hold. The first is that $\mathsf{P}(\mathsf{injury}|\mathsf{witness}) = \mathsf{P}(\mathsf{injury}|\mathsf{witness} \wedge \mathsf{call})$. This assumption is plausible because that the defendant called a firefighter for help, as opposed to someone else, does not make it more (or less) likely that he would cause injury. The second assumption is that $\mathsf{P}(\mathsf{firefighter}|\mathsf{call}) = \mathsf{P}(\mathsf{firefighter}|\mathsf{witness} \wedge \mathsf{call} \wedge \mathsf{injury})$. This assumption is plausible because that the defendant injured the victim and there is a testimony to that effect does not make it more (or less) likely that the victim was a firefighter. Unfortunately, these informal arguments are not rigorous—a point to which we will soon return. Assuming for now that the two assumptions hold, it follows that:[1]

$$\mathsf{P}(\mathsf{injury} \wedge \mathsf{firefighther}|\mathsf{witness} \wedge \mathsf{call}) = \mathsf{P}(\mathsf{injury}|\mathsf{witness}) \times \mathsf{P}(\mathsf{firefighther}|\mathsf{call}).$$

If the equality holds, even when $\mathsf{P}(\mathsf{injury}|\mathsf{witness})$ and $\mathsf{P}(\mathsf{firefighther}|\mathsf{call})$ meet the required probability threshold, $\mathsf{P}(\mathsf{injury} \wedge \mathsf{firefighther}|\mathsf{witness} \wedge \mathsf{call})$ will not.[2] This violates aggregation, and so the difficulty with conjunction persists.

There are some loose ends in this discussion, however. Why should one subscribe to the independence assumptions in the aggravated assault case? Further, do these assumptions hold in other cases? The argument should be made both more rigorous and more general. To this end, we will represent formally the relationship between claims A, B and their conjunction $A \wedge B$, as well as the supporting evidence a, b and their conjunction $a \wedge b$. A natural way to do that is to use Bayesian networks. We have already introduced Bayesian networks in an earlier chapter (**REF TO OTHER CHAPETR**). Here we will only sketch the ideas necessary to discuss the difficulty with conjunction.

[1] By the chain rule and the independence assumptions $\mathsf{P}(A|a) = \mathsf{P}(A|a \wedge b)$ and $\mathsf{P}(B|b) = \mathsf{P}(B|a \wedge b \wedge A)$, the following holds:

$$\begin{aligned}\mathsf{P}(A \wedge B|a \wedge b) &= \mathsf{P}(A|a \wedge b) \times \mathsf{P}(B|a \wedge b \wedge A)\\ &= \mathsf{P}(A|a) \times \mathsf{P}(B|b)\end{aligned}$$

[2] The only additional assumption to make here is that both $\mathsf{P}(\mathsf{injury}|\mathsf{witness})$ and $\mathsf{P}(\mathsf{firefighther}|\mathsf{call})$ are below 1, as is usually the case given that the evidence offered in a trial is fallible.

2.3 Independent hypotheses

A Bayesian network is a formal model that consists of a graphical part (a directed acyclic graph, DAG) and a numerical part (a probability table). The nodes in the graph represent random variables that can take different values. For ease of exposition, we will use 'nodes' and 'variables' interchangeably. The nodes are connected by directed edges (arrows). No loops are allowed, hence the name acyclic.

The simplest evidential relation, one of evidence bearing on a hypothesis of interest, can be represented by the following directed graph:

Alicja: M: Can you make this DAG and the others folowing smaller so that they look better on the page?

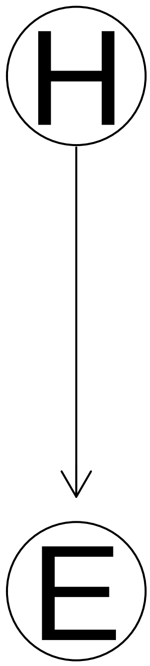

Figure 1: DAG of the simplest evidential relation

The arrow need not have a causal interpretation. The direction of the arrow indicates which conditional probabilities should be supplied in the probability table. Since the arrow goes from H to E, we should specify the probabilities of the different values of E conditional on the different values of H. In addition, since the hypothesis node has no parents, we should simply specify the prior probabilities of the difference values of H.

Back to the difficulty with conjunction. Two items of evidence, a and b, each support their own hypothesis, A and B. This set up can be represented by two directed graphs, as follows:

A: Is this what you meant?

Alicja: M: No. The two DAGS A–>a and B–>b should be one next to the other as part of the same figure. They should also look smaller and be positioned in the appropriate places.

The two directed graphs should be joined to represent the conjunction $A \wedge B$. How can this be done? Suppose for now that claims A and B are independent. We will relax this assumption later. The conjunction can be represented by adding a node AB in the directed graph in Figure 4. The arrows go from node A and node B into the conjunction node AB. This arrangement makes it possible to express the meaning of $A \wedge B$ via a probability table (Table 1). This table looks, essentially, like the truth table for the conjunction in propositional logic.[3]

The structure of the directed graph satisfies the desired independence assumptions. First, the two claims A and B are probabilistically independent of one another. Their independence is guaranteed by the fact that the conjunction node AB is a collider and thus no information flows through it.[4] Second, the supporting items of evidence a and b are also probabilistically independent of one another. The

[3]The difference is that the values 1 and 0 stand for two different things depending on where they are in the table. In the columns corresponding to the nodes they represent node states: true and false; in the Pr column they represent the conditional probability of a given state of AB given the states of A and B listed in the same row. For instance, take a look at row two. It says: if A and B are both in states 1, then the probability of AB being in state 0 is 0. In principle we could use 'true' and 'false' instead of 1 and 0 to represent states, but the numeric representation is easier to use in programming, which we do quite a bit in the background, so the reader might as well get used to this harmless ambiguity. For binary nodes, we will consistently use '1' and '0' for the states, it's just probabilities that in this case end up being extreme.

[4]A more formal treatment of this point is provided in **REFER TO OTHER CHAPTER**.

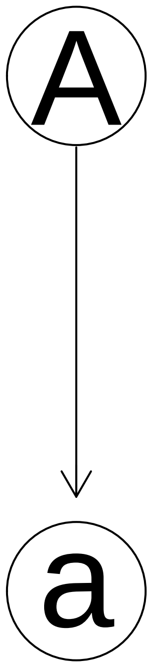

Figure 2: DAG of the a support of the A hypotheses

AB	A	B	Pr
1	1	1	1
0	1	1	0
1	0	1	0
0	0	1	1
1	1	0	0
0	1	0	1
1	0	0	0
0	0	0	1

Table 1: Conditional probability table for the conjunction node.

reason is the same: node AB blocks any flow of information between the evidence nodes. Notably, the independence of the items of evidence is not always explicitly stated in the formulation of the conjunction paradox. The Bayesian network forces us to make this explicit. This is a good thing.

With this set-up in place, the conjunction paradox arises again because aggregation is violated. By the theory of Bayesian networks, the directed graph in Figure 1 ensures that:[5]

$$\begin{aligned}\mathsf{P}(A \wedge B|a \wedge b) &= \mathsf{P}(A|a \wedge b) \times \mathsf{P}(B|a \wedge b \wedge A)\\ &= \mathsf{P}(A|a) \times \mathsf{P}(B|b)\end{aligned}$$

If, as is normally the case, neither $\mathsf{P}(A|a)$ nor $\mathsf{P}(B|b)$ equal 1, then

$$\mathsf{P}(A \wedge B|a \wedge b) < \mathsf{P}(A|a) \;\;\&\;\; \mathsf{P}(A \wedge B|a \wedge b) < \mathsf{P}(B|b).$$

Thus, even when claims A and B are sufficiently probable given their supporting evidence a and b (for a fixed threshold t)—in symbols, $\mathsf{P}(A|a) > t$ and $\mathsf{P}(B|b) > t$—it does not generally follow that $A \wedge B$ is sufficiently probable given combined evidence $a \wedge b$. One again, the conjunction principle fails because aggregation fails. The difficulty with conjunction persists.

[5] EXPLAIN THIS POINT MORE FORMALLY

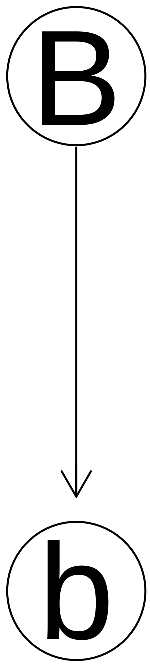

Figure 3: DAG of the b support of B the hypotheses

The argument here is general and goes beyond the specific example about aggravated assault in the previous section. The argument only assumes that the directed graph in Figure 4 is an adequate representation of a situation in which two items of evidence, a and b, support their own hypothesis, A and B. The graph encodes two plausible relations of probabilistic dependence: between hypotheses A and B and between items of evidence a and b. The theory of Bayesian network does the rest of the work.

2.4 Dependent hypotheses

What happens if the probabilistic independence between claims A and B is dispensed with? To this end, it is enough to draw an arrow between A and B in our directed graph. The result is displayed in Figure 5. There is an open path between the hypotheses nodes A and B and the evidence nodes a and b. Thus, the new graph no longer guarantees the probabilistic independence of A and B, or that of a and b. Items of evidence a and b are still probabilistically independent of one another *conditional* on their respective hypothesis. That is, $\mathsf{P}(a|A) = \mathsf{P}(a|A \wedge b)$ and $\mathsf{P}(b|B) = \mathsf{P}(b|B \wedge a)$. So a and b still counts as independent lines of evidence despite not being (unconditionally) probabilistically independent.[6]

Does the difficulty with conjunction arise even without the independence of hypotheses A and B? It does in a number of circumstances. Suppose evidence $a \wedge b$ establishes claim A and also claim B, separately, right above the probability threshold t. The graph in Figure 5 ensures that:

Alicja: M: This new DAG should have the same structure as the previous one, with an extra arrow between A and B added, but all else kept the same.

R: looks fin, although moving from P(a & b|A& B) to the numerator in the second line requires a bit of comment in a gloss after the derivation.

[6]Here is an illustration of the idea of independent lines of evidence without unconditional independence. Suppose the same phenomenon (say blood pressure) is measured by two instruments. The reading of the two instruments (say 'high' blood pressure) should be *probabilistically dependent* of one another. After all, if the instruments were both infallible and they were measuring the same phenomenon, they should give the exact same reading. On the other hand, the two instruments measuring the same phenomenon should count as *independent lines of evidence*. This fact is rendered in probabilistic terms by means of probabilistic independence conditional on the hypothesis of interest. These ideas can be worked out more systematically in the language of Bayesian networks. Roughly, two variables are probabilistically dependent if there is an open path between them. On the other hand, an open path can be closed by conditioning on one of the variables along the path. For a more rigorous exposition of the notions of open and closed paths, see **CITE EARLIER CHAPTERS**.

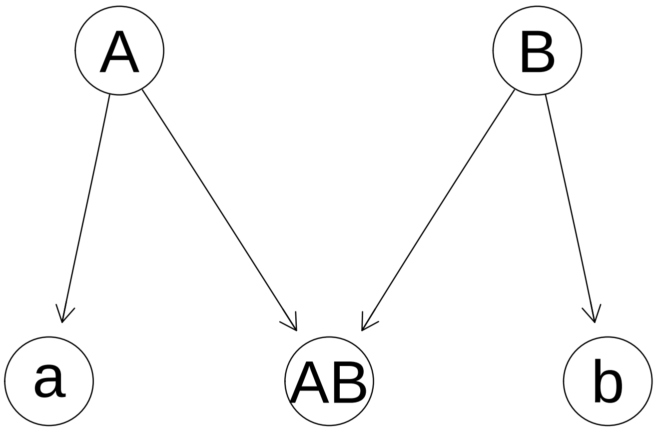

Figure 4: DAG of the conjunction set-up, with the usual independence assumptions built in.

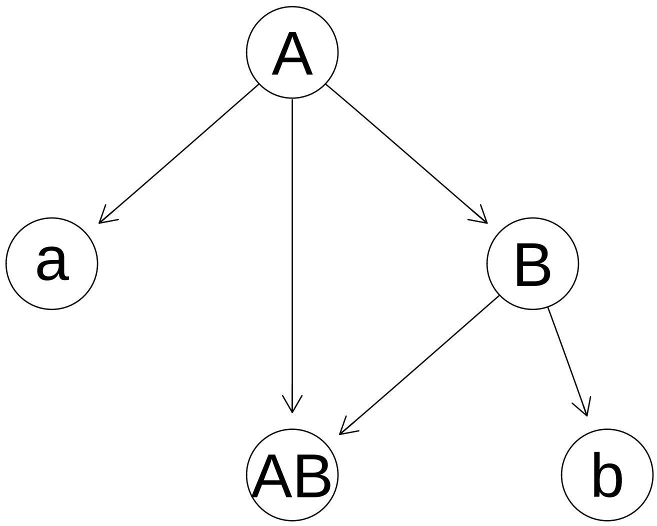

Figure 5: DAG of the conjunction set-up, without independence between A and B.

$$
\begin{aligned}
\mathsf{P}(A \wedge B|a \wedge b) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)}\mathsf{P}(A \wedge B)\\
&= \frac{\mathsf{P}(a|A)\mathsf{P}(b|B)}{\mathsf{P}(a)\mathsf{P}(b|a)} \times \mathsf{P}(A)\mathsf{P}(B|A)\\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}\mathsf{P}(A) \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}\mathsf{P}(B|A)\\
&= \mathsf{P}(A|a) \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}\mathsf{P}(B|A)
\end{aligned}
$$

M: Can you check the derivation here?

Note that, in number of cases, $\mathsf{P}(b|B)/\mathsf{P}(b|a)\mathsf{P}(B|A)$ will be less than one, or equivalently:

$$\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} < \frac{1}{\mathsf{P}(B|A)}$$

The expression $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}$ will typically be greater than one because evidence b should provide positive support for B even in combination with a.[7] The inequality can come out true for a range of probabilities. For instance, if $\mathsf{P}(b|B) = .9$, $\mathsf{P}(b|a) = \mathsf{P}(B|A) = .55$, the left-hand side is ≈ 1.63 and the right-hand

[7]Because of the relations of independence in Figure 5, $\mathsf{P}(b|B)/\mathsf{P}(b|a) = \mathsf{P}(b|B \wedge a)/\mathsf{P}(b|a)$. We are assuming that $\mathsf{P}(B|awedgea) > \mathsf{P}(B|a) > 1$, which is equivalent to $\mathsf{P}(b|B \wedge a)/\mathsf{P}(b|a) > 1$, ir in other words, b provides positive supports for B even in light of a. Hence, also $\mathsf{P}(b|B)/\mathsf{P}(b|a) > 1$.

side is ≈ 1.81. If ${}^{\mathsf{P}(b|B)}/_{\mathsf{P}(b|a)}\mathsf{P}(B|A) < 1$, then $\mathsf{P}(A \wedge B|a \wedge b) < \mathsf{P}(A|a)$.[8] There are plenty of such cases. The general pattern is that the higher the right side (for values above one), the lower the probability of $\mathsf{P}(A|A)$ for the inequality to hold. So aggregation does not hold generally even assuming that A and B are probabilistically dependent.

R: we still need the simulation here

R: check calculations in chunk

2.5 Indepedencies more generally

M: Move to appendix?

The directed graphs in Figure 4 and 5 provide a compact representation of two plausible scenarios one might have in mind in formulating the conjunction paradox. In one scenario (Figure 4), hypotheses A and B, as well as items of evidence a and b, are unconditionally independent. In another scenario (Figure 5), A and B need not be independent any longer, and a and b are only conditionally independent.

The two directed graphs encode several relationships of probabilistic independence which it would be difficult to derive without the theory of Bayesian networks. We list these relationships of independence below as we will appeal to them in the rest of the chapter. The fist graph encodes all the relationship below and the second graph encodes only a subset of them (those marked by [*]). The symbols $\perp\!\!\!\perp$ stands for independence.

Alicja: confirm all are d-sep in the first BN, check which are d-sep in the second one and whether the d-separated in the second DAG are in fact those marked by [*].

$$A \perp\!\!\!\perp B \tag{1}$$
$$A \perp\!\!\!\perp b|a \tag{2}$$
$$B \perp\!\!\!\perp a \wedge A|b \tag{3}$$
$$a \perp\!\!\!\perp b|A \wedge B \tag{4}$$
$$a \perp\!\!\!\perp b|A[*] \tag{5}$$
$$a \perp\!\!\!\perp B|A[*] \tag{6}$$
$$a \perp\!\!\!\perp B|\neg A[*] \tag{7}$$
$$a \perp\!\!\!\perp \neg B|A[*] \tag{8}$$
$$a \perp\!\!\!\perp \neg B|\neg A[*] \tag{9}$$
$$b \perp\!\!\!\perp A \wedge a|B[*] \tag{10}$$
$$b \perp\!\!\!\perp \neg A \wedge a|B[*] \tag{11}$$
$$b \perp\!\!\!\perp A \wedge a|\neg B[*] \tag{12}$$
$$b \perp\!\!\!\perp \neg A \wedge a|\neg B[*] \tag{13}$$
$$b \perp\!\!\!\perp a|B[*] \tag{14}$$
$$b \perp\!\!\!\perp a \tag{15}$$

M: D-separtation is between variables (not values of variables). Do we need to list negations?

R: in conditions, yeh; my getting a good grade and me working hard in class are dependent if the teacher doesn't assign grades randomly, but are independent if they do.

Double check if we in fact use I3.

M: Added the independence of a and b unconditionally. Check!

3 Evidential Strength

The failure of the conjunction principles encountered so far are failures of aggregation. When the probability of A and the probability of B are both above a given threshold, the probability of the conjunction $A \wedge B$ usually is not. This can happen whether or not A and B are probabilistically independent. It can also happen whether or not the probability of A and B are conditional on the supporting evidence. These failures of aggregation occur assuming the standard of proof is understood as a posterior probability threshold. Thus, aggregation cannot be justified by equating the standard of proof to such threshold.

As an alternative, legal probabilists can think of proof standards as decision criteria of evidential strength, specifically, how strong strong the evidence should be in order to justify a finding of liability. Instead of a posterior probability threshold, the standard of proof can be modeled using a probabilistic measure of evidential strength. Does this alternative way of modeling proof standards succeed in solving the conjunction paradox work? To some extent, it does, but not completely. To understand why is the task of this section.

M: We need to agree on the notation here can be confusing. We need to distinguish between variables and values of variables. Any thoughts on how to make teh notation uniform?

[8] By symmetric reasoning, the analogous conclusion that $\mathsf{P}(A \wedge B|a \wedge b) < \mathsf{P}(B|b)$ follows.

3.1 A dilemma

Two common probabilistic measures of evidential strength are the Bayes factor and the likelihood ratio. We discussed them in earlier chapters (**REFER TO EARLIER CHAPTERS**). As will become clear later, under plausible assumptions, these measures of evidential strength validate one direction of the conjunction principle: aggregation. If a is sufficiently strong evidence in favor of A and b is sufficiently strong evidence in favor of B, then $a \wedge b$ is sufficiently strong evidence in favor of the conjunction $A \wedge B$. In fact, the evidential support for the conjunction will often exceed that for the individual claims, a point already made by Dawid (1987):

> suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents.

That is, probability theory justifies this claim: if distinct items of evidence a and b constitute sufficiently strong evidence for claims A and B, so does the conjunction $a \wedge b$ for the composite claim $A \wedge B$ (although, there are some caveats and extra assumptions for this to hold; more on this later).

Dawid thought that vindicating aggregation was enough for the conjuction paradox to 'evaporate.' Unfortunately, we will show that on the evidential strength interpretation of the standard of proof, the other direction of the conjunction principle, distribution, does not hold. If $a \wedge b$ is sufficiently strong evidence in favor of $A \wedge B$, it does not follow that a is sufficiently strong evidence in favor of A or b sufficiently strong evidence in favor of B. It is not even true that, if $a \wedge b$ is sufficiently strong evidence in favor of $A \wedge B$, then $a \wedge b$ is sufficiently strong evidence in favor of A or B. This is odd. It would mean that, given a body of evidence, one can establish beyond a reasonable doubt that $A \wedge B$ (say the defendant killed the victim *and* acted intentionally) while failing to establish one of the conjuncts.

We face a dilemma. If the standard of proof is understood as a posterior probability threshold, the conjunction principle fails because aggregation fails while distribution succeeds. If, on the other hand, the standard of proof is understood as a threshold relative to evidential strength, the conjunction principle fails because distribution fails while aggregation succeeds. From a probabilistic perspective, it seems impossible to capture both directions of the conjunction principle.

In what follows, we develop more precisely the argument that, on the evidential strength approach, (a) aggregation succeeds but (b) distribution fails. The argument for these two claims is tedious. The reader should arm themselves with patience or take our word for it and jump ahead to the next section.

3.2 Combined support by the Bayes factor

The first step in the argument shows that the combined support supplied by multiple pieces of evidence typically exceeds the individual support supplied by individual pieces of evidence. This claim holds for both the Bayes factor and the likelihood ratio. We start with the Bayes factor $\mathsf{P}(E|H)/\mathsf{P}(E)$ as our measure of the support of E in favor of H. Since by Bayes' theorem

$$\mathsf{P}(H|E) = \frac{\mathsf{P}(E|H)}{\mathsf{P}(E)} \times \mathsf{P}(H),$$

the Bayes factor measures the extent to which a piece of evidence increases the probability of a hypothesis. The greater the Bayes factor (for values above one), the stronger the support of E in favor of H. Putting aside reservations about this measure of evidential support (discussed earlier in **REFER TO EARLIER CHAPTER**), the Bayes factor $\mathsf{P}(E|H)/\mathsf{P}(E)$, unlike the conditional probability $\mathsf{P}(H|E)$, offers a potential way to overcome the difficulty with conjunction by vindicating aggregation.

3.2.1 Independent hypotheses

Suppose items of evidence a and b positively support A and B, separately. In other words, both Bayes factors ${}^{\mathsf{P}(a|A)}/_{\mathsf{P}(a)}$ (abbreviated BF_A) and ${}^{\mathsf{P}(b|B)}/_{\mathsf{P}(b)}$ (abbreviated BF_B) are greater than one. Does the combined evidence $a \wedge b$ provide at least as much support in favor of the joint claim $A \wedge B$ as the individual support by a and b in favor of A and B considered separately? The combined support here is measured by the combined Bayes factor ${}^{\mathsf{P}(a \wedge b|A \wedge B)}/_{\mathsf{P}(a \wedge b)}$ (abbreviated BF_{AB}). The latter, under suitable independence assumptions, equals the product of the individual Bayes factors BF_A and BF_B. That is:[9]

Alicja: M: Check reference to independence assumptions

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)}$$
$$BF_{AB} = BF_A \times BF_B$$

This claim holds assuming (roughly) that hypotheses A and B are independent and that items of evidence a and b are independent. As noted earlier, these assumptions are plausible insofar as the Bayesian network in Figure 4 is a plausible representation of the situation at hand. Thus, the combined support BF_{AB} will always be higher than the individual support so long as BF_A and BF_B are greater than one, that is, if the individual piece of evidence positively support their respective hypotheses.

This result generalizes beyond two pieces of evidence. Figure 6 compares the Bayes factor of one item of evidence, say $\mathsf{P}(a|A)/\mathsf{P}(a)$ with the combined Bayes factor for five items of evidence, say $\mathsf{P}(a_1 \wedge \cdots \wedge a_5|A_1 \wedge \cdots \wedge A_5)/\mathsf{P}(a_1 \wedge \cdots \wedge a_5)$, for different values of sensitivity and specificity of the evidence.[10] The combined Bayes factor always exceeds the individual Bayes factors provided, as usual, the individual pieces of evidence positively support the individual hypotheses.[11] In these circumstances, Dawid's claim that 'the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents' is vindicated.

3.2.2 Dependent hypotheses

If A and B are not necessarily probabilistically independent as in the Bayesian network in Figure 5, the combined Bayes factor BF_{AB} is still greater than both the individual Bayes factor BF_A and BF_B in a number of circumstances. To see why, first note that the following holds:[12]

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}$$
$$BF'_{AB} = BF_A \times BF'_B$$

[9]Here is the argument:

$$\begin{aligned}
\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} &= \frac{\mathsf{P}(A \wedge B|a \wedge b)}{\mathsf{P}(A \wedge B)} && \text{(Bayes's theorem)}\\
&= \frac{\frac{\mathsf{P}(A \wedge B \wedge a \wedge b)}{\mathsf{P}(a \wedge b)}}{\mathsf{P}(A \wedge B)} && \text{(definition of conditional probability)}\\
&= \frac{\frac{\mathsf{P}(A)\times\mathsf{P}(B|A)\times\mathsf{P}(a|A\wedge B)\times\mathsf{P}(b|A\wedge B\wedge a)}{\mathsf{P}(a)\times\mathsf{P}(b|a)}}{\mathsf{P}(A \wedge B)} && \text{(chain rule)}\\
&=^{*} \frac{\frac{\mathsf{P}(A)\times\mathsf{P}(B)\times\mathsf{P}(a|A)\times\mathsf{P}(b|B)}{\mathsf{P}(a)\times\mathsf{P}(b)}}{\mathsf{P}(A)\times\mathsf{P}(B)} && \text{(independencies in Figure 4)}\\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} && \text{(algebraic manipulation)}\\
BF_{AB} &= BF_A \times BF_B
\end{aligned}$$

The step marked by asterisk rests on the independence assumptions codified in Figure 4, namely: (3.5), (6), and (10).

[10]The sensitivity of a piece of evidence e relative to a hypothesis H is $\mathsf{P}(e|H)$, while its specificity is $\mathsf{P}(\neg e|\neg H)$.

[11]The order is reversed if the items of evidence oppose the individual hypotheses. Neutral evidence results in a combined Bayes factor of 1, no matter the prior or the number of items of evidence

[12]Here is the derivation:

$$\begin{aligned}
\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} &= \frac{\frac{\mathsf{P}(A\wedge B\wedge a\wedge b)}{\mathsf{P}(A\wedge B)}}{\mathsf{P}(a \wedge b)}\\
&= \frac{\frac{\mathsf{P}(A)\times\mathsf{P}(B|A)\times\mathsf{P}(a|A\wedge B)\times\mathsf{P}(b|A\wedge B\wedge a)}{\mathsf{P}(A)\times\mathsf{P}(B|A)}}{\mathsf{P}(a)\times\mathsf{P}(b|a)}\\
&= \frac{\mathsf{P}(a|A\wedge B)\times\mathsf{P}(b|A\wedge B\wedge a)}{\mathsf{P}(a)\times\mathsf{P}(b|a)}\\
&=^{*} \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}\\
BF'_{AB} &= BF_A \times BF'_B
\end{aligned}$$

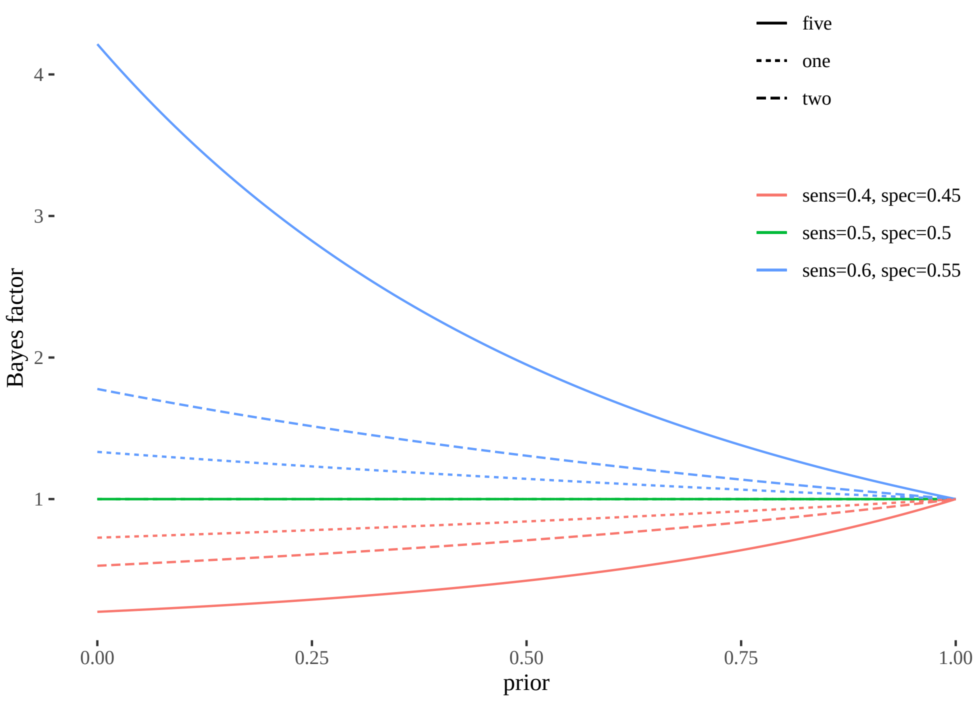

Figure 6: Bayes factor for one, two and five items of evidence and the corresponding claims, given different degrees of specificity and sensitivity of the evidence. The independence assumptions in Figure 4 hold.

The difference from the case of independent hypotheses is that $BF_B = \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)}$ was replaced by $BF_B^{'} = \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}$. Since b need not be probabilistically independent of a, there is no guarantee that $\mathsf{P}(b|a) = \mathsf{P}(b)$. However, $BF_B^{'}$ is usually greater than one so long as b raises the probability of B even in conjunction with a. This is a plausible assumption to make, or else b (in conjunction with a) would be useless evidence.[13] Since $BF_{AB}^{'} = BF_A \times BF_B^{'}$ and $BF_B^{'}$ is greater than one, $BF_{AB}^{'}$ should be greater than BF_A.

The argument is, in fact, symmetric. So we have:

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \underbrace{\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}}_{>1} = \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} \times \underbrace{\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|b)}}_{>1}$$

As before, since $BF_{AB}^{'} = BF_B \times BF_A^{'}$ and $BF_A^{'}$ is greater than one, $BF_{AB}^{'}$ should be greater than BF_B. So, putting everything together, $BF_{AB}^{'}$ will usually be greater than both BF_A and BF_B even when hypotheses A and B are dependent.

3.3 *MOVE TO APPENDIX*

Fact 1. *Assuming* (4), (6) *and* (14), *if* $\mathsf{P}(B|b \wedge a) > \mathsf{P}(B|a)$ *and* $\mathsf{P}(A|a \wedge b) > \mathsf{P}(A|b)$, *we have:*

$$BF_{AB} > \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}, \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)}$$

Alicja: double check, are these d-sep in the second network?

M: One assumption does not hold. Conditionally on A-and-B, the evidence is not independent in the DAG in Figure 2 since A-and-B is a collider node. The derivation is more complicated. See footnote.

M: What does comma mean? Fact 1 is too much formalism.

[13] Note that $BF_B^{'}$ is usually lower than BF_B because $\mathsf{P}(b|a) > \mathsf{P}(b)$ (assuming, at least, a and b are convergent pieces of evidence; **REFER TO EARLIER CHAPTER ON CROSS-EXAMINATION AND CORROBORATION**. At the same time, $BF_B^{'}$ should still be greater than one if b positively supports B even conditional on a. Note that $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} = \frac{\mathsf{P}(b|B \wedge a)}{\mathsf{P}(b|a)}$, by independence (14). Further, by Bayes' theorem, $\mathsf{P}(B|b \wedge a) = \frac{\mathsf{P}(b|B \wedge a)}{\mathsf{P}(b|a)} \times \mathsf{P}(B|a)$, so $\mathsf{P}(B|b \wedge a)$ is obtained by multiplying $\mathsf{P}(B|a)$ by $s_B^{'}$. In other words, the claim that $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} > 1$ is equivalent to the claim that $\mathsf{P}(B|b \wedge a) > \mathsf{P}(B|a)$. The latter is plausible in this context, since b should still raise the probability of B even in conjunction with a, or else b would be useless evidence.

Argument. First, notice the following holds given (4) and (6):

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} \tag{16}$$

$$BF'_{AB} = s_A \times BF'_B, \tag{17}$$

where factor $BF_B = \mathsf{P}(b|B)/\mathsf{P}(b)$ was replaced by $BF'_B = \mathsf{P}(b|B)/\mathsf{P}(b|a)$.[14] Hence, BF'_{AB} should be greater than either BF_A if BF'_B is

Alicja: check if d-sep in this fn holds in the second bn

greater than one. Note that the reasoning behind (17) is symmetric and so the assumed inequalities give the result marked by the braces:

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \underbrace{\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)}}_{>1} = \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} \times \underbrace{\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|b)}}_{>1}$$

So, symmetrically, we have $BF'_{AB} = BF_B \times BF'_A$, and so BF'_{AB} should be greater than either BF_B if BF'_A is greater than one.

Argument ends.

M: Need to introduce simulations, illustrate results for aggregation and BF without independence here

3.4 Combined support by the likelihood ratio

The Bayes factor can be replaced by the likelihood ratio, another probabilistic measure of evidential support, extensively discussed in **REFER TO Chapter XXX** . The likelihood ratio compares the probability of the evidence on the assumption that a hypothesis of interest is true and the probability of the evidence on the assumption that the negation of the hypothesis is true, that is, $\mathsf{P}(E|H)/\mathsf{P}(E|\neg H)$. We can think of the the likelihood ratio as the following:

REF

$$\frac{\textit{sensitivity}}{\textit{1- specificity}}$$

The greater the likelihood ratio (for values above one), the stronger the evidential support in favor of the hypothesis (as contrasted to the its negation). Unlike the Bayes factor, the likelihood ratio is not sensitive to the prior probability of the hypothesis of interest so long as sensitivity and specificity are not sensitive to the prior.[15]

Does the combined support measured by the combined likelihood ratio $\mathsf{P}(a \wedge b|a \wedge B)/\mathsf{P}(a \wedge b|\neg(A \wedge B))$ exceed the individual support measured by the individual likelihood ratios $\mathsf{P}(a|A)/\mathsf{P}(a|\neg A)$ and $\mathsf{P}(b|B)/\mathsf{P}(b|\neg B)$? Under suitable assumptions, the answer is positive. So, details aside, Bayes factor and likelihood ratio agree on this point. The argument for the likelihood ratio, however, is more laborious.

3.4.1 Combined sensitivity and specificity

First, we compute the numerator and denominator of the combined likelihood ratio. The numerator is easy. It results from multiplying the sensitivity of the individual items of evidence, a and b, relative to

[14] The difference between BF_B and BF'_B is that b need not be probabilistically independent of a, and thus there is no guarantee that $\mathsf{P}(b|a) = \mathsf{P}(b)$. In fact, BF'_B is usually lower than BF_B because $\mathsf{P}(b|a) > \mathsf{P}(b)$ (assuming, at least, a and b are convergent pieces of evidence; SEE DISCUSSION IN EARLIER CHAPTERS ON CROSS-EXAMINATION AND CORROBORATION). At the same time, BF'_B should still be greater than one if b positively supports B even conditional on a. Note that $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} = \frac{\mathsf{P}(b|B \wedge a)}{\mathsf{P}(b|a)}$, by independence (14). Further, by Bayes' theorem, $\mathsf{P}(B|b \wedge a) = \frac{\mathsf{P}(b|B \wedge a)}{\mathsf{P}(b|a)} \times \mathsf{P}(B|a)$, so $\mathsf{P}(B|b \wedge a)$ is obtained by multiplying $\mathsf{P}(B|a)$ by BF'_B. In other words, the claim that $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} > 1$ is equivalent to the claim that $\mathsf{P}(B|b \wedge a) > \mathsf{P}(B|a)$. The latter is plausible in this context, since b should still raise the probability of B even in conjunction with a, or else b would be useless evidence.

[15] This point is debated in the literature **CITE**. Further, we will see that specificity is sensitive to prior probabilities in the case of combined hypotheses.

their respective hypotheses, A and B:[16]

$$\mathsf{P}(a \wedge b | A \wedge B) = \mathsf{P}(a|A) \times \mathsf{P}(b|B)$$

Call this *combined sensitivity*. The denominator of the combined likelihood ratio is more complicated, mostly because of the presence of $\neg(A \wedge B)$ as a condition:[17]

$$\mathsf{P}(a \wedge b | \neg(A \wedge B)) = \frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) + \mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A) + \mathsf{P}(A)\mathsf{P}(\neg B|A) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}$$

Call this *combined specificity*. The two equalities holds whether or not A and B are probabilistically independent. So both directed graphs in Figure 4 and Figure 5 validate these claims.

3.4.2 Equal probability and independence

Many variables are at play here. Unlike with the Bayes factor, it is not easy to compare the combined evidential support and the individual support. For illustrative purpose, we simplify. Let the sensitivity and specificity for the items of evidence involved be the same and equal x. Let also A and B be probabilistically independent in agreement with the directed graph in Figure 4. In this simplified set-up, the combined likelihood ratio reduces to the following, where $\mathsf{P}(A) = k$ and $\mathsf{P}(B) = t$:

$$\frac{\mathsf{P}(a \wedge b | A \wedge B)}{\mathsf{P}(a \wedge b | \neg(A \wedge B))} = \frac{x^2}{\frac{(1-k)t(1-x)x + k(1-t)x(1-x) + (1-k)(1-t)(1-x)(1-x)}{(1-k)t + (1-t)k + (1-k)(1-t)}}$$

The combined likelihood ratio can now be easily plotted. As Figure 7 shows, the combined likelihood ratio always exceeds the individual likelihood ratios whenever they are greater than one (or in other words, as is usually assumed, the two pieces of evidence provide positive support for their respective hypotheses). Interestingly, we notice that the combined likelihood ratio varies deepening on the prior probabilities $\mathsf{P}(A)$ and $\mathsf{P}(B)$.

As with the Bayes factor, the combined likelihood ratio exceeds the individual likelihood ratios. However, the graph only covers cases in which the two pieces of evidence have the same sensitivity and specificity and hypotheses A and B are probabilistically independent. What happens when these two assumptions are relaxed?

16

$$\begin{aligned}
\mathsf{P}(a \wedge b | A \wedge B) &= \frac{\mathsf{P}(A \wedge B \wedge a \wedge b)}{\mathsf{P}(A \wedge B)} && \text{(conditional probability)} \\
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A \wedge B) \times \mathsf{P}(b|A \wedge B \wedge a)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} && \text{(chain rule)} \\
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} && \text{(independencies (5) and (10))} \\
&= \mathsf{P}(a|A) \times \mathsf{P}(b|B) && \text{(algebraic manipulation)}
\end{aligned}$$

17

$$\begin{aligned}
\mathsf{P}(a \wedge b | \neg(A \wedge B)) &= \frac{\mathsf{P}(a \wedge b \wedge \neg(A \wedge B))}{\mathsf{P}(\neg(A \wedge B))} && \text{(conditional probability)} \\
&= \frac{\mathsf{P}(a \wedge b \wedge \neg A \wedge B) + \mathsf{P}(a \wedge b \wedge A \wedge \neg B) + \mathsf{P}(a \wedge b \wedge \neg A \wedge \neg B)}{\mathsf{P}(\neg A \wedge B) + \mathsf{P}(A \wedge \neg B) + \mathsf{P}(\neg A \wedge \neg B)} && \text{(logic \& additivity)}
\end{aligned}$$

Now consider the first summand from the numerator:

$$\begin{aligned}
\mathsf{P}(a \wedge b \wedge \neg A \wedge B) &= \mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A \wedge B)\mathsf{P}(b|a \wedge \neg A \wedge B) && \text{(chain rule)} \\
&= \mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) && \text{(independencies (7) and (11))}
\end{aligned}$$

The simplification of the other two summanda is analogous (albeit with slightly different independence assumptions—(8) and (12) for the second one and (9) and (13) for the third. Once we plug these into the denominator formula we get:

$$\mathsf{P}(a \wedge b | \neg(A \wedge B)) = \frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) + \mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A) + \mathsf{P}(A)\mathsf{P}(\neg B|A) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}$$

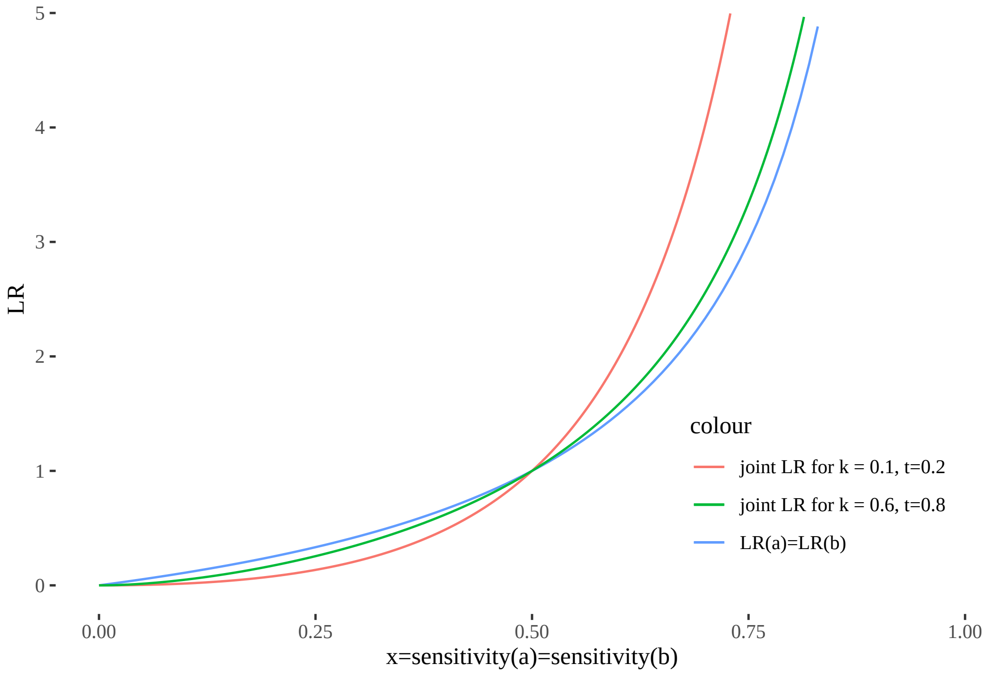

Figure 7: Combined likelihood ratios exceeds individual Likelihood ratios as soon as sensitivity is above .5. Changes in the prior probabilities t and k do not invalidate this result.

3.4.3 More complex scenarios – NEEDS SIMULATION

If the items of evidence have different levels of sensitivity and specificity, the combined likelihood ratio never goes below the lower of the two individual likelihood ratios, but can be lower than the higher one. If the composite likelihood ratio behaves differently than the Bayes factor in that it is greater than the lower of the individual likelihood ratios, rather than being greater than both of them, Dawid's claim that 'the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents' still holds.

M: Need to add simulation results to make this argument fully general and drop all the simplifying assumptions (.e.g. independence or equiprobability)

3.5 *MOVE TO APPENDIX*

Fact 2. *If independence assumptions (5), (7), (8), (9), (10), (11), (12) and (13) hold, the combined likelihood ratio is:*

Alicja: check whic of these d-sep hold in the first and in the second BN

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))} = \frac{\mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B)+\mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B)+\mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)+\mathsf{P}(A)\mathsf{P}(\neg B|A)+\mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}} \tag{18}$$

If, further the hypotheses are independent in the sense of this reduces to:

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))} = \frac{\mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\frac{\mathsf{P}(\neg A)\mathsf{P}(B)\mathsf{P}(a|\neg A)\mathsf{P}(b|B)+\mathsf{P}(A)\mathsf{P}(\neg B)\mathsf{P}(a|A)\mathsf{P}(b|\neg B)+\mathsf{P}(\neg A)\mathsf{P}(\neg B)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B)+\mathsf{P}(A)\mathsf{P}(\neg B)+\mathsf{P}(\neg A)\mathsf{P}(\neg B)}} \tag{19}$$

Argument. The numerator can be computed as follows:

$$\begin{aligned}
\mathsf{P}(a \wedge b|A \wedge B) &= \frac{\mathsf{P}(A \wedge B \wedge a \wedge b)}{\mathsf{P}(A \wedge B)} && \text{(conditional probability)}\\
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A \wedge B) \times \mathsf{P}(b|A \wedge B \wedge a)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} && \text{(chain rule)}\\
&= \frac{\mathsf{P}(A) \times \mathsf{P}(B|A) \times \mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\mathsf{P}(A) \times \mathsf{P}(B|A)} && \text{(independencies (5) and (10))}\\
&= \mathsf{P}(a|A) \times \mathsf{P}(b|B) && \text{(algebraic manipulation)}
\end{aligned}$$

The numerator clearly does not depend on the prior probability of $A \wedge B$. We call it *combined sensitivity*, and given the independence assumptions it simply results from multiplying the sensitivity of the individual items of evidence, a and b, relative to their respective hypotheses, A and B.

The denominator is more involved:

$$\begin{aligned}
\mathsf{P}(a \wedge b|\neg(A \wedge B)) &= \frac{\mathsf{P}(a \wedge b \wedge \neg(A \wedge B))}{\mathsf{P}(\neg(A \wedge B))} && \text{(conditional probability)}\\
&= \frac{\mathsf{P}(a \wedge b \wedge \neg A \wedge B) + \mathsf{P}(a \wedge b \wedge A \wedge \neg B) + \mathsf{P}(a \wedge b \wedge \neg A \wedge \neg B)}{\mathsf{P}(\neg A \wedge B) + \mathsf{P}(A \wedge \neg B) + \mathsf{P}(\neg A \wedge \neg B)} && \text{(logic \& additivity)}
\end{aligned}$$

Now consider the first summand from the numerator:

$$\begin{aligned}
\mathsf{P}(a \wedge b \wedge \neg A \wedge B) &= \mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A \wedge B)\mathsf{P}(b|a \wedge \neg A \wedge B) && \text{(chain rule)}\\
&= \mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) && \text{(independencies (7) and (11))}
\end{aligned}$$

The simplification of the other two summanda is analogous (albeit with slightly different independence assumptions—(8) and (12) for the second one and (9) and (13) for the third. Once we plug these into the denominator formula we get:

$$\mathsf{P}(a \wedge b|\neg(A \wedge B)) = \frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B) + \mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A) + \mathsf{P}(A)\mathsf{P}(\neg B|A) + \mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}$$

(19) is easily obtained from (18) by .
Argument ends.

3.6 Vindicating aggregation

The fact that—for both the Bayes faxctor and the likelihood ratio–the support supplied by the conjunction usually exceeds the support supplied by individual items of evidence can be used to justify aggregation. The argument, however, is not straightforward. Aggregation is a principle about the standard of proof. So, in order to vindicate aggregation, the standard of proof should be understood in terms of evidential strength. But how this should be done is not obvious.

Here is what should be obvious. If the decision criterion is formulated in terms of evidential strength instead of posterior probabilities,the standard of proof should no longer be formalized as a posterior probability threshold, but rather as a threshold for the Bayes factor or the likelihood ratio. The threshold should no longer be a probability between 0 and 1, but a number somewhere above one. The greater this number, the more stringent the standard of proof, for any value above one. In criminal trials, for example, the rule of decision would be: guilt is proven beyond a reasonable doubt if and only if the evidential support in favor of H—as measured by the Bayes factor $\mathsf{P}(E|H)/\mathsf{P}(E)$ (or by the likelihood ratio $\mathsf{P}(E|H)/\mathsf{P}(E|\neg H)$)—meets a suitably high threshold t_{BF} (t_{LR}). The question at this point is, how do we identify the appropriate threshold? The answer to this question is not obvious.

will at some point take a look at "Bayesian Choice"

3.6.1 Variable threhsold

We first articulate an approach that is a non-starter. It is useful to understand why this appropach does not work to formulate a more promising approach. A threshold on evidential strength can be derived from the threshold on posterior probability. The advantage of the posterior probability threshold is that its stringency can be determined in a decision-theoretic manner via the minimization of expected costs. (**REFER TO ERALIER CHAPPER**}) Something analogous can be done for the Bayes factor or the likelihood ratio. One option is to derive a threshold for the Bayes factor and the likelihood ratio, call these thresholds t_{BF} or t_{LR}, from a thresholds t on the posterior probability. This is easy to do.

Consider threshold t_{BF} first. Since $\mathsf{posterior} = \mathsf{Bayesfactor} \times \mathsf{prior}$, the Bayes factor threshold can be determined as follows:

$$t_{BF} = \frac{t}{\mathsf{prior}}$$

The higher the prior probability, the lower t_{BF}. So threshold t_{BF} would be dependent on the prior probability of the hypothesis of interest. Whether this is a desirable property for a decision threshold can be questioned, but a similar point holds about the standard posterior threshold t: the higher the prior probability, the easier to meet this threshold.

The same strategy works for the threshold t_{LR}. By the odds version of Bayes' theorem,

$$\text{posterior odds} = \text{likelihood odds} \times \text{prior odds},$$

and thus

$$\frac{\text{posterior odds}}{\text{prior odds}} = \text{likelihood ratio}.$$

If the posterior ratio is fixed at, say $t/1-t$, t_{LR} can be obtained as follows:

$$t_{LR} = \frac{{}^{t}\!/_{1-t}}{\text{prior odds}}.$$

Just as the Bayes factor, the likelihood ratio threshold will vary with the prior. The higher the prior, the lower the likelihood ratio threshold.

This approach incur two shortcoming. First, the variable threshold for Bayes factor or the likelihood ratio is parasitic on the posterior probability threshold. So if there are reasons to reject the posterior probability threshold, these reasons would also apply to the other thresholds. Second—and more to the point—aggregation still fails for t_{BF} and t_{LR} just like it failed in case of the threshold for posterior probabilities. There will be cases in which the conjuncts taken separately satisfy the decision standard t_{BF} or t_{LR}, while the conjunction does not. The culprit here is the fact that these thresholds are prior-dependent. So t_{BF} and t_{LR} will have different absolute values when applied to to individual claims A and B compared to the composite claim $A \wedge B$, since the prior probability of a composite claim differs from that of individual claims. The formal details can be found in **REFER TO APPENDIX**.

Question: how does Kaplow think about it? Does he only derive threshold for the ultimate claim? I don't remember now.

Perhaps, this is not the path that the proponent of the evidential strength approach would take anyway. If the evidential strength threshold mirrors the posterior threshold, it should not be surprising that it runs into similar problems. So what would happen if, instead, the evidential strength threshold was kept fixed and did not depend on priors?

3.6.1.1 TO BE MOVED TO Appendix Now, we would like a standard of proof to be in principle applicable to all factual claims under consideration. In particular, if we consider a conjunction $A \wedge B$, we should have a standard that makes sense not only when applied to $A \wedge B$, but also when applied to $A \wedge B$. But now notice that in the current set-up evidential strength thresholds vary with priors, and clearly the priors for A and B will differ from the priors on $A \wedge B$. This suggests that as long as we want t_{BF} and t_{LR} to be decision-theoretically justified by being derived from a decision-theoretically justified posterior probability threshold, the thresholds for individual claims (t^{A}_{BF} and t^{A}_{LR}) will differ from the thresholds for the composite claim, $t^{A\wedge B}_{BF}$ and $t^{A\wedge B}_{LR}$.

The conjunction principles formulated in terms of BF and LR would boil down to:

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} > t^{A}_{BF} \text{ and } \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} > t^{B}_{BF} \text{ iff } \frac{\mathsf{P}(a\wedge b|A\wedge B)}{\mathsf{P}(a\wedge b)} > t^{A\wedge B}_{BF}$$

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)} > t^{A}_{LR} \text{ and } \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|B)} > t^{B}_{LR} \text{ iff } \frac{\mathsf{P}(a\wedge b|A\wedge B)}{\mathsf{P}(a\wedge b|\neg(A\wedge B))} > t^{A\wedge B}_{LR}$$

Now, consider the individual claims A and B and their conjunction $A \wedge B$, assuming the independence relations required for Facts and hold. Consider a posterior threshold of .95 as might be appropriate in a criminal case.

Alicja: REF

Alicja: REF

please use nicefrac for inline fractions

If A and B both have a prior probability of, say .1, the threshold $t^{A}_{BF} = t^{B}_{BF} = {}^{.95}\!/_{.1} = 9.5$ for A or B individually. The composite claim $A \wedge B$ will be associated with the threshold $t^{A\wedge B}_{BF} = {}^{.95}\!/_{(.1\times.1)} = 95$, a much higher value. But if each individual claim meets its Bayes factor threshold of 9.5 and the two claims are independent, the joint Bayes factor would result from the multiplication of the individual Bayes factors, that is, $9.5 \times 9.5 = 90.25$. This is not quite enough to meet $t^{A\wedge B}_{BF} = 95$.

The difference grows as (i) the prior probability of the individual claims decreases, (ii) as the posterior threshold t decreases, and (iii) as the number of constituents claims increases. For two constituents, the combined Bayes factor remains only 5% below the value needed to meet $t_{BF}^{A\wedge B}$. The difference at here is between $t_{BF}^{A\wedge B} = {.95}/{p^2}$ and $t_{BF}^{A} * t_{BF}^{B} = ({.5}/{p})^2$. Note that $\frac{.95/p^2-(.5/p)^2}{.95/p^2} = .05$, for any value of the prior p. Given five constituent claims, $\frac{.95/p^5-(.95/p)^5}{.95/p^5} = .18$. Now, say $t = .5$. Even with just two claims, $t_{BF}^{A\wedge B} = .5/(.1 * .1) = 50$, but $t_{BF}^{A} * t_{BF}^{B} = (.5/.1) * (.5/.1) = 25$, only half of the required value.

An analogous point will hold for the likelihood ratio. Say A and B have prior probabilities of .2 and .3 respectively. On this approach, the likelihood ratio threshold for A and B will be $t_{LR}^{A} \approx 76$ and $t_{LR}^{B}a \approx 44$. The likelihood ratio threshold for the composite claim $A \wedge B$ will be $t_{LR}^{A\wedge B} \approx 297$. Now suppose the individual likelihood ratios meet their threshold and respective sensitivities and specificities are identical. For t_{LR}^{A} to be met, evidence a should have sensitivity of at least 0.988. For t_{LR}^{B} to be met, evidence b should have sensitivity 0.978. Now with these separate sensitivies, The combined likelihood ratio equals about 145, far short that what the threshold $t_{LR}^{A\wedge B}$ requires, namely a likelihood ratio of 297.

Things don't get better with a lower posterior threshold. Say $t = .5$, as might be appropriate in a civil case. The likelihood ratio thresholds for A and B will be $t_{LR}^{A} \approx 4$ and $t_{LR}^{B}a \approx 2.3$. The likelihood ratio threshold for the composite claim $A \wedge B$ will be $t_{LR}^{A\wedge B} \approx 15.6$. To ensure that t_{LR}^{A} is met, evidence a should have a sensitivity of at least 0.8, and to ensure that t_{LR}^{B} is met, evidence b should have a sensitivity of at least 0.7. To ensure that t_{LR}^{B} is met, evidence b should have a sensitivity of at least 0.7. With these parameters, the combined likelihood ratio equals about 5, far short that what the threshold $t_{LR}^{A\wedge B}$ requires, namely a likelihood ratio of 15.

3.6.2 Fixed threshold

The alternative here is to fix the evidential strength threshold regardless of the prior probability of the claim of interest. This raises the difficult question of how to fix the threshold irrespective of the priors. Standard decision theory can no longer be used. Still, assuming this question can be satisfactorily answered, the fixed threshold approach manages to vindicate aggregation. This is a significant improvement.

If the standard of proof is formalized using a fixed threshold for the Bayes factor or for the likelihood ratio, the conjunction principle would boil down to one of these:

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} > t_{BF} \text{ and } \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} > t_{BF} \text{ iff } \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} > t_{BF}$$
$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)} > t_{LR} \text{ and } \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|B)} > t_{LR} \text{ iff } \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))} > t_{LR}$$

The left-to-right direction—aggregation—holds for any thresholds t_{BF} or t_{LR} greater than one. As shown previously, the combined evidential support is usually greater than the individual evidential support, whether it is measured in terms of the Bayes factor or the likelihood ratio. This is an improvement. Aggregation could not be justified using posterior probabilities $\mathsf{P}(A|a)$ and $\mathsf{P}(B|b)$ nor could it be justified generally using a variable Bayes factor threshold. So it is an advantage of the fixed evidential strength threshold approach that it can justify this direction of the conjunction principle.

However, the right-to-left direction—distribution—has now become problematic. Two versions of the distribution principle can be distinguished:

$$\text{If } \mathrm{S}[a \wedge b, A \wedge B], \text{ then } \mathrm{S}[a, A] \text{ and } \mathrm{S}[b, B]. \tag{DIS1}$$
$$\text{If } \mathrm{S}[a \wedge b, A \wedge B], \text{ then } \mathrm{S}[a \wedge b, A] \text{ and } \mathrm{S}[a \wedge b, B]. \tag{DIS2}$$

Let us start with (DIS1) first, still working with the independence assumptions in the background for the sake of simplicity. Suppose the combined Bayes factor, $\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)}$, barely meets the threshold. The individual support, say $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}$, could be still below the threshold unless $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} = 1$ (which should not happen if b positively supports B).

The problem for likelihood ratio is analogous. For suppose evidence $a \wedge b$ supports $A \wedge B$ to the required threshold t. The threshold in this case should be some order of magnitude greater than one. If the combined likelihood ratio meets the threshold t_{LR}, one of the individual likelihood ratios may well be below t_{LR}. So—if the standard of proof is interpreted using evidential support measured by the likelihood ratio—even though the conjuction $A \wedge B$ was proven according to the desired standard, one of individual claims might not. (DIS1) in some cases fails.

Perhaps, the problem was with (DS1). Could this principle be rejected? Maybe it is not as essential as we thought at first. Since the evidence is not held constant, the support supplied by $a \wedge b$ could be stronger than that supplied by a and b individually. So even when the conjunction $A \wedge B$ is established to the requisite standard given evidence $a \wedge b$, it might still be that A does not meet the requisite standard (given a) nor does B (given b). Or at least one might try to argue so.

(DS2) is less controversial, as it holds the evidence constant. This principle is harder to deny: one would not want to claim that, holding fixed evidence $a \wedge b$, establishing the conjunction might not be enough for establishing one of the conjuncts. It seems that any formalization of the standard of proof should obey (DIS2). Yet, (DIS2) also fails both for Bayes factor and likelihood ratio. After all, if A and B are probabilistically independent, (DIS1) and (DIS2) are in fact equivalent, so the counterexamples to (DIS1) work also against (DS2).

Curiously, on the fixed evidential strength threshold approach, no matter whether one uses Bayes factor or likelihood ratio, there would be cases in which, even though the conjunction $A \wedge B$ is established by the desired standard of proof, one of the individual claims fails to meet the standard. This is odd, to say the least.

R:I don't think this argument worked, we can discuss this. Commented out

M: What exactly did not work in the argument that you commented out? It is just a restatement of the earlier point.

4 The comparative strategy

Instead of thinking in terms of absolute thresholds—whether relative to posterior probabilities, the Bayes factor or the likelihood ratio—the standard of proof can be understood comparatively. This suggestion has been advanced by Cheng (2012) following the theory of relative plausibility by **REFERENCE TO ALLEN AND PARDO HERE**. Say the prosecutor or the plaintiff puts foward a hypothesis H_p about what happened. The defense offers an alternative hypothesis, call it H_d. On this approach, rather than directly evaluating the support of H_p given the evidence and comparing it to a threshold, we compare the support that the evidence provides for two competing hypotheses H_p and H_d, and decide for the one for which the evidence provides better support.

M: This section seems to be missing a lot of the reasoning that was commented out. Is this still comprehensible? Not sure I can follow. Seems too brief.

Which one?

It is controversial whether this is what happens in all trial proceedings, especially in criminal trials, if one thinks of H_d as a substantial account of what has happened. The defense may elect to challenge the hypothesis put foward by the other party without proposing one of its own. For example, in the O.J. Simpson trial the defense did not advance its own story about what happened, but simply argued that the evidence provided by the prosecution, while significant on its face to establish OJ's guilt, was riddled with problems and deficencies. This defense strategy was enough to secure an acquittal. So, in order to create a reasonable doubt about guilt, the defense does not always provide a full-fledged alternative hypothesis. The supporters of the comparative approach, however, will respond that this could happen in a small number of cases, even though in general—especially for tactical reasons—the defense will provide an alternative hypothesis. And even such cases can be construed as involving a defense hypothesis, one that is simply equivalent to the negation of H_p.

Setting aside this controversy for the time being, let's first work out the comparative strategy using posterior probabilities. More specifically, given a body of evidence E and two competing hypotheses H_p and H_d, the probability $\mathsf{P}(H_p|E)$ should be suitably higher than $\mathsf{P}(H_d|E)$, or in other words, the ratio $\mathsf{P}(H_p|E)/\mathsf{P}(H_d|E)$ should be above a suitable threshold. Presumably, the ratio threshold shoud be higher for criminal than civil cases. In fact, in civil cases it seems enough to require that the ratio $\mathsf{P}(H_p|E)/\mathsf{P}(H_d|E)$ be above 1, or in other words, that $\mathsf{P}(H_p|E)$ should be higher than $\mathsf{P}(H_d|E)$.[18]

I'm not convinced this footnote is needed and what its point is.

One advantage of this approach—as Cheng shows—is that expected utility theory can set the appropriate comparative threshold t as a function of the costs and benefits of trial decisions. For simplicity, suppose that if the decision is correct, no costs result, but incorrect decisions have their price. The costs of a false positive is c_{FP} and that of a false negative is C_{FN}, both greater than zero. Intuitively, the decision rule should minimize the expected costs. That is, a finding against the defendant would be acceptable whenever its expected costs—$\mathsf{P}(H_d|E) \times c_{FP}$—are smaller than the expected costs of an acquittal—$\mathsf{P}(H_p|E) \times c_{FN}$— or in other words:

REFERENCE TO EARLIER CHAPTER FOR MORE COMPLEX COST STRUCTURE

[18]Note that H_p and H_d need not be one the negation of the other. Whenever two hypotheses are exclusive and exhaustive, $\mathsf{P}(H_p|E)/\mathsf{P}(H_d|E) > 1$ implies that $\mathsf{P}(H_p|E) > .5$, the standard probabilistic interpretation of the preponderance standard.

$$\frac{\mathsf{P}(H_p|E)}{\mathsf{P}(H_d|E)} > \frac{c_{FP}}{c_{FN}}.$$

In civil cases, it is customary to assume the costs ratio of false positives to false negatives equals one. So the rule of decision would be: find against the defendant whenever $\frac{\mathsf{P}(H_p|E)}{\mathsf{P}(H_d|E)} > 1$ or in other words $\mathsf{P}(H_p|E)$ is greater than $\mathsf{P}(H_d|E)$. In criminal trials, the costs ratio is usually considered higher, since convicting an innocent (false positive) should be more harmful or morally objectionable than acquitting a guilty defendant (false negative). Thus, the rule of decision in criminal proceedings would be: convict whenever $\mathsf{P}(H_p|E)$ is appropriately greater than $\mathsf{P}(H_d|E)$.

Does the comparative strategy just outlined solve the difficulty with conjunction? We will work through a stylized case used by Cheng himself. Suppose, in a civil case, the plaintiff claims that the defendant was speeding (S) and that the crash caused her neck injury (C). Thus, the plaintiff's hypothesis H_p is $S \wedge C$. Given the total evidence E, the conjuncts, taken separately, meet the decision threshold:

$$\frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \qquad\qquad \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1$$

The question is whether $\mathsf{P}(S \wedge C|E)/\mathsf{P}(H_d|E) > 1$. To answer it, we have to decide what the defense hypothesis H_d should be. Cheng reasons that there are three alternative defense scenarios: $H_{d_1} = S \wedge \neg C$, $H_{d_2} = \neg S \wedge C$, and $H_{d_3} = \neg S \wedge \neg C$. How does the hypothesis H_p compare to each of them? Assuming independence between C and S, we have

$$\begin{aligned}
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(S \wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(S|E)\mathsf{P}(\neg C|E)} = \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1 \\
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(C|E)} = \frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \\
\frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)} > 1
\end{aligned} \tag{20}$$

So, whatever the defense hypothesis, the plaintiff's hypothesis is more probable. At least in this case, whenever the elements of a plaintiff's claim satisfy the decision threshold, so does their conjunction. The left-to-right direction of the conjunction principle—what we called aggregation—has been vindicated, at least for simple cases involving independence.

What about the opposite direction, distribution? If the threshold to be met just 1—as might be appropriate in civil cases—distribution would be satisfied. Suppose $\mathsf{P}(S \wedge C|E)/\mathsf{P}(H_d|E) > 1$, or in other words, the combined hypothesis $S \wedge C$ has been established by preponderance of the evidence. The question is whether the individual hypotheses have been established by the same standard, specifically, whether $\frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1$ and $\frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1$. If $\mathsf{P}(S \wedge C|E)/\mathsf{P}(H_d|E) > 1$, the combined hypothesis is assumed to be more probable than any of the competing hypotheses, in particular, $\mathsf{P}(S \wedge C|E)/\mathsf{P}(\neg S \wedge C|E) > 1$, $\mathsf{P}(S \wedge C|E)/\mathsf{P}(S \wedge \neg C|E) > 1$ and $\mathsf{P}(S \wedge C|E)/\mathsf{P}(\neg S \wedge \neg C|E) > 1$. We have

$$\begin{aligned}
1 &< \frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(S \wedge \neg C|E)} = \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(S|E)\mathsf{P}(\neg C|E)} = \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} \\
1 &< \frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge C|E)} = \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(C|E)} = \frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} \\
1 &< \frac{\mathsf{P}(S \wedge C|E)}{\mathsf{P}(\neg S \wedge \neg C|E)} = \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)}
\end{aligned} \tag{21}$$

In the first two cases, clearly, if the composite hypothesis meets the threshold, so do the individual claims. But now consider the third case. $\mathsf{P}(S|E)\mathsf{P}(C|E)/\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)$ might be strictly greater than $\mathsf{P}(C|E)/\mathsf{P}(\neg C|E)$ or $\mathsf{P}(S|E)/\mathsf{P}(\neg S|E)$. So it possible that $\mathsf{P}(S|E)\mathsf{P}(C|E)/\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)$ is greater than one, while either$\mathsf{P}(C|E)/\mathsf{P}(\neg C|E)$ or $\mathsf{P}(S|E)/\mathsf{P}(\neg S|E)$ are not, say when they are 3 and 0.5, respectively. Once again, distribution fails. And the same problem would arise with a more stringent threshold as might be appropriate in criminal cases.

There is a more general problem with the comparative approach worth flagging here. Much of the heavy lifting here is done by the strategic splitting of the defense line into multiple scenarios. Now suppose, for illustrative purposes, $\mathsf{P}(H_p|E) = 0.37$ and the probability of each of the defense lines given E is 0.21. This means that H_p wins with each of the scenarios, so on this approach we should find against the defendant. But should we? Given the evidence, the accusation is very likely to be false, because $\mathsf{P}(\neg H_p|E) = 0.63$. The problem generalizes. If, as here, we individualize scenarios by Boolean combinations of elements of a case, the more elements there are, into more scenarios $\neg H_p$ needs to be divided. This normally would lead to the probability of each of them being even lower (because now $\mathsf{P}(\neg H_p)$ needs to be "split' ' between more scenarios). So, if we take this approach seriously, the more elements a case has, the more at a disadvantage the defense is. This seems undesirable.

MENTION THAT SINCE CHENG THOUGHT THE COMPARATIVE STRATEGY WOULD CAPTURE INFERENCE TO THE BEST EXPLANATION IN PROBABILISTIC TERMS, THE FAILURE OF THE COMPARATIVE STRATEGY TO CAPTURE THE CONJUNCTION PRINCIPLE IS ALSO A FAILURE OF INFERENCE TO THE BEST EXPLANATION IN CAPTURING THE CONJUNCTION PRINCIPLE SO LONG AS WE ASSUME THAT PLAUSIBILITY DOES NOT CONTRADICT PROBABILITY (WHICH ALLEN AND PARDO DO GRANT)

5 Rejecting the conjunction principle

We have seen that various strategies that a legal probabilist can try to use to handle the difficulty with conjunction are all problematic. Perhaps, a different perspective should be taken here? After all, the problem would not arise without the conjunction principle. So could legal probabilists simply reject this principle?

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5.1 Risk accumulation

In current discussions in epistemology about knowledge or justification, a principle similar to the conjunction principle has been contested because it appears to deny the fact that risks of error accumulate (Kowalewska, 2021). If one is reasonably sure about the truth of each claim considered separately, one should not be equally reasonably sure of their conjunction. You have checked each page of a book and found no error. So, for each page, you are reasonably sure there is no error. Having checked each page and found no error, can you be equally reasonably sure that the book as a whole contains no error? Not really. As the number of pages grow, it becomes virtually certain that there is at least one error in the book you have overlooked, although for each page you are reasonably sure there is no error (Makinson, 1965). A reasonable doubt about the existence of an error, in one page or another, creeps up as one considers more and more pages. The same observation applies to other contexts, say product quality control. You may be reasonably sure, for each product you checked, that it is free from defects. But you cannot, on this basis alone, be equally reasonably sure that all products you checked are free from defects. Since the risks of error accumulate, you must have missed at least one defective product.

Risk accumulation challenges aggregation: even if the probability of several claims, considered individually, is above a threshold t, their conjunction need not be above t. It does not, however, challenge distribution. If, all risks considered, you have good reasons to accept a conjunction, no further risk is involved in accepting any of the conjuncts separately. This is also mirrored by what happens with probabilities. If the probability of the conjunction of several claims is above t, so is the probability of each individual claim.

R: What's our take on risk accummulation?

The standard of proof in criminal or civil cases can be understood as a criterion concerning the degree of risk that judicial decisions should not exceed. If this understanding of the standard of proof is correct, the phenomenon of risk accumulation would invalidate the conjunction principle, specifically, it would invalidate aggregation. It would longer be correct to assume that, if each element is proven according to the applicable standard, the case as a whole is proven according to the same standard. And, in turn, if the conjunction principle no longer holds, the conjunction paradox would disappear. Success.

5.2 Atomistic and holistic approches

Matters are not so straightfoward, however. Suppose legal probabilists do away with the conjunction principle. Now what? How should they define standards of proof? Two immediate options come to mind, but neither is without problems.

Let's stipulate that, in order to establish the defendant's guilt beyond a reasonable doubt (or civil liability by preponderance of the evidence or clear and convincing evidence), the party making the accusation should establish each claim, separately, to the requisite probability, say at least .95 (or .5 in a civil case), without needing to establish the conjunction to the requisite probability. Call this the *atomistic account*. On this view, the prosecution could be in a position to establish guilt beyond a reasonable doubt without establishing the conjunction of different claims with a sufficiently high probability. This account would allow convictions in cases in which the probability of the defendant's guilt is relatively low, just because guilt is a conjunction of several independent claims that separately satisfy the standard of proof. For example, if each constituent claim is established with .95 probability, a composite claim consisting of five subclaims—assuming, as usual, probabilistic independence between the subclaims—would only be established with probability equal to .77, a far cry from proof beyond a reasonable doubt. This is counterintuitve, as it would allow convictions when the defendant is not very likely to have committed the crime. A similar argument can be run for the civil standard of proof 'preponderance of the evidence.' Under the atomistic account, the composite claim representing the case as a whole would often be established with a probability below the required threshold. The atomistic approach is a non-starter.

Another option is to require that the prosecution in a criminal case (or the plaintiff in a civil case) establish the accusation as a whole—say the conjunction of A and B—to the requisite probability. Call this the *holistic account*. This account is not without problems either.

The standard that applies to one of the conjuncts would depend on what has been achieved for the other conjuncts. For instance, assuming independence, if $\mathsf{P}(A)$ is .96, then $\mathsf{P}(B)$ must be at least .99 so that $\mathsf{P}(A \wedge B)$ is above a .95 threshold. But if $\mathsf{P}(A)$ is .9999, then $\mathsf{P}(B)$ must only be slightly greater than .95 to reach the same threshold. Thus, the holistic account might require that the elements of an accusation be proven to different probabilities—and thus different standards—depending on how well other claims have been established. This result runs counter to the tacit assumption that each element should be established to the same standard of proof.

Fortunately, this challenge can be addressed. It is true that different elements will be established with different probabilities, depending on the probabilities of the other elements. But this follows from the fact that the prosecution or the plaintiff may choose different strategies to argue their case. They may decide that, since they have strong evidence for one element and weaker evidence for the other, one element should be established with a higher probability than the other. What matters is that the case as a whole meets the required threshold, and this objective can be achieved via different means. What will never happen is that, while the case as a whole meets the threshold, one of the constituent elements does not. As seen earlier, the probability of the conjunction never exceeds the probability of one of the conjunct, or in other words, distribution is never violated.

A more difficult challenge is the observation that the proof of $A \wedge B$ would impose a higher requirement on the separate probabilities of the conjuncts. If the conjunction $A \wedge B$ is to be proven with at least .95 probability, the individual conjuncts should be established with probability higher than .95. So the more constituent claims, the higher the posterior probability for each claim needed for the conjunction to meet the requisite probability threshold.

This difficulty is best appreciated by running some numbers. Assume, for the sake of illustration, the independence and equiprobability of the constituent claims. If a composite claim consists of k individual claims, these individual claims will have to be established with probability of at least $t^{1/k}$, where t is the threshold to be applied to the composite claim.[19] For example, if there are ten constituent claims, they will have to be proven with $.5^{1/10} = .93$ even if the probability threshold is only .5. If the threshold is more stringent, as is appropriate in criminal cases, say .95, each individual claim will have to be proven with near certainty. This would make the task extremely demanding on the prosecution, if not downright impossible. If there are ten constituent claims, they will have to be proven with $.95^{1/10} = .995$. So the plaintiff or the prosecution would face the demanding task of establishing each element of the accusation beyond what the standard of proof would seem to require.

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## [1] 0.933033
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## [1] 0.9948838
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[19]Let p the probability of each constituent claim. To meet threshold t, the probability of the composite claim, p^k, should satisfy the constraint $p^k > t$, or in other words, $p > t^{1/k}$.

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## [1] 0.5987369
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It is true that the individual elements (the individual conjuncts) should be established with a higher probability than the case as a whole (the conjunction). This would seem to impose an unreasonably stringent burden of proof on the prosecution or the plaintiff. But the burden might not be as unreasonable as it appears at first. As Dawid (1987) pointed out, in one of the earliest attempts to solve the conjunction paradox from a probabilistic perspective, the prior probabilities of the conjuncts will also be higher than the prior probability of their conjunction:

> . . . it is not asking too much of the plaintiff to establish the case as a whole with a posterior probability exceeding one half, even though this means that the several component issues must be established with much larger posterior probabilities; for the *prior* probabilities of the components will also be correspondingly larger, compared with that of their conjunction (p. 97).

Dawid's proposal is compelling. Still, why exactly is it 'not asking too much' to establish the individual conjuncts by a higher threshold than the case as a whole? The prior probabilities of the conjuncts are surely higher than the prior probability of the conjunction. But what is the notion of 'not asking too much' at work here? Dawid might be recommending—as the rest of his paper suggests—that the standard of proof no longer be understood in terms of just posterior probabilities. Measures of how strongly each claim is being supported by the evidence, such as the Bayes' factor of the likelihood ratio, account for the difference between prior and posterior probabilities. So, presumably, Dawid is recommending these measures as better suited to formalize the standard of proof.

Now, as the reader will have realized, we have pursued Dawid's strategy already. This strategy can justify, on purely probabilistic grounds, one direction of the conjunction principle: aggregation. The providential support—measured by the Bayes' factor or the likelihood ratio—for the conjunction often exceeds the individual support for the individual claims. This is a success, especially because the failure of aggregation motivated Cohen's formulation of the conjunction paradox. Unfortunately, we have also seen that this strategy invalidates a previously unchallenged direction of the conjunction principle, distribution. This outcome is very counterintuitive. Clearly, we cannot appeal to risk accumulation to challenge distribution since no additional risk arises after making an inference from a conjunction to one of its conjunct.

MAYBE ADD STUFF ABOUT FACTORING OPUT PRIORS

We reached an impasse. Under the atomistic approach, the standard is too lax because it allows for findings of liability when the defendant quite likely committed no wrong. Under the holistic approach, the standard is too demanding on the prosecution (or the plaintiff) because it requires the individual claims to be established with extremely high probabilities. Dawid's approach is compelling, but switching from posterior probability to measures of evidential support invalidate distribution, a seemingly unobjectionable principle. Bummer.

5.3 The holistic approach revised

It is time to take a fresh start on the holistic approach. When suitably formulated, we believe the holistic approach can satisfactorily address the difficulty with conjunction. Our proposal is inspired by the story model of adjudication (HASTIE, VAN KOPPENS, MACKROR) and the relative plausibility theory (ALLEN AND PARDO). It posits that prosecutors and plaintiffs should aim to establish a unified narrative of what happened or explanation of the evidence, not establish each individual element of wrongdoing separately. As we shall see, any attempt to proceed in a piecemeal manner implicitly requires, sooner or later, to weave the different elements together into a unified whole. Our argument consists of two parts. First, the guilt or civil liability of a defendant on trial cannot be equated with a generic claim of guilt or civil liability as defined in the law. Call this the specificity argument. Second, it is erroneous to think of someone's guilt or civil liability as the mere conjunction of separate claims. The separate claims must be unified, not just added up in a conjunction. Call this the unity argument.

R: can you very briefly indicate why?

5.3.1 The specificity argument

Let's start with the specificity argument. The probabilistic interpretation of proof standards usually posits a threshold that applies to the posterior probability of a *generic* hypothesis, such as the defendant is guilty of a crime, call it G, or civilly liable, call it L. In criminal cases, the requirement is formulated as follows: the evidence E presented at trial establishes guilt beyond a reasonable doubt provided $\mathsf{P}(G|E)$ is above a suitable threshold, say .95. The threshold is lower in civil trials. Civil liability is proven by preponderance provided $\mathsf{P}(L|E)$ is above a suitable threshold, say .5. In case of the clear and convincing standard, the threshold is assumed to be more stringent than the preponderance standard, but not as stringent as proof beyond a reasonable doubt.

While seemingly innocuous, this formulation conflates two things. The wrongdoing as defined in the applicable law is one thing. The way in which the wrongdoing is established in court is quite another thing. The wrongdoing is defined in a generic manner and is applicable across a large class of situations, whereas the way it is established in court is specific to a unique situation and tailored to the individual defendant. A prosecutor in a criminal case does not just establish that the defendant assaulted the victim in some way or another, but rather, that the defendant behaved in such and such a manner in this time and place, and that the specific behavior in question fulfills the legal definition of assault.

R: can you say a bit more about why the specific narration is needed? Or do you do this later on?

If this is correct, the probabilistic interpretation of proof standard should be revised. The generic claim that the defendant is guilty or civilly liable should be replaced by a more fine-grained hypothesis, call it H_p, the hypothesis put forward by the prosecutor (or the plaintiff in a civil case), for example, that the defendant, given reasonably well-specified circumstances, approached the victim, pushed and kicked the victim to the ground, and then run away. Hypothesis H_p is a more precise description of what happened that entails the defendant committed the wrong. In defining proof standards, instead of saying—generically—that $\mathsf{P}(G|E)$ or $\mathsf{P}(L|E)$ should be above a suitable threshold, a probabilistic interpretation should read: civil or criminal liability is proven by the applicable standard provided $\Pr(H_p|E)$ is above a suitable threshold, where $H_p$is a reasonably specific description of what happened according to the prosecutor or the plaintiff.

This revision of the probabilistic interpretation of standards of proof may appear inconsequential, but it is not. It is the revision we invoked to address the puzzles of naked statistical evidence in another chapter **REFERENCE TO EARLIER HAPER**. Recall the gist of the argument. Consider the prisoner hypothetical, a standard example of naked statistical evidence. The naked statistics E_s make the prisoner on trial .99 likely to be guilty, that is, $\mathsf{P}(G|E_s) = .99$. It is .99 likely that the prisoner on trial is one of those who attacked and killed the guard. This is a generic claim. It merely asserts that the prisoner was – with very high probability – one of those who killed the guard, without specifying what he did, how he partook in the killing, what role he played in the attack, etc. If the prosecution offered a more specific incriminating hypothesis, call it H_p, the probability $\mathsf{P}(H_p|E_s)$ of this hypothesis based on the naked statistical evidence E_s would be well below .99, even though $\mathsf{P}(G|E_s) = .99$. That the prisoner on trial is most likely guilty is an artifact of the choice of a generic hypothesis G. When this hypothesis is made more specific—as should be—this probability drops significantly. And the puzzle of naked statistical evidence disappears.

5.3.2 The unity argument

The specificity argument addresses the problem of naked statistical evidence, but also provides the necessary background for addressing the difficultly about conjunction. In the traditional formulation, not only are G and L understood as generic claims. They are also understood as *mere conjunctions* of simpler claims that correspond to the elements of wrongdoing in the applicable law. Since the probability of a conjunction is often lower than the probability of its conjuncts, the individual claims can be established with a suitably high probability that meets the required threshold even though the conjunction as a whole fails to meet the same threshold. The mismatch between the probability of the conjunction and the probability of its conjuncts gives rise to the difficulty with conjunction.

What triggers the difficulty is that the prosecutor or plaintiff are assumed to establish each element in isolation. If they manage to prove each element to the desired standard, they presumably have meet their burden, or at least this is what the conjunction principle suggests. But here lies another conflation. It is one thing to establish that the defendant committed A and B, where the wrongdoing in question, say assault, is defined in the law as comprising two elements, A and B. It is another thing to establish that the defendant—*this* defendant—committed assault. Someone's guilt (and the same applies to civil liability) cannot be the mere conjunction of generic claims corresponding to the elements of wrongdoing as defined in the law. What is it, then? Someone's guilt is a state of affairs that is described by a well-specified series of events that possess a coherent, structured unity. These events, taken as a whole, can be subsumed under the legal definition that consists of several discrete elements.

The conjunction paradox, as is commonly formulated, is oblivious to the fact that the different elements of a crime should be part of the same episode of wrongdoing. The paradox assumes that criminal or civil wrongdoing have a simple structure and consist in a collection of elements to be proven by the required standard. The law is more complicated. The elements of wrongdoing, in a criminal or civil setting, often possess a complex structure and are not merely separate items each to be proven by the required standard. Consider, for example, the allegation of negligent misrepresentation to be established by clear and convincing evidence. These are the elements a plaintiff should establish:[20]

(E1) defendant supplied false information to guide plaintiff in their business transactions;
(E2) defendant knew the false information was supplied to guide plaintiff;
(E3) defendant was negligent in obtaining or communicating the false information;
(E4) plaintiff relied on the false information;
(E5) plaintiff's reliance on the false information was reasonable; and
(E6) the false information proximately caused damages to the plaintiff.

For the plaintiff to prevail in this type of case, they should prove each element by the required standard. What does that actually require? The plaintiff should first establish, at a minimum, that the defendant supplied false information for the guidance of the plaintiff in the course of a business transaction. Say the defendant, a owner of a vacation resort, told plaintiff, a travel agent, that the resort included amenities that were not actually there, and the plaintiff decided to book several clients at the defendant's resort instead of other resorts. The plaintiff could offer copies of emails communication or screenshots from the resort's website. This would take care of the first element. The second element qualifies the first, in that the defendant *knew* they were supplying false information to guide the business transaction. Clearly, establishing the second element presupposes having already established the first since the second element is a qualification of the first. The truth of the second element entails the truth of the first. If the defendant knowingly supplied false information (second element), they did supply false information (first element). The third element requires to show that the defendant was negligent in obtaining or communicating the false information. This is another qualification that applies to the first element and cannot be proven without having already proven the first element. The fourth and fifth element should be understood together. The plaintiff relied on the false information (fourth element) and such reliance was reasonable (fifth element). In turn, establishing the fourth and fifth element presupposes that the plaintiff did supply false information to begin with (first element). Finally, the sixth element concerns causation of harm. This is again a qualification of the first element and cannot be established without having already established the first element.

As we can see, in the case of negligent misrepresentation, the legal definition already imposes a structure on how the different elements relate to one another. They are not merely separate items that should be proven one by one.

In addition to the legal definition itself, an important theoretical, conceptual point should be made that holds regardless of how wrongdoings are actually defined in the law. The different elements should be part of the same episode or unit of wrongdoing. Even though the different elements may concern actions performed by different people, at different times, in different places, they must be part of the same wrongdoing. The false information that the defendant supplied to the plaintiff must be the information that caused damages to the plaintiff; it must be the information that the plaintiff reasonably relied on during the business transaction in question; it must be the information knowingly supplied by the defendant for the purpose of influencing the business transaction; and so on. Even though each element could be proven separately, establishing the wrongdoing requires weaving them together.

Some might object that, in principle, establishing the individual elements of wrongdoing separately might be enough, at least in some cases, for establish the wrongdoing as a whole. If there are such cases, they are likely to be more the exception than the norm. To see how fundamental it is to establish that the different elements belong to the same unit of wrongdoing, consider a much simpler example than negligent misrepresentation. This example does not impose any structure on how the elements of wrongdoing should relate to one another. Because of its simplicity, this examples is sometimes used to illustrate the conjunction paradox (SEE CHENG). Suppose only two elements must be proven. Element 1: the defendant's conduct caused a bodily injury to victim. Element 2: the defendant's conduct consisted in reckless driving. Call this criminal offense "vehicular assault."[21] The two elements are not independent, but they each add novel information. It could be that the defendant's driving caused an injury to victim, but the driving was not reckless, or the driving was reckless, but no injury ensued.

[20] As an illustration, we follow the jury instructions of the State of Washington. EXACT REFERENCE
[21] CITE RELATED LAWS

Neither element is presupposed by the other. The law does not impose any structure between the elements, unlike the example of negligent misrepresentation.

How could one go about establishing the claim of reckless driving that caused injury to the victim? One option, following the revised holistic approach, is to offer a detailed reconstruction of what happened. The reconstruction could go something like this. The defendant was driving above the speed limit, veering left and right. The defendant's reached a school crosswalk when children were getting out of school. The defendant hit a child on the crosswalk who was then pushed against a light pole on the sidewalk incurring a head injury. This story is supported by plenty of evidence: other children, people standing around, police officers, paramedics. There is plenty of supporting evidence as the incident occurred in the middle of the day. Taken at face value, this story does establish both elements: reckless driving and cause of injury. Parts of the story are relevant for element 1 (reckless driving) and others are relevant for element 1 (cause of injury). The two cannot be neatly separated, however. But what is crucial is that the different parts of the story are part of the same episode, the same unit of wrongdoing.

Could the prosecutor prove vehicular assault in a piecemeal manner? This would be implausible from a conceptual point of view, let alone being at odds with trial practice. But suppose, for the sake of argument, the prosecutor attempted to establish vehicular assault in a piecemeal manner, by establishing, first, that the defendant drove recklessly, and second—*separately from the first element*—that the defendant's reckless driving caused injury. Once both items are established by the required standard, there would still be something left to establish.

As noted before, it is not enough to establish that the defendant drove recklessly at some point in time somewhere. Nor is it enough to establish that the defendant's driving caused injury. The prosecutor should offer a specific story detailing what happened, a story relevant for the first element and a story relevant for the second element. Say this expectation of more specificity is met. Suppose the prosecutor did not simply establish generic element 1 and generic element 2 of the charge, but rather, a reasonably detailed story which, if true, would establish element 1 and a story which, if true, would establish element 2. Wouldn't that be enough? Each element—more precisely, each story associated with each element—has been established by the required standard. Still, rhere would be something missing here.

The prosecutor should establish that the two elements—reckless driving and injury, or the two stories associated with the two elements—are part of the same unity of wrongdoing. It must me *this* reckless driving that caused *this* injury. So, under the piecemeal approach, the prosecutor would be tasked with establishing three claims: (1) the defendant, in some well-specified circumstances, was driving reckless; (2) the defendant, in some well-specified circumstances, caused injury to the victim; and (3) the well-specified circumstances in (1) and (2) are part of the same episode. But once (3) is established, the prosecutor would have effectively established the charge by the required standard in accordance with the holistic approach. The prosecutor did not only establish each separate element (two separate stories) but also wove the two elements (the two stories) together. Once the piecemeal approach is pursued to its logical conclusion, it coincides with the holistic approach.[22]

Let's recapitulate the argument in schematic form. If the prosecutor or the plaintiff is expected to establish claim A and B by the required standard, what the law actually requires—even in terms of the piecemeal approach—is (1) to establish A; (2) to establish B; and (3) to establish A and B are part of the same unit of wrongdoing by the required standard. Item (3) is often implicit, which leaves the impression that the law only requires to establish (1) and (2) separately. Interestingly, (3) entails (1) and (2). In fact, (3) amounts to establishing a unified story, narrative or explanation about what happened. Such a narrative should be subsumed under the different elements of wrongdoing as defined in the law. The piecemeal approach and the revised holistic approach, therefore, converge.

Not all wrongful acts, in civil or criminal cases, require the prosecutor or the plaintiff to establish a unified *spatio-temporal* narrative. It might not be necessary to show that all elements of an offense occurred at the same point in time or in close succession one after the other. Some wrongful acts may consist of a pattern of acts that stretches for several days, months or even years. There may be temporal and spatial gaps that cannot not be filled. We consider several of these examples in our discussion of naked statistical evidence **SEE PREVIOUS CHAPTER**. Be that as it may, an accusation of wrongdoing in a criminal or civil case should still have a degree of cohesive unity. The acts and occurrences that

[22] We should be clear that it is not enough for the prosecutor or plaintiff to provide well-specified narrative in support of their allegations, even when they are well-supported by the evidence. When the two narrative are combined into one narrative, its probability could well be below the threshold. If we only require that each element-specific narrative be proven, a defendant could be found criminally or civilly liable even though it is unlikely that they committed the alleged wrongful act. This counter-intuitive result is similar to the one that arose with the atomistic approach.

constitute the wrongdoing should belong to the same wrongful act. It is this unity which the plaintiff and the prosecution must establish when they make their case. One way to establish this unity is by providing a unifying narrative, but this need not be the only way. Perhaps the expressions 'theory' or 'explanation' are more apt than 'narrative' or 'story.'

5.3.3 Probability, specificity and completeness

There is a distinction between a narrative (or theory, story, explanation, account) and a mere conjunctions of elements of wrongdoing $E_1 \wedge E_1 \wedge \cdots \wedge E_k$. The narrative describes one way among many of instantiating the conjunction. This distinction is important. The claims that constitute a narrative or unified explanation need not map neatly onto the elements of the wrongdoing. The narrative will comprise claims about the evidence itself and how the evidence supports other claims in the narrative, say that witnesses were standing around when the defendant's car hit the child. The narrative or explanation will not only comprise a description of what happened but also of how we know this is what happened.

The distinction between narrative and the mere conjunction of elements matters for how we should understand the standard of proof. Other things being equal, the conjunction is more probable on the evidence than the narrative, and each conjunct even more probable. But this does not mean that the conjunction is established by a higher standard of proof than the narrative. As we argued in the chapter on naked statistical evidence **REFERENCE TO EARLIER CHAPTERS**, a highly probable narrative that nevertheless lacks the desired degree of specificity will fail to meet the standard of proof, even though a more specific narrative that is otherwise less probable might well meet the standard. On this account, the standard of proof consists of two criteria: 1. the posterior probability of the proposed narrative (or theory, story, explanation) given the evidence presented at trial; and 2. the degree of specificity, coherence and unity of the narrative or explanation.

R: remember to add your paper to the folder.

Are we giving up on legal probabilism? We are giving up on *traditional* legal probabilism. Even though ideas such as specificity, coherence and unity cannot be captured by the posterior probability alone, they can be formalized as properties of Bayesian networks. **NEED BRIEF DISCUSSION OF HOW THIS WORKS IN GENEREAL AND REFERENCES** Further, this analysis of the standard of proof – which combines two criteria, posterior probability and specificity – can be evaluated using concepts from probability theory that are not posterior probabilities. To illustrate, compare a trial system that convicts defendants on the basis of accusatory claims that generic but highly probable, as opposed to a trial system that convicts defendants on the basis of accusatory claims that are more specific but less probable. A natural question to ask at this point is, which trial system will make fewer mistakes – fewer false convictions and false acquittals – in the long run? The answer is not obvious. But the question can be made precise in the language of probability. The question concerns the diagnostic properties of the two trial systems, specifically their rate of false positives and false negatives. We examine this question in a later chapter. **REFERENCE TO LATER CHAPTERS** To anticipate, we argue that more specific claims are liable to more extensive *direct* adversarial scrutiny than generic claims. The more specific someone's claim, the more liable to be attacked. At the same time, if a specific claim resists adversarial scrutiny, it becomes more firmly established than a less specific claim that survived scrutiny by evading the questions. So specificity plays an accuracy-conducive role even though more specific claims are, other things being equal, less probable than more generic claims.[23] That is why specificity should be an important ingredient in any theory of the standard of proof.

R: fill in with a short description.

R: talk a bit about accuracy-wise consideratios even prior to scrutiny.

Another ingredient worth adding to posterior probability and specificity is the completeness of the evidence presented at trial. Could the probability of someone's guilt be extremely high just because the evidence presented is one-sided and missing crucial pieces of information? It certainly can. If the probability of guilt is high because of the evidence is partial, guilt was not proven beyond a reasonable doubt. It is a matter of dispute whether knowledge about the partiality of the evidence should affect the posterior probability. After all, if we know that evidence about a hypothesis is missing, shouldn't we revise the assessment of the posterior probability of the hypothesis? This may be true, but the problem is that the content of the missing evidence is unknown. The missing evidence might increase or decrease this probability. We cannot know that without knowing what the evidence turns out to be. If we knew how the missing evidence would affect our judgment about the defendant's guilt, the evidence would no longer be — strictly speaking – missing.

THIGNS TO ADD: 0. POINT ABOUT UNITY 1. ROLE OF COMPLETENESS OF EVIDENCE,

[23] CITE POPPER HERE

SEE OREGON DNA CASE, MEANT TO SHOW THAT HOLISTIC AOPPROACH IS NOT AD HOC, WEIGHT, RESILIENCE IMPORTANT FOR ASESSING STRENGHT OF EVIDENCE AND STANDART OF PROOF, GUILT COULD BE HIGHLY PROBABLE GIVEN AVAILABLE EVIDEMCE, BUT THIS NEED NOT BE ENOUGH IF EVIDENCE THAT SHOULD BE THERE IS MISSING 2. NOT DENYING THAT NARRATIVES ARE CONJUCTION, BUT THE CONJUCTS DO NOT MAP NEATLY ONTO THE ELEMENTS OF A CRIME.

5.3.4 The connjuction principle revised

What does this all tell us about the conjunction paradox? Say we take seriously the idea that the standard of proof is a legal device to optimize the satisfaction of the following three criteria:

1. The defendant's civil or criminal liability must be sufficiently high
2. The narrative, story, theory, explanation that details the defendant's civil or criminal liability should be sufficiently (reasonably) specific
3. The supporting evidence should be sufficiently (reasonably) complete

The conjunction paradox could hardly arise given this conception of the standards of proof. It could hardly arise because the individual elements of the crime are established via the mediation of a reasonably specific and sufficiently probable narrative whose supporting evidence is reasonably complete.

What about the conjunction principle? Taking seriously the idea that prosecutors and plaintiffs should aim to establish a well-specified, unified account of the wrongdoing trivializes this principle and thus dissolves the difficult about conjunction. Suppose the prosecutor has established a narrative N to a very high probability, say above the required threshold for proof beyond a reasonable doubt. Denote the elements of wrongdoing by $E_1, E_2, \ldots$. Then,

$$\mathsf{P}(E_1 \wedge E_1 \wedge \cdots \wedge E_k | N) = \mathsf{P}(C_i | N) = 1 \text{ for any } i = \{1, 2, ..., k\}.$$

Both direction of the conjunction principle, aggregation and distribution, are now trivially satisfied. Once we condition on the narrative N, each individual claim has a probability of one and thus their conjunction also has a probability of one. The narrative, however, has a probability short of one, up to whatever value is required to meet the governing standard of proof. This is the way we suggest that we make sense of the conjunction principle.

R: and the trivialization is unsuprising and desirable given that no lawyer every was concerned with the reliability of conjunction elimination or introduction.

5.4 Notes on the argument

- Need to insists on risk accumulation. Many reasons why probability of conjunction is less than probability of conjunct. This cannot be denied. Examples. Illustrations.
- What is the role of defeaters? Say A has defeater Da (undercutter) and not-A (rebutter) and B has defeater Db (undercutter) and not-B (rebutter). If we can rule out Da and Db and not-A and not-B, could we say that the we have ruled out all defeaters for AB? Not necessarily. The rebutter not-AB need not be a rebutter for A or B, and there could be an undercutter Dab that does not undercut A or B. Need to give examples.
- Key move. We do not establish claims A and B in a vacuum. They need to be part of a narrative, explanation, story. The story gives the broader context, but also explains how the evidence support the claim and how challenges are addressed. Call this Na and Nb. The task then is to put together Na and Nb.
- Could whole narrative N be more coherent (and thus more probable) than Na and Nb? Boost of probability due to coherence.
- Elements of an accusation are structured in different ways. Often we see an "accumulation" structure, where each elements is intended to provide an even more specific description of the crime. So the structure is something like this. Element 1: A. Element 2: A + B. Element 3: (A + B) + C. Etc. I such cases the conjunction paradox does not apply.
- Suppose we isolate parts of the story that are relevant for Element 1, call it S1, and parts of the story that are relevant for Element 2, call it S2. Since both sub-stories are indexed through a time and space, it is clear that they form the same unity. Now S1 and S2 will be more likely than S. There is no doubt about that. This is a probabilistic fact. And fact about the world. But S1 and

S2 will also be much less detailed than S. There would be several question that S1 and S2 do not address. S1 will not address whether the driver hit anyone or got on the sidewalk, while S2 presumably would not address what the car was driving before hitting the child. Now specificity is part of the standard of proof. The standard of proof require adequate probability and adequate specificity. These two direction cannot be directly compared. Sp, in a sense S1 and S2 are more firmly established because they are more likely, but S is more firmly established because it is more specific. The standard of proof includes probability and specificity.

- Does that mean that specificity is not a probabilstic notion? Cite Popper. How does the notion of specificity relate to coherence?
- Another dimension of the standard of proof is however resistance to challenges, or else one could just come up with a very specific story that one has completely made up. A more specific story is more liable to be proven wrong. This is basically why it is less probable, other things being equal. There are any more ways it could be false compared to a less specific story. But now compare a very specific story – that ahs survived all reasonable challenges directed at the story – to a less specific story that – that has survived all reasonable challenges directed at the story. Which one should we be more impressed by? It seems taht the more specific story that has survived all the challenges should be at least as well (if not more) established than the less specific story that has survived all the challenges? Here is an analogy. Suppose you have gone out in a mountaineering expedition, while your friend has just gone out to stroll. You both comeback home safe (you survived the challenges), but which one is more indicative of strong survival skill? Clearly, the fact that your survived a mountaineer expedition, not just a stroll. The initial probability of survival was lower in the former than in the latter.
- But the problem recurs. What does it mean to survive challenges? Can this be explicated in probabilistic terms?
- Suppose we are understanding the standard of proof as consisting of: probability, specificity, resistance to challenge, completeness of evidence. The objection here might be that this criteria could violate accuracy maximization. Perhaps we need a simulation to show that following these four criteria is actually better for accuracy than just probability. Probability does not capture all sources of uncertainty at play and thus it is an inadequate criterion for decision.

5.5 What is our argument? [M:sketch, not intended to be part of chapter]

OPTION 1. Given risk accumulation, we reject the conjunction principle (specifically, we reject aggregation). But if we reject aggregation, we are still left with two problems: (1) individual claims will have to be proven with extremely high probability, which seems to make the task too demanding for prosecutors and (2) claims will have to be proven with different standard of proof not equally, not standard would not apply uniformly across claims. Problem (1) can be addressed by noting that individual claims will have higher prior probabilities, so this would not make the task too demanding on prosecutors. We need here to develop Dawid's argument in the direction of factoring out the priors. Problem (2). Not sure.

CHALLENGE We need to come up with a formal way to factor out priors. See some attempts below. But since measures of evidential support one way or another depend on priors, it's not clear how this could be done properly. So basically, we would need to come up with a formal measure of "how hard it is prove something" and this formal measures should capture the "burden of proof" and should return the intuitive results we expect.

OPTION 2. We switch to narratives. Prosecutor establishes a narrative to the required probability, $\mathsf{P}(N|E)$. The narratives entails all the separate claims that needs to be proven. Conditional on the narrative, the conjunction principle is trivially satisfied.

PROS. The switch to narratives captures the complexity of proof, different structured claims in relationships of dependence or independence.

PROBLEM. Not clear what this move actually accomplishes. Seems it changes the topic.

OPTION 3. We attack the conjunction principle upfront and show there are two different principles, each captured by probability theory – i.e. posterior probability captures distribution, while strength of evidence captures aggregation. The conjunction principle runs these two direction together as though they must be part of the same things. Perhaps not. Perhaps the standard of proof can be understood as a mixture of these two approaches.

OPTION 4. The standard of proof cannot just be based on a single criterion, say a threshold on posterior probability or a threshold on evidential strength. While these are important, most likely necessary conditions to be met, they are not sufficient. What else? The trial is an adversarial process, in which the evidence and the story or explanations proposed by one of the two parties or both are scrutinized, tested, challenged. Rebutting and undercutting defeaters must be ruled out. The most complete body of evidence, for and against the defendant, must be considered. No evidence should be left out. It is true that the posterior probability of a conjunction will tend to be lower than posterior probability of individual conjuncts. But a conjunction – the case as a whole – will typically also have a much greater degree of coherence than individual conjuncts; it will have survived more challenges as it is liable to more challenges, and the completeness of the evidence is more easily ascertained relative to the overall case. So given this multidimensional theory of the burden of proof, it is not obvious that, other things being equal, the conjunction receives a weaker support than the the conjuncts or that it should receive a stronger support than the conjuct. This will depend on a variety of factors.

See Spottswood paper, Unraveling the Conjunction Paradox

OPTION 5. p-values, what about the conjucntion of p-values?

CHAPTER ENDS HERE

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6 Extra unstructured materials

6.1 Prior Probabilities

It is worth examining the holistic account more closely, focusing in particular on the role of prior probabilities, an aspect that has gone unnoticed so far. The main problem with the holistic approach is that it would require, especially in criminal cases, individual claims to be established with a very high probability, often making the task unsurmountable for the prosecution. Or so it would seem. But a composite claim such as $A \wedge B$ will have, other things being equal, a lower prior probability than any individual claim A or B. Say a composite claim consists of k individual claims. If its prior probability is π, each constituent claim, assuming they are independent and equiprobable, will have a prior probability of $\pi^{1/k}$. The prior probability of the individual claims will approach one as the number of constituent claims increases.

Cohen worried that, as the number of constituent claims increases, the prosection or the plaintiff would see their case against the defendant become progressively weaker and it would become impossisble for them to establish liability. But this worry is an exaggeration. The paradox, as is commonly formulated, starts by assuming that the constituent claims are established by the required probability threshold and then shows that the probability of the conjuction may fall below the threshold. However, following the holistic approach, the order of presentation can be reversed. Start by assuming that the composite claim is established by the required probability threshold. No doubt the individual claims will have to be established with a higher probability, a violation of the conjuction principle. Yet, this violation is not as counterintuitive as it might first appear for two reasons. First, since risks aggregate, it is natural that the probability of a conjunction would be lower than the probability of the conjuncts. Second, the prior probabilities of the conjuncts will be higher than the prior probability of the conjunction. Thus, establishing the conjuncts with a higher probability will not be exceedingly demanding.

Along this lines, Dawid (1987), in one of the earliest attempts to solve the conjunction paradox from a probabilistic perspective, wrote:

> ...it is not asking too much of the plaintiff to establish the case as a whole with a posterior probability exceeding one half, even though this means that the several component issues must be established with much larger posterior probabilities; for the *prior* probabilities of the components will also be correspondingly larger, compared with that of their conjunction (p. 97).

The price of this strategy is the denial of the conjunction principle, specifically aggregation, the very motivation behind the conjunction paradox. Cohen could insist that this solution amounts to denying the paradox. To address the paradox, legal probabilists should offer a justification of the conjunction principle in probabilistic terms, something Cohen maintains cannot be done. Or can it be done?

6.2 The Conjunction Principle Is False

Neither the Bayes factor nor the likelihood ratio managed to fully justify both directions of the conjunction principle. One direction, aggregation, was justified. So the original concern that was driving Cohen's formulation of the conjuction paradox was addressed. But the other direction, distribution, failed. The failure of distribution creates a paradox of its own, what we called distribution paradox. It is odd that one could have sufficiently strong evidence in support of $A \wedge B$, while not having sufficiently strong evidence for A or B. This occurs even when A and B are probabilistically independent. If they were dependent of one another—say A and B were mutually reinforcing—it is possible the evidence would strongly support the conjunction, but not one of the conjuncts in isolation (becuase the additional support from the other claim, A or B, would be missing). But the failure of distribution manifests itself even when A and B are independent. What should we make of this? This problem exists for both the Bayes factor and the likelihood ratio.

6.3 Factoring Out Prior Probabilities

Let us return to the role of prior probabilities and their effect on measures of evidential strength. Dawid observed that the prior probabilities of the conjuncts are correspondingly higher than the prior probability of the conjunction. The conjunction principle, instead, ignores the role of prior probabilities and treat the conjuncts and the conjuction only in relation to the evidence, irrespective of the prior probabilities. So, in order to capture the conjunction principle, legal probabilists should rely on probabilistic measures that are not heavily depend on prior probabilities. But, as we have seen, neither the Bayes factor nor the likelihood ratio are such measures.

We have seen that the joint Bayes factor $\mathsf{P}(a \wedge b|A \wedge B)/\mathsf{P}(a \wedge b)$, under suitable independence assumptions, is greter than the individual Bayes factors $\mathsf{P}(a|A)/\mathsf{P}(a)$ and $\mathsf{P}(b|B)/\mathsf{P}(b)$. This inequality holds even if the evidence is held constant. The joint Bayes factor $\mathsf{P}(a \wedge b|A \wedge B)/\mathsf{P}(a \wedge b)$, under suitable independence assumptions, is still greter than the individual Bayes factors (for composite evidence) $\mathsf{P}(a \wedge b|A)/\mathsf{P}(a \wedge b)$ and $\mathsf{P}(a \wedge b|B)/\mathsf{P}(a \wedge b)$. But the larger Bayes factor associated with the composite claim, holding the evidence fixed, need not be a sign of stronger evidence, but merely an artifact of the lower prior probability of the composite claim. The same can be said for the combined likelihood ratio. We have seen that, holding fixed the sensitivty and specificity of a and b, the combined likelihood ratio can be changed by varying the priors of A and B. The lower the priors, the stronger the likelihood ratio. Perhpas, the same body of evidence may strongly suport the composite claim $A \wedge B$, while failing to strongly support A or B simply because $A \wedge B$ has a lower prior probability and this lower prior probability, everything else being equal, inflates the likelihood ratio or the Bayes factor *qua* measures of evidential strength.

Intuitively, the strength of the evidence should not depend on the prior probability of the hypothesis, but solely on the quality of the evidence itself. We will later see that this intuition is not completely correct, but it has a great deal of plausibility, so it is worth taking it seriously. The prior probability of the hypothesis seems extrinsic to the quality of the evidence since the latter should solely depend on the sensitivity and specificity of the evidence relative to the hypothesis of interest. Strength of evidence determines how much the evidence changes, upwards or downwards, the probability of a hypothesis. However, as the prior probability increases, the smaller the impact that the evidence will have on the probability of the hypothesis. If the prior is close to one, the evidence would have marginal if not null impact. But this does not mean that the evidence weakens as the prior probability of the hypothesis goes up. For consider the same hypothesis which in one context has a very high prior probability and in another has a moderate prior probability (say a disease is common in a population but rare in another). The outcome of the same diagnostic test (say a positive test result) performed on two people, each drawn from two populations, should not count as stronger evidence in one case than in the other. After all, it is the same test that was performed and thus the quality of the evidence should be the same. For just one item of evidence, Bayes factor does not capture this intuition, but the likelihood ratio does, which can be considered an argument in favor of the latter and against the former measure of evidenetial support. However, we have seen that, for more than one item of evidence, the Bayes factor as well as the likelihood ratio are prior dependent.

To circumvent the phenomenon of prior dependency, evidential strength can be thought as a relationship between prior and posterior probabilities. The graph in Figure 8 below represents to what extent the evidence changes the probability of a select hypothesis for any value of the prior probability of the hypothesis. The graph compares the 'base line' (representing no change in probability) and the 'posterior line' (representing the posterior probability of the hypothesis as a function of the prior for a given assignment of sensitivity and specificity of the evidence). Roughly, the larger the area between the base line and the posterior line, the stronger the evidence. Crucially, this area does not depend on the prior probability of the hypothesis, but solely on the sensitivity and specificity of the evidence. As expected, any improvement in sensitivity or specificity will increase the area between the base line and the posterior line. To be sure, what matters is the ratio of sensitivity to 1-specificity, not their absolute value. So evidence with sensitivity and specificity of 0.9 and 0.9 would be equally strong as evidence with sensitivity and specificity at 0.09 and 0.09 becaue $0.9/(1-0.9) = 0.09/(1-0.09)$.

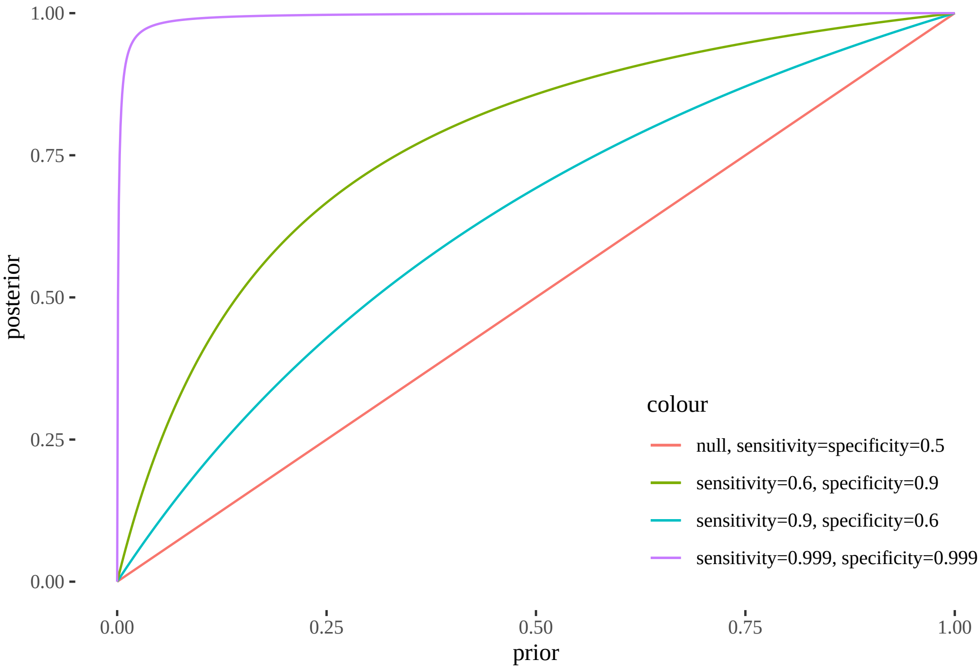


Figure 8: The further away the posterior line from the base line, the stronger the evidence irrespective of the prior probability of the hypothesis.

The same approach can model the joint evidential strength of two items of evidence, $a \wedge b$, relative to the combined hypothesis, $A \wedge B$. For simplicity, assume a and b are independent lines of evidence supporting their respective hypothesis A and B. Further, assume A and B are probabilistically independent of the other, as in the Bayesian network in Figure **??** (top). The graph in Figure 10 (top) shows how the prior probabilities are impacted by evidence in support of a single hypothesis—say $a \wedge b$ supports A[24]—versus evidence in support of a joint hypothesis—say $a \wedge b$ supports $A \wedge B$. The base line is lower in the latter than in the former case because the prior probability of $A \wedge B$ is lower than the prior probability of A. The prior of A equals x and the prior of $A \wedge B$ equals x^2 (assuming A and B have the same prior probability, and as noted before, are probabilistically independent of one another).

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 1)$val
```

```
## [1] 0.9628366
```

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 2)$val
```

```
## [1] 0.9815782
```

[24]Given the assumptions of independence we are working with, the strength of a in support of A is the same the strength of $a \wedge b$ in support of A since $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)\mathsf{P}(b|A)}{\mathsf{P}(a)\mathsf{P}(b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)}$.

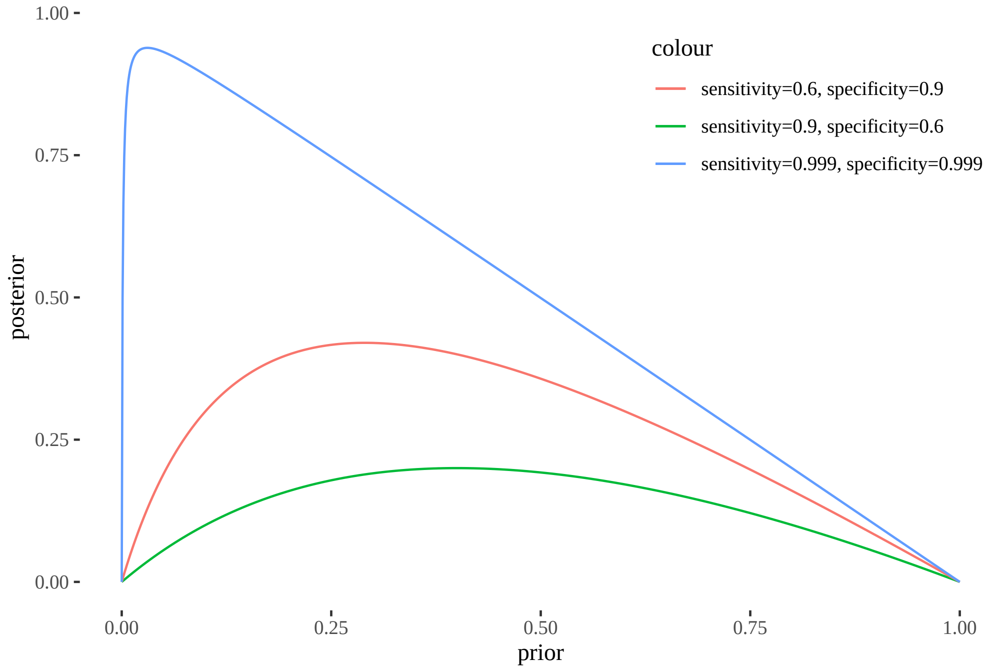


Figure 9: Difference between priors and posteriors as a measure of te strength of evidence.

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 5)$val
```

```
## [1] 0.9876204
```

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 10)$val
```

```
## [1] 0.9889299
```

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 20)$val
```

```
## [1] 0.989491
```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 1)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 2)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 5)$val
```

```
## [1] 0.5
```

```
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 10)$val
```

```
## [1] 0.5
```

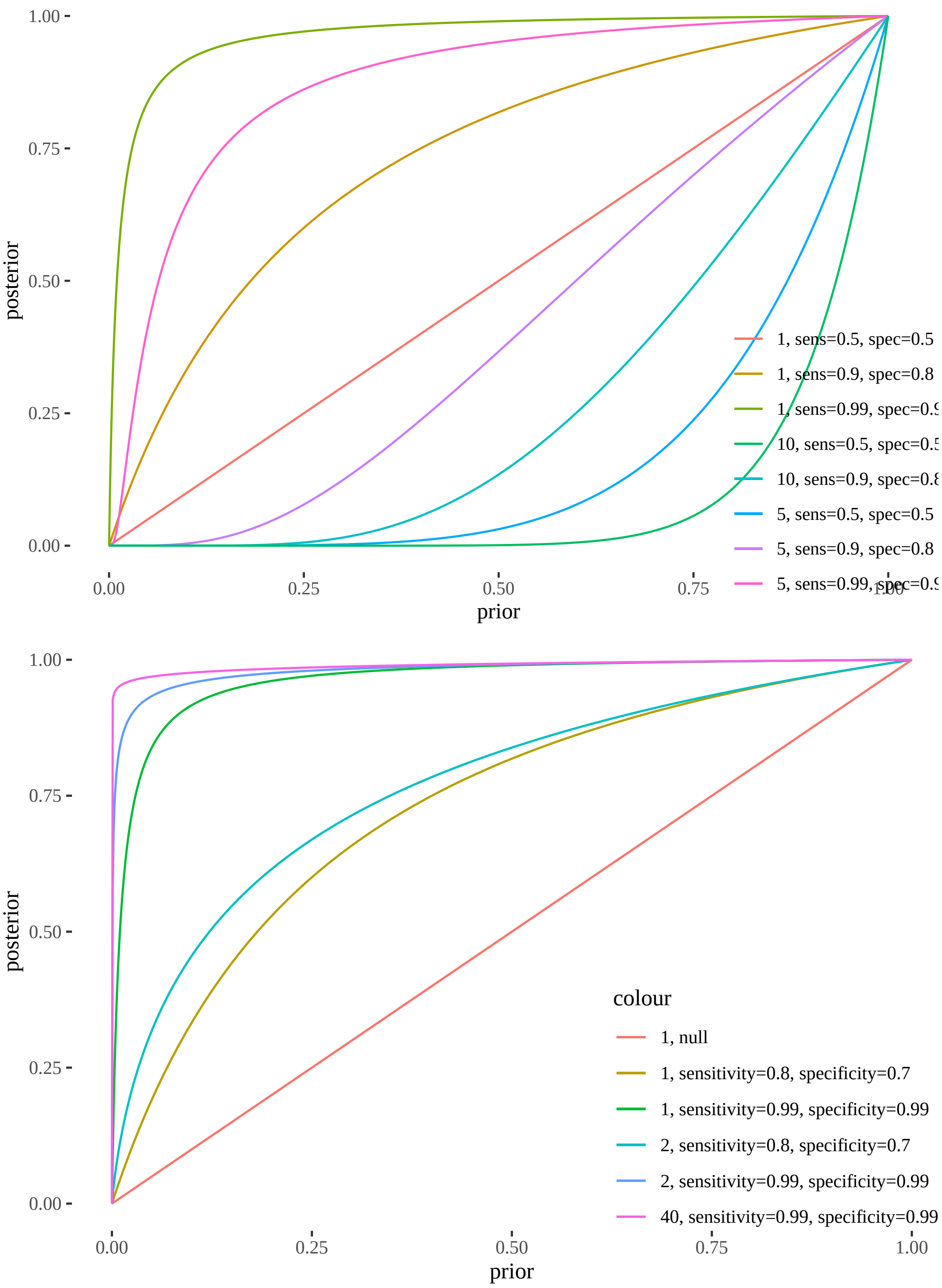


Figure 10: The comparison is between individual support (marked by 1, for one individual hypothesis) and joint support (marked by 2, for a two-claim composite claim). Top graph: The base line for joint support ($y = x * x$) is below the base line for individual support ($y = x$). Bottom graph: the two base lines are equalized and the posterior lines adjusted accordingly. The posterior lines for individual and joint support get closer especially for high posterior probability values.

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 20)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 1)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =
```

```
## [1] 0.4628366
```

```r
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 2)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =
```

```
## [1] 0.4815782
```

```r
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 5)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =
```

```
## [1] 0.4876204
```

```r
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 10)$val - integrate(postn, 0, 1, s1 =0.5 , s2
```

```
## [1] 0.4889299
```

```r
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 20)$val - integrate(postn, 0, 1, s1 =0.5 , s2
```

```
## [1] 0.489491
```

```r
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 25)$val - integrate(postn, 0, 1, s1 =0.5 , s2
```

```
## [1] 0.4895968
```

```r
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99, n = 40)$val - integrate(postn, 0, 1, s1 =0.5 , s2
```

```
## [1] 0.4897516
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 1)$val
```

```
## [1] 0.6137056
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 2)$val
```

```
## [1] 0.6355324
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 5)$val
```

```
## [1] 0.65284
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 10)$val
```

```
## [1] 0.6595075
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 20)$val
```

```
## [1] 0.663025
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 1)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 2)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 5)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 10)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 20)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 1)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.
```

```
## [1] 0.1137056
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 2)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.
```

```
## [1] 0.1355324
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 5)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.
```

```
## [1] 0.15284
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 10)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0
```

```
## [1] 0.1595075
```

```r
integrate(postn, 0, 1, s1 =0.6 , s2 =0.7, n = 20)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0
```

```
## [1] 0.163025
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 1)$val
```

```
## [1] 0.7331961
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 2)$val
```

```
## [1] 0.7717311
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 5)$val
```

```
## [1] 0.7990992
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 10)$val
```

```
## [1] 0.8086466
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 20)$val
```

```
## [1] 0.8134273
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 1)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 2)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 5)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 10)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.5 , s2 =0.5, n = 20)$val
```

```
## [1] 0.5
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 1)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.
```

```
## [1] 0.2331961
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 2)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.
```

```
## [1] 0.2717311
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 5)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0.
```

```
## [1] 0.2990992
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 10)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0
```

```
## [1] 0.3086466
```

```r
integrate(postn, 0, 1, s1 =0.9 , s2 =0.8, n = 20)$val - integrate(postn, 0, 1, s1 =0.5 , s2 =0
```

```
## [1] 0.3134273
```

```r
integrate(post, 0, 1, s1 =0.99 , s2 =0.99)$val
```

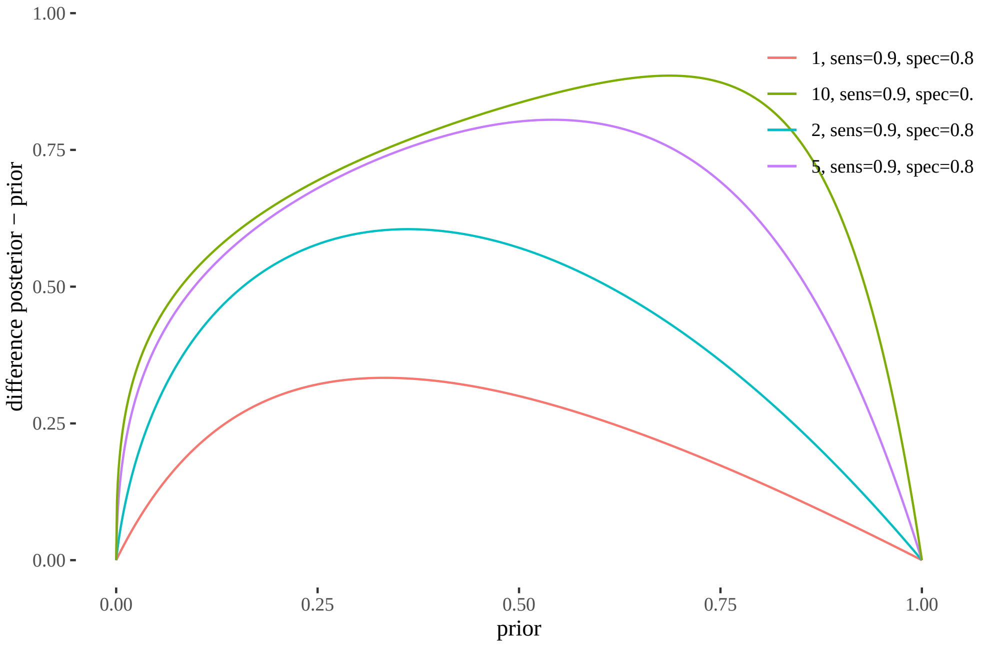
```
## [1] 0.9628366
```

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99 , n = 2)$val
```

```
## [1] 0.9815782
```

```
integrate(postn, 0, 1, s1 =0.99 , s2 =0.99 , n = 5)$val
```

```
## [1] 0.9876204
```




```
prior_a <- 0.6
prior_b <- 0.4
sen_a <- 0.85  # Pr(a given A)
spe_a <- 0.75  #  Pr(not-a given not-A)
sen_b <- 0.7
spe_b <- 0.75

prior_ab <- prior_a*prior_b

bf_a <- sen_a/((sen_a*prior_a)+((1-spe_a)*(1-prior_a)))
bf_b <- sen_b/((sen_b*prior_b)+((1-spe_b)*(1-prior_b)))
bf_ab <- bf_a*bf_b

post_a <- bf_a*prior_a
post_b <- bf_b*prior_b
post_ab <- bf_ab*prior_ab

prior_a
```

```
## [1] 0.6
```

```
prior_b
```

```
## [1] 0.4
```

```
prior_ab
```

```
## [1] 0.24
```

```
post_a
```

```
## [1] 0.8360656
```

```
post_b
```

```
## [1] 0.6511628
```

```
post_ab
```

```
## [1] 0.5444148
```

```
prior_a <- 0.24
prior_b <- 0.24

bf_a <- sen_a/((sen_a*prior_a)+((1-spe_a)*(1-prior_a)))
bf_b <- sen_b/((sen_b*prior_b)+((1-spe_b)*(1-prior_b)))
bf_ab <- bf_a*bf_b

post_a <- bf_a*prior_a
post_b <- bf_b*prior_b
post_a
```

```
## [1] 0.5177665
```

```
post_b
```

```
## [1] 0.4692737
```

What happens if we make the same comparison between individual and composite claims by equalizing their prior probability? If the claims are independent and equiprobable, let x be the prior probability of an individual claim (when it is considered in isolation) and let $x^{1/k}$ the prior probability of the same individual claim when it is part of a composite claim that consists of k claims. In this way—and again, assuming independence and equiprobability of the hypotheses—the prior probability of the composite claim equals the prior probability of the individual claim since $(x^{1/k})^k = x$, as desired. These different claims are then plotted having the same priors. Here we are explicitly factoring out the role of prior probabilities. Figure 10 (bottom) shows the result of this process of equalization.

We observe two things. First, the difference in posterior probability, though still present, is less significant, especially for values above the 75% threshold and even more clearly above the 95% threshold. Second, whatever remaining difference in posterior probability is now reversed, that is, a composite claim supported by several items of evidence has a higher posterior probability compared to an individual claim supported by one item of evidence. This second observation agrees with the analysis based on the Bayes factor and the likelihood ratio in the earlier section. That analysis showed that the support for a composite claim by a joint body of evidence often exceeds the support for an individual claim.

These two observations establish that, by factorig out prior probabilities and under certain independence assumptions, whenever the individual claims meet the applicable posterior threhsold, so does the composite claim. This verifies aggregation. Conversely, whenever the composite claim, say $A \wedge B$, meets the applicable posterior threshold, so do the individual claims insofar as the threshold is about 75% or higher. This verifies—to some approximation and in a limited class of cases—the other direction of the conjunction principle, what we called distribution.

Has the distribution paradox been eliminated then? The approach we have just described—equalizing the prior probabilities across individual and composite claims—does not entirely eliminate the paradox. There are still cases in which a composite hypothesis, say $A \wedge B$, receives stronger support than an individual hypothesis, given the same body of evidence. Sensitivity to priors seems to play a role. But it cannot be the only factor at play, or else the equalization of the prior probabilities would have eliminated the paradox entirely. So what else is going on?

6.4 Weaker Claims Weaken Sensitivity

Let's examine more closely the Bayes factor and the likelihood ratio as measures of evidential strength. Likelihood ratios are comparative in nature. Suppose we compare claim A and claim $A \wedge B$ relative to the same body of evidence $a \wedge b$. Thinks of A as 'the defendant physically injured the victim' while B as 'the defendant knew the victim was a firefighter.' Think of a and b as testimonies each supporting one of the two hypotheses. We are still working with the Bayesian networks in Figure **??**.

Consider the combined body of evidence $a \wedge b$. Which claim between A and $A \wedge B$ will receive more support by evidence $a \wedge b$? Intuitively, one might think that A should receive more or at least equal support compared to $A \wedge B$. After all, $A \wedge B$ is a stronger claim than A and thus more difficult to establish than A, other things being equal. In terms of posterior probabilities, this is true. The posterior probability of $A \wedge B$ should not be higher than the posterior probability of A alone given evidence $a \wedge b$.

Let's now think in terms of evidential support. Formally, the question is whether $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ is less than one or greater than one. Given the customary independencies between evidence and hypotheses,

$$\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} = \frac{\mathsf{P}(a|A)\mathsf{P}(b|A)}{\mathsf{P}(a|A)\mathsf{P}(b|B)} = \frac{\mathsf{P}(b|A)}{\mathsf{P}(b|B)} < 1.$$

The reason is that $\mathsf{P}(b|A) < \mathsf{P}(b|B)$ since the sensitivity of b relative to B should be higher than the sensitivity of b relative to A.[25] This is obvious if A and b are independent claims, as in the Bayesian networks in Figure **??** (bottom). In this case, $\mathsf{P}(b|A) = \mathsf{P}(b)$.
So $\frac{\mathsf{P}(b)}{\mathsf{P}(b|B)} < 1$ since b, by assumption, positively supports B, or in other words, $\frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} > 1$. Thus, $a \wedge b$ supports $A \wedge B$ more than it supports A alone. This is not what one would expect intuitively.

The same conslusion holds using the Bayes factor. If $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} < 1$, then

$$\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b)} < \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)}$$

Thus, evidence $a \wedge b$ better supports, even on an absolute scale, $A \wedge B$ compared to A. Note that, even if the Baye factor depends on the priors, the difference here is not due to the difference between the priors of A and the priors of $A \wedge B$ since the denominator is simply $\mathsf{P}(a \wedge b)$.[26]

What we have said so far agrees with the claim defended in the previous section. That is, even when $a \wedge b$ strongly supports $A \wedge B$, the same evidence need *not* strongly support A. Formally,

$$\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|\neg A)} < \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}$$

In other words, even if $a \wedge b$ favors $A \wedge B$ over its negation to a very high degree, it need not favor equally strongly A over its negation. This is a comparative claim about two compartive claims, and as such, it may not be easy to parse. Evidential support, when it is formalized by the likelihoo ratio, is always relative to a contrast class. In comparing the support that the same body of evidence provides to A as contrasted to $A \wedge B$, it might be better to include these two hypotheses in the contrast class. So the expression $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ is more straightforward.

No matter the formulation, the same conclusion holds. Evidence $a \wedge b$ supports the composite claim $A \wedge B$ more than it supports the weaker claim A, even assuming that A and B are independent of one another and thus not mutually reinforcing. This conclusion seems paradoxical. How should we make sense of it? At first, we thought the paradox could be due to prior dependency since the combined

[25] GIVE PROOF OF THIS

[26] Note that $\mathsf{P}(a \wedge b|A)\mathsf{P}(A) + \mathsf{P}(a \wedge b|\neg A)\mathsf{P}(\neg)A$ is the same as $\mathsf{P}(a \wedge b|(A \wedge B))\mathsf{P}(A \wedge B) + \mathsf{P}(a \wedge b|\neg(A \wedge B))\mathsf{P}(\neg(A \wedge B))$. **NEED A PROOF FOR THIS BUT IT SHOULD HOLD, RIGHT?**

likelihood ratio $\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}$ varies depending on the priors of A and B. But this argument seems to no longer holds since in case of $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ any prior dependency seems to have been eliminated. After all, if $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} < 1$, the sensitivity of $a \wedge b$ must be worse relative to A than the composite claim $A \wedge B$.

Sensitivity is a crucial property of the quality of the evidence. Everything else being equal, the lower the sensitivity of the evidence, the lower its evidential strength. The importance of sensitivity as a factor for assessing the strength of the evidence is hard to dispute. So why is the sensitivity of $a \wedge b$ worse relative to A than $A \wedge B$? Suppose A is the case. If A is the case, in order for $a \wedge b$ to arise, both a and b should pick up on A. If b fails to pick up on A, then $A \wedge b$ would not arise even if a pick up on A.[27] Suppose instead $A \wedge B$ holds. In this case, $a \wedge b$ would arise even if b fails to pick up on A so long a picks up on A and b picks up on B. Now of course b could also fail to pick on B just like a could fail to pick up on B. But we are assuming that b is better than a at tracking B. So b will fail less often than a at picking up on B. Thus, the sensitivity of $a \wedge b$ relative to $A \wedge B$ is better than the sensitivity of a relative to A alone. This is a subtle point that probability theory helps to bring out clearly.

How big are these variations? **PLOT GRAPH TO GET A SENSE OF VARIATIONS OF SENSITIVITY**

One explanation of the paradox, then, is the difference in sensitivity. The sensitivity of $a \wedge b$ relative to $A \wedge B$ is better than the sensitivity of the same evidence relative to A. Consequently, other thigns being equal, evidence $a \wedge b$ supports $A \wedge B$ better than A. However counterintuitive this might seem, we should accept this fact and admit that our intuitions were wrong. So the fallacy seems to be to assume that the sensitivity of $a \wedge b$ relative to A cannot be lower than the senstivity of $a \wedge b$ relative to $A \wedge B$. The thought would be something like this: if $a \wedge b$ tracks $A \wedge B$ to some degree, it surely must be able to track A alone, at least as well. But we have just shown that we cannot assume that $\mathsf{P}(a \wedge b|A) \geq \mathsf{P}(a \wedge b|A \wedge B)$ and in fact the opposite is the case, $\mathsf{P}(a \wedge b|A) < \mathsf{P}(a \wedge b|A \wedge B)$.

Another way to convince ourselves this is the case is to run a simulation. Suppose we are deciding about the truth of A and the the truth of $A \wedge B$, and we have a fixed body of evidence, say, $a \wedge b$ that speaks in favor of both claims.

M: Run simulation to show that same diagnostic test for composite claim would perform better then when applied to individual claim (worse LR). How do to do this? HELP!

We should circumscribe the point we just made since it does not always hold. Suppose, as in Figure 11, that H is a claim unrelated to A and evidence a. Evidence a supports A. Would the composite claim $A \wedge H$ be better supported by a than A alone? It would not. Mere tagging an unrelated hypothesis does not strengthen the evidence. Note that $\mathsf{P}(a|A) = \mathsf{P}(a|A \wedge H)$ because H is indepedent from everything else. Thus,

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge H)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A)} = 1.$$

Tagging an unrelated claim H does not strengthen the evidence, but leaves it unchanged. Similarly, suppose B constitutes one element of a crime and A constitutes the other element. The two claims are independent, each supported by items of evidence a and b respectively. This is our standard set up. If a supports A, would a support $A \wedge B$ more strongly than it supports A alone? Here we no longer have $a \wedge b$, but instead, a alone. The question is whether $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge B)} > 1$ or $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge B)} < 1$. Given the usual independencies between evidence and hypotheses,

$$\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A \wedge B)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|A)} = 1$$

Evidence a supports A and $A \wedge B$ to the same extent. One might complain that this is counterintuitive. How can it be that a supports A to the same degree that it supports the more demanding claim $A \wedge H$ or $A \wedge B$? For suppose we have evidence a in favor of A and then wonder whether we could use that evidence in support of another claim H or B. By tagging H or B to A, we can at least say that we have evidence a for $A \wedge H$ or $A \wedge B$ that is at least as strong as evidence a in support of A. But note that $\frac{\mathsf{P}(a|A)}{\mathsf{P}(a|negA)} > 1$ even though $\frac{\mathsf{P}(a|H)}{\mathsf{P}(a|negH)=1}$ and $\frac{\mathsf{P}(a|B)}{\mathsf{P}(a|negB)=1}$ (assuming A and B are independent). So a does not support H or B to the same degree that it supports A. However, a supports A to the same degree that it supports $A \wedge H$ or $A \wedge B$.

[27]The occurrance of $a \wedge b$ is less likely to occur than just a alone picking up on A because b may fail—and fail more often than a would—in picking up on A.

What should we make of this? The commonality between H and B is that they are irrelevant for a. Evidence a does not increase nor decrease their probability so the evidence is irrelevant for them. So the upshot here is that, tagging an irrelevant hypothesis does not change evidential support. The other upshot is that tagging a relevant hypothesis—a hypthesis that does bear on the evidence in one way or another, such as B relative to $a \wedge b$—does increase evidential support.

M: How to explain this better? Can we make this plausible? HELP!

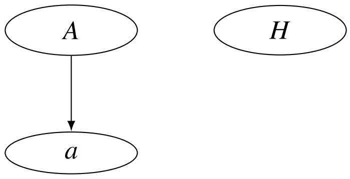


Figure 11: Bayesian network with wholly unrelated claim H.

The intuition that the same evidence tracks the conjunction $A \wedge B$ better than one of the conjuncts A and B might rest on another model of what is going on. Say ab is an item of evidence that arises when A or B occurs with 60% probability. That is, $\mathsf{P}(ab|A) = \mathsf{P}(a|B) = 60\%$. What would be the sensitivity of ab relative to the conjunction $A \wedge B$, $\mathsf{P}(ab|A \wedge B)$? We can represent this set up in Figure 12. Intuitvely, one might reason as follows. There are two paths leading to ab, one path starts with A and another path starts with B. When both these pathst are active, since we are assuming $A \wedge B$, then the probability of ab must be higher than if just one of the two paths is active. Hence, $\mathsf{P}(ab|A \wedge B) \geq 60\%$.

M: Is this true probabilistically? If not, what assumptions are required? HELP!

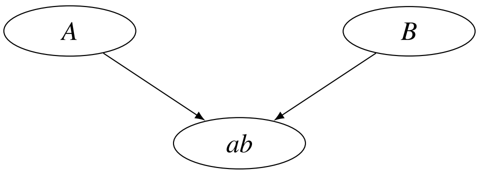


Figure 12: Bayesian network with ab resulting from A and B.

6.5 But Sensitivity Depends On Prior Probabilities

We have shown, given a suitable number of assumptions, that the sensitivity of $a \wedge b$ can be greater relative to $A \wedge B$ than A alone. This explains why the evidential support of $a \wedge b$ is greater in favor of $A \wedge B$ than alone A. Therefore—one might conclude—even factoring out differences in priors probabiliies could still lead to differences in evidential strength simply due to difference in sensitivity. But note that this argument assumes that sensitivity—or specificity—have nothing to do with prior probabilities.

Things are somewhat more complicated. After all, $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)} = \frac{\mathsf{P}(b|A)}{\mathsf{P}(b|B)}$. The denominator, which tracks the sensitivity of $a \wedge b$ relative to $A \wedge B$ does not depend on the priors of $A \wedge B$, but solely on the sensitivty of a and b relative to A and B. However, the numerator, which tracks the sensitivity of $a \wedge b$ relative to a, does depend on the priors of A (or whatvere other hypothesis one choses to computer $\mathsf{P}(b|A)$ or $\mathsf{P}(b)$). This means that the denominator can vary depeding on the prior probability of a chosen hypothesis. Thus, the lower the prior of A, the lower the probability of b. One could still insist that here we are not comparing the prior of a hypothesis, say B, and the prior of another hypothesis, say $A \wedge B$. Whatever the difference in sensitivity, it cannot be due to the difference in prior probabilities of the hypotheses.
On other hand hand, $\frac{\mathsf{P}(a \wedge b|A)}{\mathsf{P}(a \wedge b|A \wedge B)}$ is comparing two quantitities, one dependent on the priors and the other not dependent on the priors.

More generally, the intuition that characteristics of the evidence such as sensitivity and specificity should be indepedent of the probability of the hypothesis of iterest turns out to be incorrect empirically. One study in the medical literature has shown, surprisingly, that the sensitivity of a diagnostic test is independent of the prior of the hypothesis beign tested—say whether the patient has a medical condition. However, specificity is dependent on the prior of the hypothesis:

> Overall, specificity tended to be lower with higher disease prevalence; there was no such systematic effect for sensitivity (page E537). Source: Variation of a test's sensitivity and specificity with disease prevalence Mariska M.G. Leeflang, Anne W.S. Rutjes, Johannes B. Reitsma, Lotty Hooft and Patrick M.M. Bossuyt CMAJ August 06, 2013 185 (11) E537-E544; DOI: https://doi.org/10.1503/cmaj.121286

The authors of the study, however, caution that

> Because sensitivity is estimated in people with the disease of interest and specificity in people without the disease of interest, changing the relative number of people with and without the disease of interest should not introduce systematic differences. Therefore, the effects that we found may be generated by other mechanisms that affect both prevalence and accuracy.

So, according to the authors, changes in prevalence need not directly affect specificity since variations in prevalence and variation in specificity may have a common cause. Our earlier calculations about combined specificity and sensitivity agree with experimental results, namely, only specificity depends on the priors. Our calculations, in fact, show that different priors for the individual claim do affect specificity. The variation of specificity in the result of of splitting the negation of the composite hypthesis $\neg(A \wedge B)$ into three further scenarios, $\neg A \wedge B$, $B \wedge \neg A$ and $\neg A \wedge negB$. This prior sensitivity, of course, only applies to composite hyotheses, but to some extent, any hypothesis can be analyzed as a composite hypothesis. The claim that the defedant was running down 5th avnue can be broken down in the conjuction that the defendant was running and that the defendant was at 5th avenue. Any claim, under some level of description, is a composite hypothesis. So, perhpas, the quality or strength of the evidence should depend on the the priors whether the hypothesis is composite or not. Is this another example of base rate neglect?

M: Might be good to have a simulation here that makes vidid why combined specificity is in fact dependent on the priors. Maybe it is, after all, a fallacy to think that the quality/strength of the evidence should be independent of the priors. HELP!

Let's grant that the quality of the evidence should depend, contrary to our initial intuition, on the prior probability of the hypotheses. If that is so, it would not be natural to see that evidence – the same evidence – strongly favors $A \wedge B$ without strongly favoring A or B. Perhaps we can make sense of this if we keep in mind the comparison between hypothesis we are making here.

NOT SURE HOW TO CONTINUE HERE THOUGH!

TO DO:

1. NOTE THAT EVEN BY EQUALIZING PRIORS, TE DISTRBUTION PARADOX DOES NOT GO AWAY. SO WHAT ELSE IS GOING ON HERE? NEED TO MAKE COMPATISON BETWEEN HYPOTHESES. NEED TO FIGURE THIS OUT!
2. TRY TO MAKE SENSE OF THIS, IT IS INTITVELY ACCEPTABLE THAT SUPPORT FOR COMBINED CLAIM, EVEN HOLDING FIXED THE SAME EVIDENCE, SHOULD BE STRONGER THEN SUPPORT FOR INDIVIDUAL CLAIM? THAT IS CLEARLY ODD AND GOES AGAINST COMMON ASSUMPTIONS.
3. **HYPOTHESIS: WHEN THE HYPTHESIS IS NOT MEDIATED (AS IN a toward A OR b TOWARD B, AS OPPOSED TO a TOWARD A-AND-B), THEN SENSITIVITY OR SPECICITIY IS NOT DEPEDENT ON PRIORS**

6.6 Which Measure of (Combined) Evidential Support?

THIGNS TO ADD:

1. THE MIN SEEMS TO BE THE MEASURE FOR COMPOSITE CLAIMS THAT CAPTURES AGGREGATION AND DISTRIBUTION BEST. SO THE QUESTION IS WHAT PROBABILISTIC MEASURE CAPTURES MIN?
2. IS DEPEDENCY ON PRIOR ANOTHER EXAMPLE OF BASE RATE NEGLECT? WE NEGLECT BAE RATE IN CALCULATING POSTERIOR BUT ALSO IN CALCULATING STRENGTH OF EVIDENCE? WE JUST NEED TO LIVE WITH THE FACT THAT WE HAVE A POOR UNDERSTANDING OF EVIDENCE BUT PERHPAS GOOD ENOUGH TO GET BY IN THE WORLD. CONNECT TO POINT 1 AND MIN FUNCTION.

6.7 The likelihood strategy

Focusing on posterior probabilities is not the only approach that legal probabilists can pursue. By Bayes' theorem, the following holds, using G and I as competing hypotheses:

$$\frac{\Pr(G|E)}{\Pr(I|E)} = \frac{\Pr(E|G)}{\Pr(E|I)} \times \frac{\Pr(G)}{\Pr(I)},$$

or using H_p and H_d as competing hypotheses,

$$\frac{\Pr(H_p|E)}{\Pr(H_d|E)} = \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)},$$

or in words

$$\textit{posterior odds} = \textit{likelihood ratio} \times \textit{prior odds}.$$

A difficult problem is to assign numbers to the prior probabiliteis such as $\Pr(G)$ or $\Pr(H_p)$, or priors odds such as $\frac{\Pr(G)}{\Pr(I)}$ or $\frac{\Pr(H_p)}{\Pr(H_d)}$.

DISCUSS DIFFICULTIES ABOUT ASSIGNING PRIORS! WHERE? CAN WE USE IMPRECISE PROBABILKITIES T TALK ABOUT PRIORS – I.E. LOW PRIORS = TOTAL IGNORANCE = VERY IMPRECISE (LARGE INTERVAL) PRIORS? THE PROBLME WITH THIS WOULD BE THAT THERE IS NO UPDSATING POSSIBLE. ALL UPDATING WOULD STILL GET BACK TO THE STARTING POINT. DO YOU HAVE AN ANSWER TO THAT? WOULD BE INTERETSING TO DISCUSS THIS!

Given these difficulties, both practical and theoretical, one option is to dispense with priors altogether. This is not implausible. Legal disputes in both criminal and civil trials should be decided on the basis of the evidence presented by the litigants. But it is the likelihood ratio – not the prior ratio – that offers the best measure of the overall strength of the evidece presented. So it is all too natural to focus on likekihood ratios and leave the priors out of the picture. If this is the right, the question is, how would a probabilistic interpretation of standards of proof based on the likelihood rato look like? At its simplest, this stratgey will look as follows. Recall our discussion of expected utility theory:

$$\text{convict provided} \frac{cost(CI)}{cost(AG)} < \frac{\Pr(H_p|E)}{\Pr(H_d|E)},$$

which is equivalent to

$$\text{convict provided} \frac{cost(CI)}{cost(AG)} < \frac{\Pr(E|H_p)}{\Pr(E|H_d)} \times \frac{\Pr(H_p)}{\Pr(H_d)}.$$

By rearraing the terms,

$$\text{convict provided} \frac{\Pr(E|H_p)}{\Pr(E|H_d)} > \frac{\Pr(H_d)}{\Pr(H_p)} \times \frac{cost(CI)}{cost(AG)}.$$

Then, on this intepretation, the likelihood ratio should be above a suitable threshold that is a function of the cost ratio and the prior ratio. The outstanding question is how this threshold is to be determined.

6.7.1 Kaplow

Quite independently, a similar approach to juridical decisions has been proposed by Kaplow (2014) – we'll call it **decision-theoretic legal probabilism (DTLP)**. It turns out that Cheng's suggestion is a particular case of this more general approach. Let $LR(E) = \mathsf{P}(E|H_\Pi)/\mathsf{P}(E|H_\Delta)$. In whole generality, DTLP invites us to convict just in case $LR(E) > LR^\star$, where $LR^\star$ is some critical value of the likelihood ratio.

Say we want to formulate the usual preponderance rule: convict iff $\mathsf{P}(H_\Pi|E) > 0.5$, that is, iff $\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} > 1$. By Bayes' Theorem we have:

$$\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} = \frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)} \times \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > 1 \Leftrightarrow$$
$$\Leftrightarrow \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)}$$

So, as expected, $LR^\star$ is not unique and depends on priors. Analogous reformulations are available for thresholds other than 0.5.

Kaplow's point is not that we can reformulate threshold decision rules in terms of priors-sensitive likelihood ratio thresholds. Rather, he insists, when we make a decision, we should factor in its consequences. Let G represent potential gain from correct conviction, and L stand for the potential loss resulting from mistaken conviction. Taking them into account, Kaplow suggests, we should convict if and only if:

$$\mathsf{P}(H_\Pi|E) \times G > \mathsf{P}(H_\Delta|E) \times L \tag{22}$$

Now, (22) is equivalent to:

$$\frac{\mathsf{P}(H_\Pi|E)}{\mathsf{P}(H_\Delta|E)} > \frac{L}{G}$$
$$\frac{\mathsf{P}(H_\Pi)}{\mathsf{P}(H_\Delta)} \times \frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > \frac{L}{G}$$
$$\frac{\mathsf{P}(E|H_\Pi)}{\mathsf{P}(E|H_\Delta)} > \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)} \times \frac{L}{G}$$
$$LR(E) > \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)} \times \frac{L}{G} \tag{23}$$

This is the general format of Kaplow's decision standard.

6.7.2 Dawid

Here is a slightly different perspective, due to Dawid (1987), that also suggests that juridical decisions should be likelihood-based. The focus is on witnesses for the sake of simplicity. Imagine the plaintiff produces two independent witnesses: W_A attesting to A, and W_B attesting to B. Say the witnesses are regarded as 70% reliable and A and B are probabilistically independent, so we infer $\mathsf{P}(A) = \mathsf{P}(B) = 0.7$ and $\mathsf{P}(A \wedge B) = 0.7^2 = 0.49$.

But, Dawid argues, this is misleading, because to reach this result we misrepresented the reliability of the witnesses: 70% reliability of a witness, he continues, does not mean that if the witness testifies that A, we should believe that $\mathsf{P}(A) = 0.7$. To see his point, consider two potential testimonies:

A_1	The sun rose today.
A_2	The sun moved backwards through the sky today.

Intuitively, after hearing them, we would still take $\mathsf{P}(A_1)$ to be close to 1 and $\mathsf{P}(A_2)$ to be close to 0, because we already have fairly strong convictions about the issues at hand. In general, how we should revise our beliefs in light of a testimony depends not only on the reliability of the witness, but also on our prior convictions.[28] And this is as it should be: as indicated by Bayes' Theorem, one and the same testimony with different priors might lead to different posterior probabilities.

So far so good. But how should we represent evidence (or testimony) strength then? Well, one pretty standard way to go is to focus on how much it contributes to the change in our beliefs in a way independent of any particular choice of prior beliefs. Let a be the event that the witness testified that A.

[28] An issue that Dawid does not bring up is the interplay between our priors and our assessment of the reliability of the witnesses. Clearly, our posterior assessment of the credibility of the witness who testified A_2 will be lower than that of the other witness.

It is useful to think about the problem in terms of *odds, conditional odds (O)* and *likelihood ratios (LR)*:

$$
\begin{aligned}
O(A) &= \frac{\mathsf{P}(A)}{\mathsf{P}(\neg A)}\\
O(A|a) &= \frac{\mathsf{P}(A|a)}{\mathsf{P}(\neg A|a)}\\
LR(a|A) &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)}.
\end{aligned}
$$

Suppose our prior beliefs and background knowledge, before hearing a testimony, are captured by the prior probability measure $\mathsf{P}_{prior}(\cdot)$, and the only thing that we learn is a. We're interested in what our *posterior* probability measure, $\mathsf{P}_{posterior}(\cdot)$, and posterior odds should then be. If we're to proceed with Bayesian updating, we should have:

$$\frac{\mathsf{P}_{posterior}(A)}{\mathsf{P}_{posterior}(\neg A)} = \frac{\mathsf{P}_{prior}(A|a)}{\mathsf{P}_{prior}(\neg A|a)} = \frac{\mathsf{P}_{prior}(a|A)}{\mathsf{P}_{prior}(a|\neg A)} \times \frac{\mathsf{P}_{prior}(A)}{\mathsf{P}_{prior}(\neg A)}$$

that is,

$$O_{posterior}(A) = O_{prior}(A|a) = \underbrace{LR_{prior}(a|A)}_{\text{conditional likelihood ratio}} \times O_{prior}(A) \tag{24}$$

The conditional likelihood ratio seems to be a much more direct measure of the value of a, independent of our priors regarding A itself. In general, the posterior probability of an event will equal to the witness's reliability in the sense introduced above only if the prior is $1/2$.[29]

6.8 Likelihood and DAC

But how does our preference for the likelihood ratio as a measure of evidence strength relate to DAC? Let's go through Dawid's reasoning.

A sensible way to probabilistically interpret the 70% reliability of a witness who testifies that A is to take it to consist in the fact that the probability of a positive testimony if A is the case, just as the probability of a negative testimony (that is, testimony that A is false) if A isn't the case, is 0.7:[30]

$$\mathsf{P}_{prior}(a|A) = \mathsf{P}_{prior}(\neg a|\neg A) = 0.7.$$

$\mathsf{P}_{prior}(a|\neg A) = 1 - \mathsf{P}_{prior}(\neg a|\neg A) = 0.3$, and so the same information is encoded in the appropriate likelihood ratio:

$$LR_{prior}(a|A) = \frac{\mathsf{P}_{prior}(a|A)}{\mathsf{P}_{prior}(a|\neg A)} = \frac{0.7}{0.3}$$

[29] Dawid gives no general argument, but it is not too hard to give one. Let $rel(a) = \mathsf{P}(a|A) = \mathsf{P}(\neg a|\neg A)$. We have in the background $\mathsf{P}(a|\neg A) = 1 - \mathsf{P}(\neg a|\neg A) = 1 - rel(a)$. We want to find the condition under which $\mathsf{P}(A|a) = \mathsf{P}(a|A)$. Set $\mathsf{P}(A) = p$ and start with Bayes' Theorem and the law of total probability, and go from there:

$$
\begin{aligned}
\mathsf{P}(A|a) &= \mathsf{P}(a|A)\\
\frac{\mathsf{P}(a|A)p}{\mathsf{P}(a|A)p + \mathsf{P}(a|\neg A)(1-p)} &= \mathsf{P}(a|A)\\
\mathsf{P}(a|A)p &= \mathsf{P}(a|A)[\mathsf{P}(a|A)p + \mathsf{P}(a|\neg A)(1-p)]\\
p &= \mathsf{P}(a|A)p + \mathsf{P}(a|\neg A) - \mathsf{P}(a|\neg A)p\\
p &= rel(a)p + 1 - rel(a) - (1 - rel(a))p\\
p &= rel(a)p + 1 - rel(a) - p + rel(a)p\\
2p &= 2rel(a)p + 1 - rel(a)\\
2p - 2rel(a)p &= 1 - rel(a)\\
2p(1 - rel(a)) &= 1 - rel(a)\\
2p &= 1
\end{aligned}
$$

First we multiplied both sides by the denominator. Then we divided both sides by $\mathsf{P}(a|A)$ and multiplied on the right side. Then we used our background notation and information. Next, we manipulated the right-hand side algebraically and moved $-p$ to the left-hand side. Move $2rel(a)p$ to the left and manipulate the result algebraically to get to the last line.

[30] In general setting, these are called the *sensitivity* and *specificity* of a test (respectively), and they don't have to be equal. For instance, a degenerate test for an illness which always responds positively, diagnoses everyone as ill, and so has sensitivity 1, but specificity 0.

Let's say that *a provides (positive) support* for A in case

$$O_{posterior}(A) = O_{prior}(A|a) > O_{prior}(A)$$

that is, a testimony a supports A just in case the posterior odds of A given a are greater than the prior odds of A (this happens just in case $\mathsf{P}_{posterior}(A) > \mathsf{P}_{prior}(A)$). By (24), this will be the case if and only if $LR_{prior}(a|A) > 1$.

One question that Dawid addresses is this: assuming reliability of witnesses 0.7, and assuming that a and b, taken separately, provide positive support for their respective claims, does it follow that $a \wedge b$ provides positive support for $A \wedge B$?

Assuming the independence of the witnesses, this will hold in non-degenerate cases that do not involve extreme probabilities, on the assumption of independence of a and b conditional on all combinations: $A \wedge B$, $A \wedge \neg B$, $\neg A \wedge B$ and $\neg A \wedge \neg B$.[31],[32]

Let us see why the above claim holds. The calculations are my reconstruction and are not due to Dawid. The reader might be annoyed with me working out the mundane details of Dawid's claims, but it turns out that in the case of Dawid's strategy, the devil is in the details. The independence of witnesses gives us:

$$\begin{aligned}
\mathsf{P}(a \wedge b|A \wedge B) &= 0.7^2 = 0.49\\
\mathsf{P}(a \wedge b|A \wedge \neg B) &= 0.7 \times 0.3 = 0.21\\
\mathsf{P}(a \wedge b|\neg A \wedge B) &= 0.3 \times 0.7 = 0.21\\
\mathsf{P}(a \wedge b|\neg A \wedge \neg B) &= 0.3 \times 0.3 = 0.09
\end{aligned}$$

Without assuming A and B to be independent, let the probabilities of $A \wedge B$, $\neg A \wedge B$, $A \wedge \neg B$, $\neg A \wedge \neg B$ be $p_{11}, p_{01}, p_{10}, p_{00}$. First, let's see what $\mathsf{P}(a \wedge b)$ boils down to.

By the law of total probability we have:

$$\begin{aligned}
\mathsf{P}(a \wedge b) = {} & \mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B)+\\
& + \mathsf{P}(a \wedge b|A \wedge \neg B)\mathsf{P}(A \wedge \neg B)\\
& + \mathsf{P}(a \wedge b|\neg A \wedge B)\mathsf{P}(\neg A \wedge B)+\\
& + \mathsf{P}(a \wedge b|\neg A \wedge \neg B)\mathsf{P}(\neg A \wedge \neg B)
\end{aligned} \tag{25}$$

which, when we substitute our values and constants, results in:

$$= 0.49p_{11} + 0.21(p_{10} + p_{01}) + 0.09p_{00}$$

Now, note that because p_{ii}s add up to one, we have $p_{10} + p_{01} = 1 - p_{00} - p_{11}$. Let us continue.

$$\begin{aligned}
&= 0.49p_{11} + 0.21(1 - p_{00} - p_{11}) + 0.09p_{00}\\
&= 0.21 + 0.28p_{11} - 0.12p_{00}
\end{aligned}$$

Next, we ask what the posterior of $A \wedge B$ given $a \wedge b$ is (in the last line, we also multiply the numerator and the denominator by 100).

$$\begin{aligned}
\mathsf{P}(A \wedge B|a \wedge b) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B)}{\mathsf{P}(a \wedge b)}\\
&= \frac{49p_{11}}{21 + 28p_{11} - 12p_{00}}
\end{aligned}$$

In this particular case, then, our question whether $\mathsf{P}(A \wedge B|a \wedge b) > \mathsf{P}(A \wedge B)$ boils down to asking whether

$$\frac{49p_{11}}{21 + 28p_{11} - 12p_{00}} > p_{11}$$

[31] Dawid only talks about the independence of witnesses without reference to conditional independence. Conditional independence does not follow from independence, and it is the former that is needed here (also, four non-equivalent different versions of it).

[32] In terms of notation and derivation in the optional content that will follow, the claim holds if and only if $28 > 28p_{11} - 12p_{00}$. This inequality is not true for all admissible values of p_{11} and p_{00}. If $p_{11} = 1$ and $p_{00} = 0$, the sides are equal. However, this is a rather degenerate example. Normally, we are interested in cases where $p_{11} < 1$. And indeed, on this assumption, the inequality holds.

that is, whether $28 > 28p_{11} - 12p_{00}$ (just divide both sides by p_{11}, multiply by the denominator, and manipulate algebraically).

Dawid continues working with particular choices of values and provides neither a general statement of the fact that the above considerations instantiate nor a proof of it. In the middle of the paper he says:

> Even under prior dependence, the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability... When the problem is analysed carefully, the 'paradox' evaporates [pp. 95-7]

where he still means the case with the particular values that he has given, but he seems to suggest that the claim generalizes to a large array of cases.

The paper does not contain a precise statement making the conditions required explicit and, *a fortriori*, does not contain a proof of it. Given the example above and Dawid's informal reading, let us develop a more precise statement of the claim and a proof thereof.

Fact 3. *Suppose that $rel(a), rel(b) > 0.5$ and witnesses are independent conditional on all Boolean combinations of A and B (in a sense to be specified), and that none of the Boolean combinations of A and B has an extreme probability (of 0 or 1). It follows that $\mathsf{P}(A \wedge B|a \wedge b) > \mathsf{P}(A \wedge B)$. (Independence of A and B is not required.)*

Roughly, the theorem says that if independent and reliable witnesses provide positive support of their separate claims, their joint testimony provides positive support of the conjunction of their claims.

Let us see why the claim holds. First, we introduce an abbreviation for witness reliability:

$$\begin{aligned}\mathbf{a} &= rel(a) = \mathsf{P}(a|A) = \mathsf{P}(\neg a|\neg A) > 0.5\\ \mathbf{b} &= rel(b) = \mathsf{P}(b|B) = \mathsf{P}(\neg b|\neg A) > 0.5\end{aligned}$$

Our independence assumption means:

$$\begin{aligned}\mathsf{P}(a \wedge b|A \wedge B) &= \mathbf{ab}\\ \mathsf{P}(a \wedge b|A \wedge \neg B) &= \mathbf{a(1-b)}\\ \mathsf{P}(a \wedge b|\neg A \wedge B) &= \mathbf{(1-a)b}\\ \mathsf{P}(a \wedge b|\neg A \wedge \neg B) &= \mathbf{(1-a)(1-b)}\end{aligned}$$

Abbreviate the probabilities the way we already did:

$$\begin{array}{ll}\mathsf{P}(A \wedge B) = p_{11} & \mathsf{P}(A \wedge \neg B) = p_{10}\\ \mathsf{P}(\neg A \wedge B) = p_{01} & \mathsf{P}(\neg A \wedge \neg B) = p_{00}\end{array}$$

Our assumptions entail $0 \neq p_{ij} \neq 1$ for $i, j \in \{0, 1\}$ and:

$$p_{11} + p_{10} + p_{01} + p_{00} = 1 \tag{26}$$

So, we can use this with (25) to get:

$$\begin{aligned}\mathsf{P}(a \wedge b) &= \mathbf{ab}p_{11} + \mathbf{a(1-b)}p_{10} + \mathbf{(1-a)b}p_{01} + \mathbf{(1-a)(1-b)}p_{00}\\ &= p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})\end{aligned} \tag{27}$$

Let's now work out what the posterior of $A \wedge B$ will be, starting with an application of the Bayes' Theorem:

$$\begin{aligned}\mathsf{P}(A \wedge B|a \wedge b) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B)}{\mathsf{P}(a \wedge b)}\\ &= \frac{\mathbf{ab}p_{11}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})}\end{aligned} \tag{28}$$

To answer our question we therefore have to compare the content of (28) to p_{11} and our claim holds just in case:

$$\frac{\mathbf{ab}p_{11}}{p_{11}\mathbf{ab} + p_{10}(\mathbf{a} - \mathbf{ab}) + p_{01}(\mathbf{b} - \mathbf{ab}) + p_{00}(1 - \mathbf{b} - \mathbf{a} + \mathbf{ab})} > p_{11}$$

$$\frac{\mathbf{ab}}{p_{11}\mathbf{ab}+p_{10}(\mathbf{a}-\mathbf{ab})+p_{01}(\mathbf{b}-\mathbf{ab})+p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})} > 1$$

$$p_{11}\mathbf{ab}+p_{10}(\mathbf{a}-\mathbf{ab})+p_{01}(\mathbf{b}-\mathbf{ab})+p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab}) < \mathbf{ab} \tag{29}$$

Proving (29) is therefore our goal for now. This is achieved by the following reasoning:[33]

1.	$\mathbf{b} > 0.5,\ \mathbf{a} > 0.5$	assumption
2.	$2\mathbf{b} > 1,\ 2\mathbf{a} > 1$	from 1.
3.	$2\mathbf{ab} > \mathbf{a},\ 2\mathbf{ab} > \mathbf{b}$	multiplying by **a** and **b** respectively
4.	$p_{10}2\mathbf{ab} > p_{10}\mathbf{a},\quad p_{01}2\mathbf{ab} > p_{01}\mathbf{b}$	multiplying by p_{10} and p_{01} respectively
5.	$p_{10}2\mathbf{ab}+p_{01}2\mathbf{ab} > p_{10}\mathbf{a}+p_{01}\mathbf{b}$	adding by sides, 3., 4.
6.	$1-\mathbf{b}-\mathbf{a} < 0$	from 1.
7.	$p_{00}(1-\mathbf{b}-\mathbf{a}) < 0$	From 6., because $p_{00} > 0$
8.	$p_{10}2\mathbf{ab}+p_{01}2\mathbf{ab} > p_{10}\mathbf{a}+p_{01}\mathbf{b}+p_{00}(1-\mathbf{b}-\mathbf{a})$	from 5. and 7.
9.	$p_{10}\mathbf{ab}+p_{10}\mathbf{ab}+p_{01}\mathbf{ab}+p_{01}\mathbf{ab}+p_{00}\mathbf{ab}-p_{00}\mathbf{ab} > p_{10}\mathbf{a}+p_{01}\mathbf{b}+p_{00}(1-\mathbf{b}-\mathbf{a})$	8., rewriting left-hand side
10.	$p_{10}\mathbf{ab}+p_{01}\mathbf{ab}+p_{00}\mathbf{ab} > -p_{10}\mathbf{ab}-p_{01}\mathbf{ab}+p_{00}\mathbf{ab}+p_{10}\mathbf{a}+p_{01}\mathbf{b}+p_{00}(1-\mathbf{b}-\mathbf{a})$	9., moving from left to right
11.	$\mathbf{ab}(p_{10}+p_{01}+p_{00}) > p_{10}(\mathbf{a}-\mathbf{ab})+p_{01}(\mathbf{b}-\mathbf{ab})+p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	10., algebraic manipulation
12.	$\mathbf{ab}(1-p_{11}) > p_{10}(\mathbf{a}-\mathbf{ab})+p_{01}(\mathbf{b}-\mathbf{ab})+p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	11. and equation (26)
13.	$\mathbf{ab}-\mathbf{ab}p_{11} > p_{10}(\mathbf{a}-\mathbf{ab})+p_{01}(\mathbf{b}-\mathbf{ab})+p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	12., algebraic manipulation
14.	$\mathbf{ab} > \mathbf{ab}p_{11}+p_{10}(\mathbf{a}-\mathbf{ab})+p_{01}(\mathbf{b}-\mathbf{ab})+p_{00}(1-\mathbf{b}-\mathbf{a}+\mathbf{ab})$	13., moving from left to right

%\end{adjustbox}

The last line is what we have been after.

OPTIONAL CONTENT ENDS

Now that we have as a theorem an explication of what Dawid informally suggested, let's see whether it helps the probabilist handling of DAC.

6.8.1 Kaplow

On RLP, at least in certain cases, the decision rule leads us to (**??**), which tells us to decide the case based on whether the likelihood ratio is greater than 1.

[34] While Kaplow did not discuss DAC or the gatecrasher paradox, it is only fair to evaluate Kaplow's proposal from the perspective of these difficulties.

Add here stuff from Marcello's Mind paper about the prisoner hypothetical. Then, discuss Rafal's critique of the likelihood ratio threshold and see where we end up.

6.9 Dawid's likelihood strategy doesn't help

Recall that DAC was a problem posed for the decision standard proposed by TLP, and the real question is how the information resulting from Fact 3 can help to avoid that problem. Dawid does not mention any decision standard, and so addresses quite a different question, and so it is not clear that '`theparadox`' evaporates' ', as Dawid suggests.

What Dawid correctly suggests (and we establish in general as Fact 3) is that the support of the conjunction by two witnesses will be positive as soon as their separate support for the conjuncts is positive. That is, that the posterior of the conjunction will be higher that its prior. But the critic of probabilism never denied that the conjunction of testimonies might raise the probability of the conjunction if the testimonies taken separately support the conjuncts taken separately. Such a critic can still insist that Fact 3 does nothing to alleviate her concern. After all, at least *prima facie* it still might be the case that:

- the posterior probabilities of the conjuncts are above a given threshold,
- the posterior probability of the conjunction is higher than the prior probability of the conjunction,
- the posterior probability of the conjunction is still below the threshold.

That is, Fact 3 does not entail that once the conjuncts satisfy a decision standard, so does the conjunction.

At some point, Dawid makes a general claim that is somewhat stronger than the one already cited:

> When the problem is analysed carefully, the 'paradox' evaporates: suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents.

[33] Thanks to Pawel Pawlowski for working on this proof with me.

[34] Again, the name of the view is by no means standard, it is just a term I coined to refer to various types of legal probabilism in a fairly uniform manner.

[p. 97]

This is quite a different claim from the content of Fact 3, because previously the joint probability was claimed only to increase as compared to the prior, and here it is claimed to increase above the level of the separate increases provided by separate testimonies. Regarding this issue Dawid elaborates (we still use the p_{ij}-notation that we've already introduced):

> "More generally, let $\mathsf{P}(a|A)/\mathsf{P}(a|\neg A) = \lambda$, $\mathsf{P}(b|B)/\mathsf{P}(b|\neg B) = \mu$, with $\lambda, \mu > 0.7$, as might arise, for example, when there are several available testimonies. If the witnesses are independent, then
>
> $$\mathsf{P}(A \wedge B|a \wedge b) = \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00})$$
>
> which increases with each of λ and μ, and is never less than the larger of $\lambda p_{11}/(1-p_{11}+\lambda p_{11}), \mu p_{11}/(1-p_{11}1+\mu p_{11})$, the posterior probabilities appropriate to the individual testimonies." [p. 95]

This claim, however, is false.

Optional Content Starts

Let us see why. The quoted passage is a bit dense. It contains four claims for which no arguments are given in the paper. The first three are listed below as (30), the fourth is that if the conditions in (30) hold, $\mathsf{P}(A \wedge B|a \wedge b) > max(\mathsf{P}(A|a), \mathsf{P}(B|b))$. Notice that $\lambda = LR(a|A)$ and $\mu = LR(b|B)$. Suppose the first three claims hold, that is:

$$\begin{aligned}\mathsf{P}(A \wedge B|a \wedge b) &= \lambda\mu p_{11}/(\lambda\mu p_{11} + \lambda p_{10} + \mu p_{01} + p_{00}) \\ \mathsf{P}(A|a) &= \frac{\lambda p_{11}}{1-p_{11}+\lambda p_{11}} \\ \mathsf{P}(B|b) &= \frac{\mu p_{11}}{1-p_{11}+\mu p_{11}}\end{aligned} \tag{30}$$

Is it really the case that $\mathsf{P}(A \wedge B|a \wedge b) > \mathsf{P}(A|a), \mathsf{P}(B|b)$? It does not seem so. Let $\mathbf{a} = \mathbf{b} = 0.6$, $pr = \langle p_{11}, p_{10}, p_{01}, p_{00}\rangle = \langle 0.1, 0.7, 0.1, 0.1\rangle$. Then, $\lambda = \mu = 1.5 > 0.7$ so the assumption is satisfied. Then we have $\mathsf{P}(A) = p_{11} + p_{10} = 0.8$, $\mathsf{P}(B) = p_{11} + p_{01} = 0.2$. We can also easily compute $\mathsf{P}(a) = \mathbf{a}\mathsf{P}(A) + (1-\mathbf{a})\mathsf{P}(\neg A) = 0.56$ and $\mathsf{P}(b) = \mathbf{b}\mathsf{P}(B) + (1-\mathbf{b})\mathsf{P}(\neg B) = 0.44$. Yet:

$$\begin{aligned}\mathsf{P}(A|a) &= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(a)} = \frac{0.6 \times 0.8}{0.6 \times 0.8 + 0.4 \times 0.2} \approx 0.8571 \\ \mathsf{P}(B|b) &= \frac{\mathsf{P}(b|B)\mathsf{P}(B)}{\mathsf{P}(b)} = \frac{0.6 \times 0.2}{0.6 \times 0.2 + 0.4 \times 0.8} \approx 0.272 \\ \mathsf{P}(A \wedge B|a \wedge b) &= \frac{\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B)}{\begin{array}{l}\mathsf{P}(a \wedge b|A \wedge B)\mathsf{P}(A \wedge B) + \mathsf{P}(a \wedge b|A \wedge \neg B)\mathsf{P}(A \wedge \neg B) + \\ \quad + \mathsf{P}(a \wedge b|\neg A \wedge B)\mathsf{P}(\neg A \wedge B) + \mathsf{P}(a \wedge b|\neg A \wedge \neg B)\mathsf{P}(\neg A \wedge \neg B)\end{array}} \\ &= \frac{\mathbf{ab}p_{11}}{\mathbf{ab}p_{11} + \mathbf{a}(1-\mathbf{b})p_{10} + (1-\mathbf{a})\mathbf{b}p_{01} + (1-\mathbf{a})(1-\mathbf{b})p_{00}} \approx 0.147\end{aligned}$$

The posterior probability of $A \wedge B$ is not only lower than the larger of the individual posteriors, but also lower than any of them!

So what went wrong in Dawid's calculations in (30)? Well, the first formula is correct. However, let us take a look at what the second one says (the problem with the third one is pretty much the same):

$$\mathsf{P}(A|a) = \frac{\frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A \wedge B)}{\mathsf{P}(\neg(A \wedge B)) + \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A \wedge B)}$$

Quite surprisingly, in Dawid's formula for $\mathsf{P}(A|a)$, the probability of $A \wedge B$ plays a role. To see that it should not take any B that excludes A and the formula will lead to the conclusion that *always* $\mathsf{P}(A|a)$ is undefined. The problem with Dawid's formula is that instead of $p_{11} = \mathsf{P}(A \wedge B)$ he should have used

$\mathsf{P}(A) = p_{11} + p_{10}$, in which case the formula would rather say this:

$$\begin{aligned}
\mathsf{P}(A|a) &= \frac{\frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A)}{\mathsf{P}(\neg A) + \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \times \mathsf{P}(A)} \\
&= \frac{\frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)}}{\frac{\mathsf{P}(\neg a|A)\mathsf{P}(\neg A)}{\mathsf{P}(\neg a|A)} + \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)}} \\
&= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(\neg a|A)\mathsf{P}(\neg A) + \mathsf{P}(a|A)\mathsf{P}(A)}
\end{aligned}$$

Now, on the assumption that witness' sensitivity is equal to their specificity, we have $\mathsf{P}(a|\neg A) = \mathsf{P}(\neg a|A)$ and can substitute this in the denominator:

$$= \frac{\mathsf{P}(a|A)\mathsf{P}(A)}{\mathsf{P}(a|\neg A)\mathsf{P}(\neg A) + \mathsf{P}(a|A)\mathsf{P}(A)}$$

and this would be a formulation of Bayes' theorem. And indeed with $\mathsf{P}(A) = p_{11} + p_{10}$ the formula works (albeit its adequacy rests on the identity of $\mathsf{P}(a|\neg A)$ and $\mathsf{P}(\neg a|A)$), and yields the result that we already obtained:

$$\begin{aligned}
\mathsf{P}(A|a) &= \frac{\lambda(p_{11} + p_{10})}{1 - (p_{11} + p_{10}) + \lambda(p_{11} + p_{10})} \\
&= \frac{1.5 \times 0.8}{1 - 0.8 + 1.5 \times 0.8} \approx 0.8571
\end{aligned}$$

The situation cannot be much improved by taking **a** and **b** to be high. For instance, if they're both 0.9 and $pr = \langle 0.1, 0.7, 0.1, 0.1 \rangle$, the posterior of A is ≈ 0.972, the posterior of B is ≈ 0.692, and yet the joint posterior of $A \wedge B$ is 0.525.

The situation cannot also be improved by saying that at least if the threshold is 0.5, then as soon as **a** and **b** are above 0.7 (and, *a fortriori*, so are λ and μ), the individual posteriors being above 0.5 entails the joint posterior being above 0.5 as well. For instance, for $\mathbf{a} = 0.7$ and $\mathbf{b} = 0.9$ with $pr = \langle 0.1, 0.3, 0.5, 0.1 \rangle$, the individual posteriors of A and B are ≈ 0.608 and ≈ 0.931 respectively, while the joint posterior of $A \wedge B$ is ≈ 0.283.

OPTIONAL CONTENT ENDS

The situation cannot be improved by saying that what was meant was rather that the joint likelihood is going to be at least as high as the maximum of the individual likelihoods, because quite the opposite is the case: the joint likelihood is going to be lower than any of the individual ones.

OPTIONAL CONTENT STARTS

Let us make sure this is the case. We have:

$$\begin{aligned}
LR(a|A) &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|\neg A)} \\
&= \frac{\mathsf{P}(a|A)}{\mathsf{P}(\neg a|A)} \\
&= \frac{\mathbf{a}}{\mathbf{1} - \mathbf{a}}.
\end{aligned}$$

where the substitution in the denominator is legitimate only because witness' sensitivity is identical to their specificity.

With the joint likelihood, the reasoning is just a bit more tricky. We will need to know what $\mathsf{P}(a \wedge b|\neg(A \wedge B))$ is. There are three disjoint possible conditions in which the condition holds: $A \wedge \neg B, \neg A \wedge B$, and $\neg A \wedge \neg B$. The probabilities of $a \wedge b$ in these three scenarios are respectively $\mathbf{a}(\mathbf{1} - \mathbf{b}), (\mathbf{1} - \mathbf{a})\mathbf{b}, (\mathbf{1} - \mathbf{a})(\mathbf{1} - \mathbf{b})$ (again, the assumption of independence is important), and so on the assumption $\neg(A \wedge B)$ the probability of $a \wedge b$ is:

$$\begin{aligned}
\mathsf{P}(a \wedge b|\neg(A \wedge B)) &= \mathbf{a}(1 - \mathbf{b}) + (1 - \mathbf{a})\mathbf{b} + (1 - \mathbf{a})(1 - \mathbf{b}) \\
&= \mathbf{a}(1 - \mathbf{b}) + (1 - \mathbf{a})(\mathbf{b} + 1 - \mathbf{b}) \\
&= \mathbf{a}(1 - \mathbf{b}) + (1 - \mathbf{a}) \\
&= \mathbf{a} - \mathbf{ab} + 1 - \mathbf{a} = 1 - \mathbf{ab}
\end{aligned}$$

So, on the assumption of witness independence, we have:

$$LR(a \wedge b|A \wedge B) = \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))} = \frac{\mathbf{ab}}{\mathbf{1} - \mathbf{ab}}$$

With $0 < \mathbf{a}, \mathbf{b} < 1$ we have $\mathbf{ab} < \mathbf{a}$, $1 - \mathbf{ab} > 1 - \mathbf{a}$, and consequently:

$$\frac{\mathbf{ab}}{\mathbf{1} - \mathbf{ab}} < \frac{\mathbf{a}}{\mathbf{1} - \mathbf{a}}$$

which means that the joint likelihood is going to be lower than any of the individual ones.

OPTIONAL CONTENT ENDS

Fact 3 is so far the most optimistic reading of the claim that if witnesses are independent and fairly reliable, their testimonies are going to provide positive support for the conjunction,\footnote{And this is the reading that Dawid in passing suggests: "the combined support is always positive, in the sense that the posterior probability of the case always exceeds its prior probability.' ' (Dawid, 1987: 95) and any stronger reading of Dawid's suggestions fails. But Fact 3 is not too exciting when it comes to answering the original DAC. The original question focused on the adjudication model according to which the deciding agents are to evaluate the posterior probability of the whole case conditional on all evidence, and to convict if it is above a certain threshold. The problem, generally, is that it might be the case that the pieces of evidence for particular elements of the claim can have high likelihood and posterior probabilities of particular elements can be above the threshold while the posterior joint probability will still fail to meet the threshold. The fact that the joint posterior will be higher than the joint prior does not help much. For instance, if $\mathbf{a} = \mathbf{b} = 0.7$, $pr = \langle 0.1, 0.5, 0.3, 0.1 \rangle$, the posterior of A is ≈ 0.777, the posterior of B is ≈ 0.608 and the joint posterior is ≈ 0.216 (yes, it is higher than the joint prior $= 0.1$, but this does not help the conjunction to satisfy the decision standard).

To see the extent to which Dawid's strategy is helpful here, perhaps the following analogy might be useful.

Imagine it is winter, the heating does not work in my office and I am quite cold. I pick up the phone and call maintenance. A rather cheerful fellow picks up the phone. I tell him what my problem is, and he reacts:

— Oh, don't worry.
— What do you mean? It's cold in here!
— No no, everything is fine, don't worry.
— It's not fine! I'm cold here!
— Look, sir, my notion of it being warm in your office is that the building provides some improvement to what the situation would be if it wasn't there. And you agree that you're definitely warmer than you'd be if your desk was standing outside, don't you? Your, so to speak, posterior warmth is higher than your prior warmth, right?

Dawid's discussion is in the vein of the above conversation. In response to a problem with the adjudication model under consideration Dawid simply invites us to abandon thinking in terms of it and to abandon requirements crucial for the model. Instead, he puts forward a fairly weak notion of support (analogous to a fairly weak sense of the building providing improvement), according to which, assuming witnesses are fairly reliable, if separate fairly reliable witnesses provide positive support to the conjuncts, then their joint testimony provides positive support for the conjunction.

As far as our assessment of the original adjudication model and dealing with DAC, this leaves us hanging. Yes, if we abandon the model, DAC does not worry us anymore. But should we? And if we do, what should we change it to, if we do not want to be banished from the paradise of probabilistic methods?

Having said this, let me emphasize that Dawid's paper is important in the development of the debate, since it shifts focus on the likelihood ratios, which for various reasons are much better measures of evidential support provided by particular pieces of evidence than mere posterior probabilities.

Before we move to another attempt at a probabilistic formulation of the decision standard, let us introduce the other hero of our story: the gatecrasher paradox. It is against DAC and this paradox that the next model will be judged.

OPTIONAL CONTENT STARTS

In fact, Cohen replied to Dawid's paper (Cohen, 1988). His reply, however, does not have much to do with the workings of Dawid's strategy, and is rather unusual. Cohen's first point is that the calculations of posteriors require odds about unique events, whose meaning is usually given in terms of potential wagers – and the key criticism here is that in practice such wagers cannot be decided. This is not a convincing criticism, because the betting-odds interpretations of subjective probability do not require that on each occasion the bet should really be practically decidable. It rather invites one to imagine a possible situation in which the truth could be found out and asks: how much would we bet on a certain claim in such a situation? In some cases, this assumption is false, but there is nothing in principle wrong with thinking about the consequences of false assumptions.

Second, Cohen says that Dawid's argument works only for testimonial evidence, not for other types thereof. But this claim is simply false – just because Dawid used testimonial evidence as an example that he worked through it by no means follows that the approach cannot be extended. After all, as long as we can talk about sensitivity and specificity of a given piece of evidence, everything that Dawid said about testimonies can be repeated *mutatis mutandis*.

Third, Cohen complaints that Dawid in his example worked with rather high priors, which according to Cohen would be too high to correspond to the presumption of innocence. This also is not a very successful rejoinder. Cohen picked his priors in the example for the ease of calculations, and the reasoning can be run with lower priors. Moreover, instead of discussing the conjunction problem, Cohen brings in quite a different problem: how to probabilistically model the presumption of innocence, and what priors of guilt should be appropriate? This, indeed, is an important problem; but it does not have much to do with DAC, and should be discussed separately.

6.10 Problem's with Kaplow's stuff

Kaplow does not discuss the conceptual difficulties that we are concerned with, but this will not stop us from asking whether DTLP can handle them (and answering to the negative). Let us start with DAC.

Say we consider two claims, A and B. Is it generally the case that if they separately satisfy the decision rule, then so does $A \wedge B$? That is, do the assumptions:

$$\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} > \frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)} \times \frac{L}{G}$$
$$\frac{\mathsf{P}(E|B)}{\mathsf{P}(E|\neg B)} > \frac{\mathsf{P}(\neg B)}{\mathsf{P}(B)} \times \frac{L}{G}$$

entail

$$\frac{\mathsf{P}(E|A \wedge B)}{\mathsf{P}(E|\neg(A \wedge B))} > \frac{\mathsf{P}(\neg(A \wedge B))}{\mathsf{P}(A \wedge B)} \times \frac{L}{G}?$$

Alas, the answer is negative.

OPTIONAL CONTENT STARTS

This can be seen from the following example. Suppose a random digit from 0-9 is drawn; we do not know the result; we are told that the result is < 7 ($E =$'the result is < 7'), and we are to decide whether to accept the following claims:

A	the result is < 5.
B	the result is an even number.
$A \wedge B$	the result is an even number < 5.

Suppose that $L = G$ (this is for simplicity only — nothing hinges on this, counterexamples for when this condition fails are analogous). First, notice that A and B taken separately satisfy (23). $\mathsf{P}(A) = \mathsf{P}(\neg A) = 0.5$, $\mathsf{P}(\neg A)/\mathsf{P}(A) = 1$ $\mathsf{P}(E|A) = 1$, $\mathsf{P}(E|\neg A) = 0.4$. (23) tells us to check:

$$\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)} > \frac{L}{G} \times \frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)}$$
$$\frac{1}{0.4} > 1$$

so, following DTLP, we should accept A.
For analogous reasons, we should also accept B. $\mathsf{P}(B)=\mathsf{P}(\neg B)=0.5$, $\mathsf{P}(\neg B)/\mathsf{P}(B)=1$ $\mathsf{P}(E|B)=0.8$, $\mathsf{P}(E|\neg B)=0.6$, so we need to check that indeed:

$$\frac{\mathsf{P}(E|B)}{\mathsf{P}(E|\neg B)} > \frac{L}{G}\times\frac{\mathsf{P}(\neg B)}{\mathsf{P}(B)}$$
$$\frac{0.8}{0.6} > 1$$

But now, $\mathsf{P}(A\wedge B)=0.3$, $\mathsf{P}(\neg(A\wedge B))=0.7$, $\mathsf{P}(\neg(A\wedge B))/\mathsf{P}(A\wedge B)=2\frac{1}{3}$, $\mathsf{P}(E|A\wedge B)=1$, $\mathsf{P}(E|\neg(A\wedge B))=4/7$ and it is false that:

$$\frac{\mathsf{P}(E|A\wedge B)}{\mathsf{P}(E|\neg(A\wedge B))} > \frac{L}{G}\times\frac{\mathsf{P}(\neg(A\wedge B))}{\mathsf{P}(A\wedge B)}$$
$$\frac{7}{4} > \frac{7}{3}$$

The example was easy, but the conjuncts are probabilistically dependent. One might ask: are there counterexamples that involve claims which are probabilistically independent?[35]

Consider an experiment in which someone tosses a six-sided die twice. Let the result of the first toss be X and the result of the second one Y. Your evidence is that the results of both tosses are greater than one ($E =: X>1\wedge Y>1$). Now, let A say that $X<5$ and B say that $Y<5$.

The prior probability of A is $2/3$ and the prior probability of $\neg A$ is $1/3$ and so $\frac{\mathsf{P}(\neg A)}{\mathsf{P}(A)}=0.5$. Further, $\mathsf{P}(E|A)=0.625$, $\mathsf{P}(E|\neg A)=5/6$ and so $\frac{\mathsf{P}(E|A)}{\mathsf{P}(E|\neg A)}=0.75$ Clearly, $0.75>0.5$, so A satisfies the decision standard. Since the situation with B is symmetric, so does B.

Now, $\mathsf{P}(A\wedge B)=(2/3)^2=4/9$ and $\mathsf{P}(\neg(A\wedge B))=5/9$. So $\frac{\mathsf{P}(\neg(A\wedge B))}{\mathsf{P}(A\wedge B)}=5/4$. Out of 16 outcomes for which $A\wedge B$ holds, E holds in 9, so $\mathsf{P}(E|A\wedge B)=9/16$. Out of 20 remaining outcomes for which $A\wedge B$ fails, E holds in 16, so $\mathsf{P}(E|\neg(A\wedge B))=4/5$. Thus, $\frac{\mathsf{P}(E|A\wedge B)}{\mathsf{P}(E|\neg(A\wedge B))}=45/64<5/4$, so the conjunction does not satisfy the decision standard.

OPTIONAL CONTENT ENDS

Let us turn to the gatecrasher paradox.

Suppose $L=G$ and recall our abbreviations: $\mathsf{P}(E)=e$, $\mathsf{P}(H_\Pi)=\pi$. DTLP tells us to convict just in case:

$$LR(E) > \frac{1-\pi}{\pi}$$

From (**??**) we already now that

$$LR(E) = \frac{0.991-0.991\pi}{0.009\pi}$$

so we need to see whether there are any $0<\pi<1$ for which

$$\frac{0.991-0.991\pi}{0.009\pi} > \frac{1-\pi}{\pi}$$

Multiply both sides first by 009π and then by π:

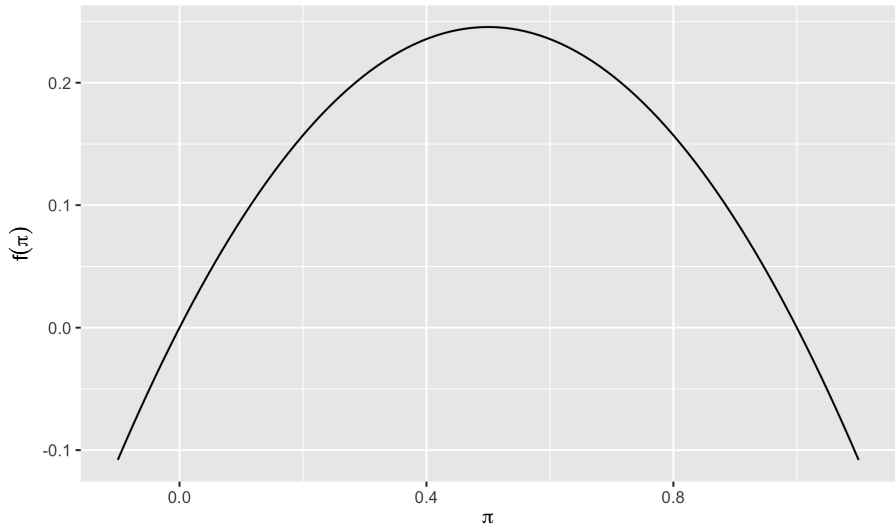
$$0.991\pi-0.991\pi^2 > 0.09\pi-0.009\pi^2$$

Simplify and call the resulting function f:

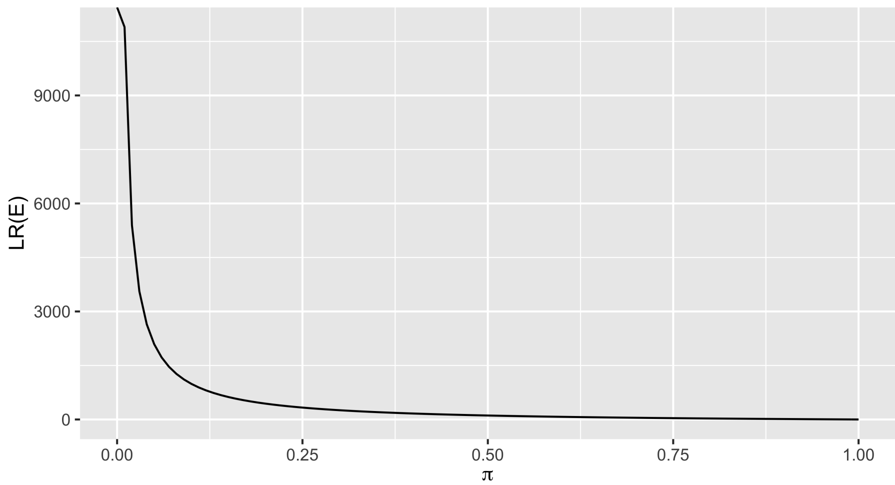
$$f(\pi) = -0.982\pi^2+0.982\pi > 0$$

The above condition is satisfied for any $0<\pi<1$ (f has two zeros: $\pi=0$ and $\pi=1$). Here is a plot of f:

[35] Thanks to Alicja Kowalewska for pressing me on this.



Similarly, $LR(E) > 1$ for any $0 < \pi < 1$. Here is a plot of $LR(E)$ against π:



Notice that $LR(E)$ does not go below 1. This means that for $L = G$ in the gatecrasher scenario DTLP wold tell us to convict for any prior probability of guilt $\pi \neq 0, 1$.

One might ask: is the conclusion very sensitive to the choice of L and G? The answer is, not too much.

Optional Content Starts

How sensitive is our analysis to the choice of L/G? Well, $LR(E)$ does not change at all, only the threshold moves. For instance, if $L/G = 4$, instead of f we end up with

$$f'(\pi) = -0.955\pi^2 + 0.955\pi > 0$$

and the function still takes positive values on the interval $(0, 1)$. In fact, the decision won't change until L/G increases to ≈ 111. Denote L/G as ρ, and let us start with the general decision standard, plugging in our calculations for $LR(E)$:

$$\begin{aligned}
LR(E) &> \frac{\mathsf{P}(H_\Delta)}{\mathsf{P}(H_\Pi)}\rho \\
LR(E) &> \frac{1-\pi}{\pi}\rho \\
\frac{0.991-0.991\pi}{0.009\pi} &> \frac{1-\pi}{\pi}\rho \\
\frac{0.991-0.991\pi}{0.009\pi}\frac{\pi}{1-\pi} &> \rho \\
\frac{0.991\pi-0.991\pi^2}{0.009\pi-0.009\pi^2} &> \rho \\
\frac{\pi(0.991-0.991\pi)}{\pi(0.009-0.009\pi)} &> \rho \\
\frac{0.991-0.991\pi}{0.009-0.009\pi} &> \rho \\
\frac{0.991(1-\pi)}{0.009(1-\pi)} &> \rho \\
\frac{0.991}{0.009} &> \rho \\
110.1111 &> \rho
\end{aligned}$$

OPTIONAL CONTENT ENDS

So, we conclude, in usual circumstances, DTLP does not handle the gatecrasher paradox.

6.11 Conclusions

Where are we, how did we get here, and where can we go from here? We were looking for a probabilistically explicated condition Ψ such that the trier of fact, at least ideally, should accept any relevant claim (including G) just in case $\Psi(A,E)$.

From the discussion that transpired it should be clear that we were looking for a Ψ satisfying the following desiderata:

conjunction closure If $\Psi(A,E)$ and $\Psi(B,E)$, then $\Psi(A\wedge B,E)$.

naked statistics The account should at least make it possible for convictions based on strong, but naked statistical evidence to be unjustified.

equal treatment the condition should apply to any relevant claim whatsoever (and not just a selected claim, such as G).

Throughout the paper we focused on the first two conditions (formulated in terms of the difficulty with conjunction (DAC), and the gatecrasher paradox), going over various proposals of what Ψ should be like and evaluating how they fare. The results can be summed up in the following table:

View	Convict iff	DAC	Gatecrasher
Threshold-based LP (TLP)	Probability of guilt given the evidence is above a certain threshold	fails	fails
Dawid's likelihood strategy	No condition given, focus on $\frac{\mathrm{P}(H\vert E)}{\mathrm{P}(H\vert\neg E)}$	- If evidence is fairly reliable, the posterior of $A \wedge B$ will be greater than the prior. - The posterior of $A \wedge B$ can still be lower than the posterior of any of A and B. - Joint likelihood, contrary do Dawid's claim, can also be lower than any of the individual likelihoods.	fails
Cheng's relative LP (RLP)	Posterior of guilt higher than the posterior of any of the defending narrations	The solution assumes equal costs of errors and independence of A and B conditional on E. It also relies on there being multiple defending scenarios individualized in terms of combinations of literals involving A and B.	Assumes that the prior odds of guilt are 1, and that the statistics is not sensitive to guilt (which is dubious). If the latter fails, tells to convict as long as the prior of guilt < 0.991.
Kaplow's decision-theoretic LP (DTLP)	The likelihood of the evidence is higher than the odds of innocence multiplied by the cost of error ratio	fails	convict if cost ratio $<$ 110.1111

Thus, each account either simply fails to satisfy the desiderata, or succeeds on rather unrealistic assumptions. Does this mean that a probabilistic approach to legal evidence evaluation should be abandoned? No. This only means that if we are to develop a general probabilistic model of legal decision standards, we have to do better. One promising direction is to go back to Cohen's pressure against **Requirement 1** and push against it. A brief paper suggesting this direction is (Di Bello, 2019), where the idea is that the probabilistic standard (be it a threshold or a comparative wrt. defending narrations) should be applied to the whole claim put forward by the plaintiff, and not to its elements. In such a context, DAC does not arise, but **equal treatment** is violated. Perhaps, there are independent reasons to abandon it, but the issue deserves further discussion. Another strategy might be to go in the direction of employing probabilistic methods to explicate the narration theory of legal decision standards (Urbaniak, 2018), but a discussion of how this approach relates to DAC and the gatecrasher paradox lies beyond the scope of this paper.

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