User guide for the **GNE** package

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As usual, the **GNE** package is loaded via the `library` function. We suppose in the following that the line below is called

```
> library(GNE)
```

1 Introduction

Definition 1 (GNEP) *We define the generalized Nash equilibrium problem GNEP(N, θ_ν, X_ν) as the solutions $x^\star$ of the N sub problems*

$$\forall \nu = 1, \dots, N, x^\star_\nu \text{ solves } \min_{y_\nu} \quad \theta_\nu(y_\nu, x^\star_{-\nu}) \text{ such that } x^\star_\nu \in X_\nu(x^\star_{-\nu}),$$

where $X_\nu(x_{-\nu})$ is the action space of player ν given others player actions $x_{-\nu}$.

If we have parametrized action space $X_\nu(x_{-\nu}) = \{y_\nu, g_\nu(y_\nu, x_{-\nu}) \leq 0\}$, we denote the GNEP by GNEP(N, θ_ν, g_ν).

We denote by $X(x)$ the action set $X(x) = X_1(x_{-1}) \times \cdots \times X_N(x_{-N})$. For standard NE, this set does not depend on x.

The following example seems very basic, but in fact it has particular features, one of them is to have four solutions, i.e. four GNEs. Let $N = 2$. The objective functions are

$$\theta_1(x) = (x_1 - 2)^2(x_2 - 4)^4 \text{ and } \theta_2(x) = (x_2 - 3)^2(x_1)^4,$$

for $x \in \mathbb{R}^2$, while the constraint function are

$$g_1(x) = x_1 + x_2 - 1 \leq 0 \text{ and } g_2(x) = 2x_1 + x_2 - 2 \leq 0.$$

2 GNEP as a nonsmooth equation

From Facchinei et al. (2009), assuming differentiability and a constraint qualification hold, the first-order necessary conditions of player ν's subproblem state there exists a Lagrangian multiplier NA

$$\nabla_{x_\nu}\theta_{x_\nu}(x) + \sum_{1\leq j\leq m_\nu} \lambda_{\nu j}\nabla_{x_\nu} g_{\nu j}(x) = 0 \qquad (\in \mathbb{R}^{n_\nu})$$

$$0 \leq \lambda_\nu \perp -g_\nu(x) \leq 0 \qquad (\in \mathbb{R}^{m_\nu})$$

Regrouping the N subproblems, we get the following system.

Definition 2 (eKKT) *For the N optimization subproblems for the functions $\theta_i : \mathbb{R}^n \to \mathbb{R}$, with constraints $g_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$, the KKT conditions can be regrouped such that it exists $\lambda \in \mathbb{R}^m$ and*

$$\tilde{L}(x,\lambda) = 0 \quad and \quad 0 \leq \lambda \perp -G(x) \leq 0,$$

where L and G are given by

$$\tilde{L}(x,\lambda) = \begin{pmatrix} \nabla_{x_1}\theta_1(x) + Jac g_1(x)^T\lambda_1 \\ \vdots \\ \nabla_{x_N}\theta_N(x) + Jac g_N(x)^T\lambda_N \end{pmatrix} \in \mathbb{R}^n \quad and \quad G(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_N(x) \end{pmatrix} \in \mathbb{R}^m,$$

with $Jac g_\nu(x)^T\lambda_\nu = \sum_{1\leq j\leq m_\nu} \lambda_{\nu j}\nabla_{x_\nu} g_{\nu j}(x)$. The extended KKT system is denoted by eKKT(N, θ_i, g_i).

Using complementarity function $\phi(a,b)$ (e.g. $\min(a,b)$), we get the following nonsmooth equation

$$\Phi(z) = \begin{pmatrix} \tilde{L}(x,\lambda) \\ \phi_.(-G(x),\lambda) \end{pmatrix} = 0,$$

where $\phi_.$ is the component-wise version of the function ϕ and $\tilde{L}$ is the Lagrangian function of the extended system.

2.1 A classic example

Returning to our example, we define the Φ as

$$\Phi(x) = \begin{pmatrix} 2(x_1-2)(x_2-4)^4 + \lambda_1 \\ 2(x_2-3)(x_1)^4 + \lambda_2 \\ \phi(\lambda_1, 1-x_1-x_2) \\ \phi(\lambda_2, 2-2x_1-x_2) \end{pmatrix},$$

where ϕ denotes a complementarity function. In R, we use

```
> F <- function(z, phi=phiMin, ...)
+ {
```

```
+         x <- z[1:2]
+         lambda <- z[3:4]
+         # cat("x", x, "\n")
+         c(        2*(x[1] - 2)*(x[2]-4)^4 + lambda[1],
+                 2*(x[2] - 3)*x[1]^4 + lambda[2],
+                 phi(lambda[1], 1-sum(x), ...),
+                 phi(lambda[2], 2-2*x[1]-x[2], ...)
+         )
+ }
```

Note that the triple dot arguments ... is used to pass arguments to the complementarity function.

Elements of the generalized Jacobian of Φ have the following form

$$\partial\Phi(x) = \left\{ \begin{pmatrix} 2(x_2-4)^4 & 8(x_1-2)(x_2-4)^3 & 1 & 0 \\ 8(x_2-3)(x_1)^3 & 2(x_1)^4 & 0 & 1 \\ -\phi_b'(\lambda_1, 1-x_1-x_2) & -\phi_b'(\lambda_1, 1-x_1-x_2) & \phi_a'(\lambda_1, 1-x_1-x_2) & 0 \\ -2\phi_b'(\lambda_2, 2-2x_1-x_2) & -\phi_b'(\lambda_2, 2-2x_1-x_2) & 0 & \phi_a'(\lambda_2, 2-2x_1-x_2) \end{pmatrix} \right\},$$

where ϕ_a' and ϕ_b' denote elements of the generalized gradient of the complementarity function. The corresponding R code is

```
> JacF <- function(z, gphia, gphib, ...)
+ {
+         x <- z[1:2]
+         lambda <- z[3:4]
+         idga1 <- gphia(lambda[1], 1- sum(x), ...)
+         idgb1 <- gphib(lambda[1], 1- sum(x), ...)
+         idga2 <- gphia(lambda[2], 2-2*x[1]-x[2], ...)
+         idgb2 <- gphib(lambda[2], 2-2*x[1]-x[2], ...)
+
+ rbind(
+ c(2*(x[2]-4)^4, 8*(x[1] - 2)*(x[2]-4)^3, 1, 0),
+ c(8*(x[2] - 3)*x[1]^3, 2*x[1]^4, 0, 1),
+ c(-idgb1, -idgb1, idga1, 0),
+ c(-2*idgb2, -idgb2, 0, idga2)
+ )
+ }
```

2.1.1 Usage example

Therefore, to compute a generalized Nash equilibrium, we use

```
> z0 <- c(10, 10, 1, 1)
> GNE.nseq(z0, F, JacF, list(phi=phiMin), list(gphia= GrAphiMin, gphib= GrBphiMin))
```

```
$par
[1] 1.000000e+00 3.582012e-13 5.120000e+02 6.000000e+00

$value
[1] 1.431037e-12

$counts
phicnt jaccnt
    14      5

$iter
[1] 5

$code
[1] 1

$message
[1] "Function criterion near zero"

$fvec
[1] -4.547474e-13 -1.347367e-12 -1.409983e-13  7.611805e-14
```

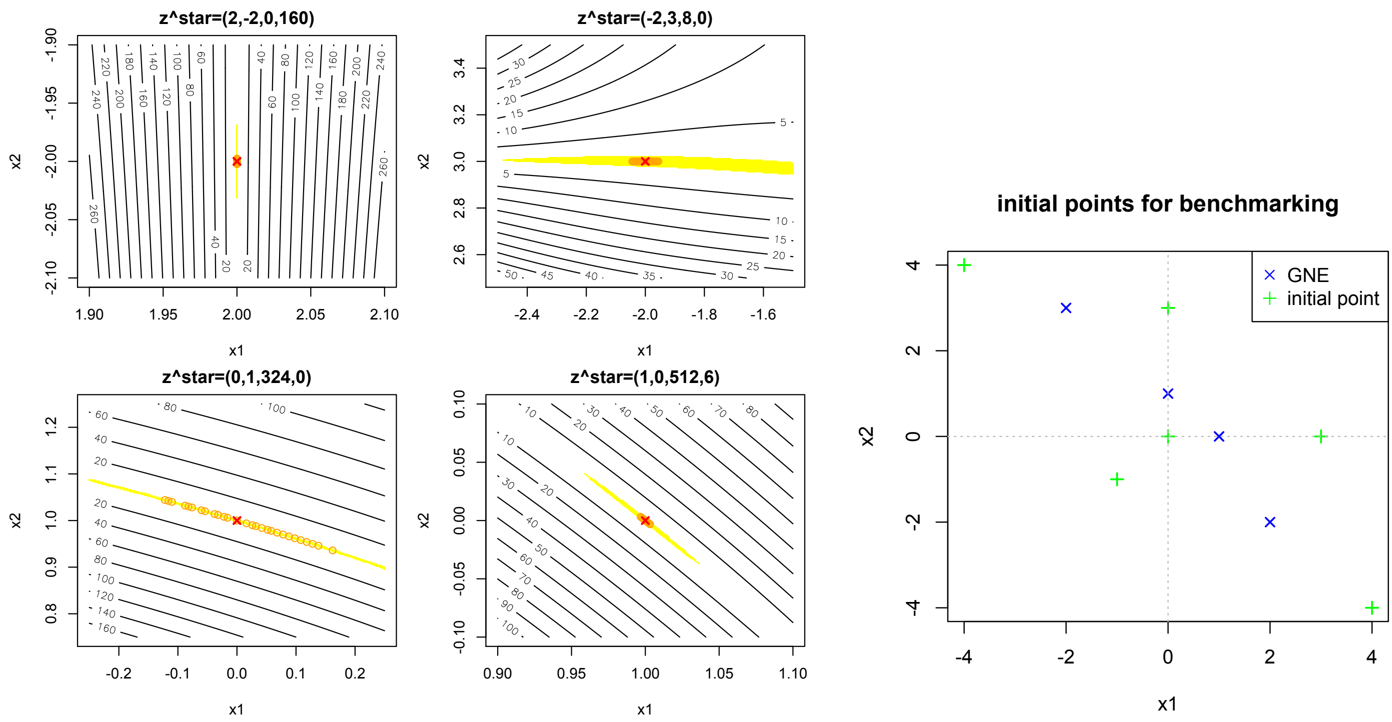
Recalling that the true GNEs are

```
> #list of true GNEs
> trueGNE <- rbind(c(2, -2, 0, 5*2^5),
+         c(-2, 3, 8, 0),
+         c(0, 1, 4*3^4, 0),
+         c(1, 0, 2^9, 6))
> colnames(trueGNE) <- c("x1", "x2", "lam1", "lam2")
> rownames(trueGNE) <- 1:4
> print(trueGNE)
```

```
  x1 x2 lam1 lam2
1  2 -2    0  160
2 -2  3    8    0
3  0  1  324    0
4  1  0  512    6
```

2.1.2 Localization of the GNEs

On figure 1a, we draw contour plots of the function $\frac{1}{2}||\Phi(z)||^2$ with respect to x_1 and x_2, given λ_1 and λ_2. The second figure 1b just plots the initial points and the 6 GNEs.



(a) The 4 GNEs

(b) The 6 initial points

Figure 1: Contour plots of the norm of Φ

2.2 Benchmark of the complementarity functions and the computation methods

Using the following function, we compare all the different methods with different initial points and different complementarity functions. We consider the following complementarity functions.

- $\phi_{Min}(a, b) = \min(a, b)$,
- $\phi_{FB}(a, b) = \sqrt{a^2 + b^2} - (a + b)$,
- $\phi_{Man}(a, b) = f(|a - b|) - f(a) - f(b)$ and $f(t) = t^3$,
- $\phi_{LT}(a, b) = (a^q + b^q)^{\frac{1}{q}} - (a + b)$ and $q = 4$,
- $\phi_{KK}(a, b) = (\sqrt{(a - b)^2 + 2\lambda ab} - (a + b))/(2 - \lambda)$ and $\lambda = 3/2$.

Firstly, we define a function calling the benchmark function for the five complementarity functions under consideration.

```
> wholebench <- function(z0)
+ {
+ #min function
+ resMin <- bench.GNE.nseq(z0, F, JacF, argPhi=list(phi=phiMin), argjac=list(gphia= GrAphiMin, gph
```

```
+
+ #FB function
+ resFB <- bench.GNE.nseq(z0, F, JacF, argPhi=list(phi=phiFB), argjac=list(gphia= GrAphiFB, gphib=
+
+ #Mangasarian function
+ resMan <- bench.GNE.nseq(z0, F, JacF, argPhi=list(phi=phiMan, f=function(t) t^3), argjac=list(gp
+
+ #LT function
+ resLT <- bench.GNE.nseq(z0, F, JacF, argPhi=list(phi=phiLT, q=4), argjac=list(gphia= GrAphiLT, g
+
+ #KK function
+ resKK <- bench.GNE.nseq(z0, F, JacF, argPhi=list(phi=phiKK, lambda=3/2), argjac=list(gphia= GrAp
+
+ list(resMin=resMin, resFB=resFB, resMan=resMan, resLT=resLT, resKK=resKK)
+ }
```

Then the following call give us a list of result tables.

```
> initialpt <- cbind(c(4, -4), c(-4, 4), c(3, 0), c(0, 3), c(-1, -1), c(0, 0))
> mytablelist <- list()
> for(i in 1: NCOL(initialpt))
+ {
+         z0 <- c(initialpt[, i], 1, 1)
+         mybench <- wholebench(z0)
+
+         cat("z0", z0, "\n")
+
+         mytable12 <- data.frame(method=mybench[[1]]$compres[, 1],
+         round(
+                 cbind(mybench[[1]]$compres[,c(-1, -4)], mybench[[2]]$compres[,c(-1, -4)])
+                 , 3) )
+
+         mytable35 <- data.frame(method=mybench[[1]]$compres[, 1],
+         round(
+                 cbind(mybench[[3]]$compres[,c(-1, -4)], mybench[[5]]$compres[,c(-1, -4)])
+                 , 3) )
+
+         mytablelist <- c(mytablelist, z0=list(z0), MINFB=list(mytable12), MANKK=list(mytable35))
+ }
```

Note that one result table given by the function `bench.GNE.nseq` reports the computation results for 10 methods given an initial point and a complementarity function. Below an example

```
> z0 <- c(-4, 4, 1, 1)
> bench.GNE.nseq(z0, F, JacF, argPhi=list(phi=phiMin), argjac=list(gphia= GrAphiMin, gphib= GrBphi
```

	method	fctcall	jaccall	comptime	X1	X2
1	Newton - pure	7	7	0.003	-2.0000000	3.000000
2	Newton - geom. line search	7	7	0.003	-2.0000000	3.000000
3	Newton - quad. line search	7	7	0.002	-2.0000000	3.000000
4	Newton - Powell trust region	8	7	0.002	-2.0000000	3.000000
5	Newton - Dbl. trust region	8	7	0.002	-2.0000000	3.000000
6	Broyden - pure	35	1	0.003	-2.0000000	3.000000
7	Broyden - geom. line search	30	3	0.003	-2.0000000	3.000000
8	Broyden - quad. line search	26	3	0.003	-2.0000000	3.000000
9	Broyden - Powell trust region	169	6	0.010	0.2145962	1.333424
10	Broyden - Dbl. trust region	182	6	0.010	0.1056156	1.554309

	X3	X4	normFx
1	7.999999	0.0000000	6.760931e-07
2	7.999999	0.0000000	6.760931e-07
3	7.999999	0.0000000	6.760931e-07
4	7.999999	0.0000000	6.760931e-07
5	7.999999	0.0000000	6.760931e-07
6	8.000000	0.0000000	8.801237e-09
7	7.999999	0.0000000	6.459239e-07
8	7.999999	0.0000000	6.459239e-07
9	180.486012	0.2689677	6.546819e-01
10	135.444351	0.4597433	8.443860e-01

The following subsections report the computation for 4 complementarity functions, the Luo-Tseng being discarded due to non convergence. We also remove the final estimates z_n when the method has not converged, $||\Phi(z_n)||^2 \neq 0$. Tables are put in appendix, except the first one.

2.2.1 Initial point $z_0 = (4, -4, 1, 1)$

We work on the initial point $z_0 = (4, -4, 1, 1)$, close the GNE $(2, -2, 0, 160)$. Clearly, we observe the Mangasarian complementarity function ϕ_{Man} does not converge except in the pure Newton method, for which the sequence converges to $(-2, 3, 8, 0)$ quite far from the initial point. So the "Man" sequence converged by a chance! For ϕ_{Min} function, when it converges, the GNEs found are $(2, -2, 0, 160)$ or $(1, 0, 512, 6)$. ϕ_{FB} and ϕ_{KK} associated sequences converge mostly to $(2, -2, 0, 160)$. In terms of function/Jacobian calls, ϕ_{FB} is significantly better when used with the Newton scheme.

2.2.2 Initial point $z_0 = (-4, 4, 1, 1)$

We work on the initial point $z_0 = (-4, 4, 1, 1)$, close the GNE $(-2, 3, 8, 0)$. Again, we observe the Mangasarian complementarity function ϕ_{Man} does not converge. All other sequences converge the closest GNE $(-2, 3, 8, 0)$. ϕ_{Min} sequence with Newton scheme is particularly good, then comes ϕ_{FB} and finally ϕ_{KK}.

	$\phi_{Min}(a,b) = \min(a,b)$							$\phi_{FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$						
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	5	5	1	0	512	6	0	6	6	2	-2	0	160	0
Newton - geom. LS	343	67	1	0	512	6	0	6	6	2	-2	0	160	0
Newton - quad. LS	292	100					2	6	6	2	-2	0	160	0
Newton - Powell TR	64	57	1	0	512	6	0	12	6	2	-2	0	160	0
Newton - Dbl. TR	63	58	1	0	512	6	0	12	6	2	-2	0	160	0
Broyden - pure	100	1					164	100	1					188
Broyden - geom. LS	403	6	1	0	512	6	0	1079	26					2
Broyden - quad. LS	291	6					1	467	3					1
Broyden - Powell TR	22	2	2	-2	0	160	0	114	2					1
Broyden - Dbl. TR	20	2	2	-2	0	160	0	115	2					1
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	113	113	-2	3	8	0	0	48	48	0	1	325	0	0
Newton - geom. LS	203	25					33	727	100					2
Newton - quad. LS	91	27					37	85	39	2	-2	0	160	0
Newton - Powell TR	75	67					3	152	100	0	1	309	0	0
Newton - Dbl. TR	62	53					3	147	100	0	1	304	0	0
Broyden - pure	200	1					506	49	1	1	0	512	6	0
Broyden - geom. LS	167	6					82	29	3	2	-2	0	160	0
Broyden - quad. LS	86	5					78	20	3	2	-2	0	160	0
Broyden - Powell TR	215	14					3	28	2	2	-2	0	160	0
Broyden - Dbl. TR	246	15					3	29	2	2	-2	0	160	0
	$\phi_{Man}(a,b) = f(\|a-b\|) - f(a) - f(b)$ and $f(t) = t^3$							$\phi_{KK}(a,b) = (\sqrt{(a-b)^2 + 2\lambda ab} - (a+b))/(2-\lambda)$ and $\lambda = 3/2$						

Table 1: With initial point $z_0 = (4, -4, 1, 1)$ close to $(2, -2, 0, 160)$

2.2.3 Initial point $z_0 = (3, 0, 1, 1)$

We work on the initial point $z_0 = (3, 0, 1, 1)$ close to the GNE $(1, 0, 512, 6)$. As always, the "Man" sequence converges by chance with the pure Newton method to a GNE $(-2, 3, 8, 0)$. Otherwise the other sequences, namely "Min", "FB" and "KK" converges to the expected GNE. As the previous subsection, Broyden updates of the Jacobian is less performant than the true Jacobian (i.e. Newton scheme). The convergence speed order is preserved.

2.2.4 Initial point $z_0 = (0, 3, 1, 1)$

We work on the initial point $z_0 = (0, 3, 1, 1)$ close to the GNE $(0, 1, 324, 0)$. As always, the "Man" sequence converges by chance with the pure Newton method to a GNE $(-2, 3, 8, 0)$. Others sequences have difficulty to converge the closest GNE. Local methods (i.e. pure) find the GNE $(0, 1, 324, 0)$, while global version converges to $(1, 0, 512, 6)$. It is logical any method will have difficulty to choose between these two GNEs, because they are close.

2.2.5 Initial point $z_0 = (-1, -1, 1, 1)$

We work on the initial point $z_0 = (-1, -1, 1, 1)$ equidistant to the GNEs $(0, 1, 324, 0)$ and $(1, 0, 512, 6)$. Despite being closer to these GNEs, the pure Newton version of the "Man" sequence converges unconditionally to the GNE $(-2, 3, 8, 0)$. All other sequences converges to the GNE $(0, 1, 324, 0)$ except for the Broyden version of the "KK" sequence, converging to the farthest GNEs. In terms of function calls, the Newton line search version of the "Min" sequence is the best, followed by the Newton trust region version of the "FB" sequence.

2.2.6 Initial point $z_0 = (0, 0, 1, 1)$

We work on the initial point $z_0 = (0, 0, 1, 1)$ equidistant to the GNEs $(0, 1, 324, 0)$ and $(1, 0, 512, 6)$. Both the "Man" and the "Min" sequences do not converge. The "Min" sequence diverges because the Jacobian at the initial point is exactly singular. Indeed, we have

```
> z0 <- c(0, 0, 1, 1)
> JacF(z0, gphia= GrAphiMin, gphib= GrBphiMin)
```

```
     [,1] [,2] [,3] [,4]
[1,]  512 1024    1    0
[2,]    0    0    0    1
[3,]   -1   -1    1    0
[4,]    0    0    0    1
```

For the "FB" and "KK" sequences, we do not have this problem.

```
> JacF(z0, gphia= GrAphiFB, gphib= GrBphiFB)
```

```
            [,1]         [,2]       [,3]       [,4]
[1,] 512.0000000 1024.0000000  1.0000000  0.0000000
[2,]   0.0000000    0.0000000  0.0000000  1.0000000
[3,]   0.2928932    0.2928932 -0.2928932  0.0000000
[4,]   0.2111456    0.1055728  0.0000000 -0.5527864
```

```
> JacF(z0, gphia= GrAphiKK, gphib= GrBphiKK, lambda=3/2)
```

```
            [,1]         [,2]       [,3]       [,4]
[1,] 512.0000000 1024.0000000  1.0000000  0.0000000
[2,]   0.0000000    0.0000000  0.0000000  1.0000000
[3,]   0.2679492    0.2679492 -0.2679492  0.0000000
[4,]   0.2203553    0.1101776  0.0000000 -0.4881421
```

So the sequence converge to a GNE, either $(0, 1, 324, 0)$ or $(-2, 3, 8, 0)$. Again the "KK" sequence converges faster.

2.2.7 Conclusions

In conclusion to this analysis with respect to initial point, the computation method and the complementarity function, we observe the strong difference in terms of convergence, firstly and in terms of convergence speed.

Clearly the choice of the complementarity function is crucial, the Luo-Tseng and the Mangasarian are particularly inadequate in our example. Regarding the remaining three complementarity functions (the minimum, the Fisher-Burmeister and the Kanzow-Kleinmichel functions) generally converge irrespectively of the computation method. However, the “KK” sequences are particularly efficient and most of the time the Newton trust region method is the best in terms of function/Jacobian calls.

3 GNEP as a fixed point equation

4 GNEP as a gap minimization problem

References

Facchinei, F., Fischer, A. & Piccialli, V. (2009), 'Generalized Nash equilibrium problems and Newton methods', Math. Program., Ser. B **117**, 163–194. 2

A Tables for the nonsmooth reformulation

	$\phi_{Min}(a,b) = \min(a,b)$							$\phi_{FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$						
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	7	7	-2	3	8	0	0	9	9	-2	3	8	0	0
Newton - geom. LS	7	7	-2	3	8	0	0	10	9	-2	3	8	0	0
Newton - quad. LS	7	7	-2	3	8	0	0	11	10	-2	3	8	0	0
Newton - Powell TR	8	7	-2	3	8	0	0	13	10	-2	3	8	0	0
Newton - Dbl. TR	8	7	-2	3	8	0	0	13	10	-2	3	8	0	0
Broyden - pure	35	1	-2	3	8	0	0	35	1	-2	3	8	0	0
Broyden - geom. LS	30	3	-2	3	8	0	0	39	4	-2	3	8	0	0
Broyden - quad. LS	26	3	-2	3	8	0	0	30	4	-2	3	8	0	0
Broyden - Powell TR	169	6					1	26	2	-2	3	8	0	0
Broyden - Dbl. TR	182	6					1	28	2	-2	3	8	0	0
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	200	200					53	11	11	-2	3	8	0	0
Newton - geom. LS	66	10					4	11	10	-2	3	8	0	0
Newton - quad. LS	25	9					3	19	14	-2	3	8	0	0
Newton - Powell TR	47	40					3	10	10	-2	3	8	0	0
Newton - Dbl. TR	44	36					3	10	10	-2	3	8	0	0
Broyden - pure	200	1					73	39	1	-2	3	8	0	0
Broyden - geom. LS	1045	25					3	75	3	-2	3	8	0	0
Broyden - quad. LS	253	11					4	42	3	-2	3	8	0	0
Broyden - Powell TR	156	12					3	33	3	-2	3	8	0	0
Broyden - Dbl. TR	108	8					3	36	2	-2	3	8	0	0
	$\phi_{Man}(a,b) = f(\|a-b\|) - f(a) - f(b)$ and $f(t) = t^3$							$\phi_{KK}(a,b) = (\sqrt{(a-b)^2 + 2\lambda ab} - (a+b))/(2-\lambda)$ and $\lambda = 3/2$						

Table 2: With initial point $z_0 = (-4,4,1,1)$ close to $(-2,3,8,0)$

	$\phi_{Min}(a,b) = \min(a,b)$							$\phi_{FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$						
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	22	22	0	1	325	0	0	21	21	2	-2	0	160	0
Newton - geom. LS	25	24	0	1	325	0	0	25	24	0	1	325	0	0
Newton - quad. LS	13	9	2	-2	0	160	0	197	100	0	1	345	0	0
Newton - Powell TR	26	24	0	1	325	0	0	12	8	1	0	512	6	0
Newton - Dbl. TR	26	24	0	1	325	0	0	13	9	1	0	512	6	0
Broyden - pure	6	1					2e+25	100	1					4
Broyden - geom. LS	58	3	1	0	512	6	0	639	4					12
Broyden - quad. LS	389	3	1	0	512	6	0	187	3	1	0	512	6	0
Broyden - Powell TR	164	2	0	1	376	0	0	133	2					1
Broyden - Dbl. TR	144	3	0	1	343	0	0	138	2					1
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	76	76	-2	3	8	0	0	66	66	-2	3	8	0	0
Newton - geom. LS	176	17					50	45	17	2	-2	0	160	0
Newton - quad. LS	1121	200					113	130	45	2	-2	0	160	0
Newton - Powell TR	72	61					3	38	25	2	-2	0	160	0
Newton - Dbl. TR	64	55					3	41	26	2	-2	0	160	0
Broyden - pure	200	1					5.9e6	85	1					393
Broyden - geom. LS	349	9					64	806	3					3
Broyden - quad. LS	123	9					58	202	3	1	0	512	6	0
Broyden - Powell TR	101	6	-2	3	8	0	0	121	4	1	0	512	6	0
Broyden - Dbl. TR	180	14					3	157	6	2	-2	0	160	0
	$\phi_{Man}(a,b) = f(\|a-b\|) - f(a) - f(b)$ and $f(t)=t^3$							$\phi_{KK}(a,b) = (\sqrt{(a-b)^2 + 2\lambda ab} - (a+b))/(2-\lambda)$ and $\lambda = 3/2$						

Table 3: With initial point $z_0 = (3,0,1,1)$ close to $(1,0,512,6)$

	$\phi_{Min}(a,b) = \min(a,b)$							$\phi_{FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$						
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	22	22	0	1	325	0	0	5	5	1	0	512	6	0
Newton - geom. LS	522	82	1	0	512	6	0	779	100					1
Newton - quad. LS	300	100					2	366	100					1
Newton - Powell TR	88	67	1	0	512	6	0	94	73	1	0	512	6	0
Newton - Dbl. TR	102	80	1	0	512	6	0	87	80	0	1	323	0	0
Broyden - pure	8	1					2e+20	100	1					617
Broyden - geom. LS	746	3					3	844	3					1
Broyden - quad. LS	312	3					3	345	3					1
Broyden - Powell TR	169	4					1	40	2	-2	3	8	0	0
Broyden - Dbl. TR	158	1					2	35	2	-2	3	8	0	0
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	136	136	-2	3	8	0	0	24	24	0	1	325	0	0
Newton - geom. LS	156	14					14	762	100					2
Newton - quad. LS	33	8					14	358	100					1
Newton - Powell TR	31	25					3	90	72	1	0	512	6	0
Newton - Dbl. TR	35	29					3	89	76	1	0	512	6	0
Broyden - pure	30	1					9e+33	18	1					4e+21
Broyden - geom. LS	327	10					13	659	4					1
Broyden - quad. LS	139	10					13	326	3					1
Broyden - Powell TR	175	14					3	35	2	-2	3	8	0	0
Broyden - Dbl. TR	130	11					3	53	2	-2	3	8	0	0
	$\phi_{Man}(a,b) = f(\|a-b\|) - f(a) - f(b)$ and $f(t)=t^3$							$\phi_{KK}(a,b) = (\sqrt{(a-b)^2 + 2\lambda ab} - (a+b))/(2-\lambda)$ and $\lambda = 3/2$						

Table 4: With initial point $z_0 = (0,3,1,1)$ close to $(0,1,324,0)$

	$\phi_{Min}(a,b) = \min(a,b)$							$\phi_{FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$						
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	21	21	0	1	323	0	0	9	9	0	1	324	0	0
Newton - geom. LS	21	21	0	1	323	0	0	194	52	0	1	323	0	0
Newton - quad. LS	21	21	0	1	323	0	0	154	68	0	1	323	0	0
Newton - Powell TR	84	70	0	1	323	0	0	32	30	0	1	323	0	0
Newton - Dbl. TR	78	70	0	1	323	0	0	32	30	0	1	323	0	0
Broyden - pure	100	1	0	1	307	0	0	37	1					408
Broyden - geom. LS	49	3	0	1	323	0	0	407	6	1	0	512	6	0
Broyden - quad. LS	56	4	0	1	302	0	0	324	6					1
Broyden - Powell TR	169	5					1	183	4					1
Broyden - Dbl. TR	172	5					1	191	4					1
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	59	59	-2	3	8	0	0	18	18	0	1	323	0	0
Newton - geom. LS	67	11					5	170	48	0	1	323	0	0
Newton - quad. LS	42	14					5	132	60	0	1	323	0	0
Newton - Powell TR	55	49					3	87	72	1	0	512	6	0
Newton - Dbl. TR	46	40					3	87	79	0	1	323	0	0
Broyden - pure	200	1					6	100	1					168
Broyden - geom. LS	836	14					5	50	4	-2	3	8	0	0
Broyden - quad. LS	89	9					3	43	3	-2	3	8	0	0
Broyden - Powell TR	113	10					3	146	6	2	-2	0	160	0
Broyden - Dbl. TR	98	7					3	136	7	2	-2	0	160	0
	$\phi_{Man}(a,b) = f(\|a-b\|) - f(a) - f(b)$ and $f(t) = t^3$							$\phi_{KK}(a,b) = (\sqrt{(a-b)^2 + 2\lambda ab} - (a+b))/(2-\lambda)$ and $\lambda = 3/2$						

Table 5: With initial point $z_0 = (-1,-1,1,1)$

	$\phi_{Min}(a,b) = \min(a,b)$							$\phi_{FB}(a,b) = \sqrt{a^2+b^2} - (a+b)$						
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	0	1					1023	10	10	-2	3	8	0	0
Newton - geom. LS	0	1					1023	9	8	-2	3	8	0	0
Newton - quad. LS	0	1					1023	10	9	-2	3	8	0	0
Newton - Powell TR	0	1					1023	140	100					1
Newton - Dbl. TR	0	1					1023	150	94	0	1	324	0	0
Broyden - pure	0	1					1023	100	1					1
Broyden - geom. LS	0	1					1023	21	2	-2	3	8	0	0
Broyden - quad. LS	0	1					1023	16	2	-2	3	8	0	0
Broyden - Powell TR	0	1					1023	40	2	-2	3	8	0	0
Broyden - Dbl. TR	0	1					1023	133	1					2
	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$	fctcall	jaccall	x_1	x_2	λ_1	λ_2	$\|\|\Phi(z)\|\|$
Newton - pure	200	200					39	43	43	0	1	325	0	0
Newton - geom. LS	91	11					6	32	26	0	1	325	0	0
Newton - quad. LS	25	9					6	77	46	0	1	325	0	0
Newton - Powell TR	43	39					3	70	52	0	1	325	0	0
Newton - Dbl. TR	42	39					3	137	100	0	1	333	0	0
Broyden - pure	200	1					489	35	1	-2	3	8	0	0
Broyden - geom. LS	276	7					3	274	8	0	1	325	0	0
Broyden - quad. LS	124	6					3	253	7	0	1	352	0	0
Broyden - Powell TR	135	12					3	157	3					2
Broyden - Dbl. TR	169	13					3	163	2					1
	$\phi_{Man}(a,b) = f(\|a-b\|) - f(a) - f(b)$ and $f(t) = t^3$							$\phi_{KK}(a,b) = (\sqrt{(a-b)^2 + 2\lambda ab} - (a+b))/(2-\lambda)$ and $\lambda = 3/2$						

Table 6: With initial point $z_0 = (0,0,1,1)$