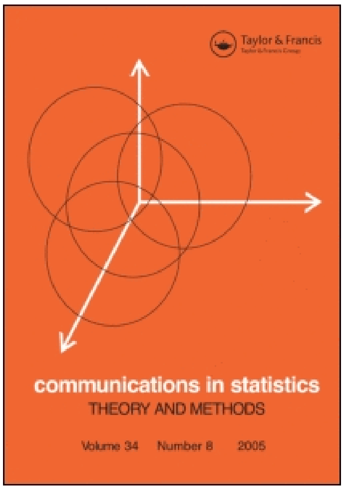

Communications in Statistics - Theory and Methods
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597238

MAXIMUM LIKELIHOOD ESTIMATION FOR GENERALISED LOGISTIC DISTRIBUTIONS

Quanxi Shao[a]
[a] CSIRO Mathematical and Information Sciences, Wembley, WA, U.S.A.

Online publication date: 11 April 2002

To cite this Article Shao, Quanxi(2002) 'MAXIMUM LIKELIHOOD ESTIMATION FOR GENERALISED LOGISTIC DISTRIBUTIONS', Communications in Statistics - Theory and Methods, 31: 10, 1687 — 1700
To link to this Article: DOI: 10.1081/STA-120014908
URL: http://dx.doi.org/10.1081/STA-120014908

COMMUNICATIONS IN STATISTICS
Theory and Methods
Vol. 31, No. 10, pp. 1687–1700, 2002

DISTRIBUTIONS AND APPLICATIONS

MAXIMUM LIKELIHOOD ESTIMATION FOR GENERALISED LOGISTIC DISTRIBUTIONS

Quanxi Shao

CSIRO Mathematical and Information Sciences,
Private Bag No. 5, Wembley, WA 6913, Australia
E-mail: quanxi.shao@csiro.au

ABSTRACT

Maximum likelihood estimation for the type I generalised logistic distributions is investigated. We show that the maximum likelihood estimation usually exists, except when the so-called embedded model problem occurs. A full set of embedded distributions is derived, including Gumbel distribution and a two-parameter reciprocal exponential distribution. Properties relating the embedded distributions are given. We also provide criteria to determine when the embedded distribution occurs. Examples are given for illustration.

Key Words: Embedded distribution; Gumbel distribution; Lehmann's power generalisation; Limiting distribution; Reciprocal exponential distribution

DOI: 10.1081/STA-120014908 0361-0926 (Print); 1532-415X (Online)

1. INTRODUCTION

Some properties of the generalised logistic distribution with cumulative distribution function

$$F_{\mathrm{GL1}}(x;\theta,\sigma,\alpha)=\{1+\exp[-(x-\theta)/\sigma]\}^{-\alpha},\quad (\sigma>0 \text{ and } \alpha>0), \tag{1.1}$$

are discussed in Zelterman.[1] Johnson et al.[2] called this distribution the type I generalised logistic distribution (GL1). The density function is given by

$$f_{\mathrm{GL1}}(x;\theta,\sigma,\alpha)=\frac{\alpha}{\sigma}\cdot\frac{e^{-(x-\theta)/\sigma}}{\{1+e^{-(x-\theta)/\sigma}\}^{\alpha+1}},\quad (\sigma>0 \text{ and } \alpha>0). \tag{1.2}$$

Although it is not mentioned, this generalisation does in fact fall into Lehmann's[3] power generalisation of a distribution. When α is a positive integer, this generalisation can also be treated as the distribution of the largest order statistics of α independent variables from a distribution. The skewness is determined by the power α. This generalised logistic distribution has been investigated widely in literature; see, for example, Ahuja and Nash,[4] Dubey,[5] George and Ojo,[6] Zelterman,[1,7,8] Balakrishnan and Leung[9,10] and Gerstenkorn.[11]

Zelterman (Ref. [1], Sec. 3) proved that maximum likelihood estimates for the type I generalised logistic distributions do not exist. An unintended exchange of logarithm and summation leads to this result. In fact, there are several possibilities about the maximum likelihood estimates and we derive all of these here.

In this paper we provide a full solution for the maximum likelihood estimates of the type I generalised logistic distribution. The paper is organised as follows. In Sec. 2 a full set of embedded distributions (or limiting distributions under an appropriate reparameterisation) is given, including Gumbel distribution and a two-parameter reciprocal exponential distribution. In Sec. 3 results are given in order to know whether or not an embedded distribution occurs. Examples are given in Sec. 4 for illustration. Conclusions and a discussion are given in Sec. 5.

2. EMBEDDED DISTRIBUTIONS

Like many other generalised distributions, the difficulties with respect to maximum likelihood estimation for the type I generalised logistic distribution are due to the so-called embedded model problem; see Cheng and Iles,[12] Cheng et al.[13] and Shao.[14] An embedded distribution can be treated as a special case of the original distribution but involving fewer parameters and is obtained by letting certain of the parameters in the original distribution tend to their boundary values (normally the boundary values are zero or infinity which are not allowed in the original parameter space). As we can see below, under an appropriate reparameterisation, an embedded distribution can be viewed as a limiting distribution. Therefore, it should not cause any difficulty to understand the paper if a reader does not know embedded model problems.

To derive the limiting distributions of the type I generalised logistic distribution, we consider a reparameterisation $(\theta, \sigma, \alpha) \to (\mu, \lambda, \alpha)$ with

$$\begin{cases} \mu = \theta + \sigma \ln(\alpha + 1), \\ \lambda = \sigma(\alpha + 1)/\alpha, \end{cases} \tag{2.1}$$

whose inverse transformation is

$$\begin{cases} \theta = \mu - \sigma \ln(\alpha + 1), \\ \sigma = \lambda\alpha/(\alpha + 1). \end{cases} \tag{2.2}$$

Then the cumulative distribution function of the type I generalised logistic distribution given in Eq. (1.1) can be rewritten as

$$F_{\mathrm{GL1}}(x; \mu, \lambda, \alpha) = \left\{1 + \left[e^{-(x-\mu)/\lambda}\right]^{(\alpha+1)/\alpha} /(\alpha + 1)\right\}^{-\alpha}. \tag{2.3}$$

The density function is as

$$f_{\mathrm{GL1}}(x; \mu, \lambda, \alpha) = \frac{1}{\lambda} \cdot \frac{[e^{-(x-\mu)/\lambda}]^{(\alpha+1)/\alpha}}{\left\{1 + [e^{-(x-\mu)/\lambda}]^{(\alpha+1)/\alpha}/(\alpha + 1)\right\}^{\alpha+1}}. \tag{2.4}$$

It can be easily seen from the reparameterised form that

$$\begin{cases} \lim\limits_{\alpha\to+\infty} F_{\mathrm{GL1}}(x; \mu, \lambda, \alpha) = F_G(x; \mu, \lambda), \\ \lim\limits_{\alpha\to+\infty} f_{\mathrm{GL1}}(x; \mu, \lambda, \alpha) = f_G(x; \mu, \lambda), \end{cases} \tag{2.5}$$

and

$$\begin{cases} \lim\limits_{\alpha\to+0} F_{\mathrm{GL1}}(x; \mu, \lambda, \alpha) = F_{\mathrm{RE}}(x; \mu, \lambda), \\ \lim\limits_{\alpha\to+0} f_{\mathrm{GL1}}(x; \mu, \lambda, \alpha) = f_{\mathrm{RE}}(x; \mu, \lambda), \end{cases} \tag{2.6}$$

where $F_G(x; \mu, \lambda)$ and $f_G(x; \mu, \lambda)$ are the cumulative distribution function and density function of the Gumbel distribution respectively given by

$$\begin{cases} F_G(x; \mu, \lambda) = \exp\left\{ -\mathrm{e}^{-(x-\mu)/\lambda} \right\}, \\ F_G(x; \mu, \lambda) = \lambda^{-1} e^{-(x-\mu)/\lambda} \exp\left\{ -\mathrm{e}^{-(x-\mu)/\lambda} \right\}, \end{cases} \tag{2.7}$$

and $F_{\mathrm{RE}}(x; \mu, \lambda)$ and $f_{\mathrm{RE}}(x; \mu, \lambda)$ are the cumulative distribution function and density function of the reciprocal two-parameter exponential distribution respectively given by

$$\begin{cases} F_{\mathrm{RE}}(x; \mu, \lambda) = \exp\big\{(x-\mu)/\lambda\big\} I_{(-\infty,\mu]}(x) + I_{(\mu,+\infty)}(x), \\ f_{\mathrm{RE}}(x; \mu, \lambda) = \lambda^{-1} \exp\big\{(x-\mu)/\lambda\big\} I_{(-\infty,\mu]}(x). \end{cases} \tag{2.8}$$

We call the above distribution the reciprocal exponential distribution because it can be obtained from the exponential distribution via the reciprocal transformation $X \to 1/X$. When $x = \mu$ the second limit of Eq. (2.6) is in fact $0.5\lambda^{-1}$. However, we modify the value to be λ^{-1} to have a proper distribution function.

By extending the parameter space of the type I generalised logistic distribution to allow $\alpha = +\infty$ and $\alpha = 0$, we can view the two-parameter reciprocal exponential distribution and the Gumbel distribution as special cases of the type I generalised logistic distribution. Using the method given by Shao,[14] we know that there is no more non-degenerated limiting distribution for the type I generalised logistic distribution.

From the reverse transformation (2.2), we know that in the original parameterised form (1.1) and (1.2) of the type I generalised logistic distribution, $\alpha = +\infty$ corresponds to $\theta \to -\infty$ with $\theta + \sigma \ln \alpha$ remaining finite and $\alpha = 0$ corresponds to $\sigma \to +\infty$ with σ/α remaining finite. This is, α and θ can be highly correlated, and α and σ can be highly negatively correlated. These correlations can also be revealed by directly examining its log-likelihood function as below.

Consider a sample of n independent observations $\{x_i; i = 1, 2, \ldots, n\}$. In the original parameterised form (1.1) and (1.2), the log-likelihood function takes the form

$$\begin{aligned} l_{\mathrm{GL1}}(\theta, \sigma, \alpha) = n \log(\alpha/\sigma) - \sum_{i=1}^{n} (x_i - \theta)/\sigma \\ - (\alpha + 1) \sum_{i=1}^{n} \log\{1 + \exp[-(x_i - \theta)/\sigma]\} \end{aligned} \tag{2.9}$$

Let the maximum likelihood estimate of (θ, σ, α) be denoted by $(\hat{\theta}, \hat{\sigma}, \hat{\alpha})$. Setting $\partial l_{\text{GL1}}/\partial\alpha = 0$ gives

$$\hat{\alpha} = n \Big/ \sum_{i=1}^{n} \log\{1 + \exp[-(x_i - \hat{\theta})/\hat{\sigma}]\}, \tag{2.10}$$

from which we can easily see that $\hat{\alpha} \to \infty$ if $\hat{\theta} \to -\infty$ and that $\hat{\alpha} \to 0$ if $\hat{\sigma} \to 0$.

3. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimates $(\hat{\theta}, \hat{\sigma}, \hat{\alpha})$, of the parameters (θ, σ, α) in the original form can be founded by numerical methods such as Newton-Raphson procedure. However, the numerical methods can collapse if $\hat{\alpha} = +\infty$ and $\hat{\theta} = -\infty$ (which could happen if the Gumbel is the true distribution) or $\hat{\alpha} = 0$ and $\hat{\sigma} = 0$ (which could happen if the two-parameter reciprocal exponential is the true distribution). We now investigate the behaviours of maximum likelihood estimation for the type I generalised logistic distribution at these boundaries. Bear in mind the invariance property of maximum likelihood estimation under reparameterisation (parameter transformation). Therefore, simpler reparameterisations will be used in the proofs of theorem.

Lemma 1. *For the type I generalised logistic distribution given by Eqs.* (1.1) *and* (1.2), $\hat{\theta}$ *is bounded above with* $-\infty < \hat{\theta} < \max(x_i)$ *and conditional on* $\hat{\theta} = \max(x_i)$, *the reciprocal two-parameter exponential distribution is fitted.*

Proof. Under a simpler reparameterisation $(\theta, \sigma, \alpha) \to (\theta, \lambda, \alpha)$ with

$$\lambda = \sigma/\alpha, \tag{3.1}$$

the density function of the type I generalised logistic distribution is given as

$$f_{\text{GL1}}(x; \theta, \lambda, \alpha) = \frac{1}{\lambda} \cdot \frac{\{e^{-(x-\mu)/\lambda}\}^{1/\alpha}}{[1 + \{e^{-(x-\mu)/\lambda}\}^{1/\alpha}]^{\alpha+1}}. \tag{3.2}$$

Define

$$l^*_{\text{GL1}}(\theta) = \max_{\sigma, \alpha}\{l_{\text{GL1}}(\theta, \sigma, \alpha)\} = \max_{\lambda, \alpha}\{l_{\text{GL1}}(\theta, \lambda, \alpha)\},$$

which is the conditional maximum of $l_{\mathrm{GL1}}(\theta, \sigma, \alpha)$ with respect to θ. We firstly show that provided that $\mu \geq x$, $f_{\mathrm{GL1}}(x; \theta, \lambda, \alpha)$ is a decreasing function of α. Let $y = e^{-(x-\theta)/\lambda} \geq 1$ and $h(y; \alpha) = \ln[y^{1/\alpha}/(1 + y^{1/\alpha})^{\alpha+1}]$. Simple calculations give that

$$\begin{aligned}\frac{dh(y;\alpha)}{d\alpha} &= -\alpha^{-2}(1 + y^{1/\alpha}) \\ &\quad \times \left\{\ln y + \alpha^2 \ln(1 + y^{1/\alpha}) + \alpha^2 y^{1/\alpha} \ln(1 + y^{1/\alpha}) - \alpha y^{1/\alpha} \ln y\right\}.\end{aligned}$$

Note that $\alpha^2 y^{1/\alpha} \ln(1 + y^{1/\alpha}) > \alpha y^{(\alpha+1)/\alpha} \ln y$. Therefore, $dh(y; \alpha)/d\alpha < 0$. Note that

$$l_{\mathrm{GL1}}(\theta, \lambda, \alpha) = -n \ln \lambda + \sum_{i=1}^{n} h(y_i; \alpha)$$

with $y_i = e^{-(x_i-\theta)/\lambda}$. Therefore, given $\theta \geq \max(x_i)$, $l^*_{\mathrm{GL1}}(\theta)$ is always maximised at $\alpha = 0$, which corresponds to the two-parameter reciprocal exponential distribution because

$$\lim_{\alpha \to 0} f_{\mathrm{GL1}}(x; \theta, \lambda, \alpha) = f_{\mathrm{RE}}(x; \theta, \lambda). \tag{3.3}$$

Note that for the two-parameter reciprocal exponential distribution, the maximum likelihood estimate of θ is $\hat{\theta} = \max(x_i)$. The proof is completed.

Theorem 1. *A local maximum at $\alpha = 0$ always exist and the reciprocal two-parameter exponential distribution is fitted.*

Proof. Note that the maximum likelihood estimates of the two-parameter exponential distribution $f_{\mathrm{RE}}(x; \theta, \lambda)$ are

$$\hat{\theta}_{\mathrm{RE}} = \max_{i=1,\ldots,n}(x_i) \quad \text{and} \quad \hat{\lambda}_{\mathrm{RE}} = \sum_{i=1}^{n}(x_i - \hat{\theta}_{\mathrm{RE}})/n. \tag{3.4}$$

For the reparameterised form given in Eq. (3.1), we know from the proof of Lemma 1 that for each $i \in \{1, \ldots, n\}$, $f_{\mathrm{GL1}}(x_i; \hat{\theta}, \hat{\lambda}, \alpha)$ is a decreasing function of α because $\hat{\theta} \geq x_i (i = 1, \ldots, n)$. Therefore,

$$l_{\mathrm{GL1}}(\hat{\theta}_{\mathrm{RE}}, \hat{\lambda}_{\mathrm{RE}}, \alpha) < l_{\mathrm{RE}}(\hat{\theta}_{\mathrm{RE}}, \hat{\lambda}_{\mathrm{RE}}), \tag{3.5}$$

where $l_{\mathrm{RE}}(\hat{\theta}_{\mathrm{RE}}, \hat{\lambda}_{\mathrm{RE}})$ is the maximum log-likelihood value of the two-parameter reciprocal exponential distribution. Therefore, a local maximum of $l_{\mathrm{GL1}}(\theta, \lambda, \alpha)$ always exists at $\alpha = 0$, which is associated with the two-parameter reciprocal exponential distribution. The proof is completed.

Next we provide the condition for the existence of a local maximum at $\alpha = +\infty$ corresponding to the Gumbel distribution.

Theorem 2. *Let*

$$\Delta_G(\mu, \sigma) = \sum_{i=1}^{n} \left\{ e^{-2y_i}/2 + y_i e^{-y_i} - 2y_i \right\} \tag{3.6}$$

with

$$y_i = (x_i - \mu)/\sigma,$$

and $(\hat{\mu}_G, \hat{\sigma}_G)$ being the maximum likelihood estimate of the Gumbel distribution $f_G(x; \mu, \sigma)$. If $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) > 0$, then a local maximum at $\alpha = +\infty$ does not exist. If $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) < 0$, then a maximum at $\alpha = +\infty$ exists and Gumbel distribution is reduced.

Proof. Under another simpler reparameterisation $(\theta, \sigma, \alpha) \to (\mu, \sigma, \alpha)$ with

$$\mu = \theta + \sigma \ln \alpha, \tag{3.7}$$

the density function of the type I generalised logistic distribution is given as

$$f_{\text{GL1}}(x; \mu, \sigma, \alpha) = \frac{1}{\sigma} \cdot \frac{e^{-(x-\mu)/\lambda}}{\left[1 + e^{-(x-\mu)/\lambda}/\alpha\right]^{\alpha+1}}. \tag{3.8}$$

We know by the Taylor series expansion that the log-likelihood function of $f_{\text{GL1}}(x; \mu, \sigma, \alpha)$ can be written as

$$l_{\text{GL1}}(\theta, \sigma, \alpha) = l_G(\mu, \sigma) + \Delta_G(\mu, \sigma)/\alpha + O(1/\alpha^2) \tag{3.9}$$

where $l_G(\mu, \sigma)$ is the log-likelihood function of Gumbel distribution $f_G(x; \mu, \sigma)$. Note that $\alpha > 0$. The sign of $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G)$ determines whether or not Gumbel distribution occurs in distribution fitting. The proof is completed.

Based on Theorems 1 and 2, the behaviour of the maximum likelihood estimation for the type I generalised logistic distribution is clear. There is always a local maximum at $\alpha = 0$ corresponding to the two-parameter reciprocal exponential distribution. Whether or not $\alpha = +\infty$ gives a local maximum depends on the sign of $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G)$ defined in Eq. (3.6). Let $l^*_{\text{GL1}}(\alpha)$ be the conditional maximum log-likelihood function of the type I generalised logistic distribution with respect to α. It is expected that $l^*_{\text{GL1}}(\alpha)$ is

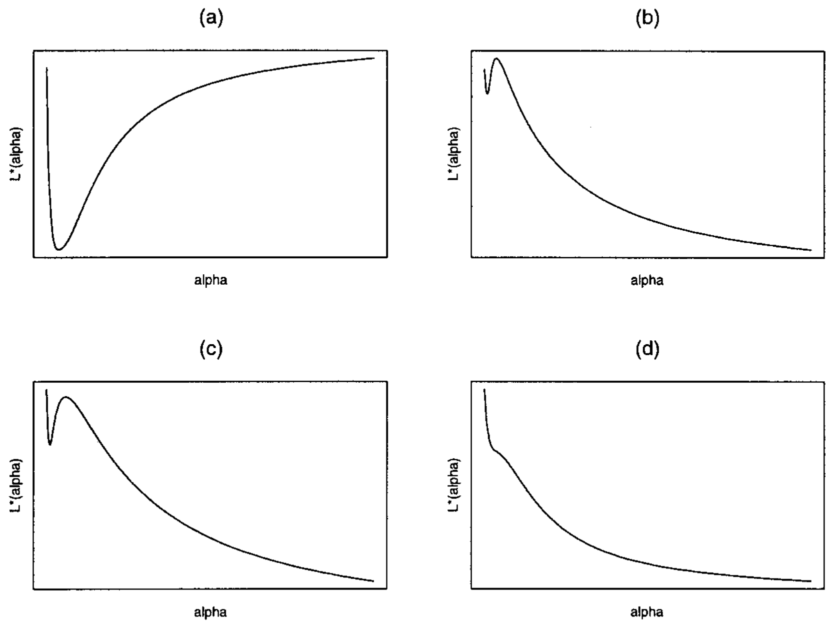

Figure 1. Typical shapes of $l^*_{\mathrm{GL1}}(\alpha)$, the conditional maximum log-likelihood values with respect to α for the type I generalised logistic distribution.

bell-shaped around $\hat{\alpha}$ (the maximum likelihood estimate of α) when the maximum likelihood estimates exist for the type I generalised logistic distribution. Various combinations of these local maxima may happen. A simulation study indicates that $l^*_{\mathrm{GL1}}(\alpha)$ have four types of shape as illustrated in Fig. 1. When the maximum likelihood estimation does not exist for the type I generalised logistic distribution (Fig. 1(a)) and $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) < 0$, there are two competing local maxima at extrema $\alpha = 0$ (the two-parameter reciprocal exponential distribution) and $\alpha = +\infty$ (Gumbel distribution). The final selection is determined by comparing their maximum (log-) likelihood values. When maximum likelihood estimation exists for the type I generalised logistic distribution (Fig. 1(b) and (c)), we know that $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) > 0$. In this case, the maximum likelihood value of the type I generalised logistic distribution needs also comparing with the local maximum at $\alpha = 0$ (the two-parameter reciprocal exponential distribution). Finally, if maximum likelihood estimation does not exist for the type I generalised logistic distribution and $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) > 0$, the maximum likelihood estimates occur at $\alpha = 0$ (the two-parameter reciprocal exponential distribution).

4. NUMERICAL EXAMPLES

In this section we use some examples to illustrate the behaviours of the maximum likelihood estimation for the type I generalised logistic distribution. All results are produced in S-Plus.[15] We first consider the example of residuals in the experiment of breaking strength of cured concrete used in Zelterman.[1] The parameter estimates of Gumbel distribution are $\hat{\mu}_G = -35.607$ and $\hat{\sigma}_G = 63.407$ with the log-likelihood -108.512. The criterion $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) = -0.3115 < 0$ confirms that Gumbel distribution can be selected. The parameter estimates of the reciprocal exponential distribution are $\hat{\theta}_{\mathrm{RE}} = 187.50$ and $\hat{\lambda}_{\mathrm{RE}} = 187.50$ with the log-likelihood value of -118.442. Overall, Gumbel distribution is selected. The profile log-likelihood $l^*_{\mathrm{GL1}}(\alpha)$ with respect to α is given in Fig. 2.

The second example is a simulated dataset from the standard logistic distribution (i.e., $\theta = 0$, $\sigma = 1$ and $\alpha = 1$). Twenty observations were generated and are listed in Table 1. The criterion $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) = 1.941 > 0$ confirms that the Gumbel distribution, whose maximum likelihood estimates are $\hat{\mu}_G = -0.303$ and $\hat{\sigma}_G = 1.428$ with maximum log-likelihood -37.717, does not occur. Numerical optimisation gives the maximum likelihood

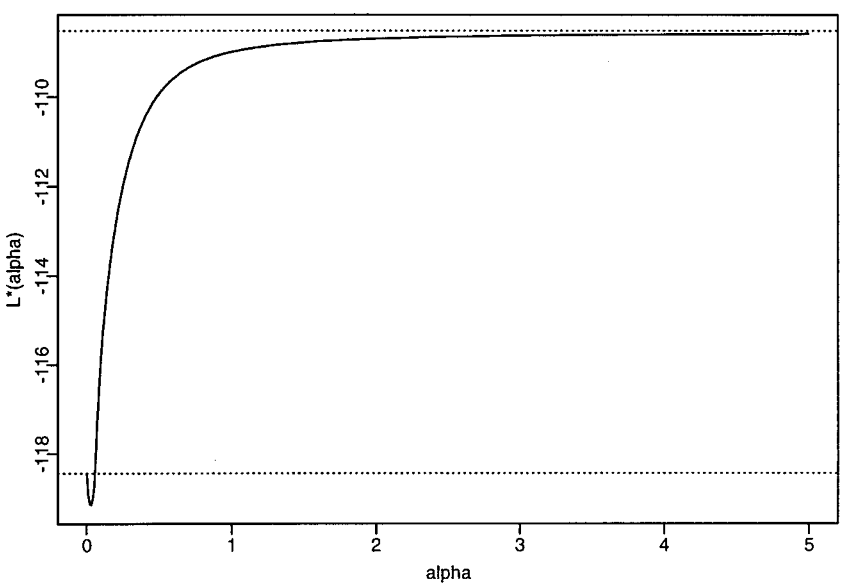

Figure 2. Plot of $l^*_{\mathrm{GL1}}(\alpha)$ for residuals in the experiment of breaking strength of cured concrete: type I generalised logistic distribution.

Table 1. Simulated Data of Size 20 from Standard Logistic Distribution (i.e., $\theta = 0$, $\sigma = 1$ and $\alpha = 1$)

−2.500	−1.740	−1.330	−0.873	−0.840
−0.823	−0.630	−0.135	0.350	0.398
0.781	0.815	1.140	1.230	1.380
1.750	1.800	2.360	2.440	3.490

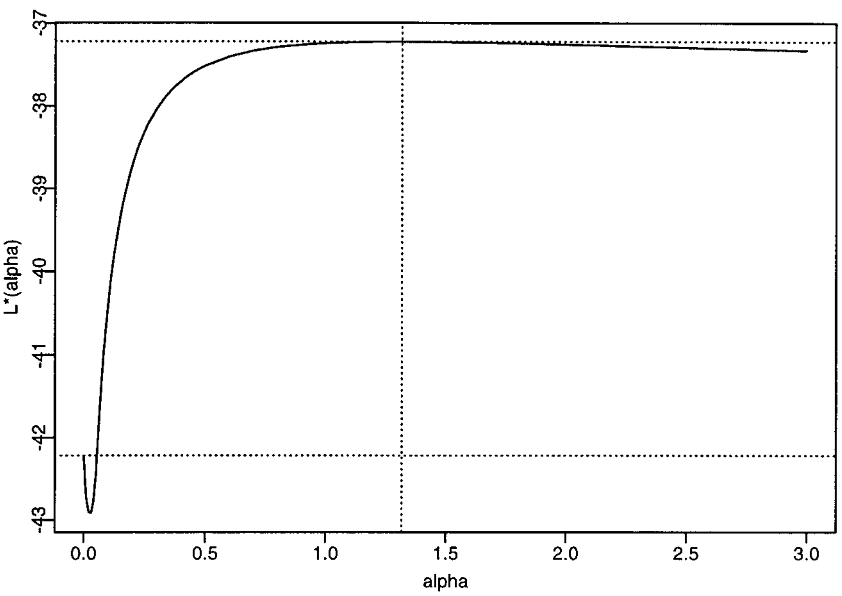

Figure 3. Plot of $l^*_{\mathrm{GL1}}(\alpha)$ for 20 simulated data from standard logistic distribution (i.e., $\theta = 0$, $\sigma = 1$ and $\alpha = 1$).

estimates of the type I generalised logistic distribution as $\hat{\theta} = 0.0613$, $\hat{\sigma} = 0.976$ and $\hat{\alpha} = 1.317$ with log-likelihood -37.222. The parameter estimates of the two-parameter reciprocal exponential distribution are $\hat{\theta}_{\mathrm{RE}} = 3.490$ and $\hat{\lambda}_{\mathrm{RE}} = 3.337$ with maximum log-likelihood -42.216. The profile log-likelihood $l^*_{\mathrm{GL1}}(\alpha)$ is given in Fig. 3, and a maximum likelihood solution clearly exist for the type I generalised logistic distribution.

The third example is a simulated dataset from the two-parameter reciprocal exponential distribution with $\theta = \lambda = 1$. Twenty observations were simulated and are listed in Table 2. The maximum likelihood estimates of Gumbel distribution are $\hat{\mu}_G = -0.4404$ and $\hat{\sigma}_G = 0.8375$ with maximum log-likelihood -26.837. The criterion $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) = 1.2980 > 0$ suggests that

Table 2. Simulated Data of Size 20 from the Two-Parameter Exponential Distribution $x_0 = \lambda = 1$

−1.4700	−1.3600	−1.3300	−1.2500	−0.8970
−0.4890	−0.3130	−0.1650	0.0168	0.3710
0.3990	0.4010	0.4260	0.5280	0.5730
0.6100	0.9160	0.9330	0.9940	0.9940

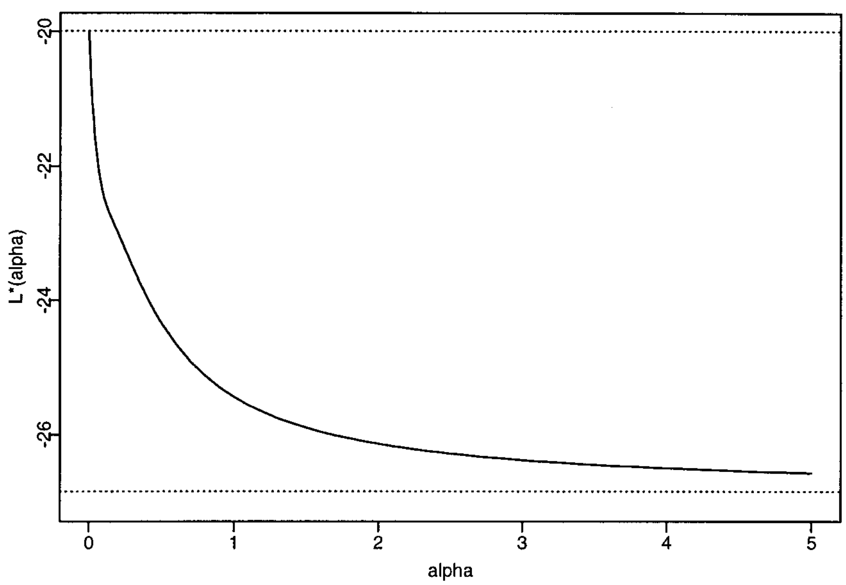

Figure 4. Plot of $l^*_{\mathrm{GL1}}(\alpha)$ for 20 simulated data from the two-parameter exponential distribution $x_0 = \lambda = 1$.

the Gumbel distribution does not occur. However, numerical optimisation cannot find maximum for the original type I generalised logistic distribution as the solution always occurs at the pre-defined lower bound of α. Therefore, the two-parameter reciprocal exponential distribution must provide the final fit. The maximum likelihood estimates are $\hat{\theta}_{\mathrm{RE}} = 0.994$ and $\hat{\lambda}_{\mathrm{RE}} = 1.000$ with maximum log-likelihood -19.992. The profile log-likelihood $l^*_{\mathrm{GL1}}(\alpha)$ is given in Fig. 4.

The final example is again simulated data selected from our simulation study when we investigated the typical shapes of $l^*_{\mathrm{GL1}}(\alpha)$ and listed in Table 3. The parameter estimates of Gumbel distribution are $\hat{\mu}_G = -3.234$ and $\hat{\sigma}_G = 3.328$ with the log-likelihood -26.940. The criterion $\Delta_G(\hat{\mu}_G, \hat{\sigma}_G) = 2.149 > 0$ confirms that Gumbel distribution cannot be selected. Numerical

Table 3. Simulated Data for Size 10 Selected from a Simulation Study

−8.1230734	−4.8773361	−3.7457921	−2.0466915	−1.2090857
−1.1443515	0.2108284	0.5199428	1.6482366	2.7843219

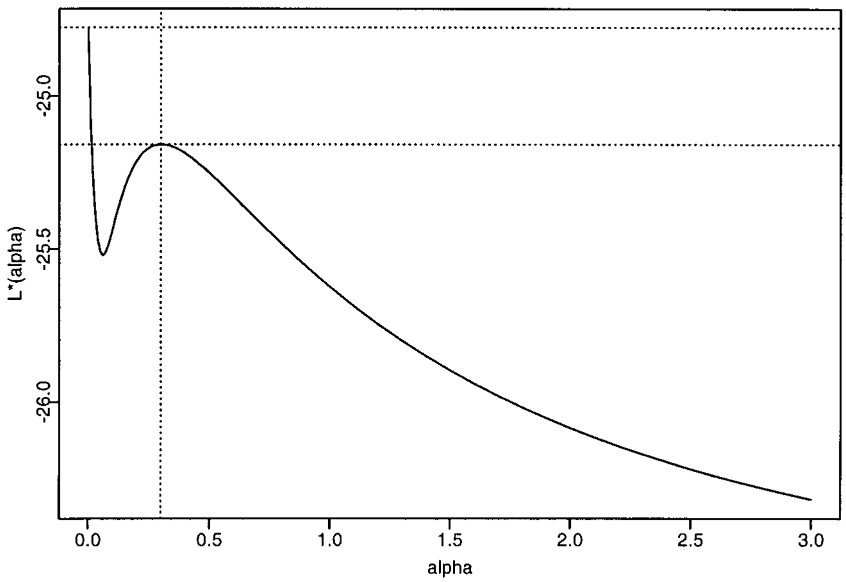

Figure 5. Plot of $l^{*}_{\mathrm{GL1}}(\alpha)$ for 10 simulated data selected from our simulation study.

optimisation provides the maximum likelihood estimates of the type I generalised logistic distribution are $\hat{\theta} = 1.1178$, $\hat{\sigma} = 0.9224$ and $\hat{\alpha} = 0.2997$ with log-likelihood -25.1578. The parameter estimates of the reciprocal exponential distribution are $\hat{\theta}_{\mathrm{RE}} = 2.7843219$ and $\hat{\lambda}_{\mathrm{RE}} = 4.382622$ with the log-likelihood value of -24.77647. Finally, the reciprocal exponential distribution is selected. The profile log-likelihood $l^{*}_{\mathrm{GL1}}(\alpha)$ is given in Fig. 5.

5. CONCLUSIONS

The type I generalised logistic distribution Eq. (1.1) has been proposed and used in the literature. This generalisation was due to Lehmann's[3] power generalisation. Zelterman's[1] conclusion that maximum likelihood estimates do not exist for the type I generalised logistic distribution is not true in general. We show that the maximum likelihood estimation does not exist only when the so-called embedded model problem occurs. A full set of embedded distributions (or limiting distributions under appropriate reparameterisations) is also derived, including Gumbel distribution and a two-parameter reciprocal exponential distribution. The behaviours of maximum likelihood estimates were investigated. We show that there is always a local maximum for the profile log-likelihood function $l^*_{\mathrm{GL1}}(\alpha)$ as $\alpha \to 0$ corresponding to the two-parameter reciprocal exponential distribution. We also provide criterion by which we can know whether or not there is a local maximum for $l^*_{\mathrm{GL1}}(\alpha)$ as $\alpha \to +\infty$ corresponding to Gumbel distribution.

While the generalisation given in Eq. (1.1) is called the type I generalised logistic distribution by Johnson et al.,[2] a parallel generalisation, called the type II generalised logistic distribution, is given by

$$F_{\mathrm{GL1}}(x; \theta, \sigma, \alpha) = 1 - \{1 + \exp[(x-\theta)/\sigma]\}^{-\alpha}, \quad (\sigma > 0 \text{ and } \alpha > 0),$$

This generalisation is formed by $1 - \{F_L[-(x-\theta)/\sigma]\}^{\alpha}$ while the type I generalisation is given by $\{F_L[(x-\theta)/\sigma]\}^{\alpha}$, where $F_L(x) = (1+e^{-x})^{-1}$ is the standard logistic distribution. The type II generalisation is the reciprocal of the type I generalisation via the transformation $((X-\theta)/\sigma) \to (\sigma/(X-\theta))$. Similar results can be derived for the type II generalised logistic distribution.

We should also note, but do not pursue here, that some possible tests can be addressed. For instance, we may assess the improvement in the likelihood at the cost of introducing the skewness parameter α. We may also wish to assess the improvement in the likelihood l_{GL1} over that in the limiting cases.

ACKNOWLEDGMENTS

The author would like to thank Drs George Brown and Eddy Campbell for their careful reading and comments on the draft of the paper. He also thanks the editor for the valuable comments.

REFERENCES

1. Zelterman, D. Parameter Estimation in the Generalized Logistic Distribution. Comput. Stat. Data An. **1987a**, *5*, 177–184.
2. Johnson, N.L.; Kotz, S.; Balakrishnan, N. *Continuous Univariate Distributions*; Vol. 2, 2nd Ed.; John Wiley: New York, 1995.
3. Lehmann, E.L. The Power of Rank Tests. Ann. Math. Statist. **1953**, *24*, 23–42.

4. Ahuja, J.C.; Nash, S.W. The Generalized Gompertz-Verhulst Family of Distributions. Sankhya Ser. A **1967**, *29*, 141–156.
5. Dubey, S.D. A New Derivation of the Logistic Distribution. Nav. Res. Log. **1969**, 16, 37–40.
6. George, E.O.; Ojo, M.O. On a Generalization of the Logistic Distribution. Ann. Inst. Statist. Math. **1980**, *32*, 161–169.
7. Zelterman, D. Estimation of Percentage Points by Simulation. J. Statist. Comput. Simul. **1987b**, *5*, 107–125.
8. Zelterman, D. Order Statistics of the Generalized Logistic Distribution. Comput. Stat. Data An. **1989**, *7*, 69–77.
9. Balakrishnan, N.; Leung, M.Y. Order Statistics from the Type I Generalized Logistic Distribution. Commun. Stat. Simul. C **1988a**, *17*, 25–50.
10. Balakrishnan, N.; Leung, M.Y. Means, Variances and Covariances of Order Statistics, BLUE's for the Type I Generalized Logistic Distribution, and Some Applications. Commun. Stat. Simul. C **1988b**, *17*, 51–84.
11. Gerstenkorn, T. Estimation of a Parameter of the Logistic Distribution. *Translations of the Eleventh Prague Conference on Information Theory*, *Statistical Decision Function*, *Random Processes*, Prague: Academic Publishing House of the Czechoslovak Academy of Sciences, 1992, 441–448.
12. Cheng, R.C.H.; Iles, T.C. Embedded Models in Three-Parameter Distributions and Their Estimation. J. R. Statist. Soc.B **1990**, *52*, 135–149.
13. Cheng, R.C.H.; Evans, B.E.; Iles, T.C. Embedded Models in Non-Linear Regression. J. R. Statist. Soc. B **1992**, *54*, 877–888.
14. Shao, Q. A Reparameterisation Method for Embedded Models. Commun. Stat. Theor. M. **2002**, *31*(5), 683–697.
15. S-Plus 4.5. Data Analysis Products Division, MathSoft, Inc., 1988–1998.