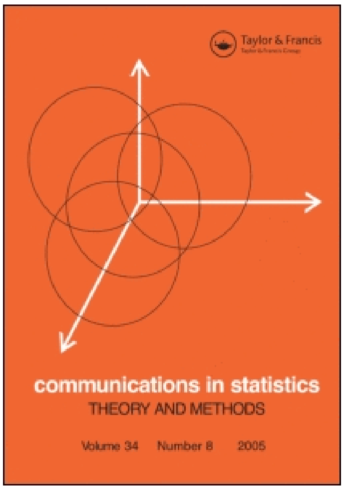

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Time series models with asymmetric innovations

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TIME SERIES MODELS WITH ASYMMETRIC INNOVATIONS

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Key Words: Time series; gamma distribution; generalized logistic; nonnormality; robustness; modified likelihood; hypothesis testing; power function.

ABSTRACT

We consider $AR(q)$ models in time series with asymmetric innovations represented by two families of distributions: (i) gamma with support $\mathbb{R} : (0, \infty)$, and (ii) generalized logistic with support $\mathbb{R} : (-\infty, \infty)$. Since the ML (maximum likelihood) estimators are intractable, we derive the MML (modified maximum likelihood) estimators of the parameters and show that they are remarkably efficient besides being easy to compute. We investigate the efficiency properties of the classical LS (least squares) estimators. Their efficiencies relative to the proposed MML estimators are very low.

AMS Classification. 62M10

1 INTRODUCTION

In time series models of the type $y_t = \sum_{j=1}^{q} \phi_j y_{t-j} + \varepsilon_t$, it is usual practice to assume that the innovations ε_t are iid (identically and independently distributed) normal $N(0, \sigma^2)$. In recent years, however, it has been recognized that this normality assumption is too restrictive from applications point of view; see, for example, Huber (1981), Tiku et al. (1986), Choi and Wette (1968), Davies et al. (1980), Granger (1979), Greenwood and Durand (1960), Ledolter (1979) and Li and McLeod (1988). In this paper we consider two important families of asymmetric distributions (i) gamma and (ii) generalized logistic. The method proposed in this paper can, however, be used for any other location-scale distribution of the type $(1/\sigma)f((y-\mu)/\sigma)$. Since the ML (maximum likelihood) estimators are intractable under these distributional assumptions, we derive MML (modified maximum likelihood) estimators; see, for example, Tiku (1967, 1968, 1988) and Tiku and Suresh (1992). In the classical framework of iid random observations, the MML estimators are known to be asymptotically fully efficient (Tiku 1970, Bhattacharyya 1985, Vaughan and Tiku 1999) and almost fully efficient for small sample sizes (Smith et al. 1973, Lee et al. 1980, Tan 1985, Tiku and Suresh 1992, Vaughan 1992). We derive the MML estimators in the present context; they are explicit functions of sample observations and are, therefore, easy to compute. We also show that they are remarkably efficient. We study the LS (least squares) estimators and their efficiencies. The efficiencies of the LS estimators relative to the proposed MML estimators are very low. We also develop procedures for testing the null hypothesis $H_0 : \phi = 0$. Two-moment normal and three-moment chi-square approximations are given and shown to be very effective in providing accurate values for the percentage points of the null distributions of the proposed test statistics. Testing $\phi = 1$ (unit root problem) is also of great importance. That will be considered in a future paper. See also Tiku and Wong (1998) who give a test of $\phi = 1$ for symmetric distributions.

2 AR(1) MODEL

In the first place, consider the model

$$y_i = \phi y_{i-1} + \varepsilon_i \qquad (i = 1, 2, 3, \ldots, n) \tag{1}$$

where $0 \le \phi < 1$ and ε_i's are iid and have the gamma distribution

$$f(\varepsilon; k) \propto \frac{1}{\sigma^k}\, e^{-\varepsilon/\sigma} \varepsilon^{k-1} \qquad (0 < \varepsilon < \infty); \tag{2}$$

$y_0 = \varepsilon_0/\sqrt{(1-\phi^2)}$, where ε_0 has the distribution (2), and is independent of ε_i $(i = 1, 2, \ldots, n)$. This is, in fact, Model 2 of Vinod and Shenton (1996). We assume that k is known. Conditional on y_0, the likelihood function of $y_1, y_2, \ldots, y_n$ is

$$L \propto \frac{1}{\sigma^n} \exp\left\{-\frac{1}{\sigma}\sum_{i=1}^{n}(y_i - \phi y_{i-1})\right\}\left\{\prod_{i=1}^{n}\left(\frac{y_i - \phi y_{i-1}}{\sigma}\right)^{k-1}\right\}. \tag{3}$$

For numerous advantages of conditional likelihoods, see Hamilton (1994, p. 123). From expression (3), we obtain

$$\ln L = \text{Const} - n \ln \sigma - \sum_{i=1}^{n} z_i + (k-1)\sum_{i=1}^{n} \ln z_i \tag{4}$$

where $z_i = (y_i - \phi y_{i-1})/\sigma$. The ML estimators of ϕ and σ are the solutions of the likelihood equations

$$\frac{\partial \ln L}{\partial \phi} = \frac{1}{\sigma}\sum_{i=1}^{n} y_{i-1} - \frac{k-1}{\sigma}\sum_{i=1}^{n} y_{i-1} z_i^{-1} = 0 \tag{5}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{nk}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_i = 0. \tag{6}$$

Note that for $1/\{-E(\partial^2 \ln L/\partial\phi^2)\}$ to be finite asymptotically, k has to be greater than 2. The asymptotics of (5)–(6) is discussed in Li and McLeod (1988). However, the fact is that (5)–(6) have no explicit solutions and have to be solved by iterative methods which can be problematic (Ross 1990, Tiku et al. 1986); see also Barnett

(1966) and Lee et al. (1980) who point out some fundamental difficulties with iterative solutions.

To formulate modified likelihood equations which have explicit solutions, we order z_i (for a given ϕ) in order of increasing magnitude and denote the ordered z-values by $z_{(i)} = (y_{[i]} - \phi y_{[i]-1})/\sigma$ $(1 \le i \le n)$. It may be noted that $(y_{[i]}, y_{[i]-1})$ is that pair of (y_i, y_{i-1}) observations which determines $z_{(i)}$; $y_{[i]}$ may be called concomitants of $z_{(i)}$ $(1 \le i \le n)$. Since complete sums are invariant to ordering

$$\frac{\partial \ln L}{\partial \phi} = \frac{1}{\sigma}\sum_{i=1}^{n} y_{[i]-1} - \frac{k-1}{\sigma}\sum_{i=1}^{n} y_{[i]-1} z_{(i)}^{-1} \tag{7}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{nk}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_{(i)}. \tag{8}$$

Let $t_{(i)} = E\{z_{(i)}\}$ $(1 \le i \le n)$ be the expected values of the ith standardized order statistics in a random sample of size n from the gamma distribution (2). For large n (≥ 20), $t_{(i)}$ may be obtained from the equation

$$\frac{1}{\Gamma(k)}\int_0^{t_{(i)}} e^{-z} z^{k-1} dz = \frac{i}{n+1} \qquad (1 \le i \le n).$$

For $n \le 20$, the values of $t_{(i)}$ are available in Gupta (1960), Harter (1970) and Prescott (1974). Modified likelihood equations are obtained by linearizing the intractable terms in likelihood equations (Tiku 1967, Tiku et al. 1986). Since $z_{(i)}$ converges to $t_{(i)}$ as n tends to infinity, we use the first two terms of a Taylor expansion to obtain $(k > 2)$

$$z_{(i)}^{-1} \simeq \alpha_i - \beta_i z_{(i)} \qquad (1 \le i \le n) \tag{9}$$

where $\alpha_i = 2t_{(i)}^{-1}$ and $\beta_i = t_{(i)}^{-2}$. Incorporating (9) in (7)–(8), we obtain the following modified likelihood equations:

$$\frac{\partial \ln L}{\partial \phi} \simeq \frac{\partial \ln L^*}{\partial \phi} = \frac{1}{\sigma}\sum_{i=1}^{n} y_{[i]-1} - \frac{k-1}{\sigma}\sum_{i=1}^{n} y_{[i]-1}\{\alpha_i - \beta_i z_{(i)}\} = 0 \tag{10}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = \frac{\partial \ln L^*}{\partial \sigma} = -\frac{nk}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_{(i)} = 0. \tag{11}$$

The solutions of (10) and (11) are the MML estimators:

$$\hat{\phi} = K - D\hat{\sigma} \quad \text{and} \quad \hat{\sigma} = \frac{\sum_{i=1}^{n} y_{[i]} - K\sum_{i=1}^{n} y_{[i]-1}}{nk - D\sum_{i=1}^{n} y_{[i]-1}} \tag{12}$$

where

$$K = \frac{\sum_{i=1}^{n} \beta_i y_{[i]} y_{[i]-1}}{\sum_{i=1}^{n} \beta_i y_{[i]-1}^2}, \qquad D = \frac{\sum_{i=1}^{n} \left(\alpha_i - \frac{1}{k-1}\right) y_{[i]-1}}{\sum_{i=1}^{n} \beta_i y_{[i]-1}^2}.$$

Note that $\hat{\phi}$ is scale invariant, because both K and $D\hat{\sigma}$ are scale invariant. The differences $\{z_{(i)}^{-1} - (\alpha_i - \beta_i z_{(i)})\}$ converge to zero as n tends to infinity. Consequently, the difference $(1/n)\{(\partial \ln L/\partial\phi) - (\partial \ln L^*/\partial\phi)\} = 0$ asymptotically. For a rigorous proof of this, see Vaughan and Tiku (1999). Thus, the MML estimators are asymptotically equivalent to the ML estimators; see also Tiku (1970), Bhattacharyya (1985), Tiku and Suresh (1992), and Vaughan (1992).

Least squares estimators: Since the distribution (2) has nonzero mean, the LS estimators are given by

$$\hat{\phi}_0 = \frac{\sum_{t=1}^{n} y_t(y_{t-1} - \bar{y})}{\sum_{t=1}^{n} (y_{t-1} - \bar{y})^2}$$

and

$$\hat{\sigma}_0 = \sqrt{\frac{\sum_{t=1}^{n} \{y_t - \bar{y} - \hat{\phi}_0(y_{t-1} - \bar{y})\}^2}{nk}}$$

where $\bar{y} = \sum_{t=1}^{n} y_t/n$; $\hat{\phi}_0$ and $\hat{\sigma}_0$ are also the ML estimators if the innovations are normal $N(\mu, \sigma^2)$. However, numerous authors have used them irrespective of the nature of the underlying distribution; see, for example, Weiss (1977) and Yakowitz (1973). We will show that this can result in substantial loss of efficiency.

Computations: In the first place we compute $\hat{\phi}$ from (12) using the order statistics of $y_i - \hat{\phi}_0 y_{i-1}$ $(1 \le i \le n)$. We then replace $\hat{\phi}_0$ by $\hat{\phi}$ and use the order statistics of $y_i - \hat{\phi} y_{i-1}$ $(1 \le i \le n)$ to compute $\hat{\phi}$ (from 12). Thus, the MML estimates are obtained in two iterations. In all our computations, partly presented in this paper, no more than two iterations were needed for the estimates to stabilize. The estimator $\hat{\phi}$ hardly ever assumes a value greater than or equal to 1 but it if does, it is equated to 0.9999, and similarly for the LS estimator $\hat{\phi}_0$.

3 ASYMPTOTIC PROPERTIES

Proceeding exactly along the same lines as Propositions 1 and 2 in Vaughan and Tiku (1999, Appendix A), it follows that ($k > 2$)

$$\lim_{n\to\infty} \frac{1}{n} E\left[\sum_{i=1}^{n} y_{[i]-1}\left\{z_{(i)}^{-1} - (\alpha_i - \beta_i z_{(i)})\right\}\right] = 0 \tag{13}$$

and

$$\lim_{n\to\infty} \frac{1}{n} E\left[\sum_{i=1}^{n} y_{[i]-1}^2 (z_{(i)}^{-2} - \beta_i)\right] = 0, \quad \beta_i = t_{(i)}^{-2}; \quad t_{(i)} = E(z_{(i)}). \tag{14}$$

Now, from Taylor expansions we have (Kendall and Stuart 1979, p. 52)

$$E(\hat{\phi}) \simeq \phi - E\left(\frac{\partial \ln L^*}{\partial \phi}\right) \Big/ E\left(\frac{\partial^2 \ln L^*}{\partial \phi^2}\right)$$

and

$$E(\hat{\sigma}) \simeq \sigma - E\left(\frac{\partial \ln L^*}{\partial \sigma}\right) \Big/ E\left(\frac{\partial^2 \ln L^*}{\partial \sigma^2}\right).$$

From derivations exactly similar to those in the following Lemma 1, it easily follows from (13) that asymptotically $(1/n)E(\partial \ln L^*/\partial\phi) = 0$. Also from (11), $E(\partial \ln L^*/\partial\sigma) = 0$ since $\sum_{i=1}^{n} z_{(i)} = \sum_{i=1}^{n} z_i$ and $E(z_i) = k$. Thus, $\hat{\phi}$ and $\hat{\sigma}$ are asymptotically unbiased.

Lemma 1 *Asymptotically* ($k > 2$)

$$\frac{1}{n}\left\{-E\left(\frac{\partial^2 \ln L^*}{\partial \phi^2}\right)\right\} = \frac{k}{k-2}\left\{\frac{1}{1-\phi^2} + \frac{k}{(1-\phi)^2}\right\}.$$

Proof. Since $z_i = \varepsilon_i/\sigma$ and y_{i-1} are independent of each other and complete sums are invariant to ordering, it follows from (14) that asymptotically

$$(1/n)\left\{-E\left(\frac{\partial^2 \ln L^*}{\partial \phi^2}\right)\right\} = \frac{k-1}{n\sigma^2} E\left(\sum_{i=1}^{n} t_{(i)}^{-2} y_{[i]-1}^2\right)$$

$$
\begin{aligned}
&= \frac{k-1}{n\sigma^2} E\left(\sum_{i=1}^{n} z_{(i)}^{-2} y_{[i]-1}^2\right) \\
&= \frac{k-1}{n\sigma^2} E\left(\sum_{i=1}^{n} z_i^{-2} y_{i-1}^2\right) \\
&= \frac{k(k-1)}{n\sigma^2} \sum_{i=1}^{n} E(z_i^{-2}) \left\{\frac{\sigma^2}{1-\phi^2} + \frac{k\sigma^2}{(1-\phi)^2}\right\} \\
&= \frac{k}{k-2} \left\{\frac{1}{1-\phi^2} + \frac{k}{(1-\phi)^2}\right\}
\end{aligned}
$$

since

$$
\begin{aligned}
E(z_i^{-2}) &= \frac{1}{\Gamma(k)} \int_0^\infty e^{-z} z^{k-3} dz \\
&= \Gamma(k-2)/\Gamma(k).
\end{aligned}
$$

Remark. Exactly along the same lines it follows that asymptotically

$$(1/n)\left\{-E\left(\frac{\partial^2 \ln L^*}{\partial\phi\partial\sigma}\right)\right\} = \frac{k}{(1-\phi)\sigma}$$

and

$$(1/n)\left\{-E\left(\frac{\partial^2 \ln L^*}{\partial\sigma^2}\right)\right\} = \frac{k}{\sigma^2}\,.$$

Thus, we have the following result:

Lemma 2 *The asymptotic variance-covariance matrix of $\hat{\phi}$ and $\hat{\sigma}$ is* $(k > 2)$

$$V = \frac{1}{n}\left\{\frac{1}{1-\phi^2} + \frac{2}{(1-\phi)^2}\right\}^{-1} \begin{bmatrix} \frac{k-2}{k} & -\frac{(k-2)\sigma}{k(1-\phi)} \\ -\frac{(k-2)\sigma}{k(1-\phi)} & \frac{\sigma^2}{k}\left(\frac{1}{1-\phi^2} + \frac{k}{(1-\phi)^2}\right) \end{bmatrix}.$$

Note that the variance of $\hat{\phi}$ is free of σ.

We worked out the expected values and variances and covariances of $\hat{\phi}$ and $\hat{\sigma}$ for n = 20, 30, 40, 50 and 60 using the asymptotic equations above. The values are given in Table VI for $n = 30$ and 50, for illustration; σ was taken to be equal to 1 without any loss of generality. Also given are the corresponding simulated values (based on 10,000 Monte Carlo runs). For larger values of n, of course, the agreement between the two is even closer. It can be seen that the asymptotic

TABLE I
Simulated values of means, variances and covariances.

	Mean		$n\times$ Variance		$n\times$ Covariance
ϕ	$\hat{\phi}_0$	$\hat{\sigma}_0/\sigma$	$\hat{\phi}_0$	$\hat{\sigma}_0/\sigma$	$(\hat{\phi}_0, \hat{\sigma}_0/\sigma)$
			$k = 4$		
0.0	−0.033	0.954	0.918	0.795	0.021
0.5	0.419	0.953	0.804	0.795	0.063
0.9	0.762	0.946	0.555	0.789	0.111
			$k = 8$		
0.0	−0.032	0.962	0.900	0.645	0.021
0.5	0.418	0.954	0.813	0.645	0.066
0.9	0.761	0.947	0.549	0.639	0.114

equations above give close approximations even for small n, since the two sets of values are so close to each other. This also implies that $\hat{\phi}$ and $\hat{\sigma}$ are highly efficient. Asymptotically, they are unbiased and are the MVB (minimum variance bound) estimators (i.e., their variances are exactly equal to the diagonal elements of the matrix V). Hence, $\hat{\phi}$ and $\hat{\sigma}$ are fully efficient (asymptotically). Note that the MVB estimators of ϕ and σ do not exist for small n. Therefore, no estimator of ϕ or σ will have variance equal to the MVB.

Efficiency of the LS estimators: The simulated values of the means, variances and covariances of $\hat{\phi}_0$ and $\hat{\sigma}_0$ are given in Table I for $n = 30$.

Comparing these values with the corresponding values in Table VI, it is seen that $\hat{\phi}_0$ and $\hat{\sigma}_0$ have very low efficiencies as compared to the proposed estimators $\hat{\phi}$ and $\hat{\sigma}$; see also Li and McLeod (1988) who showed that the LS estimators have very low efficiencies (as compared to the ML estimators) if the innovations have a log-normal distribution. Their asymptotic and simulated variances of the ML estimators, however, have quite a bit of discrepancy, even for n as large as 200. In the experience of the senior author of this paper, this is often due to the iterative nature of the ML estimators which are computed to satisfy a threshold constraint $|\partial \ln L/\partial\theta| < e$ and not $|\partial \ln L/\partial\theta| = 0$. Consequently, simulations based on iterative solutions of likelihood equations can cumulate errors resulting in such discrepancies as mentioned above.

4 HYPOTHESIS TESTING

For hypothesis testing, we need the distributions of $\hat{\phi}$ and $\hat{\sigma}$. In that regard, we have the following results.

Lemma 3 *Conditionally (for known σ), $\hat{\phi}(\sigma) = K - D\sigma$ is asymptotically the MVB estimator of ϕ and is normally distributed (asymptotically).*

Proof. This follows from the fact that asymptotically $(1/n)\{\partial \ln L/\partial\phi - \partial \ln L^*/\partial\phi\}$ $= 0$ and

$$\frac{1}{n}\frac{\partial \ln L^*}{\partial\phi} = \frac{k-1}{\sigma^2}\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i y_{[i]-1}^2\right)\{(K - D\sigma) - \phi\}.$$

Since $(1/n)\sum_{i=1}^{n}\beta_i y_{[i]-1}^2$ converges to its expected value as n tends to infinity (Lemma 1), $K - D\sigma$ is the MVB estimator (asymptotically). The result immediately follows from the fact that $\partial \ln L/\partial\phi$ is asymptotically normal (Bartlett 1953); see also Kendall and Stuart (1979, pp. 46–47)

Lemma 4 *Conditionally (for known ϕ), $\hat{\sigma}(\phi)$ is the MVB estimator of σ and the distribution of $2nk\hat{\sigma}(\phi)/\sigma$ is chi-square with $2nk$ df (degrees of freedom).*

Proof. For known ϕ, $K = \phi + D\hat{\sigma}$ and as a result $\hat{\sigma}(\phi) = (1/nk)\sum_{i=1}^{n}(y_i - \phi y_{i-1})$. Since

$$\frac{\partial \ln L}{\partial\sigma} = \frac{\partial \ln L^*}{\partial\sigma} = \frac{nk}{\sigma^3}\{\hat{\sigma}(\phi) - \sigma\},$$

$\hat{\sigma}(\phi)$ is the MVB estimator of σ and $2nk\hat{\sigma}(\phi)/\sigma$ is distributed as chi-square with $2nk$ df since $2\varepsilon/\sigma$ in (2) has a chi-square distribution with $2k$ df.

In this paper, we are primarily interested in $k > 2$ and $n \geq 20$ in which case the df of the chi-square is large, i.e., $2nk \geq 80$. It will, therefore, suffice to regard the distribution of $\sqrt{nk}(\hat{\sigma} - \sigma)/\sigma$ as normal $N(0,1)$.

TABLE II
Pearson Coefficients for $\hat{\phi}$.

	$n = 20$			$n = 30$			$n = 50$		
ϕ	0.0	0.5	0.9	0.0	0.5	0.9	0.0	0.5	0.9
					$k = 3$				
β_1^*	0.248	0.130	0.093	0.231	0.098	0.052	0.178	0.092	0.056
β_2^*	3.283	3.149	3.122	3.254	3.046	3.058	3.172	3.050	3.056
					$k = 6$				
β_1^*	0.093	0.022	0.007	0.086	0.017	0.002	0.058	0.014	0.003
β_2^*	3.066	2.968	2.940	3.145	3.053	3.034	3.092	3.033	2.984

Since $\hat{\phi}$ and $\hat{\sigma}$ converge to ϕ and σ, respectively, as n tends to infinity, we have the following result:

Corollary *The marginal distributions of $\sqrt{n}(\hat{\phi} - \phi)$ and $\sqrt{n}(\hat{\sigma} - \sigma)/\sigma$ are asymptotically normal.*

Small sample distributions: To verify the accuracy of the normal distributions above for small samples, we simulated the Pearson coefficients of skewness and kurtosis $\beta_1^* = \mu_3^2/\mu_2^3$ and $\beta_2^* = \mu_4/\mu_2^2$ of $\hat{\phi}$ and $\hat{\sigma}$ for $k > 2$ and $n \geq 20$. The β_1^* and β_2^* values of $\hat{\sigma}$ were found to be very close to 0 and 3 respectively, for all values of ϕ, k and n. For $\phi = 0$, $k = 4$ and $n = 30$, for example, the β_1^* and β_2^* values of $\hat{\sigma}$ are 0.034 and 3.094 respectively. To test $H_0 : \sigma = \sigma_0$ the distribution of $\sqrt{n}(\hat{\sigma} - \sigma_0)/\sigma_0$ is, therefore, approximated by normal with mean 0 and variance (estimated)

$$\frac{1}{k}\left\{\frac{1}{1-\hat{\phi}^2} + \frac{2}{(1-\hat{\phi})^2}\right\}^{-1}\left(\frac{1}{1-\hat{\phi}^2} + \frac{k}{(1-\hat{\phi})^2}\right).$$

The distribution of $\hat{\phi}$, however, has a moderate amount of skewness for $\phi = 0$ (and small values of ϕ) and small k (≤ 6). A representative set of values of β_1^* and β_2^* is given in Table II.

A correction for skewness can, however, be incorporated as follows:

Three-moment chi-square approximation: For small k and $\phi = 0$, β_1^* and β_2^* satisfy the condition

$$|\beta_2^* - (3 + 1.5\beta_1^*)| \leq 0.5.$$

A 3-moment chi-square approximation will, therefore, be effective (Tiku 1963, 1966). Let

$$\hat{\phi} \simeq c + d\ \chi^2$$

where χ^2 is a chi-square variate having ν df; c, d and ν are determined such that the first three moments on both sides agree. That gives

$$\nu = \frac{8}{\beta_1^*}\ ,\ d = \sqrt{\frac{V(\hat{\phi})}{2\nu}}\ ,\ \text{and } c = E(\hat{\phi}) - \nu d.$$

Thus, the approximate $100(1-\alpha)\%$ point of the distribution of $\hat{\phi}$ when $\phi = 0$ is

$$u_{1-\alpha} = c + d\ \chi^2_{1-\alpha}$$

where $\chi^2_{1-\alpha}$ is the $100(1-\alpha)\%$ point of a chi-square distribution with ν df.

The simulated values of the first four moments of $\hat{\phi}$ are given in Table VII which may be used to evaluate the above 3-moment chi-square approximation. For $\phi = 0$, $k = 3$ and $n = 20$, for example, $E(\hat{\phi}) = 0.0662$, $V(\hat{\phi}) = 0.0096$, $\beta_1^* = 0.2485$, $\beta_2^* = 3.2827$; $\beta_2^* - (3 + 1.5\beta_1^*) = -0.09$. Thus, $\nu = 32.1932$, $d = 0.01221$, $c = -0.3269$. The 95% point of the distribution of $\hat{\phi}$ when $\phi = 0$ is $u_{0.95} = -0.3269 + 0.01221(56.4268) = 0.240$. Given below are the values of $u_{1-\alpha}$ so obtained and the corresponding simulated values of the probability $P(\hat{\phi} > u_{1-\alpha}|\phi = 0)$; $\alpha = 0.05$:

	$n = 20$		$n = 30$		$n = 50$	
k	$u_{1-\alpha}$	Prob	$u_{1-\alpha}$	Prob	$u_{1-\alpha}$	Prob
2.5	0.236	0.049	0.179	0.051	0.131	0.053
3	0.240	0.052	0.184	0.048	0.134	0.050
3.5	0.243	0.052	0;189	0.052	0.139	0.052
4	0.244	0.051	0.190	0.051	0.139	0.049

It is seen that the above 3-moment chi-square approximation is remarkably accurate. It is also interesting to note that the values of $u_{1-\alpha}$ do not change much when k changes. To test $H_0 : \phi = 0$ one calculates $\hat{\phi}$ and rejects H_0 in favour of $H_1 : \phi > 0$ if $\hat{\phi}$ is greater than $u_{1-\alpha}$; α is usually chosen to be 0.05.

Power function: Calculations reveal that the power

$$1 - \beta = P(\hat{\phi} \geq u_{1-\alpha} \mid \phi > 0)$$

of the test based on $\hat{\phi}$ for testing $H_0 : \phi = 0$ against $H_1 : \phi > 0$ increases very rapidly with ϕ. For $k = 2.5$ and $n = 20$, for example, we have the following simulated values; $\alpha = 0.05$:

$\phi =$	0.0	0.10	0.20	0.30	0.40
$1 - \beta :$	0.049	0.194	0.580	0.979	1.000

Since for values of $\phi > 0$ (but not too small) $\hat{\phi}$ is almost unbiased and approximately normal, the power-function is (approximately)

$$1 - \beta = P\left\{ Z \geq \frac{u_{1-\alpha} - \phi}{\sqrt{V(\hat{\phi})}} \right\} \qquad (k > 2)$$

where Z is a standard normal variate and $V(\hat{\phi})$ is the first element of the matrix V with $\phi = \hat{\phi}$. For $\phi \geq u_{1-\alpha}$, this equation gives accurate values of $1 - \beta$. We do not give details for conciseness.

5 ROBUSTNESS PROPERTIES

In practice, the shape parameter k in (2) might be somewhat misspecified or the sample might contain outliers. From a practical point of view, therefore, it is very important for an estimator to have efficiency robustness; see, for example, Ledolter (1979), Huber (1981) and Tiku et al. (1986). Such an estimator is fully efficient (or nearly so) for an assumed model but maintains high efficiency for plausible alternatives to the assumed model. The model we assume is (2) with $k = 4$ and unknown scale σ. The value $k = 4$ was chosen for illustration but, of course, any other value of k can as well be chosen with similar results. The alternatives to this model will be called sample models. Out of a large number of sample models considered by various authors, we choose a representative set as follows:

(a) The family of distributions (2), i.e., *Gamma*(k): (1) $k = 2.5$, (2) $k = 3.0$, (3) $k = 5.0$, (4) $k = 5.5$.

(b) Dixon outlier models: (5) $(n - r)$ observations come from *Gamma*(4) but r observations (we do not know which ones) come from the same distribution but with a different scale $\lambda\sigma$: (5) $\lambda = 2$, (6) $\lambda = 4$; $r = [\frac{1}{2} + 0.1n]$.

(c) Mixture models: (7) $0.90Gamma(4) + 0.10Gamma(1)$, (8) $0.90Gamma(4) + 0.10Gamma(7)$.

The models (b) and (c) are very important from a practical point of view. In fact, Huber (1981) stated that the occurrence of five to ten percent outliers in a sample is a rule and not an exception.

It may be noted that $\hat{\sigma}$ is an estimator of the scale of the sample model, i.e., $\theta\sigma$ $(\theta > 0)$. When the sample model is exactly the same as the assumed model $\theta = 1$. For the sample models (a)–(c), the values of θ are given in Table VIII. It may be noted that the value of θ has absolutely no role to play in the computation of $\hat{\phi}$ and $\hat{\sigma}$. Their values are given only for bias and mean square error calculations. In fact, θ is the ratio of the standard deviation of the sample model to the standard deviation of the assumed model.

The simulated values of the means and variances of the LS estimators $\hat{\phi}_0$ and $\hat{\sigma}_0$ and the MML estimators $\hat{\phi}$ and $\hat{\sigma}$ are given in Table VIII. These may be compared with the values given in Table III where the sample model is the same as the assumed model with $k = 4$ $(\theta = 1)$; $n = 20$:

TABLE III
Simulated values of the means and variances.

	$\phi = 0.0$		$\phi = 0.3$		$\phi = 0.6$		$\phi = 0.9$	
	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
$\hat{\phi}_0$	−0.050	0.045	0.207	0.044	0.459	0.039	0.693	0.033
$\hat{\phi}$	0.061	0.011	0.344	0.006	0.626	0.002	0.907	0.001
$\hat{\sigma}_0/\sigma$	0.932	0.043	0.931	0.043	0.928	0.044	0.918	0.045
$\hat{\sigma}/\sigma$	0.940	0.026	0.938	0.027	0.935	0.029	0.927	0.030

It can be seen that $\hat{\phi}$ is remarkably robust and $\hat{\phi}$ and $\hat{\sigma}$ are substantially more efficient than $\hat{\phi}_0$ and $\hat{\sigma}_0$.

Remark. The reason for the remarkable robustness of $\hat{\phi}$ is that it is essentially the solution of equation (10). This equation assigns the weight β_i to $z_{(i)} = \varepsilon_{(i)}/\sigma$ $(1 \le i \le n)$. Since $\beta_i = t_{(i)}^{-2}$ $(1 \le i \le n)$ have half-umbrella ordering and constitute a decreasing sequence of positive numbers, the large residuals are automatically assigned small weights. This depletes the influence of outliers and long tails.

Robust test: A robust test is most powerful (or nearly so) for an assumed model but for plausible alternatives its type I error is never substantially higher than the presumed level (criterion robustness) and its power is high (efficiency robustness); see, for example, Huber (1981) and Tiku et al. (1986). Since the estimator $\hat{\phi}$, like the sample correlation coefficient $\hat{\rho}$ (Tiku 1988b), is the ratio of two second order statistics, it is difficult to achieve its robustness especially for skew distributions. It may be possible to achieve it through a robust Bayesian procedure (Bian and Tiku 1997a,b). We did, however, verify the robustness to random choices of k as follows:

We assumed that the true model is (2) with $k = 4$. We generated a value at random in the interval (3,5); denote this value by k^*. We then generated a random sample of size n from (2) with $k = k^*$ (sample model) and computed $\hat{\phi}$ (with $k = 4$ in (5)–(12)). We repeated this procedure 10,000 times and simulated the type I error and power $P(\hat{\phi} \ge 0.244 \mid \phi \ge 0)$ of the $\hat{\phi}$-test; $0.244 = u_{1-\alpha}$ for $k = 4$, $n = 20$ and $\alpha = 0.05$. The values are given below. Also given are the corresponding values when $k^* = k = 4$, $n = 20$:

$\phi =$	0.0	0.1	0.2	0.3	0.4
Random k^* :	0.036	0.128	0.409	0.847	0.998
$k^* = k = 4$:	0.051	0.177	0.504	0.901	1.000

The $\hat{\phi}$-test is reasonably robust to such deviations. Alternatively, k and the parameter b of the generalized logistic of Section 7 may be estimated by graphical interpolation as in Tiku (1968, p. 137). This will be the subject matter of a future paper.

6 UNKNOWN LOCATION

In certain situations one may like to work with the model

$$y_i = \mu + \phi y_{i-1} + \varepsilon_i \qquad (i = 1, 2, 3, \ldots, n). \tag{15}$$

Assuming the distribution (2), the modified likelihood equation for estimating μ is

$$\frac{\partial \ln L^*}{\partial \mu} = \frac{n}{\sigma} - \frac{k-1}{\sigma} \sum_{i=1}^{n} \left\{ \alpha_i - \beta_i \frac{y_{[i]} - \mu - \phi y_{[i]-1}}{\sigma} \right\} = 0. \tag{16}$$

Since $(1/n) \sum_{i=1}^{n} \alpha_i \simeq 1/(k-1)$ for large n (≥ 10), (16) gives

$$\mu \simeq \frac{1}{m} \sum_{i=1}^{n} \beta_i \{ y_{[i]} - \phi y_{[i]-1} \} \qquad \left(m = \sum_{i=1}^{n} \beta_i \right). \tag{17}$$

Incorporating (17) in (10)-(11) and writing

$$w_{[i]} = y_{[i]} - \frac{1}{m} \sum_{i=1}^{n} \beta_i y_{[i]}$$

the MML estimators of ϕ and σ are given by (12) with $y_{[i]}$ and $y_{[i]-1}$ replaced by $w_{[i]}$ and $w_{[i]-1}$, respectively. The MML estimator of μ is then obtained:

$$\hat{\mu} = \frac{1}{m} \sum_{i=1}^{n} \beta_i \{ y_{[i]} - \hat{\phi} y_{[i]-1} \}. \tag{18}$$

Asymptotically, $\hat{\mu}$ is unbiased and the variance-covariance matrix of $(\hat{\mu}, \hat{\phi}, \hat{\sigma})$ is given by $I^{-1}(\mu, \phi, \sigma)$ where I is the information matrix:

$$I(\mu, \phi, \sigma) = \begin{bmatrix} \frac{n}{(k-2)\sigma^2} & \frac{nk}{(k-2)(1-\phi)\sigma} & \frac{n}{\sigma^2} \\ " & \frac{nk}{(k-2)} \left\{ \frac{1}{1-\phi^2} + \frac{k}{(1-\phi)^2} \right\} & \frac{nk}{(1-\phi)\sigma} \\ " & " & \frac{nk}{\sigma^2} \end{bmatrix}. \tag{19}$$

The MML estimators $\hat{\mu}$, $\hat{\phi}$ and $\hat{\sigma}$ are considerably more efficient than the corresponding LS estimators $\hat{\mu}_0$, $\hat{\phi}_0$ and $\hat{\sigma}_0$. We omit details for conciseness.

7 GENERALIZATION TO $AR(q)$ MODEL

The method immediately generalizes to the $AR(q)$ model $(n > q)$

$$y_i = \sum_{j=1}^{q} \phi_j y_{i-j} + \varepsilon_i \qquad (i = 1, 2, \ldots, n). \tag{20}$$

Here, the modified likelihood equations are $(j = 1, 2, \ldots, q)$

$$\frac{\partial \ln L}{\partial \phi_j} \simeq \frac{\partial \ln L^*}{\partial \phi_j} = \frac{1}{\sigma}\sum_{i=1}^{n} y_{[i]-j} - \frac{k-1}{\sigma}\sum_{i=1}^{n} y_{[i]-j}\{\alpha_i - \beta_i z_{(i)}\} = 0 \tag{21}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = \frac{\partial \ln L^*}{\partial \sigma} = -\frac{nk}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_{(i)} = 0 \tag{22}$$

where $z_{(i)} = (y_{[i]} - \phi_1 y_{[i]-1} - \cdots - \phi_q y_{[i]-q})/\sigma$ $(1 \le i \le n)$, and $(y_{[i]}, y_{[i]-1}, \ldots, y_{[i]-q})$ are the concomitants of $z_{(i)}$. The solutions of (21)–(22) are the following MML estimators:

Write

$$\begin{aligned}
b_{jk} &= \sum_{i=1}^{n} \beta_i y_{[i]-j} y_{[i]-k},\ b_{jk} = b_{kj} \qquad (j, k = 0, 1, 2, \ldots, q) \\
d_j &= \sum_{i=1}^{n} y_{[i]-j},\ a_j = \sum_{i=1}^{n} \alpha_i y_{[i]-j} \qquad (j = 0, 1, 2, \ldots, q) \\
\mathbf{B} &= (b_{jk})_{j,k=1,2,\ldots,q}
\end{aligned}$$

$$\mathbf{B}_0 = \begin{pmatrix} b_{01} \\ b_{02} \\ \vdots \\ b_{0q} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_q \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_q \end{pmatrix}, \quad \hat{\mathbf{\Phi}} = \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_q \end{pmatrix};$$

then

$$\hat{\mathbf{\Phi}} = \mathbf{B}^{-1}\mathbf{B}_0 - \mathbf{B}^{-1}\left(\mathbf{A} - \frac{1}{k-1}\,\mathbf{D}\right)\hat{\sigma} \tag{23}$$

and

$$\hat{\sigma} = \frac{d_0 - \mathbf{D}'\mathbf{B}^{-1}\mathbf{B}_0}{nk - \mathbf{D}'\mathbf{B}^{-1}\mathbf{A} + \frac{1}{k-1}\,\mathbf{D}'\mathbf{B}^{-1}\mathbf{D}}\,. \tag{24}$$

For $q = 1$, (23)–(24) reduce to (12).

Let γ_s be the autocovariance $\mathrm{Cov}(y_t, y_{t-s})$ and ρ_s be the autocorrelation such that $\rho_s = \gamma_s/\gamma_0$. Then, the asymptotic variance-covariance matrix of $\hat{\mathbf{\Phi}}$ and $\hat{\sigma}$ is $I^{-1}(\mathbf{\Phi}, \sigma)$ with elements $(1 \leq j, \ell \leq q)$:

$$\begin{aligned}
-E\left(\frac{\partial^2 \ln L^*}{\partial \phi_j^2}\right) &= \frac{n}{(k-2)\sigma^2}\,\{\gamma_0 + E^2(y)\} \\
-E\left(\frac{\partial^2 \ln L^*}{\partial \phi_j \partial \phi_\ell}\right) &= \frac{n}{(k-2)\sigma^2}\,\{\gamma_{|j-\ell|} + E^2(y)\} \\
-E\left(\frac{\partial^2 \ln L^*}{\partial \sigma^2}\right) &= \frac{nk}{\sigma^2} \\
-E\left(\frac{\partial^2 \ln L^*}{\partial \phi_j \partial \sigma}\right) &= \frac{n}{\sigma^2}\,E(y).
\end{aligned} \tag{25}$$

Here

$$\begin{aligned}
E(y) &= \frac{k\sigma}{1 - \phi_1 - \phi_2 - \cdots - \phi_q}\,, \\
V(y) = \gamma_0 &= \frac{k\sigma^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_q\rho_q}
\end{aligned} \tag{26}$$

and

$$\mathrm{Cov}(y_j, y_\ell) = \gamma_{|j-\ell|} = \rho_{|j-\ell|}\gamma_0 \qquad (j \neq \ell). \tag{27}$$

To compute $\gamma_{|j-\ell|}$, we first let $\rho = (\rho_1, \rho_2, \ldots, \rho_q)'$, $\mathbf{\Phi} = (\phi_1, \phi_2, \ldots, \phi_q)'$ and $\mathbf{P} = (p_{j\ell})_{j,\ell=1,2,\ldots,q}$ where

$$p_{j\ell} = -(\phi_{j-\ell} + \phi_{j+\ell})$$

in which $\phi_0 = -1$ and $\phi_j = 0$ for $j < 0$ or $j > q$. Then

$$\rho = \mathbf{P}^{-1}\mathbf{\Phi}. \tag{28}$$

One can first obtain ρ_j from (28) and then γ_0 from (26) and $\gamma_{|j-\ell|}$ from (27). For example when $q = 2$,

$$\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{bmatrix} 1-\phi_2 & 0 \\ -\phi_1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \frac{\phi_1}{1-\phi_2} \\ \frac{\phi_1^2+(1-\phi_2)\phi_2}{1-\phi_2} \end{pmatrix}.$$

We then obtain the value of γ_0 from (26)

$$\gamma_0 = \frac{(1-\phi_2)k\sigma^2}{1-\phi_2-\phi_1^2-\phi_2^2-\phi_2\phi_1^2+\phi_2^3} \qquad (q=2)$$

and the value of γ_1 from (27)

$$\gamma_{|1-2|} = \gamma_{|2-1|} = \gamma_1 = \rho_1\gamma_0 = \frac{\phi_1 k\sigma^2}{1-\phi_2-\phi_1^2-\phi_2^2-\phi_2\phi_1^2+\phi_2^3} \qquad (q=2).$$

The estimators $\hat{\mathbf{\Phi}}$ and $\hat{\sigma}$ have the same efficiency properties as in the $AR(1)$ model: (i) they are asymptotically MVB estimators and (ii) for small samples, as numerical simulations show, their variances and covariances are closely approximated by (25).

8 GENERALIZED LOGISTIC

Suppose that in the $AR(1)$ model (1), the innovations ε_i have the generalized logistic distribution ($b > 0$)

$$f(w;b) = \frac{b}{\sigma}\,\frac{e^{-w/\sigma}}{(1+e^{-w/\sigma})^{b+1}}\,, \qquad -\infty < w < \infty \tag{29}$$

with

$$F(w;b) = \int_{-\infty}^{w} f(u;b)du = \frac{1}{(1+e^{-w/\sigma})^b}\,.$$

For $b = 1$, (29) is the logistic distribution and is symmetric; for $b < 1$ it is negatively skewed and for $b > 1$ it is positively skewed. The likelihood equations are expressions in terms of the awkward functions

$$\frac{e^{-z_{(i)}}}{1+e^{-z_{(i)}}}\,, \quad z_{(i)} = \frac{y_{[i]}-\phi y_{[i]-1}}{\sigma} \qquad (1 \le i \le n) \tag{30}$$

and are, therefore, intractable. The modified likelihood equations are obtained by linearizing (30) and are given by

$$\frac{\partial \ln L}{\partial \phi} \simeq \frac{\partial \ln L^*}{\partial \phi} = \frac{1}{\sigma}\sum_{i=1}^{n} y_{[i]-1} - \frac{b+1}{\sigma}\sum_{i=1}^{n} y_{[i]-1}\{\alpha_i - \beta_i z_{(i)}\} = 0 \tag{31}$$

and

$$\frac{\partial \ln L}{\partial \sigma} \simeq \frac{\partial \ln L^*}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_{(i)} - \frac{b+1}{\sigma}\sum_{i=1}^{n} z_{(i)}\{\alpha_i - \beta_i z_{(i)}\} = 0 \tag{32}$$

where

$$\beta_i = \frac{e^{t_{(i)}}}{(1+e^{t_{(i)}})^2} \quad \text{and} \quad \alpha_i = \frac{1}{1+e^{t_{(i)}}} + \beta_i t_{(i)} \qquad (1 \le i \le n).$$

The coefficients $t_{(i)}$ are determined by the equations $F(t_{(i)}; b) = q_i$, $q_i = i/(n+1)$, i.e.,

$$t_{(i)} = -\ln(q_i^{-1/b} - 1) \qquad (1 \le i \le n).$$

The solutions of (31)–(32) are the MML estimators:

$$\hat{\phi} = K + D\hat{\sigma} \qquad \text{and} \qquad \hat{\sigma} = \frac{B + \sqrt{B^2 + 4nC}}{2n} \tag{33}$$

where

$$K = \frac{\sum_{i=1}^{n} \beta_i y_{[i]} y_{[i]-1}}{\sum_{i=1}^{n} \beta_i y_{[i]-1}^2}, \qquad D = \frac{\sum_{i=1}^{n} \left(\frac{1}{b+1} - \alpha_i\right) y_{[i]-1}}{\sum_{i=1}^{n} \beta_i y_{[i]-1}^2};$$

$$B = (b+1)\sum_{i=1}^{n}\left(\frac{1}{b+1} - \alpha_i\right)\{y_{[i]} - K y_{[i]-1}\}$$

and

$$\begin{aligned} C &= (b+1)\sum_{i=1}^{n} \beta_i \{y_{[i]} - K y_{[i]-1}\}^2 \\ &= (b+1)\left\{\sum_{i=1}^{n} \beta_i y_{[i]}^2 - K \sum_{i=1}^{n} \beta_i y_{[i]} y_{[i]-1}\right\}. \end{aligned}$$

Asymptotic properties: The estimators $\hat{\phi}$ and $\hat{\sigma}$ are asymptotically fully efficient, and their asymptotic variance-covariance matrix is given by I^{-1}, where $I(\phi, \sigma)$ is the information matrix with elements

TABLE IV
Comparison of LS and MML estimators.

	$\phi = 0$			$\phi = 0.5$			$\phi = 0.9$		
	Mean	$(\text{Bias})^2$	Variance	Mean	$(\text{Bias})^2$	Variance	Mean	$(\text{Bias})^2$	Variance
					$b = 0.5$				
$\hat{\phi}_0$	−0.037	0.001	0.033	0.412	.008	0.035	0.796	0.011	0.024
$\hat{\phi}$	−0.001	0.000	0.021	0.479	0.000	0.013	0.883	0.000	0.003
$\hat{\sigma}_0/\sigma$	1.348	0.121	0.545	1.345	0.119	0.543	1.348	0.121	0.551
$\hat{\sigma}/\sigma$	0.999	0.000	0.025	0.999	0.000	0.025	0.999	0.000	0.025
					$b = 4$				
$\hat{\phi}_0$	−0.036	0.001	0.033	0.417	0.007	0.033	0.857	0.002	0.006
$\hat{\phi}$	0.008	0.000	0.012	0.499	0.000	0.005	0.899	0.000	0.0004
$\hat{\sigma}_0/\sigma$	0.477	0.273	0.287	0.476	0.274	0.288	0.503	0.247	0.260
$\hat{\sigma}/\sigma$	0.992	0.000	0.018	0.996	0.000	0.019	0.998	0.000	0.020

$$\begin{aligned}
-E\left(\frac{\partial^2 \ln L^*}{\partial \phi^2}\right) &= \frac{nb}{b+2}\left\{\frac{\psi'(b)+\psi(1)}{1-\phi^2} + \frac{[\psi(b)-\psi(1)]^2}{(1-\phi)^2}\right\} \\
-E\left(\frac{\partial^2 \ln L^*}{\partial \phi \partial \sigma}\right) &= \frac{nb}{(b+2)\sigma}\,\frac{\psi(b)-\psi(1)}{1-\phi} \qquad (34) \\
-E\left(\frac{\partial^2 \ln L^*}{\partial \sigma^2}\right) &= \frac{n}{\sigma^2} + \frac{nb}{(b+2)\sigma^2}\,\{\psi'(b+1)+\psi'(2)+[\psi(b+1)-\psi(2)]^2\}.
\end{aligned}$$

A few relevant details about the ψ-functions (and their numerical values) are given in the Appendix. In particular, $\psi(2) - \psi(1) = 1$.

The elements of I^{-1} give, as calculations reveal, close approximations to the true values for all $n \geq 20$. For $\phi = 0$, $b = 2$ and $n = 30$, for example, I^{-1} gives $V(\hat{\phi}) = 0.023$ and $(1/\sigma^2)V(\hat{\sigma}) = 0.027$. The corresponding simulated values (based on 10,000 Monte Carlo runs) are 0.021 and 0.022, respectively.

Least squares estimators: As for the gamma family, the LS estimators $\hat{\phi}_0$ and $\hat{\sigma}_0$ have very low efficiencies for the generalized logistic family as compared to the MML estimators (33). Given in Table IV are the simulated values of the mean, $(\text{bias})^2$ and the mean square error for comparison, $n = 30$; $\hat{\sigma}_0$ was adjusted for the fact that it is an estimator of the standard deviation of the generalized logistic.

Therefore, $\hat{\sigma}_0$ was divided by an appropriate constant $\sqrt{E(Z^2) - E^2(Z)}$ given in the Appendix.

It can be seen that the LS estimators have very low efficiencies. For the generalized logistic such efficiencies for negative ϕ are similar to those for positive ϕ but we do not reproduce them for brevity. For the gamma distribution since y_i in model (1) are all positive, ϕ cannot be negative.

Asymptotic distributions: As for the gamma family, the marginal distributions of $\sqrt{n}(\hat{\phi} - \phi)$ and $\sqrt{n}(\hat{\sigma} - \sigma)$ are asymptotically normal. These results are quite natural since the MML estimators are asymptotically equivalent to the ML estimators.

Small sample distributions: For $n \geq 50$, the distribution of $\hat{\sigma}$ is very close to normal for all values of the shape parameter b as was clear from the Pearson coefficients of skewness and kurtosis $\beta_1^* = \mu_3^2/\mu_2^3$ and $\beta_2^* = \mu_4/\mu_2^2$ which we simulated from 10,000 Monte Carlo runs. For $n \geq 50$ the distribution of $\sqrt{n}(\hat{\sigma} - \sigma)$ is, therefore, referred to a normal distribution with mean zero and variance (estimated) given by the last element in the matrix

$$[I^{-1}(\phi, \sigma)]_{\phi=\hat{\phi}, \sigma=\hat{\sigma}}\,.$$

For smaller values of n (< 50), as is clear from the values of β_1^* and β_2^* given in Table V, the condition $\nabla = |\beta_2^* - (3 + 1.5\beta_1^*)| \leq 0.5$ is satisfied. A 3-moment chi-square approximation is, therefore, very effective (as illustrated earlier):

TABLE V
Skewness and Kurtosis of $\hat{\sigma}$.

ϕ	n	β_1^*	β_2^*	∇	β_1^*	β_2^*	∇	β_1^*	β_2^*	∇
			$b = 0.5$			$b = 2$			$b = 6$	
0.0	20	0.184	3.402	0.13	0.103	3.195	0.04	0.039	3.023	0.03
	30	0.114	3.211	0.04	0.076	3.120	0.01	0.028	3.063	0.00
0.5	20	0.180	3.397	0.13	0.107	3.201	0.04	0.064	3.056	0.04
	30	0.112	3.204	0.04	0.080	3.136	0.02	0.044	3.064	0.00
0.9	20	0.172	3.344	0.09	0.103	3.171	0.02	0.085	3.100	0.03
	30	0.112	3.207	0.04	0.079	3.120	0.00	0.058	3.052	0.03

We also simulated the β_1^* and β_2^* value of $\hat{\phi}$. For $\phi = 0$ (and small values of ϕ) they are very close to 0 and 3, respectively. For $\phi = 0.0$ and $n = 20$, for example, we have $\beta_1^* = 0.010$ and $\beta_2^* = 3.096$ when $b = 0.5$, $\beta_1^* = 0.001$ and $\beta_2^* = 2.985$ when $b = 4$, etc. For testing $H_0 : \phi = 0$ the null distribution of $\hat{\phi}/\sqrt{V(\hat{\phi})}$ is, therefore, referred to a normal distribution $N(0,1)$. The power of the test increases very rapidly to 1 as $|\phi|$ increases. We do not, however, reproduce details for conciseness.

Robustness properties: As for the gamma family, $\hat{\phi}$ has very good robustness properties for deviations from an assumed model (i.e., a particular value of b) and for outlier and mixture models. The reason is that $\hat{\phi}$ is essentially the solution of equation (31). This equation assigns the weight β_i to $z_{(i)} = \varepsilon_{(i)}/\sigma$ where $\varepsilon_{(i)}$ is the ith ordered residual. But β_i $(1 \le i \le n)$ have umbrella ordering. Thus, extreme residuals on both sides are automatically assigned small weights. This depletes the influence of outliers and long tails on either side. See also Tiku and Kumar (1998).

9 DISCUSSION

As said earlier, it is generally recognized that nonnormal distributions occur so frequently in practice. We have assumed here that the innovations are asymmetric and of the type $(1/\sigma)f((x-\mu)/\sigma)$ (location-scale model). The likelihood equations are intractable and solving them by iterative methods is tedious and time consuming and the results obtained might even be unreliable. We have, therefore, used the methodology of modified likelihood estimation. In the context of iid random sampling and in survey sampling, this method is known to yield estimators which are asymptotically fully efficient (Tiku 1970, Bhattacharyya 1985) and almost fully efficient for small n (Tiku 1967, Smith et al. 1973, Tan 1985, Schneider 1986, Tiku and Suresh 1992, Vaughan 1992, Tiku and Vellaisamy 1996). An attractive feature of the method is that it yields estimators (MML) which are explicit functions of sample observations and are, therefore, easy to compute. Being explicit functions, the MML estimators can also be studied analytically. We have derived the MML

TABLE VI
Asymptotic and simulated values of the means, variances and the covariance.

		Mean		$n\times$ variance		$n\times$ covariance
		$\hat{\phi}$	$\hat{\sigma}/\sigma$	$\hat{\phi}$	$\hat{\sigma}/\sigma$	$(\hat{\phi}, \hat{\sigma}/\sigma)$
		$n = 30$ and $\phi = 0$				
$k = 3$	Asymp	0.000	1.000	0.111	0.444	−0.111
	Simul	0.048	0.952	0.177	0.486	−0.177
$k = 4$	Asymp	0.000	1.000	0.168	0.417	−0.168
	Simul	0.044	0.956	0.213	0.444	−0.210
$k = 8$	Asymp	0.000	1.000	0.249	0.375	−0.249
	Simul	0.037	0.962	0.264	0.384	−0.267
$k = 12$	Asymp	0.000	1.000	0.279	0.360	−0.279
	Simul	0.034	0.965	0.291	0.369	−0.291
		$n = 30$ and $\phi = 0.5$				
$k = 3$	Asymp	0.500	1.000	0.036	0.477	−0.072
	Simul	0.526	0.948	0.051	0.513	−0.099
$k = 4$	Asymp	0.500	1.000	0.054	0.465	−0.108
	Simul	0.523	0.953	0.063	0.486	−0.126
$k = 8$	Asymp	0.500	1.000	0.080	0.446	−0.161
	Simul	0.519	0.961	0.081	0.447	−0.165
$k = 12$	Asymp	0.500	1.000	0.089	0.440	−0.179
	Simul	0.518	0.965	0.090	0.442	−0.187
		$n = 30$ and $\phi = 0.9$				
$k = 3$	Asymp	0.900	1.000	0.002	0.496	−0.016
	Simul	0.906	0.943	0.003	0.525	−0.021
$k = 4$	Asymp	0.900	1.000	0.002	0.494	−0.024
	Simul	0.905	0.949	0.003	0.513	−0.027
$k = 8$	Asymp	0.900	1.000	0.004	0.490	−0.036
	Simul	0.904	0.957	0.004	0.489	−0.036
$k = 12$	Asymp	0.900	1.000	0.004	0.489	−0.004
	Simul	0.904	0.960	0.004	0.487	−0.004
		$n = 50^*$ and $\phi = 0.0$				
$k = 3$	Asymp	0.000	1.000	0.111	0.444	−0.111
	Simul	0.032	0.966	0.165	0.480	−0.165
$k = 4$	Asymp	0.000	1.000	0.168	0.417	−0.168
	Simul	0.029	0.970	0.200	0.430	−0.200
$k = 8$	Asymp	0.000	1.000	0.249	0.375	−0.249
	Simul	0.024	0.975	0.265	0.380	−0.265
$k = 12$	Asymp	0.000	1.000	0.279	0.360	−0.279
	Simul	0.022	0.978	0.285	0.365	−0.285

* We have not given the values for $\phi = 0.5$ and 0.9 for conciseness. The agreement between the asymptotic and simulated values is much closer here than for $n = 30$.

TABLE VII

Values of $E(\hat{\phi})$, $V(\hat{\phi})$ and the Pearson coefficients of $\hat{\phi}$ when $\phi = 0$.

k	$10 \times E(\hat{\phi})$	$10^2 \times V(\hat{\phi})$	β_1^*	β_2^*	k	$10 \times E(\hat{\phi})$	$10^2 \times V(\hat{\phi})$	β_1^*	β_2^*
					$n = 20$				
2.5	0.690	0.8588	0.3640	3.420	3.0	0.662	0.9594	0.2485	3.283
3.5	0.635	1.0432	0.1987	3.215	4.0	0.605	1.1096	0.1442	3.131
4.5	0.589	1.1341	0.1403	3.109	5.0	0.565	1.1785	0.1088	2.998
5.5	0.564	1.2316	0.1129	3.067	6.0	0.553	1.2607	0.0926	3.066
					$n = 25$				
2.5	0.582	0.6557	0.3326	3.321	3.0	0.552	0.7415	0.2600	3.305
3.5	0.533	0.8087	0.1937	3.251	4.0	0.507	0.8663	0.1603	3.134
4.5	0.490	0.9008	0.1511	3.137	5.0	0.469	0.9293	0.1017	2.966
5.5	0.467	0.9738	0.0884	3.053	6.0	0.459	1.0004	0.0898	3.169
					$n = 30$				
2.5	0.505	0.5178	0.3212	3.366	3.0	0.477	0.5899	0.2307	3.254
3.5	0.459	0.6580	0.1934	3.213	4.0	0.438	0.7052	0.1216	3.082
4.5	0.427	0.7400	0.1302	3.125	5.0	0.404	0.7766	0.1063	3.042
5.5	0.402	0.7988	0.1094	3.179	6.0	0.395	0.8249	0.0863	3.145
					$n = 35$				
2.5	0.453	0.4403	0.3471	3.330	3.0	0.430	0.5034	0.2335	3.303
3.5	0.409	0.5567	0.2027	3.321	4.0	0.390	0.5913	0.1294	3.097
4.5	0.379	0.6345	0.1496	3.113	5.0	0.361	0.6655	0.1165	3.109
5.5	0.357	0.6785	0.0714	2.995	6.0	0.351	0.7001	0.0677	3.051
					$n = 40$				
2.5	0.409	0.3708	0.2964	3.260	3.0	0.386	0.4318	0.1939	3.098
3.5	0.371	0.4847	0.1893	3.241	4.0	0.346	0.5047	0.1363	3.173
4.5	0.336	0.5390	0.1204	3.075	5.0	0.320	0.5609	0.0797	3.006
5.5	0.320	0.5992	0.0817	3.094	6.0	0.314	0.6058	0.0607	3.031
					$n = 50$				
2.5	0.345	0.2921	0.3171	3.389	3.0	0.325	0.3344	0.1779	3.172
3.5	0.310	0.3807	0.1509	3.241	4.0	0.291	0.4034	0.1115	3.117
4.5	0.281	0.4296	0.1165	3.113	5.0	0.265	0.4581	0.1082	3.186
5.5	0.258	0.4660	0.0611	3.019	6.0	0.258	0.4765	0.0576	3.092
					$n = 60$				
2.5	0.298	0.2311	0.3019	3.358	3.0	0.282	0.2772	0.1876	3.270
3.5	0.268	0.3118	0.1157	3.083	4.0	0.252	0.3318	0.0917	3.114
4.5	0.245	0.3549	0.1180	3.202	5.0	0.227	0.3679	0.0890	3.124
5.5	0.218	0.3856	0.0626	3.117	6.0	0.218	0.3937	0.0400	3.103

estimators here in the context of $AR(q)$ models. These estimators are as attractive as in the classical framework of iid random observations. We have demonstrated their very high efficiencies not shared by the LS estimators.

Future work: We are presently in the process of extending the methodology above to autoregressive models

TABLE VIII
Simulated values* of means and variances, $n = 20$.

	(1) Mean	(1) Variance	(2) Mean	(2) Variance	(3) Mean	(3) Variance	(4) Mean	(4) Variance
				$\phi = 0.0$				
	$\theta = 0.791$		$\theta = 0.866$		$\theta = 1.118$		$\theta = 1.173$	
$\hat{\phi}_0$	−0.049	0.044	−0.049	0.044	−0.048	0.044	−0.047	0.044
$\hat{\phi}$	−0.045	0.010	−0.005	0.010	0.114	0.011	0.139	0.011
$\hat{\sigma}_0/\sigma$	0.736	0.029	0.805	0.032	1.045	0.044	1.096	0.047
$\hat{\sigma}/\sigma$	0.653	0.012	0.754	0.015	1.107	0.029	1.184	0.034
				$\phi = 0.3$				
$\hat{\phi}_0$	0.208	0.042	0.207	0.043	0.209	0.044	0.210	0.043
$\hat{\phi}$	0.251	0.006	0.287	0.006	0.388	0.006	0.408	0.006
				$\phi = 0.6$				
$\hat{\phi}_0$	0.461	0.038	0.459	0.039	0.461	0.040	0.462	0.038
$\hat{\phi}$	0.562	0.002	0.588	0.002	0.655	0.002	0.667	0.002
				$\phi = 0.9$				
$\hat{\phi}_0$	0.695	0.032	0.690	0.033	0.694	0.034	0.696	0.032
$\hat{\phi}$	0.889	0.0002	0.896	0.0002	0.915	0.0001	0.918	0.0001
	(5)		(6)		(7)		(8)	
				$\phi = 0.0$				
	$\theta = 1.140$		$\theta = 1.581$		$\theta = 1.062$		$\theta = 1.130$	
$\hat{\phi}_0$	−0.051	0.042	−0.051	0.034	−0.049	0.045	−0.047	0.044
$\hat{\phi}$	0.026	0.010	−0.039	0.007	−0.042	0.014	0.045	0.010
$\hat{\sigma}_0/\sigma$	1.192	0.111	2.184	0.684	0.990	0.039	1.050	0.054
$\hat{\sigma}/\sigma$	1.074	0.030	1.354	0.054	0.962	0.022	1.027	0.027
				$\phi = 0.3$				
$\hat{\phi}_0$	0.208	0.041	0.214	0.031	0.207	0.044	0.210	0.043
$\hat{\phi}$	0.315	0.006	0.251	0.005	0.254	0.008	0.331	0.006
				$\phi = 0.6$				
$\hat{\phi}_0$	0.462	0.037	0.474	0.028	0.458	0.040	0.461	0.039
$\hat{\phi}$	0.606	0.002	0.557	0.002	0.565	0.003	0.617	0.002
				$\phi = 0.9$				
$\hat{\phi}_0$	0.696	0.032	0.712	0.030	0.670	0.033	0.695	0.034
$\hat{\phi}$	0.902	0.0002	0.884	0.0002	0.890	0.0002	0.905	0.0002

* The means and variances of $\hat{\sigma}$ for non-zero values of ϕ are essentially the same as those for $\phi = 0$ and are not, therefore, reproduced.

$$y_t - \phi y_{t-1} = \mu + \beta(x_t - \phi x_{t-1}) + e_t \qquad (t = 1, 2, \ldots, n)$$

where x_t are the values of a design variable and e_t are random innovations and have a skew distribution. We will make our findings known at a future time; see also Tan and Lin (1993) who use Tiku's modified likelihood function based on censored normal samples for estimating the parameters in such models. See also Tiku, Wong and Bian (1999).

APPENDIX

Consider a random variable Z which has the generalized logistic distribution ($b > 0$)

$$f(z) = \frac{be^{-z}}{(1+e^{-z})^{b+1}} \qquad (-\infty < z < \infty).$$

The moments of Z are expressions in terms of $\psi(b) = \Gamma'(b)/\Gamma(b)$. In particular,

$$E(Z) = \psi(b) - \psi(1) \qquad \text{and} \qquad V(Z) = \psi'(b) + \psi'(1).$$

The expressions for the ψ-function $\psi(u)$ and its derivative $\psi'(u)$ are given in Abramowitz and Stegun (1965). The elements of the information matrix $I(\psi, \sigma)$ work out in terms of $\psi(b)$ and $\psi(b+1)$ and their derivatives, and $E(Z)$ and $E(Z^2)$. We tabulate their values below for easy accessibility:

b	$\psi(b)$	$\psi(b+1)$	$\psi'(b)$	$\psi'(b+1)$	$E(Z)$	$E(Z^2)$
0.1	−10.4238	−0.4238	101.4316	1.4331	−9.8466	200.0313
0.2	−5.2891	−0.2891	26.2665	1.2672	−4.7119	50.1132
0.5	−1.9635	0.0365	4.9348	0.9348	−1.3863	8.5015
1.0	−0.5772	0.4228	1.6449	0.6449	0.0000	3.2899
2.0	0.4228	0.9228	0.6449	0.3949	1.0000	3.2899
4.0	1.2561	1.5061	0.2838	0.2213	1.8333	5.2899
6.0	1.7061	1.8728	0.1813	0.1536	2.2833	7.0399
8.0	2.0156	2.1406	0.1331	0.1175	2.5929	8.5010

The asymptotic information matrix for the MML estimators is exactly the same as that for the ML estimators and is obtained from the following expressions:

$$g(z) = \frac{e^{-z}}{1+e^{-z}}, \qquad h(z) = \frac{e^{-z}}{(1+e^{-z})^2}:$$

$$E\{g(Z)\} = (b+1)^{-1}, \quad E\{Z\,g(Z)\} = (b+1)^{-1}[\psi(b) - \psi(2)]$$

$$
\begin{aligned}
E\{h(Z)\} &= b(b+1)^{-1}(b+2)^{-1} \\
E\{Z\,h(Z)\} &= b(b+1)^{-1}(b+2)^{-1}[\psi(b+1)-\psi(2)] \\
E\{Z^2 h(Z)\} &= b(b+1)^{-1}(b+2)^{-1}[\psi'(b+1)+\psi'(2)+\{\psi(b+1)-\psi(2)\}^2]
\end{aligned}
$$

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