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Tests of fit for the logistic distribution based on the empirical distribution function

By M. A. STEPHENS
Department of Mathematics, Simon Fraser University, Burnaby, British Columbia

Summary

Goodness-of-fit tests are given for the logistic distribution, based on statistics calculated from the empirical distribution function. Emphasis is on the statistics W^2, U^2 and A^2, for which asymptotic percentage points are given, for each of the three cases where one or both of the parameters of the distribution must be estimated from the data. Slight modifications of the calculated statistics are given to enable the points to be used with small samples. Monte Carlo results are included also for statistics D^+, D^-, D and V.

Some key words: Cramér–von Mises statistic; Empirical distribution function; Goodness-of-fit; Kolmogorov–Smirnov statistic; Logistic distribution.

1. Introduction

In this paper we discuss the test of fit of H_0: that a random sample of n values of x comes from the logistic distribution

$$F(x) = [1+\exp\{-(x-\alpha)/\beta\}]^{-1} \quad (-\infty<x<\infty), \tag{1}$$

with possibly one or both of the parameters α and β unknown.

The tests are based on statistics which measure the discrepancy between the theoretical distribution function (1), with estimates inserted for any unknown parameters, and the empirical distribution function of the sample. Emphasis is placed on the statistics W^2, U^2 and A^2, for which asymptotic theory is derived; the theory is supported by Monte Carlo results to give percentage points for finite n. These tests are described in §§ 2–4. In § 5, some Monte Carlo results are given for the Kolmogorov and Kuiper statistics D^+, D^-, D and V, for which no asymptotic theory is available.

2. The goodness-of-fit tests

The null hypothesis H_0 is that the sample of x values is a random sample from the logistic distribution (1). It will be convenient to suppose the sample is labelled in ascending order, that is $x_1<\ldots<x_n$.

Four test situations may be distinguished, parallel to those of Stephens (1974, 1976, 1977): case 0, where both α and β are known, so that $F(x)$ is completely specified; case 1, where β is known and α is to be estimated; case 2, where α is known and β is to be estimated; and case 3, where both α and β are unknown, and must be estimated.

Suppose that the parameters are estimated from the sample by maximum likelihood; the estimates, for case 3, are given by the equations

$$n^{-1}\Sigma_i[1+\exp\{(x_i-\hat{\alpha})/\hat{\beta}\}]^{-1} = \tfrac{1}{2}, \tag{2}$$

$$n^{-1}\Sigma_i\left(\frac{x_i-\hat{\alpha}}{\hat{\beta}}\right)\frac{1-\exp\{(x_i-\hat{\alpha})/\hat{\beta}\}}{1+\exp\{(x_i-\hat{\alpha})/\hat{\beta}\}} = -1. \tag{3}$$

These two equations may be solved iteratively; suitable starting values for $\hat{\alpha}$ and $\hat{\beta}$ are $\alpha_0 = \bar{x}$ and $\beta_0 = s$, where $\bar{x}$ and s^2 are respectively the sample mean and variance. In case 1, β is known, and (2) is used for $\hat{\alpha}$ with β replacing $\hat{\beta}$. In case 2, α is known; $\hat{\beta}$ is given by solving (3) with α replacing $\hat{\alpha}$.

When the parameters have been estimated as necessary, the steps in testing H_0 are as follows.

(a) Calculate $z_i = F(x_i)$, where $F(x)$ is given in (1), with the appropriate estimates inserted for unknown parameters in cases 1, 2 or 3. The z_i will now be in ascending order. Let $\bar{z}$ be the mean of the z_i.

(b) Calculate the test statistic desired:

$$W^2 = \Sigma_i\{z_i - \tfrac{1}{2}(2i-1)/n\}^2 + (12n)^{-1}, \quad U^2 = W^2 - n(\bar{z}-\tfrac{1}{2})^2,$$

$$A^2 = -n^{-1}[\Sigma_i(2i-1)\{\log z_i + \log(1-z_{n+1-i})\}] - n.$$

(c) Refer to Table 1, first calculating the modified statistic and then comparing the result with the upper tail points given in the table, for the appropriate case. For example, suppose that with a sample of size 20, case 2, the value of A^2 is 2·150; the modification involves calculating (0·6) (20) (2·150) − 1·8 for the numerator, giving the value 24·0 and (0·6) (20) − 1·0 for the denominator, giving the value 11·0; the resulting modified A^2 is 2·182, which would be significant at about the 6% level. The modifications make only slight changes to the given value of a statistic, but they make it possible to dispense with tables of points for each n.

Table 1. *Percentage points for modified statistics* W^2, U^2, A^2

Statistic	Case	Modification	Upper tail percentage points, α 0·75	0·90	0·95	0·975	0·99	0·995
W^2	0	$(W^2-0{\cdot}4/n+0{\cdot}6/n^2)\,(1{\cdot}0+1{\cdot}0/n)$	0·209	0·347	0·461	0·581	0·743	0·869
	1	$(1{\cdot}9nW^2-0{\cdot}15)/(1{\cdot}9n-1{\cdot}0)$	0·083	0·119	0·148	0·177	0·218	0·249
	2	$(0{\cdot}95nW^2-0{\cdot}45)\,(0{\cdot}95n-1{\cdot}0)$	0·184	0·323	0·438	0·558	0·721	0·847
	3	$(nW^2-0{\cdot}08)/(n-1{\cdot}0)$	0·060	0·081	0·098	0·114	0·136	0·152
U^2	0	$(U^2-0{\cdot}1/n+0{\cdot}1/n^2)\,(1{\cdot}0+0{\cdot}8/n)$	0·105	0·152	0·187	0·221	0·267	0·304
	2	$(1{\cdot}6nU^2-0{\cdot}16)/(1{\cdot}6n-1{\cdot}0)$	0·080	0·116	0·145	0·174	0·214	0·246
A^2	0	None	1·248	1·933	2·492	3·070	3·857	4·500
	1	$A^2+0{\cdot}15/n$	0·615	0·857	1·046	1·241	1·505	1·710
	2	$(0{\cdot}6nA^2-1{\cdot}8)/(0{\cdot}6n-1{\cdot}0)$	1·043	1·725	2·290	2·880	3·685	4·308
	3	$A^2(1{\cdot}0+0{\cdot}25/n)$	0·426	0·563	0·660	0·769	0·906	1·010

For U^2 cases 1 and 3 use modifications and percentage points for W^2 cases 1 and 3 respectively: see §3.

3. Asymptotic theory of the tests

The percentage points in Table 1 are those of the asymptotic distributions of the test statistics. These are found from the theory of the empirical process, discussed, for example, by Durbin (1973). The statistics may be written as functionals of appropriate Gaussian processes, over the interval [0, 1], each with mean zero, and covariance depending on the statistic, the case and the distribution tested. The type of calculation is given by Stephens (1976), applied there to the normal distribution, and will not be repeated. The covariances for the different statistics, for the empirical process associated with the logistic distribution (1), are now set forth.

Statistic W^2. The covariances depend on functions

$$\phi_1(s) = -s(1-s)\sqrt{3}, \quad \phi_2(s) = \{3/(3+\pi^2)^{\frac{1}{2}}\}s(s-1)\log\{(1-s)/s\};$$

the covariances, $\rho_j(s,t)$ for case j, are then for $0 \leqslant s \leqslant t \leqslant 1$

$$\text{Case } 0: \rho_0(s,t) = s - st, \quad \text{Case } 1: \rho_1(s,t) = \rho_0(s,t) - \phi_1(s)\,\phi_1(t),$$

$$\text{Case } 2: \rho_2(s,t) = \rho_0(s,t) - \phi_2(s)\,\phi_2(t), \quad \text{Case } 3: \rho_3(s,t) = \rho_0(s,t) - \phi_1(s)\,\phi_1(t) - \phi_2(s)\,\phi_2(t).$$

The covariance in case 3 takes a relatively simple form because the estimates $\hat{\alpha}$ and $\hat{\beta}$ are asymptotically uncorrelated; for calculations when this is not so, concerning tests for the extreme value distribution, see Stephens (1977).

Statistic A^2. The covariances for A^2 are the same as those for W^2, for the corresponding case, multiplied by $w(s,t) = \{(s-s^2)(t-t^2)\}^{-\frac{1}{2}}$.

Statistic U^2. For U^2, we need also $a(s,t) = \frac{1}{12} + \frac{1}{2}\{s(s-1) + t(t-1)\}$; the covariances are then for $0 \leqslant s \leqslant t \leqslant 1$

$$\text{Case } 0: \rho_0(s,t) + a(s,t), \quad \text{Case } 1: \rho_0(s,t) + a(s,t) - \{\phi_1(s) + \sqrt{3}/6\}\{\phi_1(t) + \sqrt{3}/6\},$$

$$\text{Case } 2: \rho_0(s,t) + a(s,t) - \phi_2(s)\,\phi_2(t),$$

$$\text{Case } 3: \rho_0(s,t) + a(s,t) - \{\phi_1(s) + \sqrt{3}/6\}\{\phi_1(t) + \sqrt{3}/6\} - \phi_2(s)\,\phi_2(t).$$

The asymptotic distribution of a typical statistic, for illustration W^2, may then be expressed as

$$W^2 = \sum_{i=1}^{\infty} u_i^2/\lambda_i, \tag{4}$$

where u_i are independent standard normal variables, and where, for case j, the weights λ_i are the eigenvalues of the integral equation

$$f(s) = \lambda \int_0^1 \rho_j(s,t) f(t)\, dt. \tag{5}$$

Asymptotic points for W^2 are therefore found by solving (5) for the weights λ_i, and then calculating the percentage points of the distribution (4). The weights λ_i are found by a standard method described in detail by Stephens (1976, 1977). These weights have been found for the three statistics W^2, U^2 and A^2, for the cases 1, 2 and 3 above. In general, it is difficult to find exact percentage points for distribution (4), and approximations are needed. It is then useful to have the exact value of the mean. From (4) the mean μ is $\mu = \Sigma_i 1/\lambda_i$; μ can also be written as $\mu = \int \rho_j(s,s)\, ds$, where the limits of integration are over 0 to 1. The first expression for μ is not very useful since the series usually converges extremely slowly, but the second expression fortunately admits of direct calculations. These give for the four means of W^2: $\mu_0 = \frac{1}{6}$, $\mu_1 = 0{\cdot}06667$, $\mu_2 = 0{\cdot}14825$ and $\mu_3 = 0{\cdot}048253$. For U^2, the means are $\mu_0 = \frac{1}{12}$, $\mu_1 = 0{\cdot}06667$, $\mu_2 = 0{\cdot}06492$ and $\mu_3 = 0{\cdot}048253$ and for A^2 they are $\mu_0 = 1$, $\mu_1 = 0{\cdot}5$, $\mu_2 = 0{\cdot}84966$ and $\mu_3 = 0{\cdot}34966$.

There is an interesting special result which for the logistic distribution (1) makes it unnecessary to perform some of the calculations for U^2. This follows because (2) may be written $\Sigma_i F(x_i) = \frac{1}{2}n$; in the notation of step (a), § 2, we have $\bar{z} = \frac{1}{2}$, and thus $W^2 = U^2$. Equation (2) is used for estimation in cases 1 and 3, so that for these cases $W^2 = U^2$ for all n, and therefore they have the same distributions. Thus the theory for U^2 needs to be worked out independently only for case 2.

4. Calculation of percentage points

For all three cases the percentage points were calculated by Imhof's (1961) approximation, adapted for an infinite sequence of terms by Durbin & Knott (1972), and also by fitting

Pearson curves using the first four moments. In terms of values of α the points given by these methods differ negligibly; the Imhof points are those quoted in Table 1. Points in case 0 are included to show how much the percentage points drop in the other cases; they demonstrate how important it is to use the correct values for the appropriate case.

For the percentage points of the various statistics for finite n, Monte Carlo studies were made, with $n = 5$, 8, 10, 20 and 50; 10,000 samples were used for each case. Previous experience with these statistics (Stephens, 1974, 1977) suggests that convergence to the asymptotic points will be rapid, and a plot of percentage points against $1/n$ proves this to be so. The Monte Carlo points were used to calculate the modified forms given in Table 1; it can be seen that several models were employed to connect the percentage point of a test statistic for sample size n with its asymptotic point. It is believed that use of these modifications and the appropriate asymptotic point will give an error in α always less than 0·5%, for the upper tail given, and for $n > 5$.

5. Kolmogorov–Smirnov statistics

Another important group of statistics based on the distribution function contains those associated with the names of Kolmogorov, Smirnov and Kuiper; these are statistics D^+, D^-, D and V defined in terms of the z_i of § 2 as follows:

$$D^+ = \max_i(i/n - z_i), \quad D^- = \max_i\{z_i - (i-1)/n\}, \quad D = \max(D^+, D^-), \quad V = D^+ + D^-.$$

Although asymptotic distribution theory cannot be offered for these statistics for the cases 1, 2 and 3 of this paper, Monte Carlo results were obtained at the same time as for W^2, U^2 and A^2. The smoothed percentage points of $D^+\sqrt{n}$, $D\sqrt{n}$ and $V\sqrt{n}$ are given in Table 2. Points for $D^+\sqrt{n}$ can be used also for $D^-\sqrt{n}$. The asymptotic values are found by extrapolation, and the accuracy is therefore somewhat difficult to determine. Statistic D is well established, with a pictorial appeal to many statisticians; also D^+ and D^- enable one-sided tests to be made.

Table 2. *Upper tail percentage points for statistics $D^+\sqrt{n}$, $D\sqrt{n}$ and $V\sqrt{n}$, for cases 1, 2 and 3*

		Statistic $D^+\sqrt{n}$				Statistic $D\sqrt{n}$				Statistic $V\sqrt{n}$			
Case	n \ α	0·90	0·95	0·975	0·99	0·90	0·95	0·975	0·99	0·90	0·95	0·975	0·99
1	5	0·702	0·758	0·805	0·854	0·736	0·791	0·845	0·883	1·369	1·471	1·580	1·658
	10	0·730	0·792	0·846	0·913	0·777	0·837	0·895	0·953	1·410	1·520	1·630	1·741
	20	0·744	0·809	0·867	0·944	0·800	0·865	0·926	0·997	1·433	1·550	1·659	1·790
	50	0·752	0·819	0·880	0·962	0·808	0·874	0·937	1·011	1·447	1·564	1·675	1·815
	∞	0·757	0·826	0·888	0·974	0·816	0·883	0·947	1·025	1·454	1·574	1·685	1·832
2	5	0·971	1·120	1·239	1·380	1·108	1·236	1·349	1·474	1·314	1·432	1·547	1·674
	10	0·990	1·143	1·268	1·423	1·148	1·274	1·388	1·521	1·372	1·483	1·587	1·711
	20	0·999	1·150	1·282	1·444	1·167	1·294	1·406	1·545	1·400	1·510	1·607	1·730
	50	1·005	1·161	1·290	1·456	1·179	1·305	1·419	1·559	1·417	1·525	1·619	1·741
	∞	1·009	1·166	1·297	1·464	1·187	1·313	1·427	1·568	1·429	1·535	1·627	1·748
3	5	0·603	0·650	0·690	0·735	0·643	0·679	0·723	0·751	1·170	1·246	1·299	1·373
	10	0·636	0·687	0·736	0·789	0·679	0·730	0·774	0·823	1·230	1·311	1·381	1·466
	20	0·653	0·705	0·758	0·816	0·698	0·755	0·800	0·854	1·260	1·344	1·422	1·514
	50	0·663	0·716	0·773	0·832	0·708	0·770	0·817	0·873	1·277	1·364	1·448	1·542
	∞	0·669	0·723	0·781	0·842	0·715	0·780	0·827	0·886	1·289	1·376	1·463	1·560

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