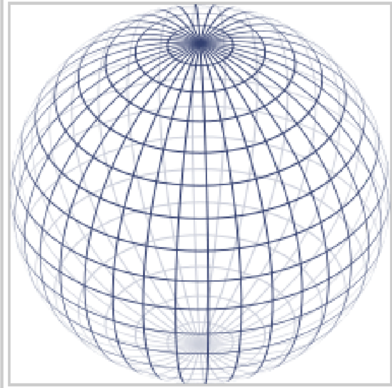
n-sphere

From Wikipedia, the free encyclopedia

In mathematics, the ***n*-sphere** is the generalization of the ordinary sphere to spaces of arbitrary dimension. It is an n-dimensional manifold that can be embedded in Euclidean $(n + 1)$-space.



2-sphere wireframe as an orthogonal projection

For any natural number n, an n-sphere of radius r may be defined in terms of an embedding in $(n + 1)$-dimensional Euclidean space as the set of points that are at distance r from a central point, where the radius r may be any positive real number. Thus, the n-sphere would be defined by:

$$S^n = \left\{ x \in \mathbb{R}^{n+1} : \|x\| = r \right\}.$$

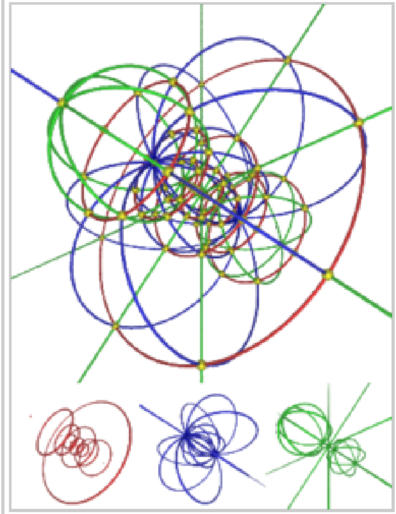
In particular:

- the pair of points at the ends of a (one-dimensional) line segment is 0-sphere,
- the circle, which is the one-dimensional circumference of a (two-dimensional) disk in the plane is a 1-sphere,
- the two-dimensional surface of a (three-dimensional) ball in three-dimensional space is a 2-sphere, often simply called a sphere,
- the three-dimensional boundary of a (four-dimensional) 4-ball in four-dimensional Euclidean is a 3-sphere, also known as a **glome**.

An n-sphere embedded in an $(n + 1)$-dimensional Euclidean space is called a **hypersphere**. The n-sphere of unit radius is called the **unit *n*-sphere**, denoted S^n. The unit n-sphere is often referred to as *the* n-sphere.

When embedded as described, an n-sphere is the surface or boundary of an $(n + 1)$-dimensional ball. For $n \geq 2$, the n-spheres are the simply connected n-dimensional manifolds of constant, positive curvature. The n-spheres admit several other topological descriptions: for example, they can be constructed by gluing two n-dimensional Euclidean spaces together, by identifying the boundary of an n-cube with a point, or (inductively) by forming the suspension of an $(n - 1)$-sphere.

Contents



Just as a stereographic projection can project a sphere's surface to a plane, it can also project the surface of a 3-sphere into 3-space. This image shows three coordinate directions projected to 3-space: parallels (red), meridians (blue) and hypermeridians (green). Due to the conformal property of the stereographic projection, the curves intersect each other orthogonally (in the yellow points) as in 4D. All of the curves are circles: the curves that intersect <0,0,0,1> have an infinite radius (= straight line).

Description

For any natural number n, an n-sphere of radius r is defined as the set of points in $(n + 1)$-dimensional Euclidean space that are at distance r from some fixed point $\mathbf{c}$, where r may be any positive real number and where $\mathbf{c}$ may be any point in $(n + 1)$-dimensional space. In particular:

- a 0-sphere is a pair of points $\{c - r, c + r\}$, and is the boundary of a line segment (1-ball).
- a 1-sphere is a circle of radius r centered at $\mathbf{c}$, and is the boundary of a disk (2-ball).
- a 2-sphere is an ordinary 2-dimensional sphere in 3-dimensional Euclidean space, and is the boundary of an ordinary ball (3-ball).
- a 3-sphere is a sphere in 4-dimensional Euclidean space.

Euclidean coordinates in (*n* + 1)-space

The set of points in $(n + 1)$-space: $(x_1,x_2,\ldots,x_{n+1})$ that define an n-sphere (S^n), is represented by the equation:

$$r^2 = \sum_{i=1}^{n+1}(x_i - c_i)^2.$$

where c is a center point, and r is the radius.

The above n-sphere exists in $(n + 1)$-dimensional Euclidean space and is an example of an n-manifold. The ppl volume form ω of an n-sphere of radius r is given by

$$\omega = \frac{1}{r}\sum_{j=1}^{n+1}(-1)^{j-1}x_j\,dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1} = {*dr}$$

where * is the Hodge star operator; see Flanders (1989, §6.1) for a discussion and proof of this formula in the case $r = 1$. As a result,

$$dr \wedge \omega = dx_1 \wedge \cdots \wedge dx_{n+1}.$$

n-ball

The space enclosed by an *n*-sphere is called an (*n* + 1)-ball. An (*n* + 1)-ball is closed if it includes the *n*-sphere, and it is open if it does not include the *n*-sphere.

Specifically:

- A 1-*ball*, a line segment, is the interior of a 0-sphere.
- A 2-*ball*, a disk, is the interior of a circle (1-sphere).
- A 3-*ball*, an ordinary ball, is the interior of a sphere (2-sphere).
- A 4-*ball* is the interior of a 3-sphere, etc.

Topological description

Topologically, an *n*-sphere can be constructed as a one-point compactification of *n*-dimensional Euclidean space. Briefly, the *n*-sphere can be described as $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$, which is *n*-dimensional Euclidean space plus a single point representing infinity in all directions. In particular, if a single point is removed from an *n*-sphere, it becomes homeomorphic to $\mathbb{R}^n$. This forms the basis for stereographic projection.[1]

Volume and surface area

In general, the volumes of the *n*-ball in *n*-dimensional Euclidean space, and the *n*-sphere in (*n* + 1)-dimensional Euclidean, of radius *R*, are proportional to the *n*th power of the radius, *R*. We write $V_n(R) = V_n R^n$ for the volume of the *n*-ball and $S_n(R) = S_n R^n$ for the surface of the *n*-sphere, both of radius R.

Interestingly, given the radius *R*, the volume and the surface area of the *n*-sphere reaches a maximum and then decrease towards zero as the dimension *n* increases. In particular, the volume $V_n(R)$ of the *n*-sphere of constant radius *R* in *n*-dimensions reaches a maximum for dimension $n^* = n$ if $V_{n-1} < R < V_n$ and $n^* = \{n, n+1\}$ if $R = V_n$ where $V_n = \dfrac{\Gamma(n/2 + 3/2)}{\sqrt{\pi}\Gamma(n/2+1)}$ for $n = 1, 2, \ldots$. Similarly, defining the sequence $\{A_n = V_{n-2}\}$, the surface area $S_n(R)$ of the *n*-sphere of constant radius *R* in *n* dimensions reaches a maximum for dimension $n^* = n$ if $A_{n-1} < R < A_n$ and $n^* = \{n, n+1\}$ if $R = A_n$.[2]

Examples

The 0-ball consists of a single point. The 0-dimensional Hausdorff measure is the number of points in a set, so

$$V_0 = 1.$$

The unit 1-ball is the interval $[-1, 1]$ of length 2. So,

$$V_1 = 2.$$

The 0-sphere consists of its two end-points, $\{-1, 1\}$. So,

$$S_0 = 2.$$

The unit 1-sphere is the unit circle in the Euclidean plane, and this has circumference (1-dimensional measure)

$$S_1 = 2\pi.$$

The region enclosed by the unit 1-sphere is the 2-ball, or unit disc, and this has area (2-dimensional measure)

$$V_2 = \pi.$$

Analogously, in 3-dimensional Euclidean space, the surface area (2-dimensional measure) of the unit 2-sphere is given by

$$S_2 = 4\pi.$$

and the volume enclosed is the volume (3-dimensional measure) of the unit 3-ball, given by

$$V_3 = \frac{4}{3}\pi.$$

$V_n(R)$ and $S_n(R)$ are the n-dimensional volume and surface area of the n-ball and n-sphere of radius R, respectively.

The constants V_n and S_n (for the unit ball and sphere) are related by the recurrences:

$$\begin{aligned} V_0 &= 1 & V_{n+1} &= S_n/(n+1) \\ S_0 &= 2 & S_{n+1} &= 2\pi V_n \end{aligned}$$

The surfaces and volumes can also be given in closed form:

$$S_n(R) = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} R^n$$

$$V_n(R) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} R^n$$

where Γ is the gamma function. Derivations of these equations are given in this section.

Recurrences

The *surface area*, or properly the n-dimensional volume, of the n-sphere at the boundary of the $(n + 1)$-ball of radius R is related to the volume of the ball by the differential equation

$$S_n R^n = \frac{dV_{n+1} R^{n+1}}{dR} = (n+1) V_{n+1} R^n,$$

or, equivalently, representing the unit n-ball as a union of concentric $(n - 1)$-sphere *shells*,

$$V_{n+1} = \int_0^1 S_n r^n \, dr$$

So,

$$V_{n+1} = \frac{S_n}{n+1}.$$

We can also represent the unit $(n + 2)$-sphere as a union of tori, each the product of a circle (1-sphere) with an n-sphere. Let $r = \cos\theta$ and $r^2 + R^2 = 1$, so that $R = \sin\theta$ and $dR = \cos\theta \, d\theta$. Then,

$$\begin{aligned} S_{n+2} &= \int_0^{\pi/2} S_1 r \,.\, S_n R^n \, d\theta = \int_0^{\pi/2} S_1 \,.\, S_n R^n \cos\theta \, d\theta \\ &= \int_0^1 S_1 \,.\, S_n R^n \, dR = S_1 \int_0^1 S_n R^n \, dR \\ &= 2\pi V_{n+1} \end{aligned}$$

Since $S_1 = 2\pi V_0$, the equation $S_{n+1} = 2\pi V_n$ holds for all n.

This completes our derivation of the recurrences:

$$\begin{aligned} V_0 &= 1 \qquad & V_{n+1} &= S_n/(n+1) \\ S_0 &= 2 \qquad & S_{n+1} &= 2\pi V_n \end{aligned}$$

Closed forms

Combining the recurrences, we see that $V_{n+2} = 2\pi V_n/(n+2)$. So it is simple to show by induction on k that,

$$\begin{aligned} V_{2k} &= \frac{\pi^k}{k!} \\ V_{2k+1} &= \frac{2(2\pi)^k}{(2k+1)!!} = \frac{2k!(4\pi)^k}{(2k+1)!} \end{aligned}$$

where $!!$ denotes the double factorial, defined for odd integers $2k + 1$ by $(2k + 1)!! = 1 \cdot 3 \cdot 5 \cdots (2k - 1) \cdot (2k + 1)$.

In general, the volume, in n-dimensional Euclidean space, of the unit n-ball, is given by

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

where Γ is the gamma function, which satisfies $\Gamma(1/2) = \sqrt{\pi}; \Gamma(1) = 1; \Gamma(x+1) = x\Gamma(x)$.

By multiplying V_n by R^n, differentiating with respect to R, and then setting $R = 1$, we get the closed form

$$S_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

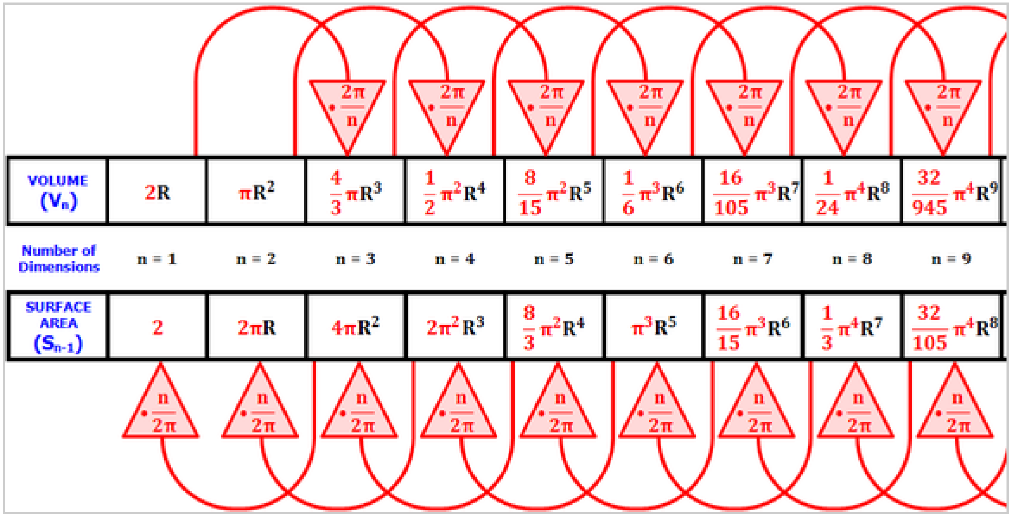
Other relations

The recurrences can be combined to give a "reverse-direction" recurrence relation for surface area, as depicted in the diagram:

$$S_{n-1} = \frac{n}{2\pi} S_{n-1+2}$$

Index-shifting n to $n-2$ then yields the recurrence relations:

$$V_n = \frac{2\pi}{n} V_{n-2}$$



(Note that n refers to the dimension of the ambient Euclidean space, which is also the intrinsic dimension of the solid whose volume is listed here, but which is 1 more than the intrinsic dimension of the sphere whose surface area is listed here.) The curved red arrows show the relationship between formulas for different n. The formula coefficient at each arrow's tip equals the formula coefficient at that arrow's tail times the factor in the arrowhead. If the direction of the bottom arrows were reversed, their arrowheads would say to multiply by $\frac{2\pi}{n-2}$. Alternatively said, the surface area S_{n+1} of the sphere in $n+2$ dimensions is exactly $2\pi R$ times the volume V_n enclosed by the sphere in n dimensions.

$$S_{n-1} = \frac{2\pi}{n-2} S_{n-1-2}$$

where $S_0 = 2$, $V_1 = 2$, $S_1 = 2\pi$ and $V_2 = \pi$.

The recurrence relation for V_n can also be proved via integration with 2-dimensional polar coordinates:

$$\begin{aligned} V_n &= \int_0^1 \int_0^{2\pi} V_{n-2}\left(\sqrt{1-r^2}\right)^{n-2} r\, d\theta\, dr \\ &= \int_0^1 \int_0^{2\pi} V_{n-2}(1-r^2)^{n/2-1} r\, d\theta\, dr \\ &= 2\pi V_{n-2} \int_0^1 (1-r^2)^{n/2-1} r\, dr \\ &= 2\pi V_{n-2}\left[-\frac{1}{n}(1-r^2)^{n/2}\right]_{r=0}^{r=1} \\ &= 2\pi V_{n-2}\frac{1}{n} = \frac{2\pi}{n} V_{n-2}. \end{aligned}$$

Spherical coordinates

We may define a coordinate system in an *n*-dimensional Euclidean space which is analogous to the spherical coordinate system defined for 3-dimensional Euclidean space, in which the coordinates consist of a radial coordinate, r, and $n - 1$ angular coordinates $\phi_1, \phi_2, \ldots, \phi_{n-1}$ where ϕ_{n-1} ranges over $[0, 2\pi)$ radians (or over [0, 360) degrees) and the other angles range over $[0, \pi]$ radians (or over [0, 180] degrees). If x_i are the Cartesian coordinates, then we may compute $x_1, \ldots, x_n$ from $r, \phi_1, \ldots, \phi_{n-1}$ with:

$$\begin{aligned}
x_1 &= r\cos(\phi_1) \\
x_2 &= r\sin(\phi_1)\cos(\phi_2) \\
x_3 &= r\sin(\phi_1)\sin(\phi_2)\cos(\phi_3) \\
&\vdots \\
x_{n-1} &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\cos(\phi_{n-1}) \\
x_n &= r\sin(\phi_1)\cdots\sin(\phi_{n-2})\sin(\phi_{n-1})\,.
\end{aligned}$$

Except in the special cases described below, the inverse transformation is unique:

$$\begin{aligned}
r &= \sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_2}^2 + {x_1}^2} \\
\phi_1 &= \operatorname{arccot}\frac{x_1}{\sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_2}^2}} = \arccos\frac{x_1}{\sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_1}^2}} \\
\phi_2 &= \operatorname{arccot}\frac{x_2}{\sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_3}^2}} = \arccos\frac{x_2}{\sqrt{{x_n}^2 + {x_{n-1}}^2 + \cdots + {x_2}^2}} \\
&\vdots \\
\phi_{n-2} &= \operatorname{arccot}\frac{x_{n-2}}{\sqrt{{x_n}^2 + {x_{n-1}}^2}} = \arccos\frac{x_{n-2}}{\sqrt{{x_n}^2 + {x_{n-1}}^2 + {x_{n-2}}^2}} \\
\phi_{n-1} &= 2\operatorname{arccot}\frac{x_{n-1} + \sqrt{x_n^2 + x_{n-1}^2}}{x_n} = \begin{cases} \arccos\frac{x_{n-1}}{\sqrt{{x_n}^2 + {x_{n-1}}^2}} & x_n \geq 0 \\ 2\pi - \arccos\frac{x_{n-1}}{\sqrt{{x_n}^2 + {x_{n-1}}^2}} & x_n < 0 \end{cases}.
\end{aligned}$$

where if $x_k \neq 0$ for some k but all of $x_{k+1}, \ldots, x_n$ are zero then $\phi_k = 0$ when $x_k > 0$, and $\phi_k = \pi$ radians (180 degrees) when $x_k < 0$.

There are some special cases where the inverse transform is not unique; ϕ_k for any k will be ambiguous whenever all of $x_k, x_{k+1}, \ldots, x_n$ are zero; in this case ϕ_k may be chosen to be zero.

Spherical volume element

Expressing the angular measures in radians, the volume element in *n*-dimensional Euclidean space will be found from the Jacobian of the transformation:

$$\begin{pmatrix} \cos(\phi_1) & -r\sin(\phi_1) & 0 & 0 & \cdots & 0 \\ \sin(\phi_1)\cos(\phi_2) & r\cos(\phi_1)\cos(\phi_2) & -r\sin(\phi_1)\sin(\phi_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ & & & & & 0 \\ \sin(\phi_1)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1}) & \cdots & \cdots & & & -r\sin(\phi_1)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1}) \\ \sin(\phi_1)\ldots\sin(\phi_{n-2})\sin(\phi_{n-1}) & r\cos(\phi_1)\ldots\sin(\phi_{n-1}) & \cdots & & & r\sin(\phi_1)\ldots\sin(\phi_{n-2})\cos(\phi_{n-1}) \end{pmatrix}$$

$$\begin{aligned} d^nV &= \left|\det \frac{\partial(x_i)}{\partial(r,\phi_j)}\right| dr\,d\phi_1\,d\phi_2\cdots d\phi_{n-1} \\ &= r^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\cdots\sin(\phi_{n-2})\,dr\,d\phi_1\,d\phi_2\cdots d\phi_{n-1} \end{aligned}$$

and the above equation for the volume of the n-ball can be recovered by integrating:

$$V_n = \int_{\phi_{n-1}=0}^{2\pi}\int_{\phi_{n-2}=0}^{\pi}\cdots\int_{\phi_1=0}^{\pi}\int_{r=0}^{R} d^nV.$$

The volume element of the (n-1)–sphere, which generalizes the area element of the 2-sphere, is given by

$$d_{S^{n-1}}V = \sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\cdots\sin(\phi_{n-2})\,d\phi_1\,d\phi_2\cdots d\phi_{n-1}.$$

The natural choice of an orthogonal basis over the angular coordinates is a product of ultraspherical polynomials,

$$\begin{aligned} &\int_0^{\pi}\sin^{n-j-1}(\phi_j)C_s^{((n-j-1)/2)}(\cos\phi_j)C_{s'}^{((n-j-1)/2)}(\cos\phi_j)\,d\phi_j \\ &= \frac{\pi 2^{3-n+j}\Gamma(s+n-j-1)}{s!(2s+n-j-1)\Gamma^2((n-j-1)/2)}\delta_{s,s'} \end{aligned}$$

for $j = 1, 2, \ldots, n-2$, and the $e^{is\varphi_j}$ for the angle $j = n-1$ in concordance with the spherical harmonics.

Stereographic projection

Just as a two-dimensional sphere embedded in three dimensions can be mapped onto a two-dimensional plane by a stereographic projection, an n-sphere can be mapped onto an n-dimensional hyperplane by the n-dimensional version of the stereographic projection. For example, the point $[x,y,z]$ on a two-dimensional sphere of radius 1 maps to the point $\left[\frac{x}{1-z},\frac{y}{1-z}\right]$ on the xy plane. In other words,

$$[x,y,z]\mapsto\left[\frac{x}{1-z},\frac{y}{1-z}\right].$$

Likewise, the stereographic projection of an n-sphere $\mathbf{S}^{n-1}$ of radius 1 will map to the $n-1$ dimensional hyperplane $\mathbf{R}^{n-1}$ perpendicular to the x_n axis as

$$[x_1, x_2, \ldots, x_n] \mapsto \left[\frac{x_1}{1-x_n}, \frac{x_2}{1-x_n}, \ldots, \frac{x_{n-1}}{1-x_n}\right].$$

Generating random points

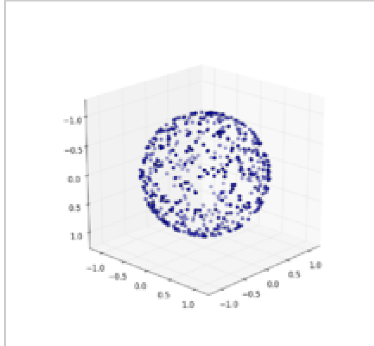
Uniformly at random from the $(n - 1)$-sphere

To generate uniformly distributed random points on the $(n - 1)$-sphere (*i.e.*, the surface of the n-ball), Marsaglia (1972) gives the following algorithm.

Generate an n-dimensional vector of normal deviates (it suffices to use N(0, 1), although in fact the choice of the variance is arbitrary), $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Now calculate the "radius" of this point, $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

The vector $\frac{1}{r}\mathbf{x}$ is uniformly distributed over the surface of the unit n-ball.



A set of uniformly distributed points on the surface of a unit 2-sphere generated using Marsaglia's algorithm.

Examples

For example, when $n = 2$ the normal distribution $\exp(-x_1{}^2)$ when expanded over another axis $\exp(-x_2{}^2)$ after multiplication takes the form $\exp(-x_1{}^2 - x_2{}^2)$ or $\exp(-r^2)$ and so is only dependent on distance from the origin.

Alternatives

Another way to generate a random distribution on a hypersphere is to make a uniform distribution over a hypercube that includes the unit hyperball, exclude those points that are outside the hyperball, then project the remaining interior points outward from the origin onto the surface. This will give a uniform distribution, but it is necessary to remove the exterior points. As the relative volume of the hyperball to the hypercube decreases very rapidly with dimension, this procedure will succeed with high probability only for fairly small numbers of dimensions.

Wendel's theorem gives the probability that all of the points generated will lie in the same half of the hypersphere.

Uniformly at random from the n-ball

With a point selected from the surface of the n-ball uniformly at random, one needs only a radius to obtain a point uniformly at random within the n-ball. If u is a number generated uniformly at random from the interval [0, 1] and $\mathbf{x}$ is a point selected uniformly at random from the surface of the n-ball then $u^{1/n}\mathbf{x}$ is uniformly distributed over the entire unit n-ball.

Specific spheres

0-sphere
The pair of points $\{\pm R\}$ with the discrete topology for some $R > 0$. The only sphere that is not path-connected. Has a natural Lie group structure; isomorphic to O(1). Parallelizable.

1-sphere
Also known as the circle. Has a nontrivial fundamental group. Abelian Lie group structure U(1); the circle group. Topologically equivalent to the real projective line, $\mathbf{R}\mathrm{P}^1$. Parallelizable. SO(2) = U(1).

2-sphere
Also known as the sphere. Complex structure; see Riemann sphere. Equivalent to the complex projective line, $\mathbf{C}\mathrm{P}^1$. SO(3)/SO(2).

3-sphere
Also known as the glome. Parallelizable, Principal U(1)-bundle over the 2-sphere, Lie group structure Sp(1), where also
$\mathrm{Sp}(1) \cong \mathrm{SO}(4)/\mathrm{SO}(3) \cong \mathrm{SU}(2) \cong \mathrm{Spin}(3)$.

4-sphere
Equivalent to the quaternionic projective line, $\mathbf{H}\mathrm{P}^1$. SO(5)/SO(4).

5-sphere
Principal U(1)-bundle over $\mathbf{C}\mathrm{P}^2$. SO(6)/SO(5) = SU(3)/SU(2).

6-sphere
Almost complex structure coming from the set of pure unit octonions. SO(7)/SO(6) = G_2/SU(3).

7-sphere
Topological quasigroup structure as the set of unit octonions. Principal Sp(1)-bundle over S^4. Parallelizable. SO(8)/SO(7) = SU(4)/SU(3) = Sp(2)/Sp(1) = Spin(7)/G_2 = Spin(6)/SU(3). The 7-sphere is of particular interest since it was in this dimension that the first exotic spheres were discovered.

8-sphere
Equivalent to the octonionic projective line $\mathbf{O}\mathrm{P}^1$.

23-sphere
A highly dense sphere-packing is possible in 24-dimensional space, which is related to the unique qualities of the Leech lattice.

See also

- Affine sphere
- Conformal geometry
- Homology sphere
- Homotopy groups of spheres
- Homotopy sphere
- Hyperbolic group
- Hypercube
- Inversive geometry
- Loop (topology)
- Manifold
- Möbius transformation
- Orthogonal group
- Spherical cap
- Volume of an *n*-ball

Notes

1. James W. Vick (1994). *Homology theory*, p. 60. Springer
2. Loskot, Pavel (November 2007). "On Monotonicity of the Hypersphere Volume and Area". *Journal of Geometry*. **87** (1-2): 96–98. doi:10.1007/s00022-007-1891-1.

References

- Flanders, Harley (1989). *Differential forms with applications to the physical sciences*. New York: Dover Publications. ISBN 978-0-486-66169-8..
- Moura, Eduarda; Henderson, David G. (1996). *Experiencing geometry: on plane and sphere*. Prentice Hall. ISBN 978-0-13-373770-7 (Chapter 20: 3-spheres and hyperbolic 3-spaces.)
- Weeks, Jeffrey R. (1985). *The Shape of Space: how to visualize surfaces and three-dimensional manifolds*. Marcel Dekker. ISBN 978-0-8247-7437-0 (Chapter 14: The Hypersphere)
- Marsaglia, G. (1972). "Choosing a Point from the Surface of a Sphere". *Annals of Mathematical Statistics*. **43** (2): 645–646. doi:10.1214/aoms/1177692644.
- Huber, Greg (1982). "Gamma function derivation of n-sphere volumes". *Am. Math. Monthly*. **89** (5): 301–302. doi:10.2307/2321716. JSTOR 2321716. MR 1539933.
- Barnea, Nir (1999). "Hyperspherical functions with arbitrary permutational symmetry: Reverse construction". *Phys. Rev. A*. **59** (2): 1135–1146. doi:10.1103/PhysRevA.59.1135.

External links

- Exploring Hyperspace with the Geometric Product (http://www.bayarea.net/~kins/thomas_briggs/)
- Properties of Spherical Coordinates in n Dimensions (http://faculty.madisoncollege.edu/alehnen/sphere/Apendxa/Appendixa.htm)
- Weisstein, Eric W. "Hypersphere". *MathWorld*.

Retrieved from "https://en.wikipedia.org/w/index.php?title=N-sphere&oldid=754246570"

Categories: Multi-dimensional geometry | Spheres

- This page was last modified on 11 December 2016, at 17:06.
- Text is available under the Creative Commons Attribution-ShareAlike License; additional terms may apply. By using this site, you agree to the Terms of Use and Privacy Policy. Wikipedia® is a registered trademark of the Wikimedia Foundation, Inc., a non-profit organization.