Lista 6 - MC I

Exercício 1

b) $Y_i | X_i, \theta \sim \text{Pois}(\theta . X_i)$, X_i conhecido.

$$l(\theta | \underset{\sim}{y}, \underset{\sim}{x}) = \log[L(\theta | \underset{\sim}{y}, \underset{\sim}{x})] = \sum_{i=1}^{n} l(\theta | y_i, x_i)$$

$$= \sum_{i=1}^{n} \left\{ -\theta x_i + y_i[\log(\theta) + \log(x_i)] - \log(y_i!) \right\}$$

$$\frac{d\, l(\theta | \underset{\sim}{y}, \underset{\sim}{x})}{d\theta} = \sum_{i=1}^{n} \left[-x_i + \frac{y_i}{\theta} \right]$$

Tradicionalmente, o EMV é obtido ao igualar o escore a 0.

$$\frac{dl}{d\theta} = 0 \Rightarrow -n\bar{x} + \frac{n\bar{y}}{\theta} = 0 \Rightarrow \hat{\theta} = \frac{\bar{y}}{\bar{x}}.$$

Portanto, o método escore de Fisher tende a $\frac{\bar{y}}{\bar{x}}$.

$$I(\theta) = -E\left[\frac{d^2 l}{d\theta^2}\right] = \sum_{i=1}^{n} \frac{E[y_i]}{\theta^2} = \sum_{i=1}^{n} \frac{\theta x_i}{\theta^2} = \frac{n\bar{x}}{\theta}$$

Exercício 2

c) $$P(Z_i = 1 | X_i = x_i, \underset{\sim}{\theta}) = \frac{f_{X_i}(x_i | Z_i = 1, \underset{\sim}{\theta}) . \overbrace{P(Z_i = 1 | \underset{\sim}{\theta})}^{\text{não depende de } \underset{\sim}{\theta}}}{\sum_{j=0}^{1} f_{X_i}(x_i | Z_i = j, \underset{\sim}{\theta}) . P(Z_i = j | \underset{\sim}{\theta})}$$

$$= \frac{p . \exp\left\{-\frac{1}{2\sigma^2}(x_i - \theta_1)^2\right\}}{(1-p)\exp\left\{-\frac{1}{2\sigma^2}(x_i - \theta_0)^2\right\} + p . \exp\left\{-\frac{1}{2\sigma^2}(x_i - \theta_1)^2\right\}}$$

$$= \frac{1}{\frac{(1-p)}{p} \exp\left\{-\frac{1}{2\sigma^2}\left[(x_i - \theta_0)^2 - (x_i - \theta_1)^2\right]\right\} + 1}$$

Para $\sigma \to 0$, i) Se $|x_i - \theta_1| < |x_i - \theta_0|$, $P(Z_i = 1 | X_i, \underset{\sim}{\theta}) = 1$

ii) Se $|x_i - \theta_1| > |x_i - \theta_0|$, $P(Z_i = 1 | X_i, \underset{\sim}{\theta}) = 0$

$$\log\left[f(\underset{\sim}{X},\underset{\sim}{Z}\mid\underset{\sim}{\theta})\right] = \sum_{i=1}^{m} \log f(X_i, Z_i \mid \theta)$$

$$= -\frac{m}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{m}\left\{ z_i \cdot \left[\log(p) - \frac{1}{2\sigma^2}(x_i-\theta_1)^2\right] + (1-z_i)\cdot\left[\log(1-p) - \frac{1}{2\sigma^2}(x_i-\theta_0)^2\right]\right\}$$

$$E_{\underset{\sim}{Z}\mid \underset{\sim}{X}\underset{\sim}{\theta}^{(t)}}\left\{\log\left[f(\underset{\sim}{X},\underset{\sim}{Z}\mid\underset{\sim}{\theta})\right]\right\} = -\frac{m}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{m}\left\{ \underbrace{\boxed{P(Z_i=1\mid x_i, \underset{\sim}{\theta}^{(t)})}}_{p_{1i}} \cdot \left[\log p - \frac{1}{2\sigma^2}(x_i-\theta_1)^2\right] + \underbrace{\boxed{P(Z_i=0\mid x_i, \underset{\sim}{\theta}^{(t)})}}_{p_{0i}} \cdot \left[\log(1-p) - \frac{1}{2\sigma^2}(x_i-\theta_0)^2\right]\right\}$$

$$\frac{dE}{d\theta_0} = \sum_{i=1}^{m} \frac{p_{0i}^{(t)}(x_i-\theta_0)}{\sigma^2} = 0 \Rightarrow \theta_0^{(t+1)} = \frac{\sum_{i=1}^{m} p_{0i}^{(t)} x_i}{\sum_{i=1}^{m} p_{0i}^{(t)}}$$

$$\frac{dE}{d\theta_1} = \sum_{i=1}^{m} \frac{p_{1i}^{(t)}(x_i-\theta_1)}{\sigma^2} = 0 \Rightarrow \theta_1^{(t+1)} = \frac{\sum_{i=1}^{m} p_{1i}^{(t)} x_i}{\sum_{i=1}^{m} p_{1i}^{(t)}}$$

Quando $\sigma \to 0$, $\theta_K^{(t+1)} = \dfrac{\sum_{i=1}^{m} \mathbb{I}[Z_i^{(t)}=K]\, x_i}{\sum_{i=1}^{m} \mathbb{I}[Z_i^{(t)}=K]}$

Exercício 3

$$P(Z_i=1\mid x_i, \underset{\sim}{\theta}) = \frac{f_{X_i}(x_i\mid\theta_1)\, p}{f_{X_i}(x_i\mid\theta_0)\cdot(1-p) + f_{X_i}(x_i\mid\theta_1)\cdot p}$$

$$= \frac{p\,\theta_1 \exp\{-\theta_1 x_i\}}{(1-p)\,\theta_0 \exp\{-\theta_0 x_i\} + p\,\theta_1\exp\{-\theta_1 x_i\}}$$

$$= \frac{1}{\frac{(1-p)\theta_0}{p\,\theta_1}\cdot\exp\{-(\theta_0-\theta_1)x_i\} + 1} = p_{1i}$$

$$\log\left[f(\underset{\sim}{X},\underset{\sim}{Z}\mid\underset{\sim}{\Theta})\right]=\sum_{i=1}^{m}\log f(x_i,z_i\mid\underset{\sim}{\Theta})$$

$$=\sum_{i=1}^{m}\Big\{z_i\cdot\left[\log(p)+\log(\Theta_1)-\Theta_1 x_i\right]+$$

$$+(1-z_i)\cdot\left[\log(1-p)+\log(\Theta_0)-\Theta_0 x_i\right]\Big\}$$

$$\underset{\underset{\sim}{Z}\mid\underset{\sim}{X},\Theta^{(t)}}{E}[\ell]=\sum_{i=1}^{m}\left\{p_{1i}^{(t)}\cdot\left[\log(p)+\log(\Theta_1)-\Theta_1 x_i\right]+p_{0i}^{(t)}\cdot\left[\log(1-p)+\log(\Theta_0)-\Theta_0 x_i\right]\right\}$$

$$\frac{dE}{d\Theta_0}=\sum_{i=1}^{m}p_{0i}^{(t)}\cdot\left[\frac{1}{\Theta_0}-x_i\right]=0\Rightarrow\Theta_0^{(t+1)}=\frac{\sum_{i=1}^{m}p_{0i}^{(t)}}{\sum_{i=1}^{m}p_{0i}^{(t)}x_i}$$

$$\frac{dE}{d\Theta_1}=0\Rightarrow\Theta_1^{(t+1)}=\frac{\sum_{i=1}^{m}p_{1i}^{(t)}}{\sum_{i=1}^{m}p_{1i}^{(t)}x_i}$$