MAP555 : Signal Processing [1]

Rémi Flamary

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[1] **Warning** : This document is currently being written and should be considered unfinished and full of mistakes and typos. It should not be used yet as a pedagogical support for a course.

Contents

Chapter 1

Introduction

In this chapter we will introduce signal processing and discuss briefly the numerous fundamental problems of signal processing.

1.1 Signal processing

Signal processing is everywhere Signal processing is a field that aim at modeling signals and providing automatic processing of those signals. It has been heavily researched for several decades and signal processing methods are central part of numerous technologies in telecommunications, multi-media processing, compression and storage. In recent years, tremendous results have been obtained by using modern machine learning and artificial intelligence techniques.

Objective of this course The objective of this course is to provide an introduction to the very large field of signal processing. One fascinating aspect of signal processing is that it is at the crossroad between Physics (to generate the signals), Electronics (to measure the signals), Mathematics (to model the signals) and Computer Science (to process the signals). In this sense, Signal processing is a perfect example of a multi-disciplinary field and a lot thee existing methods are known with other names in other fields. An effort will be made to provide vocabulary coming from the signal processing community but also statistics, machine learning and computer science.

We plan on introducing in this documents both the mathematical models, the numerical algorithms used for their processing and several examples of real life applications. The implementation of the signal processing methods in Python will also be discussed with example code and existing toolboxes. Note that most of the methods are introduced very briefly, but we will always provide detailed references for a more in-depth study.

Content of the document The course begins with a short introduction of signal processing containing a few definitions and problems formulations followed by bibliographical notes. Chapter 3 provides a presentation of Fourier analysis and analog filtering with some applicative examples such as modulation and Fourier optics in astronomy. Chapter 4 introduces signal sampling and digital signal filtering that has become the de-facto standard in practical applications. It also presents the very important Fast Fourier Transform (FFT) algorithm and discuss some examples of filtering in image processing. Chapter 5 discuss the random/stochastic aspects of signals and their optimal linear filtering when modeled as as stochastic processes. The modeling of speech is taken as an example for the study of auto-regressive models. Chapter 6 briefly introduces several signal representations commonly used such as the Discrete Cosine Transform (DCT), and wavelet transforms used in JPEG encoding and image reconstruction. The short time Fourier transform will also be introduced to model non-stationary signals. Finally some recent approaches based on machine learning such as dictionary learning and deep learning signal reconstruction will be presented.

1.2 Bibliographical notes

This document was strongly inspired by a number of outstanding references books that have been published over the years. In this section we discuss a few of those strongly recommended references. Suggestions to the author are welcome to provide a curated list of "awesome" references for signal processing similar to the lists available on GitHub.

Signal processing

- Signals and Systems [Haykin and Van Veen, 2007].
- Signals and Systems [Oppenheim et al., 1997].
- Signal Analysis [Papoulis, 1977].

- Polycopiés from Stéphane Mallat and Éric Moulines [Mallat et al., 2015].
- Théorie du signal [Jutten, 2018].

Analog signal processing and Fourier Transform

- Fourier Analysis and its applications [Vretblad, 2003]
- Distributions et Transformation de Fourier [Roddier, 1985]

Digital signal processing

- https://www.numerical-tours.com/
- Discrete-time signal processing [Oppenheim and Shafer, 1999].

Random signals, stochastic processes

- Random variables and stochastic processes [Papoulis, 1965].
- [Ross et al., 1996]
- [Kay, 1993]

Signal representations

- A Wavelet tour of signal processing [Mallat, 1999].
- Wavelets and sub-band coding [Vetterli and Kovacevic, 1995].

1.3 About this document

This document contains lecture notes of MAP555 Signal Processing Course from the Applied Mathematics Department at École Polytechnique. It is currently being written and should be considered unfinished and full of mistakes and typos. It should not be used yet as a pedagogical support for a course.

The document is available in [PDF format] and [HTML format] compiled automatically when the source is modified in the GitHub repository. All the scripts that were used to generate the figures are available here.

 Reader are encouraged to report typos and mistakes in the mathematical formulas and proposed correction as Pull Requests on Github. Contributors will be listed in the HTML and PDF documents.

Chapter 2

Signals and convolution

2.1 Signals and properties

2.1.1 Properties of analog signals

Analog signal We define a signal is this course as a function of time or space. For instance $x : \mathbb{R} \rightarrow \mathbb{C}$ is a complex 1D signal of time $t \in \mathbb{R}$. $x : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a 2D image of space $\mathbf{p} \in \mathbb{R}^2$.

Causality A signal $x(t)$ is causal if

$$x(t) = 0, \quad \forall x < 0$$

Example: $x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \sin(t) \exp\left(-\frac{t^2}{2}\right) & \text{for } t \geq 0 \end{cases}$

Periodicity A signal $x(t)$ is periodic of period T_0 is

$$x(t - kT_0) = x(t), \forall t \in \mathbb{R}, \forall k \in \mathbb{N}$$

Example: $x(t) = \exp\left(-\frac{(t-kT_0-1)^2}{2}\right)$ for $kT_0 < t < (k+1)T_0, \quad \forall k \in \mathbb{N}$

Signal in L_p space $L_p(S)$ is the set of functions whose absolute value to the power of p has a finite integral or equivalently that

$$\|x\|_p = \int_S |x(t)|^p dt < \infty \tag{2.1}$$

- $L_1(\mathbb{R})$ is the set of absolute integrable functions
- $L_2(\mathbb{R})$ is the set of quadratically integrable functions (finite energy)
- $L_\infty(\mathbb{R})$ is the set of bounded functions

Instantaneous power The instantaneous power of signal $x(t)$

$$p_x(t) = |x(t)|^2 \tag{2.2}$$

Unit : Watt (W).

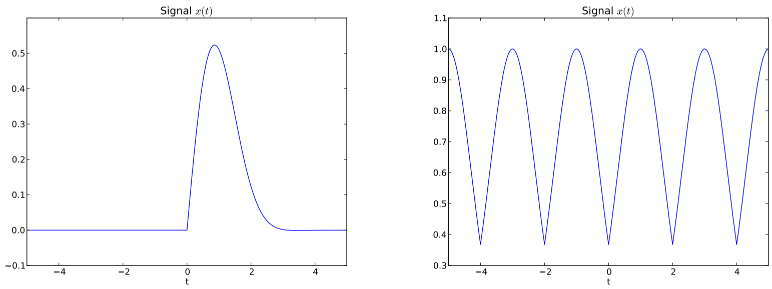

Figure 2.1: Examples of Causal signal (left) and periodic signal (right).

Energy of a signal We define the energy of a signal $x(t)$ as :

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt \tag{2.3}$$

the signal is said to be of finite energy if $E < \infty$ ($\|x\|_2 < \infty$ means $x \in L_2(\mathbb{R})$).

Unit: Joule, Calorie or Watt-hour (J, Cal ou Wh, 1 calorie = 4.2 J).

Average power of a signal The average power of a signal is defined as

$$P_m = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 dt \tag{2.4}$$

- For a periodic signal, the average power can be computed on a unique period.
- Power is homogeneous to an energy divided by time.
- $P_{RMS} = \sqrt{P_m}$ is called the Root Mean Square power ("valeur efficace" in french).
- A finite energy signal has a n average power $P_m = 0$.
- Unit : Watt (W).

Additive noise Additive noise is a kind of noise that is added to the signal of interest.

$$y(t) = x(t) + b(t)$$

$y(t)$ is the observed signal, $x(t)$ the signal of interest and $b(t)$ is the noise.

Signal-to-Noise ratio (SNR) The Signal to Noise Ratio is defined as:

$$SNR = \frac{P_S}{P_N} \quad \text{ou} \quad SNR(dB) = 10\log_{10}(SNR) \tag{2.5}$$

where P_S is the power of the signal and P_N the power of the noise.

- An Analog-to-Digital conversion process should have the best possible SNR.

- The SNR is often used for additive noise models.

- Other measures such as Peak Signal to Noise Ratio (PSNR) can be used on specific data (images).

- One of the objective of filtering is to get a better SNR when the signal and the noise have different frequency contents..

2.1.2 Common signals

Heaviside function

$$\Gamma(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases} \tag{2.6}$$

Also known as the step function.

Rectangular function

$$\Pi_T(t) = \begin{cases} 1/T & \text{if } |t| < T/2 \\ 1/2T & \text{if } |t| = T/2 \\ 0 & \text{else} \end{cases} \tag{2.7}$$

- $\Pi(t) = \frac{1}{T}(\Gamma(t - \frac{T}{2}) - \Gamma(t + \frac{T}{2}))$.

- Finite energy signal (finite support).

Complex exponential let $e_z(t)$ be the following function $\mathbb{R} \to \mathbb{C}$

$$e_z(t) = \exp(zt) \tag{2.8}$$

where z is a complex number. When $z = \tau + wi$ the,

$$e_z(t) = (\cos(wt) + i\sin(wt))\exp(\tau t)$$

Special cases:

- $z = \tau$ real, then we recover the classical exponential.

$$e_z(t) = \exp(\tau t)$$

- $z = wi$ imaginary then

$$e_z(t) = \cos(wt) + i\sin(wt)$$

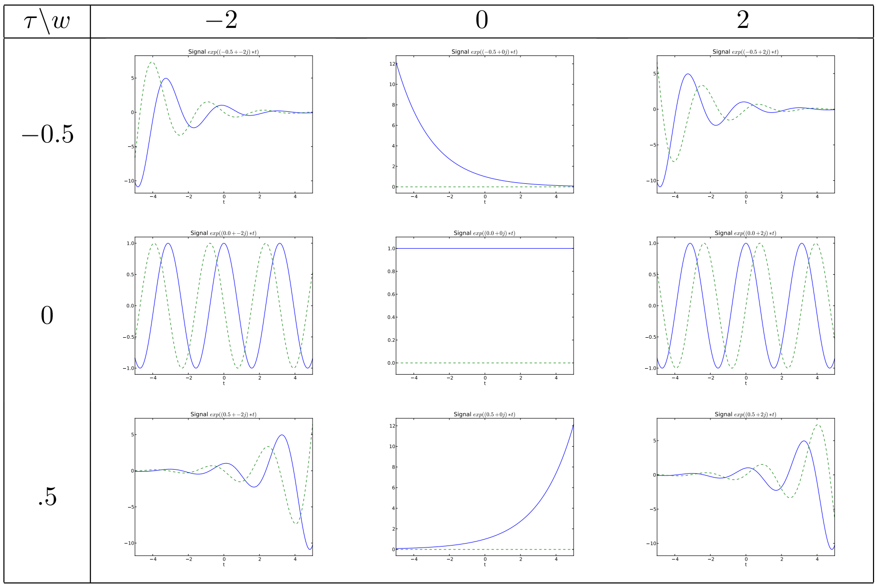

Figure 2.2: Example of complex exponential for different values of z

2.1.2.1 Dirac delta

Main properties of Dirac delta

- Model point mass at 0.
- Value outside 0 : $\delta(t) = 0, \forall t \neq 0$
- δ is a tempered distribution.
- Very useful tool in signal processing
- Can be seen as the derivative of the Heavyside function $1_{t\geq 0}(t)$
- Integral

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1, \qquad \int_{-\infty}^{+\infty} x(t)\delta(t)dt = x(0) \tag{2.9}$$

- Dirac and function evaluation for signal $x(t)$ and $t_0 \in \mathbb{R}$:

$$\delta(t-t_0)x(t) = \delta(t-t_0)x(t_0)$$

$$\langle x(t), \delta(t-t_0)\rangle = \int_{-\infty}^{+\infty} x(t)\delta(t-t_0)dt = x(t_0) \tag{2.10}$$

Dirac delta definition

- Let ϕ a function supported in $[-1,1]$ of unit mass: $\int_{-\infty}^{\infty} \phi(u)du = 1$
- $\phi_T(t) = \frac{1}{T}\phi(\frac{t}{T})$ has support on $[-T,T]$ and unit mass.
- We can define the dirac delta δ as

$$\delta(t) = \lim_{T\to 0} \phi_T(t)$$

Delta dirac in practice

- Theoretical object in signal processing (impulse).
- Used to model signal sampling for digital signal processing.
- Used to model point source in Astronomy/image processing, point charge in Physics.
- Has a bounded discrete variant.

2.1.3 Discrete time and digital signals

2.2 Convolution and filtering

2.2.1 Convolution and properties

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Convolution Let two signals $x(t)$ and $h(t)$. The convolution between the two signals is defined as

$$x(t) \star h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \tag{2.11}$$

- Convolution is a bilinear mapping between x and h.
- It models the relation between the input and the output of a Linear Time Invariant system.
- If $f \in L_1(\mathbb{R})$ and $h \in L_p(\mathbb{R}), p \geq 1$ then

$$\|f \star h\|_p \leq \|f\|_1 \|h\|_p$$

- The dirac delta δ is the neutral element for the convolution operator:

$$x(t) \star \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = x(t) \tag{2.12}$$

$$x(t) \star \delta(t-t_0) = x(t-t_0) \tag{2.13}$$

Example of convolution

- $x(t) = \Gamma(t)$ the Heaviside step function.
- $h(t) = e^{-t}\Gamma(t)$ the positive part of the decreasing exponential.
- $x(t) \star h(t) = (1-e^{-t})\Gamma(t)$

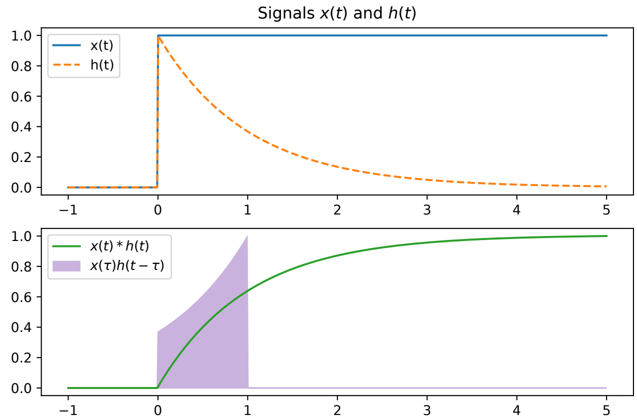


Figure 2.3: Illustration of the convolution operator between the Heaviside step function and a causal decreasing exponential.

2.2.2 Linear Time Invariant (LTI) systems

Linear Time Invariant (LTI) system

- A system describes a relation between an input $x(t)$ and an output $y(t)$.
- Properties of LTI systems:
  - Linearity $$x_1(t) + ax_2(t) \rightarrow y_1(t) + ay_2(t)$$
  - Time invariance $$x(t-\tau) \rightarrow y(t-\tau)$$
- A LTI system can most of the time be expressed as a convolution of the form:
  $$y(t) = x(t) \star h(t)$$
  where $h(t)$ is called the impulse response (the response of the system to an input $x(t) = \delta(t)$)

Examples

- Passive electronic systems (resistor/capacitor/inductor) .
- Newtonian mechanics, Fluid mechanics, Fourier Optics.

Ordinary Differential Equation (ODE) The system is defined by a linear equation of the form:

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + a_n \frac{d^n y(t)}{dt^n} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_m \frac{d^m x(t)}{dt^m} \quad (2.14)$$

- ODE based system with linear relations are an important class of LTI systems.
- Also called homogeneous linear differential equation.

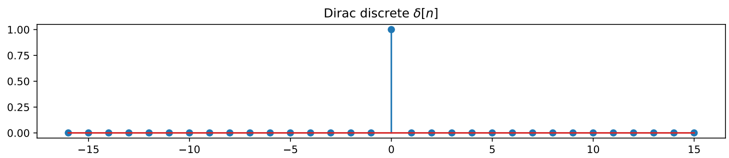
- n is the number of derivatives for $y(t)$ and m for $x(t)$.
- $\max(m, n)$ is the order of the system.
- The output of the system can be computed from the input by solving Eq. (2.14).
- Linearity and time invariance are obvious from equation.

2.3 Discrete time and digital signals

2.3.1 Discrete time

Notations

- $x(t)$ with $t \in \mathbb{R}$ is the analog signal.
- $x_T(t)$ with $t \in \mathbb{R}$ is the sampled signal of period (T) but still continuous time:
$$x_T(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$
- $x[n]$ with $n \in \mathbb{Z}$ is the discrete signal sampled with period T such that:
$$x[n] = x(nT)$$
- Obviously one can recover $x_T(t)$ from $x[n]$ with
$$x_T(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$
- In order to simplify notations we will suppose $T = 1$ in the following.
- In this course we suppose that $|x[n]|$ is bounded.



Discrete dirac We note the discrete dirac $\delta[n]$ defined as
$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{else} \end{cases} \tag{2.15}$$

Discrete signal Any discrete signal $x[n]$ can be decomposed as a sum of translated discrete diracs:
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k] \tag{2.16}$$

The discrete diracs are an orthogonal basis of $L_2(\mathbb{Z})$ of scalar product and corresponding norm
$$< x[n], h[n] >= \sum_{k=-\infty}^{\infty} x[k]h^*[k], \qquad \|x[n]\|^2 =< x[n], x[n] >= \sum_{k=-\infty}^{\infty} |x[k]|^2.$$

Convolution between discrete signals Let $x[n]$ and $h[n]$ two discrete signals. The convolution between them is expressed as:

$$x[n] \star h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2.17}$$

Digital filter properties Let the discrete system/operator/filter L described by its impulse response $h[n]$.

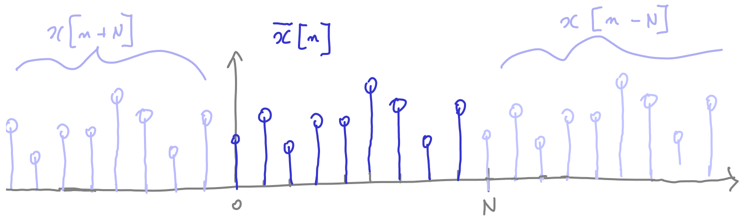
- **Causality** L is causal if $h[n] = 0,\ \forall n \leq 0$. L is causal if

$$h[n] = h[n]\Gamma[n], \qquad \text{where} \qquad \Gamma[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{else} \end{cases} \tag{2.18}$$

- **Stability** A system is stable if the output of a bonded input is bounded. A necessary and sufficient condition is that

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{2.19}$$

2.3.2 Finite signals



Finite discrete signals

- Most of the theoretical results seen up to now correspond to signals $x[n]$ where $n \in \mathbb{Z}$.
- In practice recordings are only done for a finite amount of time resulting to only N samples.
- We defined $\tilde{x}[n]$ a finite signal of N samples with $n \in \{0, \dots, N-1\}$.
- We use in the following the periodization of $\bar{x}[n]$

$$x[n] = \tilde{x}[n \mod N]$$

  where $\mod$ is the modulo operator.

Discrete convolution of finite signals The convolution between $\tilde{x}[n]$ and $\tilde{h}[n]$ both finite signals of n samples can be expressed as:

$$\tilde{y}[n] = \tilde{x}[n] \star \tilde{h}[n] = \sum_{p=-\infty}^{+\infty} \tilde{x}[p]\tilde{h}[n-p] \tag{2.20}$$

- It requires values for the signals outside of the sampling widow.
- One common approach consists in having $\tilde{x}[n]$ and $\tilde{h}[n]$ equal to 0 outside the sampling interval. Other choices can be done (see next slides)

Circular convolution When using the periodic version of the signals the circular convolution can be computed on a unique period of size N:

$$x \circledast h[n] = \sum_{p=0}^{N-1} x[p]h[n-p].$$

The circular convolution is rarely appropriate in real life images due to border effects.

Vector representation and convolution matrix

- Finite signal x of N samples can be represented as a vector $\mathbf{x} \in \mathbb{C}^N$.
- The convolution operator is linear and can be expressed as:

$$\mathbf{y} = \mathbf{x} \star \mathbf{h} = \mathbf{C_h x}$$

  Where $\mathbf{C_h} \in \mathcal{M}_\mathbb{C}(N, N)$ is a convolution matrix parametrized by vector $\mathbf{h}$.

Discrete convolution The convolution operator when the values outside the support are 0 can be expressed as

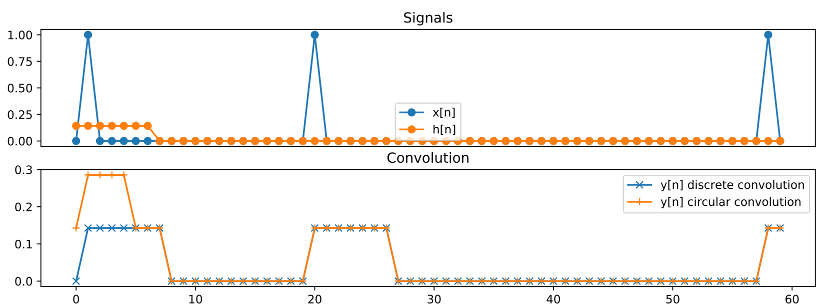
$$\mathbf{C_h} = \begin{bmatrix} h[0] & 0 & \cdots & 0 \\ h[1] & h[0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & \dots & h[0] \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h[N-1] \end{bmatrix}$$

where $\mathbf{C_h} \in \mathcal{M}_\mathbb{C}(2 * N - 1, N)$ is a Toeplitz matrix.

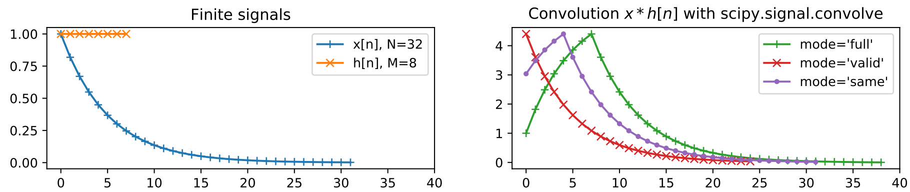
Circular convolution The circular convolution operator can be expressed as

$$\mathbf{C_h} = \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & \cdots & h[2] \\ \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & \dots & h[0] \end{bmatrix}$$

where $\mathbf{C_h} \in \mathcal{M}_\mathbb{C}(N, N)$ is a circulant Toeplitz matrix.

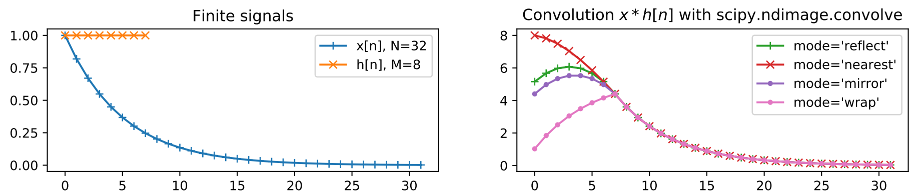


- Convolution between diracs $\tilde{x}[n]$ and a shape $h[n]$ will repeat the shape at the diracs position.
- A dirac at the end of the signal will cut the shape for discrete convolution where the outside of the sampling is 0.
- With circular convolution the shape is repeated t the beginning of the signal.
- One can remove border effects by creating virtual periodic signal with zeros (zero padding, see fast convolution).



The Scipy `scipy.signal.convolve` function:

- Convolution between two signals of support respectively N and M samples supposing that their values are 0 outside of the support.
- The third parameter is `mode` that change the size of the output :
  - `mode='full'` returns a signal of support $N + M - 1$ (default).
  - `mode='valid'` returns a signal of support $|N - M| + 1$ with only the samples that do not rely on zeros padding of the larger signal.
  - `mode='same'` returns a signal of the same size as the first input.
- Parameter `method` allows to choose between `'direct'` computation and `'fft'` and selects the most efficient by default.



The Scipy `scipy.ndimage.convolve` function:

- Always return the same size as the first parameter by default.
- The `mode` parameter allows selecting the borders of a signal $x = (abcd)$:
  - `mode='reflect'` : $(dcba|abcd|dcba)$ (default)
  - `mode='constant'` : $(kkkk|abcd|kkkk)$
  - `mode='nearest'` : $(aaaa|abcd|dddd)$
  - `mode='mirror'` : $(dcb|abcd|cba)$
  - `mode='wrap'` : $(abcd|abcd|abcd)$ (circular convolution)
- Parameter `origin` allows to select the origin of the filter h.

2.3.3 Quantization and storage

2.4 Fundamental signal processing problems

2.4.1 Filtering

2.4.2 Deconvolution, unmixing and regression

2.4.3 Blind source separation and deconvolution

Chapter 3

Fourier analysis and analog filtering

3.1 Fourier transform

3.2 Frequency response and filtering

3.3 Applications of analog signal processing

Chapter 4

Digital signal processing

4.1 Sampling and Analog/Digital conversion

4.2 Digital filtering and transfer function

4.3 Finite signals and Fast Fourier Transform

4.4 Applications of DSP

Chapter 5

Random signals

5.1 Random Signals and Correlations

5.2 Frequency representation of random signals

5.3 AR modeling and linear prediction

Chapter 6

Signal representations

6.1 Short Time Fourier Transform

6.2 Common signal representations

6.3 Source separation and dictionary learning

6.4 Machine learning for signal processing

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