Generalized Empirical Likelihood with R

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Abstract

This an extention of the gmmS4 vignette, to explain how to use the package for generalized empirical likelihood estimation.

Contents

1 A very brief review of the GEL method

We present how to use the package to estimate models by the Generalized Empirical Likelihood method (GEL) (see Newey and Smith (2004) for the iid case and Anatolyev (2005) for weakly dependent processes). We assume that the reader has read the gmmS4 vignette in which many classes and methods needed are defined. We first describe the method without going into too much details. The author can refer to the above papers for a detailed description, or Chaussé (2010) who explains GEL estimation using the gmm (Chaussé (2019))package.

The estimation is based on the following moment conditions

$$\mathrm{E}[g_i(\theta)] = 0,$$

For the iid case, the estimator is defined as the solution to either

$$\hat{\theta} = \arg\min_{\theta, p_i} \sum_{i=1}^{n} h_n(p_i)$$

subject to,

$$\sum_{i=1}^{n} p_i g_i(\theta) = 0$$

and

$$\sum_{i=1}^{n} p_i = 1,$$

where $h_n(p_i)$ belongs to the following Cressie-Read family of discrepancies:

$$h_n(p_i) = \frac{[\gamma(\gamma+1)]^{-1}[(np_i)^{\gamma+1} - 1]}{n},$$

or

$$\hat{\theta} = \arg\min_{\theta} \left[\max_{\lambda} \frac{1}{n} \sum_{i=1}^{n} \rho\left(\lambda' g_i(\theta)\right) \right] \tag{1}$$

The first is the primal and the second is the dual problem, the latter being preferred in general to define GEL estimators. The vector λ is the Lagrange multiplier associated with the first constraint in the primal problem. Its estimator plays an important role in testing the validity of the moment conditions. $\rho(v)$ is a strictly concave function normalized so that $\rho'(0) = \rho''(0) = -1$. It can be shown that $\rho(v) = \ln(1-v)$ corresponds to Empirical Likelihood (EL) of Owen (2001) , $\rho(v) = -\exp(v)$ to the Exponential Tilting (ET) of Kitamura and Stutzer (1997), and $\rho(v) = (-v - v^2/2)$ to the Continuous Updated GMM estimator (CUE) of Hansen et al. (1996). In the context of GEL, the CUE is also known at the Euclidean Empirical Likelihood (EEL), because it corresponds to $h_n(p_i)$ being the Euclidean distance.

Once the solution is obtained for θ and λ, the implied probabilities can be computed as follows

$$\hat{p}_i = \frac{\rho'(\hat{\lambda}' g_i(\hat{\theta}))}{\sum_{j=1}^{n} \rho'(\hat{\lambda}' g_j(\hat{\theta}))} \tag{2}$$

If we relax the iid assumption, the problem is identical, but the moment function must be smoothed using a kernel method. Smith (2001) proposes to replace $g_i(\theta)$ by:

$$g_i^w(\theta) = \sum_{s=-m}^{m} w(s) g_{i-s}(\theta)$$

where $w(s)$ are kernel based weights that sum to one (see also Kitamura and Stutzer (1997) and Smith (2001).

2 An S4 class object for GEL models

All classes for GEL models inherit directly from "gmmModels" classes. It is just a gmmModel class with two additional slots: "gelType" and "wSpec". The first is a list with the name of the GEL method and a function for $\rho(v)$, if not already available in the package. The second slot is used to store information about the smoothing of $g_i(\theta)$ in the case of non-iid observations. As for gmmModels, "gelModels" is a union class for "linearGel", "nonlinearGel", "formulaGel" and "functionGel". They inherit directly from the associated GMM model class. For example, "linearGel" is a class that contains a "linearGmm" class object and the two additional slots. The constructor is the gelModel() function. We use the same models presented in the gmmS4 vignette:

```
library(gmm4)

## Loading required package:  sandwich

data(simData)
```

- Linear models:

```
lin <- gelModel(y~x1+x2, ~x2+z1+z2, data=simData, vcov="iid", gelType="EL")
lin

## GEL Model: Type  EL
## ******************************
## Moment type: linear
## No Smoothing required
##
## Number of regressors: 3
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50
```

- Formula-type models:

```
theta0=c(mu=1,sig=1)
x <- simData$x1
dat <- data.frame(x=x, x2=x^2, x3=x^3, x4=x^4)
gform <- list(x~mu,
              x2~mu^2+sig,
              x3~mu^3+3*mu*sig,
              x4~mu^4+6*mu^2*sig+3*sig^2)
form <- gelModel(gform, NULL, gelType="EEL", theta0=theta0, vcov="MDS", data=dat)
form

## GEL Model: Type  EEL
## ******************************
## Moment type: formula
## No Smoothing required
##
## Number of regressors: 2
## Number of moment conditions: 4
## Number of Endogenous Variables: 0
## Sample size:  50
```

- function:

```
fct <- function(theta, x)
    cbind(x-theta[1], (x-theta[1])^2-theta[2],
          (x-theta[1])^3, (x-theta[1])^4-3*theta[2]^2)
```

```
dfct <- function(theta, x)
    {
        m1 <- mean(x-theta[1])
        m2 <- mean((x-theta[1])^2)
        m3 <- mean((x-theta[1])^3)
        matrix(c(-1, -2*m1, -3*m2, -4*m3,
                 0, -1, 0, -6*theta[2]), 4, 2)
    }
theta0=c(mu=1,sig2=1)
func <- gelModel(fct, simData$x3, theta0=theta0, grad=dfct, vcov="iid", gelType="ET")
func

## GEL Model: Type  ET
## ******************************
## Moment type: function
## No Smoothing required
##
## Number of regressors: 2
## Number of moment conditions: 4
## Sample size:  50
```

- Non-linear models:

```
theta0=c(b0=1, b1=1, b2=1)
gform <- y~exp(b0+b1*x1+b2*x2)
nlin <- gelModel(gform, ~z1+z2+z3+x2, theta0=theta0, vcov="MDS", data=simData,
                 gelType="HD")
nlin

## GEL Model: Type  HD
## ******************************
## Moment type: nonlinear
## No Smoothing required
##
## Number of regressors: 3
## Number of moment conditions: 5
## Number of Endogenous Variables: 2
## Sample size:  50
```

It is also posssible to convert an existing gmmModels to gelModels:

```
nlin <- gmmModel(gform, ~z1+z2+z3+x2, theta0=theta0, vcov="MDS", data=simData)
nlin <- gmmToGel(nlin, gelType="HD")
```

2.1 The rhoXXX functions

The package provides $\rho(v)$ for ET, EL, EEL and HD. The function names are respectively "rhoET", "rhoEL", "rhoEEL" and "rhoHD". For any other GEL, users can pass user specific $\rho(v)$ function to the rhoFct argument of the gelModel(). The following example shows how the function must be built:

```
rhoEL

## function (gmat, lambda, derive = 0, k = 1)
## {
##     lambda <- c(lambda) * k
##     gmat <- as.matrix(gmat)
##     gml <- c(gmat %*% lambda)
```

```
##     switch(derive + 1, log(1 - gml), -1/(1 - gml), -1/(1 - gml)^2)
## }
## <bytecode: 0x55cf503bb528>
## <environment: namespace:gmm4>
```

Therefore, the function must be in the form $f(X, \lambda, d, k)$, where the first argument is an $n \times q$ matrix, the second argument is a vector of q elements, the third is an integer that indicates the order of derivative to return, and the last is a scalar that depends on the kernel used to smooth the moment function. We will discuss that in more details below. The function must return the vector $\rho(kX\lambda)$, $\rho'(kX\lambda)$ or $\rho''(kX\lambda)$, when derive is 0, 1, or 2, where $\rho'(v)$ and $\rho''(v)$ are the first and second derivative of $\rho(v)$.

2.2 The Lagrange multiplier solver

The function getLambda() function solve the maximization problem

$$\max_{\lambda} \frac{1}{n} \sum_{i=1}^{n} \rho(\lambda' X_i)$$

It is used to solve problem (1). The arguments are:

```
args(getLambda)

## function (gmat, lambda0 = NULL, gelType = NULL, rhoFct = NULL,
##     tol = 1e-07, maxiter = 100, k = 1, method = "BFGS", algo = c("nlminb",
##         "optim", "Wu"), control = list())
## NULL
```

The first argument is the $n \times q$ matrix X. The second argument is the starting value for λ. If set to NULL, a vector of zeroes is used. The third argument is the GEL type, which is either "ET", "EL", "EEL", "REEL" or "HD". If the rhoFct is provided, getType is ignored. The argument control is used to pass a list of arguments to optim(), nlminb() or constrOptim(). The argument "k" is the scalar described in the previous subsection. To describe all other arguments, we are presenting the options by type of GEL:

- EL: There are three possible options for EL. The default is "nlminb", in which case, the arguments tol, maxiter and method are ignored. If algo is set to "optim", the solution is obtained using constrOptim(), with the restriction $(1 - \lambda' X_i) > 0$ for all i. If algo is set to "Wu", the Wu (2005) algorithm is used. The argument tol and maxit are used to control the stopping rule.
- HD: Identical to EL, except that the "Wu" algorithm is not available for that type of GEL.
- EEL: There is an analytical solution, so all other arguments are ignored.
- REEL: This is EEL with a non-negativity constraint on all implied probabilities. In that case, the maxiter can be used to control the number of iterations.
- Others: When rhoFct is provided or the type is ET, the solution is obtained by either "nlminb" or "optim". In that case, the algorithms are controlled through the control argument.

Here are a few example using the simulated dataset (convergence code of 0 means that the algorithm converged with no detected problem):

```
X <- simData[c("x3","x4","z5")]
(res <- getLambda(X, gelType="EL"))$lambda

## [1] -0.04178415 -0.06745195 -0.02075063

res$convergence$convergence

## [1] 0
```

```
(res <- getLambda(X, gelType="EL", algo="Wu"))$lambda

## [1] -0.04178415 -0.06745195 -0.02075063

res$convergence$convergence

## [1] 0
```

```
(res <- getLambda(X, gelType="ET",control=list(maxit=2000)))$lambda

## [1] -0.04171694 -0.06804520 -0.02120255

res$convergence$convergence

## [1] 0
```

The following shows that we can provide getLambda() with a rhoFct instead:

```
(res <- getLambda(X, rhoFct=rhoEEL))$lambda

## [1] -0.04119158 -0.06769691 -0.02148683

res$convergence$convergence

## [1] 0
```

Although we rarely call the function directly, it is good to understand how it works, because it is an important part of the estimation procedure.

3 Methods for gelModels Classes

We saw above that any "gelModels" is a class that contains one of the "gmmModels" class object. Therefore, many "gmmModels" methods can be applied to "gelModels" through this direct inheritance. when it is the case, we will specify "gmmModels inherited method".

- *kernapply*: In the case of weakly dependent moment conditions, we saw above that the moment function must be smoothed using the following expression:

$$g_t^w(\theta) = \sum_{s=-m}^{m} w(s) g_{t-s}(\theta)$$

When a GEL model is defined with vcov="HAC", the specification of the kernel is stored in the "wSpec" slot of the object. For example, we can define the linear model above with the HAC specification:

```
linHAC <- gelModel(y~x1+x2, ~x2+z1+z2, data=simData, vcov="HAC", gelType="EL")
linHAC

## GEL Model: Type  EL
## ******************************
## Moment type: linear
## Smoothing: Truncated kernel and Andrews bandwidth (1.413)
## Number of regressors: 3
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50
```

The optimal bandwidth is computed when the model is created, and remains the same during the estimation process, unless another one is specified. The above model shows that the default

kernel is the "Truncated" one, and the default bandwidth is based on Andrews (1991). The bandwidth is not based on the smoothing kernel, but on the implied kernel for the HAC estimation. Smith (2001) shows that when $g_t(\theta)$ is replaced by $g_t^w(\theta)$, $V = \sum_{i=1}^n g_t^w(\theta) g_t^w(\theta)'/n$ is an HAC estimator of the covariance matrix of $\sqrt{n}\bar{g}(\theta)$, with Bartlett kernel when the smoothing kernel is the Truncated, and with Parzen kernel when the smoothing kernel is the Bartlett. The optimal bandwidth above is therefore based on the Bartlett kernel.

It is possible to modify the specifications of the kernel and bandwidth through the argument vcovOptions (See help(vcovHAC) from the sandwich package for all possible options). Notice that the kernel type that is passed is the kernel used for the HAC estimation, not the smoothing of $g_t(\theta)$. See in the following example that the Parzen kernel is selected, which implies a Bartlett kernel for the smoothing of $g_t(\theta)$.

```
linHAC2 <- gelModel(y~x1+x2, ~x2+z1+z2, data=simData, vcov="HAC",
                    gelType="EL",
                    vcovOptions=list(kernel="Parzen", bw="NeweyWest", prewhite=1))
linHAC2

## GEL Model: Type  EL
## ******************************
## Moment type: linear
## Smoothing: Bartlett kernel and NeweyWest bandwidth (4.736)
## Number of regressors: 3
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50
```

It is also possible to set the bandwidth to a fix number:

```
linHAC3 <- gelModel(y~x1+x2, ~x2+z1+z2, data=simData, vcov="HAC",
                    gelType="EL",
                    vcovOptions=list(kernel="Parzen", bw=3, prewhite=1))
linHAC3

## GEL Model: Type  EL
## ******************************
## Moment type: linear
## Smoothing: Bartlett kernel and Fixed  bandwidth (3)
## Number of regressors: 3
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50
```

The *kernapply* method, which is defined as an S3 method in the stats package, uses the information contained in the "wSpec" slot to compute the $n \times q$ matrix of moment with the ith row being $g_t^w(\theta)'$. A theta is required, because we need to evaluate $g_t(\theta)$.

```
gw <- kernapply(linHAC, theta=c(1,1,1))$smoothx
head(gw)

##   (Intercept)         x2         z1         z2
## 2  -10.098571 -124.39829 -10.587446 -30.814727
## 3   -9.941592 -122.40371 -12.084930 -30.151628
## 4   -6.403760  -48.78897  -1.999816 -17.523432
## 5   -6.022626  -37.63422  -1.821833  -8.477109
## 6   -8.604733  -62.82294  -6.329339 -12.859888
## 7   -9.562394  -78.47840  -7.311563 -15.964863
```

The function also returns the weights, bandwidth, kernel name and the scalars k_1 and k_2 that are needed for the asymptotic properties of the estimators (see Anatolyev (2005)). If the argument

smooth is set to FALSE, the optimal bandwidth is computed and no smoothing is done. By default, the first step GMM estimator with the identity matrix is used, unless theta is provided.

```
kernapply(linHAC, smooth=FALSE)
```

```
## $k
## [1] 2 2
##
## $w
## unknown
## coef[-1] = 0.3333
## coef[ 0] = 0.3333
## coef[ 1] = 0.3333
##
## $bw
## [1] 1.412821
##
## $kernel
## [1] "Truncated"
```

There is also a *kernapply* method for "gmmModels" classes. For now, it only returns the optimal bandwidth, the weights, the kernel names and the k_i's.

```
kernapply(as(linHAC, "gmmModels"))
```

```
## $k
## [1] 2 2
##
## $w
## unknown
## coef[-1] = 0.3333
## coef[ 0] = 0.3333
## coef[ 1] = 0.3333
##
## $bw
## [1] 1.412821
##
## $kernel
## [1] "Truncated"
```

- *residuals (gmmModels inherited method)*: Only defined for linearGel and nonlinearGel. It returns $\varepsilon(\theta)$:
- *Dresiduals (gmmModels inherited method)*: Only for "linearGel" and "nonlinearGel", it returns the $n \times k$ matrix $d\varepsilon(\theta)/d\theta$:
- *model.matrix (gmmModels inherited method)*: For linearGel and nonlinearGel only. For both classes, it ca be used to get the matrix of instruments:
- *modelResponse (gmmModels inherited method)*: For linear model only, it returns the vector of response. It is not defined for nonlinearGel classes because the left hand side is not always defined.
- *"[" (gmmModels inherited method)*: It creates a new object of the same class with a subset of moment conditions:
- *as*: "linearGel" can be converted into a "nonlinearGel" or "functionGmm". Also, "nonlinearGel" and "formulaGel" can be converted to "functionGel"

```
as(lin, "nonlinearGel")
```

```
## GEL Model: Type  EL
```

```
## ******************************
## Moment type: nonlinear
## No Smoothing required
##
## Number of regressors: 3
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50

as(nlin, "functionGel")

## GEL Model: Type  HD
## ******************************
## Moment type: function
## No Smoothing required
##
## Number of regressors: 3
## Number of moment conditions: 5
## Sample size:  50
```

- *subset (gmmModels inherited method)*: As for the S3 method, it creates the same class of object with a subset of the sample.
- *evalMoment*: It computes the $n \times q$ matrix of moments, with the i^{th} row being $g_i(\theta)'$, when the "vcov" slot is not "HAC", and $g_i^w(\theta)'$ when it is.

```
gt <- evalMoment(linHAC, theta=1:3)
```

- *evalDMoment*: It computes the $p \times k$ matrix of derivatives of the sample mean of $g_i(\theta)$. It calls the next method when the slot "vcov" is not "HAC". Otherwise, a numerical derivative is computed using numericDeric().
- *momentVcov*: It calls the next method when the slot "vcov" is not "HAC". Otherwise, it returns

$$V = \frac{1}{n}\sum_{i=1}^{n} g_t^w(\theta) g_t^w(\theta)'$$

- *update*: This method is used to modify existing objects. For now, only the covariance structure, the gelType and rhoFct can be modified.

```
update(lin, vcov="HAC", gelType="ET")

## GEL Model: Type  ET
## ******************************
## Moment type: linear
## Smoothing: Truncated kernel and Andrews bandwidth (1.413)
## Number of regressors: 3
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50
```

4 Restricted models

As for "gmmModels", restrictions can be imposed on the coefficients. The union class for all restricted GEL models is "rgelModels". The classes are "rlinearGel", "rformulaGel", "rfunctionGel" and "rnonlinearGel". Each one contains its unrestricted model plus additional slots that specify the constraints. See the gmmS4 vignette for more details. The constructor is the *restModel* method.

```
lin2 <- gelModel(y~x1+x2+x3+z1, ~x1+x2+z1+z2+z3+z4, data=simData,
                 gelType="EL", vcov="MDS")
rlin2 <- restModel(lin2, c("x1=x2", "x3=0"))
rlin2
```

```
## GEL Model: Type  EL
## ******************************
## Moment type: rlinear
## No Smoothing required
##
## Number of regressors: 3
## Number of moment conditions: 7
## Number of Endogenous Variables: 1
## Sample size:  50
## Constraints:
##   x1 - x2 = 0
##   x3 = 0
## Restricted regression:
##  y = (Intercept)+(x1+x2)+z1
```

All methods described in the previous section also apply to the restricted models. For example,

```
e <- residuals(rlin2, theta=c(1,1,1)) ## we now have 3 coefficients
gt <- evalMoment(rlin2, theta=c(1,1,1))
```

To recover the full coefficient vector, we use the *coef* method:

```
coef(rlin2, theta=1:3)
```

```
## (Intercept)          x1          x2          x3          z1
##           1           2           2           0           3
```

5 The *solveGel* Method

The main method to estimate a model by GEL methods is *solveGel*. The available signatures are:

```
showMethods("solveGel")
```

```
## Function: solveGel (package gmm4)
## object="gelModels"
```

The arguments are

- theta0: The initial value for the minimization algorithm. It is required if the model does not have a theta0 slot. If it has a theta0 slot, it is used by default unless a new theta0 is provided.
- lambda0: The initial value to pass to the lambda solver. By default, it is set to 0, which is its asymptotic value in case of correctly specified models.
- lamSlv: An optional function to solve for lambda. By default, getLambda() is used. The function must have the following form:

```
mylSolve <- function(gmat, lambda0, gelType=NULL, rhoFct=NULL, k=1, ...)
    {
        lambda <-  rep(0,ncol(gmat))
        obj <-  sum(colMeans(gmat)^2)
        list(lambda=lambda, convergence=0, obj=obj)
    }
```

Therefore, it must return a list with lambda, convergence and obj. In the above example, λ is set to a vector of zeros and the returned obj is $\bar{g}(\theta)'\bar{g}(\theta)$. The solution will therefore be the one step GMM with the weighting matrix equals to the identity.

```
solveGel(lin,c(0,0,0), lamSlv=mylSolve)

## $theta
##  (Intercept)           x1           x2
## -0.226153103  1.009952150  0.002381384
##
## $convergence
## [1] 0
##
## $lambda
## (Intercept)          x2          z1          z2
##           0           0           0           0
##
## $lconvergence
## [1] 0
```

To present a more realistic example, suppose we want to estimate the model using the exponentially tilted empirical likelihood (ETEL) method of Schennach (2007), we could write a function that solves the lambda using ET, and returns the empirical log-likelihood ratio:

```
mylSolve <- function(gmat, lambda0=NULL, gelType=NULL, rhoFct=NULL, k=1, ...)
    {
        gmat <- as.matrix(gmat)
        res  <-  getLambda(gmat, lambda0, gelType="ET", k=k)
        gml <- c(gmat%*%res$lambda)
        w <- exp(gml)
        w <- w/sum(w)
        n <- nrow(gmat)
        res$obj <- mean(-log(w*n))
        res
    }
etelFit <- solveGel(lin,c(1,1,0), lamSlv=mylSolve)
etelFit$theta

## (Intercept)          x1          x2
##   1.3266917   0.8504703  -0.1030545
```

That's equivalent to setting gelType to "ETEL":

```
solveGel(update(lin, gelType="ETEL"), c(1,1,0))$theta

## (Intercept)          x1          x2
##   1.3266917   0.8504703  -0.1030545
```

- coefSlv: A character string that indicates the name of the minimization solver used to obtain $\hat{\theta}$. By default, "optim" is use. The other options are "nlminb" and "constrOptim".
- lControl: A list of options for the lambda solver. It is passed to getLambda() or lamSlv() if provided. For example, we can use the Wu method for EL as follows:

```
solveGel(lin, c(1,1,0), lControl=list(algo="Wu"))$theta

## (Intercept)          x1          x2
##   1.3272620   0.8505003  -0.1031604
```

- tControl: A list of control for the coefSlv function. We could, for example, use the following options:

```
solveGel(lin, c(1,1,0), lControl=list(algo="Wu"),
         tControl=list(method="BFGS", control=list(maxit=2000, reltol=1e-9)))$theta

## (Intercept)          x1          x2
##   1.3271920   0.8505091  -0.1031434
```

In that particular case, the list is directly passed to optim().

The method returns a list with: theta=$\hat{\theta}$, lambda=$\hat{\lambda}$, convergence = convergence message and code for θ, and lconvergence = convergence message and code for λ.

6 The *modelFit* Method

This is the main estimation method. It returns an object of class "gelfit". The arguments are:

- object: Any object that belongs to the union class "gelModels"
- getType: Optional type of GEL if we want to estimate the model with a type that is different from the one defined in the object.
- rhoFct: Optional $\rho(v)$ function if we want to estimate the model with a function that is different from the one defined in the object.
- initTheta: Method to obtain the starting value for θ. By default, the one step GMM is used. The other option is to use the one included in the object.
- theta0: The stating value for θ. If provided, the argument initTheta is ignored.
- lambda0: Starting value for λ. By default, a vector of zeros is used.
- vcov: Logical argument that specifes if the covariance matrices of $\hat{\theta}$ and $\hat{\lambda}$ should be computed? It is FALSE by default.
- ... : Additional argument that is passed to the *gelSolve* method.

In general, it works fine with the default arguments:

```
fit <- modelFit(lin)
```

The following slots are available for the "gelfit" object:

```
showClass("gelfit")

## Class "gelfit" [package "gmm4"]
##
## Slots:
##
## Name:         theta  convergence       lambda lconvergence         call
## Class:      numeric      numeric      numeric      numeric   callORNULL
##
## Name:          type         vcov        model
## Class:    character         list    gelModels
```

6.1 Methods for "gelfit" classes

- *print*: It prints the model, $\hat{\theta}$ and $\hat{\lambda}$. The arguments "model" and "lambda" can be set to FALSE to avoid printing them.

```
fit <- modelFit(lin, lControl=list(algo="Wu"))
print(fit, lambda=FALSE, model=FALSE)
```

```
##
## Estimation:  EL
## Convergence Theta:  0
## Convergence Lambda:  0
## coefficients:
## (Intercept)           x1           x2
##   1.3270715    0.8505249   -0.1031374
```

- *residuals*: For "linearGel" and "nonlinearGel", it returns the difference between the left hand side and right hand side.
- *getImpProb*: It computes the vector of implied probabilities $\hat{p}_i$'s. By default, they are defined as:

$$\hat{p}_i = \frac{-\rho'(g_i(\hat{\theta}))/n}{\sum_{j=1}^{n} -\rho'(g_j(\hat{\theta}))/n}$$

The options are

  - posProb: This option is only considered for EEL type of GEL. If set to TRUE, the method of Antoine et al. (2007) is used to compute the probabilities:

$$\tilde{p}_i = \frac{\hat{p}_i + \varepsilon/n}{1+\varepsilon},$$

where $\varepsilon = -n \min(\min_i(\hat{p}_i), 0)$, and $\hat{p}_i = -\rho'(g_i(\hat{\theta}))/n$, which is the unormalized version of the above definition. It is then normalized to sum to 1. The method is only needed when we want to compute moments based on them. It is therefore set to FALSE by default.
  - normalize: If TRUE, the default, the above normalized version is returned. If FALSE, it returns $-\rho'(g_i(\hat{\theta}))/n$.

The method returns a list with pt, the vector $\{\hat{p}_i\}$, convMom $= \sum_{i=1}^{n} \hat{p}_i g_i(\hat{\theta})$, and convProb $= |(\sum_{i=1}^{n} \hat{p}_i) - 1|$. convMom is important, because it indicates if the estimation went well. It should be a vector close to zero.

```
pt <- getImpProb(fit)
pt$convMom

##   (Intercept)            x2            z1            z2
## -1.441989e-17 -1.114560e-16  1.700842e-17 -4.645806e-17
```

- *vcov*: It returns the covariance matrices of $\hat{\theta}$ and $\hat{\lambda}$ in a list. The returned covariance matrix for $\hat{\theta}$ is:

$$\hat{\Sigma} = \frac{1}{n}\left[\hat{G}'\hat{\Omega}^{-1}\hat{G}\right]^{-1}$$

when the "vcov" component of the object is not "HAC", with

$$\hat{G} = \frac{1}{n}\sum_{i=1}^{n} \frac{dg_i(\hat{\theta})}{d\theta},$$

and

$$\hat{\Omega} = \frac{1}{n}\sum_{i=1}^{n} g_i(\hat{\theta}) g_i(\hat{\theta})'.$$

For HAC specification, the covariance matrix is:

$$\hat{\Sigma}_w = \frac{1}{n}\left[\hat{G}_w'\hat{\Omega}_w^{-1}\hat{G}_w\right]^{-1},$$

with

$$\hat{G}_w = \frac{1}{nk_1}\sum_{i=1}^{n}\frac{dg_i^w(\hat{\theta})}{d\theta}$$

and

$$\hat{\Omega}_w = \frac{b}{nk_2}\sum_{i=1}^{n} g_i^w(\hat{\theta})g_i^w(\hat{\theta})',$$

where b is the bandwidth and the scalars k_1 and k_2 are determined by the kernel. The values of $\{k_1, k_2\}$ are $\{2, 2\}$ when the smoothing is based on the Truncated kernel, which implies Bartlett for the HAC estimator, and $\{1, 2/3\}$ when the smoothing is based on the Bartlett kernel, which implies Parzen for the HAC estimator.
Using the above definitions, the returned covariance matrix for $\hat{\lambda}$ is

$$\widehat{Var(\hat{\lambda})} = \frac{1}{n}\left[\hat{\Omega}^{-1} - \hat{\Omega}^{-1}\hat{G}\hat{\Sigma}\hat{G}'\hat{\Omega}^{-1}\right]$$

for non-smoothed moments, and

$$\widehat{Var(\hat{\lambda})}_w = \frac{b^2}{n}\left[\hat{\Omega}_w^{-1} - \hat{\Omega}_w^{-1}\hat{G}_w\hat{\Sigma}_w\hat{G}'_w\hat{\Omega}_w^{-1}\right]$$

for smoothed moments. The method has two additional arguments. The first is "withImpProb", which is FALSE by default. if set to TRUE, all moment estimations in the form $\sum_{i=1}^{n}[]_i/n$ are replaced by $\sum_{i=1}^{n}\hat{p}_i[]_i$. The second argument is tol, which is a tolerance level for too small diagonal elements of the covariance matrix of $\hat{\lambda}$. When some moment conditions are not binding, the estimated variance of the associated Lagrange multiplier may be too small or even negative, which is a numerical problem. To avoid negative values, tol is the minimum value the diagonal elements can take.

- *summary*: It returns an object of class "summaryGel". It returns the usual tables in estimation methods. The only argument is ... which is passed to the *(*vcov) method.

```
summary(fit)

## GEL Model: Type  EL
## ******************************
## Moment type: linear
## No Smoothing required
##
## Number of regressors: 3
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50
## Convergence Theta: 0
## Convergence Lambda: 0
## Average |Sum of pt*gt()]|: 4.7336e-17
## |Sum of pt - 1|: 0
##
## coefficients:
##             Estimate Std. Error t value  Pr(>|t|)
## (Intercept)  1.32707    0.86365  1.5366   0.12440
## x1           0.85052    0.11121  7.6480 2.041e-14 ***
## x2          -0.10314    0.06054 -1.7036   0.08845 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
##
## Lambdas:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.031751   0.070088  0.4530   0.6505
## x2           0.016730   0.046190  0.3622   0.7172
## z1          -0.027515   0.063545 -0.4330   0.6650
## z2          -0.075503   0.204461 -0.3693   0.7119
##
##  Test E(g)=0
##       Statistics  df   pvalue
## LR:      0.13377   1  0.71455
## LM:      0.13681   1  0.71147
##  J:      0.13212   1  0.71624
```

- specTest It returns the likelihood ratio (LR), Lagrange multiplier (LM) and J test, for the hypothesis $H_0 : \mathrm{E}(g(\theta)) = 0$. The signature is ("gelfit", "missing"), so the second argument, which, is not used. The argument type is set to "ALL" by default, which means that the three test are returned. Alternatively we can specify a single test. It returns an object of class "specTest" which possesses a *print* method.

```
specTest(fit)
```

```
##
##  Test E(g)=0
##       Statistics  df   pvalue
## LR:      0.13377   1  0.71455
## LM:      0.13681   1  0.71147
##  J:      0.13212   1  0.71624
```

- *confint*: It returnd confidence itervals for θ or λ By default, a simple Wald type interval is constructed. The parm argument allows to select just a few parameters. By default, it is done for all.

```
confint(fit, parm=1:2, level=0.90)
```

```
##
##  Wald type confidence interval
##                   0.05   0.95
## (Intercept)  -0.09351  2.748
## x1            0.66760  1.033
```

```
confint(fit, level=0.90, lambda=TRUE)
```

```
##
##  Wald confidence interval for Lambda
##                   0.05     0.95
## (Intercept)  -0.08353  0.14704
## x2           -0.05925  0.09271
## z1           -0.13204  0.07701
## z2           -0.41181  0.26081
```

The "vcov" argument is used to provide the method with another covariance matrix. The ... is passed to the *vcov* method.

Confidence interval can also be semi-parametric, by inverting any of the specification test. To undertand how it is constructed, let us define θ_{-i} as the vector θ without θ_i, and $\tilde{\theta}_{-i}(\theta_i)$ its estimate for a given θ_i. Let $S(\theta_i, \theta_{-i})$ be one of the specification test evaluated at $\{\theta_i, \theta_{-i}\}$. Under the null that θ_i is the true value,

$$T(\theta_i) = S(\theta_i, \tilde{\theta}_{-i}(\theta_i)) - S(\hat{\theta}_i, \hat{\theta}_{-i}) \xrightarrow{d} \chi^2_1$$

The following $(1-\alpha)$ confidence interval for θ_i is therefore asymptotically valid (given some regularity conditions):

$$\{\theta_i | T(\theta_i) \leq C_{1-\alpha}\}$$

where $C_{1-\alpha}$ is the $(1-\alpha)$ quantile of the χ^2_1 distribution. To get $\tilde{\theta}_{-i}$, we need to estimate a restricted model for each θ_i and search for $T(\theta_i) = C_{1-\alpha}$. This is done by setting type "invLR", "invLM", or "invJ". A function that returns $[T(\theta_i) - C_{i-\alpha}]$ is passed to the uniroot() function using intervals on both sides of $\hat{\theta}_i$. The argument "fact" is used to control the wideness of the interval passed to uniroot(). By default, the two intervals are (one for the left value and one for the right value of the confidence interval) $[\hat{\theta}_i - 3s_i, \hat{\theta}_i]$ and $[\hat{\theta}_i, \hat{\theta}_i + 3s_i]$, where s_i is the standard error of $\hat{\theta}_i$. If the method fails, it is possible to adjust the intervals with "fact".

```
confint(fit, 1:2, type="invLR", level=0.90)

##
##  Confidence interval based on the inversion of the LR test
##                 0.05   0.95
## (Intercept)  -0.2466  2.664
## x1            0.6722  1.060
```

A common application in introductory course on EL, is to compute EL confidence interval for a mean. Here, it is done by first creating a model with only one intercept and an intercept as instrument:

```
muModel <- gelModel(x1~1, ~1, gelType="EL", data=simData)
```

We then fit the model:

```
muFit <- modelFit(muModel, tControl=list(method="Brent", lower=-10, upper=10),
                  lControl=list(algo="Wu"))
muFit@theta

## (Intercept)
##    5.072075
```

The solution is just the sample mean:

```
mean(simData$x1)

## [1] 5.072075
```

We can then compute the EL confidence interval:

```
confint(muFit, type="invLR")

##
##  Confidence interval based on the inversion of the LR test
##               0.025  0.975
## (Intercept)  4.294  5.927
```

- *update*: The method is used to re-estimate a model with different specifications, keeping all the others equal. Suppose we start with the estimation using the Wu (2005) algorithm:

```
fit <- modelFit(lin, lControl=list(algo="Wu"))
fit@theta

## (Intercept)          x1          x2
##   1.3270715   0.8505249  -0.1031374
```

We want to re-estimate the model with the same algorithm for λ but a different one for θ:

```
fit2 <- update(fit, coefSlv="nlminb")
fit2@theta

## (Intercept)          x1          x2
##   1.3270608   0.8505255  -0.1031361
```

We can also change the type of GEL (here we reset the lControl because "Wu" is only for EL):

```
fit3 <- update(fit2, gelType="ET", lControl=list())
fit3@theta

## (Intercept)          x1          x2
##   1.3324427   0.8498311  -0.1034840
```

It is also possible to pass a new model to *update*. This is particularly useful to estimate a restricted model with the same numerical methods. This is used by *confint* to compute the interval by the inverse of the specification test.

```
rlin <- restModel(fit3@model, "x1=1")
update(fit3, newModel=rlin)

## GEL Model: Type  ET
## ******************************
## Moment type: rlinear
## No Smoothing required
##
## Number of regressors: 2
## Number of moment conditions: 4
## Number of Endogenous Variables: 1
## Sample size:  50
## Constraints:
##   x1 = 1
## Restricted regression:
##  (y-x1) = (Intercept)+x2
##
## Estimation:  ET
## Convergence Theta:  0
## Convergence Lambda:  0
## coefficients:
## (Intercept)           x2
##  0.23219597  -0.05177569
## lambdas:
## (Intercept)           x2           z1           z2
## -0.23451899   0.02899777   0.16810567  -0.09212368
```

Here, fit3 is an ET estimation. If we restrict the model lin, which is EL, the *update* will also use EL. That's why *restModel* is applied to the model in fit3.

7 The *evalModel* Method

This method create a "gelfit" object without estimation. It is possible to simply fix λ and θ:

```
print(evalModel(lin, theta=c(1,2,3), lambda=rep(.1,4)), model=FALSE)

##
## Estimation:  Eval-EL with fixed lambda
## Convergence Theta:  1
```

```
## Convergence Lambda:  1
## coefficients:
## (Intercept)           x1           x2
##           1            2            3
## lambdas:
## (Intercept)           x2           z1           z2
##         0.1          0.1          0.1          0.1
```

If the λ is not provided, then it is estimated for the given θ. It is like calling getLambda() with $g(\theta)$ as "gmat". The difference is that a "gelfit" object is returned. It is particularly usefull when we estimate restricted models in which the number of restriction is equal to the number of coefficients. It is use by *confint* method for the confidence interval on the mean (see above):

```
specTest(evalModel(muModel, theta=4), type="LR")

##
##  Test E(g)=0
##       Statistics  df  pvalue
## LR:       7.5096   0      NA
```

The problem with the *evalModel* is that not much effort is put on adjusting the parameters of the model. Above, it says that the degrees of freedom is 0. It is just not an estimation method. A proper estimation of the above would be done with *modelFit*:

```
rmuModel <- restModel(muModel, R=matrix(1,1,1), rhs=4)
specTest(modelFit(rmuModel))

##
##  Test E(g)=0
##       Statistics  df     pvalue
## LR:       7.5096   1  0.0061372
## LM:      10.6270   1  0.0011145
##  J:       5.8924   1  0.0152066
```

In that case *modelFit* is calling *evalModel*.

8 One function to fit them all: gel4

Most users will prefer to avoid going through the steps of building a model and fitting it. The gel4() function build the model, fit it and return the "gelfit" object. The arguments are similar to the ones described above for *modelFit*, *gelSolve* and gelModel().

- Arguments for the model: "g", "x", "theta0", "gelType","rhoFct", "vcov", "grad", "vcovOptions", "centeredVcov", and "data". See gelModel() above for details. The additional arguments "cstLHS" and "cstRHS" are to estimate restricted models. "chstLHS" is passed to *restModel* as "R" and "cstRHS" is passed as "rhs". The default getType is "EL", and the default "vcov" is "MDS"
- Arguments for the estimation: "lamSlv", "coefSlv", "initTheta", "lControl" and "tControl". The default coefSlv is "optim", the default lamSlv is getLambda(), and the default initTheta is "gmm". See *solveGel* and *modelFit* above for more details.
- The argument "getVcov" is passed to *modelFit* as vcov. Therefore, if it is set to TRUE, the covariance matrices are computed.

We can estimate the models from Section 2 as follows.

```
fit <- gel4(y~x1+x2, ~x2+z1+z2, data=simData, vcov="iid", gelType="EL",
            lControl=list(algo="Wu"))
print(fit, model=FALSE)
```

```
##
## Estimation:  EL
## Convergence Theta:  0
## Convergence Lambda:  0
## coefficients:
## (Intercept)           x1           x2
##   1.3270715    0.8505249   -0.1031374
## lambdas:
## (Intercept)           x2           z1           z2
##  0.03175148   0.01673016  -0.02751524  -0.07550335
```

To change the initial values in the linear case, we have to provide it and set initTheta to "theta0".

```
fit <- gel4(y~x1+x2, ~x2+z1+z2, data=simData, vcov="iid", gelType="EL",
            lControl=list(algo="Wu"), theta0=c(1,1,0), initTheta="theta0")
coef(fit)

## (Intercept)          x1          x2
##   1.3272620   0.8505003  -0.1031604
```

To estimate the model with the restiction $x1 = x2$, we set $cstLHS$ as R in the *restModel* method:

```
fit <- gel4(y~x1+x2, ~x2+z1+z2, data=simData, vcov="iid", gelType="EL",
            lControl=list(algo="Wu"), cstLHS="x2=x1")
print(fit, model=FALSE)

##
## Estimation:  EL
## Convergence Theta:  0
## Convergence Lambda:  0
## coefficients:
## (Intercept)      (x1+x2)
##    8.835489    -0.313172
## lambdas:
##   (Intercept)             x2             z1             z2
##  0.2705212100  -0.0004247482  -0.2049316654  -0.0190353209
```

For the formula type, a named theta0 is required for the same reason it is required in gelModel(). The names are used to locate the coefficients in the formulas.

```
theta0=c(mu=1,sig=1)
x <- simData$x1
dat <- data.frame(x=x, x2=x^2, x3=x^3, x4=x^4)
gform <- list(x~mu,
              x2~mu^2+sig,
              x3~mu^3+3*mu*sig,
              x4~mu^4+6*mu^2*sig+3*sig^2)
fit <- gel4(g=gform, gelType="EEL", theta0=theta0, vcov="MDS", data=dat)
print(fit, model=FALSE)

##
## Estimation:  EEL
## Convergence Theta:  0
## Convergence Lambda:  0
## coefficients:
##       mu      sig
## 4.708788  4.532422
## lambdas:
##        Mom_1         Mom_2         Mom_3         Mom_4
## -1.068346459   0.427092184  -0.055895513   0.002262411
```

For the function type, theta0 is required because the moment function is evaluated when the model is created. It is also the starting value if initTheta is set to "theta0". the argument x must be provided, because it is the second argument of the moment function

```
fct <- function(theta, x)
   cbind(x-theta[1], (x-theta[1])^2-theta[2],
         (x-theta[1])^3, (x-theta[1])^4-3*theta[2]^2)
dfct <- function(theta, x)
    {
        m1 <- mean(x-theta[1])
        m2 <- mean((x-theta[1])^2)
        m3 <- mean((x-theta[1])^3)
        matrix(c(-1, -2*m1, -3*m2, -4*m3,
                 0, -1, 0, -6*theta[2]), 4, 2)
    }
fit <- gel4(g=fct, x=simData$x3, theta0=c(1,1), grad=dfct, vcov="iid", gelType="ET")
print(fit, model=FALSE)

##
## Estimation:  ET
## Convergence Theta:  0
## Convergence Lambda:  0
## coefficients:
##    theta1     theta2
## 0.1141211  0.8202440
## lambdas:
##          h1           h2           h3           h4
##  0.24454964  -0.79959184  -0.09938776   0.16246546
```

The last type is non-linear models, which requires a named theta0 as for the formula type.

```
theta0=c(b0=1, b1=0, b2=0)
gform <- y~exp(b0+b1*x1+b2*x2)
fit <- gel4(gform, ~z1+z2+z3+x2, theta0=theta0, vcov="MDS", data=simData,
                gelType="HD")
print(fit, model=FALSE)

##
## Estimation:  HD
## Convergence Theta:  0
## Convergence Lambda:  0
## coefficients:
##          b0           b1           b2
##  0.57889968   0.19054605  -0.01927983
## lambdas:
##  (Intercept)            z1            z2            z3            x2
##  0.022346736  -0.014311581  -0.044300403  -0.010552163   0.008644042
```

9 GEL confidence intervals for the mean

The method *confint* can also be used to construct confidence intervals for the mean, ot confidence region for two means. The first method is for "numeric" objects:

```
x2 <- simData$x2
confint(x2, gelType="EL")

##
##  Wald type confidence interval
```

```
##     0.025  0.975
## x2  5.209  6.927
```

Which can also be done using the method for "data.frame":

```
confint(simData, "x2", gelType="EL")

##
##  Wald type confidence interval
##     0.025  0.975
## x2  5.209  6.927
```

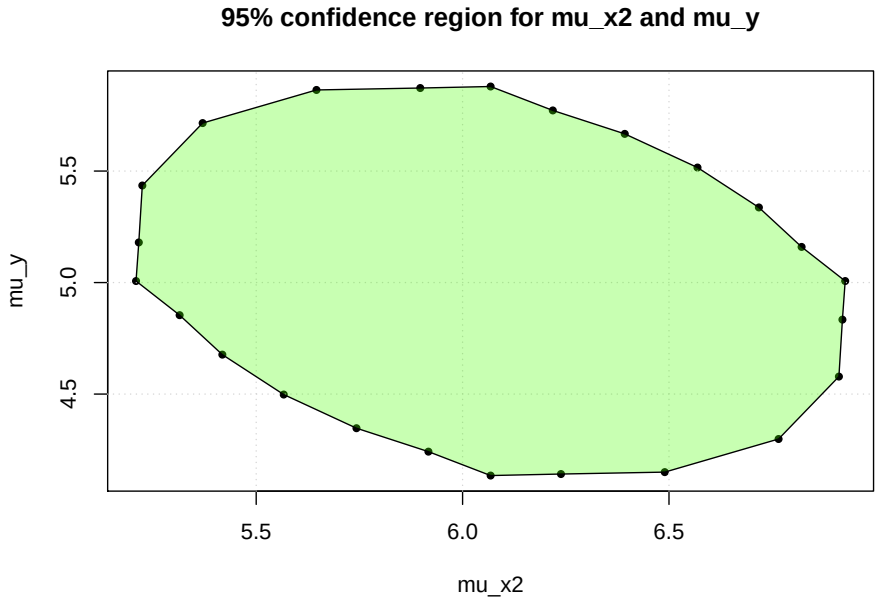
The arguments "level", "fact" and "type" are as for the method with "gelfit" signature. The "parm" is only used with "data.frame" in which case, it is used to select one or two columns. The argument "vcov" is used to specify the type of data is stored in the object. The options are "iid", the default, or "HAC" for weakly dependent processes.

For "data.frame" and two variables, an object of class "mconfint" is created. The argument "n" is the number of lines to use to construct the region. The number of equally spaced points around the region is $2(n + 2)$ because a vertical and horizontal lines are added. The arguments "cores" is for the number of cores to use with mclapply(). For Windows operating systems, it is set to 1. The following shows how to use the *print* and *plot* methods for that class. In the following, the Wald region is computed instead, to avoid having a vignette that takes to much time to compile.

```
res <- confint(simData, c("x2","y"))
res

## 2D Wald confidence region
## ************************
## Level: 0.95
## Number of points: 24
## Variables:
##  Range for mu_x2: [5.209, 6.927]
##  Range for mu_y: [4.135, 5.879]
```

```
plot(res, pch=20, bg=2, col=rainbow(7, alpha=.3)[3])
```



The arguments “main”, “xlab” “ylab”, “pch” and “bg” are passed to plot.default(), and all other arguments are passed to polygon().

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Appendix

A Some extra codes

The following *extract* is used with the "texreg" package of Leifeld (2013) to produce nice latex tables.

```
library(texreg)
setMethod("extract", "gelfit",
          function(model, includeSpecTest=TRUE,
                   specTest=c("LR","LM","J"), include.nobs=TRUE,
                   include.obj.fcn=TRUE, ...)
              {
                  specTest <- match.arg(specTest)
                  s <- summary(model, ...)
                  wspecTest <- grep(specTest, rownames(s@specTest@test))
                  spec <- modelDims(model@model)
                  coefs <- s@coef
                  names <- rownames(coefs)
                  coef <- coefs[, 1]
                  se <- coefs[, 2]
                  pval <- coefs[, 4]
                  n <- model@model@n
                  gof <- numeric()
                  gof.names <- character()
                  gof.decimal <- logical()
                  if (includeSpecTest) {
                      if (spec$k == spec$q)
                          {
                              obj.fcn <- NA
                              obj.pv <- NA
                          } else {
                              obj.fcn <- s@specTest@test[wspecTest,1]
                              obj.pv <- s@specTest@test[wspecTest,3]
                          }
                      gof <- c(gof, obj.fcn, obj.pv)
                      gof.names <- c(gof.names,
                                     paste(specTest,"-test Statistics", sep=""),
                                     paste(specTest,"-test p-value", sep=""))
                      gof.decimal <- c(gof.decimal, TRUE, TRUE)
                  }
                  if (include.nobs == TRUE) {
                      gof <- c(gof, n)
                      gof.names <- c(gof.names, "Num.\\ obs.")
                      gof.decimal <- c(gof.decimal, FALSE)
                  }
                  tr <- createTexreg(coef.names = names, coef = coef, se = se,
                                     pvalues = pval, gof.names = gof.names, gof = gof,
                                     gof.decimal = gof.decimal)
                  return(tr)
              })
```

Here is an example

	Model 1	Model 2
(Intercept)	−1.0507	−0.4054
	(0.6155)	(0.9789)
x1	1.1968***	1.1129***
	(0.1254)	(0.1505)
x2		−0.0355
		(0.0504)
LR-test Statistics	0.5866	0.1901
LR-test p-value	0.7458	0.6628
Num. obs.	50	50

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: Statistical models

```
fit1 <- gel4(y~x1, ~x2+x3+x4, data=simData)
fit2 <- gel4(y~x1+x2, ~x2+x3+x4, data=simData)
texreg(list(fit1,fit2), digits=4)
```