

Higher-order (Precise) Probabilism

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Abstract. Rational agents are often uncertain about the truth of many propositions. To represent this uncertainty, it is natural to rely on probability theory. Two options are typically on the table, precise and imprecise probabilism, but both fall short in some respect. Precise probabilism is not expressive enough, while imprecise probabilism suffers from belief inertia and the impossibility of proper scoring rules. We put forward a novel version of probabilism, higher-order probabilism, and we argue that it outperforms existing alternatives.

Keywords: Probabilism; Imprecise probabilities; Evidence; Probability; Belief inertia; Bayesian networks; Proper scores.

1 Introduction

As rational agents, we are uncertain about the truth of many propositions since our evidence is often fallible. To represent this uncertainty, it is natural to rely on probabilities. Two options are typically on the table: precise and imprecise probabilism. Precise probabilism models an agent's state of uncertainty (or credal state) with a single probability measure: each proposition in the algebra is assigned a probability value between 0 and 1 (a sharp credence). The problem is that a single probability measure is not expressive enough to distinguish between intuitively different states of uncertainty (§2). To avoid this problem, a *set* of probability measures can be used to represent the uncertainty of a rational agent. This approach is known as imprecise probabilism. It outperforms precise probabilism in some respects, but also runs into its own problems, such as belief inertia and the impossibility of defining proper scoring rules (§3).

To make progress, this paper argues that a rational agent's uncertainty should be represented by a set of precise probability measures defined over an algebra of propositions, and in addition, a probability distribution over parameter values interpreted as probability measures. We call this proposal *high-order (precise) probabilism*. The theory we propose is not mathematically novel, nor are philosophers unfamiliar with higher-order probabilities **CITE SOME: Skyrms, Gaifmann, Domotor, Uchii, Pearl, Peijnenburg and Atkinson**. Here we address specifically how higher-order probabilism can overcome the problems that plague precise and imprecise probabilism (§4 and §??). We also argue that higher-order probabilism fares better than existing versions of probabilism when the probability of multiple propositions, dependent

or independent, is to be assessed (§?? and ??). Many of the examples in this paper are about coin tosses, but in the final two sections, we discuss legal examples and illustrate the broader applicability of higher-order probabilism.

2 Precise probabilism

Precise probabilism holds that a rational agent's uncertainty about a proposition is to be represented as a single, precise probability measure. Bayesian updating regulates how the prior probability measure should change in light of new evidence that the agent learns. This is an elegant and simple theory, but representing our uncertainty about a proposition with a single, precise probability measure fails to capture an important dimension of how our fallible beliefs reflect the evidence we have (or have not) obtained. A couple of stylized examples featuring coin tosses should make the point clear.

No evidence v. fair coin You are about to toss a coin but have no evidence about its bias. You are completely ignorant. Compare this to the situation in which you know, based on overwhelming evidence, that the coin is fair.

On precise probabilism, both scenarios are represented by assigning a probability of .5 to the proposition *the coin will land heads on the next toss* (or *heads* for short). If you are completely ignorant, the principle of insufficient evidence suggests that you assign .5 to this proposition. Similarly, if you are sure the coin is fair, assigning again .5 seems the best way to quantify your uncertainty about the outcome. The agent's evidence in the two scenarios is quite different, but the precise probability of *heads* cannot capture this difference.

Learning from ignorance You toss a coin with unknown bias. You toss it 10 times and observe 5 heads. You toss it further and observe 50 heads in 100 tosses.

Since the coin initially had an unknown bias, you should presumably assign a probability of .5 to the proposition *heads*, if you stick with precise probabilism. After the 10 tosses, you again assess the probability to be .5. You must have learned something, but whatever that is, it is not reflected in the precise probability of *heads*. When you toss the coin 100 times and observe 50 heads, you also learn something new, but your precise probability does not change.¹

These examples show that the precise probability of *heads* is not appropriately responsive to evidence. But instead of focusing on this probability, the precise probabilist might respond that we should extend the algebra and include propositions about the bias of the coin. As evidence accumulates that the coin is fair, the (precise) probability of the proposition *the coin has a .5 bias* should go up, even though the (precise) probability of *heads* should remain .5. We think this response is on the right track, but how does it generalize beyond cases of coin tosses?

¹Another problem for precise probabilism is known as *sweetening* (Hare, 2010). Imagine a rational agent who does not know the bias of the coin. For precise probabilism, this state of uncertainty is represented by a .5 probability assignment to heads. Next, the agent learns that the bias towards heads, whatever the bias is, has been slightly increased, say by .001. Intuitively, the new information should leave the agent equally undecided about betting on heads or tails. After sweetening, the agent still does not know much about the actual bias of the coin. But, according to precise probabilism, sweetening should make the agent bet on heads: if the probability of heads was initially .5, it should now be slightly above .5.

Suppose that, given a certain stock of evidence, A is more likely than B . Further, suppose that the acquisition of new evidence does not change the probabilities. Admittedly, something has changed in the agent's state of uncertainty: the quantity of evidence on which the agent can assess whether A is more probable than B has become larger. And yet, this change is not reflected in the precise probabilities assigned to A and B .² The precise probabilist might recommend we extend the algebra and include propositions of the form *the probability of event X is such and such*. Since such propositions contain a probability, assigning a probability to them effectively amounts to using higher-order probabilities.³ Before going higher-order, however, we should explore another view in the literature.

3 Imprecise probabilism

Imprecise probabilism holds that a rational agent's credal stance towards a hypothesis is to be represented by a set of probability measures, typically called a representor \mathbb{P} , rather than a single measure P . The representor should include all and only those probability measures which are compatible with the evidence (more on this point later).⁴ Modeling an agent's credal state by sets of probability measures can easily accommodate the scenarios in the previous section without adding propositions about coin biases. For instance, if an agent is sure the coin is fair, their credal state would be represented by the singleton set $\{P\}$, where P is a probability measure that assigns .5 to *heads*. If, on the other hand, the agent knows nothing about the coin's bias, their credal state would be represented by the set of all probabilistic measures, since none of them is excluded by the available evidence.

So far so good. But now consider this scenario:

Even v. uneven bias: You have two coins and you are sure the probability of *heads* is .4, if you toss one coin, and .6, if you toss the other. But you cannot tell which is which. You pick one coin at random and toss it. Contrast this with an uneven case. You have four coins and you are sure three of them have bias .4 and one of them bias .6. You pick a coin at random and toss it. You should be three times more confident that the probability of getting heads is .4. rather than .6.

The first scenario can be easily modeled by imprecise probabilism. The representor would contain two probability measures; one assigns .4. and the other assigns .6 to the proposition *the coin will land heads*. Yet imprecise probabilism cannot model the second scenario. Since the probability measures in the set are all compatible with the agent's evidence, no probability

²The distinction here is sometimes formulated in terms of the *balance* of the evidence (that is, whether the evidence available tips in favor a hypothesis or another) as opposed to its *weight* (that is, the overall quantity of evidence regardless of its balance); see Keynes (1921) and Joyce (2005).

³Interestingly, **PERAL ADD REFERENCE** argues that the semantics of higher-order probability statements can still be expressed in the language of first-order probability.

⁴For the development of imprecise probabilism, see Keynes (1921); Levi (1974); Gärdenfors & Sahlin (1982); Kaplan (1968); Joyce (2005); Fraassen (2006); Sturgeon (2008); Walley (1991). Bradley (2019) is a good source of further references. Imprecise probabilism is closely related to what we might call interval probabilism (Kyburg, 1961; Kyburg Jr & Teng, 2001). In interval probabilism, precise probabilities are replaced by intervals of probabilities. Imprecise probabilism is more general since the representor set need not be an interval.

measure can be assigned a greater (higher-order) probability than any other.⁵

These scenarios show that imprecise probabilism is not always expressive enough. Worst still, imprecise probabilism suffers from three shortcomings that do not affect precise probabilism: first, the idea of compatibility between the available evidence and the probability measures in the representator set is not clearly defined; second, updating imprecise probabilities can run into belief inertia; and third, no proper scoring rules exist for imprecise probabilities.

The first shortcoming has not received extensive discussion in the literature. For imprecise probabilism, an agent's state of uncertainty is represented by those probability measures that are *compatible* with the agent's evidence, but the notion of compatibility lacks a clear definition in the literature. If we think of it as the fact that the agent's evidence does not outright rule out the probability measure in question, the problem is that observations, evidence and data will hardly rule out any probability measure. Admittedly, there will be clear-cut cases: if you see the outcome of a coin toss is heads, you reject the measure with $P(H) = 0$, and similarly for tails. Another class of cases might arise while randomly drawing objects from a finite set where the objective chances are known.⁶ But clear-cut cases aside, what else?

A second, related problem for imprecise probabilism is belief inertia (Levi, 1980). Precise probabilism offers an elegant model of learning from evidence: Bayesian updating. Imprecise probabilism, at least *prima facie*, offers an equally elegant model of learning from evidence, richer and more nuanced. When faced with new evidence E between time t_0 and t_1 , the representor set should be updated point-wise, running the standard Bayesian updating on each probability measure in the representor:

$$\mathbb{P}_{t_1} = \{P_{t_1} | \exists P_{t_0} \in \mathbb{P}_{t_0} \forall H [P_{t_1}(H) = P_{t_0}(H|E)]\}.$$

Belief inertia arises in situations in which no amount of evidence can lead the agent to change their initial belief state, according to a given modeling strategy. Consider a situation in which you start tossing a coin knowing nothing about its bias, so the range of possibilities is $[0, 1]$. After a few tosses, if you observed at least one tail and one head, you can exclude the measures assigning 0 or 1 to *heads*. But else what can you learn? For any sequence of outcomes that you can obtain and any probability value in $(0, 1)$, there will exist a probability measure (conditional on the outcomes) that assigns that probability to *heads*. Consequently, the edges of your resulting interval will remain the same. Some downplay the problem of belief inertia and blame the uniform prior $[0, 1]$.⁷ Just as contingent propositions should not be assigned (precise) probabilities of 0 or 1 since these extreme values are unrevisable, by the same token

⁵Other scenarios can be constructed in which imprecise probabilism fails to capture distinctive intuitions about evidence and uncertainty; see, for example, (Rinard, 2013).

⁶Probability measures can be inconsistent with evidential constraints that agents believe to be true (Bradley, 2012), for example, *structural constraints* such as “ X and Y are independent”. But, unless they come from an oracle, there will usually be some degree of uncertainty about the acceptability of these constraints.

⁷Another strategy is to say that, in a state of ignorance, a special updating rule should be deployed. Elkin (2017) suggests the rule of *credal set replacement* that recommends that upon receiving evidence the agent should drop measures rendered implausible, and add all non-extreme plausible probability measures. This, however, is tricky. One needs a separate account of what makes a distribution plausible from a principled account of why one should use a separate special update rule when starting with complete ignorance.

the uniform prior $[0, 1]$ should not be used whenever it is impervious to revision (Moss, 2020). More generally, it is unsurprising you will not learn anything when you start from a place of ignorance (Joyce, 2010). But, as we will see in the next section, uniform priors are not necessarily unrevisable and can be a starting point for learning. The problem lies with imprecise probabilities, not with uniform priors as such.

The third problem for imprecise probabilism is to find suitable proper scoring rules. In the precise case, scoring rules measure the distance between a rational agent's probability measure and the actual value. The Brier score is the most common scoring rule for precise probabilities.⁸ A requirement of scoring rules is *propriety*: any rational agent will expect their probability measure to be more accurate than any other. A bit more formally, the expected inaccuracy of p from the perspective of p should always be smaller than the expected inaccuracy of p from the perspective of another distribution q . After all, if an agent thought a different probability measure was more accurate, they should switch to it. Well-known results demonstrate the strict propriety of the Brier score for precise probabilities.⁹ Can similar results be established for imprecise probabilities? The answer is likely to be negative.

In the precise case, let $I(p, w)$ be an inaccuracy score of a probability distribution p relative to the true state w . Its expected inaccuracy, from the perspective of p equals

$$\sum_{w \in W} p(w) I(p, w).$$

In the imprecise case, let $I([p_-, p_+], w)$ be the inaccuracy score for the interval $[p_-, p_+]$. What is its expected inaccuracy from the perspective of, say, the interval $[p_-, p_+]$ itself? There is no straightforward answer to this question because $I([p_-, p_+], w)$ cannot be multiplied by $[p_-, p_+]$, in the way in which $I(p, w)$ can be multiplied by $p(w)$. The expected inaccuracy of $[p_-, p_+]$ can be evaluated from the perspective of the precise probabilities at its edges.¹⁰ The problem is that, in this case, finding a proper inaccuracy score that is also continuous is mathematically impossible (Seidenfeld, Schervish, & Kadane, 2012).

Proper scoring rules are often used to formulate accuracy-based arguments for precise probabilism. These arguments show (roughly) that, if your precise measure follows the axioms of probability theory, no other non-probabilistic measure is going to be more accurate than yours whatever the facts are. But, without proper scoring rules for imprecise probabilities, the prospects for an accuracy-based argument for imprecise probabilism look dim (Campbell-Moore, 2020; Mayo-Wilson & Wheeler, 2016).¹¹

⁸The Brier score is defined as the squared distance between the true state and the probability forecast, or formally, $(p(x) - V(x, w))^2$, where $p(x)$ is the probability forecast and $V(x, w)$ determines if a proposition obtains at w ($V(x, w) = 1$) or not ($V(x, w) = 0$). If, for example, the proposition 'rain' obtains at w , the forecast 'rain with .9 probability' would be more accurate at w than the forecast 'rain with .8 probability'. If, on the other hand, the proposition 'not rain' obtained at w , the latter forecast would be more accurate.

⁹Besides propriety, other common requirements (which the Brier score also satisfies) are: the score $I(p, w)$ should be a function of the probability distribution p and the true state w (extensionality); and the score should be a continuous function around p (continuity).

¹⁰So the expected inaccuracy would equal $\sum_{w \in W} p_-(w) I([p_-, p_+], w)$ or $\sum_{w \in W} p_+(w) I([p_-, p_+], w)$.

¹¹Moreover, as shown by Schoenfield (2017), if an accuracy measure satisfies certain plausible formal constraints, it will never strictly recommend an imprecise stance, as for any imprecise stance there will be a precise one with at least the same accuracy.

4 Higher-order probabilism

Let us take stock. Imprecise probabilism is expressive enough to model the difference between a state in which there is no evidence about a proposition (or its negation) and a state in which the evidence for and against a proposition is in equipoise. However, imprecise probabilism has its own expressive limitations, for example, it cannot model the case of uneven bias. In addition, imprecise probabilism faces difficulties that do not affect precise probabilism: the notion of compatibility between a probability measure and the evidence is too permissive; belief inertia makes it impossible for a rational agent to learn via Bayesian updating; and no proper scoring rules exist for imprecise probabilism. In this section and the next, we show that higher-order probabilism overcomes the expressive limitations of imprecise probabilism without falling prey to any such difficulties.

Proponents of imprecise probabilism already hinted at the need to rely on higher-order probabilities. For instance, Bradley compares the measures in a representor to committee members, each voting on a particular issue, say the true chance or bias of a coin. As they acquire more evidence, the committee members will often converge on a chance hypothesis.

...the committee members are “bunching up”. Whatever measure you put over the set of probability functions—whatever “second order probability” you use—the “mass” of this measure gets more and more concentrated around the true chance hypothesis. (Bradley, 2012, p. 157)

But such bunching up cannot be modeled by imprecise probabilism alone: a probability distribution over chance hypotheses is needed.¹² That one should use higher-order probabilities has also been suggested by critics of imprecise probabilism. Carr (2020) argues that sometimes evidence requires uncertainty about what credences to have. Carr, however, does not articulate this suggestion more fully; does not develop it formally; and does not explain how her approach would fare against the difficulties affecting precise and imprecise probabilism. We now set out to do precisely that.

The central idea of higher-order probabilism is this: instead of a number between 0 and 1, rational agents assign a probability density to each proposition of interest. For example, consider the scenario in which the agent knows that the bias of the coin is either .4 or .6 but the former is three times more likely. This scenario—which imprecise probabilism cannot model—can be easily modeled within high-order probabilism by assigning different probabilities to the two biases, as illustrated in Figure 1. So, an agent’s state of uncertainty (or credal stance) towards a proposition x will not be represented by a single probability value $p(x)$ between 0 and 1, but by a probability density $f_x(p(x))$, where the first-order probability of x is the parameter in question and is treated as a random variable.¹³ More generally, let \mathbf{X} be the set of first-order propositions an agent is interested in (more on this restriction soon). So, for

¹²In a similar vein, Joyce (2005), in a paper defending imprecise probabilism, explicates the notion of weight of evidence using a probability distribution over chance hypotheses. Oddly, representor sets play no central role in Joyce’s account of the weight of evidence.

¹³For computational ease, we will use a higher-order density that is discretized and ranges over 1000 first-order probabilities.

each proposition $x \in \mathbf{X}$, the rational agent will assign a probability density $f_x(p(x))$, and not simply a probability value $p(x)$. This mapping from a set of propositions to densities models the agent's overall state of uncertainty.

Note that the set \mathbf{X} is restricted to first-order propositions. There is an intuitive distinction between propositions more directly about events in the world, such as *the coin will land heads on the next toss* and more abstract constructs such as the probabilities of these propositions. So we call propositions about the former events first-order, and propositions about the latter events second-order. Now, instead of going higher-order, the set of \mathbf{X} can be extended to include second-order propositions, denoted \mathbf{X}^+ . Then, an agent's state of uncertainty will be modeled by a single, precise probability measure defined over all members of the extended set \mathbf{X}^+ . This proposal can easily model the scenario of uneven coins by assigning different probabilities to the (second-order) propositions about the two possible biases of the coin, again as illustrated in Figure 1.

Higher-order probabilist and this variation of precise probabilism in which the event space is suitably extended are quite similar. But a few differences are worth pointing out. Consider the (first-order) proposition *the coin will land heads on the next toss* (in short, *heads*) and the (second-order) *the coin has a bias of .5*. We expected them to be related in precise ways. For example, it is inconsistent to assign a probability of one (or an extremely high probability) to both *heads* and the coin having a fair bias, or to assign an extremely high probability to *heads*. Clearly, the two propositions are not probabilistically independent and their dependence is conceptual or formal, not an empirical question. The simplest way to model this formal probabilistic dependence is to posit that the expectation of the density $f_{heads}(p(heads))$ equals $p(heads)$.

An agent's uncertainty about a proposition could—perhaps, should—sometimes be represented by a single probability value. Higher-order probabilism does not prohibit that. For example, there may well be cases in which an agent's uncertainty is aptly represented by the expectation.¹⁴ But this need not always be the case. If the probability distribution is not sufficiently concentrated around a single value, a one-point summary will fail to do justice to the nuances of the agent's credal state.¹⁵

Representing the agent's credal state with the expectation $.75 \times .4 + .25 \times .6 = .45$ would fail to capture the agent's different epistemic attitudes towards the two biases. The agent believes the two biases have different probabilities, and is also certain the bias is *not* .45.

The precise probabilist will note that, by suitable extending the set of propositions of interest to include also second-order propositions—such as propositions about the probability of first-order event—

This higher-order approach at the technical level is by no means novel. Bayesian probabilistic programming languages embrace the well-known idea that parameters can be stacked

¹⁴The expectation is usually defined as $\int_0^1 x f(x) dx$. In the context of our approach here, x is the first-order probability of a given proposition and f is the density representing the agent's uncertainty about x .

¹⁵This approach lines up with common practice in Bayesian statistics, where the primary role of uncertainty representation is assigned to the whole distribution. Summaries such as the mean, mode standard deviation, mean absolute deviation or highest posterior density intervals are only succinct ways of representing uncertainty.

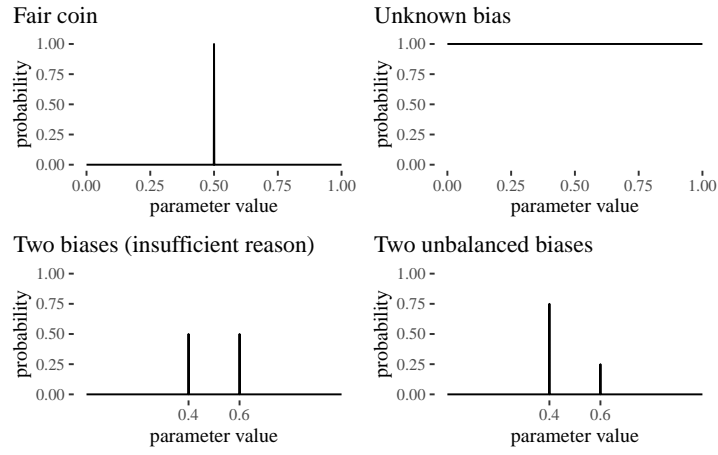


Figure 1: Examples of higher-order distributions for a few scenarios problematic for both precise and imprecise probabilism.

and depend on each other (Bingham et al., 2021). But, while the technical machinery has been around for a while, it has not been deployed by philosophers to model a rational agent’s uncertainty or credal state.

We should distinguish our version of higher-order (precise) probabilism from a couple of other variants. First, consider what we might call higher-order (imprecise) probabilism. On this account, an agent’s state of uncertainty is modeled by (a) a set of probability measures defined over a fixed algebra of first-order propositions (just like imprecise probabilism does), and in addition, (b) a probability distribution over those first-order probability measures. Our higher-order (precise) probabilism instead models an agent state of uncertainty

- (a) a single, precise probability measure defined over a fixed algebra of first-order propositions (just like precise probabilism does), and addition, (b) by a probability density

How should these second-order probabilities be understood? It is helpful to think of higher-order probabilism as a generalization of imprecise probabilism. Imprecisers already admit that some probability measures are compatible and others incompatible with the agent’s evidence at some point. Compatibility is a coarse notion; it is an all-or-nothing affair. But, as seen earlier, evidence can hardly exclude a probability measure in a definitive manner except in clear-cut cases. Just as it is often a matter of degrees whether the evidence supports a proposition, the compatibility between evidence and a probability measure can itself be a matter of degrees. In this picture, the evidence justifies different values of first-order probability to various degrees. So, second-order probabilities express the extent to which the first-order probabilities are supported by the evidence.

Besides its greater expressive power in modeling uncertainty, higher-order probabilism does not fall prey to belief inertia. Consider a situation in which you have no idea about the bias of a coin. You start with a uniform distribution over $[0, 1]$ as your prior. Observing any non-zero number of heads will exclude 0 and observing any non-zero number of tails will exclude 1 from the basis of the posterior. The posterior distribution will become more centered as the observations come in. This result is a straightforward application of Bayesian updating. Instead of

plugging sharp probability values into the formula for Bayes's theorem, the factors to be multiplied in the theorem will be probability densities (or ratios of densities as needed). Figure 2 illustrates—starting with a uniform prior distribution—how the posterior (beta) distribution changes after successive observations.¹⁶

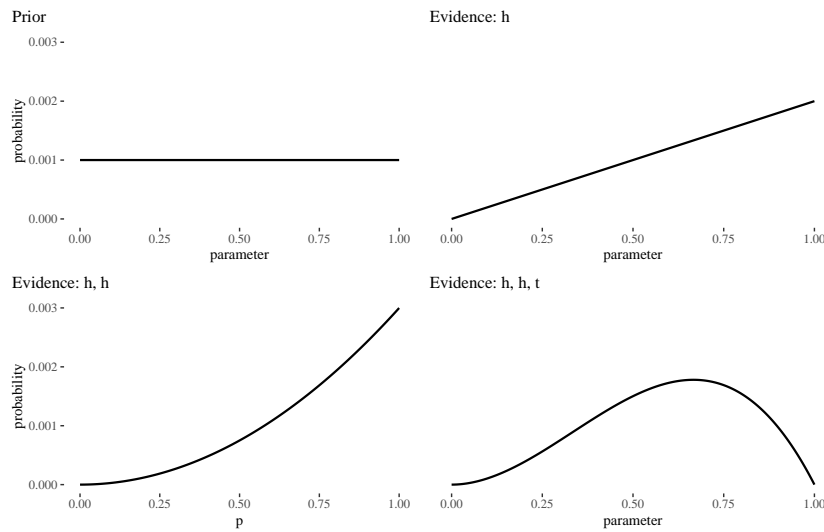


Figure 2: As observations of heads, heads and tails come in, extreme parameter values drop out of the picture and the posterior is shaped by the evidence.

The impossibility of defining proper scoring rules was another weakness of imprecise probabilism. This is a significant shortcoming, especially because proper scores do exist for precise probabilism. Fortunately, one can show that there exist proper scoring rules for higher-order probabilism. We defend this claim in the next section. The argument, however, will be more formal and can be skipped upon first reading.

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¹⁶Assuming independence and constant probability for all the observations, learning is modeled the Bayesian way. You start with some prior density p over the parameter values. If you start with a complete lack of information, p should be uniform. Then, you observe the data D which is the number of successes s in a certain number of observations n . For each particular possible value θ of the parameter, the probability of D conditional on θ follows the binomial distribution. The probability of D is obtained by integration. That is:

$$\begin{aligned} p(\theta|D) &= \frac{p(D|\theta)p(\theta)}{p(D)} \\ &= \frac{\theta^s(1-\theta)^{(n-s)}p(\theta)}{\int (\theta')^s(1-\theta')^{(n-s)}p(\theta') d\theta'}. \end{aligned}$$

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