The Difficulty With Conjunction

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The standard of proof in criminal or civil trials is a criterion of decision. If the evidence meets the standard of proof, the defendant should be judged criminally or civilly liable, and otherwise the accusation should be dismissed. But when does the evidence meet the standard of proof and thus warrant a judgement of liability? According to some legal probabilists, a verdict against the defendant is warranted in case the evidence establishes, with a probability above a suitable threshold, that the defendant is criminally or civilly liable (see Bernoulli, 1713; Dekay, 1996; Kaplan, 1968; Kaye, 1979; Laplace, 1814; Laudan, 2006). In criminal cases, the bar is high, so the threshold will be, say, "at least .95 probability." In civil cases, the bar is lower, so the threshold will be, say, "at least .5 probability."

This threshold interpretation of the standard of proof is simple and elegant. Chapters XX showed how the .95 and .5 thresholds can be formally justified. But, unfortunately, this interpretation is not without problems. Chapter ZZ examined a first theoretical problem: the paradox of naked statistical evidence. The present chapter examines a second theoretical problem: **the difficulty with conjunction**, also known as **the conjunction paradox**. Section 1 describes in some detail the difficulty with conjunction. Sections 2 and 3 articulate two broad strategies that legal probabilists have pursued as a response to this difficulty. These strategies are promising and worth examining, but we show that they are ultimately unsatisfactory. Sections 4 and 5 put forward our own proposal.

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Our proposal for addressing the difficulty with conjunction complements our solution to the problem of naked statistical evidence. As we view them, both problems suggest that criminal and civil liability should not be understood as focused on general propositions, but as related to case-specific stories, theories or explanations tailored to the defendant on trial. This perspective—already formalized to some extent using Bayesian networks in chapter XY —affords a better understanding of how standards of proof operate in a trial.

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1 Formulating the difficulty

First formulated by Cohen (1977), the difficulty with conjunction has enjoyed a great deal of scholarly attention every since (Allen, 1986; Allen & Pardo, 2019; Allen & Stein, 2013; Cheng, 2012; Clermont, 2012; Dawid, 1987; Haack, 2014; Schwartz & Sober, 2017; Spottswood, 2016; Stein, 2005). This difficulty arises when a claim of wrongdoing, in a civil or criminal proceeding, is broken down into its constituent elements. By the probability calculus, the probability of a conjunction is often lower than the probability of the conjuncts. So, according to the probabilistic interpretation of the standard of proof, even if each constituent element (each individual conjunct) is established by the required standard of proof, the overall claim of wrongdoing (the conjunction) might not meet the required standard. Cohen and others after him believe that this outcome is counter-intuitive and runs contrary to trial practice.

1.1 Simple formulation

An example will help to fix ideas. Suppose the prosecution should establish two claims: first, that the defendant caused the victim's death; and second, that the defendant's action was premeditated. Cohen (1977) argues that common law systems subscribe to a **conjunction principle**, according to which if two claims, A and B, are established according to the governing standard of proof, so is their conjunction $A \wedge B$ (and *vice versa*). If the conjunction principle holds, the following must be equivalent:

Separate	A is established by standard S and B is established by standard S
Overall	The conjunction $A \wedge B$ is established by standard S

If we generalize to more than just two constituent claims, the conjunction principle requires that:

$$S[C_1 \wedge C_2 \wedge \cdots \wedge C_k] \Leftrightarrow S[C_1] \wedge S[C_2] \wedge \cdots \wedge S[C_k],$$

where $S[C_i]$ means that claim or hypothesis C_i is established according to standard S.

The principle goes in both directions. The implication from right to left, call it **aggregation**, posits that establishing the individual claims by the requisite standard is enough to establish the conjunction by the same standard. The implication in the opposite direction, call it **distribution**, posits that establishing the conjunction by the requisite standard is enough to establish the individual claims by the same standard. The difficulty with conjunction is traditionally concerned with the failure of aggregation, but as will see later on, on some probabilistic explications of the standard of proof, distribution can also fail.

We will ultimately question the tenability of the conjunction principle (more on this toward the end of this chapter). For the time being, however, let us assume the principle correctly captures two desired features of the standard of proof. The principle has some degree of plausibility and is consistent with the case law. For example, the United States Supreme Court writes that in criminal cases

> the accused [is protected] against conviction except upon proof beyond a reasonable doubt of *every fact* necessary to constitute the crime with which he is charged. (In re Winship, 397 U.S. 358, 364, 1970)

A plausible way to interpret this quotation is to assume the following identity: to establish someone's guilt beyond a reasonable doubt *just is* to establish each element of the crime beyond a reasonable doubt (abbreviated as BARD). Thus,

$$\mathsf{BARD}[C_1 \wedge C_2 \wedge \cdots \wedge C_k] \Leftrightarrow \mathsf{BARD}[C_1] \wedge \mathsf{BARD}[C_2] \wedge \cdots \wedge \mathsf{BARD}[C_k],$$

where the conjunction $C_1 \wedge C_2 \cdots \wedge C_k$ comprises all the material facts that, according to the applicable law, constitute the crime with with the accused is charged. A similar argument could be run for the standard of proof in civil cases, preponderance of the evidence or clear and convincing evidence.

The problem for the legal probabilist is that the conjunction principle conflicts with a threshold-based probabilistic interpretation of the standard of proof. For suppose the prosecution in a criminal case presents evidence that establishes claims A and B, separately, by the required probability, say about .95 each. If each claim is established by the requisite probability threshold, each claim is established by the requisite standard (assuming the threshold-based interpretation of the standard of proof). And if each claim is established by the requisite standard, then liability as a whole is established by the requisite standard (assuming the conjunction principle). And yet, if the claims are independent, the probability of their conjunction will be only $.95 \times .95 \approx .9$, below the required .95 threshold. So liability as a whole is *not* established by the requisite standard (assuming a threshold-based probabilistic interpretation of the standard). This contradicts the conjunction principle. Even though aggregation posits that establishing each conjunct by the required standard of proof is enough to establish the conjunction as a whole, the probability of each conjunct, in isolation, can meet the required probability threshold without the conjunction as a whole meeting the threshold.

This difficulty is persistent. It does not subside as the number of constituent claims increases. If anything, the difficulty becomes more apparent. Say the prosecution has established three separate claims by .95 probability. Their conjunction—again if the claims are independent—would be about .85 probable, even further below the .95 probability threshold. Nor does the difficulty with conjunction generally subside if the claims are no longer regarded as independent. The probability of the conjunction $A \wedge B$, without the assumption of independence, equals $\mathsf{P}(A|B) \times \mathsf{P}(B)$ or $\mathsf{P}(B|A) \times \mathsf{P}(A)$. But if claims A and B, separately, are established with .95 probability, enough for each to meet the threshold, the probability of $A \wedge B$ should still be below the .95 threshold unless $\mathsf{P}(A|B)$ or $\mathsf{P}(B|A)$ equal one.[1]

[1] For example, that someone premeditated a harmful act against another (call it premed) makes it more likely that they did cause harm in the end (call it harm). Since $\mathsf{P}(\mathsf{harm}|\mathsf{premed}) > \mathsf{P}(\mathsf{harm})$, the two claims are not independent. Still, premeditation does not always lead to harm, so $\mathsf{P}(\mathsf{harm}|\mathsf{premed})$ will often be below 1. If both claims are established with .95, the probability of the conjunction $\mathsf{harm} \wedge \mathsf{premed}$ should still be below the .95 threshold so long as $\mathsf{P}(\mathsf{harm}|\mathsf{premed})$ is still below 1.

1.2 Adding the evidence

So far we proceeded without mentioning the evidence in support of the claims that constitute the wrongdoing. This is a simplification. As we will see, it is crucial to pay attention to the supporting evidence. With this in mind, the conjunction principle for two claims can be formulated, as follows:

$$\mathrm{S}[a,A] \text{ and } \mathrm{S}[b,B] \Leftrightarrow \mathrm{S}[a \wedge b, A \wedge B].$$

In the case of more than two claims, the formulation of the principle can be extended accordingly. Note that a and b denote the evidence for claims A and B respectively, and S denotes the standard by which the evidence establishes the claim in question. So, for example, the expression $\mathrm{S}[a,A]$ should be read as 'evidence a supports claim A by standard S.' If we adopt the threshold interpretation of the standard of proof, the expression $\mathrm{S}[a,A]$ should be interpreted as 'evidence a establishes claim A with a probability above a suitable threshold t corresponding to standard S' or in symbols $\mathsf{P}(A|a) > t$. An analogous probabilistic reading applies to the expressions $\mathrm{S}[b,B]$ and $\mathrm{S}[a \wedge b, A \wedge B]$.

Does a threshold-based probabilistic interpretation of the standard of proof also conflict with this revised version of the conjunction principle? The answer is positive, but seeing why requires a bit more work. We should check whether, if both $\mathsf{P}(A|a)$ and $\mathsf{P}(B|b)$ meet the threshold, say .95, then so does $\mathsf{P}(A \wedge B|a \wedge b)$. We are no longer just comparing the probability of $A \wedge B$ to the probability of A and the probability of B as such. Rather, we are comparing the probability of $A \wedge B$ given the combined evidence $a \wedge b$ to the probability of A given evidence a and the probability of B given evidence b.

Consider an aggravated assault case. Suppose the prosecution should establish two claims: first, that the defendant injured the victim; and second, that the defendant knew he was interacting with a public official. A witness testimony (call it $\mathsf{witness}$) is offered in support of the proposition that the defendant injured the victim (call this proposition injury). In addition, the defendant's call to an emergency number (call it $\mathsf{emergency}$) is offered as evidence for the proposition that the defendant knew he was interacting with a firefighter (call this proposition $\mathsf{firefighther}$). If $\mathsf{P}(\mathsf{injury}|\mathsf{witness})$ and $\mathsf{P}(\mathsf{firefighter}|\mathsf{emergency})$ both meet the required probability threshold, does $\mathsf{P}(\mathsf{injury} \wedge \mathsf{firefighter}|\mathsf{witness} \wedge \mathsf{emergency})$ also necessarily meet the threshold?

The answer is negative, at least provided two independence assumptions hold. The first is that $\mathsf{P}(\mathsf{injury}|\mathsf{witness}) = \mathsf{P}(\mathsf{injury}|\mathsf{witness} \wedge \mathsf{emergency})$. This assumption is plausible: that the defendant called an emergency number should not make it more (or less) likely that the defendant would cause injury to someone. The second assumption is that $\mathsf{P}(\mathsf{firefighther}|\mathsf{emergency}) = \mathsf{P}(\mathsf{firefighther}|\mathsf{witness} \wedge \mathsf{emergency} \wedge \mathsf{injury})$. This assumption is also plausible: that the defendant injured the victim and there is a testimony to that effect does not make it more (or less) likely that the victim was a firefighter. Admittedly, more is needed to justify these assumptions than an appeal to plausibility, a point to which we will soon return. But, granting for now that the two assumptions hold, it follows that:[2]

$$\mathsf{P}(\mathsf{injury} \wedge \mathsf{firefighther}|\mathsf{witness} \wedge \mathsf{emergency}) = \mathsf{P}(\mathsf{injury}|\mathsf{witness}) \times \mathsf{P}(\mathsf{firefighther}|\mathsf{emergency}).$$

If the equality holds, even when $\mathsf{P}(\mathsf{injury}|\mathsf{witness})$ and $\mathsf{P}(\mathsf{firefighther}|\mathsf{emergency})$ meet the required probability threshold, $\mathsf{P}(\mathsf{injury} \wedge \mathsf{firefighther}|\mathsf{witness} \wedge \mathsf{emergency})$ generally will not. An assumption to make here is that both $\mathsf{P}(\mathsf{injury}|\mathsf{witness})$ and $\mathsf{P}(\mathsf{firefighther}|\mathsf{emergency})$ are below 1, as is usually the case given that the evidence offered in a trial is fallible. The conclusion is that aggregation is violated.

So, in at least one case and under seemingly plausible assumptions, the revised conjunction principle fails if the standard of proof is interpreted as a probability threshold. But can we say something more general? To address this question, we will now represent more formally—specifically, using Bayesian networks—the relationship between claims A, B and the conjunction $A \wedge B$, as well as the supporting evidence a, b and the conjunction $a \wedge b$.

[2] By the chain rule and the independence assumptions $\mathsf{P}(A|a) = \mathsf{P}(A|a \wedge b)$ and $\mathsf{P}(B|b) = \mathsf{P}(B|a \wedge b \wedge A)$, the following holds:

$$\begin{aligned} \mathsf{P}(A \wedge B|a \wedge b) &= \mathsf{P}(A|a \wedge b) \times \mathsf{P}(B|a \wedge b \wedge A) \\ &= \mathsf{P}(A|a) \times \mathsf{P}(B|b) \end{aligned}$$

1.3 Independent hypotheses

We already studied Bayesian networks in Chapter ZZ. Here we only sketch the essential ideas. A Bayesian network is a formal model that consists of a graphical part (a directed acyclic graph, DAG) and a numerical part (a probability table).

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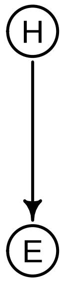
The nodes in the graph represent random variables that can take different values. For ease of exposition, we will use 'nodes' and 'variables' interchangeably. The nodes are connected by directed edges (arrows). No loops are allowed, hence the name acyclic.

The simplest evidential relation, one of evidence bearing on a hypothesis, can be represented by the directed graph displayed in Figure 1. The arrow need not have a causal interpretation. The direction of the arrow indicates which conditional probabilities should be supplied in the probability table. Since the arrow goes from H to E, we should specify the probabilities of the different values of E conditional on the different values of H.

Return now to the difficulty with conjunction. We will first examine the case in which the two hypotheses A and B are probabilistically independent. We will relax this assumption later. The two directed graphs visualized in Figure 2 represent two items of evidence each supporting its own hypothesis: a supports A and b supports B. To represent the conjunction $A \wedge B$, a conjunction node AB is added and arrows are drawn from nodes A and B into node AB (Figure 3). This arrangement makes it possible to express the meaning of $A \wedge B$ via a probability table that mirrors the truth table for the conjunction in propositional logic (see Table 1).[3]



	H	$\neg H$
E	$\mathsf{P}(E\|H)$	$\mathsf{P}(E\|\neg H)$
$\neg E$	$\mathsf{P}(\neg E\|H)$	$\mathsf{P}(\neg E\|\neg H)$

Figure 1: DAG of the simplest evidential relation along with a probability table

Figure 2: DAGs of a supporting A and b supporting B.

[3] The difference is that the values 1 and 0 stand for two different things depending on where they are in the table. In the columns corresponding to the nodes they represent node states: true and false; in the Pr column they represent the conditional probability of a given state of AB given the states of A and B listed in the same row. For instance, take a look at row two. It says: if A and B are both in states 1, then the probability of AB being in state 0 is 0. In principle we could use 'true' and 'false' instead of 1 and 0 to represent states, but the numeric representation is easier to use in programming, which we do quite a bit in the background, so the reader might as well get used to this harmless ambiguity. For binary nodes, we will consistently use '1' and '0' for the states, it's just probabilities that in this case end up being extreme.

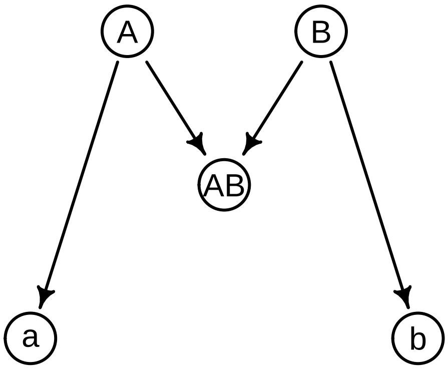


Figure 3: DAG of the conjunction set-up, with the usual independence assumptions built in (DAG 1).

AB	A	B	Pr
1	1	1	1
0	1	1	0
1	0	1	0
0	0	1	1
1	1	0	0
0	1	0	1
1	0	0	0
0	0	0	1

Table 1: Conditional probability table for the conjunction node.

The resulting graph, call it DAG 1, satisfies the desired independence assumptions. First, the two claims A and B are probabilistically independent of one another. Their independence is guaranteed by the fact that the conjunction node AB is a collider and thus no information flows through it.[4] Second, the supporting items of evidence a and b are also probabilistically independent of one another. The reason is the same: node AB blocks any flow of information between the evidence nodes. Notably, the independence of the items of evidence is not always explicitly stated in the formulation of the conjunction paradox. The good thing is that the Bayesian network makes such assumptions explicit.

With this set-up in place, the conjunction paradox arises because aggregation is violated. By the theory of Bayesian networks, DAG 1 in Figure 3 ensures the following:[5]

$$
\begin{aligned}
\mathsf{P}(A \wedge B | a \wedge b) &= \mathsf{P}(A | a \wedge b) \times \mathsf{P}(B | a \wedge b \wedge A) \\
&= \mathsf{P}(A | a) \times \mathsf{P}(B | b)
\end{aligned}
$$

Thus, even when claims A and B are sufficiently probable given their supporting evidence a and b (for a fixed threshold t)—in symbols, $\mathsf{P}(A|a) > t$ and $\mathsf{P}(B|b) > t$—it does not generally follow that $A \wedge B$ is sufficiently probable given combined evidence $a \wedge b$ provided (as is normally the case) neither $\mathsf{P}(A|a)$ nor $\mathsf{P}(B|b)$ equal 1. As before, the conjunction principle fails because aggregation fails.

The argument here goes beyond the specific example about aggravated assault in the previous section. The argument only assumes that the directed graph in Figure 3 is an adequate representation of a situation in which two items of evidence, a and b, support their own hypothesis, A and B. The graph encodes two plausible relations of probabilistic independence: between hypotheses A and B and between items of evidence a and b. The theory of Bayesian networks does the rest of the work.

1.4 Dependent hypotheses

Consider now what happens if claims A and B are not regarded as probabilistically independent. To represent this, it is enough to draw an arrow between nodes A and B. The modified graph is displayed in Figure 4, call it DAG 2. The open path between nodes A and B no longer guarantees the probabilistic independence of A and B or the independence of evidence nodes a and b. Note, however, that there is still no *direct* dependence between the items of evidence. The items of evidence are still probabilistically

[4] A more formal treatment of this point is provided in **REFER TO OTHER CHAPTER**.

[5] This is because the only path between A and B goes through AB, which is a collider; as long as we do not condition on it, all paths between A and B remain blocked. See our chapter introducing Bayesian networks for details on this issue. **REFER TO APPROPRIATE CHAPTER**

independent of one another *conditional* on their respective hypothesis. That is, $\mathsf{P}(a|A) = \mathsf{P}(a|A \wedge b)$ and $\mathsf{P}(b|B) = \mathsf{P}(b|B \wedge a)$. So a and b still count as independent lines of evidence despite not being (unconditionally) probabilistically independent.[6]

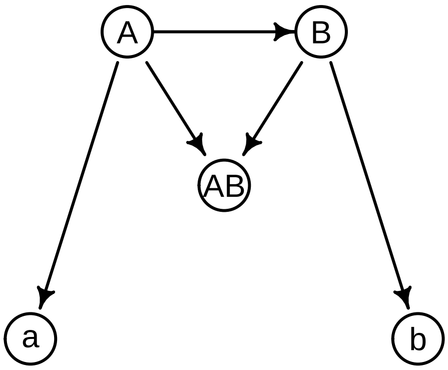


Figure 4: DAG of the conjunction set-up, without independence between A and B (DAG 2).

The difficulty with conjunction arises even without the independence of hypotheses A and B, at least in a number of circumstances. Using simulation we can also can study how often, in principle, the joint posterior $\mathsf{P}(A \wedge B|a \wedge b)$ is below both of the individual posteriors $\mathsf{P}(A|a)$ and $\mathsf{P}(B|b)$. To this end, we simulated 10,000 random Bayesian networks based on DAG 1 and DAG 2. Assuming that each possible Bayesian network has an equal probability of occurring, the joint posterior is lower than both individual posteriors 68% of the time for DAG 1, and around 60% for DAG 2 (see the appendix for details). This result agrees with Schwartz & Sober (2017) who pointed out that, if hypotheses A and B are probabilistically dependent on one another, aggregation will fail less often. Still, postulating a dependence between hypotheses does little for solving the difficulty with conjunction since a drop in the failure rate from 68% to 60% is limited.

2 Evidential Strength

The previous section demonstrated that aggregation fails in a large class of cases if the standard of proof is understood as a posterior probability threshold. Some believe that such failures are strong enough ground to advocate a different conception of probability, along the lines of Baconian probability (Cohen, 2008), fuzzy logic or belief functions (Clermont, 2013), or reject legal probabilism altogether (Allen & Stein, 2013). But posterior probability thresholds are not the only way of understanding standards of proof from a probabilistic perspective. Standards of proof can also be modeled using probabilistic measures of *evidential strength*. This is the approach we explore in this section.

Two common probabilistic measures of evidential strength are the Bayes factor and the likelihood ratio. We discussed them earlier in Chapter XX. As we show below, under plausible assumptions, these measures of evidential strength validate one direction of the conjunction principle: aggregation. If a is sufficiently strong evidence in favor of A and b is sufficiently strong evidence in favor of B, then $a \wedge b$ is sufficiently strong evidence in favor of the conjunction $A \wedge B$. In fact, the evidential support for the conjunction will often exceed that for the individual claims, a point already made by Dawid (1987) who wrote that 'suitably measured, the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents' (97).

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Dawid thought that vindicating aggregation was enough for the conjunction paradox to 'evaporate.'

[6]Here is an illustration of the idea of independent lines of evidence without unconditional independence. Suppose the same phenomenon (say blood pressure) is measured by two instruments. The reading of the two instruments (say 'high' blood pressure) should be *probabilistically dependent* of one another. After all, if the instruments were both infallible and they were measuring the same phenomenon, they should give the exact same reading. On the other hand, the two instruments measuring the same phenomenon should count as *independent lines of evidence*. This fact is rendered in probabilistic terms by means of probabilistic independence conditional on the hypothesis of interest. These ideas can be worked out more systematically in the language of Bayesian networks. Roughly, two variables are probabilistically dependent if there is an open path between them. On the other hand, an open path can be closed by conditioning on one of the variables along the path. For a more rigorous exposition of the notions of open and closed paths, see **CITE EARLIER CHAPTERS**.

But, as we show below, this is not quite right. On the evidential strength interpretation of the standard of proof, the other direction of the conjunction principle, distribution, fails. If $a \wedge b$ is sufficiently strong evidence in favor of $A \wedge B$, it does not follow that $a \wedge b$ is sufficiently strong evidence in favor of A or that $a \wedge b$ is sufficiently strong evidence in favor of B. This is odd. If we interpret the standard of proof using evidential strength, one could establish beyond a reasonable doubt that $A \wedge B$ (say the defendant killed the victim *and* acted intentionally) while failing to establish one of the conjuncts.

The prospects for legal probabilism look dim. If the standard of proof is understood as a posterior probability threshold, the conjunction principle fails because aggregation fails (previous section). If, instead, the standard of proof is understood as a threshold relative to evidential strength, the conjunction principle still fails because distribution fails (this section). From a probabilistic perspective, it seems impossible to capture both directions of the conjunction principle. In what follows, we defend more precisely the claim that, on the evidential strength approach, aggregation succeeds but distribution fails. The argument is laborious. The reader should arm themselves with patience or take our word for it and jump ahead. The curious reader is welcome to read the appendix for the technical details.

2.1 Combined support: Bayes factor

The first part of the argument shows that the combined support supplied by multiple pieces of evidence (e.g. $a \wedge b$) for a conjunctive claim (e.g. $A \wedge B$) typically exceeds the individual support supplied by individual pieces of evidence for individual claims. This claim holds for the Bayes factor and to some extent for the likelihood ratio. We start with the Bayes factor $\mathsf{P}(E|H)/\mathsf{P}(E)$ as our measure of the support of E in favor of H. As is apparent from Bayes' theorem,

$$\mathsf{P}(H|E) = \frac{\mathsf{P}(E|H)}{\mathsf{P}(E)} \times \mathsf{P}(H),$$

the Bayes factor measures the extent to which a piece of evidence increases the probability of a hypothesis, as compared to its prior probability. The greater the Bayes factor (for values above one), the stronger the support of E in favor of H.

Suppose Bayes factors $\mathsf{P}(a|A)/\mathsf{P}(a)$ (abbreviated BF_A) and $\mathsf{P}(b|B)/\mathsf{P}(b)$ (abbreviated BF_B) are greater than one. That is, items of evidence a and b positively support A and B, separately. The combined support here is (see the appendix for a proof):

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} = \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)}$$
$$BF_{AB} = BF_A \times BF_B$$

Consequently, the combined support BF_{AB} is always higher than the individual support so long as the individual pieces of evidence positively support their respective hypothesis. This claim holds assuming hypotheses A and B are independent and items of evidence a and b are independent. These assumptions are validated by DAG 1 in Figure 3. Under these circumstances, Dawid's claim that 'the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents' (if support is to be measured in terms of Bayes factor) is verified.[7]

Even when A and B are not probabilistically independent, the combined Bayes factor BF_{AB} will still be greater than both the individual Bayes factor BF_A and BF_B if the probabilistic measure fits DAG 2 in Figure 4. To see why, first note that the following holds (see the appendix for a proof):

$$\begin{aligned} BF_{AB} = \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b)} &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b|a)} && = BF_A \times BF_B' \\ &= \frac{\mathsf{P}(a|A)}{\mathsf{P}(a|b)} \times \frac{\mathsf{P}(b|B)}{\mathsf{P}(b)} && = BF_A' \times BF_B \end{aligned}$$

The difference from the case of independent hypotheses is that $BF_B = \mathsf{P}(b|B)/\mathsf{P}(b)$ is now replaced by $BF_B' = \mathsf{P}(b|B)/\mathsf{P}(b|a)$, or alternatively $BF_A = \mathsf{P}(a|A)/\mathsf{P}(a)$ by $BF_A' = \mathsf{P}(a|A)/\mathsf{P}(a|b)$.[8] Still, if the probabilistic

[7] The result holds beyond two pieces of evidence; see Figure 5). Note that the order is reversed if the items of evidence oppose the individual hypotheses. Neutral evidence results in a combined Bayes factor of 1, no matter the prior or the number of items of evidence.

[8] Since b need not be probabilistically independent of a, there is no guarantee that $\mathsf{P}(b|a) = \mathsf{P}(b)$ or $\mathsf{P}(a|b) = \mathsf{P}(a)$.

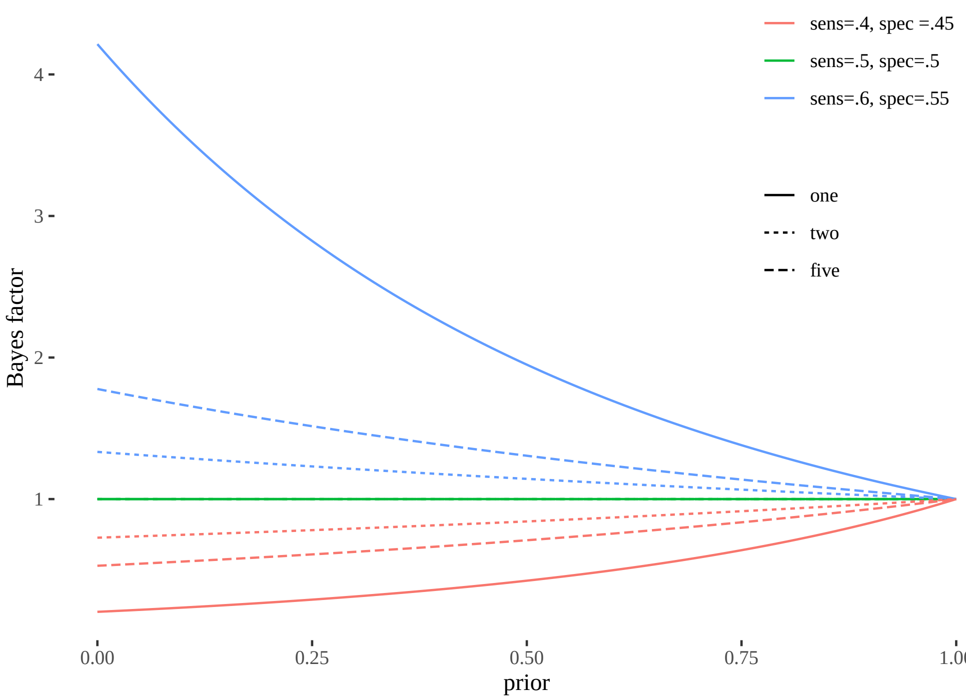

Figure 5: Bayes factor for one, two and five items of evidence and the corresponding claims, given different degrees of specificity and sensitivity of the evidence. The independence assumptions in Figure 3, DAG 1, hold.

measure fits DAG 2, whenever BF_B is greater than 1, so is $BF_B^{'}$, and whenever BF_A is greater than 1, so is $BF_A^{'}$ (see the appendix for a proof). Thus, the joint Bayes factor BF_{AB} will be greater than any of the individual Bayes factors.[9]

2.2 Combined support: likelihood ratio

We now run a similar argument for the likelihood ratio, another probabilistic measure of evidential strength. The likelihood ratio—extensively discussed in Chapter XX —compares the probability of REFER TO EARLIER CHAPTER the evidence on the assumption that a hypothesis of interest is true (*sensitivity*) and the probability of the evidence on the assumption that the negation of the hypothesis is true (*1- specificity*). That is,

$$\frac{\mathsf{P}(E|H)}{\mathsf{P}(E|\neg H)} = \frac{\textit{sensitivity}}{\textit{1- specificity}}$$

The greater the likelihood ratio (for values above one), the stronger the evidential support in favor of the hypothesis (as contrasted to its negation).

The question is whether the combined support measured by the combined likelihood ratio

$$\frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))}$$

exceeds the individual support measured by the individual likelihood ratios $\mathsf{P}(a|A)/\mathsf{P}(a|\neg A)$ and $\mathsf{P}(b|B)/\mathsf{P}(b|\neg B)$. Under suitable assumptions, the answer is positive. So, details aside, Bayes factor and likelihood ratio agree here. The argument for the likelihood ratio, however, is more laborious.

We start with the fact that, in a large class of cases (see the appendix for a proof),

$$LR_{AB} = \frac{\mathsf{P}(a \wedge b|A \wedge B)}{\mathsf{P}(a \wedge b|\neg(A \wedge B))} = \frac{\mathsf{P}(a|A) \times \mathsf{P}(b|B)}{\frac{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|B)+\mathsf{P}(A)\mathsf{P}(\neg B|A)\mathsf{P}(a|A)\mathsf{P}(b|\neg B)+\mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)\mathsf{P}(a|\neg A)\mathsf{P}(b|\neg B)}{\mathsf{P}(\neg A)\mathsf{P}(B|\neg A)+\mathsf{P}(A)\mathsf{P}(\neg B|A)+\mathsf{P}(\neg A)\mathsf{P}(\neg B|\neg A)}}$$

[9] In contrast, if the underlying DAG were to contain a direct dependence between the items of evidence, the joint Bayes factor could be lower than either of the individual Bayes factors (see the appendix for a proof).

This equality is fairly general. It holds whether or not hypotheses A and B are probabilistically independent. Given the many unknowns here, it pays to make some simplifications if only for illustrative purposes.

The first temporary simplification we make is to assume that the sensitivity and specificity of the items of evidence are both equal to the same value x. Second, we assume that A and B are probabilistically independent in agreement with DAG 1 (Figure 3). The combined likelihood ratio can now be plotted as a function of x. Figure 6 shows that the combined likelihood ratio always exceeds the individual likelihood ratios whenever they are greater than one (or in other words, as is usually assumed, the two pieces of evidence provide positive support for their respective hypotheses).[10] As with the Bayes factor, the combined likelihood ratio exceeds the individual likelihood ratios.

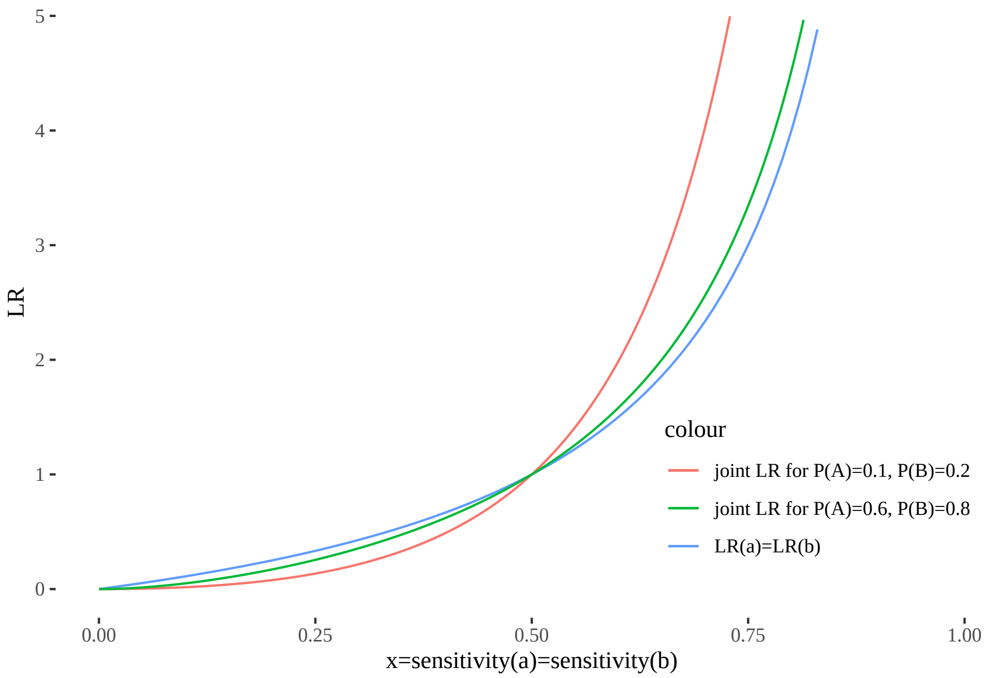

Figure 6: Combined likelihood ratios exceeds individual Likelihood ratios as soon as sensitivity is above .5. Changes in the prior probabilities $\mathsf{P}(A)$ and $\mathsf{P}(B)$ do not invalidate this result.

What happens if the two simplifying assumptions are relaxed? If the items of evidence have different levels of sensitivity and specificity, the combined likelihood ratio never goes below the lower of the two individual likelihood ratios, but can be lower than the higher individual likelihood ratio. We established this claim by means of a computer simulation (see the appendix for details). This holds if the probabilistic measure fits DAG 1 (independent hypotheses) or DAG 2 (dependent hypotheses), but fails if there is direct dependence between the pieces of evidence.[11] In this sense, the joint likelihood ratio behaves differently than the joint Bayes factor since it is greater than the lowest of the individual likelihood ratios, rather than being greater than both of them. Consequently, Dawid's claim that 'the support supplied by the conjunction of several independent testimonies exceeds that supplied by any of its constituents' should be weakened and restricted to cases in which there is no direct probabilistic dependence between the items of evidence. In such cases, the evidential support (when it is measured by the likelihood ratio) supplied by the conjunction of several independent items of evidence exceeds the support supplied by at least one individual item of evidence but possibly not all.

[10]Interestingly, the combined likelihood ratio varies depending on the prior probabilities $\mathsf{P}(A)$ and $\mathsf{P}(B)$.

[11]The simplified set-up from before does not contradict this claim but follows from it. In the simplified set-up, both individual likelihood ratios were the same, so whenever the joint likelihood was higher than the minimum of the individual likelihood ratios, it was higher than the both of them.

2.3 Vindicating aggregation

We have seen that under suitable assumptions the combined evidential support is never below the lowest individual support. More precisely, let $\mathsf{str}[E,H]$ stand for the strength of the evidential support of a piece of evidence E toward a hypothesis H, measured by the Bayes factor or the likelihood ratio. We have established the following fact:

$$\mathsf{str}[a \wedge b, A \wedge B] \geq \min(\mathsf{str}[a,A], \mathsf{str}[b,B]),$$

where the function min returns the lowest of its arguments.

This fact can be used to justify aggregation. To be sure, aggregation is not directly a principle about evidential strength. It is a principle about the standard of proof. So, to complete the argument, the standard of proof should be tied to evidential strength. How? The rule of decision could be: liability is proven according to the governing standard of proof in case the strength of the evidential support—measured by the Bayes factor or the likelihood ratio—meets a suitably high threshold. On this reading, the threshold should no longer be a posterior probability between 0 and 1, but a number somewhere above one. The greater this number, the more stringent the standard of proof, for any value above one.

The only question left at this point is, how to identify the appropriate evidential strength threshold? The answer isn't obvious. Below we examine two possible approaches. First, the threshold for the Bayes factor and the threshold for the likelihood ratio can be derived from the threshold for the posterior probability by manipulating two simple equations. Consider first the Bayes factor threshold. Since

$$\text{Bayes factor} = \frac{\text{posterior}}{\text{prior}},$$

the Bayes factor threshold, call it t_{BF}, can be defined as follows:

$$t_{BF} = \frac{t}{\text{prior}},$$

where t is the posterior probability threshold. The value of t can be determined in a decision-theoretic manner by minimizing expected costs (see Chapter XX). Once t is fixed in this way, the value t_{BF} can be easily derived by the equations above. The same strategy works for the likelihood ratio threshold, call it t_{LR}. By the odds version of Bayes' theorem, REFER TO ER-ALIER CHAPPER

$$\text{likelihood ratio} = \frac{\text{posterior odds}}{\text{prior odds}}.$$

If the posterior ratio is fixed at, say $t/1-t$, then the value of t_{LR} can be easily obtained as follows:

$$t_{LR} = \frac{t/1-t}{\text{prior odds}}.$$

Note that thresholds t_{BF} and t_{LR} will depend on the prior probability of the hypothesis. The higher the prior probability, the lower the threshold. Whether this is a desirable property of a decision threshold can be questioned, but a similar point applies to the posterior threshold t: the higher the prior probability, the easier it is to meet the threshold.

This approach is simple and elegant, but incurs a major shortcoming: aggregation still fails. There will be cases in which the conjuncts taken separately satisfy the decision standard t_{BF} or t_{LR}, while the conjunction does not.

To illustrate, consider a posterior probability threshold of .95 as might be appropriate in a criminal case. If the individual claims A and B both have a prior probability of, say .1, the Bayes factor threshold $t^A_{BF} = t^B_{BF} = {.95}/{.1} = 9.5$ for A or B individually. Note the superscripts A and B. The Bayes factor threshold t_{BF} is indexed to the claim of interest because the threshold is prior-dependent and thus claim-dependent (since different claims have different prior probabilities). If, as is often assumed, claims A and B are probabilistically independent, the composite claim $A \wedge B$ will be associated with the Bayes factor threshold $t^{A\wedge B}_{BF} = {.95}/{(.1\times.1)} = 95$, a much higher value. Suppose each claim barely meet the 9.5 Bayes factor threshold. Given independence, the joint Bayes factor results from multiplying the individual Bayes factors, that is, $9.5 \times 9.5 = 90.25$. This is not quite enough to meet $t^{A\wedge B}_{BF} = 95$.

So aggregation fails. An analogous point holds for the likelihood ratio threshold.[12] The culprit is the fact that t_{BF} and t_{LR} have different values when applied to individual claims A and B as opposed to the composite claim $A \wedge B$.

But not all hope is lost. Perhaps it is unsurprising that deriving the evidential strength threshold from the posterior probability threshold did not solve the problem. If there are reasons to reject the posterior probability threshold, these reasons also apply, *mutatis mutandis*, to thresholds that are parasitic to it. The second approach we consider is more straightforward. It consists in fixing the evidential strength threshold independently of the posterior probability threshold and regardless of the prior probability of the hypothesis. Criminal cases would still have a higher decision threshold than civil cases, but the threshold would be constant across individual and composite claims.[13] The good news is that the fixed threshold approach vindicates aggregation. The argument is easy. As seen previously, the combined evidential support is usually greater than at least one, if not all, individual evidential supports, whether measured by the Bayes factor or the likelihood ratio. So, for a fixed value of the threshold, whenever BF_A and BF_B meet the threshold t_{BF}, usually also the combined Bayes factor BF_{AB} meets t_{BF}. The same applies for the likelihood ratio threshold. Whenever LR_A and LR_B meet the threshold t_{LR}, then usually also LR_{AB} meets t_{LR}. Aggregation is finally vindicated!

2.4 The failure of distribution

But it is too soon to declare victory. The trouble is that, if the evidential strength threshold is held fixed across individual and composite claims, the principle of distribution becomes problematic. This is the other direction of the conjunction principle. Distribution posits that establishing the conjunction, by a governing standard of proof, is sufficient for establishing the individual conjuncts. This is a seemingly uncontroversial principle. How can it possibly fail? To be clear, the principle of distribution does not fail if the standard of proof is understood as a posterior probability threshold. After all, the probability of a conjunction cannot be higher than the probability of its conjuncts. But distribution does fail if the standard of proof is understood as an evidential strength threshold.

For suppose the combined Bayes factor, ${}^{\mathsf{P}(a \wedge b|A \wedge B)}/_{\mathsf{P}(a \wedge b)}$, barely meets the threshold. The individual support, say ${}^{\mathsf{P}(a|A)}/_{\mathsf{P}(a)}$, could still be below the threshold unless ${}^{\mathsf{P}(b|B)}/_{\mathsf{P}(b)} = 1$ (which should not happen if b positively supports B). The problem for likelihood ratio is analogous. Looking at the graphs in Figure 5 and Figure 6 should be enough to convince oneself that this is the case. So—if the standard of proof is interpreted as a fixed evidential strength threshold—even though the conjunction $A \wedge B$ was proven according to the desired standard, one of the individual claims might not.

We have just shown that the following distribution principles fails:

$$S[a \wedge b, A \wedge B] \Rightarrow S[a, A] \wedge S[b, B], \tag{DIS1}$$

where S is a placeholder for the standard of proof. But, perhaps, principle (DIS1) is not as obviously intuitive as one might have thought. Since the evidence is not held constant throughout, the support supplied by $a \wedge b$ could be stronger than that supplied by a and b individually. After all, the individual evidence a or b is weaker evidence than the combined evidence $a \wedge b$. At least one might argue this way.

To accommodate this line of reasoning, here is a weaker distribution principle:

$$S[a \wedge b, A \wedge B] \Rightarrow S[a \wedge b, A] \wedge S[a \wedge b, B]. \tag{DIS2}$$

This principle might seem less controversial because it holds the evidence constant. One would not want to claim that, while holding fixed evidence $a \wedge b$, establishing the conjunction might not be enough

R: revised this bit in light of the stuff I added in the appendix.

[12] Say A and B have prior probabilities of .2 and .3 respectively. On this approach, the likelihood ratio threshold for A and B will be $t^A_{LR} \approx 76$ and $t^B_{LR}a \approx 44$, assuming posterior probability threshold of 0.95. The likelihood ratio threshold for the composite claim $A \wedge B$ will be $t^{A \wedge B}_{LR} \approx 297$. Suppose the individual likelihood ratios meet their threshold. And, for simplicity, suppose sensitivity and specificity of the individual items of supporting evidence are the same. In this case, for t^A_{LR} to be met, evidence a should have sensitivity (and specificity) of at least 0.988. For t^B_{LR} to be met, evidence b should have sensitivity (and specificity) of 0.978. Given these values, the combined likelihood ratio equals about 145, far short from the threshold $t^{A \wedge B}_{LR} \approx 297$.

[13] This approach raises the question of how the threshold should be fixed in the first place. Since the threshold is no longer derived from the posterior probability threshold, standard decision theory cannot help here. We do not further examine this question, since this approach is not part of our ultimate solution to the conjunction paradox.

for establishing one of the conjuncts. Yet, as explained in the appendix, (DIS2) is in no better position when it comes to the fate of aggregation and distribution. Distribution fails even without direct dependence between hypothesis, and aggregation starts to fail as soon as such a dependence is allowed.

3 The comparative strategy

We hit two dead ends. The posterior probability threshold is subject to the difficulty with conjunction, specifically, the failure of aggregation (Section 1). The evidential strength threshold is subject to another kind of difficulty, the failure of distribution (Section 2). As a third approach, the standard of proof can be understood comparatively. This approach has been advanced by Cheng (2012) following the theory of relative plausibility by Pardo & Allen (2008). Say the prosecutor or the plaintiff puts forward a hypothesis H_p about what happened. The defense offers an alternative hypothesis, call it H_d. On this approach, rather than directly evaluating the support of H_p given the evidence and comparing it to a threshold, we compare the support that the evidence provides for two competing hypotheses H_p and H_d, and decide for the one that garners greater evidential support.

R: revised this bit in light of our discussion

It is controversial whether this is what happens in all trial proceedings, especially in criminal trials, if one thinks of the defense hypothesis H_d as a substantial account of what has happened. The defense may elect to challenge the hypothesis put forward by the other party without proposing one of its own. It is also quite clear that the defense hypothesis needs not to be proven more plausible than the prosecution hypothesis: it is enough that it is sufficiently plausible to cast reasonable doubt on the prosecution hypothesis, but the required plausibility level does not have to be high enough to warrant belief.
In the O.J. Simpson trial, for example, the defense did not advance its own story about what happened, but simply argued that the evidence provided by the prosecution, while significant on its face to establish OJ's guilt, was riddled with problems and deficiencies, and that the alternative explanations of the evidence were at least to some extent plausible. This defense strategy was enough to secure an acquittal. So, in order to create a reasonable doubt about guilt, the defense does not always provide a full-fledged alternative hypothesis. The supporters of the comparative approach, however, respond that this could happen in a small number of cases, even though in general—especially for tactical reasons—the defense will provide an alternative hypothesis.

3.1 Comparing posteriors

Setting aside these qualms, we first work out the comparative strategy using posterior probabilities. Here, the standard of proof is understood as follows: given a body of evidence E and two competing hypotheses H_p and H_d, the probability $\mathsf{P}(H_p|E)$ should be suitably higher than $\mathsf{P}(H_d|E)$, or in other words, the ratio $\mathsf{P}(H_p|E)/\mathsf{P}(H_d|E)$ should be above a suitable threshold. Presumably, the ratio threshold should be higher for criminal than civil cases, for example, greater than one for civil cases and substantively above one for criminal cases.[14] Note that H_p and H_d need not be one the negation of the other.[15]

We will work through a stylized case used by Cheng himself. Suppose, in a civil case, the plaintiff claims that the defendant was speeding (S) and that the crash caused her neck injury (C). Here the plaintiff's hypothesis is $S \wedge C$. Suppose, given evidence E, the conjuncts, taken separately, meet the decision threshold:

$$\frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \qquad\qquad \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1$$

[14] In the comparative approach—as Cheng (2012) shows—expected utility theory can set the appropriate threshold as a function of the costs and benefits of trial decisions. Suppose the costs of a false positive is c_{FP} and that of a false negative is c_{FN}, both greater than zero. Intuitively, the decision rule should minimize the expected costs (see Chapter YYY). That is, a finding against the defendant would be acceptable whenever its expected costs—$\mathsf{P}(H_d|E) \times c_{FP}$—are smaller than the expected costs of an acquittal—$\mathsf{P}(H_p|E) \times c_{FN}$—or in other words: $\frac{\mathsf{P}(H_p|E)}{\mathsf{P}(H_d|E)} > \frac{c_{FP}}{c_{FN}}$. In civil cases, it is customary to assume the costs ratio of false positives to false negatives equals one. So the rule of the decision would be: find against the defendant whenever $\mathsf{P}(H_p|E)$ is greater than $\mathsf{P}(H_d|E)$. In criminal trials, the costs ratio is considered higher, since convicting an innocent (false positive) should be more harmful or morally objectionable than acquitting a guilty defendant (false negative). So the rule of decision would be: convict whenever $\mathsf{P}(H_p|E)$ is substantively greater than $\mathsf{P}(H_d|E)$.

[15] If two hypotheses are exclusive and exhaustive, $\mathsf{P}(H_p|E)/\mathsf{P}(H_d|E) > 1$ implies that $\mathsf{P}(H_p|E) > .5$, the standard probabilistic interpretation of the preponderance standard for civil cases, and $\mathsf{P}(H_p|E)/\mathsf{P}(H_d|E) > 19$ implies that $\mathsf{P}(H_p|E) > .95$, a common interpretation of proof beyond a reasonable doubt.

To see whether aggregation is satisfied, we check whether ${\mathsf{P}(S\wedge C|E)}/{\mathsf{P}(H_d|E)} > 1$. The key is to decide on the defense hypothesis H_d. Cheng reasons that there are three alternative defense scenarios: $H_{d_1} = S \wedge \neg C$, $H_{d_2} = \neg S \wedge C$, and $H_{d_3} = \neg S \wedge \neg C$. How does H_p compare to each of them? Assuming independence between C and S, we have:[16]

$$
\begin{aligned}
\frac{\mathsf{P}(S\wedge C|E)}{\mathsf{P}(S\wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(S|E)\mathsf{P}(\neg C|E)} = \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1 \\
\frac{\mathsf{P}(S\wedge C|E)}{\mathsf{P}(\neg S\wedge C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(C|E)} = \frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1 \\
\frac{\mathsf{P}(S\wedge C|E)}{\mathsf{P}(\neg S\wedge \neg C|E)} &= \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)} > 1
\end{aligned} \tag{1}
$$

So, whatever the defense hypothesis, the plaintiff's hypothesis is more probable. At least in this case, whenever the elements of a plaintiff's claim satisfy the decision threshold, so does their conjunction. The left-to-right direction of the conjunction principle—what we are calling aggregation—has been vindicated, at least for simple cases involving independence. Success.

What about the opposite direction, distribution? Distribution is not generally satisfied. Suppose ${\mathsf{P}(S\wedge C|E)}/{\mathsf{P}(H_d|E)} > 1$, or in other words, the combined hypothesis $S \wedge C$ has been established by preponderance of the evidence. The question is whether the individual hypotheses have been established by the same standard, specifically, whether $\frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} > 1$ and $\frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} > 1$. If ${\mathsf{P}(S\wedge C|E)}/{\mathsf{P}(H_d|E)} > 1$, the combined hypothesis is assumed to be more probable than any of the competing hypotheses, in particular, ${\mathsf{P}(S\wedge C|E)}/{\mathsf{P}(\neg S\wedge C|E)} > 1$, ${\mathsf{P}(S\wedge C|E)}/{\mathsf{P}(S\wedge \neg C|E)} > 1$ and ${\mathsf{P}(S\wedge C|E)}/{\mathsf{P}(\neg S\wedge \neg C|E)} > 1$. We have:

$$
\begin{aligned}
1 &< \frac{\mathsf{P}(S\wedge C|E)}{\mathsf{P}(S\wedge \neg C|E)} = \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(S|E)\mathsf{P}(\neg C|E)} = \frac{\mathsf{P}(C|E)}{\mathsf{P}(\neg C|E)} \\
1 &< \frac{\mathsf{P}(S\wedge C|E)}{\mathsf{P}(\neg S\wedge C|E)} = \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(C|E)} = \frac{\mathsf{P}(S|E)}{\mathsf{P}(\neg S|E)} \\
1 &< \frac{\mathsf{P}(S\wedge C|E)}{\mathsf{P}(\neg S\wedge \neg C|E)} = \frac{\mathsf{P}(S|E)\mathsf{P}(C|E)}{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)}
\end{aligned} \tag{2}
$$

In the first two cases, clearly, if the composite hypothesis meets the threshold, so do the individual claims. But consider the third case. ${\mathsf{P}(S|E)\mathsf{P}(C|E)}/{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)}$ might be strictly greater than ${\mathsf{P}(C|E)}/{\mathsf{P}(\neg C|E)}$ or ${\mathsf{P}(S|E)}/{\mathsf{P}(\neg S|E)}$. It is possible that ${\mathsf{P}(S|E)\mathsf{P}(C|E)}/{\mathsf{P}(\neg S|E)\mathsf{P}(\neg C|E)}$ is greater than one, while either${\mathsf{P}(C|E)}/{\mathsf{P}(\neg C|E)}$ or ${\mathsf{P}(S|E)}/{\mathsf{P}(\neg S|E)}$ are not, say when they are 3 and 0.5, respectively. Distribution fails. And the same problem would arise with a more stringent threshold as might be appropriate in criminal cases.

There is a more general problem with Cheng's comparative approach. Much of the heavy lifting here is done by the strategic splitting of the defense line into multiple scenarios. Suppose, for illustrative purposes, $\mathsf{P}(H_p|E) = 0.37$ and the probability of each of the defense lines given E is 0.21. This means that H_p wins with each of the scenarios. We should then find against the defendant. But should we? Given the evidence, the accusation is likely to be false, because $\mathsf{P}(\neg H_p|E) = 0.63$. The problem generalizes. If, as here, we individualize scenarios by Boolean combinations of elements of a case, the more elements, the more alternative scenarios into which $\neg H_p$ needs to be divided. This normally would lead to lowering even further the probability of each of them (because now $\mathsf{P}(\neg H_p)$ needs to be split between more scenarios). If we take this approach seriously, the more elements a case has, the more at a disadvantage the defense is. This seems undesirable.

3.2 Comparing likelihoods

Instead of posterior probabilities, we could consider comparing likelihoods and their ratios. The standard of proof would then be as follows: the ratio between the likelihoods ${\mathsf{P}(E|H_p)}/{\mathsf{P}(E|H_d)}$ should be above a suitable threshold. Note that the posterior ratio ${\mathsf{P}(H_p|E)}/{\mathsf{P}(H_d|E)}$ from before was replaced by the likelihood ratio ${\mathsf{P}(E|H_p)}/{\mathsf{P}(E|H_d)}$ where H_d and H_p, as before, need not be exhaustive hypotheses. In civil cases, the likelihood ratio should perhaps be just be above 1, meaning that the evidence supports H_p

[16] We are assuming that E is the conjunction of two items of evidence, $s \wedge c$, where s supports S and c supports C. The inequalities holds on DAG 1 in Figure 3 after replacing A and a by S and s, and B and b by C and c, respectively.

more strongly than it supports H_d. In criminal cases, the ratio should be several orders of magnitude above one. This approach runs into the same problem as Cheng's. It cannot justify distribution.

We do not provide all the details of the argument. The reasoning is analogous. Consider the car crash example from before, where S stands for the defendant's speeding, C stands for the statement that the crash caused neck injury, and E stands for the total evidence. The plaintiff's hypothesis H_p is $S \wedge C$. Suppose $\mathsf{P}(E|S \wedge C)/\mathsf{P}(E|H_d) > 1$, or in other words, the combined hypothesis $S \wedge C$ has been established by preponderance of the evidence. The question is whether the individual hypotheses have been established by the same standard, specifically, whether $\frac{\mathsf{P}(E|C)}{\mathsf{P}(E|\neg C)} > 1$ and $\frac{\mathsf{P}(E|S)}{\mathsf{P}(E|\neg S)} > 1$. Focusing on a specific defense hypothesis, $\neg S \wedge \neg C$, the following holds:[17]

$$1 < \frac{\mathsf{P}(E|S \wedge C)}{\mathsf{P}(E|\neg S \wedge \neg C)} = \frac{\mathsf{P}(E|S)\mathsf{P}(E|C)}{\mathsf{P}(E|\neg S)\mathsf{P}(E|\neg C)} \tag{3}$$

Note that $\mathsf{P}(E|S)\mathsf{P}(E|C)/\mathsf{P}(E|\neg S)\mathsf{P}(E|\neg C)$ might be strictly greater than $\mathsf{P}(E|C)/\mathsf{P}(E|\neg C)$ or $\mathsf{P}(E|S)/\mathsf{P}(E|\neg S)$. It is possible that $\mathsf{P}(E|S \wedge C)/\mathsf{P}(E|H_d)$ is greater than one, while either $\frac{\mathsf{P}(E|C)}{\mathsf{P}(E|\neg C)}$ and $\frac{\mathsf{P}(E|S)}{\mathsf{P}(E|\neg S)}$ are not, say when they are 3 and 0.5, respectively. Once again, distribution fails.

A more general worry lingers, related to how this comparative likelihood strategy is sensitive to the choice of the hypotheses. There might be pairs of hypotheses that one wishes to compare, say H_1 and H_2, such that $\mathsf{P}(E|H_1)$ is much (say, at least a few times) larger than $\mathsf{P}(E|H_2)$. And yet, $\mathsf{P}(E|H_1)$ is still smaller than $\mathsf{P}(E|\neg H_1)$. In such circumstances, the comparative likelihood strategy would be recommending the acceptance of H_1 (because it enjoys stronger evidential support than H_2) even though, in absolute terms, the evidence supports the negation of H_1 to a greater extent.

4 Rejecting the conjunction principle?

A number of strategies the legal probabilist can pursue to theorize about the standard of proof have proven problematic: posterior probability (Section 1), evidential strength (Section 2), comparing posteriors or likelihoods (Section 3). It seems impossible, on probabilistic grounds, to justify both directions of the conjunction principle. Of course, these strategies do not exhaust the entire space of possibilities. The legal probabilist could pursue other strategies, but the ones examined so far provide good *prima facie* evidence that perseverance will not pay off. It is time to try a different approach.

Observe that the difficulty with conjunction would not arise without endorsing the conjunction principle. Should legal probabilists simply reject this principle? So far we have not challenged it, but it is time to scrutinize it more closely. In this section, we provide an epistemic argument and a legal argument to question the conjunction principle. At the same time, we caution that merely rejecting the conjunction principle will not automatically dissolve the difficulty with conjunction. More work needs to be done. We take it on in the final section.

4.1 The legal argument

Before moving further, it is worth asking what the law says about the conjunction principle. The answer, perhaps unsurprisingly, is that the law does say not very much about it. We have been assuming that the law agrees with the conjunction principle. At least, this is what Cohen thought. Matters, however, are not so clear-cut. Looking at legal practice, the conjunction principle is an uncertain principle at best.

17

$$\begin{aligned}
\frac{\mathsf{P}(E|S \wedge C)}{\mathsf{P}(E|\neg S \wedge \neg C)} &= \frac{\mathsf{P}(s \wedge c|S \wedge C)}{\mathsf{P}(s \wedge c|\neg S \wedge \neg C)} \\
&= \frac{\mathsf{P}(s|S)\mathsf{P}(c|C)}{\mathsf{P}(s|\neg S)\mathsf{P}(c|\neg C)} \\
&= \frac{\mathsf{P}(s \wedge c|S)\mathsf{P}(s \wedge c|C)}{\mathsf{P}(s \wedge c|\neg S)\mathsf{P}(s \wedge c|\neg C)} \\
&= \frac{\mathsf{P}(E|S)\mathsf{P}(E|C)}{\mathsf{P}(E|\neg S)\mathsf{P}(E|\neg C)}
\end{aligned}$$

There will be of course differences across countries. We cannot provide a comprehensive analysis here. We shall be content with just a few examples. The best place to look is how jury instructions are formulated. Do they obey the conjunction principle? To some extent, they do. For example, here are sample jury instructions about negligence in civil cases:

> A negligence claim has three elements:
>
> 1. [Defendant] did not use ordinary care;
> 2. [Defendant's] failure to use ordinary care caused [Plaintiff's] harm; and
> 3. [Plaintiff] is entitled to damages as compensation for that harm.
>
> [Plaintiff] must prove each element by a preponderance of the evidence—that each element is more likely so than not so. If [Plaintiff] proves each element, your verdict must be for [Plaintiff]. If [Plaintiff] does not prove each element, your verdict must be for [Defendant].[18]

The elements are explicitly separated and the standard of proof is applied to each element separately. This seems to confirm the conjunction principle. Other jury instructions are more ambiguous:

> In order to find that the plaintiff is entitled to recover, you must decide it is more likely true than not true that:
>
> 1. the defendant was negligent;
> 2. the plaintiff was harmed; and
> 3. the defendant's negligence was a substantial factor in causing the plaintiff's harm.[19]

The elements are still separated, but the standard of proof ('more likely than not') applies to the conjunction as a whole, not the individual claims. This second set of jury instruction is at best ambiguous between an atomistic reading (the standard of proof applies to each claim separately) and a holistic reading (the standard of proof applies to the conjunction). Only the atomistic reading would justify the conjunction principle.

This quick survey of jury instructions gives us some reassurance that, should we decide to reject the conjunction principle, we would not violate a well-entrenched, indispensable legal principle.[20]

4.2 Risk accumulation

Beside legal uncertainty about the tenability of the conjunction principle, there are also independent theoretical reasons to question the principle. In current discussions in epistemology about knowledge or justification, a principle similar to the conjunction principle has been contested, because it appears to deny the fact that risks of error accumulate (Kowalewska, 2021). If one is reasonably sure about the truth of each claim considered separately, one should not be equally reasonably sure of their conjunction. You have checked each page of a book and found no error. So, for each page, you are reasonably sure there is no error. Having checked each page and found no error, can you be equally reasonably sure that the book as a whole contains no error? Not really. As the number of pages grow, it becomes virtually certain that there is at least one error in the book you have overlooked, although for each page you are reasonably sure there is no error (Makinson, 1965). A reasonable doubt about the existence of an error, in one page or another, creeps up as one considers more and more pages. The same observation applies to other contexts, say product quality control. You may be reasonably sure, for each product you checked, that it is free from defects. But you cannot, on this basis alone, be equally reasonably sure that all products you checked are free from defects. Since the risks of error accumulate, you must have missed at least one defective product.[21]

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[18] Standardized Civil Jury Instructions for the District of Columbia, Sec. 5.01 (Civil Jury Instructions, revised edition 2017).

[19] Alaska Civil Pattern Jury Instructions, Sec. 3.01 (Civil Pattern Jury Instructions 2017).)

[20] For a detailed analysis of whether jury instructions obey the conjunction principle, see (Schwartz & Sober, 2017). Their review of the empirical evidence shows a variety of formulations, not all compliant with the conjunction principle.

[21] The phenomenon of risk accumulation can also be formulated without using an explicitly probabilistic language. Say a claim is established if all reasonable defeaters have been ruled out. You have checked the quality of one product and it appears free from defects, and you have done the same with many other products. They all appear to be free from defects. In this sense, for each product, the claim 'this product is free from defects' has been established, but the conjunctive claim 'every product examined so far is free from defects' might not. After all, you know for sure you made at least one mistake. You know that from numerous past experiences. Your track record supplies a reasonable defeater to the claim 'every product examined so far is free from defects' that is not a defeater for the individual claims of the form 'this product is free from defects'.

Risk accumulation challenges aggregation: even if the probability of several claims, considered individually, is above a threshold t, their conjunction need not be above t. It does not, however, challenge distribution. If, all risks considered, you have good reasons to accept a conjunction, no further risk is involved in accepting any of the conjuncts separately. This is also mirrored by what happens with probabilities. If the probability of the conjunction of several claims is above t, so is the probability of each individual claim. The standard of proof in criminal or civil cases can be understood as a criterion concerning the degree of risk that judicial decisions should not exceed. If this understanding of the standard of proof is correct, the phenomenon of risk accumulation would invalidate the conjunction principle, specifically, it would invalidate aggregation. It would no longer be correct to assume that, if each element is proven according to the applicable standard, the case as a whole is proven according to the same standard. And, in turn, if the conjunction principle no longer holds, the conjunction paradox will disappear. Or will it? As we shall now see, matters are not so straightforward.

4.3 Atomistic and holistic approches

Suppose legal probabilists do away with the conjunction principle. Now what? How should they define standards of proof? Two immediate options come to mind, but neither is without problems.

Let's stipulate that, in order to establish the defendant's guilt beyond a reasonable doubt (or civil liability by preponderance of the evidence or clear and convincing evidence), the party making the accusation should establish each claim, separately, to the requisite probability, say at least .95 (or .5 in a civil case), without needing to establish the conjunction to the requisite probability. Call this the *atomistic account*. On this view, the prosecution could be in a position to establish guilt beyond a reasonable doubt without establishing the conjunction of different claims with a sufficiently high probability. This account would allow convictions in cases in which the probability of the defendant's guilt is relatively low, just because guilt is a conjunction of several independent claims that separately satisfy the standard of proof. For example, if each constituent claim is established with .95 probability, a composite claim consisting of five subclaims—assuming, as usual, probabilistic independence between the subclaims—would only be established with probability equal to .77, a far cry from proof beyond a reasonable doubt. This is counterintuitve, as it would allow convictions when the defendant is not very likely to have committed the crime. A similar argument can be run for the civil standard of proof 'preponderance of the evidence.' Under the atomistic account, the composite claim representing the case as a whole would often be established with a probability below the required threshold. The atomistic approach is a non-starter.

Another option is to require that the prosecution in a criminal case (or the plaintiff in a civil case) establish the accusation as a whole—say the conjunction of A and B—to the requisite probability. Call this the *holistic account*. This account is not without problems either.

The standard that applies to one of the conjuncts would depend on what has been achieved for the other conjuncts. For instance, assuming independence, if $\mathsf{P}(A)$ is .96, then $\mathsf{P}(B)$ must be at least .99 so that $\mathsf{P}(A \wedge B)$ is above a .95 threshold. But if $\mathsf{P}(A)$ is .9999, then $\mathsf{P}(B)$ must only be slightly greater than .95 to reach the same threshold. Thus, the holistic account might require that the elements of an accusation be proven to different probabilities—and thus different standards—depending on how well other claims have been established. This result runs counter to the tacit assumption that each element should be established to the same standard of proof (Urbaniak, 2019).

Fortunately, this challenge can be addressed. It is true that different elements will be established with different probabilities, depending on the probabilities of the other elements. But this follows from the fact that the prosecution or the plaintiff may choose different strategies to argue their case. They may decide that, since they have strong evidence for one element and weaker evidence for the other, one element should be established with a higher probability than the other. What matters is that the case as a whole meets the required threshold, and this objective can be achieved via different means. What will never happen is that, while the case as a whole meets the threshold, one of the constituent elements does not. As seen earlier, the probability of the conjunction never exceeds the probability of one of the conjunct, or in other words, distribution is never violated.

A more difficult challenge is the observation that the proof of $A \wedge B$ would impose a higher requirement on the separate probabilities of the conjuncts. If the conjunction $A \wedge B$ is to be proven with at least .95 probability, the individual conjuncts should be established with probability higher than .95. So the more constituent claims, the higher the posterior probability for each claim needed for the conjunction to meet the requisite probability threshold.

This challenge is best appreciated by running some numbers. Assume, for the sake of illustration, the independence and equiprobability of the constituent claims. If a composite claim consists of k individual claims, these individual claims will have to be established with probability of at least $t^{1/k}$, where t is the threshold to be applied to the composite claim.[22] For example, if there are ten constituent claims, they will have to be proven with $.5^{1/10} = .93$ even if the probability threshold is only .5. If the threshold is more stringent, as is appropriate in criminal cases, say .95, each individual claim will have to be proven with near certainty. This would make the task extremely demanding on the prosecution, if not downright impossible. If there are ten constituent claims, they will have to be proven with $.95^{1/10} = .995$. So the plaintiff or the prosecution would face the demanding task of establishing each element of the accusation beyond what the standard of proof would seem to require.

We reached an impasse. Under the atomistic approach, the standard is too lax because it allows for findings of liability when the defendant quite likely committed no wrong. Under the holistic approach, the standard is too demanding on the prosecution (or the plaintiff) because it requires the individual claims to be established with extremely high probabilities.

4.4 Not asking too much

Consider again the holistic approach. It is true that the individual elements (the individual conjuncts) should be established with a higher probability than the case as a whole (the conjunction). This would seem to impose an unreasonably stringent burden of proof on the prosecution or the plaintiff. But the burden might not be as unreasonable as it appears at first. As Dawid (1987) pointed out, in one of the earliest attempts to solve the conjunction paradox from a probabilistic perspective, the prior probabilities of the conjuncts will also be higher than the prior probability of their conjunction:

> ...it is not asking too much of the plaintiff to establish the case as a whole with a posterior probability exceeding one half, even though this means that the several component issues must be established with much larger posterior probabilities; for the *prior* probabilities of the components will also be correspondingly larger, compared with that of their conjunction [p. 97].

Dawid's proposal seems compelling. The prior probabilities of the conjuncts are surely higher than the prior probability of the conjunction. But why, exactly, is it 'not asking too much' to establish the individual conjuncts by a higher threshold than the case as a whole? Perhaps, Dawid is pointing out that the *difference* between prior and posterior probabilities of the individual claims which the evidence should bring about will not be unreasonably large. In other words, Dawid might be recommending—as the rest of his paper suggests—that the standard of proof not be understood solely in terms of posterior probabilities. Measures of how strongly each claim issupported by the evidence, such as the Bayes' factor or the likelihood ratio, account for the difference between prior and posterior probabilities. So, presumably, Dawid is recommending these measures as better suited to formalize the standard of proof.

Now, as the reader will have realized, we have pursued Dawid's strategy already in Section 2. This strategy can justify, on purely probabilistic grounds, one direction of the conjunction principle: aggregation. The evidential support—measured by the Bayes' factor or the likelihood ratio—for the conjunction often exceeds the individual support for (at least one of) the individual claims. This is a success, especially because the failure of aggregation motivated Cohen's formulation of the conjunction paradox. Unfortunately, we have already seen that this strategy invalidates a previously unchallenged direction of the conjunction principle: distribution.

5 The proposal

Here is where we have gotten so far. There might be good reasons to reject the conjunction principle, but rejecting it does not automatically solve the difficulty with conjunction. We still need a theory that explains how individual claims are combined, together with the available evidence, to form more complex claims. The conjunction principle provides a recipe—a very simple one at that—to combine

[22] Let p the probability of each constituent claim. To meet threshold t, the probability of the composite claim, p^k, should satisfy the constraint $p^k > t$, or in other words, $p > t^{1/k}$.

individual claims and form conjunctive claims. If that recipe is not right, a good theory of the standard of proof should provide an alternative recipe for combining individual claims.

Our proposal is inspired by the story model of adjudication (Pennington & Hastie, 1993; Wagenaar, Van Koppen, & Crombag, 1993) and the relative plausibility theory (Allen & Pardo, 2019; Pardo & Allen, 2008). It posits that prosecutors and plaintiffs should aim to establish a unified narrative of what happened or explanation of the evidence, not establish each individual element of wrongdoing separately. As we shall see, any attempt to proceed in a piecemeal manner implicitly requires, sooner or later, to weave the different elements together into a unified whole. Our argument consists of two key ideas. First, the guilt or civil liability of a defendant cannot be equated with a generic statement of guilt or civil liability as defined in the law. The allegations against the defendant facing trial should always be grounded in specific details. Call this the specificity argument. Second, it is erroneous to think of someone's guilt or civil liability as the mere conjunction of separate claims. The separate claims must be coherently unified, not just added up in a conjunction. Call this the unity argument.

5.1 The specificity argument

We start with the specificity argument. The probabilistic interpretation of proof standards usually posits a threshold that applies to the posterior probability of a *generic* hypothesis, such as the defendant is guilty of a crime, call it G, or civilly liable, call it L. In criminal cases, the requirement is formulated as follows: the evidence E presented at trial establishes guilt beyond a reasonable doubt provided $\mathsf{P}(G|E)$ is above a suitable threshold, say .95. The threshold is lower in civil trials. Civil liability is proven by preponderance provided $\mathsf{P}(L|E)$ is above a suitable threshold, say .5.

This formulation conflates two things. The act of wrongdoing as defined in the applicable law is one thing. The way in which the wrongdoing is established in court is another thing. The wrongdoing is defined in the law in a generic manner and its definition is applicable across a class of situations, whereas the way the wrongdoing is established in court is specific to a unique situation and tailored to the individual defendant. A prosecutor in a criminal case does not just establish that the defendant assaulted the victim in one way or another, but rather, that the defendant behaved in such and such a manner in this time and place, and that the behavior in question fulfills the legal definition of assault. The requirement of specificity is a consequence of the fact that defendants have a right to be informed with sufficient detail and be in a position to prepare a defense.[23]

If this is right, the probabilistic interpretation of proof standards should be revised. The generic statement that the defendant is guilty or civilly liable should be replaced by a more fine-grained factual hypothesis, call it H_p, the hypothesis put forward by the prosecutor (or the plaintiff in a civil case), for example, that the defendant, given reasonably specific circumstances, approached the victim, pushed and kicked the victim to the ground, and then run away. Hypothesis H_p is a more precise description of what happened and entails that the defendant committed the criminal offense or civil wrong of which they are accused. In defining the standard of proof, instead of saying—generically—that $\mathsf{P}(G|E)$ or $\mathsf{P}(L|E)$ should be above a suitable threshold, a probabilistic interpretation should read: civil or criminal liability is proven by the applicable standard provided $\Pr(H_p|E)$ is above a suitable threshold, where H_p is a reasonably specific description of what happened according to the prosecution or the plaintiff.

This revision may appear inconsequential, but it is not. It is the revision we invoked to address the puzzles of naked statistical evidence. Here is the gist of the argument. Consider the prisoner hypothetical, a standard example of naked statistical evidence. The naked statistics E_s make the prisoner on trial .99 likely to be guilty, that is, $\mathsf{P}(G|E_s) = .99$. Given the known facts in the prisoner hypothetical, it is .99 likely that the prisoner on trial was one of those who attacked and killed the guard. But this is a very generic claim. It merely asserts that the prisoner was—with very high probability—one of those who killed the guard, without specifying what he did, what role he played in the attack, how he killed the guard, etc. If the prosecution offered a more specific incriminating hypothesis H_p, the probability $\mathsf{P}(H_p|E_s)$ of this hypothesis based on the naked statistical evidence E_s would be well below .99, even

[23]How the objective of specificity is actually achieved in trial proceedings is a difficult question. Different countries and jurisdictions may use different approaches, say, through the discovery process itself or a request of a bill of particulars. For example, at the end of the 19th century, the Supreme Court of Massachusetts wrote: 'It is always open to the defendant to move the judge before whom the trial is had to order the prosecuting attorney to give a more particular description, in the nature of a specification or bill of particulars, of the acts on which he intends to rely, and to suspend the trial until this can be done; and such an order will be made whenever it appears to be necessary to enable the defendant to meet the charge against him, or to avoid danger of injustice.' (Commonwealth v. Sherman, 95 Mass. 248, 13 Allen 248, 250, 1866).

though $\mathsf{P}(G|E_s) = .99$. That the prisoner on trial is most likely guilty is an artifact of the choice of a generic hypothesis G. When this hypothesis is made more specific—as should be—this probability drops significantly. And the puzzle of naked statistical evidence disappears. For a detailed articulation of this argument, see Chapter XX.[24]

REFERENCE TO EARLIER CHAPTER

5.2 The unity argument

The specificity argument addresses the problem of naked statistical evidence, but also provides the necessary background for addressing the difficultly about conjunction. In its formulation, not only are G and L understood as generic claims, but they are also understood as *mere conjunctions* of simpler claims that correspond to the elements of wrongdoing in the applicable law. Since the probability of a conjunction is often lower than the probability of its conjuncts, the individual claims can be established with a suitably high probability that meets the required threshold even though the conjunction as a whole fails to meet the same threshold. This mismatch gives rise to the difficulty with conjunction.

Here lies another conflation. It is one thing to establish that the defendant committed A and B, where the wrongdoing in question is defined in the law as comprising two elements, A and B. It is another thing to establish that the defendant committed the wrong. The conjunction paradox—in particular, the conjunction principle—assumes that criminal or civil wrongdoings are mere collections of separate elements. But the law is more nuanced. Legal definitions often impose a structured unity on how the different elements relate to one another. This unity could take many forms. It could be a temporal unity, the unity that exists between a plan and its execution, or the unity between the *actus reus* and the *mens rea* in a criminal offense.

But what if the law, at least in simple cases, did not impose any structure onto the different elements of wrongdoing? In such cases, one might argue, the conjunction principle—by which the different claims are simply added to one another in a conjunction—would be adequate. But even in simple cases in which the different elements of wrongdoing are not structured in any explicitly way, it would still be a mistake to follow the conjunction principle. To see why, consider a case in which only two elements must be proven. Element 1: the defendant's conduct caused a bodily injury to the victim. Element 2: the defendant's conduct consisted in reckless driving. This offense is sometimes called 'vehicular assault.'[25] The two elements each add novel information. It could be that the defendant's driving caused an injury to victim, but the driving was not reckless, or the driving was reckless, but no injury ensued. Neither element is presupposed by the other. Crucially, here the law does not impose any explicit structure between the elements. But—we will now see—the unity argument still applies at a conceptual level.

Let's think about how to establish the claim of reckless driving that caused injury to the victim. One option is to offer a detailed reconstruction of what happened. The reconstruction could go something like this. The defendant was driving above the speed limit, veering left and right. The defendant's reached a school crosswalk when children were getting out of school. The defendant hit a child on the crosswalk who was then pushed against a light pole on the sidewalk incurring a head injury. Suppose this story is supported by several testimonies by other children, people standing around, police officers, paramedics. There is plenty of supporting evidence as the incident occurred in the middle of the day. Taken at face value, this story does establish both elements: reckless driving and cause of injury. Parts of the story are relevant for element 1 (reckless driving) and other parts are relevant for element 2 (cause of injury). The two cannot be neatly separated, however. Still, what is crucial is that the different parts of the story are part of the same episode, the same unit of wrongdoing.

Could the prosecutor prove vehicular assault in a piecemeal manner? Suppose the prosecutor attempted to do that, by establishing, first, that the defendant drove recklessly, and second—*separately from the first element*—that the defendant's action caused injury. As noted before in the specificity argument, it is not enough to establish that the defendant drove recklessly at some point in time somewhere. Nor is it enough to establish that the defendant's action caused injury. The prosecutor should offer a specific story detailing what happened, a story relevant for the first element and a story relevant for the second element. Say this expectation of specificity is met. Suppose the prosecutor did not just establish generic element 1 and generic element 2, but rather, a reasonably detailed story for ech

[24] An earlier version of this argument can be found in Di Bello (2013) and Di Bello (2021).

[25] See, for example, Arizona Revised Statutes Title 13. Criminal Code § 13–1204 ('A person commits aggravated assault if the person commits assault … under any of the following circumstances… [such as] … the person causes serious physical injury to another … [or]… the person uses a deadly weapon or dangerous instrument.').

element. Would that be enough? It wouldn't. Even if each element—that is, each story associated with each element—was established by the required standard, there would still be something missing here.

The prosecutor should establish that the two elements are part of the same unity of wrongdoing. It must be *this* reckless driving that caused *this* injury. So, under the piecemeal approach, the prosecutor would be tasked with establishing three claims: (a) the defendant, in some well-specified circumstances, was driving reckless; (b) the defendant, in some well-specified circumstances, caused injury to the victim; and (c) the well-specified circumstances in (a) and (b) are part of the same episode. But once (c) is established, the prosecutor would have effectively established the charge by the required standard in accordance with the holistic approach. The prosecutor did not only establish each separate element (two separate stories) but also combine the two elements (the two stories) together. Once the piecemeal approach is pursued to its logical conclusion, it coincides with the holistic approach.[26]

Let's summarize the unity argument in schematic form. If the prosecutor or the plaintiff is expected to establish claim A and B by the required standard, what the law actually requires—even in terms of the piecemeal approach—is (a) to establish A; (b) to establish B; and (c) to establish A and B are part of the same unit of wrongdoing by the required standard. Item (c) is often implicit, which leaves the impression that the law only requires to establish (a) and (b) separately. Interestingly, (c) entails (a) and (b). In fact, (c) amounts to establishing a unified story, narrative or theory about what happened. Such a narrative should be subsumed under the different elements of wrongdoing as defined in the law. The piecemeal approach and the holistic approach, therefore, converge.

To be sure, not all wrongful acts, in civil or criminal cases, require the prosecutor or the plaintiff to establish a unified *spatio-temporal* narrative. It might not be necessary to show that all elements of an offense occurred at the same point in time or in close succession one after the other. Some wrongful acts may consist of a pattern of acts that stretches for several days, months or even years. There may be temporal and spatial gaps that cannot not be filled. We consider several of these examples in our discussion of naked statistical evidence in Chapter XX. Be that as it may, an accusation of wrongdoing in a criminal or civil case should still have a degree of cohesive unity. The acts and occurrences that constitute the wrongdoing should belong to the same wrongful act. It is this unity which the plaintiff and the prosecution must establish when they make their case. One way to establish this unity is by providing a unifying narrative, but this need not be the only way. A unifying 'theory' of what happened or a cohesive 'explanation' of the evidence could all deliver the structured unity that is needed to establish the defendant's liability.

SEE PREVIOUS CHAPTER

5.3 Probability, specificity and completeness

We emphasize the distinction between a narrative (or theory, explanation) and a mere conjunction of elements of wrongdoing. The narrative describes one way among many of instantiating the conjunction. This distinction is important. The claims that constitute a narrative (or unified theory, explanation) need not map neatly onto the elements of the wrongdoing. The narrative will comprise claims about the evidence itself and how the evidence supports other claims in the narrative, say that witnesses were standing around when the defendant's car hit the child. The narrative (or theory, explanation) will not only comprise a description of what happened but also of how we know it is what happened.

The distinction between narrative and the mere conjunction of elements matters for how we should understand the standard of proof. Other things being equal, the conjunction is more probable on the evidence than the narrative, and each conjunct even more probable. But this does not mean that the mere conjunction is established by a higher standard of proof than the narrative. As we argued in Chapter XX on naked statistical evidence, a highly probable narrative that nevertheless lacks the desired degree of specificity will fail to meet the standard of proof. By contrast, a more specific narrative that is otherwise less probable than the mere conjunction might well meet the standard. On this account, the standard of proof is sensitive to two variables: (1) the posterior probability of the proposed narrative (or theory, explanation) given the evidence presented at trial; and (2) the degree of specificity and unity of the narrative (or theory, explanation). Another variable worth adding to posterior probability and specificity is (3) the completeness of the evidence presented at trial. Could the probability of someone's

REFERENCE TO EARLIER CHAPTERS

[26] We should be clear that it is not enough for the prosecutor or plaintiff to provide well-specified narrative in support of their allegations, even when they are well-supported by the evidence. When the two narratives are combined into one narrative, its probability could well be below the threshold. If we only require that each element-specific narrative be proven, a defendant could be found criminally or civilly liable even though it is unlikely that they committed the alleged wrongful act. This counter-intuitive result is similar to the one that arose with the atomistic approach.

guilt be extremely high just because the evidence presented is one-sided and missing crucial pieces of information? It surely can. If the probability of liability is high because the evidence is partial, liability was not proven beyond a reasonable doubt (and perhaps not even by a lower standard such as preponderance of the evidence).[27]

On our proposal, the standard of proof is informed by three maxims:

1. The probability of the defendant's liability should be sufficiently (or reasonably) high;
2. The narrative (or theory, explanation) of the defendant's liability should be sufficiently (or reasonably) specific; and
3. The supporting evidence should be sufficiently (or reasonably) complete.

At this point, the reader might wonder: are we ultimately giving up on legal probabilism? We are only giving up on *traditional* legal probabilism. Even though ideas such as specificity, unity and completeness of the evidence cannot be formalized in the language of posterior probability alone, they can be formalized as properties of Bayesian networks. Here is the general idea. The individual claims or hypotheses to be established and the supporting pieces of evidence are represented as nodes in a Bayesian network. Some nodes will count as 'evidence nodes' and others as 'claim nodes' or 'hypothesis nodes'. A narrative (or theory, explanation) is then a suitable collection of evidence nodes and hypothesis nodes that are connected to one another by relationships of conditional probabilistic dependence (represented by arrows in the Bayesian network), meant to capture causal or evidential relationships. This web of dependencies affords the narrative its unity and coherence. But a unified narrative (theory, or explanation) may still fall short of factual specificity if it contains gaps—that is, if there are propositions about which the narrative should make a commitment but instead remains neutral about them. The narrative may also have incomplete evidence if it contains evidential gaps—that it, if there are evidence-bearing propositions whose probability is non-negligible (given what is assumed to be true in the narrative), but are nevertheless not included in the evidence nodes. A more detailed articulation—using the formalism of Bayesian networks—of the ideas of narrative unity and coherence, factual specificity, and evidential gaps is carried out in other chapters.[28]

REF TO OTHER CHAPTERS

Probability theory plays also a second-order role in our larger argument. Specifically, our analysis of the standard of proof—which combines three ingredients: posterior probability, specificity and completeness—can be evaluated at the meta-level using concepts from probability theory. Suppose we wish to compare a trial system that convicts defendants on the basis of claims that are generic but highly probable, as opposed to a trial system that convicts defendants on the basis of claims that are more specific but less probable. Which trial system will make fewer mistakes—fewer false convictions and false acquittals—in the long run? The answer is not obvious. But the question can be made precise in the language of probability. The question concerns the diagnostic properties of the two trial systems, such as their rate of false positives and false negatives. We examine this question in Chapter YY. We conduct a simulation study of the impact of properties of narratives (as defined in Bayesian networks) on their expected accuracy. To anticipate, we argue that more specific claims are liable to more extensive adversarial scrutiny than generic claims. The more specific someone's claim, the more liable to be attacked. At the same time, if a specific claim resists adversarial scrutiny, it becomes more firmly established than a less specific claim that was not scrutinized. So specificity plays an accuracy-conducive role even though more specific claims are, other things being equal, less probable than more generic claims.[29]

REFERENCE TO LATER CHAPTER

[27] It is a matter of dispute whether knowledge about missing or partial evidence should affect the posterior probability. After all, if we know that some evidence is missing, shouldn't we revise the assessment of the posterior probability of the hypothesis? The problem is that the content of the missing evidence is unknown. The missing evidence might increase or decrease this probability. We cannot know that without knowing the content of the evidence. If we knew how the missing evidence would affect our judgment about the defendant's guilt, the evidence would no longer be—strictly speaking—missing. That is why we prefer to add the completeness of the evidence as a third variable to consider. For a recent case in which missing evidence was alleged as a reason for reversing a verdict of guilt, see Johnson v. Premo, 315 Oregon Appeal 1 (2021).

[28] An early development of this approach can be found in Urbaniak (2018).

[29] These remarks echo a point made by Karl Popper (2002) about science. He wrote: 'Science does not aim, primarily, at high probabilities. It aims at a high informative content, well backed by experience. But a hypothesis may be very probable simply because it tells us nothing, or very little. A high degree of probability is therefore not an indication of goodness' (416, Appendix *IX).

5.4 The conjuction principle trivialized

What does this discussion tell us about the conjunction paradox? That prosecutors and plaintiffs should aim to establish a well-specified, unified account of the wrongdoing ends up trivializing the conjunction principle and thus dissolving the difficulty about conjunction. How so? Suppose the prosecutor established a narrative N by a very high probability, say above the required threshold for proof beyond a reasonable doubt. Denote the elements of wrongdoing by $EL_1, EL_2, \ldots$. Then,

$$\mathsf{P}(EL_1 \wedge EL_1 \wedge \cdots \wedge EL_k | N) = \mathsf{P}(EL_i | N) = 1 \text{ for any } i = \{1, 2, ..., k\}.$$

Both directions of the conjunction principle, aggregation and distribution, are now trivially satisfied. By conditioning on the narrative N, each individual claim has a probability of one and thus their conjunction also has a probability of one. The conjunction principle is reduced to a deductive check that the elements of the wrongdoing follow from the narrative put forward. The narrative, however, has a probability short of one, up to whatever value is required to meet the governing standard of proof. The standard applies to the narrative as a whole, and only indirectly—via a deductive check—to the individual elements. This trivialization of the conjunction principle is perhaps unsurprising. If anything, it mirrors the fact that no lawyer has ever been concerned with the reliability of conjunction elimination or introduction.

Someone might object that what is stated above is not the conjunction principle we started out with it. Quite right. But—we have argued—the conjunction principle is not the right principle to combine simple, individual claims into more complex claims. Mere conjunctive addition does not get us very far in complex legal cases. More structure is needed. As such, the conjunction principle should be rejected and replaced by a more nuanced method to aggregate evidence and construct complex claims. We have gestured at what this method should look like—(roughly) it relies on Bayesian networks. A more precise articulation of this point is left to other chapters. REF TO OTHER CHAPTERS

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