MAP555 : Signal Processing [1]

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[1] **Warning** : This document is currently being written and should be considered unfinished and full of mistakes and typos. It should not be used yet as a pedagogical support for a course.

Contents

Chapter 1

Introduction

In this chapter we will introduce signal processing and discuss briefly the numerous fundamental problems of signal processing.

1.1 Signal processing

Signal processing is everywhere Signal processing is a field that aim at modeling signals and providing automatic processing of those signals. It has been heavily researched for several decades and signal processing methods are central part of numerous technologies in telecommunications, multi-media processing, compression and storage. In recent years, tremendous results have been obtained by using modern machine learning and artificial intelligence techniques.

Objective of this course The objective of this course is to provide an introduction to the very large field of signal processing. One fascinating aspect of signal processing is that it is at the crossroad between Physics (to generate the signals), Electronics (to measure the signals), Mathematics (to model the signals) and Computer Science (to process the signals). In this sense, Signal processing is a perfect example of a multi-disciplinary field and a lot thee existing methods are known with other names in other fields. An effort will be made to provide vocabulary coming from the signal processing community but also statistics, machine learning and computer science.

We plan on introducing in this documents both the mathematical models, the numerical algorithms used for their processing and several examples of real life applications. The implementation of the signal processing methods in Python will also be discussed with example code and existing toolboxes. Note that most of the methods are introduced very briefly, but we will always provide detailed references for a more in-depth study.

Content of the document The course begins with a short introduction of signal processing containing a few definitions and problems formulations followed by bibliographical notes. Chapter 3 provides a presentation of Fourier analysis and analog filtering with some applicative examples such as modulation and Fourier optics in astronomy. Chapter 4 introduces signal sampling and digital signal filtering that has become the de-facto standard in practical applications. It also presents the very important Fast Fourier Transform (FFT) algorithm and discuss some examples of filtering in image processing. Chapter 5 discuss the random/stochastic aspects of signals and their optimal linear filtering when modeled as as stochastic processes. The modeling of speech is taken as an example for the study of auto-regressive models. Chapter 6 briefly introduces several signal representations commonly used such as the Discrete Cosine Transform (DCT), and wavelet transforms used in JPEG encoding and image reconstruction. The short time Fourier transform will also be introduced to model non-stationary signals. Finally some recent approaches based on machine learning such as dictionary learning and deep learning signal reconstruction will be presented.

1.2 Bibliographical notes

This document was strongly inspired by a number of outstanding references books that have been published over the years. In this section we discuss a few of those strongly recommended references. Suggestions to the author are welcome to provide a curated list of "awesome" references for signal processing similar to the lists available on GitHub.

Signal processing

- Signals and Systems [Haykin and Van Veen, 2007].
- Signals and Systems [Oppenheim et al., 1997].
- Signal Analysis [Papoulis, 1977].

- Polycopiés from Stéphane Mallat and Éric Moulines [Mallat et al., 2015].
- Théorie du signal [Jutten, 2018].

Analog signal processing and Fourier Transform

- Fourier Analysis and its applications [Vretblad, 2003]
- Distributions et Transformation de Fourier [Roddier, 1985]

Digital signal processing

- https://www.numerical-tours.com/
- Discrete-time signal processing [Oppenheim and Shafer, 1999].

Random signals, stochastic processes

- Random variables and stochastic processes [Papoulis, 1965].
- [Ross et al., 1996]
- [Kay, 1993]

Signal representations

- A Wavelet tour of signal processing [Mallat, 1999].
- Wavelets and sub-band coding [Vetterli and Kovacevic, 1995].

1.3 About this document

This document contains lecture notes of MAP555 Signal Processing Course from the Applied Mathematics Department at École Polytechnique. It is currently being written and should be considered unfinished and full of mistakes and typos. It should not be used yet as a pedagogical support for a course.

The document is available in [PDF format] and [HTML format] compiled automatically when the source is modified in the GitHub repository. All the scripts that were used to generate the figures are available here.

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Chapter 2

Signals and convolution

2.1 Signals and properties

2.1.1 Properties of analog signals

Analog signal We define a signal is this course as a function of time or space. For instance $x : \mathbb{R} \to \mathbb{C}$ is a complex 1D signal of time $t \in \mathbb{R}$. $x : \mathbb{R}^2 \to \mathbb{R}$ is a 2D image of space $\mathbf{p} \in \mathbb{R}^2$.

Causality A signal $x(t)$ is causal if

$$x(t) = 0, \quad \forall x < 0$$

$$\text{Example: } x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \sin(t)\exp\left(-\frac{t^2}{2}\right) & \text{for } t \geq 0 \end{cases}$$

Periodicity A signal $x(t)$ is periodic of period T_0 is

$$x(t - kT_0) = x(t), \forall t \in \mathbb{R}, \forall k \in \mathbb{N}$$

$$\text{Example: } x(t) = \exp\left(-\frac{(t-kT_0-1)^2}{2}\right) \text{ for } kT_0 < t < (k+1)T_0, \quad \forall k \in \mathbb{N}$$

Signal in L_p space $L_p(S)$ is the set of functions whose absolute value to the power of p has a finite integral or equivalently that

$$\|x\|_p = \int_S |x(t)|^p dt < \infty \tag{2.1}$$

- $L_1(\mathbb{R})$ is the set of absolute integrable functions
- $L_2(\mathbb{R})$ is the set of quadratically integrable functions (finite energy)
- $L_\infty(\mathbb{R})$ is the set of bounded functions

Instantaneous power The instantaneous power of signal $x(t)$

$$p_x(t) = |x(t)|^2 \tag{2.2}$$

Unit : Watt (W).

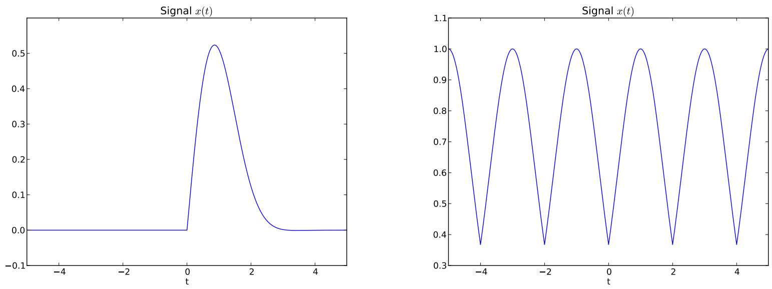

Figure 2.1: Examples of Causal signal (left) and periodic signal (right).

Energy of a signal We define the energy of a signal $x(t)$ as :

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt \tag{2.3}$$

the signal is said to be of finite energy if $E < \infty$ ($\|x\|_2 < \infty$ means $x \in L_2(\mathbb{R})$).
Unit: Joule, Calorie or Watt-hour (J, Cal ou Wh, 1 calorie = 4.2 J).

Average power of a signal The average power of a signal is defined as

$$P_m = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 dt \tag{2.4}$$

- For a periodic signal, the average power can be computed on a unique period.
- Power is homogeneous to an energy divided by time.
- $P_{RMS} = \sqrt{P_m}$ is called the Root Mean Square power ("valeur efficace" in french).
- A finite energy signal has a n average power $P_m = 0$.
- Unit : Watt (W).

Additive noise Additive noise is a kind of noise that is added to the signal of interest.

$$y(t) = x(t) + b(t)$$

$y(t)$ is the observed signal, $x(t)$ the signal of interest and $b(t)$ is the noise.

Signal-to-Noise ratio (SNR) The Signal to Noise Ratio is defined as:

$$SNR = \frac{P_S}{P_N} \quad \text{ou} \quad SNR(dB) = 10\log_{10}(SNR) \tag{2.5}$$

where P_S is the power of the signal and P_N the power of the noise.

- An Analog-to-Digital conversion process should have the best possible SNR.

- The SNR is often used for additive noise models.
- Other measures such as Peak Signal to Noise Ratio (PSNR) can be used on specific data (images).
- One of the objective of filtering is to get a better SNR when the signal and the noise have different frequency contents..

2.1.2 Common signals

Heaviside function

$$\Gamma(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases} \tag{2.6}$$

Also known as the step function.

Rectangular function

$$\Pi_T(t) = \begin{cases} 1/T & \text{if } |t| < T/2 \\ 1/2T & \text{if } |t| = T/2 \\ 0 & \text{else} \end{cases} \tag{2.7}$$

- $\Pi(t) = \frac{1}{T}(\Gamma(t - \frac{T}{2}) - \Gamma(t + \frac{T}{2}))$.
- Finite energy signal (finite support).

Complex exponential let $e_z(t)$ be the following function $\mathbb{R} \to \mathbb{C}$

$$e_z(t) = \exp(zt) \tag{2.8}$$

where z is a complex number. When $z = \tau + wi$ the,

$$e_z(t) = (\cos(wt) + i\sin(wt))\exp(\tau t)$$

Special cases:

- $z = \tau$ real, then we recover the classical exponential.

$$e_z(t) = \exp(\tau t)$$

- $z = wi$ imaginary then

$$e_z(t) = \cos(wt) + i\sin(wt)$$

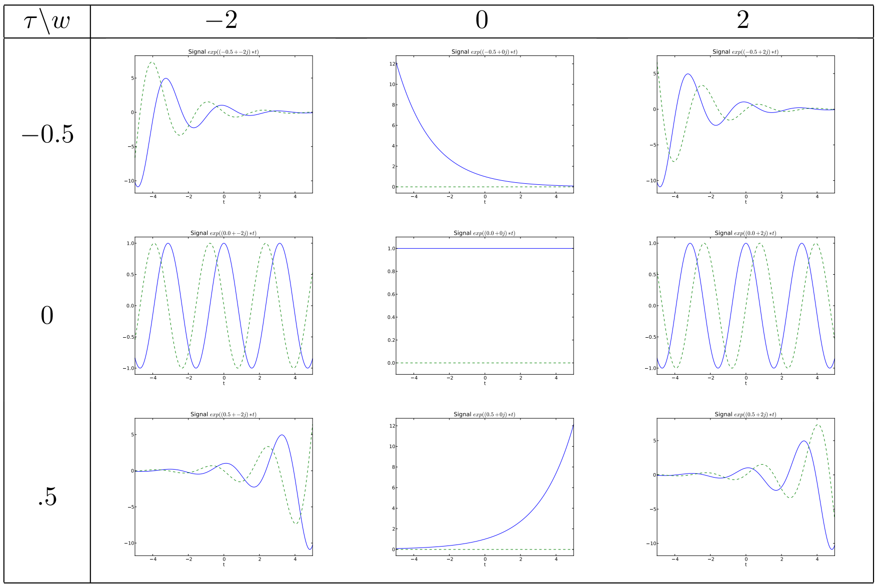

Figure 2.2: Example of complex exponential for different values of z

2.1.2.1 Dirac delta

Main properties of Dirac delta

- Model point mass at 0.
- Value outside 0 : $\delta(t) = 0, \forall t \neq 0$
- δ is a tempered distribution.
- Very useful tool in signal processing
- Can be seen as the derivative of the Heavyside function $1_{t \geq 0}(t)$
- Integral

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1, \qquad \int_{-\infty}^{+\infty} x(t)\delta(t)dt = x(0) \tag{2.9}$$

- Dirac and function evaluation for signal $x(t)$ and $t_0 \in \mathbb{R}$:

$$\delta(t - t_0)x(t) = \delta(t - t_0)x(t_0)$$

$$\langle x(t), \delta(t - t_0)\rangle = \int_{-\infty}^{+\infty} x(t)\delta(t - t_0)dt = x(t_0) \tag{2.10}$$

Dirac delta definition

- Let ϕ a function supported in $[-1, 1]$ of unit mass: $\int_{-\infty}^{\infty} \phi(u)du = 1$
- $\phi_T(t) = \frac{1}{T}\phi(\frac{t}{T})$ has support on $[-T, T]$ and unit mass.
- We can define the dirac delta δ as

$$\delta(t) = \lim_{T \to 0} \phi_T(t)$$

Delta dirac in practice

- Theoretical object in signal processing (impulse).
- Used to model signal sampling for digital signal processing.
- Used to model point source in Astronomy/image processing, point charge in Physics.
- Has a bounded discrete variant.

The dirac comb

- The dirac comb is expressed as

$$Ш_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{2.11}$$

where Ш is the Cyrilic Sha symbol.

- The Fourier Transform of the dirac comb is

$$\mathcal{F}[Ш_T(t)] = \sum_{k=-\infty}^{\infty} e^{2i\pi kTf} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) = \frac{1}{T} Ш_{\frac{1}{T}}(f) \tag{2.12}$$

where the second equality comes from the Poisson summation formula.

- The dirac comb is used to perform a regular temporal sampling.
- Multiplying a signal by the dirac comb corresponds to a convolution by a dirac comb in the Frequency domain (and vice versa).

2.1.3 Discrete time and digital signals

2.2 Convolution and filtering

2.2.1 Convolution and properties

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Convolution Let two signals $x(t)$ and $h(t)$. The convolution between the two signals is defined as

$$x(t) \star h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau \tag{2.13}$$

- Convolution is a bilinear mapping between x and h.
- It models the relation between the input and the output of a Linear Time Invariant system.
- If $f \in L_1(\mathbb{R})$ and $h \in L_p(\mathbb{R}), p \geq 1$ then

$$\|f \star h\|_p \leq \|f\|_1 \|h\|_p$$

- The dirac delta δ is the neutral element for the convolution operator:

$$x(t) \star \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = x(t) \tag{2.14}$$

$$x(t) \star \delta(t-t_0) = x(t-t_0) \tag{2.15}$$

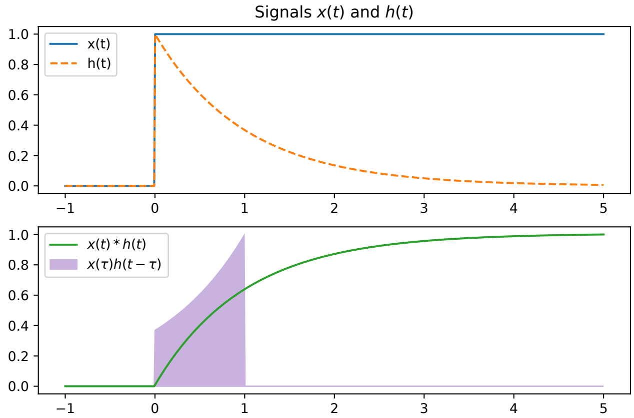


Figure 2.3: Illustration of the convolution operator between the Heaviside step function and a causal decreasing exponential.

Example of convolution

- $x(t) = \Gamma(t)$ the Heaviside step function.
- $h(t) = e^{-t}\Gamma(t)$ the positive part of the decreasing exponential.
- $x(t) \star h(t) = (1 - e^{-t})\Gamma(t)$

2.2.2 Linear Time Invariant (LTI) systems

Linear Time Invariant (LTI) system

- A system describes a relation between an input $x(t)$ and an output $y(t)$.
- Properties of LTI systems:
  - Linearity $\quad x_1(t) + ax_2(t) \rightarrow y_1(t) + ay_2(t)$
  - Time invariance $\quad x(t-\tau) \rightarrow y(t-\tau)$
- A LTI system can most of the time be expressed as a convolution of the form:

$$y(t) = x(t) \star h(t)$$

  where $h(t)$ is called the impulse response (the response of the system to an input $x(t) = \delta(t)$)

Examples

- Passive electronic systems (resistor/capacitor/inductor) .
- Newtonian mechanics, Fluid mechanics, Fourier Optics.

Ordinary Differential Equation (ODE) The system is defined by a linear equation of the form:

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + a_n \frac{d^n y(t)}{dt^n} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_m \frac{d^m x(t)}{dt^m} \quad (2.16)$$

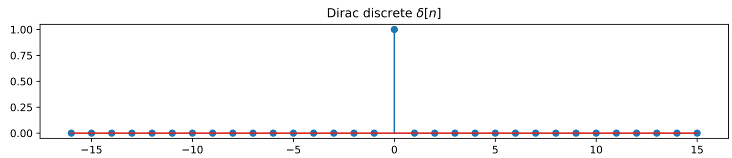
- ODE based system with linear relations are an important class of LTI systems.
- Also called homogeneous linear differential equation.
- n is the number of derivatives for $y(t)$ and m for $x(t)$.
- $\max(m, n)$ is the order of the system.
- The output of the system can be computed from the input by solving Eq. (2.16).
- Linearity and time invariance are obvious from equation.

2.3 Discrete time and digital signals

2.3.1 Discrete time

Notations

- $x(t)$ with $t \in \mathbb{R}$ is the analog signal.
- $x_T(t)$ with $t \in \mathbb{R}$ is the sampled signal of period (T) but still continuous time:
$$x_T(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$
- $x[n]$ with $n \in \mathbb{Z}$ is the discrete signal sampled with period T such that:
$$x[n] = x(nT)$$
- Obviously one can recover $x_T(t)$ from $x[n]$ with
$$x_T(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$
- In order to simplify notations we will suppose $T = 1$ in the following.
- In this course we suppose that $|x[n]|$ is bounded.



Discrete dirac We note the discrete dirac $\delta[n]$ defined as

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{else} \end{cases} \tag{2.17}$$

Discrete signal Any discrete signal $x[n]$ can be decomposed as a sum of translated discrete diracs:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{2.18}$$

The discrete diracs are an orthogonal basis of $L_2(\mathbb{Z})$ of scalar product and corresponding norm

$$< x[n], h[n] >= \sum_{k=-\infty}^{\infty} x[k]h^*[k], \qquad \|x[n]\|^2 =< x[n], x[n] >= \sum_{k=-\infty}^{\infty} |x[k]|^2.$$

Convolution between discrete signals Let $x[n]$ and $h[n]$ two discrete signals. The convolution between them is expressed as:

$$x[n] \star h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2.19}$$

Digital filter properties Let the discrete system/operator/filter L described by its impulse response $h[n]$.

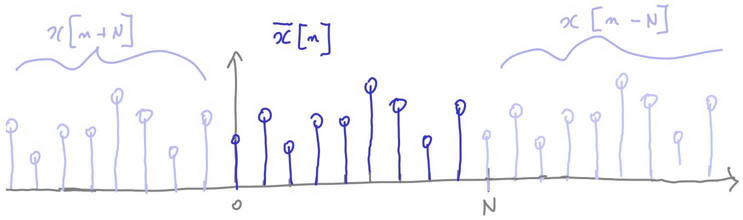
- **Causality** L is causal if $h[n] = 0,\ \forall n \leq 0$. L is causal if

$$h[n] = h[n]\Gamma[n], \qquad \text{where} \qquad \Gamma[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{else} \end{cases} \tag{2.20}$$

- **Stability** A system is stable if the output of a bonded input is bounded. A necessary and sufficient condition is that

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{2.21}$$

2.3.2 Finite signals



Finite discrete signals

- Most of the theoretical results seen up to now correspond to signals $x[n]$ where $n \in \mathbb{Z}$.
- In practice recordings are only done for a finite amount of time resulting to only N samples.
- We defined $\tilde{x}[n]$ a finite signal of N samples with $n \in \{0, \dots, N-1\}$.
- We use in the following the periodization of $\bar{x}[n]$

$$x[n] = \tilde{x}[n \mod N]$$

  where $\mod$ is the modulo operator.

Discrete convolution of finite signals The convolution between $\tilde{x}[n]$ and $\tilde{h}[n]$ both finite signals of n samples can be expressed as:

$$\tilde{y}[n] = \tilde{x}[n] \star \tilde{h}[n] = \sum_{p=-\infty}^{+\infty} \tilde{x}[p]\tilde{h}[n-p] \tag{2.22}$$

- It requires values for the signals outside of the sampling widow.
- One common approach consists in having $\tilde{x}[n]$ and $\tilde{h}[n]$ equal to 0 outside the sampling interval. Other choices can be done (see next slides)

Circular convolution When using the periodic version of the signals the circular convolution can be computed on a unique period of size N:

$$x \circledast h[n] = \sum_{p=0}^{N-1} x[p]h[n-p].$$

The circular convolution is rarely appropriate in real life images due to border effects.

Vector representation and convolution matrix

- Finite signal x of N samples can be represented as a vector $\mathbf{x} \in \mathbb{C}^N$.
- The convolution operator is linear and can be expressed as:

$$\mathbf{y} = \mathbf{x} \star \mathbf{h} = \mathbf{C_h x}$$

  Where $\mathbf{C_h} \in \mathcal{M}_{\mathbb{C}}(N, N)$ is a convolution matrix parametrized by vector $\mathbf{h}$.

Discrete convolution The convolution operator when the values outside the support are 0 can be expressed as

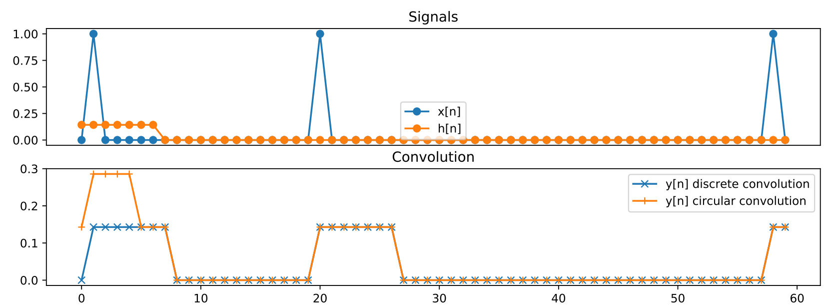
$$\mathbf{C_h} = \begin{bmatrix} h[0] & 0 & \cdots & 0 \\ h[1] & h[0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & \dots & h[0] \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h[N-1] \end{bmatrix}$$

where $\mathbf{C_h} \in \mathcal{M}_{\mathbb{C}}(2 * N - 1, N)$ is a Toeplitz matrix.

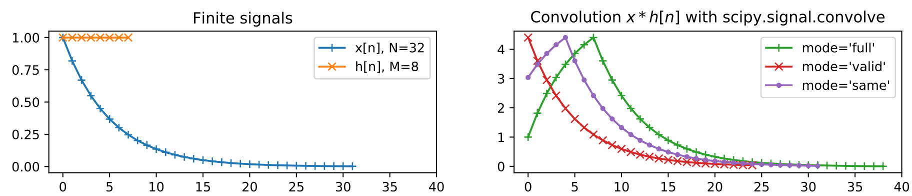
Circular convolution The circular convolution operator can be expressed as

$$\mathbf{C_h} = \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & \cdots & h[2] \\ \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & \dots & h[0] \end{bmatrix}$$

where $\mathbf{C_h} \in \mathcal{M}_{\mathbb{C}}(N, N)$ is a circulant Toeplitz matrix.



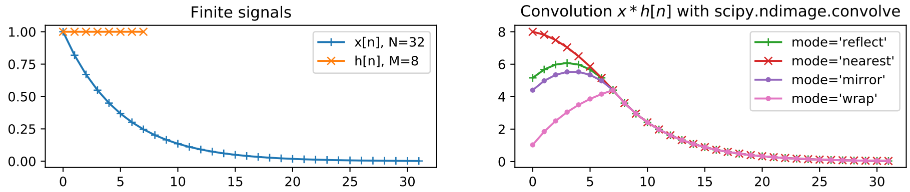
- Convolution between diracs $\tilde{x}[n]$ and a shape $h[n]$ will repeat the shape at the diracs position.
- A dirac at the end of the signal will cut the shape for discrete convolution where the outside of the sampling is 0.
- With circular convolution the shape is repeated t the beginning of the signal.
- One can remove border effects by creating virtual periodic signal with zeros (zero padding, see fast convolution).



The Scipy `scipy.signal.convolve` function:

- Convolution between two signals of support respectively N and M samples supposing that their values are 0 outside of the support.
- The third parameter is `mode` that change the size of the output :
  - `mode='full'` returns a signal of support $N + M - 1$ (default).
  - `mode='valid'` returns a signal of support $|N - M| + 1$ with only the samples that do not rely on zeros padding of the larger signal.
  - `mode='same'` returns a signal of the same size as the first input.

- Parameter `method` allows to choose between `'direct'` computation and `'fft'` and selects the most efficient by default.



The Scipy `scipy.ndimage.convolve` function:

- Always return the same size as the first parameter by default.
- The `mode` parameter allows selecting the borders of a signal $x = (abcd)$:
  - `mode='reflect'` : $(dcba|abcd|dcba)$ (default)
  - `mode='constant'` : $(kkkk|abcd|kkkk)$
  - `mode='nearest'` : $(aaaa|abcd|dddd)$
  - `mode='mirror'` : $(dcb|abcd|cba)$
  - `mode='wrap'` : $(abcd|abcd|abcd)$ (circular convolution)
- Parameter `origin` allows to select the origin of the filter h.

2.3.3 Quantization and storage

2.4 Fundamental signal processing problems

2.4.1 Filtering

2.4.2 Deconvolution, unmixing and regression

2.4.3 Blind source separation and deconvolution

Chapter 3

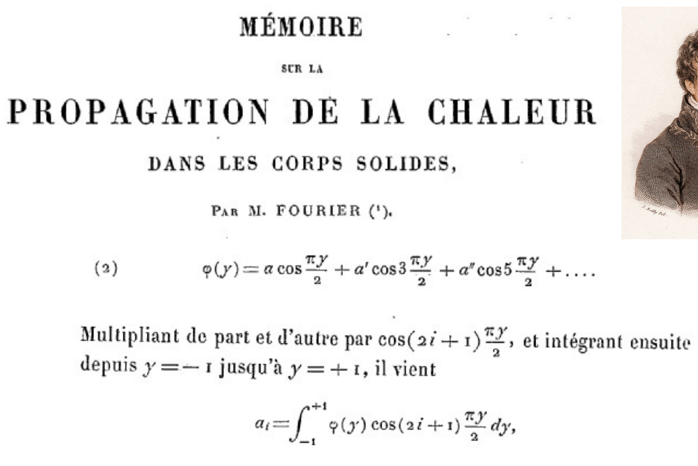
Fourier analysis and analog filtering

- A signal is $x(t)$ a function of time, an image $x(\mathbf{v})$ a function of space.
- Those functions are what we measure/observe but can be hard to interpret/process automatically.
- Another representation for a signal is in the frequency domain $(1/t)$.
- Better representation for numerous applications.

Applications

- Signal processing (biomedical, electrical).
- Image processing (2D signals), filtering, reconstruction.
- Colors are combination of waves of different frequencies.

3.1 Fourier series



MÉMOIRE

SUR LA

PROPAGATION DE LA CHALEUR

DANS LES CORPS SOLIDES,

PAR M. FOURIER (1).

$$(2) \qquad \varphi(y) = a\cos\frac{\pi y}{2} + a'\cos 3\frac{\pi y}{2} + a''\cos 5\frac{\pi y}{2} + \dots$$

Multipliant de part et d'autre par $\cos(2i+1)\frac{\pi y}{2}$, et intégrant ensuite depuis $y = -1$ jusqu'à $y = +1$, il vient

$$a_i = \int_{-1}^{+1} \varphi(y)\cos(2i+1)\frac{\pi y}{2}\,dy,$$

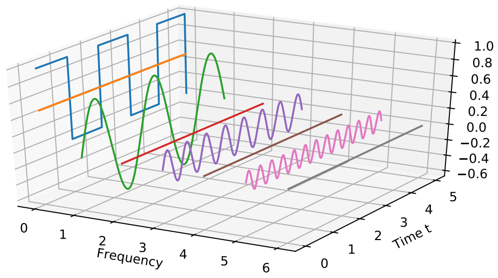


Figure 3.1: Illustration of Fourier series for Example

History

- Trigonometric series used by Euler, d'Alembert, Bernoulli and Gauss.
- Introduced by Joseph Fourier in [Fourier, 1807].
- Fourier claimed that these series could approximate any function.

Decomposition as trigonometric series One can express periodic $x(t)$ of period $T_0 = \frac{2\pi}{w_0}$ integrable on the period as

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kw_0 t) + b_k \sin(kw_0 t)]$$

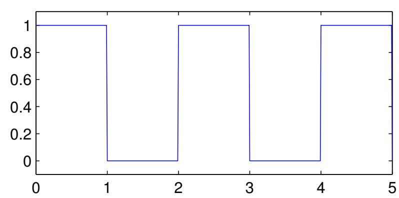
where a_k and b_k are the Fourier coefficients that can be computed as

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(kw_0 t) dt \qquad b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(kw_0 t) dt$$

- Representation of a periodic signal as an infinite number of coefficients corresponding to harmonic frequencies.
- Can be interpreted as a change of basis from temporal to frequencies.
- Functions can be approximated with a finite number N of terms.
- Gibbs phenomenon appears for discontinuous functions [Hewitt and Hewitt, 1979].

Example : Square wave 5cm

- Square wave with $T_0 = 2$



6cm

- $x(t) = \sum_{i=-\infty}^{\infty} 1_{[iT_0, iT_0+T_0/2]}(t)$
- $a_0 = 1, a_k = 0 \quad \forall k > 0$
- $b_k = \frac{2}{\pi k}$ for k odd else $b_k = 0$

Complex harmonic decomposition One can express periodic $x(t)$ of period $T_0 = \frac{2\pi}{w_0}$ integrable on the period as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \qquad \text{avec } w_0 = \frac{2\pi}{T_0}$$

where the coefficients c_k are called the **complex Fourier coefficients** and can be computed with

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jkw_0 t}dt = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jkw_0 t}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jkw_0 t}dt$$

Relations between decompositions Using the Euler formula we can show that a_k and b_k and the c_k coefficients are related by

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k})$$

Note that if $x(t)$ is an even function then the $b_k = 0$, and if $x(t)$ is odd then $a_k = 0$.

3.2 Fourier transform

3.2.1 Definition

The Fourier Transform (FT) of a signal $x(t)$ can be expressed as

$$\mathcal{F}[x(t)] = X(f) = \int_{-\infty}^{\infty} e^{-i2\pi ft}x(t)dt \tag{3.1}$$

When it exists the inverse Fourier transform is defined as

$$\mathcal{F}^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} e^{2i\pi ft}X(f)df \tag{3.2}$$

- Note that the ˆ operator is also often used for the Fourier transform $\hat{x}$ of x.
- In signal processing the references often use j instead of i for the imaginary number (i is a measure of current).
- The FT is a change of representation for the function x from the temporal representation to the harmonic (frequency) representation.

$$x(t) = \int_{-\infty}^{\infty} e^{i2\pi ft}X(f)df$$

Harmonic representation

- The FT represents the signal in the frequency domain.
- $|X(f)|$ is the magnitude of a sinusoidal signal for frequency f.
- $Arg(X(f)$ is the phase of the sinusoidal signal.
- For a real signal $x(t)$, $X(f) = X(-f)^*$ and an informal interpretation would be

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{i2\pi ft}df = \int_{-\infty}^{+\infty} |X(f)|e^{i2\pi(ft+Arg(X(f)))}df \tag{3.3}$$

$$\approx X(0) + 2\int_{0+}^{+\infty} |X(f)|\cos(2\pi(ft + Arg(X(f)))) \tag{3.4}$$

- The modulus and argument of the FT allow identification of the frequency content of the signal and its phase.

Fourier Transform in $L_p(\mathbb{R})$

- For $1 \leq p \leq 2$ the FT maps from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$.
- Consequence of the Riesz–Thorin theorem.
- The TF of an absolute integrable function is bounded (Example : rectangle).

Parseval-Plancherel identity in L_2 The TF of an L_2 function is L_2. Note that L_2 is a Hilbert space of inner product:

$$< x, y >= \int_{-\infty}^{\infty} x(t)y^*(t)dt$$

For two functions $x, y \in L_2(\mathbb{R})^2$ of respective TF $X, Y \in L_2(\mathbb{R})^2$ the Parseval-Plancherel identity states that

$$< x, y >= \int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df \tag{3.5}$$

$$< x, x >= \int_{-\infty}^{\infty} |x(t)|^2dt = \int_{-\infty}^{\infty} |X(f)|^2df \tag{3.6}$$

which means that the energy of a signal is preserved by FT.

More details in [Hunter, 2019, Chap. 5.A] and [Mallat et al., 2015, Chap. 1]

Fourier Transform in $\mathbb{R}^d$ The Fourier Transform can be naturally extended to functions in $\mathbb{R}^d$.

Fourier Tansform in $\mathbb{R}^d$ Let $x(\mathbf{v}) : \mathbb{R}^d \to \mathbb{C}$, the Fourier Transform of x can be expressed as

$$\mathcal{F}[x(\mathbf{v})] = X(\mathbf{u}) = \int_{\mathbb{R}^d} x(\mathbf{v}) e^{-2i\pi<\mathbf{v},\mathbf{u}>} d\mathbf{v} \tag{3.7}$$

When it exists the Inverse FT is defined as

$$\mathcal{F}^{-1}[X(\mathbf{u})] = x(\mathbf{v}) = \int_{\mathbb{R}^d} X(\mathbf{u}) e^{2i\pi<\mathbf{v},\mathbf{u}>} d\mathbf{u} \tag{3.8}$$

- $\mathbf{u} \in \mathbb{R}^d$ is a directional frequency.
- All the properties of the 1D FT are preserved (duality, convolution, ...)
- With $d = 2$, frequency representation of black and white images.
- With large d, approximation for efficient kernel approximation in machine learning [Rahimi and Recht, 2008].

Fourier transform and angular frequency

- The FT in this course is a function of frequency f (in Hz).
- Another common way to represent frequency is the angular frequency w (in rad/s) such that
$$w = 2\pi f, \qquad f = \frac{w}{2\pi}$$
- When using angular frequency the FT is non-unitary meaning that :
$$\tilde{\mathcal{F}}[x(t)] = \tilde{X}(w) = \int_{-\infty}^{\infty} e^{-iwt} x(t) dt$$
$$\tilde{\mathcal{F}}^{-1}[X(f)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} \tilde{X}(w) dw$$
- There exists a unitary angular frequency FT that scales both FT and IFT by $\frac{1}{\sqrt{2\pi}}$.
- In the following we will sometime use the FT as a function of the angular frequency:
$$\tilde{X}(w) = X\left(\frac{w}{2\pi}\right)$$

3.2.2 Examples of Fourier Transform

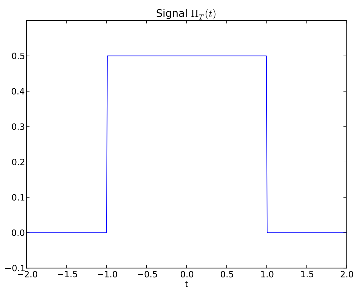
Rectangular function

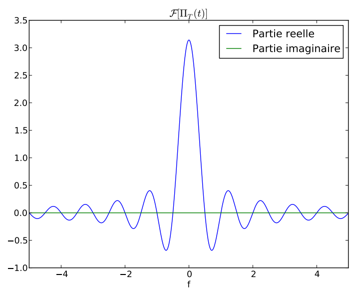
$$\Pi_T(t) = \begin{cases} 1/T & \text{if } |t| < T/2 \\ 1/2T & \text{if } |t| = T/2 \\ 0 & \text{else} \end{cases} \tag{3.9}$$

The Fourier transform is

$$\begin{aligned}
\mathcal{F}[\Pi_T(t)] =& \frac{1}{T}\int_{-T/2}^{T/2} e^{-i2\pi ft}dt \\
=& \left[\frac{-e^{-i2\pi ft}}{i2\pi fT}\right]_{-T/2}^{T/2} \\
=& \frac{e^{i\pi fT} - e^{-i\pi fT}}{i2\pi fT} \\
=& \frac{\sin(\pi fT)}{\pi fT} = \operatorname{sinc}(\pi fT)
\end{aligned}$$

with $\operatorname{sinc}(t) = \frac{\sin(t)}{t}$ and $\operatorname{sinc}(0) = 1$





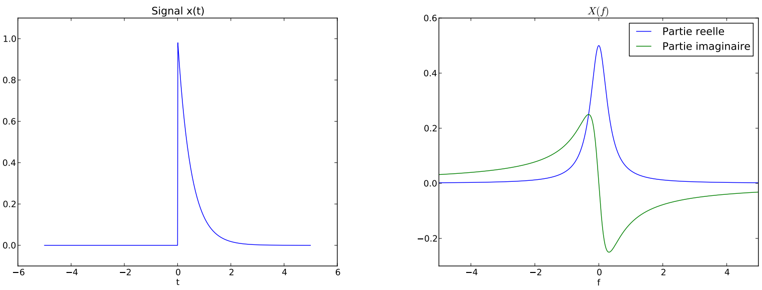
Decreasing exponential

$$x(t) = e^{-at}\Gamma(t), \quad \Gamma(t) = \begin{cases} 1 & \text{for } t > 0 \\ 1/2 & \text{for } t = 0 \\ 0 & \text{else} \end{cases}$$

with $a > 0$

The Fourier transform is

$$\begin{aligned}
\mathcal{F}[e^{-at}\Gamma(t)] =& \int_0^{\infty} e^{-at}e^{-i2\pi ft}dt \\
=& \int_0^{\infty} e^{-(a+i2\pi f)t}dt \\
=& \left[\frac{e^{-(a+i2\pi f)t}}{-(a+i2\pi f)}\right]_0^{\infty} \\
=& \frac{1}{a+i2\pi f}
\end{aligned}$$



3.2.3 Properties of the Fourier Transform

Linearity Ley $x_1(t)$ and $x_2(t)$ two signals of TF $X_1(f)$ and $X_2(f)$ respectively.
For $a \in \mathbb{R}$ and $b \in \mathbb{R}$, we have :

$$\mathcal{F}[ax_1(t) + bx_2(t)] = aX_1(f) + bX_2(f)$$

Comes from the linearity of the integration.

Time shift Let $x(t)$ be a signal of FT $X(f)$.
For $t_0 \in \mathbb{R}$, let $x(t - t_0)$ a time shift of $x(t)$ then we have:

$$\mathcal{F}[x(t - t_0)] = e^{-i2\pi t_0 f} X(f)$$

Change of variable in the integral.

Frequency shift Let $x(t)$ be a signal of FT $X(f)$ then we have

$$\mathcal{F}\left[e^{i2\pi f_0 t} x(t)\right] = X(f - f_0)$$

Multiplication by a complex exponential of frequency f_0, translates the TF by f_0.
Regroup exponentials in the integral.

Time scaling Let $x(t)$ be a signal of FT $X(f)$ and a a scaling $a \neq 0$ then we have

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Change of variable for separate cases $a > 0$ and $a < 0$.

Derivation Let $x(t)$ be a signal of FT $X(f)$ then we have

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = i2\pi f X(f)$$

Integration Let $x(t)$ be a signal of FT $X(f)$ such that $\int_{-\infty}^{\infty} x(t)dt = 0$ then we have

$$\mathcal{F}\left[\int_{-\infty}^{t} x(u)du\right] = \frac{1}{i2\pi f}X(f)$$

If $\int_{-\infty}^{\infty}(x(t) - c)dt = 0$ where c is often called the constant term, we have

$$\mathcal{F}\left[\int_{-\infty}^{t} x(u)du\right] = \frac{1}{i2\pi f}X(f) + c\delta(f)$$

where $\delta(f)$ is the Dirac delta.

Those two properties can be used to solve Ordinary Differential Equations (ODE).

Even and odd signals

$x(t)$	$X(f)$
Even real	Even real
Odd real	Odd imaginary
Even imaginary	Even imaginary
Odd imaginary	Odd real

For a real signal $x(t) : X(f) = X(-f)^*$

Conjugate signal Let $x(t)$ be a signal of FT $X(f)$ and $x^*(t)$ its complex conjugate, then we have

$$\mathcal{F}[x^*(t)] = X^*(-f)$$

Duality of the Fourier Transform Let $x(t)$ be a signal of FT $X(f)$. When the inverse Fourier transform exists we can write

$$x(-t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi f(-t)}df = \int_{-\infty}^{+\infty} X(f)e^{-j2\pi ft}df$$

- The last term is the TF of function $X(f)$.
- This means that if $\mathcal{F}[x(t)] = X(f)$ then

$$\mathcal{F}[X(t)] = x(-f)$$

- Applying twice the TF operator to $x(t)$ returns $x(-t)$: $\mathcal{F}[\mathcal{F}[x(t)]] = x(-t)$

For the rectangular function $\Pi_T(t)$:

$$\Pi_T(t) \to \text{sinc}(\pi fT)$$
$$\text{sinc}(\pi Tt) \to \Pi_T(-f) = \Pi_T(f)$$

Convolution and Fourier Transform

Convolution and Fourier Transform Let two signals $x(t)$ and $h(t)$ of respective Fourier transform $X(f)$ and $H(f)$ then

$$\mathcal{F}[x(t) \star h(t)] = X(f)H(f) \tag{3.10}$$

- The TF of a convolution is a pointwise multiplication in frequency.
- The complex exponential function is the eigenvector for the convolution operator.
- Easy interpretation of the effect of a linear filtering.

Proof

$$\begin{aligned}
\mathcal{F}[x(t) \star h(t)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2i\pi f t} x(u)h(t-u)dudt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2i\pi f (u+v)} x(u)h(v)dudv \\
&= \left\{\int_{-\infty}^{\infty} e^{-2i\pi f u} x(u)du\right\}\left\{\int_{-\infty}^{\infty} e^{-2i\pi f v} h(v)dv\right\} = X(f)H(f)
\end{aligned}$$

with the change of variable $v = t - u$.

Fourier transform and Dirac delta

- Fourier Transform of $\delta(t)$ and $\delta(t-t_0)$:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t)e^{-i2\pi f t}dt = e^0 = 1$$

$$\mathcal{F}[\delta(t-t_0)] = e^{-i2\pi f t_0}$$

- By duality of FT we have:

$$\mathcal{F}[1] = \delta(t)$$
$$\mathcal{F}[e^{i2\pi f_0 t}] = \delta(f-f_0)$$

- Convolution

$$\mathcal{F}[x(t) \star \delta(t)] = 1X(f) = X(f)$$
$$\mathcal{F}[x(t)\delta(t)] = X(f) * 1 = \int_{-\infty}^{\infty} X(f)df = x(0)$$

Fourier transform of periodic signals

Cosine

$$x(t) = \cos(2\pi f_0 t) \qquad \text{with } f_0 > 0$$

- Bounded signal with unbounded energy.
- Intuitively this signal contains only one frequency (f_0)
- Its TF can be computed thanks to the dirac distribution.

FT of trigonometric functions

$$\mathcal{F}\left[\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right] = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

$$\mathcal{F}\left[\sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2i}\right] = \frac{1}{2i}\delta(f - f_0) - \frac{1}{2i}\delta(f + f_0)$$

The FT of sine and cosine is equal to 0 everywhere except on the frequency f_0 of the functions.

Fourier transform of periodic signal Let $x(t)$ be a periodic signal of period T_0, it can be expressed as the following complex Fourier series:

$$x(t) = \sum_k c_k e^{i2\pi \frac{k}{T_0} t}$$

Its Fourier transform can be expressed as

$$X(f) = \mathcal{F}[x(t)] = \sum_k c_k \delta\left(f - \frac{k}{T_0}\right)$$

- The FT of a periodic signal of period is null except on frequencies $\frac{k}{T_0}$, $k \in \mathbb{N}$.
- $\frac{1}{T_0}$ is the fundamental frequency, $\frac{k}{T_0}$ with $|k| \geq 2$ are called the harmonics.
- The TF of a periodic function is a weighted sum of diracs.

How to compute a Fourier Transform ?

Usual steps

1. Use known FT pairs if possible.
2. Express the function as a composition of operations with known properties:
   - Linearity, time shift
   - Convolution
   - Duality
3. Use the properties of FT on the composition.
4. Check properties (FT of even/odd function) to detect easy mistakes.

As a rule : try to avoid computing the integral but sometime you have to do it.

3.3 Frequency response and filtering

3.3.1 Frequency response of an LTI system

Impulse response and frequency response

- Most LTI systems can be expressed as a convolution of the form:

$$y(t) = x(t) \star h(t)$$

  where $h(t)$ is called the impulse response (the response of the system to an input $x(t) = \delta(t)$)

- The Fourier transform of the LTI system relation is

$$Y(f) = H(f)X(f) \tag{3.11}$$

- The frequency response $H(f)$ (also called transfer function) of the LTI system is the Fourier transform of $h(t)$:

$$H(f) = \frac{Y(f)}{X(f)} \tag{3.12}$$

Response to a mono-frequency signal

- For a system of impulse response $h(t)$ with an input $x(t) = e^{2j\pi f_0 t}$

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)e^{2j\pi f_0 h(t-\tau)}d\tau \\ &= e^{2j\pi f_0 t}\int_{-\infty}^{+\infty} h(\tau)e^{-2j\pi f_0 h\tau}d\tau \\ &= e^{2j\pi f_0 t}H(f_0) = x(t)H(f_0) \end{aligned}$$

- An input signal with unique frequency f_0 is multiplied by $H(f_0)$.
- Its amplitude is multiplied by $|H(f_0)|$ and a phase $Arg(H(f_0))$ is added.
- The complex exponential is an eigenvector of the convolution operator.

Static gain The complex static gain is the constant K such that

$$K = H(0) = \int_{-\infty}^{+\infty} h(t)dt$$

Ordinary Differential Equation (ODE) The system is defined by an equation of the form:

$$a_0 y(t) + a_1\frac{dy(t)}{dt} + \cdots + a_n\frac{d^n y(t)}{dt^n} = b_0 x(t) + b_1\frac{dx(t)}{dt} + \cdots + b_m\frac{d^m x(t)}{dt^m} \tag{3.13}$$

Frequency response of an ODE

- We recall the properties of the FT for th n-th derivative of a function:

$$\mathcal{F}\left[\frac{d^{(n)}x(t)}{dt^n}\right] = (2i\pi f)^n X(f) = (iw)^n X(w)$$

- The Frequency response of the ODE can be expressed as

$$H(w) = \frac{Y(w)}{X(w)} = \frac{b_0 + b_1 jw + \cdots + b_m (jw)^m}{a_0 + a_1 jw + \cdots + a_n (jw)^n} \tag{3.14}$$

3.3.2 Representation of the frequency response

Frequency interpretation of the frequency response

- The frequency response of a system gives information on the transformations due to the system.

- Quantities that can be plotted :

$$\begin{aligned} \tilde{H}(w) &= Re(\tilde{H}(w)) + jIm(\tilde{H}(w)) \\ &= |\tilde{H}(w)| e^{jArg(\tilde{H}(w))} \end{aligned}$$

  - $|\tilde{H}(w)|$ modulus of the frequency response.
  - $Arg(\tilde{H}(w)) = \angle\tilde{H}(w) = \tan^{-1}\left(\frac{Im(\tilde{H}(w))}{Re(\tilde{H}(w))}\right)$ phase in radian.

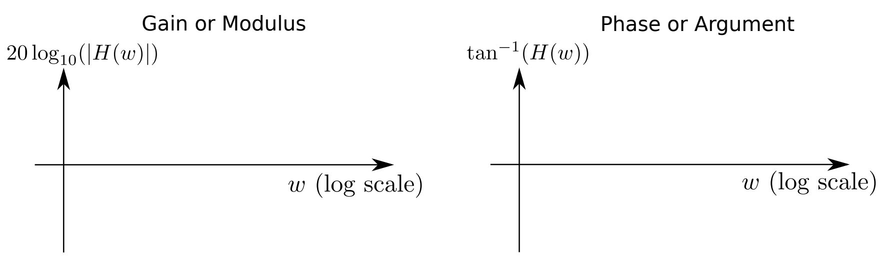
Graphical representation of systems

- Bode plot (Modulus+Argument).

- Nichols/Black plot (Modulus VS Argument).

- Nyquist plot (Real VS Imaginary)

Bode plot



Definition The Bode plot of a system is composed of two plots that are function of w:

- Magnitude (or gain) in decibels (dB)

$$\tilde{G}(w) = 20\log_{10}(|\tilde{H}(w)|)$$

- Phase in degrees or radians

$$\tilde{\Phi}(w) = Arg(H(w)|) = \angle|\tilde{H}(w)|$$

The scale of the radial frequency w is logarithmic, which means that for a rational frequency response H one will be mostly piecewise linear.

Properties of the Bode plot The logarithm and the argument allows for simple diagrams for combination of systems

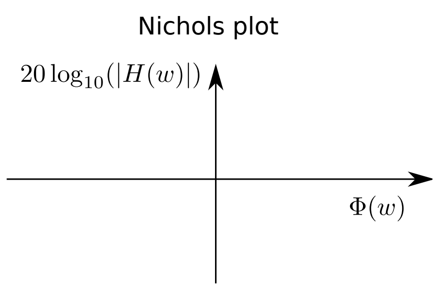
Multiplication If two LTIs $\tilde{H}_1(w)$ and $\tilde{H}_2(w)$ are in series the the equivalent system is $\tilde{H}(w) = \tilde{H}_1(w)\tilde{H}_2(w)$

- $\tilde{G}(w) = \tilde{G}_1(w) + \tilde{G}_2(w)$
- $\tilde{\Phi}(w) = \tilde{\Phi}_1(w) + \tilde{\Phi}_2(w)$

Division If and LTI can be expressed as $\tilde{H}(w) = \dfrac{\tilde{H}_1(w)}{\tilde{H}_2(w)}$ then

- $\tilde{G}(w) = \tilde{G}_1(w) - \tilde{G}_2(w)$
- $\tilde{\Phi}(w) = \tilde{\Phi}_1(w) - \tilde{\Phi}_2(w)$

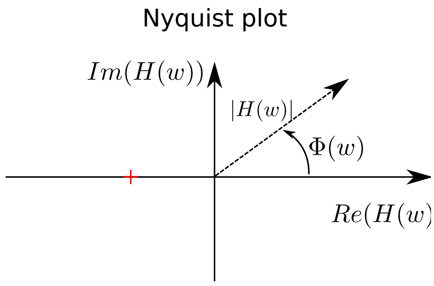
This is particularly useful for rational frequency responses such as ODE.



Nichosl plot The Nichols plot (Diagramme de Black in France) is a parametric plot of $\tilde{H}(w)$ with $20\log_{10}|\tilde{H}(w)|$ on y-axis and phase $\tilde{\Phi}(w)$ on x-axis.

- Show theM odulus/Phase trajectory as a function of w.
- Can be ploted following the Bode plot w.

Nyquist plot



Definition The Nyquist plot is a parametric plot of $\tilde{H}(w)$ with $Real(\tilde{H}(w))$ on x-axis and $Imag(\tilde{\Phi}(w))$ on y-axis.

- Show the trajectory of $\tilde{H}$ in the complex plane.
- Used in system control to study the stability of systems.

Frequency response of electronic systems

Principle Ohm's law can be extended to capacitors and inductors using what is called complex called electrical impedance. The linear system $i(t) \rightarrow u(t)$ is expressed as

$$\tilde{U}(w) = \tilde{H}(w)\tilde{I}(w) = \tilde{Z}(w)\tilde{I}(w)$$

Resistor

- $u(t) = Ri(t)$
- $\tilde{U}(w) = R\tilde{I}(w)$
- $Z_R = R$

Capacitor

- $u(t) = \frac{1}{C}\int_{-\infty}^{t} i(u)du$
- $\tilde{U}(w) = \frac{1}{jCw}\tilde{I}(w)$
- $Z_C = \frac{1}{jCw}$

Inductor

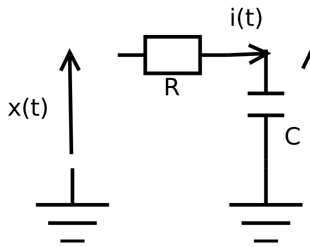
- $u(t) = L\frac{di(t)}{dt}$
- $\tilde{U}(w) = jLw\tilde{I}(w)$
- $Z_L = jLw$

The frequency response of passive electronic systems can be computed with simple computation of complex numbers.

First order system

- System

$$x(t) = Ri(t) + y(t)$$
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(v)dv$$
$$x(t) = RCy'(t) + y(t)$$



- Frequency response

$$\tilde{H}(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + RC2j\pi f}$$

- Using complex impedance

$$\tilde{Y}(w) = Z_c\tilde{I}(w) \quad \text{and} \quad \tilde{X}(w) = (Z_R + Z_C)\tilde{I}(w)$$
$$\tilde{H}(w) = \frac{\tilde{Y}(w)}{\tilde{X}(w)} = \frac{Z_c}{Z_C + Z_R} = \frac{1}{1 + \frac{Z_R}{Z_C}} = \frac{1}{1 + RCjw}$$

Normalized system We reformulate the frequency response as :

$$\tilde{H}(w) = \frac{1}{1 + j\frac{w}{w_0}} \tag{3.15}$$

where $w_0 = \frac{1}{\tau} = \frac{1}{RC}$. **Bode plot**

Modulus

1. $\tilde{H}(w) = \frac{1}{1+j\frac{w}{w_0}}$
2. $|\tilde{H}(w)| = \frac{1}{\sqrt{1+\frac{w^2}{w_0^2}}}$
3. $\tilde{G}(w) = 20\log_{10}(|H(w)|) = -10\log_{10}(1 + \frac{w^2}{w_0^2})$
4. $\lim_{w\to 0}\tilde{G}(w) = 0$
5. $\lim_{w\to\infty}\tilde{G}(w) = -10\log_{10}(\frac{w^2}{w_0^2}) = -20\log_{10}(w) + 20\log_{10}(w_0)$
6. When $w = w_0$, $\tilde{G}(w) = -10\log_{10}(2) = -3dB$

Argument

1. $\tilde{H}(w) = \frac{1}{1+j\frac{w}{w_0}}$

2. $\tilde{\Phi}(w) = \arg(H(w)) = -arg(1+jw) = -tan^{-1}(w)$

3. $\lim_{w\to 0} \tilde{\Phi}(w) = 0$

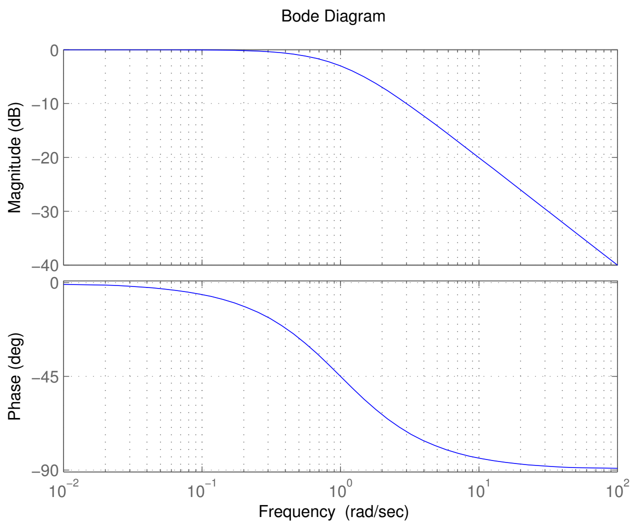
4. $\lim_{w\to\infty} \tilde{\Phi}(w) = -\pi/2$

5. When $w = w_0$, $\tilde{\Phi}(w) = -tan^{-1}(1) = -\pi/4$ $(-45°)$

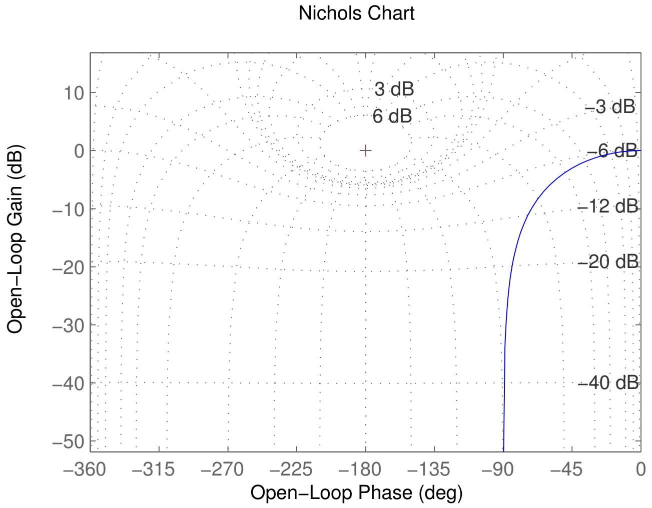
   when $w = 10w_0$, $\tilde{\Phi}(w) = -84°$

   when $w = .1w_0$ $\tilde{\Phi}(w) = -6°$

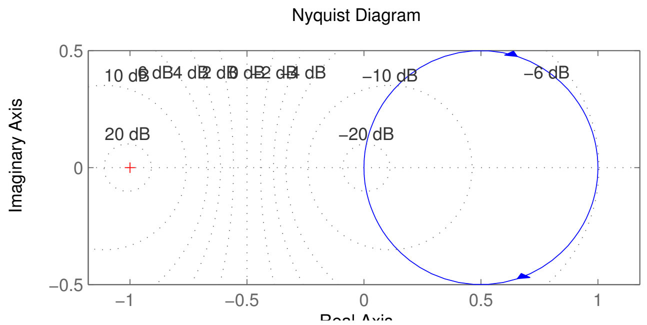
Bode plot



Nichols plot



Nyquist plot

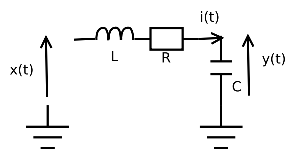


Secon order system 6cm

- Complex Impedance

$$\tilde{Y}(w) = Z_c \tilde{I}(w)$$
$$\tilde{X}(w) = (Z_L + Z_R + Z_C)\tilde{I}(w)$$

5cm



- Frequency response

$$\tilde{H}(w) = \frac{\tilde{Y}(w)}{\tilde{X}(w)} = \frac{Z_C}{Z_L + Z_R + Z_C} = \frac{\frac{1}{jCw}}{\frac{1}{jCw} + R + jLw}$$

- Normalized frequency response

$$\tilde{H}(w) = \frac{1}{1 + RCjw + LC(jw)^2} = \frac{k}{1 + 2z\frac{jw}{w_n} + (\frac{jw}{w_n})^2}$$

  - k Static gain : $k = 1$
  - z damping ratio of the system : $z = \frac{R}{2}\sqrt{\frac{C}{L}}$
  - w_n natural frequency of the system : $w_n = \frac{1}{\sqrt{LC}}$

Linear differential equation The second order differential equation corresponding to the system is

$$\frac{d^2y(t)}{dt^2} + 2zw_n\frac{dy(t)}{dt} + w_n^2 y(t) = kw_n^2 x(t) \tag{3.16}$$

Factorization The second order system can be factorized as

$$\tilde{H}(w) = \frac{kw_n^2}{(jw - c_1)(jw - c_2)} \tag{3.17}$$

with

$$c_1 = -zw_n + w_n\sqrt{z^2 - 1} \tag{3.18}$$

$$c_2 = -zw_n - w_n\sqrt{z^2 - 1} \tag{3.19}$$

c_1 and c_2 are called the poles of the transfer function.

Response of the system for $z > 1$

- c_1 and c_2 are real coefficients.
- The FT can be expressed as

$$\tilde{H}(w) = \frac{M}{jw - c_1} - \frac{M}{jw - c_2} \tag{3.20}$$

  with $M = \frac{w_n}{2\sqrt{z^2-1}}$,
- The impulse response of the system is

$$h(t) = M(e^{c_1 t} - e^{c_2 t})\Gamma(t)$$

- The step response of the system is

$$e(t) = \left(1 + M\left(\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right)\right)\Gamma(t)$$

Response of the system for $z = 1$ The FT becomes:

$$\tilde{H}(w) = \frac{kw_n^2}{(jw + w_n)^2} \tag{3.21}$$

that is the square of one first order system.

The impulse response fo the system can be expressed as

$$h(t) = w_n^2 t e^{-w_n t} \Gamma(t)$$

The step response can be expressed as

$$e(t) = (1 - e^{-w_n t} - w_n t e^{-w_n t}) \Gamma(t)$$

Response of the system for $z < 1$

- In this case the damping is weak and oscillations appear.
- This comes from teh fact that when $z < 1$ coefficients c_1 and c_2 are complex. The impulse response is

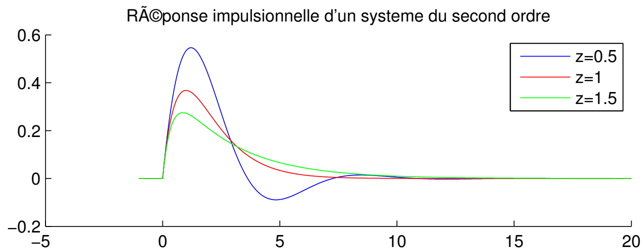
$$h(t) = M(e^{c_1 t} - e^{c_2 t}) \Gamma(t)$$

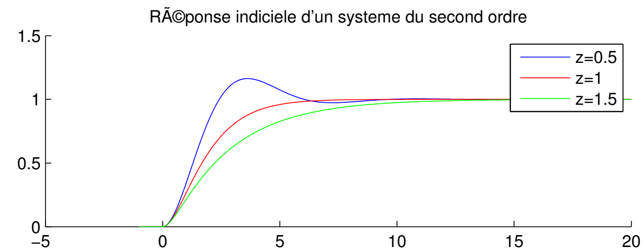
- The step response is

$$h(t) = \frac{w_n e^{-z w_n t}}{\sqrt{1 - z^2}} \sin\left(w_n t \sqrt{1 - z^2}\right) \Gamma(t)$$

  that is a sine with an exponentially decreasing magnitude.

Impulse and step responses





Bode plot We can plot the Bode plot using the normalized frequency response:

$$H(w) = \frac{k}{\left(\frac{jw}{w_n}\right)^2 + 2z\left(\frac{jw}{w_n}\right) + 1} \tag{3.22}$$

Modulus

1. $\tilde{H}(w) = \frac{k}{\left(\frac{jw}{w_n}\right)^2+2z\left(\frac{jw}{w_n}\right)+1}$.
2. $|\tilde{H}(w)| = \frac{k}{\sqrt{\left(1-\left(\frac{w}{w_n}\right)^2\right)^2+4z^2\left(\frac{w}{w_n}\right)^2}}$.
3. $\tilde{G}(w) = 20\log_{10}(|\tilde{H}(w)|) = -10\log_{10}\left(\left(1-\left(\frac{w}{w_n}\right)^2\right)^2 + 4z^2\left(\frac{w}{w_n}\right)^2\right) + 20log(k)$
4. $\lim_{w\to 0}\tilde{G}(w) = 20log(k)$
5. $\lim_{w\to\infty}\tilde{G}(w) = -10\log_{10}(\frac{w^4}{w_n^4}) = -40\log_{10}(w) + 40\log_{10}(w_n)$
6. En $w = w_0$, $\tilde{G}(w) = -20\log_{10}(2z) + 20\log(k)$.

Properties of the modulus

- The modulus of the frequency response for $z < \sqrt(2)/2$ has a maximum at the following frequency

$$w_{max} = w_n\sqrt{1-2z^2}$$

- The value of the modulus at this frequency is

$$|\tilde{H}(w_{max})| = \frac{k}{2z\sqrt{1-z^2}}$$

- The cutoff frequency at -3dB is equal to

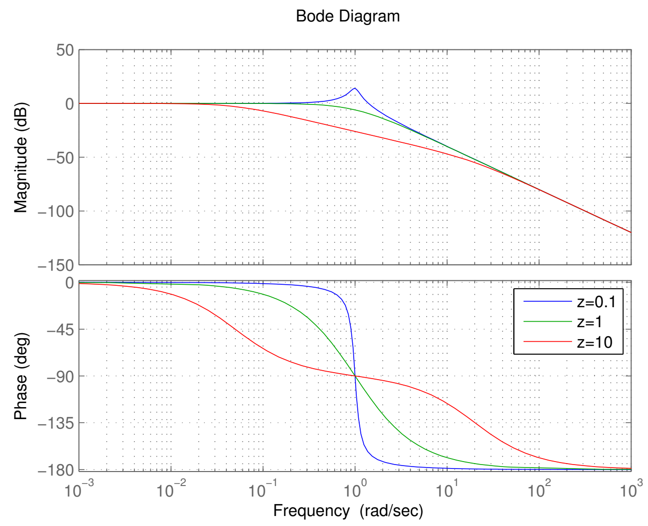
$$w_{-3} = w_n\sqrt{1+2z^2+\sqrt{2-4z^2+4z^4}}$$

Bode plot

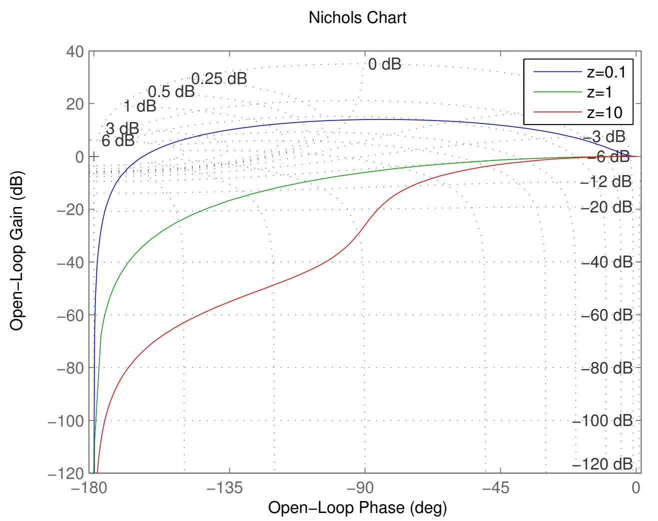
Argument

1. $\tilde{H}(w) = \frac{k}{\left(\frac{jw}{w_n}\right)^2+2z\left(\frac{jw}{w_n}\right)+1}$.
2. $\tilde{\Phi}(w) = \arg(H(w)) = -arg(\left(\frac{jw}{w_n}\right)^2 + 2z\left(\frac{jw}{w_n}\right) + 1) = -tan^{-1}\left(\frac{2z\frac{w}{w_n}}{1-\frac{w^2}{w_n^2}}\right)$.
3. $\lim_{w\to 0}\tilde{\Phi}(w) = 0$
4. $\lim_{w\to\infty}\tilde{\Phi}(w) = -\pi(-180°)$
5. En $w = w_0$, $\tilde{\Phi}(w) = -tan^{-1}(1) = -90°$,

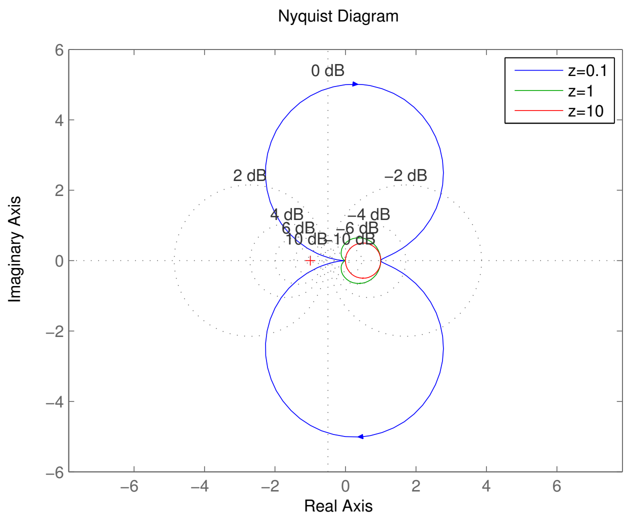
Bode plot



Nichols plot



Nyquist plot



3.4 Applications of analog signal processing

Applications of analog signal processing

- Analog signal filtering.
  - Electronic passive and active filters.
  - Modeling and filtering with physical systems.
- Telecommunications.
  - Amplitude modulation.
  - Multiplexing.
- Fourier opticsL
  - Light propagation in perfect lens/mirror systems.
  - Point spread functions of telescope and cameras.

3.4.1 Analog filtering

3.4.1.1 Properties of analog filters

Definition Signal processing system that aim at selecting part of the signal and attenuating another part (noise).

Analog filtering as opposed to digital filtering (next course)

Objectives

- Find a system that transform a signal $x(t)$ to extract pertinent information.
- Attenuate noise in a signal.
- Separate several components of a signal (when different frequency bands).

Filtering and bandwidth

Gain and Attenuation

- In order to characterize a filter one uses its Gain/Phase (Bode plot).

$$\tilde{G}_{DB}(w) = 20\log_{10}(|\tilde{H}(w)|) \quad \text{et} \quad \tilde{\Phi}(w) = Arg(\tilde{H}(w))$$

- Attenuation is also often used $\tilde{A}(w) = -\tilde{G}_{DB}(w)$

Bandwith and passband The band with of a filter is the set of frequency for which the Gain is over a reference (usually -3dB). Bandwith at $-3dB$:

$$BW = \left\{ w | 20\log\left(\frac{|\tilde{H}(w)|}{\max(|\tilde{H}(w)|)}\right) \geq -3 \right\}$$

Types of filters

- **Low-pass**, $BW = [O, f_c]$ with f_c cutoff frequency
- **High-pass**, $BW = [f_c, \infty]$
- **Band-pass**, $BW = [f_{c_1}, f_{c_2}]$
- **Band-stop**, $BW = [0, f_{c_1}] \cup [f_{c_2}, \infty]$

Filter distortion

Undistorted transmission A signal is considered undistorted when the output of the system is

$$y(t) = Cx(t - t_0)$$

With

- C a constant gain.
- $t_0 > 0$ is a delay.

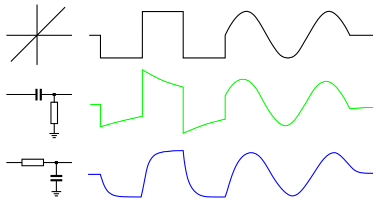
A system with no distortion has the following FT and impulse response

$$\tilde{H}(w) = \frac{\tilde{X}(w)}{\tilde{Y}(w)} = Ce^{-jwt_0} \quad \text{et} \quad h(t) = C\delta(t - t_0)$$

With

- $|\tilde{H}(w)| = C$ or else amplitude distortion.
- $Arg(\tilde{H}(w)) = -wt_0$ or else phase distortion.

Note that the argument of the frequency response varies linearly with the frequency.



Phase distortion Let a system of frequency response

$$\tilde{H}(w) = |H(w)|e^{j\phi(w)}$$

We can deduce that for

$$x(t) = \cos(\omega t)$$
$$y(t) = |\tilde{H}(\omega)|\cos(\omega t + \phi(\omega)) = |\tilde{H}(\omega)|\cos(\omega(t + \phi(\omega)/\omega))$$

The delay $\phi(\omega)/\omega$ is also called **propagation time** of **frequency delay**. For it to be independent from frequency it is necessary that

$$\frac{\phi(\omega)}{\omega} = cte = \tau \quad \rightarrow \quad \phi(\omega) = \omega\tau$$

Ideal low pass filter

Definition

- The ideal low-pass filter is often a theoretical object in signal processing.
- Perfect to use when the noise and signal have non-overlapping spectra.
- The frequency response of the ideal filter is

$$H(f) = \begin{cases} 1 & \text{if } |f| < f_c \\ 0 & \text{else} \end{cases}$$

  where f_c is the cutoff frequency.
- The impulse response of the filter is

$$h(t) = 2f_c \frac{\sin(2\pi f_c t)}{2\pi f_c t} = 2f_c \text{sinc}(2\pi f_c t)$$

Realizable filter

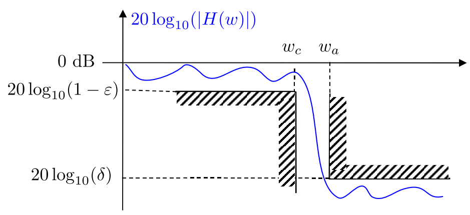
- A realizable temporal filter is **causal** and **stable** (absolute integrable).
- Ideal filter is neither of those and cannot be used for 1D (time) filtering.
- For images (2D) causality is not necessary.

3.4.1.2 Filter design

Real filter

- Ideal filters are non causal and cannot be implemented in practice .
- We search for an approximation of the ideal filter.
- the approximation has to respect **constraints** (Gabarit in french).

Constraints of a filter



Parameters:

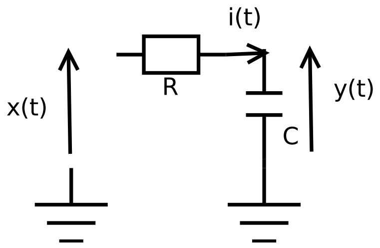
- Bandwidth BW and rejected band:
  - w_c cutoff frequency
  - w_a attenuation frequency
- Oscillations :
  - ε in passing bandwidth
  - δ in attenuated bandwidth

The constraints define the transfer function area that are acceptable for a given application.

Simple example of filter design

- Application to Brain computer interface.
- Interesting signal for event related potentials below $\approx$ 12Hz ($w_s = 2\pi * 12$).
- Electrical noise (EDF) at 50Hz ($w_{edf} = 2\pi * 50$).
- Two low power signals $A_s \approx A_n$.
- Maximum attenuation of signal at -3dB.
- Filtering with first order filter.

5cm



- Frequency response
$$\tilde{H}(w) = \frac{1}{1 + j\frac{w}{w_0}}$$

- Gain in Db
$$\tilde{G}(w) = -10\log_{10}\left(1 + \frac{w^2}{w_0^2}\right)$$

- Before filtering: SNR$= 20\log_{10}\left(\frac{A_s}{A_n}\right) = 0$

- After filtering :

  SNR$= G(w_s) - G(w_{edf})$

- Choice of w_0?

- SNR$= \tilde{G}(w_s) - \tilde{G}(w_{edf})$

- SNR is a decreasing function of w_0 .

What is the best value for w_0?

- With the maximum attenuation of 3dB constraint. $\rightarrow w_s \leq w_0 \leq \infty$.

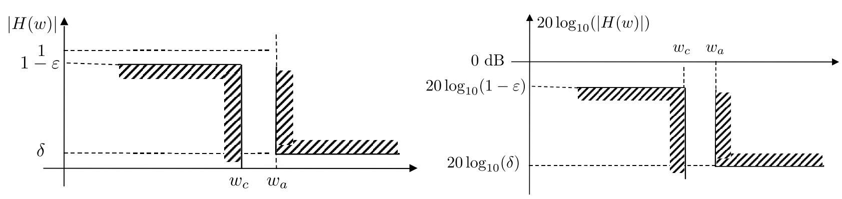
- For $w_0 = w_{edf} \rightarrow R_{S/B} = 2.76dB$

- For $w_0 = (w_{edf} + w_s)/2 = 37 * 2 * \pi \rightarrow R_{S/B} = 4.07dB$

- Pour $w_0 = w_s \rightarrow R_{S/B} = 9.63dB$

$\rightarrow w_0 = w_s$ respects the constraint and maximizes the SNR.

Approximating a low pass filter



Constraints for a low-pass filter

- Passband: $1 - \varepsilon \leq |\tilde{H}(w)| \leq 1 \quad \text{pour} \quad w < w_p$
  - w_p: passing frequency.
  - ε: passband margin parameter ($\varepsilon = 1/2 \rightarrow -3dB$).
- Stopband : $|\tilde{H}(w)| \leq \delta \quad \text{pour} \quad w > w_a$
  - w_a: attenuation frequency.
  - δ: stopband margin parameter.
- $w_a - w_c$ is the transition band.

Additional constraints

- Need for an approximation function that respects the constraints *constrained optimization*.
- Criterion is optimized (for instance maximization of SNR).
- Two approaches are usually used:

Maximally flat frequency response

- Minimal distortion is achieved when the passband is flat.
- Let $|\tilde{H}(w)|$ be the modulus of the frequency response of an order k filter.
- $|\tilde{H}(w)|$ is *maximally flat* in $w = 0$ if all the K^{th}derivatives are null

$$\frac{d^K|\tilde{H}(w)|}{dw^K} = 0$$

Equiripple filter

- A better rollof (sharper decrease) can be achieved at teh cost of oscillations.
- Oscilations can occur in the passband (leading to distortion) of cutband (limited attenuation).
- An equiripple filter has constant magnitude for its oscillations in the bandpass.

3.4.1.3 Classical Analog Filters

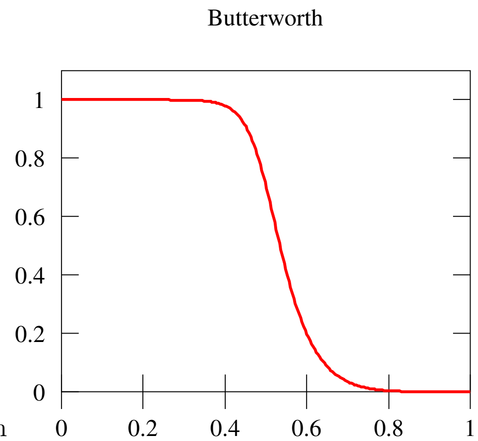
Butterworth filter

- Butterworth filters are *maximally flat* [Butterworth et al., 1930].
- The amplitude of the frequency response can be expressed as 6cm

$$|\tilde{H}(w)| = \frac{1}{\sqrt{1+\left(\frac{w}{w_c}\right)^{2n}}} \tag{3.23}$$

with

  - n: order of the filter.
  - w_c: cutoff frequency.



4cm

- The passing w_p and attenuation w_a frequencies are: 5.5cm For $|\tilde{H}(w)| = 1-\varepsilon$

$$w_p = w_c\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1/2n}$$

5.5cm For $|\tilde{H}(w)| = \delta$

$$w_a = w_c\left(\frac{1-\delta}{\delta}\right)^{1/2n}$$

- The Butterworth filter is monotonically decreasing with the frequency.
- The amplitude of the frequency response can be expressed as

$$|\tilde{H}(w)| = 1 - \frac{1}{2}\left(\frac{w}{w_c}\right)^{2n} + \frac{3}{8}\left(\frac{w}{w_c}\right)^{4n} - \frac{5}{16}\left(\frac{w}{w_c}\right)^{6n} + \ldots$$

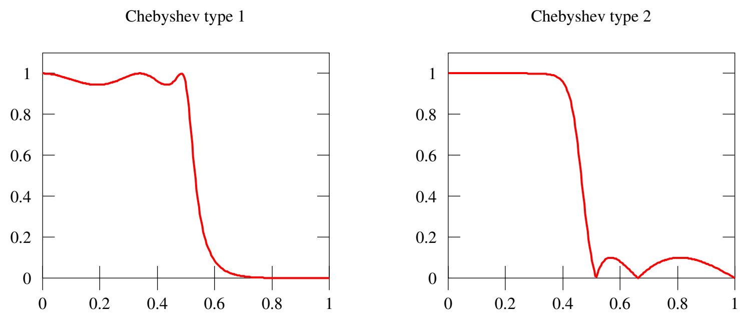
- The derivative in $w=0$ is then null up to order $k = 2n-1$.

- The frequency response of a (normalized) Butterworth filter can be expressed as $\tilde{H}(w) = \frac{1}{B_n(w)}$ where $B_n(w)$ is a Butterworth polynomial :

$$, \quad B_n(w) = \begin{cases} \prod_{k=1}^{\frac{n}{2}} \left[(jw))^2 - 2jw\cos\left(\frac{2k+n-1}{2n}\pi\right) + 1\right] & \text{if } n = \text{ even} \\ (jw+1)\prod_{k=1}^{\frac{n-1}{2}} \left[(jw))^2 - 2jw\cos\left(\frac{2k+n-1}{2n}\pi\right) + 1\right] & \text{if } n = \text{ odd} \end{cases}$$

Order	Polynomial
1	$1 + jw$
2	$(jw)^2 + \sqrt{2}jw + 1$
3	$(jw+1)((jw)^2 + jw + 1)$

Chebyshev filter



- Better rolloff than Butterworth of same order but leads to oscillations in the bandpass (type 1) or in the stopband (type 2).
- *Equiripple* filter.
- Amplitude of the frequency response: 6cm

$$|\tilde{H}(w)| = \frac{1}{\sqrt{1 + \varepsilon^2 T_n^2\left(\frac{w}{w_c}\right)}}$$

5cm

– $T_n(\cdot)$: Chebyshev polynomial of order n.

3.4.2 Filter implementation

Implementation of the filter consist in finding the physical components that recovers the selected frequency response $\tilde{H}(w)$.

Passive filter

- Only passive components (R, C, L).
- No energy source, no amplification (conservation of energy).
- The input and output impedance has an effect on the frequency response (impedance matching).

Active filter

- Use an energy source and Operational Amplifiers (OA).
- OA has near infinite impedance but limited bandwidth (typically 100KHz).
- Saturation can occur (non-linearity).
- Stability can be a problem (due to feedback)

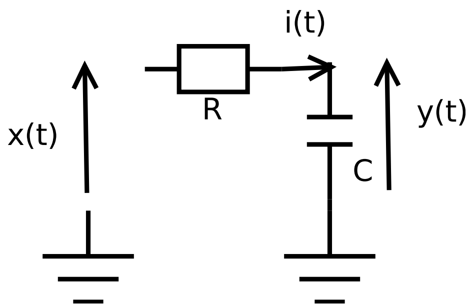
Rarely use inductors in practice (price, resistance, space, mutual inductance) !

Passive filters

Example filter 7cm

- Brain-Computer Interface application.
- $w_0 = w_s = 2\pi * 12$
- $w_0 = \frac{1}{RC} \rightarrow RC = \frac{1}{2*\pi*12} \approx 0.01326$
- What to choose for R and C ?
- Price and space constraints.

5cm



Filter transformation

- **low-pass → high-pass**

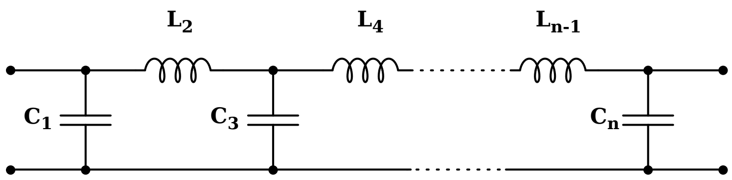
$$1/jCw \rightarrow jLw \quad \text{et} \quad jLw \rightarrow 1/jCw$$

- **low pass → band-pass**

$$1/jCw \rightarrow B/C(jw + 1/jw) \quad \text{et} \quad jLw \rightarrow L/B/(jw + 1/jw)$$

Butterworth Filter

- Corresponding frequency response with the Cauer topology.
- For an order n filter with cutoff frequency $w_c = 1$ the following structure:



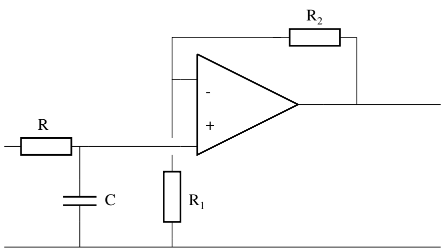
With the values :

- $C_k = 2\sin(\frac{2k-1}{2n}\pi)$ for k odd.
- $L_k = 2\sin(\frac{2k-1}{2n}\pi)$ for k even.

- Assuming the input and output have a 1 Ohm resistance.

Active filters

First order active filter (with amplification)



- Frequency response

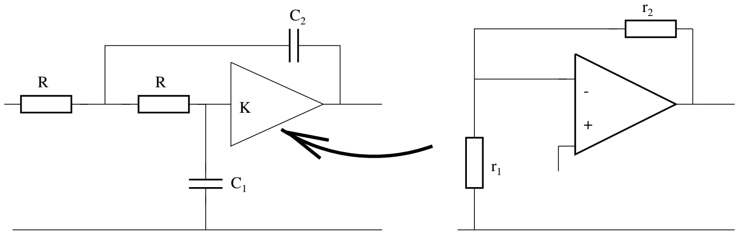
$$\tilde{H}(w) = \frac{A}{1 + \frac{jw}{w_0}}$$

where

$$A = \frac{R_1 + R_2}{R_1} \quad \text{et} \quad w_0 = \frac{1}{RC}$$

- Parameters: R, C, R_1, R_2
- Permute R and C for a high-pass filter.

Second order active filter (Structure from [Sallen and Key, 1955])



- Frequency response

$$\tilde{H}(w) = \frac{K}{1 + \frac{2zjw}{w_n} + \frac{(jw)^2}{w_n^2}}$$

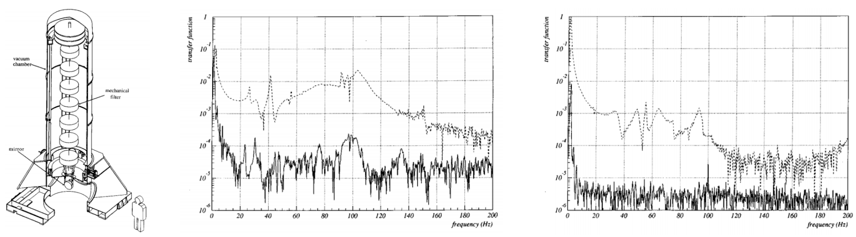
where
$$w_n = \frac{1}{R\sqrt{C_1 C_2}} \quad \text{et} \quad z = \sqrt{\frac{C_1}{C_2}} \frac{3-K}{2} \quad \text{et} \quad K = \frac{r_1 + r_2}{r_1}$$

- Parameters: R, C_1, C_2, r_1, r_2.

Analog filtering : mechanical filter

Virgo Gravitational waves detector [**Acernese et al., 2014**]

- Interferometer to detect gravitational waves.
- Attenuate vibrations from the earth [Braccini et al., 1996]
- Objective : attenuations of 10^{-9} for high frequencies.
- Use a mirror in a chamber with mechanical filters.
- Use a series of mechanical filters for the attenuation.
- Active correction for remaining low frequencies.



3.4.3 Modulation

3.4.4 Fourier optics

Chapter 4

Digital signal processing

4.1 Sampling and Analog/Digital conversion

4.2 Digital filtering and transfer function

4.3 Finite signals and Fast Fourier Transform

4.4 Applications of DSP

Chapter 5

Random signals

5.1 Random Signals and Correlations

5.2 Frequency representation of random signals

5.3 AR modeling and linear prediction

Chapter 6

Signal representations

6.1 Short Time Fourier Transform

6.2 Common signal representations

6.3 Source separation and dictionary learning

6.4 Machine learning for signal processing

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