Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Statistics & Probability Letters 71 (2005) 61–70
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Estimating parameters in autoregressive models with asymmetric innovations

Wing-Keung Wong[a,*], Guorui Bian[b]

[a]*Department of Economics, Faculty of Arts; Social Sciences, National University of Singapore, 1 Arts Link, Singapore, 117570, Singapore*
[b]*Department of Statistics, East China Normal University, China*

Received 2 November 2001; received in revised form 11 September 2004
Available online 19 November 2004

Abstract

Tiku et al. (Theory Methods 28(2) (1999) 315) considered the estimation in a regression model with autocorrelated error in which the underlying distribution be a shift-scaled Student's t distribution, developed the modified maximum likelihood (MML) estimators of the parameters and showed that the proposed estimators had closed forms and were remarkably efficient and robust.

In this paper, we extend the results to the case, where the underlying distribution is a generalized logistic distribution. The generalized logistic distribution family represents very wide skew distributions ranging from highly right skewed to highly left skewed. Analogously, we develop the MML estimators since the ML (maximum likelihood) estimators are intractable for the generalized logistic data. We then study the asymptotic properties of the proposed estimators and conduct simulation to the study.

Keywords: Autoregression; Nonnormality; Modified maximum likelihood; Least squares; Robustness; Generalized logistic distribution

*Corresponding author. Tel.: +65 6874 6014; fax: +65 6775 2646.
E-mail address: ecswwk@nus.edu.sg (W.-K. Wong).

0167-7152/$ - see front matter
doi:10.1016/j.spl.2004.10.022

1. Introduction

The estimation of coefficients in a simple regression model with autocorrelated errors is an important problem and has received a great deal of attention in the literature. Most of the work reported is, however, based on the assumption of normality; see, for example, Anderson (1949), Cochrane and Orcutt (1949), Durbin (1960), Beach and Machinnon (1978), Magee et al. (1987), Dielman and Pfaffenberger (1989), Maller (1989), Cogger (1990), Weiss (1990), Schäffler (1991), Nagaraj et al. (1992), Tan and Lin (1993). The paper by Tan and Lin (1993) is of particular interest. They assumed normality but based their estimators on censored samples. They showed that the resulting estimators are robust to plausible deviations from normality. In recent years, however, it has been recognized that the underlying distribution is, in most situations, basically not normal; see, for example, Huber (1981), Tiku et al. (1986, 1999, 2000), Wong and Miller (1990) and Bian and Wong (1997). The problem, therefore, is to develop efficient estimators of coefficients in autoregressive models when the underlying distribution is non-normal. Naturally, one would prefer closed form estimators which are fully efficient (or nearly so). Preferably, these estimators should also be robust to plausible deviations from an assumed model.

Tiku et al. (1999) studied the estimation in autoregressive models with the underlying distribution be a shift-scaled Student's t distribution. They developed the modified maximum likelihood (MML) estimators of the parameters and showed that the proposed estimators had closed forms and were remarkably efficient and robust.

In this paper, we extend the work of Tiku et al. (1999) to the case, where the underlying distribution is a generalized logistic distribution. The generalized logistic distribution family represents a very wide skew distributions ranging from highly right skewed to highly left skewed. Analogously, we develop the MML estimators since the maximum likelihood (ML) estimators are intractable for the generalized logistic data. Then we study the asymptotic properties of the proposed estimators and conduct simulation to the study.

2. Regression model with autoregressive error

Consider the autoregressive model

$$\begin{aligned} y_t &= \mu' + \delta x_t + \eta_t, \\ \eta_t &= \phi\,\eta_{t-1} + \varepsilon_t, \quad (t = 1, 2, 3, \ldots, n), \end{aligned} \tag{1}$$

where

y_t = observed value of a random variable y at time t,

x_t = value of a nonstochastic design variable x at time t, and

ϕ = autoregressive coefficient ($|\phi| < 1$).

The autoregressive model (1) has many applications. For example, in predicting future stock prices the effect of an intervention might persist for some time. Numerous other applications of the above model are in agricultural, biological and biomedical problems besides business and economics; see, for example, Anderson (1949), Durbin (1960), Beach and Machinnon (1978), Cogger (1990), Weiss (1990), Schäffler (1991) and Wong and Bian (2000).

It is assumed that the innovations e_t are independent and identically distributed according to a generalized logistic distribution. Namely, the density function of ε_t ($t = 1, 2, \ldots, n$) is

$$f(\varepsilon) = \frac{b\mathrm{e}^{-\varepsilon/\sigma}}{\sigma(1 + \mathrm{e}^{-\varepsilon/\sigma})^{b+1}}, \quad (-\infty < \varepsilon < \infty). \tag{2}$$

The cumulative distribution is given by

$$F(\varepsilon) = (1 + \mathrm{e}^{-\varepsilon/\sigma})^{-b}. \tag{3}$$

The logistic distribution is negatively skew as $b < 1$ and positively skew as $b > 1$. It is symmetric when $b = 1$.

3. Modified maximum likelihood estimators

An alternative form of the model (1) is

$$y_t - \phi y_{t-1} = \mu + \delta(x_t - \phi x_{t-1}) + \varepsilon_t, \quad (1 \leqslant t \leqslant n) \tag{4}$$

or

$$(1 - \Phi B)Y = \mu\mathbf{1} + \delta(1 - \phi D)X + \mathbf{e},$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix},$$

$\mathbf{1}$ is an $n \times 1$ of 1's and B is the backward shift operator.

Conditional on y_0, the likelihood function for the model (4) is

$$L(\mu, \delta, \phi, \sigma) \propto \sigma^{-n} \Pi_{i=1}^{n} \frac{\mathrm{e}^{-z_i}}{(1 + \mathrm{e}^{-z_i})^{b+1}},$$

where $z_t = (1/\sigma)\{(y_t - \phi y_{t-1}) - \mu - \delta(x_t - \phi x_{t-1})\}$; see Hamilton (1994, p. 123) for numerous advantages of conditional likelihoods. The log-likelihood function is

$$ln\, L(\mu, \delta, \phi, \sigma) \propto -n\, ln(\sigma) - \sum_{i=1}^{n} z_i - (b+1) \sum_{i=1}^{n} ln[1 + \mathrm{e}^{-z_i}]. \tag{5}$$

Denote $g(z) = 1/1 + e^z$ and take the derivatives of the log-likelihood, we obtain

$$\begin{aligned}
\frac{\partial \ln L}{\partial \mu} &= \frac{n}{\sigma} - \frac{b+1}{\sigma}\sum_{i=1}^{n} g(z_i), \\
\frac{\partial \ln L}{\partial \delta} &= \frac{1}{\sigma}\sum_{i=1}^{n}(x_i - \phi x_{i-1}) - \frac{b+1}{\sigma}\sum_{i=1}^{n}(x_i - \phi x_{i-1})g(z_i), \\
\frac{\partial \ln L}{\partial \phi} &= \frac{1}{\sigma}\sum_{i=1}^{n}(y_{i-1} - \delta x_{i-1}) - \frac{b+1}{\sigma}\sum_{i=1}^{n}(y_{i-1} - \delta x_{i-1})g(z_i), \\
\frac{\partial \ln L}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_i - \frac{b+1}{\sigma}\sum_{i=1}^{n} z_i\, g(z_i).
\end{aligned} \tag{6}$$

The ML estimators are solutions of the likelihood equations,

$$\frac{\partial \ln L}{\partial \mu} = 0, \quad \frac{\partial \ln L}{\partial \delta} = 0, \quad \frac{\partial \ln L}{\partial \phi} = 0, \quad \text{and} \quad \frac{\partial \ln L}{\partial \sigma} = 0. \tag{7}$$

These equations are, however, intractable. Solving them by iterative methods can be very problematic, e.g., one may encounter multiple roots, slow convergence, or converge to wrong values or even divergence; see specifically Barnett (1966) and Lee et al. (1980).

To obtain efficient closed form estimators, we invoke Tiku's method of modified likelihood estimation which is by now well established (Smith et al., 1973; Lee et al., 1980; Tan, 1985; Schneider, 1986; Vaughan, 1992; Tiku et al., 1986, 1999, 2000). For given values of μ, δ and ϕ, let $z_{(1)} \leqslant z_{(2)} \leqslant \cdots \leqslant z_{(n)}$ (arranged in ascending order) be the order statistics of z_i $(1 \leqslant i \leqslant n)$. Let $t_{(i)} = E\{z_{(i)}\}$ $(1 \leqslant i \leqslant n)$ be the expected values of the standardized order statistics. Denote $[i]$ as the concomitant index of the ith observation which corresponds to the order statistic $z_{(i)}$. Clearly,

$$[i] = j, \quad \text{if} \quad z_i = z_{(j)}. \tag{8}$$

Since $g(z)$ is almost linear in a small interval $c \leqslant z \leqslant d$ (Tiku, 1967, 1968; Tiku and Suresh, 1992) and realizing that under some very general regularity conditions $z_{(i)}$ converges to $t_{(i)}$ as n becomes large, we use the first two terms of a Taylor series expansion to obtain

$$g(z_{(i)}) \simeq a_i - b_i z_{(i)}, \quad (1 \leqslant i \leqslant n), \tag{9}$$

where

$$a_i = (1 + e^{t_i})^{-1} + b_i t_i, \quad b_i = e^{t_i}(1 + e^{t_i})^{-2}, \quad \text{and} \quad t_{(i)} = E\{z_{(i)}\} = -\ln\left[\left(\frac{i}{n+1}\right)^{-\frac{1}{b}} - 1\right].$$

Substituting (9) in (7), we obtain the modified likelihood equations which can be written as

$$\frac{\partial \ln L}{\partial \mu} \simeq \frac{\partial \ln L^*}{\partial \mu} = \frac{n}{\sigma} - \frac{b+1}{\sigma}\sum_{i=1}^{n}(a_{[i]} - b_{[i]} z_i) = 0,$$

$$\frac{\partial \ln L}{\partial \delta} \simeq \frac{\partial \ln L^*}{\partial \delta} = \frac{1}{\sigma}\sum_{i=1}^{n}(x_i - \phi x_{i-1}) - \frac{b+1}{\sigma}\sum_{i=1}^{n}(x_i - \phi x_{i-1})(a_{[i]} - b_{[i]}z_i) = 0,$$
$$\frac{\partial \ln L}{\partial \phi} \simeq \frac{\partial \ln L^*}{\partial \phi} = \frac{1}{\sigma}\sum_{i=1}^{n}(y_{i-1} - \delta x_{i-1}) - \frac{b+1}{\sigma}\sum_{i=1}^{n}(y_{i-1} - \delta x_{i-1})(a_{[i]} - b_{[i]}z_i) = 0,$$
$$\frac{\partial \ln L}{\partial \sigma} \simeq \frac{\partial \ln L^*}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n} z_i - \frac{b+1}{\sigma}\sum_{i=1}^{n} z_i(a_{[i]} - b_{[i]}z_i) = 0. \tag{10}$$

Solving the estimating Eq. (10), we obtain the MML estimators:

$$\begin{pmatrix}\hat{\mu}\\ \hat{\delta}\end{pmatrix} = (X_1' W X_1)^{-1}[X_1' W(1-\hat{\phi}B)Y + X_1'\mathbf{a}\hat{\sigma}], \tag{11}$$

$$\begin{pmatrix}\hat{\mu}\\ \hat{\phi}\end{pmatrix} = (X_2' W X_2)^{-1}[X_2' W(Y-\hat{\delta}X) + X_2'\mathbf{a}\hat{\sigma}], \tag{12}$$

$$\hat{\sigma} = \frac{B + \sqrt{B^2 + 4nC}}{2n}, \tag{13}$$

where

$$W = \text{Diagonal}(b_{[1]}, b_{[2]}, \ldots, b_{[n]}),$$
$$\mathbf{a} = \frac{1}{b+1}\mathbf{1} - \begin{pmatrix} a_{[1]} \\ \vdots \\ a_{[n]} \end{pmatrix},$$
$$X_1 = (\mathbf{1}, (1-\hat{\phi}B)X),$$
$$X_2 = (\mathbf{1}, B(Y-\hat{\delta}X)),$$
$$B = -(b+1)[(1-\hat{\phi}B)Y]'\mathbf{a},$$
$$C = (b+1)[(1-\hat{\phi}B)Y]'W[(1-\hat{\phi}B)Y - \hat{\delta}(1-\hat{\phi}B)X - \hat{\mu}\mathbf{1}].$$

It is clear that the MML estimators above have all closed form algebraic expressions. Moreover, they are asymptotically equivalent to the ML estimators.

Computations. To initialize ordering of $z(i)$, we ignore the constraint $\gamma = -\delta\phi$ (Durbin 1960; Tan and Lin, 1993; Tiku et al., 1999) and calculate the LS estimators $\hat{\mu}_0$, $\hat{\delta}_0$, $\hat{\phi}_0$ and $\hat{\gamma}_0$ such that

$$\begin{pmatrix}\hat{\mu}_0\\ \hat{\delta}_0\\ \hat{\phi}_0\\ \hat{\gamma}_0\end{pmatrix} = \begin{pmatrix} n & \sum x_i & \sum y_{i-1} & \sum x_{i-1} \\ \sum x_i & \sum x_i^2 & \sum y_{i-1}x_i & \sum x_i x_{i-1} \\ \sum y_{i-1} & \sum y_{i-1}x_i & \sum y_i^2 & \sum y_{i-1}x_{i-1} \\ \sum x_i & \sum x_i x_{i-1} & \sum y_{i-1}x_{i-1} & \sum x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum y_i x_i \\ \sum y_i y_{i-1} \\ \sum y_i x_{i-1}\end{pmatrix},$$

each sum is carried over $i = 1, 2, \ldots, n$. Initially, we set

$$z_{(i)} = (1/\sigma)\{(y_{[i]} - \hat{\phi}_0 y_{[i]-1}) - \hat{\mu}_0 - \hat{\delta}_0(x_{[i]} - \hat{\phi}_0 x_{[i]-1})\}, \quad (1 \leqslant i \leqslant n). \tag{14}$$

Using the initial concomitants $(y_{[i]}, x_{[i]})$ $(1 \leqslant i \leqslant n)$ determined by (14), the MML estimator $\hat{\sigma}$ is first calculated from (13) with $\phi = \hat{\phi}_0$ and $\delta = \hat{\delta}_0$. The MML estimator $\hat{\mu}$, $\hat{\phi}$ and $\hat{\delta}$ are then calculated from Eqs. (11), (12) with $\sigma = \hat{\sigma}$. Few more iterations are carried out till the estimates stabilize (Tiku et al., 1999, 2000). In all our computations partly presented in this paper, no more than three iterations were needed for the estimates to stabilize.

4. Asymptotic results

Since $\partial \ln L^*/\partial\mu$, $\partial \ln L^*/\partial\delta$, $\partial \ln L^*/\partial\phi$ and $\partial \ln L^*/\partial\sigma$ are, as discussed earlier, asymptotically equivalent to $\partial \ln L/\partial\mu$, $\partial \ln L/\partial\delta$, $\partial \ln L/\partial\phi$ and $\partial \ln L/\partial\sigma$ respectively, we have the following asymptotic results. Efficient estimators, typically, have these properties.

Lemma 1. *The MML estimators, $\hat{\mu}(\phi, \sigma)$ and $\hat{\delta}(\phi, \sigma)$ are asymptotically and conditionally (for known ϕ and σ) the MVB (minimum variance bound) estimator with variance*

$$\frac{\sigma^2}{b+1}(X_1' W X_1)^{-1}.$$

Proof. From (10), we have

$$\begin{pmatrix} \dfrac{\partial \ln L^*}{\partial\mu} \\ \dfrac{\partial \ln L^*}{\partial\delta} \end{pmatrix} = \frac{b+1}{\sigma^2}(X_1' W X_1)^{-1}\begin{pmatrix} \hat{\mu}(\phi,\sigma) - \mu \\ \hat{\delta}(\phi,\sigma) - \delta \end{pmatrix},$$

where

$$\begin{pmatrix} \hat{\mu}(\phi,\sigma) - \mu \\ \hat{\delta}(\phi,\sigma) - \delta \end{pmatrix} = (X_1' W X_1)^{-1}[X_1' W(1 - \hat{\phi}B)Y + X_1' \mathbf{a}\sigma].$$

When ϕ and σ are given, $(X_1' W X_1)$ is independent from observations and both $1/n\,|\partial \ln L/\partial\mu - \partial \ln L^*/\partial\mu|$ and $1/n\,|\partial \ln L/\partial\delta - \partial \ln L^*/\partial\delta|$ tend to zero as n goes to infinity (Kendall and Stuart, 1979, Chapter 18). Hence, $\hat{\mu}(\phi, \sigma)$ and $\hat{\delta}(\phi, \sigma)$ are asymptotically the MVB estimators. □

Theorem 1. *For given ϕ and σ, $\hat{\mu}$, $\hat{\delta}$ are asymptotically unbiased and normally distributed with the variance-covariance matrix*

$$\Sigma(\phi,\sigma) = \frac{\sigma^2}{n}\frac{b+2}{b}\frac{1}{m_2 - m_1^2}\begin{pmatrix} m_2 & -m_1 \\ -m_1 & 1 \end{pmatrix},$$

where $m_1 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \phi x_{i-1})$, and $m_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \phi x_{i-1})^2$.

Proof. This follows from Lemma 1 and the fact that when n goes to infinity,

$$X_1' W X_1 \longrightarrow \frac{b}{(b+1)(b+2)} n \begin{pmatrix} 1 & m_1 \\ m_1 & m_2 \end{pmatrix}. \qquad \square$$

Lemma 2. *The MML estimator $\hat{\sigma}$ is asymptotically unbiased and normally distributed with the variance*

$$\frac{\sigma^2}{n}\left[-1 + 2E(z) - 2(b+1)E\left(\frac{z}{1+e^z}\right) + (b+1)E\left(\frac{z^2 e^z}{(1+e^z)^2}\right)\right]^{-1}.$$

Proof. This follows from the fact that $1/n\,|\partial \ln L/\partial\sigma - \partial \ln L^*/\partial\sigma|$ goes to zero as n goes to infinity and

$$\frac{\partial^2 \ln L^*}{\partial\sigma^2} = -\frac{n}{\sigma^2}\left[-1 + \frac{2}{n}\sum_{i=1}^{n} z_i - \frac{b+1}{n}\sum_{i=1}^{n}(2\alpha_{[i]}z_i - 3\beta_{[i]}z_i^2)\right],$$

which gives

$$-E\left(\frac{\partial^2 \ln L^*}{\partial\sigma^2}\right) = \frac{n}{\sigma^2}[-1 + 2E(z) - \frac{b+1}{n}\sum_{i=1}^{n}(2\alpha_i t_i - 3\beta_i t_i^2)]$$

$$\longrightarrow \frac{n}{\sigma^2}\left[-1 + 2E(z) - 2(b+1)E\left(\frac{z}{1+e^z}\right) + (b+1)E\left(\frac{z^2 e^z}{(1+e^z)^2}\right)\right].$$

$$\text{as} \quad n \to \infty. \qquad \square$$

Lemma 3. *The MML estimator $\hat{\phi}$ is conditionally (known δ and σ) asymptotically unbiased with the variance given by*

$$\frac{\sigma^2}{n}(b+1)\left[\frac{1}{n}E\left(\sum_{i=1}^{n}\beta_{[i]}(y_{i-1} - \delta x_{i-1})\right)^2\right]^{-1}.$$

Proof. This follows from the fact that

$$\frac{\partial^2 \ln L^*}{\partial\phi^2} = -\frac{-(b+1)}{\sigma^2}\sum_{i=1}^{n}\beta_{[i]}(y_{i-1} - \delta x_{i-1})^2. \qquad \square$$

5. Simulation

AS the MML estimators $\hat{\mu}$, $\hat{\delta}$, $\hat{\sigma}$ and $\hat{\phi}$ are asymptotically unbiased and normally distributed with the same variance as the minimum variance bound estimators while the LS estimators are wildly used irrespective of the nature of the underlying distribution, MML estimators are expected to be more efficient than the LS estimators. In this paper, we investigate their efficiencies for sample size of 100 with the x-values (common to all y-samples) being generated from a normal distribution $N(0,1)$ (Tan and Lin, 1993). For simplicity, we only consider b to be 1 and 2 in our simulation. Without loss of generality, we chose the following settings in our simulation:

1. $\mu = 0$, $\delta = 1$, $\phi = 0.1$ and $\sigma = 1$,
2. $\mu = 0$, $\delta = 1$, $\phi = 0.5$ and $\sigma = 1$, and
3. $\mu = 0$, $\delta = 1$, $\phi = 0.8$ and $\sigma = 1$.

In the 10,000 Monte Carlo runs, we simulate the estimates of all parameters for each run and for each of the parameters μ, δ, ϕ and σ, we compute the mean, $100 \times (bias)^2$, variance and MSE for both the LS and the MML estimators and for $n = 100$ with three alternative settings and with $b = 1$ and 2. The results reported in Table 1 show that the MML estimators are considerably more

Table 1
The simulated values of mean, bias square and mean square error of the LS estimators $\hat{\phi}_0$, $\hat{\delta}_0$, $\hat{\mu}_0$ and $\hat{\sigma}_0$, and the MML estimators $\hat{\phi}$, $\hat{\delta}$, $\hat{\mu}$ and $\hat{\sigma}$; $n = 100$

		$b = 1.0$				$b = 2.0$			
		Mean	$100 \times (\text{Bias})^2$	MSE	var	Mean	$100 \times (\text{Bias})^2$	MSE	var
$\mu = 0.0$	$\hat{\mu}_0$	−0.0033	0.0011	0.0347	0.0347	1.0110	102.2169	1.0576	0.0355
	$\hat{\mu}$	−0.0022	0.0005	0.0276	0.0276	0.3415	11.6632	0.1443	0.0277
$\delta = 1.0$	$\hat{\delta}_0$	1.0014	0.0002	0.0294	0.0294	1.0010	0.0001	0.0205	0.0205
	$\hat{\delta}$	1.0006	0.0000	0.0226	0.0226	1.0008	0.0001	0.0157	0.0157
$\phi = 0.10$	$\hat{\phi}_0$	0.0879	0.0145	0.0102	0.0101	0.0876	0.0153	0.0102	0.0100
	$\hat{\phi}$	0.0916	0.0070	0.0077	0.0076	0.0917	0.0069	0.0076	0.0075
$\sigma = 1.0$	$\hat{\sigma}_0$	1.1472	2.1674	0.0546	0.0329	1.6178	38.1724	0.4088	0.0270
	$\hat{\sigma}$	1.0230	0.0529	0.0082	0.0077	1.3333	11.1074	0.1312	0.0201
$\mu = 0.0$	$\hat{\mu}_0$	−0.0034	0.0012	0.0377	0.0377	1.0445	109.0989	1.1468	0.0558
	$\hat{\mu}$	−0.0034	0.0012	0.0303	0.0303	0.3560	12.6766	0.1717	0.0449
$\delta = 1.0$	$\hat{\delta}_0$	1.0014	0.0002	0.0294	0.0294	1.0012	0.0002	0.0204	0.0204
	$\hat{\delta}$	1.0008	0.0001	0.0186	0.0186	1.0006	0.0000	0.0130	0.0130
$\phi = 0.50$	$\hat{\phi}_0$	0.4757	0.0590	0.0087	0.0081	0.4760	0.0006	0.0085	0.0079
	$\hat{\phi}$	0.4828	0.0295	0.0065	0.0062	0.4847	0.0235	0.0063	0.0060
$\sigma = 1.0$	$\hat{\sigma}_0$	1.5467	29.8860	0.3523	0.0535	2.4899	221.9896	2.3008	0.0809
	$\hat{\sigma}$	1.0287	0.0824	0.0123	0.0115	1.3218	11.1412	0.1341	0.0227
$\mu = 0.0$	$\hat{\mu}_0$	−0.0036	0.0013	0.0496	0.0496	1.1502	132.2960	1.4490	0.1260
	$\hat{\mu}$	−0.0050	0.0025	0.0381	0.0381	0.3369	11.3511	0.2153	0.1018
$\delta = 1.0$	$\hat{\delta}_0$	1.0011	0.0001	0.0295	0.0295	1.0028	0.0008	0.0205	0.0205
	$\hat{\delta}$	0.9992	0.0001	0.0142	0.0142	1.0016	0.0002	0.0097	0.0097
$\phi = 0.80$	$\hat{\phi}_0$	0.7641	0.1287	0.0061	0.0048	0.7677	0.1044	0.0053	0.0043
	$\hat{\phi}$	0.7757	0.0590	0.0043	0.0037	0.7841	0.0251	0.0033	0.0030
$\sigma = 1.0$	$\hat{\sigma}_0$	2.6449	270.5623	2.9559	0.2502	5.4392	1970.6780	20.2371	0.5303
	$\hat{\sigma}$	1.0250	0.0627	0.0131	0.0125	1.3440	11.8336	0.1457	0.0274

efficient than the LS estimators for all parameters as almost all MML estimators have smaller bias, smaller variance and smaller MSE than the LS estimators.

6. Summary

In this paper, we extend the results of Tiku et al. (1999) to the case, where the underlying distribution for the error term is a generalized logistic distribution. We develop the MML estimators and find that the MML estimators are asymptotically unbiased and normally distributed with the same variance as the minimum variance bound estimators. Further extension includes applying the work to Economics or Finance, see for example Thompson and Wong (1991, 1996), Wong and Li (1999), Wong et al. (2001), Wong and Chan (2004) and Fong et al. (2004); and incorporating Bayesian approach (Matsumura et al., 1990, Bian and Tiku, 1997a,b and Wong and Bian, 2000) in the MMLE estimation.

Acknowledgements

Our deepest thanks to Professor Moti Lal Tiku for initiating the issue and providing us constructive suggestions. We also thank Jun Du for assistance with the simulation. Special thanks also to Professor Richard Johnson and the anonymous referees for their valuable comments that have significantly improved this manuscript. The first author would like to thank Professors Robert B. Miller and Howard E. Thompson for their continuous guidance and encouragement. The research is partially supported by the grant numbered R-122-000-082-112 from National University of Singapore.

References

Anderson, R.L., 1949. The problem of autocorrelation in regression analysis. J. Amer. Statist. Assoc. 44, 113–127.

Barnett, V.D., 1966. Evaluation of the maximum likelihood estimator when the likelihood equation has multiple roots. Biometrika 53, 151–165.

Beach, C.M., Machinnon, J.G., 1978. A maximum likelihood procedure for regression with autocorrelated errors. Econometrika 46, 51–58.

Bian, G., Tiku, M.L., 1997a. Bayesian inference based on robust priors and MML estimators: Part I, symmetric location—scale distributions. Statistics 29, 317–345.

Bian, G., Tiku, M.L., 1997b. Bayesian inference based on robust priors and MML estimators: Part II, skew location—scale distributions. Statistics 29, 81–99.

Bian, G., Wong, W.K., 1997. An alternative approach to estimate regression coefficients. J. Appl. Statist. Sci. 6 (1), 21–44.

Cochrane, D., Orcutt, G.H., 1949. Application of least squares regression to relationships containing autocorrelated error terms. J. Amer. Statist. Assoc. 44, 32–61.

Cogger, K.O., 1990. Robust time series analysis—an L_1 approach. In: Lawrence, K.D., Arthur, J.L. (Eds.), Robust Regression. Marcel Dekker, New York.

Dielman, T.E., Pfaffenberger, R.C., 1989. Efficiency of ordinary least square for linear model with autocorrelation. J. Amer. Statist. Assoc. 84, 248.

Durbin, J., 1960. Estimation of parameters in time-series regression model. J. Roy. Statist. Soc. Ser. B Statist. Methodology 22, 139–153.

Fong, W.M., Lean, H.H., Wong, W.K., 2004. Stochastic dominance and the rationality of the momentum effect across markets. J. Financial Markets, forthcoming.

Hamilton, J.D., 1994. Time Series Analysis. Princeton University Press, New Jersey.

Huber, P.J., 1981. Robust Statistics. Wiley, New York.

Kendall, M.G., Stuart, A., 1979. The Advanced Theory of Statistics. Charles Griffin, London.

Lee, K.R., Kapadia, C.H., Dwight, B.B., 1980. On estimating the scale parameter of Rayleigh distribution from censored samples. Statist. Hefte 21, 14–20.

Magee, L., Ullah, A., Srivastava, V.K., 1987. Efficiency of estimators in the regression model with first-order autoregressive errors. Specification Analysis in the Linear Model, International Library of Economics, Routledge and Kegan Paul, London, pp. 81–98.

Maller, R.A., 1989. Regression with autoregressive errors—some asymptotic results. Statistics 20, 23–39.

Matsumura, E.M., Tsui, K.W., Wong, W.K., 1990. An extended multinomial-dirichlet model for error bounds for dollar-unit sampling. Contemp. Acc. Res. 6 (2), 485–500.

Nagaraj, N.K., Fuller, W.A., 1992. Least squares estimation of the linear model with autoregressive errors. New Directions in Time Series Analysis, Part I, IMA, Mathematics and its Applications, vol. 45, Springer, New York, pp. 215–225.

Schäffler, S., 1991. Maximum likelihood estimation for linear regression model with autoregressive errors. Statistics 22, 191–198.

Schneider, H., 1986. Truncated and Censored Samples from Normal Populations. Marcel Dekker, New York.

Smith, W.B., Zeis, C.D., Syler, G.W., 1973. Three parameter lognormal estimation from censored data. J. Indian Statist. Assoc. 11, 15–31.

Tan, W.Y., 1985. On Tiku's robust procedure—a Bayesian insight. J. Statist. Plann. Inference 11, 329–340.

Tan, W.Y., Lin, V., 1993. Some robust procedures for estimating parameters in an autoregressive model. Sankhya B 55, 415–435.

Thompson, H.E., Wong, W.K., 1991. On the unavoidability of 'unscientific' judgment in estimating the cost of capital. Managerial Decision Econom. 12, 27–42.

Thompson, H.E., Wong, W.K., 1996. Revisiting 'dividend yield plus growth' and its applicability. Eng. Econom. 41 (2), 123–147.

Tiku, M.L., 1967. Estimating the mean and standard deviation from censored normal samples. Biometrika 54, 155–165.

Tiku, M.L., 1968. Estimating the parameters of log-normal distribution from censored samples. J. Amer. Statist. Assoc. 63, 134–140.

Tiku, M.L., Suresh, R.P., 1992. A new method of estimation for location and scale parameters. J. Statist. Plann. Inference 30, 281–292.

Tiku, M.L., Wong, W.K., 1998. Testing for a unit root in an AR(1) model using three and four moment approximations: symmetric distributions. Comm. Statist. Simulation Comput. 27 (1), 185–198.

Tiku, M.L., Tan, W.Y., Balakrishnan, N., 1986. Robust Inference. Marcel Dekker, New York.

Tiku, M.L., Wong, W.K., Bian, G., 1999. Estimating parameters in autoregressive models in non-normal situations: symmetric innovations. Comm. Statist. Theory Methods 28 (2), 315–341.

Tiku, M.L., Wong, W.K., Vaughan, D.C., Bian, G., 2000. Time series models with nonnormal innovations: symmetric location—scale distributions. J. Time Ser. Anal. 21 (5), 571–596.

Vaughan, D.C., 1992. On the Tiku–Suresh method of estimation. Comm. Statist. Theory Methods 21, 451–469.

Weiss, G., 1990. Least absolute error estimation in the presence of serial correlation. J. Econometrics 44, 127–158.

Wong, W.K., Bian, G., 2000. Robust bayesian inference in asset pricing estimation. J. Appl. Math. Decision Sci. 4 (1), 65–82.

Wong, W.K., Chan, R., 2004. The estimation of the cost of capital and its reliability. Quant. Finance 4 (3), 365–372.

Wong, W.K., Li, C.K., 1999. A note on convex stochastic dominance theory. Econom. Lett. 62, 293–300.

Wong, W.K., Miller, R.B., 1990. Analysis of ARIMA-noise models with repeated time series. J. Business Econom. Statist. V8 (2), 243–250.

Wong, W.K., Chew, B.K., Sikorski, D., 2001. Can P/E ratio and bond yield be used to beat stock markets? Multinational Finance J. 5 (1), 59–86.