Biomass dynamic models

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Contents

1 Schaefer (1954)

1.1 Same + Growth − Removals

The Schaefer model is:

$$B_t \quad = \quad B_{t-1} \; + \; rB_t\left(1-\frac{B_t}{K}\right) \; - \; C_t$$

2 Surplus production

The second term describes the relationship between surplus production (growth) and abundance:

$$g(B) \quad = \quad rB\left(1-\frac{B}{K}\right)$$

In biological terms, the surplus production combines recruitment, body growth, and natural mortalities.

Conceptually, the model describes the simplest case of constant recruitment, where all individuals have the same body weight, and the natural mortality rate is constant. After a period of removals, the population rebuilds towards K until the number of individuals dying from natural causes equals the recruitment.

2.1 Original equation

Schaefer (1954, Eq. 2):

$$g(B) \quad = \quad k_2 B\,(L-B)$$

Replace $k_2 = \frac{r}{K}$ and $L = K$:

$$\begin{aligned} g(B) &= \frac{r}{K}\,B\,(K-B) \\ &= rB\left(\frac{K-B}{K}\right) \\ &= rB\left(1-\frac{B}{K}\right) \end{aligned}$$

2.2 $B_{\mathrm{MSY}} = 0.5K$

The surplus production is maximized where the derivative of the growth function is zero. It's easiest to find the derivative by using the elementary calculus notation,

$$
\begin{aligned}
f(x) &= rx\left(1 - \frac{x}{K}\right) \\
&= rx \, - \, \frac{rx^2}{K} \\
&= rx \, - \, \frac{r}{K}\,x^2 \\
f'(x) &= r \, - \, 2\,\frac{r}{K}\,x \\
&= r \, - \, \frac{2r}{K}\,x
\end{aligned}
$$

so:

$$
g'(B) = r \, - \, \frac{2r}{K}\,B
$$

B_{MSY} is where the derivative equals zero,

$$
0 = r \, - \, \frac{2r}{K}\,B_{\mathrm{MSY}}
$$

and we can isolate B_{MSY}:

$$
\begin{aligned}
\frac{2r}{K}\,B_{\mathrm{MSY}} &= r \\
2rB_{\mathrm{MSY}} &= rK \\
B_{\mathrm{MSY}} &= \frac{rK}{2r} \\
&= \frac{r}{2r}\,K \\
&= 0.5K
\end{aligned}
$$

2.3 $\mathrm{MSY} = 0.25\, rK$

Knowing $B_{\mathrm{MSY}} = 0.5K$ we can evaluate $\mathrm{MSY} = g(B_{\mathrm{MSY}})$:

$$
\begin{aligned}
g(B) &= rB\left(1 - \frac{B}{K}\right) \\
\mathrm{MSY} &= rB_{\mathrm{MSY}}\left(1 - \frac{B_{\mathrm{MSY}}}{K}\right) \\
&= 0.5\, rK\left(1 - \frac{0.5K}{K}\right) \\
&= 0.5\, rK\,(1 - 0.5) \\
&= 0.5\, rK\,(0.5) \\
&= 0.25\, rK
\end{aligned}
$$

2.4 Relative growth rate

The relative growth rate, as a fraction of current abundance, is $\mathrm{Rate}(B) = g(B)/B$,

$$
\begin{aligned}
g(B) &= rB\left(1 - \frac{B}{K}\right) \\
\mathrm{Rate}(B) &= rB\left(1 - \frac{B}{K}\right)\Big/ B \\
&= r\left(1 - \frac{B}{K}\right) \\
&= r - \frac{rB}{K}
\end{aligned}
$$

so the relative growth rate approaches r when the abundance is close to zero, and declines linearly with B until zero growth occurs at abundance K.

3 Variations

3.1 Gompertz (1825)

From Kingsland (1982, Eq. 5):

$$
\begin{aligned}
g(B) &= rB\,\log\!\left(\frac{K}{B}\right) \\
&= rB\,(\log K - \log B)
\end{aligned}
$$

3.2 Pella-Tomlinson (1969)

Pella and Tomlinson (1969, Eq. 5):

$$
g(B) = \mathcal{H}B^m - \mathcal{K}B
$$

Replace $\mathcal{H}=-\frac{r}{K}$, $m=p+1$, and $\mathcal{K}=-r$:

$$
\begin{aligned}
g(B) &= -\frac{r}{K}B^{\,p+1} + rB \\
&= rB - \frac{r}{K}B^{\,p+1} \\
&= rB - \frac{r}{K}B^{p} \times B \\
&= B\left(r - \frac{r}{K}B^{p}\right) \\
&= rB\left(1 - \frac{B^{p}}{K}\right)
\end{aligned}
$$

3.3 Garrod-Fox (1969/1970)

From Laloe (1995, Eq. 2):

$$
g(B) = rB\left(1 - \frac{\log B}{\log K}\right)
$$

3.4 Theta-logistic (1973)

Gilpin and Ayala (1973, Eq. 3):

$$g(B) = rB\left[1-\left(\frac{B}{K}\right)^{\theta}\right]$$

where θ gives the asymmetry of the growth.

3.5 Fletcher (1978)

From Prager (2002, Eqs 2–4):

$$
\begin{aligned}
g(B) &= \gamma m \frac{B}{K} - \gamma m \left(\frac{B}{K}\right)^n \\
&= \gamma m \left[\frac{B}{K} - \left(\frac{B}{K}\right)^n\right]
\end{aligned}
$$

where

$$
\gamma = \frac{n^{n/(n-1)}}{n-1}
$$

and $m = \frac{1}{4}\, rK$, i.e. MSY:

$$
\begin{aligned}
g(B) &= \gamma m \left[\frac{B}{K} - \left(\frac{B}{K}\right)^n\right] \\
&= \gamma \frac{rK}{4} \left[\frac{B}{K} - \left(\frac{B}{K}\right)^n\right] \\
&= \gamma \frac{rK}{4} \frac{B}{K} - \gamma \frac{rK}{4} \left(\frac{B}{K}\right)^n \\
&= \gamma \frac{r}{4} B - \gamma \frac{rK}{4} \left(\frac{B}{K}\right)^n \\
&= \gamma \frac{1}{4} rB - \gamma \frac{1}{4} rK \left(\frac{B}{K}\right)^n \\
&= \frac{\gamma}{4} rB - \frac{\gamma}{4} rK \left(\frac{B}{K}\right)^n \\
&= \frac{\gamma}{4} r \left[B - rK \left(\frac{B}{K}\right)^n\right]
\end{aligned}
$$

The Schaefer model corresponds to $n=2$ where $\gamma=4$,

$$\begin{aligned}
\gamma &= \frac{n^{n/(n-1)}}{n-1} \\
&= \frac{2^{2/(2-1)}}{2-1} \\
&= \frac{2^{2/(1)}}{1} \\
&= 2^{2/(1)} \\
&= 2^2 \\
&= 4
\end{aligned}$$

as verified here:

$$\begin{aligned}
g(B) &= \frac{\gamma}{4}\,rB \;-\; \frac{\gamma}{4}\,rK\left(\frac{B}{K}\right)^n \\
&= \frac{4}{4}\,rB \;-\; \frac{4}{4}\,rK\left(\frac{B}{K}\right)^2 \\
&= rB \;-\; rK\left(\frac{B}{K}\right)^2 \\
&= rB \;-\; rK\,\frac{B^2}{K^2} \\
&= rB \;-\; r\,\frac{B^2}{K} \\
&= rB \;-\; rB\,\frac{B}{K} \\
&= rB\left(1 \;-\; \frac{B}{K}\right)
\end{aligned}$$

Prager (2002) describes the shape of the production curve with the unitless ratio $\phi = \frac{B_{\text{MSY}}}{K}$, which has a more intuitive meaning than the n exponent. The relationship is:

$$\phi = \left(\frac{1}{n}\right)^{1/(n-1)}$$

Insert $n\!=\!2$ and note how the Schaefer model corresponds to $\phi = 0.5$, as expected:

$$\begin{aligned}\phi &= \left(\frac{1}{n}\right)^{1/(n-1)} \\ &= \left(\frac{1}{2}\right)^{1/(2-1)} \\ &= \left(\frac{1}{2}\right)^{1/(1)} \\ &= \frac{1}{2}\end{aligned}$$

3.6 Polacheck et al. (1993)

Polacheck et al. (1993, Eq. 1):

$$g(B) = \frac{r}{p}\,B\left[1-\left(\frac{B}{K}\right)^{p}\right]$$

where p controls the asymmetry of the sustainable yield versus stock biomass relationship.

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4.1 Look up later

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