EEOR 4650: Homework 3

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Question 4.3
First, the optimization problem is a covnex optimization problem. The objective function is $f_0(x) = \frac{1}{2}x^T Px + q^T x + r$, and its Hessian matrix is $\mathbf{H}f_0 = P$.
With

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 13x^2 + 17y^2 + 12c^2 + 24xy + 12yz - 4xz$$
$$= 2(\frac{12}{5}x + \frac{5}{2}y)^2 + 2(\frac{3}{2}y + 2z)^2 + (x - 2z)^2 + \frac{12}{25}x^2$$
$$\geq 0$$

P is a positive definite, so f_0 is a convex function. It is easy to see that the interval constraints also form a convex set.
Note that $\nabla f_0(x^\star) = Px^\star + q^T = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}^T$, so we have $\nabla f_0(x^\star)(y - x^\star) = 2y_3 - y_1 + 3 \geq 0$ for any feasible y. $x^\star$ is feasible, so it is the optimal solution.

Question 4.11
Denote $A = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^T$, where $a_1, \ldots a_m \in \mathbf{R}^n$.
(a)
$\|Ax - b\|_\infty = \max(|a_1^T x - b_1|, \ldots, |a_m^T x - b_m|)$, let $-t_i \leq a_i^T x - b_i \leq t_i$ for $i = 1, \ldots, m$, we have the LP form

$$\begin{aligned} \min_{x,t,z} \quad & z \\ \text{s.t.} \quad & -t_i \leq a_i^T x - b_i \leq t_i, \\ & t_i \leq z \end{aligned}$$

The optimal solution $e^\star$ of the equivalent LP must equal to the solution of the original programming, because for any given x_0, $e^\star = \|Ax_0 - b\|_\infty$, otherwise it would not be optimal, because $-t_i \le a_i^T x - b_i \le t_i$ and $t_i \le z$ guarantee that $z \ge |a_i^T x - b_i|$ and $z \ge \|Ax_0 - b\|_\infty$ and if $e^\star > \|Ax_0 - b\|_\infty$, it definitely could be smaller without violating any constraint. So if optimizing over all x, t and z, $e^\star$ would equal to the solution of the original problem. The x part optima of both programming will be the same.

(b)

$\|Ax - b\|_1 = \sum_{i=1}^m |a_i^T x - b_i|$, let $-t_i \le a_i^T x - b_i \le t_i$ for $i = 1, \ldots, m$, we have the LP form

$$\begin{array}{ll} \min\limits_{x,t} & \sum\limits_{i=1}^m t_i \\ \text{s.t.} & -t_i \le a_i^T x - b_i \le t_i \end{array}$$

For any given x_0, the optimal solution of the equivalent LP satisfies that $t_i = |a_i^T x_0 - b_i|$, otherwise it would not be optimal, because $-t_i \le a_i^T x - b_i \le t_i$ make sure that $t_i \ge |a_i^T x - b_i|$ and $\sum_{i=1}^m t_i \ge \|Ax - b\|_1$. The objective is just $\|Ax_0 - b\|_1$. So when optimizing over x and t, optima of the equivalent LP will be the optima of the original problem.

(c)

$\|Ax - b\|_1 = \sum_{i=1}^m |a_i^T x - b_i|$, let $-t_i \le a_i^T x - b_i \le t_i$ for $i = 1, \ldots, m$, and $-z_i \le x_i \le z_i$ with $z_i \le 1$ for $i = 1, \ldots, n$ we have the LP form

$$\begin{array}{ll} \min\limits_{x,t,z} & \sum\limits_{i=1}^m t_i \\ \text{s.t.} & -t_i \le a_i^T x - b_i \le t_i, \\ & -z_i \le x_i \le z_i, \\ & z_i \le 1 \end{array}$$

For the equivalen t LP, with $-z_i \le x_i \le z_i$ and $z_i \le 1$, all the feasible x will satisfy that $\|x\|_\infty \le 1$. And all the x with $\|x\|_\infty \le 1$ also satisfies these constriants, so they are equivalent. From (b), we see that constriants $-t_i \le a_i^T x - b_i \le t_i$ can make the objective to be $\|Ax - b\|_1$. Therefore, the optima x for the equivalent LP will also be the optima of the original problem.

(d)

$\|x\|_1 = \sum_{i=1}^m |x_i|$, let $-z_i \le x_i \le z_i$ for $i = 1, \ldots, n$ and $-t_i \le a_i^T x - b_i \le t_i$ with $t_i \le 1$ for $i = 1, \ldots, m$, we have the LP form

$$\begin{array}{ll} \min\limits_{x,t,z} & \sum\limits_{i=1}^n z_i \\ \text{s.t.} & -t_i \le a_i^T x - b_i \le t_i, \\ & -z_i \le x_i \le z_i, \\ & t_i \le 1 \end{array}$$

Similar to (c), $t_i \le 1$ and $-t_i \le a_i^T x - b_i \le t_i$ make the feasible x equivalent to be $\|Ax - b\|_\infty \le 1$; $-z_i \le x_i \le z_i$ and the objective is to guarantee the optimal value of the equivalent LP equal the optimal value of the original programming. And two problems have the same x optima.

(e)

Let $-z_i \le x_i \le z_i$ for $i = 1, \dots, n$ and $-t_i \le a_i^T x - b_i \le t_i$ with $t_i \le t_0$ for $i = 1, \dots, m$, we have the LP form

$$
\begin{aligned}
\min_{x,t,z} \quad & \sum_{i=1}^{n} t_i + z_0 \\
\text{s.t.} \quad & -t_i \le a_i^T x - b_i \le t_i, \\
& -z_i \le x_i \le z_i, \\
& z_i \le z_0
\end{aligned}
$$

For given x_0, the optimal solution of the equivalent LP will satisfy that $\sum_{i=1}^{n} t_i = \|Ax - b\|_1$ and $z_0 = \|x\|_\infty$, otherwise, they could be smaller and would not be optimal. So two problems have the same x optima and optimal value.

Question 3

First, $p^\star = v^T b$ if and only if there exists $v \in \mathbf{R}^m$ such that $c = A^T v$ ($c \in \text{Row}(A)$) and $b \in \text{Col}(A)$. When there exists v such that $c = A^T v$, $x^T c = x^T A^T v = (Ax)^T v = b^T v$. It satisfies the optimal condition, so $p^\star = v^T b$.

Second, $p^\star = +\infty$ is equivalent to that there is no feasible point for the programming, which could happen if and only if $Ax = b$ has no solution. This would happen if and only if $b \notin \text{Col}(A)$.

Third, $p^\star = -\infty$ if and only if $c \notin \text{Row}(A)$ and $b \in \text{Col}(A)$. $p^\star = -\infty$ means that any x is not optimal point of the programming. And thus for any x, the optimal condition that there exists v such that $c = A^T v$ is not satisfied. When $c \notin \text{Row}(A)$, we can infer that $c \ne 0$ and $\mathbf{Rank}\, A < n$. If $\mathbf{Rank}\, A = n$, it must be that $c \in \text{Row}(A)$. Therefore $Ax = b$ has infinite solutions and these solutions can be expressed as $x = z + x_0$, where $Ax_0 = b$ and $z \in \text{Nul}A$. Also note that c can be expressed uniquely as $m + n$ where $m \in \text{Row}(A)$ and $n \in \text{Row}(A)^\perp = \text{Nul}(A)$. Also, $n \ne 0$ and otherwise $c \in \text{Row}(A)$. Then we have $c^T x = c^T(z + x_0) = c^T x_0 + c^T z = c^T x_0 + (m+n)^T z = c^T x_0 + n^T z$, we can take $z = tn$ for any $t \in \mathbf{R}$, so that $c^T x = c^T x_0 + t n^T n$ can be arbitrarily small, namely, $p^\star = -\infty$.

Any condition can be divided into exactly one of these three cases.

Question 4.28

(a)

It can be expressed as

$$
\begin{aligned}
\min_{x,t} \quad & t \\
\text{s.t.} \quad & \frac{1}{2} x^T P_i x + q^T x + r \le t, \\
& Ax \preceq b
\end{aligned}
$$

For any x_0, $\frac{1}{2}x^T P_i x + q^T x + r \leq t$ makes sure that $t \geq \sup_{P\in\mathcal{E}} \frac{1}{2}x^T P x + q^T x + r$, minimizing over x and t is equivalent to the original problem. The new form is QCQP. (b)
When $\mathcal{E} = \{P \in \mathbf{S}^n \mid -\gamma I \preceq P - P_0 \preceq \gamma I\}$, $\sup_{P\in\mathcal{E}} \frac{1}{2}x^T P x + q^T x + r = \frac{1}{2}x^T(P_0+\gamma I)x + q^T x + r$. This is because for any $P \in \mathcal{E}$, $P - P_0 \preceq \gamma I$ so $P_0 + \gamma I - P$ is positive semidefinite, and thus $x^T(P_0 + \gamma I - P)x \geq 0$, namely, $x^T(P_0 + \gamma I)x \geq x^T P x$. It becomes equality when $P = P_0 + \gamma I$.
The programming can be expressed as

$$\begin{array}{ll} \min\limits_{x} & \frac{1}{2}x^T(P_0 + \gamma I)x + q^T x + r \\ \text{s.t.} & Ax \preceq b \end{array}$$

which is a QP.
(c)
When $\mathcal{E} = \{P = P_0 + \sum_{i=1}^K P_i u_i \mid \|u\|_2 \leq 1\}$, $\sup_{P\in\mathcal{E}} \frac{1}{2}x^T P x + q^T x + r = \frac{1}{2}x^T P_0 x + q^T x + r + \frac{1}{2}\max(|x^T P_1 x|, \cdots, |x^T P_K x|)$. Because $\sum_{i=1}^K x^T P_i x u_i \leq \sum_{i=1}^K |x^T P_i x||u_i| \leq \max_i |x^T P_i x|$. We can make $-t \leq x^T P_i x \leq t$ for $i = 1, \ldots, K$ and get

$$\begin{array}{ll} \min\limits_{x,t} & \frac{1}{2}x^T P_0 x + q^T x + r + \frac{1}{2}t \\ \text{s.t.} & Ax \preceq b, \\ & -t \leq x^T P_i x \leq t \end{array}$$

which is a QCQP.

Question 5.1
$f(x) = \|Ax - b\| - 2^2 = (Ax - b)^T(Ax - b) = x^T A^T A x - 2b^T A x + b^T b$, $\mathbf{H}f(x) = A^T A$ is a positive semidefinite, so $f(x)$ is convex function. Obviously $f(x) \geq 0$, it is bounded below so it must have an infimum. The optimal solution must satisfy that $\nabla f(x) = 2A^T A x - 2A^T b = 0$. And this gives only one point $x = (A^T A)^{-1} A^T b$ when $A^T A$ is full rank. Because $f(x)$ is convex, This point must be the minimum solution, otherwise $f(x)$ will have no minimum.

Question 5.2
(a)
Let $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, $A \in \mathbf{R}^{N\times 3}$ and $b \in \mathbf{R}^N$.

With $b_i = u_i^2 + v_i^2$ and row i, of A being $a_i = \begin{bmatrix} 2u_i & 2v_i & -1 \end{bmatrix}$, we have

$$\begin{aligned} \|Ax - b\|_2^2 &= \sum_{i=1}^{N} (a_i x - b_i)^2 \\ &= \sum_{i=1}^{N} (2u_i x_1 + 2v_i x_2 - x_3 - u_i^2 + v_i^2)^2 \\ &= \sum_{i=1}^{N} (2u_i u_c + 2v_i v_c + R^2 - u_c^2 - v_c^2 - u_i^2 - v_i^2)^2 \\ &= \sum_{i=1}^{N} ((u_i - u_c)^2 + (v_i - v_c)^2 - R^2)^2 \end{aligned}$$

(b)
If $x_1^2 + x_2^2 - x_3 < 0$, $R^2 = x_3 - x_1^2 - x_2^2 < 0$, which means the change of variables is invalid.
(c)
$p^\star = 43.1149257697742$, $x = \begin{bmatrix} -3.10 & 2.69 & -1.70 \end{bmatrix}^T$, $u_c = -3.099108821789537$, $v_c = 2.694099245160927$, $R = 4.3090792363282$.